Ruin Theory

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by

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Abstract

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Classical ruin theory was developed by Lundberg in 1907 and refined by Cramer in 1930. This theory describes the evolution of the surplus of an insurance company over time. It assumes that an insurance company begins with an initial surplus and then receives premiums continuously at a constant rate. It also assumes that claims of random and independent size are paid at random and independent times. Ruin occurs when the surplus becomes negative meaning that the average inflow of money (premiums) is smaller than the average outflow of money (claims). Cramer expanded on this theory to show that probability of ruin decays exponentially fast as the initial surplus grows larger. This paper will synthesize some of the key results from Ruin Theory. These results will not be proven via formula but will be conclusively demonstrated using simulation.

Chapter 1: Introduction

Ruin theory uses mathematical models to describe an insurer's vulnerability to insolvency and/or ruin. In order to ensure the sustainability of an insurance operation, one must routinely assess the risk associated with the portfolio of insurance contracts. Ruin is defined when the surplus of the policy, portfolio, or company becomes negative. The amount of surplus is equivalent to the quantity of interest of a policy or portfolio. Ruin modeling is necessary for long-run financial planning and maintenance.

Two processes, the discrete time process and the continuous time process, can be utilized when viewing the evolution of the portfolio over time. The continuous time process may be defined as the total losses paid from time 0 to time *t*. The discrete time process can be derived from the continuous time process by only viewing the loss values at integral times.

For both discrete and continuous modeling techniques, we'll need to define the following parameters. *Surplus* represents excess funds that would not be needed if the policy and/or portfolio were terminated. $U_o = u$, the *Initial Surplus,* represents the initial surplus at time 0. We are trying to measure $\{U_t : t \geq 0\}$, the *Surplus Process,* at time *t*. $\{P_t : t \geq 0\}$, the *Premium Process,* measures all premiums collected up to time *t* and

$\{S_t : t \geq 0\}$, the *Loss Process,* measures all losses paid up to time *t*. We are now able to define the Surplus Process as:

$$U_t = U_o + P_t - S_t.$$

We will first examine the discrete time model. The increment in the surplus process in year *t* may be defined as:

$$W_t = P_t - P_{t-1} - S_t + S_{t-1},\ t = 1, 2, \ldots$$

Then the *progression of surplus* is:

$$U_t = U_{t-1} + W_t\ ,\ t = 1, 2, \ldots$$

Because W_t depends on P_t, we are able to pay dividends based on the surplus at the end of the previous year. The method of computing U_t using a discrete-time model will be examined in Chapter 2.

The continuous time model presents greater difficulty because we must understand the surplus at every point in time rather than just a countable set of time points. The Compound Poisson claim process is typically used for continuous time analyses. In this process premiums are collected at a constant continuous nonrandom rate and the total loss process is:

$$S_t = X_1 + \ldots + X_{N_t},$$

where $\{N_t : t \geq 0\}$ is the Poisson process. We will examine continuous time modeling techniques in Chapter 3.

Ruin Theory helps us understand whether a portfolio will survive over time. The probability of survival can be defined in four different ways:

i. Continuous – time, infinite – horizon survival probability is depicted by:

$\phi(u) = \Pr(U_t \geq 0 \text{ for all } t \geq 0 \,;\, U_0 = u)$.

ii. Discrete – time, finite – horizon survival probability is depicted by:

$\phi(u,\ \tau) = \Pr(U_t \geq 0 \text{ for all } t = 0, 1, \ldots, \tau \,;\, U_0 = u)$.

iii. Continuous – time, finite – horizon survival probability is depicted by:

$\phi(u,\ \tau) = \Pr(U_t \geq 0 \text{ for all } 0 \leq t \leq \tau \,;\, U_0 = u)$.

iv. Discrete – time, infinite – horizon survival probability is depicted by:

$\phi(u) = \Pr(U_t \geq 0 \text{ for all } t = 0, 1, \ldots \,;\, U_0 = u)$.

The continuous – time, infinite – horizon survival probability requires that we continuously check the surplus and expect the portfolio to survive forever. Because both these requirements are unrealistic, the discrete – time, finite – horizon survival probability assessment is more practical. In this case the portfolio is required to survive for a specific number of τ periods and we only check surplus at the end of each period. However, if the Poisson process holds, infinite – horizon probabilities are also easily attainable. As the number of times per year that surplus is checked increases, the discrete – time survival probabilities converge to their continuous – time counterparts. Finally, we may define the continuous – time, infinite - horizon ruin probability as:

$$\psi(u) = 1 - \phi(u).$$

Chapter 2: Discrete Models

In this chapter we will examine discrete, finite – time ruin probabilities. Let P_t represent the premium collected in the t^{th} period, S_t represent the losses paid in the t^{th} period, and C_t represent any cash flow other than the collection of premiums and the payment of losses. The surplus at the end of the t^{th} period is:

$$U_t = U_{t-1} + P_t + C_t - S_t.$$

Assume that the random variable $W_t = P_t + C_t - S_t$ depends only on U_{t-1} and not on any other previous experience. In order to evaluate ruin probability, we'll define a new process U^*_t which begins with $U^*_o = u$ and assumes the following:

$$W^*_t = 0 \text{ if } U^*_{t-1} < 0$$

$$W^*_t = W_t \text{ if } U^*_{t-1} \geq 0$$

$$U^*_t = U^*_{t-1} + W^*_t.$$

Note that if $U^*_t < 0$, then $U^*_q < 0$ for all $q > t$. The finite – horizon survival probability is

$$\phi(u, \tau) = \Pr(U^*_\tau \geq 0).$$

Example 1:

Consider a process with initial surplus of $U_0 = 2$, a fixed annual premium of $P_t = 2$, and losses of $S_t = 0$ or $S_t = 5$ with probabilities of .7 and .3, respectively. There are no other cash flows. We will determine $\phi(2, 2)$. It is evident that surplus in year 1 equals:

$$U_1 = U_0 + P_1 - S_1 = 2 + 2 - 0 = 4, \text{ and}$$

$$U_1 = 2 + 2 - 5 = -1$$

with probabilities of .7 and .3, respectively. In every year, W_t takes the values of 2 and -3 with probabilities of .7 and .3. For example, calculations W_1 and W_2 are as follows:

$$W_1 = P_1 - P_0 - S_1 + S_0 = 2 - 0 - 0 + 0 = 2$$

$$W_1 = 2 - 0 - 5 + 0 = -3$$

$$W_2 = P_2 - P_1 - S_2 + S_1 = 4 - 2 - 0 + 0 = 2$$

$$W_2 = 4 - 2 - 10 + 5 = -3.$$

For year 2, there are four possible ways for the process to end as shown below:

Table 1: Discrete Models Example 1

Case	**U_1**	**W_2**	**W^*_2**$= W_2$ if $U^*_1 = 4$	**U^*_2** $= U^*_1 + W^*_2$	**Probability**
1	4	2	2	6	.7*.7 = .49
2	4	-3	-3	1	.7*.3 = .21

3	-1	2	0	-1	.3*.7 = .21
4	-1	-3	0	-1	.3*.3 = .09

We can see that $\phi(2, 2) = .49 + .21 = .70$. Note that we only need to check U^*_t at time τ because once ruined, the process is not allowed to become positive.

Example 2:

We will now evaluate the probability of ruin for a discrete and finite distribution. Consider the following assumptions:

1) Annual losses are 0, 1, 2, and 3 with probabilities .4, .3, .2, and .1 respectively,

2) $U_0 = 2$,

3) $P_t = .5$

4) Interest is earned at 15% on any surplus available at the beginning of the year because losses were paid at the end of the year.

We will determine the survival probability at the end of the first year. The first year ends with four possible surplus values as noted below.

Table 2: Discrete Models Example 2

$U_1 = (U_0 + P_1)(1.15) - S_1$	**Probability**
(2 + .5)(1.15) – 0 = 2.875	.4
(2 + .5)(1.15) – 1 = 1.875	.3
(2 + .5)(1.15) – 2 = .875	.2
(2 + .5)(1.15) – 3 = -.125	.1

The only case producing ruin is the last one and thus $\psi(2,1) = .1$. In the next chapter, we will begin to examine continuous distributions.

Chapter 3: Continuous Models and the Adjustment Coefficient

When working with continuous-time ruin models let us assume the number of claims have a Poisson distribution. The Poisson process $\{N_t : t \geq 0\}$ represents the number of claims on a portfolio of policies. The Poisson process has the following 3 properties: a) $N_0 = 0$, b) stationary and independent increments, and c) the number of claims in an interval of length t is Poisson distributed with mean λt. The aggregate model of the claim payments becomes the compound Poisson process. The total Loss process $\{S_t : t \geq 0\}$ is a *compound Poisson process* for fixed t if the following criteria is met: a) $\{N_t : t \geq 0\}$ is a Poisson process with rate λ, b) the individual losses $\{X_1, X_2, \ldots\}$ are independent and identically distributed positive random variables, independent of N_t, each with cumulative distribution function $F(x)$ and mean $\mu < \infty$, and c) S_t is the total loss in $(0,t]$ and it is given by $S_t = 0$ if $N_t = 0$ and $S_t = \sum X_j$ if $N_t > 0$. Note that

$$E(S_t) = E(N_t)\, E(X_j) = (\lambda t)(\mu) = \lambda \mu t.$$

We'll let c be the premium income per unit time and thus the total net premium in $(0,t]$ is ct. Assuming that the net premium has positive loading, that is, the insurer's premium income (per unit time) is greater than the expected number of outgoing claims (per unit time), we see that $c > \lambda\mu$. Thus let

$$c = (1+\theta)\lambda\mu,$$

where $\theta > 0$ is called the premium loading factor. The surplus process is now defined as:

$$U_t = u + ct - S_t,$$

where $u = U_0$ is initial surplus. Ruin occurs if U_t ever becomes negative. The infinite – time survival probability is:

$$\phi(u) = \Pr(U_t \geq 0 \text{ for all } t \geq 0 \text{ ; } U_0 = u).$$

Finally, the infinite – time ruin probability is:

$$\psi(u) = 1 - \phi(u).$$

Assuming a parameter κ > 0, which we will define later, exists, *Lundberg's Inequality* states that the probability of ruin $\psi(u)$ satisfies:

$$\psi(u) \leq e^{-\kappa u}, \; u \geq 0.$$

κ is known as the *adjustment coefficient* or *Lundberg's exponent*. If the adjustment coefficient exists, Lundberg's Inequality allows us to obtain an upper bound for the probability of ruin. It's evident that a larger adjustment coefficient value implies smaller ruin probabilities.

The value of the adjustment coefficient depends on the distribution of aggregate claims and the rate of premium income c. Formally, the adjustment coefficient exists if there is a value κ such that $t = \kappa$ is the smallest positive solution to the equation

$$\lambda M_X(t) = \lambda - ct.$$

Since we defined $c = (1 + \theta)\lambda\mu$, the above can be rewritten as:

$$M_X(t) = 1 + (1+\theta)\mu t,$$

where $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} dF(x)$ is the moment generating function of the claim severity random variable X. Note that $M_X(t)$ is strictly convex since $M^{II}_X(t) = E(X^2 e^{tX}) > 0$, $M^I_X(0) = E(Xe^{(0)X}) = E(X) < (1+\theta)\mu$ and $M_X(t)$ increases ∞ continuously. Thus if it exists, the solution $t = \kappa$ is unique and strictly positive per the illustration below.

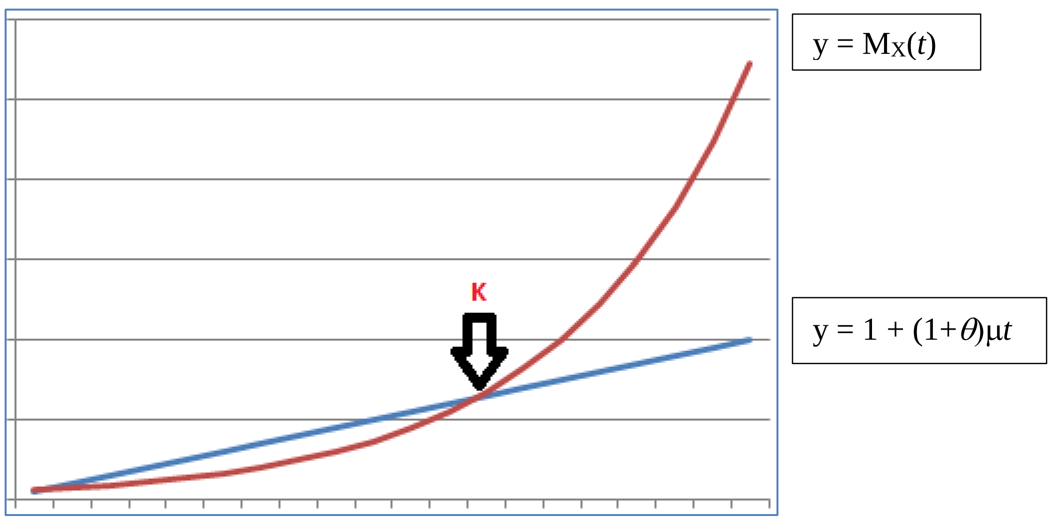

Figure 1: Adjustment Coefficient Graph

In general, it is not possible to explicitly solve for κ and we often need to form an initial guess. The following can be used to form an initial value of κ:

$$\kappa < (2\theta\mu)/E(X^2).$$

This initial value can be seen by expanding the mgf as follows:

$$1 + (1+\theta)\mu\kappa = E(e^{\kappa X}) > E(1 + \kappa X + .5\kappa^2 X^2) = 1 + \kappa\mu + .5\kappa^2 E(X^2).$$

Additionally, we'll define $H(t) = 1 + (1+\theta)\mu t - M_X(t)$. The *Newton – Raphson formula* can be used to solve $H(t) = 0$ by the iteration:

$$\kappa_{j+1} = \kappa_j - [H(\kappa_j)/H'(\kappa_j)].$$

Example 3:

We will now determine an adjustment coefficient given the following conditions: 1) Poisson parameter is $\lambda = 3$, 2) premium rate is $c = 5$, and 3) the individual loss amount distribution is given by $\Pr(X = 1) = .7$ and $\Pr(X = 2) = .3$. We have:

$$\mu = E(X) = (1)(.7) + (2)(.3) = 1.3 \text{ and}$$

$$E(X^2) = (1)^2(.7) + (2)^2(.3) = 1.9.$$

Then $\theta = c(\lambda\mu)^{-1} - 1 = 5(3.9)^{-1} - 1 = .282$. We know κ must be less than $\kappa_o = (2\theta\mu)/E(X^2) = 2(.282)(1.3)/1.9 = .3859$. Thus, our initial guess is $\kappa_o = .3859$.

Since $M_X(t) = .7e^t + .3e^{2t}$, we have the following per the definition above:

$$H(t) = 1 + (1.667t) - 7e^t - .3e^{2t}.$$

Additionally, because $M'_X(t) = (1e^t)(.7) + (2e^{2t})(.3)$, we also have:

$$H^{I}(t) = 1.667 - .7e^{t} - .6e^{2t}.$$

Then $H(\kappa_o) = -0.0355$ and $H^{I}(\kappa_o) = -.6609$. Thus by the Newton-Raphson formula, our updated estimate is $\kappa_1 = .3859 – (-.0355/-.6609) = .3322$.

Then $H(\kappa_1) = -.0050$ and $H^{I}(\kappa_1) = - 4748$. Completing another iteration of the Newton-Raphson formula, we see that κ now equals:

$$\kappa_2 = .3322 – (-.0050/-.4745) = .3215.$$

Continuing in this fashion, we get $\kappa_3 = .3212$, $\kappa_4 = .3212$, and $\kappa_5 = .3212$. Thus, the adjustment coefficient is $\kappa = .3212$ to four decimal places of accuracy.

Example 4:

Let us examine another example based on the following conditions: a) c = 3, b) $\lambda = 4$, and c) the individual loss size density is $f(x) = e^{-2x} + (3/2)e^{-3x}$, $x > 0$ (*Loss Models* 284). The individual loss size density can be re-written as $f(x) = (1/2)(2e^{-2x}) + (1/2)(3e^{-3x})$. Using integration by parts, we see that

$$E(X) = \int_0^\infty xf(x)dx = (1/2)((1/2) + (1/3)) = (5/12).$$

Then, by definition, $c = (1+\theta)\lambda\mu = (1+\theta)(4)(5/12) \Rightarrow \theta = 4/5$. Applying integration by parts again, we see

$$M_X(t) = \int_0^\infty e^{tX}f(x)dx = (1/2)[2 \int_0^\infty e^{-(2-t)x}dx + 3 \int_0^\infty e^{-(3-t)x}dx]$$

$$= (1/2)(2/(2-t)) +(1/2)(3/(3-t)),\ t < 2.$$

We know that κ is the smallest positive root of $1 + (1+\theta)\mu t = M_X(t)$. Thus,

$$1 + (1+(4/5))(5/12)\kappa = (1/2)(2/(2-\kappa)) +(1/2)(3/(3-\kappa)).$$

Solving for the roots of the above equation we have 0, 1, and 8/3. Thus the adjustment coefficient κ is 1.

This same example can be solved using the Newton – Raphson methodology that was illustrated in the first example. Using the same initial conditions as the second example,

$$E(X^2) = \int_0^\infty X^2 f(x)dx = (1/2)[2 \int_0^\infty x^2 e^{-2x}dx + 3 \int_0^\infty x^2 e^{-3x}dx]$$

$$= (1/2)[2\Gamma(3)(1/2)^3 + 3\Gamma(3)(1/3)^3] = (1/2)[4(1/8) + 6(1/27)] = 13/36.$$

We know κ must be less than $\kappa_o = (2\theta\mu)/E(X^2) = 2(4/5)(5/12)(36/13) = 1.8462$. Then,

$$H(t) = 1 + (1+\theta)\mu t - M_X(t) = 1 + (3/4)t - (1/2)(2/(2-t)) +(1/2)(3/(3-t)), \text{ and}$$

$$H^{\prime}(t) = (3/4) - (1/2)(2/(2-t)^2) +(1/2)(3/(3-t)^2).$$

Letting $t = 1.8462$, the Newton-Raphson formula gives $\kappa_1 = 1.7191$. Per the table below, we again find that $\kappa = 1$ after completing multiple iterations:

Table 3: Newton-Raphson Iterations

n	κ_n
2	1.5289

3	1.3051
4	1.1178
5	1.0217
6	1.0009
7	1

Chapter 4: Cramer's Asymptotic Ruin Formula and Tijms' Approximation

Suppose $\kappa > 0$ satisfies $1 + (1+\theta)\mu t = M_X(t)$. Then by *Cramer's Asymptotic Ruin Formula*, the ruin probability satisfies:

$$\psi(u) \sim Ce^{-\kappa u},\ u \to \infty, \text{ where}$$

$$C = \mu\theta/(M'_X(\kappa) - \mu(1+\theta))$$

and $M_x(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}dF(x)$ is the moment generating function of the claim severity random variable X. Although an asymptotic approximation, Cramer's Formula is quite accurate even for smaller values of *u*.

In order to further improve the accuracy of the estimation for a small *u*, Tijms' approximation adds an exponential term to Cramer's asymptotic ruin formula and is defined by:

$$\psi_T(u) = ((1/1+\theta) - C)e^{-u/\alpha} + Ce^{-\kappa u},\ u \geq 0,$$

where α is given by:

$$\alpha = [(E(X^2)/(2\mu\theta) - C/\kappa)]/[1/(1+\theta)) - C].$$

Tijms' approximation to ruin probability is not only able to provide an accurate solution but in some cases it is also able to provide the true value of $\psi(u)$.

Example 5:

Let's suppose that $\theta = 3/5$ and the single claim size density is $f(x) = 3e^{-4x} + e^{-2x}/2$, $x \geq 0$ (*Loss Models* 302). Using this information, we will determine Cramer's asymptotic ruin formula and Tijms' approximation to ruin probability. The moment generating function is:

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx = 3(4 - t)^{-1} + (1/2)(2-t)^{-1}, \text{ and}$$

$$M'_X(t) = 3(4 - t)^{-2} + (1/2)(2-t)^{-2}.$$

It follows that $\mu = M'_X(0) = (3/16) + (1/8) = 5/16$. By definition, the adjustment coefficient $\kappa > 0$ satisfies $1 + (1/2)\kappa = 3(4 - \kappa)^{-1} + (1/2)(2-\kappa)^{-1}$. Multiplication by $2(4 - \kappa)(2 - \kappa)$ yields:

$$2(4 - \kappa)(2 - \kappa) + \kappa(4 - \kappa)(2 - \kappa) = 6(2 - \kappa) + (4 - \kappa), \text{ that is,}$$

$$2(\kappa^2 - 6\kappa + 8) + \kappa^3 - 6\kappa^2 + 8\kappa = 16 - 7\kappa.$$

Rearrangement gives the following:

$$0 = \kappa^3 - 4\kappa^2 + 3\kappa = \kappa(\kappa - 3)(\kappa - 1),$$

and we can see that $\underline{\kappa = 1}$ because it is the smallest positive root.

We will next determine Cramer's asymptotic formula. By definition,

$C = [(5/16)(3/5)]/[(M^{I}_{X}(1) – (1/2)] = 9/16$, where $M^{I}_{X}(1) = (1/3) + (1/2)(1) = 5/6$.

Thus Cramer's asymptotic formula is:

$$\psi(u) \sim (9/16)e^{-u},\ u \to \infty.$$

The last step will be determining Tijms' approximation. By definition,

$$\psi_T(u) = ((5/8) – (9/16))e^{-u/\alpha} + (9/16)e^{-u} = (1/16)e^{-u/\alpha} + (9/16)e^{-u}.$$

To compute α, we note that $M^{II}_{X}(t) = 6(4 – t)^{-3} + (2-t)^{-3}$, and thus $E(X^2) = M^{II}_{X}(0) = (3/32) + (1/8) = 7/32$. Therefore, the mean aggregate loss is:

$$(E(X^2)/(2\mu\theta)) = (7/32)/[2(3/16)] = 7/12.$$

Finally, we are able to determine α as follows:

$$\alpha = [(7/12) – (9/16)]/[(5/8) – (9/16)] = 1/3.$$

Thus, Tijms' approximation becomes:

$$\psi_T(u) = (1/16)e^{-3u} + (9/16)e^{-u}.$$

As mentioned above, this example demonstrates that Tijms' approximation $\psi_T(u)$ is exactly equal to the true value of ruin $\psi(u)$. This holds true for all claim severity distributions with a probability density function of the form:

$$f(x) = p(B_1 e^{-B_1 x}) + (1-p)(B_2 e^{-B_2 x}),\ x \geq 0.$$

Example 6:

In the next example, let's assume that $\theta = 4/5$ and the single claim size density is $f(x) = (1+6x)e^{-3x}$, $x \geq 0$ (*Loss Models* 302). Using this information, we will again determine Cramer's asymptotic ruin formula and Tijms' approximation to ruin probability.

Note that $f(x)$ can be re-written as $f(x) = (1/3)(3e^{-3x}) + (2/3)(9xe^{-3x})$. The moment generating function is:

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx = (1/3)[3/(3-t)] + (2/3)[3/(3-t)]^2 = (3-t)^{-1} + 6(3-t)^{-2}, \text{ and}$$

$$M'_X(t) = (3-t)^{-2} + 12(3-t)^{-3}.$$

It follows that $\mu = M'_X(0) = (1/9) + (12/27) = 5/9$. By definition, the adjustment coefficient $\kappa > 0$ satisfies $1 + \kappa = (3-\kappa)^{-1} + 6(3-\kappa)^{-2}$. Rearrangement gives the following:

$$0 = \kappa^3 - 5\kappa^2 + 4\kappa = \kappa(\kappa-1)(\kappa-4),$$

and we can see that $\underline{\kappa = 1}$ because it is the smallest positive root.

We will next determine Cramer's asymptotic formula. By definition,

$C = [(5/9)(4/5)]/[(M'_X(1) - 1] = 16/27$, where $M'_X(1) = (1/4) + (3/2) = 7/4$.

Thus Cramer's asymptotic formula is:

$$\psi(u) \sim (16/27)e^{-u}, \ u \to \infty.$$

We can now determine Tijms' approximation. By definition,

$$\psi_T(u) = ((5/9) - (16/27))e^{-u/\alpha} + (16/27)e^{-u} = -(1/27)e^{-u/\alpha} + (16/27)e^{-u}.$$

To compute α, we note that $M''_X(t) = 2(3 - t)^{-3} + 36(3-t)^{-4}$, and thus $E(X^2) = M''_X(0) = (2/27) + (36/81) = 14/27$. Therefore, the mean aggregate loss is:

$$(E(X^2)/(2\mu\theta)) = (14/27)/[2(4/9)] = 7/12.$$

Finally, we are able to determine α as follows:

$$\alpha = [(7/12) - (16/27)]/[-(1/27)] = 1/4.$$

Thus, Tijms' approximation becomes:

$$\psi_T(u) = (-1/27)e^{-4u} + (16/27)e^{-u}.$$

Similar to the first example, this second example also demonstrates that Tijms' approximation $\psi_T(u)$ is exactly equal to the true value of ruin $\psi(u)$. This relationship holds true for all claim severity distributions with a probability density function of the form:

$$f(x) = p(B^{-1}e^{-x/B}) + (1-p)(B^{-2}xe^{-x/B}),\ x \geq 0.$$

Tijms' approximation doesn't always reproduce the true ruin probability like we saw in the examples above but it is able to consistently generate an approximation of good quality. Exact ruin probability values, Cramer's asymptotic values, and Tijms' approximate values will all converge as $u \to \infty$.

Chapter 5: Monte Carlo Simulation

In this chapter we will use a Monte Carlo simulation to test these theories. Let us reconsider the example that was discussed on page 16. We've assumed that $\theta = 3/5$ and the single claim size density is f(x) = $3e^{-4x} + e^{-2x}/2$, $x \geq 0$. As noted on page 17, f(x) can be re-written in the following general form f(x) = $p(B_1e^{-B_1x}) + (1-p)(B_2e^{-B_2x})$, where p=.75, B_1=4, and B_2= 2. Additionally, by integrating f(x), we see that F(x) = $1 - .75e^{-4x} - .25e^{-2x}$, $x \geq 0$. We are able to find the below values for F(x) and f(x):

Table 4: Monte Carlo Simulation - Distribution of Claim Size

Size of 1 Claim		
x	F(*x*)	f(*x*)
0	0	3.5
0.1	0.292577277	2.420325515
0.2	0.495423265	1.683146915
0.3	0.636901432	1.177988454
0.4	0.73624537	0.830354036
0.5	0.806528677	0.58994557
0.6	0.856662982	0.422750966
0.7	0.892743212	0.30572867
0.8	0.918954218	0.223234871
0.9	0.938182486	0.164620611
1	0.95242945	0.122614558
1.1	0.963091205	0.092233599

1.2	0.971148201	0.070048218
1.3	0.977294182	0.053686482
1.4	0.982024087	0.041498622
1.5	0.985694169	0.032329791
1.6	0.988563281	0.025365774
1.7	0.990821351	0.02002796
1.8	0.99260913	0.015901619
1.9	0.994031968	0.01268674
2	0.995169493	0.010164207

Based on the above data, we see the following Distribution graph based on the F(x) values and Density graph based on the f(x) values. It is evident that as claim size increases, the probability of that claim actually materializing becomes increasingly lower.

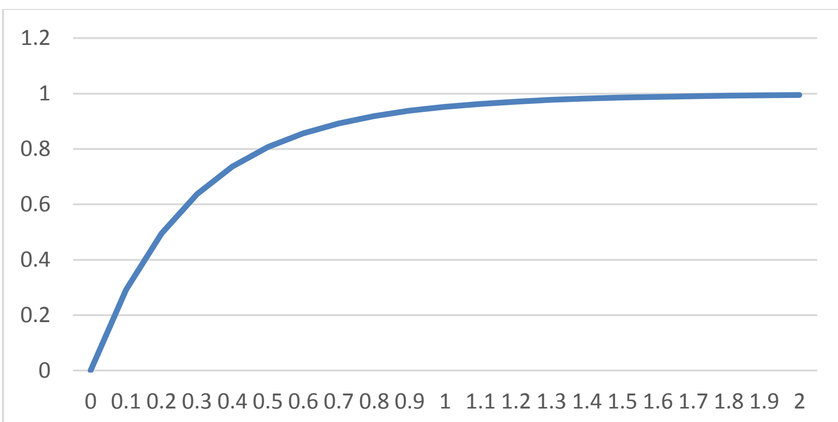

Figure 2: Distribution Function Graph

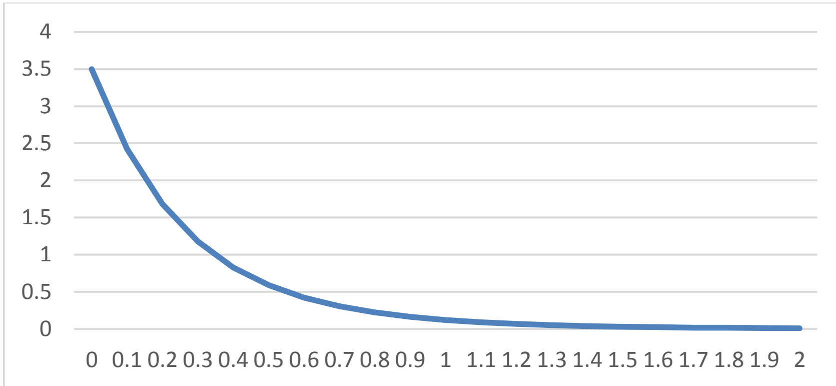

Figure 3: Density Function Graph

To clearly see this, we will simulate 100 claims and then determine the size of each claim. We'll let F(x) be a randomly generated number and then solve for x to find the claim size. In order to find x, we will first need to solve for e^{-2x}. Letting A= p =.75, B=1-p =.25, and C= F(x)-1, we can use the quadratic formula $Ax^2 + Bx + C = 0$ to solve for e^{-2x} as shown in the below table. Per the simulation below, only 7 out of 100 claims exceeded 0.7 in claim size.

Table 5: Monte Carlo Simulation - Size of 100 Claims

claim #	random # = F(x)	exp(-2x)	x ie size of claim
	rand()	(quadratic formula)	(natural log)
1	0.915922162	0.207340769	0.786695807
2	0.025495885	0.985338801	0.007384868
3	0.646757705	0.539568071	0.308493164
4	0.452891771	0.703537572	0.175816998
5	0.082197972	0.952044116	0.024571952
6	0.031810287	0.981678836	0.009245537
7	0.530537791	0.641867433	0.221686744
8	0.5586417	0.618352767	0.240348082

9	0.421257517	0.7274443	0.159108923
10	0.07793029	0.954584446	0.023239584
11	0.259788668	0.840669534	0.08677832
12	0.857819067	0.299543967	0.602747035
13	0.874035327	0.27574793	0.644134064
14	0.038743679	0.977646609	0.011303508
15	0.003075655	0.998241157	0.000880196
16	0.728112129	0.458069229	0.390367476
17	0.750853565	0.433310853	0.418149951
18	0.140653208	0.916649308	0.043515157
19	0.50896014	0.659467458	0.208161326
20	0.148237809	0.911971687	0.046073167
21	0.521181489	0.649545596	0.215741121
22	0.538154381	0.635562687	0.226622275
23	0.606341658	0.576741409	0.275180639
24	0.895042345	0.242871266	0.707611873
25	0.544826961	0.629998359	0.231019032
26	0.458488952	0.699238934	0.178881386
27	0.381591549	0.756546407	0.139495702
28	0.203159339	0.877475083	0.06535336
29	0.05739849	0.966726377	0.016919892
30	0.678541054	0.508899545	0.33775232
31	0.577744218	0.601959047	0.253782932
32	0.294859426	0.817185563	0.100944542
33	0.788716672	0.389650583	0.471252441
34	0.030820195	0.982253486	0.008952936
35	0.961860501	0.113744538	1.086900122
36	0.506946409	0.661090886	0.206931975
37	0.205113568	0.876226594	0.066065276
38	0.695582382	0.491868021	0.354772424
39	0.133929307	0.920779287	0.041267458
40	0.262484056	0.838884113	0.087841354
41	0.411965097	0.734346272	0.154387301
42	0.111131537	0.934666947	0.03378251
43	0.137862849	0.91836512	0.042580117
44	0.161576687	0.903695669	0.050631313
45	0.614272807	0.569594633	0.281415171
46	0.698642662	0.488762629	0.357939164
47	0.065418028	0.961999393	0.01937073

48	0.351319939	0.778153171	0.125415948
49	0.177230206	0.893901186	0.05608002
50	0.79704942	0.379573712	0.484353233
51	0.735097563	0.450569939	0.398620983
52	0.468315694	0.691639914	0.184344907
53	0.908079302	0.221068692	0.754640901
54	0.119826762	0.929390857	0.03661295
55	0.699762279	0.487622825	0.359106535
56	0.429908544	0.720970488	0.163578537
57	0.638320508	0.547488149	0.301207232
58	0.054401485	0.968487859	0.016009666
59	0.236925777	0.855688494	0.077924439
60	0.478657685	0.683569104	0.190213763
61	0.750092623	0.434155781	0.417175933
62	0.418347967	0.729611093	0.157621818
63	0.260203954	0.840394655	0.086941835
64	0.385426536	0.753772937	0.141332051
65	0.127835413	0.924508797	0.039246356
66	0.065424715	0.961995443	0.019372783
67	0.231424708	0.85926941	0.075836387
68	0.096743605	0.943342179	0.0291631
69	0.467879752	0.691978453	0.18410023
70	0.3829937	0.755533336	0.140165687
71	0.890046768	0.250924147	0.691302294
72	0.57043238	0.608275012	0.248564089
73	0.172282726	0.897006596	0.054346032
74	0.909975412	0.217794725	0.762101144
75	0.723561324	0.462906744	0.385114831
76	0.197801547	0.880890356	0.063411058
77	0.267985517	0.835230079	0.090024024
78	0.070797454	0.958817454	0.021027286
79	0.800037253	0.375914916	0.489196224
80	0.581798163	0.598434782	0.256718864
81	0.462803685	0.695910595	0.181267042
82	0.720943309	0.465672931	0.382135878
83	0.255550959	0.843470205	0.085115351
84	0.659071376	0.527847031	0.319474376
85	0.471321824	0.689301798	0.18603804
86	0.884849108	0.259141155	0.675191183

87	0.793017848	0.384472136	0.47794198
88	0.497153007	0.668941157	0.20102959
89	0.60643701	0.576655895	0.27525478
90	0.71466864	0.472253978	0.375119174
91	0.195716659	0.882216342	0.062658984
92	0.319504728	0.800341507	0.11135838
93	0.34347396	0.783673184	0.121881602
94	0.989401385	0.038050857	1.634415831
95	0.073967817	0.956937959	0.022008359
96	0.660210254	0.526752955	0.320511808
97	0.083420378	0.951315417	0.024954801
98	0.027933942	0.98392703	0.008101771
99	0.925939276	0.189037438	0.832905099
100	0.321333307	0.79908004	0.112147082

We will now use the Poisson process to simulate the number of claims that will materialize in a given amount of time. Letting time t = .1, we see the following Poisson distribution:

Table 6: Monte Carlo Simulation - Poisson Distribution for Number of Total Claims

Poisson Distribution for # of claims		
k	N(k)	F(k)
	$=(e^{-t})(t^k)/k!$	$=\sum N_k$
0	0.904837418	0.904837418
1	0.090483742	0.99532116
2	0.004524187	0.999845347
3	0.000150806	0.999996153
4	3.77016E-06	0.999999923
5	7.54031E-08	0.999999999
6	1.25672E-09	1

7	1.79531E-11	1
8	2.24414E-13	1
9	2.49349E-15	1
10	2.49349E-17	1

With this information, we can now expand on the simulation shown on page 22 to predict the number of expected claims in addition to the expected claim size and thus predict either the amount of surplus at time t or the time of ruin. We will call this *Trial 1*. Let us assume the following:

a) If a randomly generated number (ie, ‘Rand_1’) is less than all values of F(k) shown in above table, then the number of of claims equals 0. Otherwise, the number of claims equals the number of F(k) values greater than ‘Rand_1.’
b) Beginning Surplus at time 0 is equal to 1.
c) When there is more than one claim in a given period, each claim is assumed to be the same size.
d) At the end of Trial 1, t = 100. However, given space constraints, the below table only shows values up to t = 5.
e) Note that if End Surplus is less than 0, ruin occurs at time t. Otherwise, ruin does not occur. (Note: t = 101 signifies that time of ruin is outside of simulated time period).

Table 7: Monte Carlo Simulation - Trial 1

t = time	rand_1	# of Claims	rand_2	exp(-2x)	Size of Claim	Beg Surplus	Premium	Claim	End Surplus	Time of Ruin
0	0.7687	0	0.2034	0.8773	0.06545	1	0.05	0	1.05	101
0.1	0.5852	0	0.8246	0.3449	0.53224	1.05	0.05	0	1.1	101
0.2	0.5066	0	0.732	0.45392	0.39491	1.1	0.05	0	1.15	101
0.3	0.9096	1	0.122	0.92809	0.03732	1.15	0.05	0.0373	1.1627	101
0.4	0.7073	0	0.8157	0.35634	0.51594	1.1627	0.05	0	1.2127	101
0.5	0.7289	0	0.465	0.69423	0.18248	1.2127	0.05	0	1.2627	101
0.6	0.91	1	0.1545	0.9081	0.0482	1.2627	0.05	0.0482	1.2645	101
0.7	0.4261	0	0.953	0.13417	1.00433	1.2645	0.05	0	1.3145	101
0.8	0.1713	0	0.4346	0.71747	0.16601	1.3145	0.05	0	1.3645	101
0.9	0.7518	0	0.7914	0.38648	0.47534	1.3645	0.05	0	1.4145	101
1	0.1587	0	0.0232	0.98666	0.00671	1.4145	0.05	0	1.4645	101
1.1	0.5386	0	0.8487	0.31247	0.58162	1.4645	0.05	0	1.5145	101
1.2	0.8468	0	0.5076	0.66054	0.20735	1.5145	0.05	0	1.5645	101
1.3	0.946	1	0.9736	0.08428	1.2368	1.5645	0.05	1.2368	0.3777	101
1.4	0.974	1	0.7194	0.46734	0.38034	0.3777	0.05	0.3803	0.0473	101
1.5	0.003	0	0.1897	0.88606	0.06048	0.0473	0.05	0	0.0973	101
1.6	0.2935	0	0.6929	0.4946	0.352	0.0973	0.05	0	0.1473	101
1.7	0.0896	0	0.9582	0.12237	1.05034	0.1473	0.05	0	0.1973	101
1.8	0.4859	0	0.2373	0.85544	0.07807	0.1973	0.05	0	0.2473	101
1.9	0.4536	0	0.0284	0.98364	0.00825	0.2473	0.05	0	0.2973	101
2	0.0214	0	0.996	0.01536	2.08802	0.2973	0.05	0	0.3473	101
2.1	0.2717	0	0.2358	0.85645	0.07748	0.3473	0.05	0	0.3973	101
2.2	0.7404	0	0.011	0.99371	0.00316	0.3973	0.05	0	0.4473	101
2.3	0.0915	0	0.3481	0.78043	0.12396	0.4473	0.05	0	0.4973	101
2.4	0.3716	0	0.4135	0.73325	0.15514	0.4973	0.05	0	0.5473	101
2.5	0.3065	0	0.8324	0.33459	0.54742	0.5473	0.05	0	0.5973	101
2.6	0.8288	0	0.9039	0.22823	0.7387	0.5973	0.05	0	0.6473	101
2.7	0.2135	0	0.7879	0.39057	0.47007	0.6473	0.05	0	0.6973	101
2.8	0.9323	1	0.0842	0.95087	0.02519	0.6973	0.05	0.0252	0.7222	101
2.9	0.1627	0	0.6124	0.57133	0.2799	0.7222	0.05	0	0.7722	101
3	0.9165	1	0.5717	0.60715	0.24949	0.7722	0.05	0.2495	0.5727	101
3.1	0.1063	0	0.1152	0.9322	0.0351	0.5727	0.05	0	0.6227	101
3.2	0.9065	1	0.6375	0.54828	0.30048	0.6227	0.05	0.3005	0.3722	101
3.3	0.6342	0	0.0478	0.97234	0.01403	0.3722	0.05	0	0.4222	101
3.4	0.5777	0	0.5222	0.6487	0.21639	0.4222	0.05	0	0.4722	101
3.5	0.2353	0	0.6974	0.49001	0.35666	0.4722	0.05	0	0.5222	101
3.6	0.2706	0	0.0053	0.99698	0.00151	0.5222	0.05	0	0.5722	101

3.7	0.366	0	0.0419	0.97582	0.01224	0.5722	0.05	0	0.6222	101
3.8	0.3948	0	0.727	0.45931	0.38902	0.6222	0.05	0	0.6722	101
3.9	0.7139	0	0.6824	0.50505	0.34154	0.6722	0.05	0	0.7222	101
4	0.8415	0	0.2661	0.83647	0.08928	0.7222	0.05	0	0.7722	101
4.1	0.525	0	0.0229	0.98683	0.00663	0.7722	0.05	0	0.8222	101
4.2	0.947	1	0.6281	0.55695	0.29264	0.8222	0.05	0.2926	0.5795	101
4.3	0.1704	0	0.222	0.86536	0.0723	0.5795	0.05	0	0.6295	101
4.4	0.5391	0	0.9415	0.15851	0.92097	0.6295	0.05	0	0.6795	101
4.5	0.0878	0	0.8355	0.33041	0.55371	0.6795	0.05	0	0.7295	101
4.6	0.4866	0	0.279	0.82788	0.09444	0.7295	0.05	0	0.7795	101
4.7	0.1196	0	0.8078	0.36634	0.5021	0.7795	0.05	0	0.8295	101
4.8	0.8101	0	0.3539	0.77637	0.12657	0.8295	0.05	0	0.8795	101
4.9	0.2918	0	0.8964	0.2407	0.71211	0.8795	0.05	0	0.9295	101
5	0.0661	0	0.6801	0.50738	0.33925	0.9295	0.05	0	0.9795	101

Based on *Trial 1*, the average claim equals .03048, ruin does not occur, and final surplus equals 29.48. The theory claims that the expected average claim equals μt = (5/16)(.1) = .03125 and thus our simulated average claim is off by .00077. We will try to reduce this variance by completing 50 trials similar to the one shown above. The results for these 50 trials are shown below:

Table 8: Monte Carlo Simulation - 50 Trials

Trial #	Result (Ruin = 1)	Time of Ruin	Final Surplus	Avg Claim
1	1	0.1	21.11303065	0.029886969
2	0	101	20.53644314	0.030463557
3	1	12.4	15.34061492	0.035659385
4	0	101	17.88353739	0.033116463
5	0	101	12.95932106	0.038040679

6	0	101	26.58011666	0.024419883
7	0	101	12.15483909	0.038845161
8	0	101	10.09020161	0.040909798
9	0	101	25.03053362	0.025969466
10	0	101	22.77743306	0.028222567
11	0	101	24.77616589	0.026223834
12	0	101	24.56262804	0.026437372
13	1	0.5	12.71530671	0.038284693
14	0	101	15.0883261	0.035911674
15	0	101	13.74633577	0.037253664
16	0	101	23.44922703	0.027550773
17	0	101	14.13959926	0.036860401
18	0	101	22.68469419	0.028315306
19	0	101	13.92597543	0.037074025
20	0	101	26.0249274	0.024975073
21	0	101	18.8475669	0.032152433
22	0	101	20.82976994	0.03017023
23	0	101	25.63713892	0.025362861
24	0	101	20.46301874	0.030536981
25	1	0.1	24.22004922	0.026779951
26	0	101	13.48033858	0.037519661
27	0	101	20.33092945	0.030669071
28	0	101	20.97978879	0.030020211
29	1	15.4	12.69148086	0.038308519
30	0	101	19.89882423	0.031101176
31	0	101	24.82963675	0.026170363
32	0	101	13.57509373	0.037424906
33	0	101	21.42755882	0.029572441
34	0	101	22.25940245	0.028740598
35	1	12.5	13.01928458	0.037980715
36	0	101	19.80798268	0.031192017
37	0	101	14.88874389	0.036111256
38	0	101	16.80961709	0.034190383
39	0	101	19.87760133	0.031122399
40	0	101	29.05211191	0.021947888
41	1	0.6	22.8049957	0.028195004
42	0	101	13.50988695	0.037490113
43	0	101	7.960869309	0.043039131
44	0	101	16.67443791	0.034325562

45	1	2.7	19.02854446	0.031971456
46	0	101	26.70678611	0.024293214
47	1	18.1	12.43012662	0.038569873
48	0	101	21.23661506	0.029763385
49	0	101	22.58345977	0.02841654
50	0	101	17.92237415	0.033077626

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After completing 50 trials our average claim size becomes .0320. The .00076 variance from the theory's average expected claim value is marginally improved after completing 49 more trials. Additionally, we can see that 9 of the 50 trials ended in ruin. Cramer's Asymptotic Ruin Formula predicts 10.35 ruins after 50 trials and Tijms' approximation predicts 10.5 ruins after 50 trials.

Continuing in this fashion in an effort to yield the highest level of accuracy, 50,000 trials produces the below results:

Table 9: Monte Carlo Simulation - Average Claim Size

Trials	Avg Claim	Variance to Theory
50	0.03201	0.00076
1,000	0.03140	0.00015
5,000	0.03129	0.00004
10,000	0.03128	0.00003
20,000	0.03123	0.00002

50,000	0.03124	0.00001

Table 10: Monte Carlo Simulation - Project Number of Ruins

	# of Ruins			Variance to Experiment		% Variance	
Trials	Experiment	Cramer	Tijms	Cramer	Tijms	Cramer	Tijms
50	9.00	10.35	10.50	1.346609	1.502193872	2.69%	3.00%
1,000	218.00	206.93	210.04	11.06781	7.956122568	1.11%	0.80%
5,000	1056.00	1034.66	1050.22	21.33907	5.78061284	0.43%	0.12%
10,000	2124.00	2069.32	2100.44	54.67814	23.56122568	0.55%	0.24%
20,000	4262.00	4138.64	4200.88	123.3563	61.12245136	0.62%	0.31%
50,000	10628.00	10346.61	10502.19	281.3907	125.8061284	0.56%	0.25%

As seen above, 50,000 trials takes the average claim size to .00001 accuracy. Additionally, the second table depicts the number of ruins the experiment, Cramer, and Tijms each predict. We can see that the % variance decreases as the number of trials increases and as discussed earlier in the paper, Tijms' approximation is proven to yield a more exact calculation than Cramer's asymptotic formula.

Finally, after 50,000 trials, we can conclude that this single claim size density function has a 21.26% probability of ruin. If ruin does occur, we can expect to see ruin at time t=5.75 and if ruin does not occur, we can project that the final surplus will be 19.76.

Conclusion

Ruin theory has a broad range of applications within the field of insurance. Insurance companies utilize Ruin Theory assumptions to set risk limits and ensure that their solvency capital requirement coverage ratio stays above a certain level with a large enough probability.

The principles of this theory help actuaries create risk management plans in an effort to analyze and answer the following questions:

- What is the optimum level of initial capital?
- How much capital should a company hold given its business plans and strategies?
- For a fixed amount of capital what is the optimal level of exposures?
- How much additional premium should be charged to cover a new peril or an emerging risk?
- How much more capital should a company hold to if entering risky corporate bonds?
- Should the company invest $X in an I.T. system to reduce the chances of various operational risks?

- If a specific probability of ruin is expected and/or tolerated, what initial surplus and premium loading is required to maintain this probability?

Actuaries form initial assumptions to questions such as those above and as demonstrated within this paper, Ruin Theory is able to provide approximated and/or explicit probabilities related to the risk of insolvency based on those conditions. In this way, insurance companies are able to effectively manage risk and avoid ruin.

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Biography

Ashley Fehr grew up in Maryland. She attended the University of West Virginia, where she received her Bachelor of Science in Mathematics and Bachelor of Science in Business Administration (Finance) in 2009. She went on to receive her Master of Science in Mathematics from George Mason University in 2015.