LARGE DEVIATIONS AND RARE EVENT SIMULATION FOR PORTFOLIO CREDIT RISK

by

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Abstract

LARGE DEVIATIONS AND RARE EVENT SIMULATION FOR PORTFOLIO CREDIT RISK
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Estimating the loss distribution of a credit portfolio is an important problem in credit risk management. When dealing with credit portfolios, correlations between defaults play an important role. “Threshold based factor models” is a widely used method of modeling this phenomenon. Both the well know industry models, KMV and CreditMetrics fall into this category. We begin this thesis by deriving sharp large deviation asymptotics for a single factor model as the number of obligors go to infinity. Both the systematic and idiosyncratic risk factors are allowed to take general distributions. We derive our asymptotics under a unified framework, and they can be applied to many different probability distributions. Some of the distributions that we consider for the risk factors are Gaussian, Exponential, Pareto, Gamma, and Stretched Exponential.

Next we consider the rare event simulation problem for the same model. We develop an importance sampling estimator, and prove its logarithmic efficiency. Our importance sampling estimator can be used for computing Value-at-Risk and Expected Shortfall.

Finally, we consider a multifactor model where the dependence between the obligors is given by the Gaussian copula. For this model, we derive logarithmic large deviations, letting both
the number of obligors and the number of risk factors go to infinity.
Chapter 1: Introduction

1.1 Factor Models of Credit Risk

The risk that a given obligor may default on its credit obligation is referred to as credit risk. Estimating the loss distribution of a credit portfolio is an important problem not only in risk management, but also in pricing structured financial products. When dealing with credit portfolios, correlations between defaults play an important role. “Threshold based factor models” (see [30], [12]), is a widely used method of modeling this phenomenon. Both the well know industry models, KMV and CreditMetrics fall into this category. These models postulate that each obligor defaults if its latent variable crosses a pre-specified threshold. Correlations between the latent variables are introduced by using a common factor. To make the discussion more precise, let the latent variable of the $k^{th}$ obligor be given by

$$X_k = aZ + b\varepsilon_k.$$ (1.1)

In the above equation, $Z$ and $\varepsilon_k$ are independent random variables, $\varepsilon_k$ are i.i.d., and $b = \sqrt{1-a^2}$ with $0 < a < 1$. In financial literature, $Z$ and $\varepsilon_k$ are often referred to as the systematic and idiosyncratic risk factors respectively. Let us denote the cumulative distribution functions of $X_k$, $Z$ and $\varepsilon_k$ by $G(\cdot)$, $F(\cdot)$ and $H(\cdot)$ respectively. We set the threshold for the $k^{th}$ obligor at $G^{-1}(p_k)$ so that the $k^{th}$ obligor defaults if $X_k < G^{-1}(p_k)$. Therefore, we may denote the default indicators by

$$Y_k = \mathbb{1}_{\{X_k < G^{-1}(p_k)\}}.$$ (1.2)
Then, by (1.2), the probability that the $k^{th}$ obligor defaults is given by $P(Y_k = 1) = p_k$.

Let $m$ denote the number of obligors in the portfolio. Define the portfolio loss by $L_m = \sum_{k=1}^{m} U_k l_k Y_k$ where $U_k$’s are random variables taking values in $(0, 1]$, and $l_k$’s are deterministic. $U_k$ is the loss given default (1-recovery rate), and $l_k$ is the credit exposure, so that $U_k l_k$ is the actual loss suffered.

By assuming $U_k, l_k = 1$, $p_k = p$ and $X_k, Z, \epsilon_k \sim \mathcal{N}(0, 1)$, Vasicek established

$$\lim_{m \to \infty} P \left( \frac{L_m}{m} \leq x \right) = \Phi \left( \frac{b\Phi^{-1}(x) - \Phi^{-1}(p)}{a} \right).$$

This is referred to as the large homogeneous portfolio approximation (see [33] and [32]).

We summarize the relevant results of what is generally known as the Vasicek Single Factor Model (VSFM).

1.2 Vasicek Single Factor Model

As discussed in the previous section, VSFM (see [33] and [32]) deals with approximating the asymptotic loss distribution, assuming the latent variable model given in 1.1. Both the default probabilities and the credit exposures are assumed to be uniform ($p_k = p$ and $l_k = l$). $X_k, Z$ and $\epsilon_k$ are assumed to be standard normal. For an expository note on VSFM, see [30]. For sake of clarity, we formally state the assumptions of the VSFM below.

Assumptions VSFM

1. The default indicator $Y_k$ is given by

$$Y_k = \mathbb{1}\{X_k < \Phi^{-1}(p)\},$$

(1.3)
where

\[ X_k = aZ + b\varepsilon_k, \]

where \( Z \) and \( \varepsilon_k \) independent standard normal random variables, \( 0 < a < 1 \) and \( b = \sqrt{1 - a^2} \).

2. From (1.3) above it follows that the unconditional default probability of the \( k^{th} \) obligor \( \mathbb{P}(Y_k = 1) \) is given by

\[ \mathbb{P}(Y_k = 1) = p. \]

This means that the likelihood of default of each obligor is the same.

3. The default probability of the \( k \)-th obligor conditioned on \( Z \), denoted by \( p(Z) \), is given by

\[
p(z) = \mathbb{P}(Y_k = 1|Z = z) = \mathbb{P}(X_k < \Phi^{-1}(p)|Z = z) = \mathbb{P}(aZ + b\varepsilon_k < \Phi^{-1}(p)|Z = z) = \mathbb{P}(az + b\varepsilon_k < \Phi^{-1}(p_k)) = \mathbb{P}\left(\varepsilon_k < \frac{\Phi^{-1}(p_k) - az}{b}\right) = H\left(\frac{\Phi^{-1}(p) - az}{b}\right).
\]

4. The total loss from defaults \( L_m \) is given by
\[ L_m = \sum_{k=1}^{\infty} Y_k. \]

This means that the credit exposure is the same for every obligor.

We next analyze the distribution of the portfolio loss fraction \( \frac{L_m}{m} \).

**Theorem 1.2.1.** Suppose [Assumptions VSFM] are satisfied. Then,

\[ \frac{L_m}{m} \to p(Z) \text{ in probability as } m \to \infty. \]

**Proof.** Let \( \epsilon > 0 \).

\[
\mathbb{P} \left( \left| \frac{L_m}{m} - p(Z) \right| > \epsilon \right) \leq \frac{\text{Var} \left( \frac{L_m}{m} | Z \right)}{\epsilon^2} \]

\[
= \frac{1}{m^2 \epsilon^2} \text{Var}(L_m | Z) \]

\[
= \frac{p(Z)(1 - p(Z))}{m \epsilon^2} \]

\[
\leq \frac{1}{m \epsilon^2}. \]

By taking the expectation on both sides

\[
\mathbb{P} \left( \left| \frac{L_m}{m} - p(Z) \right| > \epsilon \right) \leq \frac{2}{m \epsilon^2} \to 0 \text{ as } m \to \infty. \]

Therefore, for every \( \epsilon > 0 \)

\[
\lim_{m \to \infty} \mathbb{P} \left( \left| \frac{L_m}{m} - p(Z) \right| > \epsilon \right) = 0. \]
The above theorem enables us to find the limiting distribution of the loss fraction \( \frac{L_m}{m} \) as \( m \to \infty \). This is given in the next theorem.

**Theorem 1.2.2.** Suppose Assumptions VSFM hold. Then,

\[
\lim_{m \to \infty} \mathbb{P} \left( \frac{L_m}{m} \leq x \right) = \Phi \left( \frac{b\Phi^{-1}(x) - \Phi^{-1}(p)}{a} \right).
\]

**Proof.**

\[
\lim_{m \to \infty} \mathbb{P} \left( \frac{L_m}{m} \leq x \right) = \mathbb{P} \left( p(Z) \leq x \right) \quad \text{(since} \frac{L_m}{m} \to p(Z) \text{ in distribution.)}
\]

\[
= \mathbb{P} \left( Z > p^{-1}(x) \right) \quad \text{(since} p(z) = \Phi \left( \frac{\Phi^{-1}(p) - az}{b} \right) \text{ is decreasing in} z \text{)}
\]

\[
= \mathbb{P} \left( -Z < -p^{-1}(x) \right)
\]

\[
= \Phi \left( -p^{-1}(x) \right) .
\]

But,

\[
p(z) = \Phi \left( \frac{\Phi^{-1}(p) - az}{b} \right) \implies p^{-1}(z) = \frac{\Phi^{-1}(p) - b\Phi^{-1}(z)}{a}.
\]

Therefore,

\[
\lim_{m \to \infty} \mathbb{P} \left( \frac{L_m}{m} \leq x \right) = \Phi \left( \frac{b\Phi^{-1}(x) - \Phi^{-1}(p)}{a} \right). \tag{1.4}
\]

In the VSFM, the distributions of \( X_k, Z \) and \( \varepsilon_k \) are assumed to be standard normal random variables. An important fact to notice is that (see [30]) if we assume that \( X_k, \varepsilon_k \) and \( Z \) have
cumulative distribution functions $H(\cdot), G(\cdot)$ and $F(\cdot)$, and $\varepsilon_k$ and $Z$ both have symmetric distributions, then the direct generalization of (1.4) is

$$\lim_{m \to \infty} \mathbb{P} \left( \frac{L_m}{m} \leq x \right) = F \left( \frac{bH^{-1}(x) - G^{-1}(p)}{a} \right).$$

(1.5)

Many extensions to the VSFM have been considered by previous authors. See for example [21], [18], [28] and [27]. In Chapter 2 we derive sharp large deviation asymptotics for a generalized version of the VSFM. We consider stochastic loss given defaults, non-homogeneous default probabilities, and non-homogeneous credit exposures.

1.3 Large Deviation Theory

Large deviation theory deals with approximating the exponential rates at which certain probabilities of rare events decay. In this section, we provide a brief introduction to large deviation theory. Standard references for large deviation theory include [9], [31] and [10].

Consider a sequence $\{X_i\}_{i \geq 1}$ of i.i.d. random variables with probability law $\mathbb{P}$ and $\mathbb{E}(X_1) = \mu < \infty$. Assume $\mathbb{E}(|X_1|) < \infty$. By the weak law of large numbers, it is known that for any $\epsilon > 0$

$$\lim_{m \to \infty} \mathbb{P} \left( \left| \frac{1}{m} \sum_{i=1}^{m} X_i - \mu \right| < \epsilon \right) = 1,$$

and hence for any $l > \mu$,

$$\lim_{m \to \infty} \mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} X_i > l \right) = 0.$$

In certain situations, we might be interested in estimating the probability $\mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} X_i > l \right)$,
when is \( m \) is large. One of the most important results in large deviation theory, Cramer’s Theorem states that

\[
P \left( \frac{1}{m} \sum_{i=1}^{m} X_i > l \right) \approx Ce^{-mI(l)} \text{ for large } m,
\]

where \( C \) is a constant. \( I(l) \) is known as the rate function. The above equation says that the probability \( P \left( \frac{1}{m} \sum_{i=1}^{m} X_i > l \right) \) decays exponentially quickly when \( l > \mu \). Before we state Cramer’s Theorem, we give below a formal definition of a sequence of probability measures satisfying a large deviation principle.

**Definition 1.3.1.** A sequence of probability measures \( \{P_m\}_{m \in \mathbb{N}} \) is said to satisfy a large deviation principle with rate function \( I(\cdot) \) if,

1. \( I(\cdot) \) is lower semi continuous.
2. \( \{x | I(x) < k\} \) is compact for every \( k \in \mathbb{R} \).
3. For every closed set \( C \)
   
   \[
   \limsup_{m \to \infty} \frac{1}{m} \log P_m(C) \leq - \inf_{x \in C} I(x).
   \]
4. For every open set \( G \)
   
   \[
   \limsup_{m \to \infty} \frac{1}{m} \log P_m(G) \geq - \inf_{x \in G} I(x).
   \]

Next we introduce some basic and recurring concepts in large deviation theory. See [9] for the proofs.

**Definition 1.3.2.** Let \( X \) be a random variable and suppose \( M(\theta) = \mathbb{E}(e^{\theta X}) < \infty \) for all
\( \theta \in \mathbb{R} \). Let \( \Lambda(\theta) = \log(M(\theta)) \). Then the Fenchel-Legendre transform of \( \Lambda(\cdot) \) is defined to be

\[
\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} (\theta x - \Lambda(\theta)) \quad \text{for} \ x \in \mathbb{R}.
\] (1.6)

The following theorem summarizes some useful properties of \( \Lambda^*(\cdot) \) defined above.

**Theorem 1.3.3.** Let \( X \) be a random variable and suppose \( M(\theta) = \mathbb{E}(e^{\theta X}) < \infty \) for all \( \theta \in \mathbb{R} \). Let \( \Lambda(\theta) = \log(M(\theta)) \). Let \( \mathbb{E}(X) = \mu \). Then,

1. \( \Lambda^*(\cdot) \) is convex and lower semi-continuous.
2. If \( \Lambda(\cdot) \) is differentiable at \( \theta_x \) and \( x = \Lambda'(\theta_x) \) then, \( \Lambda^*(x) = \theta_x x - \Lambda(\theta_x) \).
3. \( \Lambda^*(x) \geq 0 \) for \( x \in \mathbb{R} \).
4. \( \Lambda^*(m) = 0 \).
5. \( \Lambda^*(\cdot) \) is decreasing for \( x < m \) and increasing for \( x > \mu \).

We are now ready to provide a formal statement of Cramer’s theorem.

**Theorem 1.3.4 (Cramer’s Theorem).** Let \( \{X_k\}_{k \geq 1} \) be a sequence of i.i.d. random variables defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( S_m = \sum_{i=k}^m X_k \). Let \( M(\theta) = \mathbb{E}(e^{\theta X_1}) < \infty \) for all \( \theta \in \mathbb{R} \), and let \( \Lambda(\theta) = \ln(M(\theta)) \). Let \( l > \mathbb{E}(X_1) \). Then,

\[
\lim_{m \to \infty} \frac{1}{m} \log(\mathbb{P}(S_m > ml)) = -\Lambda^*(l),
\] (1.7)

where, \( \Lambda^*(l) = \sup_{\theta \in \mathbb{R}} (\theta l - \Lambda(\theta)) \). \( \Lambda^*(l) \) in (1.7) is called the large deviation rate function.

For certain random variables, \( \Lambda^*(\cdot) \) can be calculated explicitly.
For instance, if $X_1 \sim N(0, \sigma^2)$ then,

$$\Lambda^*(l) = \frac{l^2}{2\sigma^2},$$

and if $X_1 \sim Bernoulli(p)$ then,

$$\Lambda^*(l) = l \log \left( \frac{l}{p} \right) + (1 - l) \log \left( \frac{1 - l}{1 - p} \right) \text{ for } l \in [0, 1].$$

### 1.3.1 Sharp Large Deviations

Large deviation theory is concerned with estimating the rate of decay of rare events on an exponential scale. In some instances more accurate estimates are possible. These are called sharp or precise large deviations.

Cramer’s Theorem 1.7 implies that there is a sequence $f(m, l)$ such that

$$\mathbb{P}(S_m > ml) \approx f(m, l)e^{-m\Lambda^*(l)}, \quad (1.8)$$

and

$$\lim_{m \to \infty} \frac{1}{m} \log f(m, l) = 0.$$

Since Cramer’s Theorem is concerned only with finding the rate function on an exponential scale, it does not tell us what $f(m, l)$ is. If we can find $f(m, l)$, then we can estimate the probabilities more accurately. $f(m, l)$ in (1.8) was determined explicitly by Bahadur and Rao [4].

**Theorem 1.3.5 (Theorem of Bahadur and Rao).** Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables and let $S_m = \sum_{i=1}^{m} X_i$. Let $\Lambda(\theta) = \log(M(\theta))$ where $M(\theta) = \mathbb{E}(e^{\theta X_1})$. Assume $\Lambda(\theta)$ is analytic at $\theta_l$ and $\Lambda'(\theta_l) = l$. Then,
\[
\lim_{m \to \infty} J_m(l) P(S_m > ml) = 1,
\]

where
\[
J_m(l) = \theta_l \sqrt{\Lambda''(\theta_l) 2\pi m} \exp(m\Lambda^*(l)),
\]

and
\[
\Lambda^*(l) = \sup_{\theta \in \mathbb{R}} \{\theta l - \Lambda(\theta)\}.
\]

Remark:

1. By Theorem 1.3.5, \( f(m, l) \) in (1.8) is determined to be
\[
f(m, l) = C(l) \frac{1}{\sqrt{m}},
\]

where
\[
C(l) = \frac{1}{\theta_l \sqrt{\Lambda''(\theta_l) 2\pi}}.
\]

2. Note that the assumption \( \Lambda'(\theta_l) = l \) implies that \( \Lambda^*(l) \) given in (1.7) can be expressed as
\[
\Lambda^*(l) = \sup_{\theta \in \mathbb{R}} \{\theta l - \Lambda(\theta)\} = \theta l - \Lambda(\theta_l).
\]

1.4 Importance Sampling

Importance Sampling is a variance reduction technique used in Monte Carlo simulations. We discuss an importance sampling procedure for single factor credit risk models in Chapter 3.
Therefore, in this section, we discuss some basics of Monte Carlo simulation and Importance Sampling.

1.4.1 Monte Carlo Simulation

Monte Carlo methods are widely used to approximate the expectations of random variables. Suppose we are confronted with the problem of approximating the expectation of the random variable \( A(X') \) where \( X' \) is a random variable with density \( f(\cdot) \) and \( A : \mathbb{R} \to \mathbb{R} \). Let us denote the quantity we need to approximate by \( \mu' \).

\[
\mu' = \mathbb{E}_f(A(X')) = \int A(x)f(x) \, dx.
\]

Suppose we know how to simulate from the density \( f(\cdot) \) and let \( X'_k \) denote the realization of the \( k \)th simulation so that \( X'_k \) are i.i.d. random variables with common density \( f(\cdot) \). Then,

\[
\alpha_{m,f} = \frac{1}{m} \sum_{k=1}^{m} A(X'_k)
\]

is an unbiased estimator for the quantity \( \mathbb{E}_f(A(X')) \). That is

\[
\mathbb{E}_f(\alpha_{m,f}) = \mu'.
\]

The variance of the estimator \( \alpha_{m,f} \) is given by

\[
\text{Var}_f(\alpha_{m,f}) = \frac{1}{m^2} \sum_{k=1}^{m} \text{Var}_f(A(X'_k))
\]

\[
= \frac{1}{m} \text{Var}_f(A(X')).
\]

If we can find another unbiased estimator \( \beta_m \) such that
\[ E(\beta_m) = \mu' \]

and

\[ \text{Var}(\beta_m) < \text{Var}_f(\alpha_{m,f}) = \frac{1}{m} \text{Var}(A(X')) \]

then \( \beta_m \) may be preferred over \( \alpha_{m,f} \). Importance sampling is a method that is used to find such estimators \( \beta_m \).

### 1.4.2 Importance Sampling

We continue our discussion of the previous section: the problem of estimating \( E_f(A(X')) \).

Let \( g(\cdot) \) be another density such that

\[ g(x) = 0 \implies f(x) = 0, \quad \forall x \in \mathbb{R}. \]

Then

\[
E_f(A(X')) = \int A(x) \frac{f(x)}{g(x)} g(x) \, dx
= \mathbb{E}_g \left( A(Y') \frac{f(Y')}{g(Y')} \right).
\]

Therefore the estimator \( \beta_{m,g} \) given by

\[
\beta_{m,g} = \frac{1}{m} \sum_{k=1}^{m} A(Y'_k) \frac{f(Y'_k)}{g(Y'_k)}
\]

where \( Y'_k \)'s are i.i.d. with density \( g(\cdot) \) is an unbiased estimator for \( \mu' = E_f(A(X')) \). The variance of \( \beta_{m,g} \) is given by
\[
\text{Var}_g(\beta_{m,g}) = \frac{1}{m^2} \sum_{k=1}^{m} \text{Var}_g \left( A(Y'_k) \frac{f(Y'_k)}{g(Y'_k)} \right) \\
= \frac{1}{m} \text{Var}_g \left( A(Y') \frac{f(Y')}{g(Y')} \right).
\]

If \( \text{Var}_g(\beta_{m,g}) < \text{Var}_f(\alpha_{m,f}) \) (where \( \alpha_{m,f} \) is defined in (1.9)), then the estimator \( \beta_{m,g} \) will be preferred over \( \alpha_{m,f} \). Notice that both estimators have the same expected value. Therefore it suffices to compare the second moments. In order to find a better \( \beta_{m,g} \) (compared to \( \alpha_{m,f} \)) we would require that

\[
\mathbb{E}_g \left( A^2(Y') \frac{f^2(Y')}{g^2(Y')} \right) < \mathbb{E}_f \left( A^2(X') \right).
\]

Notice also that the variance of \( \beta_{m,g} \) is

\[
\text{Var}_g(\beta_{m,g}) = \frac{1}{m} \mathbb{E}_g \left( \left( A(Y') \frac{f(Y')}{g(Y')} - \mu' \right)^2 \right) \\
= \frac{1}{m} \int \left( A(x) \frac{f(x)}{g(x)} - \mu' \right)^2 dx.
\]

Therefore the density

\[
g(x) = \frac{A(x)f(x)}{\mu'}
\]

gives a zero variance estimator. This is of course not possible to find since it would require
the knowledge of the quantity $\mu'$, what we are trying to estimate in the first place. In the next section we consider the case of importance sampling for rare events.

### 1.4.3 Importance sampling for rare events

Consider the problem of estimating the probability $P(X' \in B)$ where the random variable $X'$ has density $f(\cdot)$.

$$P(X' \in B) = \mathbb{E}_{f}(\mathbb{1}_{\{X' \in B\}}(X'))$$  \hspace{1cm} (1.10)

Thus in view of the above discussion $A(x) = \mathbb{1}_{\{x \in B\}}$. Therefore the theoretically optimal importance sampling density $g(x)$ is given by

$$g(x) = \frac{\mathbb{1}_{\{x \in B\}}(x)f(x)}{P(X' \in B)}$$

which is exactly the conditional probability distribution of $X'$ given $X' \in B$. Thus when we are looking for an importance sampling density, we should look for densities $g(x)$ that assign a high value to $\int_{x \in B} g(x) \, dx$. In the next section we look into how rare event simulation is related to large deviation theory.

### 1.4.4 Relationship to Large Deviation Theory

Relationship between importance sampling and large deviation theory arises through the computation of rare event probabilities.

Let $\{X'_i\}_{i \in \mathbb{N}}$ be a i.i.d. random variables with density $f(\cdot)$. Let $S_m = \sum_{k=1}^{m} X'_k$. Suppose we are required to calculate the probability $P_{f}(S_m > lm) = \mathbb{E}_{f}(S_m > lm)$ where $l \gg \mathbb{E}_{f}(X'_1)$.

For every $\theta \in \mathbb{R}$,
\[ g_\theta(x) = e^{\theta x - \Lambda_f(\theta)} f(x) \]

where \( \Lambda_f(\theta) = \log\left( \mathbb{E}_f(e^{\theta X_1}) \right) \), defines a probability measure. In using Monte Carlo simulation to compute \( \mathbb{P}_f(S_m > lm) \), instead of sampling from density \( f(\cdot) \), we could sample from \( g_\theta(\cdot) \). But we would want to determine the optimal \( \theta \) for doing so.

\[
\mathbb{P}_f(S_m > lm) = \mathbb{E}_f(1\{S_m > ml\}) = \mathbb{E}_{g_\theta}(1\{S_m > ml\} e^{-\theta S_m + m\Lambda_f(\theta)})
\]

We wish to pick \( \theta \) so that \( \mathbb{E}_{g_\theta} ((1\{S_m > ml\} e^{-\theta S_m + m\Lambda_f(\theta)})^2) \) is minimized.

Define the second moment of the importance sampling estimator

\[
M_m^2(\theta, l) = \mathbb{E}_{g_\theta} \left((1\{S_m > ml\} e^{-\theta S_m + m\Lambda_f(\theta)})^2\right). \tag{1.11}
\]

Note that

\[
\mathbb{E}_{g_\theta} \left((1\{S_m > ml\} e^{-\theta S_m + m\Lambda_f(\theta)})^2\right) \leq e^{-2ml\theta + 2m\Lambda_f(\theta)}.
\]

\[
M_m^2(\theta, l) \leq e^{-2ml\theta + 2m\Lambda_f(\theta)}.
\]

\[
\log M_m^2(\theta, l) \leq -2ml\theta + 2m\Lambda_f(\theta).
\]

Suppose there exists \( \theta_l \) such that

\[
\Lambda^*_f(l) = \sup_{\theta \in \mathbb{R}} (\theta l - \Lambda_f(\theta))
\]

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\[ = \theta l - \Lambda_f(\theta l). \]  

(1.12)

By (1.12),

\[
\frac{1}{m} \log M_m^2(\theta l, l) \leq -2\Lambda^*_f(l).
\]

Therefore

\[
\limsup_{m \to \infty} \frac{1}{m} \log M_m^2(\theta l, l) \leq -2\Lambda^*_f(l). \tag{1.13}
\]

We also have, for any \( \theta \),

\[
(\mathbb{P}_f(S_m > ml))^2 \leq \mathbb{E}_{\theta} \left( \mathbb{1}\{S_m > ml\} e^{-\theta S_m + m\Lambda_f(\theta)} \right)^2 \] by Jensen’s inequality.

Therefore, for any \( \theta \)

\[
(\mathbb{P}_f(S_m > ml))^2 \leq M_m^2(\theta, l).
\]

\[
2 \log \mathbb{P}_f(S_m > ml) \leq \log M_m^2(\theta, l).
\]

\[
2 \liminf_{m \to \infty} \frac{1}{m} \log \mathbb{P}_f(S_m > ml) \leq \liminf_{m \to \infty} \frac{1}{m} \log M_m^2(\theta, l).
\]

Therefore

\[
-2\Lambda^*_f(l) \leq \liminf_{m \to \infty} \frac{1}{m} \log M_m^2(\theta l, l). \tag{1.14}
\]

By (1.13) and (1.14) we have
\[
\lim_{m \to \infty} \frac{1}{m} \log M_m^2(\theta_l, l) = -2\Lambda^*(l).
\]

But we also have that

\[
\lim_{m \to \infty} \frac{1}{m} \log \mathbb{P}_f(S_m > ml) = -\Lambda^*(l).
\]

\[
\lim_{m \to \infty} \frac{\log M_m^2(\theta_l, l)}{\log \mathbb{P}_f(S_m > ml)} = -2.
\]

Thus, as \(\mathbb{P}(\{S_m > ml\}) \to 0\), the ratio \(\frac{\log M_m^2(\theta_l, l)}{\log \mathbb{P}_f(S_m > ml)}\) converges to a limit. Therefore, by choosing \(\theta = \theta_l\) we have found an asymptotically optimal importance sampling distribution \(g_{\theta_l}(\cdot)\).

### 1.5 Organization of the Thesis

In this thesis we study large deviations and rare event simulation algorithms for factor models of portfolio credit risk.

In Chapter 2 and Chapter 3, we study a single factor model where the latent variable of the \(k^{th}\) obligor is given by

\[
X_k = aZ + b\varepsilon_k
\]  \hspace{1cm} (1.15)

where \(Z\) and \(\varepsilon_k\) are random variables, \(\varepsilon_k\) are i.i.d. and \(b = \sqrt{1 - a^2}\) with \(a \neq 0\). All three random variables \(X_k, Z\) and \(\varepsilon_k\) are allowed to take general distributions.
In Chapter 2 we derive with sharp large deviation asymptotics for the portfolio loss. Logarithmic large deviations have been studied for the model in 1.15 by Glasserman et al. in [17] and [15] assuming $X_k, Z$ and $\varepsilon_k$ are distributed as $N(0, 1)$. Our work is inspired by Glasserman and his group but differs in the following sense:

1. We consider general probability distributions for $X_k, Z$ and $\varepsilon_k$.

2. We derive sharp large deviation asymptotic where as Glasserman et al. (see [17], [15]) derive logarithmic large deviations.

In Chapter 3 we develop a rare event simulation algorithm and prove its logarithmic efficiency. This algorithm is the same algorithm developed by Glasserman et al. in [17] assuming all random variables are Standard Gaussian. Our contribution is that we prove its logarithmic efficiency for general distribution functions.

In Chapter 4 we derive logarithmic large deviations for the multifactor Gaussian copula model with finite number of obligor types, previously considered by Glasserman et al. (see [15]). Under their model, if the $k^{th}$ obligor is of type $j$, its latent variable is given by

$$X_k = (a_j)^T Z + b_j \varepsilon_k,$$

(1.16)

where $a_j \in \mathbb{R}^d$ with $0 < a < \|a_j\| < \bar{a} < 1$, $Z$ is a $d$ dimensional standard normal random vector, $b_j = \sqrt{1 - (a_j)^T a_j}$ and $\varepsilon_k$ are independent standard normal random variables.

Glasserman et al. (see [15]) derive logarithmic large deviations for the portfolio loss letting the number of obligors $m$ go to infinity but holding the number of factors fixed at $d$. They also divide the set of obligors into $t$ types where $t \in \mathbb{N}$. We expand on the results by Glasserman et al. (see [15]) and let the number of factors go to infinity together with the number of obligors. To this end we change the model in (1.16) to accommodate $Z$ to be an
$m$ dimensional vector. More specifically, (1.16) is replaced by

$$X_k^{(m)} = (a_j^{(m)})^T Z^{(m)} + b_j^{(m)} \varepsilon_k,$$

where $a_j^{(m)} \in \mathbb{R}^m$ with $0 < a < \|a_j^{(m)}\| < \bar{a} < 1$, $Z^{(m)}$ is a $m$ dimensional standard normal random vector, $b_j^{(m)} = \sqrt{1 - (a_j^{(m)})^T a_j^{(m)}}$ and $\varepsilon_k$ are independent standard normal random variables.

Finally, the concluding remarks are given in Chapter 5.
Chapter 2: Sharp Large Deviations for Single Factor Models

2.1 Introduction

Threshold based factor models are a widely used tool in credit risk modeling (see [30], [12]). Consider the factor model where the latent variable of the \( k \)\(^{th} \) firm is given by

\[
X_k = aZ + b\varepsilon_k.
\]

In the above equation, \( Z \) and \( \varepsilon_k \) are independent random variables, \( \varepsilon_k \) are i.i.d., and \( b = \sqrt{1-a^2} \) with \( 0 < a < 1 \). All three random variables \( X_k, Z \) and \( \varepsilon_k \) are allowed to take general distributions. Let us denote the cumulative distribution functions of \( X_k, Z \) and \( \varepsilon_k \) by \( G(\cdot), F(\cdot) \) and \( H(\cdot) \) respectively. In financial literature, \( Z \) and \( \varepsilon_k \) are referred to as systematic and idiosyncratic risk factors respectively (see [30]). \( X_k \) typically represents the value of the \( k \)\(^{th} \) firm. The \( k \)\(^{th} \) firm defaults if its value falls below a pre-specified threshold. Let

\[
Y_k = 1\{X_k < G^{-1}(p_k)\},
\]

so that the default probability of the \( k \)\(^{th} \) firm is \( p_k \). Let \( m \) be the number of obligors, and let \( L_m = \sum_{k=1}^{m} U_k l_k Y_k \) be the loss of the portfolio, where \( U_k \)'s are random variables taking values in \((0,1]\) and \( l_k \)'s are deterministic. \( U_k \) is the loss given default (LGD) and \( l_k \) is the exposure at default (EAD), so that \( U_k l_k \) is the actual loss suffered. This type of modeling is motivated by Merton’s firm value model (see [26]).
The systematic risk factor $Z$, gives rise to correlations between $X_k$’s. An important feature of (2.1) is that conditioned on $Z$, the latent variables $X_k$’s become independent. The model we study, the one given in (2.1) reduces to the widely used single factor Gaussian Copula when both $Z$ and $\varepsilon_k$ are assumed to be Gaussian. The limiting loss distribution for the Gaussian Copula was first studied by Vasicek (see [33], [32]).

By assuming $U_k, l_k = 1$, $p_k = p$, and $X_k, Z, \varepsilon_k \sim N(0, 1)$, Vasicek established that

$$
\lim_{m \to \infty} \mathbb{P}\left( \frac{L_m}{m} > x \right) = 1 - \Phi \left( \frac{b \Phi^{-1}(x) - \Phi^{-1}(p)}{a} \right). \tag{2.3}
$$

This is known as the large homogeneous portfolio approximation (see [33], [32] and [30]).

In this chapter, we examine the large deviation behavior for the Vasicek’s single factor model when both $Z$ and $\varepsilon_k$ are assumed to take general distribution functions. But unlike Vasicek, we consider heterogeneous portfolios and stochastic loss given defaults (LGD). Our model can be considered to fall under the category of threshold based factor models (see [12]). It is well known that Vasicek’s result (2.3), holds for general distributions. (See for example [30] and [29].) If we assume that $X_k, \varepsilon_k$ and $Z$ have cumulative distribution functions $H(\cdot), G(\cdot), F(\cdot)$, and $\varepsilon_k$ and $Z$ both have symmetric distributions, then the direct generalization of (2.3) is

$$
\lim_{m \to \infty} \mathbb{P}\left( \frac{L_m}{m} > x \right) = 1 - F \left( \frac{b H^{-1}(x) - G^{-1}(p)}{a} \right). \tag{2.4}
$$

It is evident from (2.4) that the event $\{L_m > xm\}$ is not a large deviation event. Large deviations for the Gaussian Copula was derived by Glasserman et al. (see [17]) for the single factor case, and Glasserman et al. (see [15]) for the multifactor case. Following their
work, we identify two large deviation regimes: large loss threshold regime, and small default probability regime. The motivation is as follows: If we replace $x$ in (2.4) by $H(s_m)$ for some $s_m \to \infty$, then we would have a rare event with the loss level getting larger. This is the large loss regime. Similarly, if we replace $p$ in (2.4) by $G(-s_m)$, then we have a rare event with the probability of default going to 0. This is the small default regime. Our work is inspired by Glasserman et al. ([17] and [15]) but is different in the following ways:

1. We consider general distributions for $Z$ and $\varepsilon_k$.

2. We derive sharp large deviations where as they derived logarithmic scale large deviation.

Our analysis reveals that the large deviation regime is always governed by the distribution of $Z$. This is to be expected by (2.4). Glasserman et al. (see [17] and [15]) noted that large loss regime produces a heavier tail compared to the small default regime. We make the same observation, even for general distributions.

The rest of the chapter is organized as follows. In section 2.2, we postulate the model assumptions and state the main theorems. The main results for the large loss threshold regime is given by Theorem 2.2.1, Theorem 2.2.2, and Theorem 2.2.4 where as the main results for the small default regime is given by Theorem 2.2.5. Next we set out to prove these theorems. Since we are dealing with two probability regimes, it turns out that some concepts recur in the proofs. Therefore, we establish some preliminary results in section 2.3 and these will be employed in proving our main theorems. Section 2.4 and section 2.5 deal with large loss threshold regime. In section 2.4 we prove two general theorems, Theorem 2.4.1 and Theorem 2.4.2. These two theorems do not assume any particular distributions for $Z$ and $\varepsilon_k$, but they rather hypothesize certain conditions that need to be satisfied by them (distributions of $Z$ and $\varepsilon_k$). In section 2.5, we apply these two general theorems (Theorem 2.4.1 and Theorem 2.4.2) to prove Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.4. Section 2.6 and 2.7 is devoted to the small default probability regime.
Similar to the large loss regime, we prove two general theorems, Theorem 2.6.1 and Theorem 2.6.2 in section 2.6. These two theorems are used to prove Theorem 2.2.5 in section 2.7.

2.2 Model Assumptions and Main Results

In this section, we state the model assumptions first and then we give the main results for this chapter. In section 2.2.1 we provide a set of assumptions that are universally applicable to both probability regimes. Then we consider each probability regime separately. In section 2.2.2 is concerned with the large loss threshold regime: we first provide the regime specific assumptions, and then we state the main results. Similarly section 2.2.3 is devoted to the small default probability regime; we provide the regime specific assumptions, and then we state the main results.

2.2.1 Model Assumptions

In this section, we provide a set of assumptions, Assumptions GEN, that are universally applicable to both probability regimes. We use the following notation for the large loss regime

\[ m = \text{Number of obligors} \]

\[ Y_k = \text{Default indicator of the k-th obligor. (1 for default and 0 otherwise.)} \]

\[ p_k = \mathbb{P}(Y_k = 1) \]

\[ p_k(z) = \mathbb{P}(Y_k = 1 | Z = z) \]

\[ L_m = \sum_{k=1}^{m} l_k U_k Y_k \text{ where } l_k \text{ is deterministic and } U_k \text{ is random} \]
where as the following is for the small default regime

\[ m = \text{Number of obligors} \]

\[ Y_k^{(m)} = \text{Default indicator of the k-th obligor. (1 for default and 0 otherwise.)} \]

\[ p^{(m)} = \mathbb{P}(Y_k^{(m)} = 1) \]

\[ p^{(m)}(z) = \mathbb{P}(Y_k^{(m)} = 1|Z = z) \]

\[ L_m = \sum_{k=1}^{m} l_k U_k Y_k^{(m)} \text{ where } l_k \text{ is deterministic and } U_k \text{ is random.} \]

The reason for the notational difference is that in the small default regime, we let the default probabilities approach 0, whereas we hold them constant in the large loss regime. \( U_k \) is a random variable taking values in \((0, 1]\). In financial literature, it is known as the loss given default (LGD). It is the loss for a dollar worth of investment, which is also equal to 1 minus the recovery rate. (See Gordy [18] and Kupiec [21].)

The following set of assumptions, Assumptions GEN, are universally applicable for both probability regimes. We then postulate regime specific assumptions Assumptions LL1 for the large loss threshold regime, and Assumptions SD1 for the small default probability regime.

**Assumptions GEN**

1. Latent variable \( X_k \) is defined as follows:

Let \( Z \) be a random variable with density \( f(\cdot) \) and distribution function \( F(\cdot) \) and let \( \varepsilon_k \) be a random variable with density \( h(\cdot) \) and distribution function \( H(\cdot) \). \( Z \) and \( \varepsilon_k \)
are independent for every \( k \). \( \varepsilon_i \) and \( \varepsilon_j \) are independent for \( i \neq j \). Assume \( f(\cdot) \) and \( h(\cdot) \) are symmetric. Let

\[
X_k = aZ + b\varepsilon_k, \tag{2.5}
\]

where \( 0 < a < 1 \) and \( b = \sqrt{1-a^2} \).

2. From (2.5) it follows that

\[
\mathbb{P}(X_k < x|Z = z) = \mathbb{P}(aZ + b\varepsilon_k < x|Z = z)
\]

\[
= \mathbb{P}\left( \varepsilon_k < \frac{x - az}{b} \right)
\]

\[
= H\left( \frac{x - az}{b} \right),
\]

and

\[
\mathbb{P}(X_k < x) = \int H\left( \frac{x - az}{b} \right) f(z) \, dz.
\]

We will denote \( G(x) = \mathbb{P}(X_k < x) \) and \( g(x) = \frac{d}{dx} G(x) \).

3. The total loss from defaults is denoted by \( L_m \). For the large loss regime

\[
L_m = \sum_{k=1}^{m} l_k U_k Y_k,
\]

where as for the small default regime
\[ L_m = \sum_{k=1}^{m} l_k U_k Y_k^{(m)}. \]

\( l_k \) is deterministic and satisfies \( 0 < \underline{l} \leq l_k \leq \bar{l} < \infty \). \( U_k \) are i.i.d. random variables that take values in \([u, 1]\) for some \( 0 < u \leq 1 \). \( U_k \) are independent of \( Z \) and \( \varepsilon_k \). The mean of \( U_k \) will be denoted by \( u_k = u \).

\[ \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} u_k l_k = \lim_{m \to \infty} \frac{1}{m} u \sum_{k=1}^{m} l_k = C \] (2.6)

for some \( 0 < C < \infty \).

Having stated Assumptions GEN which are universally applicable for both probability regimes, we need to consider the two probability regimes separately. In section 2.2.2 we provide the assumptions and the main results for the large loss threshold regime. Then, in section 2.2.3 we provide the assumptions and the main results for the small default probability regime.

### 2.2.2 Main Results for Large Loss Threshold Regime

This section devoted to the large loss threshold regime. The main results for this section is given by Theorem 2.2.1, Theorem 2.2.2, and Theorem 2.2.4. These theorems are proved in section 2.5. Before we state the main results, we provide the regime specific assumptions, Assumptions LL1. We note that the full set of assumptions for large loss threshold regime comprise of Assumptions LL1 and Assumptions GEN.

**Assumptions LL1**
1. The default indicators $Y_k$ are given by

$$Y_k = \mathbb{I}\{X_k < G^{-1}(p_k)\}. \quad (2.7)$$

By (2.7) it follows that $\mathbb{P}(Y_k = 1) = p_k$.

2. The default probabilities $p_k$ satisfy $0 < \underline{p} \leq p_k \leq \bar{p} < 1$.

3. By (2.7) above the default probability of the $k$-th obligor conditioned on $Z = z$ is given by

$$p_k(z) = \mathbb{P}(X_k < G^{-1}(p_k)|Z = z)$$

$$= \mathbb{P}(aZ + b\varepsilon_k < G^{-1}(p_k)|Z = z)$$

$$= \mathbb{P}(aZ + b\varepsilon_k < G^{-1}(p_k))$$

$$= \mathbb{P}\left(\varepsilon_k < \frac{G^{-1}(p_k) - az}{b}\right)$$

$$= H\left(\frac{G^{-1}(p_k) - az}{b}\right).$$

4. The threshold loss is given by

$$x_m = H(ss_m)\sum_{k=1}^{m} l_k u_k = H(ss_m)u\sum_{k=1}^{m} l_k \text{ where } s > 0. \quad (2.8)$$

We are interested in the finding exact asymptotic for $\mathbb{P}(L_m > x_m)$ in the following sense: we will look for a quantity $y_m$ such that

$$\lim_{m \to \infty} y_m \mathbb{P}(L_m > x_m) = 1. \quad (2.9)$$
$y_m$ will typically be given in terms of terms $a, b, s, s_m$. We will also characterize the large deviation regime in terms of $s_m$. This means that once we have assumed certain distribution for $Z$ and $\varepsilon_k$, and determined $y_m$ explicitly, (2.9) may hold, for example when $s_m = \log m$ but not when $s_m = m$. Therefore it is important to characterize the large deviation regime for which (2.9) holds in terms of $s_m$. This task is accomplished.

Before we present our main results for the large loss threshold regime, we introduce a quantity that will be recurring throughout the main results.

$$\gamma = \frac{sb}{|a|}$$

A striking feature in the large loss regime (compared to small default regime) is that even though the speed of convergence $y_m$ given in (2.9) is governed by $Z$, the large deviation regime is governed by $\varepsilon_k$. We will notice in the small default regime, $\varepsilon_k$ has no influence on the large deviation regime. This indicates that for high quality credit, the idiosyncratic risk has more of an influence on the loss distribution. We will give below some theorems that we derived by assuming specific distributions for $Z$ and $\varepsilon_k$, in addition to the main assumptions, $\text{Assumptions GEN}$ and $\text{Assumptions LL1}$.

Even though the large deviation result is always given by the distribution of $Z$, there is no one theorem that can account for all distributions. Therefore we give some results below restricting ourselves to some specific distributions. We consider Gaussian, Exponential and Stretched Exponential distributions. The reason for picking these distributions is that they have been considered as alternative models for stock price distributions (See [24] and [25]). For our sharp large deviation theorems to hold, $Z$ needs to have a heavier tail than $\varepsilon_k$. If it is not the case, it is still possible to obtain logarithmic large deviations. We first consider the case when $\varepsilon_k \sim N(0,1)$.

**Theorem 2.2.1.** Suppose $\text{Assumptions GEN}$ and $\text{Assumptions LL1}$ hold. Let $\gamma$ be
given by (2.10). Suppose $\bar{p} = p = p^*$. Suppose $\varepsilon_k \sim N(0,1)$. Suppose either

1. $\lim_{m \to \infty} \frac{s_m}{\log m} = 0$ and $s > 0$ or

2. $s_m = \sqrt{\log m}$ and $0 < s < 1$.

Then,

1. If $Z \sim N(0,1)$ then,

$$
\lim_{m \to \infty} \sqrt{2\pi\gamma s_m} e^{\frac{1}{2}\left(\gamma s_m - \frac{G^{-1}(p^*)}{|a|}\right)^2} P(L_m > x_m) = 1.
$$

2. If $Z \sim \text{Gamma}(\alpha, \beta)$ then,

$$
\lim_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta \alpha \gamma^{-1} s_m^{\alpha-1}} e^{\beta\left(\gamma s_m - \frac{G^{-1}(p^*)}{|a|}\right)} P(L_m > x_m) = 1.
$$

The admissible large deviation regime characterized by $s_m$ in Theorem 2.2.1 comes from the fact that $\varepsilon_k \sim N(0,1)$. It is clear that the large deviation result come from the distribution of $Z$.

McCaulley and Gunaratne (see [25]) considered modeling stock returns with exponential distributions. In our next theorem we let both $Z$ and $\varepsilon_k$ be Exponential distributed. That is, $Z \sim \text{Exp}(\lambda)$ and $\varepsilon_k \sim \text{Exp}(\lambda_0)$. Once again we need to make the restriction that $\lambda_0 \leq \lambda$ in order to ensure that $Z$ is heavier tailed than $\varepsilon_k$.

**Theorem 2.2.2.** Suppose [Assumptions GEN] and [Assumptions LL1] hold. Let $\gamma$ be given by (2.10). Suppose $\bar{p} = p = p^*$. Suppose $\varepsilon_k \sim \text{Exp}(\lambda)$ and $Z \sim \text{Exp}(\lambda_0)$ with $\lambda_0 \leq \lambda$. Suppose either

1. $\lim_{m \to \infty} \frac{s_m}{\log m} = 0$ and $s > 0$ or
2. \( s_m = \log m \) and \( 0 < s < 1 \).

Then,

\[
\lim_{m \to \infty} e^{\lambda_0 \left( \gamma s_m - \frac{G^{-1}(p^*)}{|\sigma|} \right)} \mathbb{P}(L_m > x_m) = 1.
\]

The admissible large deviation regime of \( s_m \) in Theorem 2.2.2 owes to the fact that \( \varepsilon_k \sim \text{Exp}(\lambda) \).

We conclude the main results for this section by considering the case when \( \varepsilon_k \sim \text{Stretched Exponential} \). Stretched Exponential distribution has recently been proposed for modeling stock prices. For example, Sornette et al. ([24]) considered Stretched Exponential and Pareto distributions for modeling stock prices.

**Definition 2.2.3** (Stretched Exponential Distribution). A random variable \( Z \) is said to be Stretched Exponentially distributed with parameters \( k, c_1(\cdot), c_2(\cdot), b(\cdot) \), denoted by \( Z \sim \text{Str Exp} (k, c_1(\cdot), c_2(\cdot), b(\cdot)) \), if there exists \( k \in (0, 1) \) and slowly varying functions \( b, c_1, c_2 : (0, \infty) \to (0, \infty) \) and \( t_0 > 0 \) such that for \( t > t_0 \):

\[
c_1(t) \exp(-b(t)t^k) \leq \mathbb{P}(Z > t) \leq c_2(t) \exp(-b(t)t^k).
\]

**Theorem 2.2.4.** Suppose **Assumptions GEN** and **Assumptions LL1** hold. Suppose \( \bar{p} = p = p^* \). Suppose \( \varepsilon_i \sim \text{Str Exp}(k, c_1(\cdot), c_2(\cdot), b(\cdot)) \). Suppose \( \lim_{m \to \infty} \frac{c_1(s_m)}{c_2(s_m)} = 1 \). Suppose either

1. \( \lim_{m \to \infty} \frac{b(s_m)s_m^k}{\log m} = 0 \) and \( s > 0 \) or

2. \( b(s_m)s_m^k = \log m \) and \( s^k < \frac{1}{2} \).

Then,
1. If \( Z \sim \text{Str Exp}(k_0, c_{10}^0(\cdot), c_{20}^0(\cdot), b^0(\cdot)) \) with either

(a) \( k_0 < k \) or

(b) \( k = k_0 \) and \( \lim_{m \to \infty} b^0(s_m) = B < \infty \) then,

\[
\limsup_{m \to \infty} \frac{1}{c_{21}(s_m)} e^{b^0(s_m) c_{20}(s_m) s_m} \mathbb{P}(L_m > x_m) \leq 1
\]

and

\[
\liminf_{m \to \infty} \frac{1}{c_{11}(s_m)} e^{b^0(s_m) c_{10}(s_m) s_m} \mathbb{P}(L_m > x_m) \geq 1.
\]

2. If \( Z \sim \text{Paretto}(\alpha, \beta) \) then,

\[
\lim_{m \to \infty} \left( \frac{\gamma s_m}{\beta} \right)^\alpha \mathbb{P}(L_m > x_m) = 1.
\]

Having stated both the assumptions and the main results for the large loss threshold regime, we next consider the small default probability regime.

### 2.2.3 Main Results for the Small Default Regime

This section devoted to the small default probability regime. The main results for this section is given by Theorem 2.2.5. This theorem is proved in section 2.7. Before we state the main results, we provide the regime specific assumptions, \textbf{Assumptions SD1} and \textbf{Assumptions SD2}. We note that the full set of assumptions for small default probability regime comprise of \textbf{Assumptions SD1}, \textbf{Assumptions SD2} and \textbf{Assumptions GEN}.

**Assumptions SD1**
1. The default indicators $Y_{k}^{(m)}$ are given by

$$Y_{k}^{(m)} = 1 \{ X_k < -s s_m \},$$  \hspace{1cm} (2.11)

for some sequence $s_m \to \infty$ and $0 < s$. This implies that default probability of the $k$-th obligor is

$$p^{(m)} = P \left( Y_{k}^{(m)} = 1 \right) = G(-s s_m).$$

2. By (2.11) above the default probability of the $k$-th obligor conditioned on $Z$ is given by

$$p^{(m)}(z) = P \left( Y_k = 1 | Z = z \right)$$

$$= P \left( X_k < -s s_m | Z = z \right)$$

$$= P \left( aZ + b \varepsilon_k < -ss_m | Z = z \right)$$

$$= P \left( \varepsilon_k < \frac{-ss_m - az}{b} \right)$$

$$= H \left( \frac{-az - ss_m}{b} \right).$$

3. The threshold loss is given by

$$x_m = q \sum_{k=1}^{m} l_k u_k = qu \sum_{k=1}^{m} l_k$$  \hspace{1cm} (2.12)

where $\frac{1}{2} < q < 1$.  

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We are interested in computing the rare event asymptotic for $\mathbb{P}(L_m > x_m)$ as $m \to \infty$. Similar to the large loss threshold regime, in the small default probability regime, the large deviation asymptotic is governed by the distribution of $Z$. But there is an important distinction. In the large loss threshold regime, the large deviation regime (characterized by $s_m$) was governed by $\varepsilon_k$. In the Small Default regime, the distribution of $\varepsilon_k$ has no effect on the large deviation regime. All that we require is that density of $\varepsilon_k$, $h(\cdot)$ is continuous at $q$ and $h(q) \neq 0$. (Recall that in this regime the threshold loss is given by $x_m = q \sum_{k=1}^{m} u_k l_k$.) We state this as Assumptions SD2. As mentioned before, the full set of assumptions for small default regime comprise of Assumptions GEN, Assumptions SD1 and Assumptions SD2.

**Assumptions SD2**

1. The density of $\varepsilon_k$, $h(\cdot)$ does not vanish at $H^{-1}(q)$: $h(H^{-1}(q)) \neq 0$.

2. $h(\cdot)$ is continuous on a neighborhood of $H^{-1}(q)$.

Before we give the main results for this section, we define an important quantity for this regime. We define

$$\gamma^* = \frac{s}{|a|}. \quad (2.13)$$

The following theorem summarizes the main results for the small default probability regime.

**Theorem 2.2.5.** Suppose Assumptions GEN, Assumptions SD1 and Assumptions SD2 hold. Let $\gamma^*$ be given by (2.13). Then,

1. If $Z \sim N(0, 1)$ and $\lim_{m \to \infty} \frac{s_m}{m^{\frac{1}{4}}} = 0$ then,

$$\lim_{m \to \infty} \sqrt{2\pi \gamma^* s_m e^{\frac{1}{2}\left(\gamma^* s_m + \frac{6}{m} H^{-1}(q)\right)^2}} \mathbb{P}(L_m > x_m) = 1.$$
2. If $Z \sim \text{Exp}(\lambda)$ and $\lim_{m \to \infty} \frac{s_m}{m^3} = 0$ then, $\lim_{m \to \infty} e^{\lambda \left(\gamma s_m + \frac{b}{c_1(s_m)} H^{-1}(q)\right)} P(L_m > x_m) = 1$.

3. If $Z \sim \text{Gamma}(\alpha, \beta)$ with $\alpha > 1$ and $\lim_{m \to \infty} \frac{s_m}{m^3} = 0$ then,

$$\lim_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta \alpha s_m^{\alpha - 1} e^{-\beta \left(\gamma s_m + \frac{b}{c_1(s_m)} H^{-1}(q)\right)}} P(L_m > x_m) = 1.$$ 

4. If $Z \sim \text{Str Exp}(k, c_1(\cdot), c_2(\cdot), b(\cdot))$ and $\lim_{m \to \infty} \frac{s_m^{3k}}{mb(s_m)} = 0$ then,

$$\limsup_{m \to \infty} \frac{1}{c_1(s_m)} e^{b(s_m) \gamma s_m^{k} \left(\gamma s_m^{k} \right)} P(L_m > x_m) \leq 1$$

and

$$\liminf_{m \to \infty} \frac{1}{c_2(s_m)} e^{b(s_m) \gamma s_m^{k} \left(\gamma s_m^{k} \right)} P(L_m > x_m) \geq 1.$$ 

5. If $Z \sim \text{Paretto}(\alpha, \beta)$ and there exists $j_0 > 0$ $\lim_{m \to \infty} \frac{s_m^{j_0}}{m} = 0$ then,

$$\lim_{m \to \infty} \left(\frac{\gamma s_m}{\beta}\right)^{\alpha} P(L_m > x_m) = 1.$$

### 2.3 Some Preliminary Results

Having stated the main results, our next task is to prove Theorem 2.2.1, Theorem 2.2.2, Theorem 2.2.4, and Theorem 2.2.5.

We are dealing with two probability regimes and it turn out there are some similarities between our main proofs. In this section we introduce some tools that we will use in common for both the probability regimes. In section 2.3.1 we introduce the tools for the
upper bound computation and in section 2.3.2 we introduce the tools for the lower bound computation.

### 2.3.1 Tools for the Upper Bound Computation

Define the cumulant generating function of $U_k$ by

$$\Lambda(\theta) = \log \mathbb{E}(e^{\theta U_k})$$  \hspace{1cm} (2.14)

which obviously holds for both regimes. (Note that $U_k$ are i.i.d random variables taking values in $[u, 1]$). Define the cumulant generating function of $L_m$ conditioned on $Z$ by

$$\psi_m(\theta, z) = \frac{1}{m} \log \mathbb{E}(e^{\theta L_m} | Z = z).$$  \hspace{1cm} (2.15)

Let

$$\theta_m(z) = \arg\min_{\theta \geq 0} \{-\theta x_m + m\psi_m(\theta, z)\}.$$  \hspace{1cm} (2.16)

Define a new measure $P_m$ by

$$\frac{dP}{d\mathbb{P}} = e^{-\theta_m(Z)L_m + m\psi_m(\theta_m(Z), Z)},$$  \hspace{1cm} (2.17)

and let $E_m$ denote the expectation under $P_m$. The next lemma shows that $P_m$ is indeed a probability measure.

**Lemma 2.3.1.** $P_m(\cdot)$ defined by (2.17) is a probability measure.

**Proof.** It suffices to show that $E \left( \left( \frac{dP}{d\mathbb{P}} \right)^{-1} \right) = E \left( e^{\theta_m(Z)L_m - m\psi_m(\theta_m(Z), Z)} \right) = 1.$
\[
E\left(e^{\theta_m(Z)L_m - m\psi_m(\theta_m(Z), Z)}\right) = E\left(e^{\theta_m(Z)L_m - \log(E(e^{\theta_m(Z)L_m}|Z))}\right) \\
= E\left(e^{\theta_m(Z)L_m}\left(E\left(e^{\theta_m(Z)L_m}\right)\right)^{-1}\right) \\
= E\left(E\left(e^{\theta_m(Z)L_m}\left(E\left(e^{\theta_m(Z)L_m}\right)\right)^{-1}|Z\right)\right) \\
= E\left(E\left(e^{\theta_m(Z)L_m}|Z\right)^{-1} E\left(e^{\theta_m(Z)L_m}|Z\right)\right) \\
= 1.
\]

\[\square\]

It should be noted that \(\psi_m(\theta, z)\) defined above takes different expressions depending on what probability regime we are dealing with. In the large loss regime:

\[
\psi_m(\theta, z) = \frac{1}{m} \log E\left(e^{\theta L_m}| Z = z\right) \\
= \frac{1}{m} \log E\left(e^{\theta \sum_{k=1}^{m} l_k U_k Y_k}| Z = z\right) \\
= \frac{1}{m} \sum_{k=1}^{m} \log E\left(e^{\theta l_k U_k Y_k}| Z = z\right) \quad \text{(by conditional independence)} \\
= \frac{1}{m} \sum_{k=1}^{m} \log E\left(e^{\theta l_k U_k Y_k}| U_k, Z = z\right)|Z = z) \\
= \frac{1}{m} \sum_{k=1}^{m} \log \left(1 + p_k(z)(e^{\theta l_k U_k} - 1)|Z = z\right) \\
= \frac{1}{m} \sum_{k=1}^{m} \log \left(1 + p_k(z)\left(E[e^{\theta l_k U_k}| Z = z]\right) - 1\right)
\]
Clearly, a similar calculation holds for the small default regime as well. We opt to use the same notation $\psi_m(\theta, z)$ for economy since it would be obvious from the context. We summarize this as follows.

$$\psi_m(\theta, z) = \begin{cases} 
\frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k(z) \left( e^{\Lambda(\theta_l)} - 1 \right) \right) & \text{for the large loss regime} \\
\frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k^{(m)}(z) \left( e^{\Lambda(\theta_l)} - 1 \right) \right) & \text{for the small default regime}
\end{cases}. \quad (2.18)$$

The following is a restatement of Lemma 4.3 in [15].

**Lemma 2.3.2.** Let $\Lambda(\cdot)$ be the cumulant generating function of $U_k$ given by $(2.14)$. Then, there exists a positive constant $D$ such that

$$\log \left( 1 + \alpha \left( e^{\Lambda(\theta)} - 1 \right) \right) \leq \alpha u_k \theta + D \theta^2,$$

for all $\theta \in [0, 1]$ and $\alpha \in [0, 1]$. ($u_k = u$ is the mean of $U_k$.)

**2.3.2 Tools for the Lower Bound Computation**

The following is a restatement of Lemma 3.10 in [15].

**Lemma 2.3.3.** Suppose a sequence of events $\{A_m\}_{m \in \mathbb{N}}$ and a sequence of positive integers $\{n_m\}_{m \in \mathbb{N}}$ with $\lim_{m \to \infty} n_m = \infty$ are given. Suppose that, given $A_m$, $T_k^{(m)}$, $k = 1, 2, ..., n_m$ are conditionally independent random variables with conditional mean $\theta$ for which,
\[
\limsup_{m \to \infty} \frac{1}{(n_m)^2} \sum_{k=1}^{n_m} \text{Var} \left( T_k^{(m)} \mid A_m \right) = 0.
\]

Let

\[ S_m = \frac{1}{m} \sum_{k=1}^{m} T_k^{(m)}. \]

Then,

\[
\lim_{m \to \infty} \mathbb{P}(|S_m| > \epsilon |A_m) = 0.
\]

(Note that \( T_k^{(m)} \) and \( T_l^{(m)} \) may have different distributions for \( k \neq l \).)

Note that

\[
\mathbb{E}(L_m|Z) = \begin{cases} 
\sum_{k=1}^{m} l_k u_k p_k(Z) & \text{for the large loss regime} \\
\sum_{k=1}^{m} l_k u_k p^{(m)}(Z) & \text{for the small default regime}
\end{cases}.
\]

But the following theorem holds for both the probability regimes nonetheless.

**Lemma 2.3.4.** Let \( \eta(s_m) : [0, \infty] \to [0, \infty] \) be a function such that

\[
\lim_{m \to \infty} \frac{\eta(s_m)}{\sqrt{m}} = 0.
\]

Let \( z^{(m)} \) be an arbitrary sequence in \( \mathbb{R} \) and let

\[ S_m = \frac{\eta(s_m)}{m} (L_m - \mathbb{E}(L_m|Z)). \]
Then,

\[
\lim_{m \to \infty} \mathbb{P} \left( |S_m| > \epsilon \bigg| Z = z^{(m)} \right) = 0.
\]

**Proof.** We give the proof for the large loss regime. It is easily seen that a similar proof holds for the small default probability regime as well.

Apply Lemma 2.3.3 with \( A_m = \{ Z = z^{(m)} \} \), \( n_m = m \)

\[
T_k^{(m)} = \eta(s_m) l_k(U_kY_k - u_kp_k(Z)).
\]

- \( T_k^{(m)} \)'s are conditionally independent given \( Z \).

- \( \mathbb{E}(T_k^{(m)} | Z = z^{(m)}) = 0 \) since \( U_k \) is independent of \( Z \) and \( Y_k \).

- We will show that

\[
\limsup_{m \to \infty} \frac{1}{m^2} \text{Var}(T_k^{(m)} | Z = z^{(m)}) = 0.
\]

Note that

\[
\text{Var} \left( T_k^{(m)} | Z = z^{(m)} \right) = (\eta(s_m))^2 l_k^2 \mathbb{E} \left( (U_kY_k - u_kp_k(Z))^2 | Z = z^{(m)} \right),
\]

\[
\leq 4(\eta(s_m))^2 \bar{l}^2
\]
and therefore

\[ \frac{1}{m^2} \sum_{k=1}^{m} \text{Var} \left( T_k^{(m)} \mid Z = z^{(m)} \right) \leq 4 m (\bar{l})^2 \frac{1}{m^2} 4(\eta(s_m))^2 \]

\[ \leq 4 (\bar{l})^2 \frac{1}{m} 4(\eta(s_m))^2 \]

\[ \to 0. \]

Therefore,

\[ \limsup_{m \to \infty} \frac{1}{m^2} \text{Var} \left( T_k^{(m)} \mid Z = z^{(m)} \right) = 0. \]

Therefore, for any \( \epsilon > 0 \),

\[ \lim_{m \to \infty} P \left( |S_m| > \epsilon \mid Z = z^{(m)} \right) = 0. \]

With these tools at our disposal, we consider the two probability regimes separately.

### 2.4 Analysis of the Large Loss Regime

In this section, we prove two general theorems, Theorem 2.4.1 and Theorem 2.4.2. Theorem 2.4.1 deals with the upper bound computation, where as Theorem 2.4.2 deals with the lower bound computation. These two theorems will be used to prove Theorem 2.2.1, Theorem 2.2.2, and Theorem 2.2.4.

We first introduce some notation for this section. Define, for \( \xi \in \mathbb{R} \),
\[ q_m^\varepsilon(\xi) = H(s(1 + \xi \epsilon_m)s_m), \quad (2.20) \]

which proves to be a useful quantity for this section. We also reserve the notation \( \theta_m^\star \) and \( \epsilon_m \) for two positive sequences such that \( \theta_m^\star, \epsilon_m \to 0 \) and \( \xi_0 > 0 \). Define
\[
\gamma = \frac{sb}{|a|}. \quad (2.21)
\]

Large losses are more likely to occur for those values of \( z \) for which \( p_k(z) \) is large. We exploit this fact in the following manner. A sequence of real numbers \( \mu^{(m)} \) and a sequence of sets \( G^{(m)} \) defined by
\[
\mu^{(m)} = \gamma(1 - \xi_0 \epsilon_m)s_m - \frac{G^{-1}(\bar{p})}{|a|}, \quad (2.22)
\]
and
\[
G^{(m)} = \{ z \in \mathbb{R} : -a.z \geq sb(1 - \xi_0 \epsilon_m)s_m - G^{-1}(\bar{p}) \}
\]
\[
= \left\{ z \in \mathbb{R} : H\left(\frac{G^{-1}(\bar{p}) - az}{b}\right) \geq H(s(1 - \xi_0 \epsilon_m)s_m) \right\}
\]
\[
= \left\{ z \in \mathbb{R} : H\left(\frac{G^{-1}(\bar{p}) - az}{b}\right) \geq q_m^\varepsilon(-\xi_0) \right\} \quad (2.23)
\]
will be used for the upper bound computation. \( \mu^{(m)} \) and \( G^{(m)} \) are related as follows:

\[
G^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq - \left( \frac{sb(1-\xi_0 \epsilon_m)s_m-G^{-1}(\bar{\rho})}{|a|} \right) = -\mu^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \frac{sb(1-\xi_0 \epsilon_m)s_m-G^{-1}(\bar{\rho})}{|a|} = \mu^{(m)} \} & \text{if } a < 0
\end{cases}
\]  

(2.24)

Similarly for the lower bound computation, we define \( \nu^{(m)} \) and \( H^{(m)} \) by

\[
\nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m)s_m - \frac{G^{-1}(\bar{\rho})}{|a|},
\]

(2.25)

and

\[
H^{(m)} = \{ z \in \mathbb{R} : -a.z \geq sb(1 + \xi_0 \epsilon_m)s_m - G^{-1}(\bar{\rho}) \}
\]

\[
= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(\bar{\rho}) - az}{b} \right) \geq H(s(1 + \xi_0 \epsilon_m)s_m) \right\}
\]

\[
= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(\bar{\rho}) - az}{b} \right) \geq q^c_m(\xi_0) \right\}.
\]

(2.26)

\( \nu^{(m)} \) and \( H^{(m)} \) are related by

\[
H^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq - \left( \frac{sb(1+\xi_0 \epsilon_m)s_m-G^{-1}(\bar{\rho})}{|a|} \right) = -\nu^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \frac{sb(1+\xi_0 \epsilon_m)s_m-G^{-1}(\bar{\rho})}{|a|} = \nu^{(m)} \} & \text{if } a < 0
\end{cases}
\]

(2.27)

As we mentioned in the introduction, the large deviation asymptotic is governed by \( Z \). But it turns out that the cdf of \( X_k \) has to satisfy certain lower bounds, namely for \( q^c_m(0) - q^c_m(-\xi_0) \) and \( q^c_m(\xi_0) - q^c_m(0) \), in order for our large deviation results to hold.
2.4.1 Upper Bound Computation

We state the main theorem for the upper bound computation below.

**Theorem 2.4.1.** Suppose Assumptions GEN and Assumptions LL1 hold. Let $\xi_0 > 0$ and $\epsilon_m > 0$ such that $\epsilon_m \to 0$. Let $\gamma$ be given by (2.21), let $q'_m(\xi)$ be given (2.20), and let $\mu^{(m)}$ be given by (2.22).

Then, there exists $D > 0$ such that for any positive sequence $\theta^*_m \to 0$

$$\mathbb{P}(L_m > x_m) \leq e^{-\theta^*_m mL_u(q'_m(0)-q'_m(-\xi_0))} + mD^2(\theta^*_m)^2 + \mathbb{P}(Z > \mu^{(m)}),$$

(2.28)

where $u = \mathbb{E}(U_k)$.

Suppose further that we can pick functions $\Psi_U(\cdot)$ and $\psi(\cdot)$, and positive sequences $\epsilon_m, \theta^*_m \to 0$ such that the three equations given by (A1), (A2), (A3) are satisfied.

1. 

$$\limsup_{m \to \infty} \Psi_U(s_m)\mathbb{P}(Z > \mu^{(m)}) \leq \psi(\xi_0) \quad \text{(A1)}$$

and

$$\lim_{\xi_0 \to 0^+} \psi(\xi_0) = 1. \quad \text{(A2)}$$

2. 

$$\lim_{m \to \infty} \Psi_U(s_m)e^{-\theta^*_m mL_u(q'_m(0)-q'_m(-\xi_0))} + mD^2(\theta^*_m)^2 = 0. \quad \text{(A3)}$$

Then,
$$\limsup_{m \to \infty} \Psi_U(s_m) \mathbb{P}(L_m > x_m) \leq 1. \quad (2.29)$$

**Proof.** First note that (2.29) would follow by (A1) and (A2) given that (2.28) is true. Hence we need only prove (2.28). We recall the following. By (2.18)

$$\psi_m(\theta, z) = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k(z) \left( e^{\Lambda(\theta_k)} - 1 \right) \right),$$

by (2.16)

$$\theta_m(z) = \argmin_{\theta \geq 0} \left\{ -\theta x_m + m\psi_m(\theta, z) \right\},$$

and by (2.17) \( \mathbb{P}_m \) was defined to be

$$\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z) L_m + m\psi_m(\theta_m(Z), Z)},$$

where \( \mathbb{E}_m \) denotes the expectation under \( \mathbb{P}_m \).

\[
\begin{align*}
\mathbb{P}(L_m > x_m) &= \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{c}}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{e}}(Z) \right) \\
&= \mathbb{E}_m \left( e^{-\theta_m(Z) L_m + m\psi_m(\theta_m(Z), Z)} \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{c}}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{e}}(Z) \right) \\
&\leq \mathbb{E}_m \left( e^{-\theta_m(Z) x_m + m\psi_m(\theta_m(Z), Z)} \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{c}}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^{(m)})_{e}}(Z) \right) \\
&\leq \mathbb{E}_m \left( e^{-\theta_m(Z) x_m + m\psi_m(\theta_m(Z), Z)} \mathbb{1}_{(G^{(m)})_{c}}(Z) \right) + \mathbb{E} \left( \mathbb{1}_{(G^{(m)})_{e}}(Z) \right)
\end{align*}
\]
\( \leq \mathbb{E}_m \left( e^{-\theta^*_m x_m + m \psi_m(\theta^*_m, Z)} \mathbb{1}_{(G^{(m)})^c}(Z) \right) + \mathbb{P}(Z \in G^{(m)}) \) for any \( \theta^*_m \geq 0. \) \hfill (2.30)

(2.30) follows since \( -\theta_m(z)x_m + m \psi_m(\theta_m, z) \leq -\theta^*_m x_m + m \psi_m(\theta^*_m, z) \) for any \( \theta^*_m \geq 0. \)

By (2.24)

\[
G^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\mu^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \mu^{(m)} \} & \text{if } a < 0
\end{cases}
\] \hfill (2.31)

Therefore, by symmetry of \( Z, \)

\[ \mathbb{P}(Z \in G^{(m)}) = \mathbb{P}(Z > \mu^{(m)}). \]

Hence, we have by (2.30)

\[ \mathbb{P}(L_m > x_m) \leq \mathbb{E}_m \left( e^{-\theta^*_m x_m + m \psi_m(\theta^*_m, Z)} \mathbb{1}_{(G^{(m)})^c}(Z) \right) + \mathbb{P}(Z < \mu^{(m)}) \) for any \( \theta^*_m \geq 0. \) \hfill (2.32)

Now we need to handle the first term of the RHS of the above equation.

Notice that by (2.23), for \( z \in (G^{(m)})^c, \)

\[
p_k(z) = H \left( \frac{G^{-1}(p_k) - az}{b} \right) \leq H \left( \frac{G^{-1}(\bar{p}) - az}{b} \right)
\]
\[ x_m = H(\mathbf{s}_m) \sum_{k=1}^{m} l_k u_k. \]

Therefore with our new notation

\[ x_m = q_m^\epsilon(0) \sum_{k=1}^{m} l_k u_k. \]

At this point we will assume \( \theta_m^* \to 0 \). We will specify \( \theta_m^* \) depending on specific distributions we are looking at.

For \( z \in (G^{(m)})^c \),

\[ -\theta_m^* x_m^* + m \psi_m(\theta_m^*, Z) = -\theta_m^* q_m^\epsilon(0) \sum_{k=1}^{m} l_k u_k + \sum_{k=1}^{m} \log \left( 1 + p_k(z)(e^{\Lambda(\theta_m^* l_k)} - 1) \right) \]

\[ = \sum_{k=1}^{m} \left( -\theta_m^* q_m^\epsilon(0) l_k u_k + \log \left( 1 + p_k(z)(e^{\Lambda(\theta_m^* l_k)} - 1) \right) \right) \]

\[ \leq \sum_{k=1}^{m} \left( -\theta_m^* q_m^\epsilon(0) l_k u_k + p_k(z) u_k l_k \theta_m^* + D\bar{l}^2(\theta_m^*)^2 \right) \]

(for large \( m \) by (2.32))

\[ \leq \sum_{k=1}^{m} \left( -\theta_m^* l_k u_k(q_m^\epsilon(0) - q_m^\epsilon(-\xi_0)) + D\bar{l}^2(\theta_m^*)^2 \right) \]

(by (2.33))

\[ \leq \sum_{k=1}^{m} \left( -\theta_m^* l_k u_k(q_m^\epsilon(0) - q_m^\epsilon(-\xi_0)) + D\bar{l}^2(\theta_m^*)^2 \right) \]

\[ = q_m^\epsilon(-\xi_0). \] (2.33)
\[ \sum_{k=1}^{m} (-\theta_m^* l u_k (q_m^*(0) - q_m^*(\xi_0)) + Dl^2(\theta_m^*)^2) \]

(since \( q_m^*(0) - q_m^*(\xi_0) < 0 \))

\[ = -\theta_m^* m l u (q_m^*(0) - q_m^*(\xi_0)) + m Dl^2(\theta_m^*)^2. \]

\[ E_m \left( e^{-\theta_m^* x_m + m \psi_m(\theta_m^*, \xi)} \mathbb{1}_{(G_m)^c}(Z) \right) \]

\[ = E_m \left( e^{-\theta_m^* x_m + m \psi_m(\theta_m^*, \xi)} \mathbb{1}_{(G_m)^c}(Z) \right) \]

\[ \leq E_m \left( e^{-\theta_m^* \xi_m(0) \sum_{k=1}^{m} l k u_k + \sum_{k=1}^{m} \log \left( 1 + p_k(z) (e^{\lambda(\theta_m^* l k)} - 1) \right) } \right) \]

\[ \leq e^{-\theta_m^* m l u (q_m^*(0) - q_m^*(-\xi_0)) + m Dl^2(\theta_m^*)^2} \]

Therefore, by (2.32)

\[ \mathbb{P}(L_m > x_m) \leq e^{-\theta_m^* m l u (q_m^*(0) - q_m^*(-\xi_0)) + m Dl^2(\theta_m^*)^2} + \mathbb{P}(Z > \mu_m). \]

\[ \square \]

### 2.4.2 Lower Bound Computation

We give below the main theorem for the computation of the lower bound below.

**Theorem 2.4.2.** Suppose **Assumptions GEN** and **Assumptions LL1** hold. Let \( \xi_0 > 0 \) and \( \epsilon_m > 0 \) such that \( \epsilon_m \to 0 \). Let \( \gamma \) be given by (2.21), let \( q_m^*(\xi) \) be given by (2.20), and let \( \nu^{(m)} \) be given by (2.25). Let \( \eta : [0, \infty] \to [0, \infty] \) be such that

- There exists and \( d_0(\xi_0) \) and \( M_1 = M_1(\xi_0) \) such that for \( m > M_1 \)
\[
q'_m(\xi_0) - q'_m(0) \geq d_0(\xi_0) \frac{1}{\eta(s_m)} \tag{B1}
\]

\[
\lim_{m \to \infty} \frac{(\eta(s_m))^2}{m} = 0 \tag{B2}
\]

Then,

\[
P(L_m > x_m) \geq k_m P(Z > \nu^{(m)}) \tag{2.34}
\]

for some sequence \(k_m \to 1\).

Suppose further that there exists functions \(\Psi_L(\cdot)\) and \(\psi(\cdot)\) such that

\[
\lim_{m \to \infty} \Psi_L(s_m)P(Z > \nu^{(m)}) \geq \psi(\xi_0) \tag{B3}
\]

and

\[
\lim_{\xi_0 \to 0} \psi(\xi_0) = 1. \tag{B4}
\]

Then,

\[
\lim_{m \to \infty} \inf \Psi_L(s_m)P(L_m > x_m) \geq 1. \tag{2.35}
\]

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Proof. First note that \(2.35\) would follow from \(B3\) and \(B4\), given that \(2.34\) is true. Therefore we need only prove \(2.34\). Let

\[ S_m = \frac{\eta(s_m)}{m} (L_m - \mathbb{E}(L_m|Z)) \tag{2.36} \]

and \(z^{(m)}\) be an arbitrary sequence in \(\mathbb{R}\). Then, by Lemma \(2.3.4\) and \(B2\), for any \(\epsilon > 0\)

\[ \lim_{m \to \infty} P(|S_m| > \epsilon|Z = z^{(m)}) = 0. \tag{2.37} \]

**Lemma 2.4.3.** Let \(z^{(m)}_{*} \in H^{(m)}\). Then,

\[ \lim_{m \to \infty} P(L_m > x_m|Z = z^{(m)}_{*}) = 1. \tag{2.38} \]

Proof. Let \(z^{(m)}_{*} \in H^{(m)}\).

\[ \mathbb{E}(L_m|Z = z^{(m)}_{*}) \geq q_m^c(\xi_0) \sum_{k=1}^{m} l_k u_k. \tag{2.39} \]

Therefore for \(m > M_1(\xi_0)\)

\[
\eta(s_m) \frac{1}{m} \left( x_m - \mathbb{E}(L_m|Z = z^{(m)}_{*}) \right) \leq \eta(s_m) \frac{1}{m} \left( x_m - q_m^c(\xi_0) \sum_{k=1}^{m} l_k u_k \right) \\
= \eta(s_m) \frac{1}{m} \left( q_m^c(0) - q_m^c(\xi_0) \right) \sum_{k=1}^{m} l_k u_k \\
= -\eta(s_m) \frac{1}{m} \left( q_m^c(\xi_0) - q_m^c(0) \right) \sum_{k=1}^{m} l_k u_k \\
\leq -\eta(s_m) \frac{1}{m} d_0(\xi_0) \frac{1}{\eta(s_m)} \sum_{k=1}^{m} l_k u_k \tag{by \(B1\)}
\]
\[ -d_0(\xi_0) \frac{1}{m} \sum_{k=1}^{m} l_k u_k \]  
\[ \rightarrow -d_0(\xi_0) C, \] (2.40)

where

\[ C = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} l_k u_k. \]

Let \( 0 < \epsilon < d_0(\xi_0) C \). There exists \( M_2 = M_2(\xi_0) \in \mathbb{N} \) such that for all \( m > M_2 \)

\[ d_0(\xi_0) \frac{1}{m} \sum_{k=1}^{m} l_k u_k > \epsilon, \]

or equivalently

\[ -d_0(\xi_0) \frac{1}{m} \sum_{k=1}^{m} l_k u_k < -\epsilon. \] (2.41)

For \( m > \max\{M_1, M_2\} \)

\[ P(L_m > x_m | Z = z^{(m)}_*) \]
\[ = P \left( L_m - \mathbb{E}(L_m | Z) > x_m - \mathbb{E}(L_m | Z) \bigg| Z = z^{(m)}_* \right) \]
\[ = P \left( \eta(s_m) \frac{1}{m} (L_m - \mathbb{E}(L_m | Z)) > \eta(s_m) \frac{1}{m} (x_m - \mathbb{E}(L_m | Z)) \bigg| Z = z^{(m)}_* \right) \]
\[ = P \left( S_m > \eta(s_m) \frac{1}{m} (x_m - \mathbb{E}(L_m | Z)) \bigg| Z = z^{(m)}_* \right) \]
\[ \geq P\left( S_m > -d_0(\xi_0) \frac{1}{m} \sum_{k=1}^{m} l_k u_k \bigg| Z = z^{(m)}_* \right) \quad \text{(for } m > M_1 \text{ by (2.40)}) \]
\[ \geq P( S_m > -\epsilon ) \quad \text{(for } m > M_2 \text{ by (2.41)}) \]
\[ \geq P\left( |S_m| \leq \epsilon \bigg| Z = z^{(m)}_* \right) \rightarrow 1 \text{ as } m \rightarrow \infty. \]

\[ p_k(z) = \mathcal{H}\left( \frac{G^{-1}(p_k) - az}{b} \right). \] Therefore \( p_k(z) \) is decreasing in \( z \) if \( a > 0 \). Analogously \( p_k(z) \) is increasing in \( z \) if \( a < 0 \).

Notice also that by (2.27)

\[ H^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\nu^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \nu^{(m)} \} & \text{if } a < 0 
\end{cases}. \]

Assume \( a < 0 \). Then,

\[ P(L_m > x_m) \geq \int_{t \in [0, \infty)} P\left( L_m > x_m \big| Z = \nu^{(m)} + t \right) P\left( Z = \nu^{(m)} + t \right) dt \]
\[ \geq P\left( L_m > x_m \big| Z = \nu^{(m)} \right) \int_{t \in [0, \infty)} P\left( Z = \nu^{(m)} + t \right) dt \]
\[ = P\left( L_m > x_m \big| Z = \nu^{(m)} \right) P\left( Z > \nu^{(m)} \right). \quad (2.42) \]

Note that \( \nu^{(m)} \in H^{(m)} \).
Similarly if \( a > 0 \)

\[
\mathbb{P}(L_m > x_m) \geq \int_{t \in [0, \infty)} \mathbb{P}(L_m > x_m | Z = -\nu^{(m)} - t) \mathbb{P}(Z = -\nu^{(m)} - t) \, dt
\]

\[
\geq \mathbb{P}(L_m > x_m | Z = -\nu^{(m)}) \int_{t \in [0, \infty)} \mathbb{P}(Z = -\nu^{(m)} - t) \, dt
\]

\[
= \mathbb{P}(L_m > x_m | Z = -\nu^{(m)}) \mathbb{P}(Z < -\nu^{(m)}).
\]

Note that \(-\nu^{(m)} \in H^{(m)}\). Notice also that by the symmetry of \( Z \)

\[
\mathbb{P}(Z > \nu^{(m)}) = \mathbb{P}(Z < -\nu^{(m)}).
\]

Therefore in either case (whether \( a > 0 \) or \( a < 0 \)), we have

\[
\mathbb{P}(L_m > x_m) \geq k_m \mathbb{P}(Z > \nu^{(m)}),
\]

for some sequence \( k_m \to 1 \).

\[\Box\]

### 2.5 Proofs of Main Theorems: Large Loss Threshold Regime

Now that we have proved the two general theorems for this regime, we will use them to prove Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.4. In these theorems, \( \varepsilon_k \) is allowed to take Gaussian, Exponential and Stretched Exponential distributions. Recall that

\[
q^k_m(\xi) = H((1 + \xi\epsilon_m)ss_m),
\]

as given by 2.20 where \( H(\cdot) \) is the cumulative distribution of \( X_k \). It turns out that our proofs
depend on obtaining lower bounds for the two quantities \( q^e_m(0) - q^e_m(-\xi) \) and \( q^e_m(\xi) - q^e_m(0) \). These lower bounds need to be obtained for each distribution that \( X_k \) is assumed to take. That is for \( N(0, 1) \) in Theorem 2.2.1, Exp(\( \lambda \)) in Theorem 2.2.2, and Stretched Exp in Theorem 2.2.4. We recall the definitions \( \mu^{(m)} = \gamma(1 - \xi_0 \epsilon_m) s_m - \frac{G^{-1}(\beta)}{|a|} \) given by 2.22 and \( \nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(\beta)}{|a|} \) given by 2.25, which we will be using throughout the rest of this section. We may choose \( \epsilon_m \) as we please, depending on the specific distributions we consider.

2.5.1 Proof of Theorem 2.2.1

\( \varepsilon_k \sim N(0, 1) \). \( Z \) is allowed to take \( N(0, 1) \) and Gamma(\( \alpha, \beta \)). Define \( \epsilon_m = \frac{1}{s_m^2} \). We give below two lemmas whose proofs are given in the supplementary proofs for the chapter, in section 2.8.

**Lemma 2.5.1.** Let \( 0 < \xi_0 \) and \( \epsilon_m = \frac{1}{s_m^2} \). Then, there exists \( M_1 = M_1(\xi_0) \) \( \in \mathbb{N} \) such that for \( m > M_1 \)

\[
q^e_m(0) - q^e_m(-\xi) \geq \frac{e^{-\frac{1}{2}(1-\xi_0 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1 - e^{-\xi_0 s^2}}{2} \right). \tag{2.43}
\]

**Lemma 2.5.2.** Let \( 0 < \xi_0 \) and \( \epsilon_m = \frac{1}{s_m^2} \). Then, there exists \( M_1 = M_1(\xi_0) \) such that for \( m > M_1 \)

\[
q^e_m(\xi) - q^e_m(0) \geq \frac{e^{-\frac{1}{2}s^2 s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1 - e^{-\xi_0 s^2}}{2} \right). \tag{2.44}
\]

**Lower Bound**

We wish to employ Theorem 2.4.2. We need to prove (B1), (B2), (B3) and (B4). Let
$$\epsilon_m = \frac{1}{s_m^2}.$$ 

By (2.44), there exists \(M_1 = M_1(\xi_0)\) such that for \(m > M_1\): 

$$q_m'(\xi_0) - q_m'(0) \geq \frac{e^{-\frac{1}{2} s_m^2}}{\sqrt{2\pi s_m}} \left( \frac{1 - e^{-\xi_0 s_m^2}}{2} \right).$$

Define \(\eta(s_m) = ss_m e^{\frac{1}{2} s_m^2} s_m^2\) and \(d_0(\xi_0) = \frac{1}{\sqrt{2\pi}} \left( \frac{1 - e^{-\xi_0 s_m^2}}{2} \right).\)

- **(B1)** There exists and \(d_0(\xi_0)\) and \(M = M(\xi_0)\) such that for \(m > M\): 
  $$q_m'(\xi_0) - q_m'(0) \geq d_0(\xi_0) \frac{1}{\eta(s_m)}.$$

- **(B2)** \(\lim_{m \to \infty} \frac{(\eta(s_m))^2}{m} = 0\) since

  $$\log \left( \frac{(\eta(s_m))^2}{m} \right) = \log \left( \frac{(ss_m)^2 e^{s_m^2 s_m^2}}{m} \right)$$

  $$= \log \left( (ss_m)^2 e^{s_m^2 s_m^2} \right) - \log m$$

  $$= s_m^2 s_m^2 - \log m + o(s_m)$$

  $$= s_m^2 \left( s_m^2 - \frac{\log m}{s_m^2} + o(1) \right)$$

  \(\to -\infty.\)

Therefore by Theorem 2.4.2, 

$$P(L_m > x_m) \geq k_m P(Z > \nu^{(m)})$$

where \(\lim_{m \to \infty} k_m = 1.\) It remains to check (B3) and (B4). In order to do that we need to consider separate cases for \(Z.\)

**Case 1:** \(Z \sim N(0, 1)\)

Define \(\Psi_L(s_m) = \sqrt{2\pi} \gamma s_m e^{\frac{1}{2} \left( \gamma s_m - \frac{G^{-1}(\mathbb{P})}{|a|} \right)^2}.\) Recall that \(\nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(\mathbb{P})}{|a|}\)

and \(\epsilon_m = \frac{1}{s_m^2}.\)
\[(\nu^{(m)})^2 = \left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2 + 2\left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)\gamma \xi_0 \epsilon_m s_m + \gamma^2 \xi_0^2 \epsilon_m^2 s_m^2\]

\[= \left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2 + 2\gamma^2 \xi_0 \epsilon_m s_m^2 + \frac{G^{-1}(\mathbf{P})}{|a|}\gamma \xi_0 \epsilon_m s_m + \gamma^2 \xi_0^2 \epsilon_m^2 s_m^2\]

\[= \left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2 + 2\gamma^2 \xi_0 + o(1). \tag{2.45}\]

By noticing that

\[P(Z > \nu^{(m)}) \sim e^{-\frac{1}{2}\left(\left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2 + 2\gamma^2 \xi_0 + o(1)\right)}\sqrt{\frac{2\pi}{\gamma s_m(1 + o(1))}},\]

(B3) and (B4) hold with \(\Psi_L(s_m) = \sqrt{2\pi} \gamma s_m e^{\frac{1}{2}\left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2}\) and \(\psi(\xi_0) = e^{-\gamma^2 \xi_0}\). Therefore by Theorem 2.4.2

\[\liminf_{m \to \infty} \frac{\sqrt{2\pi} \gamma s_m e^{\frac{1}{2}\left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2}}{P(L_m > x_m)} \geq 1.\]

**Case 2: \(Z \sim \text{Gamma}(\alpha, \beta)\) with \(\alpha > 1\)**

Define \(\Psi_L(s_m) = \beta \Gamma(\alpha) \beta^{-1} s_m^{-\alpha} \frac{1}{\beta} e^{\frac{\beta}{\beta - 1} \left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2}\). Recall that \(\nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\), and \(\epsilon_m = \frac{1}{s_m^2}\). \(Z \sim \text{Gamma}(\alpha, \beta)\) satisfies the following tail asymptotic: \(P(Z > x) \sim \frac{1}{\beta} \frac{\Gamma(\alpha)}{\beta^{\alpha - 1}} x^{\alpha - 1} e^{-\beta x}\) (See the supplementary proofs in section 2.8) Therefore (B3) and (B4) are satisfied with \(\Psi_L(s_m)\) and \(\psi(\xi_0) = 1\). Therefore by Theorem 2.4.2

\[\liminf_{m \to \infty} \beta \Gamma(\alpha) \beta^{-1} s_m^{-\alpha} e^{\frac{\beta}{\beta - 1} \left(\gamma s_m - \frac{G^{-1}(\mathbf{P})}{|a|}\right)^2} P(L_m > x_m) \geq 1.\]
Upper Bound

Case 1: \( Z \sim N(0,1) \)

Define \( \Psi_U(s_m) = \frac{1}{\sqrt{2\pi s_m}} e^{-\frac{1}{2} \left( \frac{\gamma s_m - G^{-1}(\bar{p})}{s_{m}} \right)^2} \). Recall that \( \epsilon_m = \frac{1}{s_m} \) and \( \mu^{(m)} = \gamma(1 - \xi_0 \epsilon_m)s_m - \frac{G^{-1}(\bar{p})}{s_{m}} \). A similar calculation as we did for the lower bound in (2.45) shows that (A1) and (A2) hold with \( \Psi_U(s_m) = \sqrt{2\pi \gamma s_m} e^{-\frac{1}{2} \left( \frac{\gamma s_m - G^{-1}(\bar{p})}{s_{m}} \right)^2} \) and \( \psi(\xi_0) = e^{\gamma^2 \xi_0} \). It remains to show that:

\[
\lim_{m \to \infty} \Psi_U(s_m) e^{-\theta_m^* m L u(q^*_m(0) - q^*_m(-\xi_0)) + m D^2(\theta_m^*)^2}
= \lim_{m \to 0} e^{-\theta_m^* m L u(q^*_m(0) - q^*_m(-\xi_0)) + m D^2(\theta_m^*)^2 + \frac{1}{2} \left( \frac{\gamma s_m - G^{-1}(\bar{p})}{s_{m}} \right)^2} \geq 0.
\]

The logarithm of the quantity in question is

\[
-\theta_m^* m L u(q^*_m(0) - q^*_m(-\xi_0)) + m D^2(\theta_m^*)^2 + \frac{1}{2} \left( \frac{\gamma s_m - G^{-1}(\bar{p})}{s_{m}} \right)^2 s_m^2 + o(s_m^2)
= s_m^2 \left( -\theta_m^* m L u(q^*_m(0) - q^*_m(-\xi_0)) + m D^2(\theta_m^*)^2 + \frac{1}{2} \left( \frac{\gamma s_m - G^{-1}(\bar{p})}{s_{m}} \right)^2 \right) + o(1).
\]

Define \( \theta_m^* = \frac{s_m^2}{\sqrt{m}} \) so that \( m \theta_m^* = 1 \). By (2.43), there exists \( M_1 = M_1(\xi_0) \in \mathbb{N} \) such that for \( m > M_1 \):

\[
q^*_m(0) - q^*_m(-\xi_0) \geq \frac{e^{-\frac{1}{2} \left( \frac{(1-\xi_0)}{s_m^2} \right)^2 s_m^2}}{\sqrt{2\pi s_m^2}} \left( \frac{1 - e^{-\xi_0 s_m^2}}{2} \right). \]

Therefore for \( m > M_1 \),

\[
\frac{\theta_m^* m}{s_m^2} (q^*_m(0) - q^*_m(-\xi_0)) \geq \frac{\theta_m^* m e^{-\frac{1}{2} \left( \frac{(1-\xi_0)}{s_m^2} \right)^2 s_m^2}}{\sqrt{2\pi s_m^2}} \left( \frac{1 - e^{-\xi_0 s_m^2}}{2} \right).
\]

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\[ \theta^*_m me^{-\frac{1}{2}(1 - \xi_0 e_m)s^2 s_m^2} = \frac{m^\frac{1}{2}}{s_m^2} e^{-\frac{1}{2}(1 - \xi_0 e_m)s^2 s_m^2}. \]

\[
\log \left( \frac{m^\frac{1}{2}}{s_m^2} e^{-\frac{1}{2}(1 - \xi_0 e_m)s^2 s_m^2} \right) = \frac{1}{2} \log m - \frac{1}{2} (1 - \xi_0 e_m)^2 s^2 s_m^2 + o(s_m)
\]

\[
= s_m^2 \left( \frac{1}{2} \log m - \frac{1}{2} (1 - \xi_0 e_m)^2 s^2 + o(1) \right)
\]

\[
= \frac{1}{2} s_m^2 \left( \frac{\log m}{s_m^2} - s^2 + o(1) \right)
\]

\[
\to \infty.
\]

Therefore by Theorem 2.4.1, \( \limsup_{m \to \infty} \sqrt{2\pi \gamma s_m} e^{\frac{1}{2} \left( \gamma s_m - \frac{e^{-1}(\rho)}{|a|} \right)^2} \mathbb{P}(L_m > x_m) \leq 1. \)

**Case 2:** \( Z \sim \text{Gamma}(\alpha, \beta) \)

Define \( \Psi_U(s_m) = \beta \frac{\Gamma(\alpha)}{\beta^\alpha \gamma^\alpha} e^{\frac{\beta}{\gamma^\alpha} \left( \gamma s_m - \frac{e^{-1}(\rho)}{|a|} \right)} \). By a calculation similar to that of the lower bound, it can be shown that (A1) and (A2) are satisfied with \( \psi(\xi_0) = 1 \). It remains to verify that

\[
\lim_{m \to \infty} \Psi_U(s_m)e^{-\theta^*_m m \ell_u(q_m(0) - q_m(-\xi_0)) + m D\mathcal{F}(\theta^*_m)^2} = 0.
\]

The logarithm of the quantity in question is
\[-\theta^*_m m L u(q^e_m(0) - q^e_m(-\xi_0)) + m D l^2 (\theta^*_m)^2 + \beta \left( \gamma s_m - \frac{G^{-1}(\bar{\mu})}{|a|} \right) + o(s_m)\]

\[= s_m \left(-\theta^*_m m L u(q^e_m(0) - q^e_m(-\xi_0)) s_m^{-1} + \frac{m D l^2 (\theta^*_m)^2}{s_m} + \beta \gamma + o(1) \right).\]

Define \(\theta^*_m = \sqrt{\frac{s_m}{m}}\) so that \(\frac{m(\theta^*_m)^2}{s_m} = 1\). By (2.43) there exists \(M_1 = M_1(\xi_0) \in \mathbb{N}\) such that for \(m > M_1: q^e_m(0) - q^e_m(-\xi_0) \geq \frac{\varepsilon}{2} (1 - \xi_0^2) s^2 s_m \left(1 - e^{-\xi_0^2} \right) \). Therefore for \(m > M_1\),

\[\frac{\theta^*_m m}{s_m} (q^e_m(0) - q^e_m(-\xi_0)) \geq \frac{\theta^*_m m e^{-\frac{1}{2}(1 - \xi_0^2) s^2 s_m}}{s_m} \left(1 - e^{-\xi_0^2} \right).\]

\[\therefore \frac{\theta^*_m m e^{-\frac{1}{2}(1 - \xi_0^2) s^2 s_m}}{s_m} = \frac{m^\frac{3}{2}}{s_m^\frac{3}{2}} e^{-\frac{1}{2}(1 - \xi_0^2) s^2 s_m}.\]

\[\log \left( \frac{m^\frac{3}{2}}{s_m^\frac{3}{2}} e^{-\frac{1}{2}(1 - \xi_0^2) s^2 s_m} \right) = \frac{1}{2} \log m - \frac{1}{2} (1 - \xi_0^2) s^2 s_m + o(s_m)\]

\[= s_m \left( \frac{1}{2} \log m - \frac{1}{2} (1 - \xi_0^2) s^2 + o(1) \right)\]

\[= \frac{1}{2} s_m \left( \log m - s^2 + o(1) \right)\]

\[\to \infty.\]

Therefore by Theorem 2.4.1, \(\limsup_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma^{\alpha-1} s_m^{\alpha-1}} e^{\beta \left( \gamma s_m - G^{-1}(\bar{\mu}) \right)} \mathbb{P}(L_m > x_m) \leq 1\).
2.5.2 Proof of Theorem 2.2.2

In this case, \( \varepsilon_k \sim \text{Exp}(\lambda) \) and \( H(\cdot) \) denotes the cdf of \( \varepsilon_k \). Define \( \epsilon_m = \frac{1}{s_m} \). First we state two lemmas whose proofs can be found in the appendix.

**Lemma 2.5.3.** Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( \xi_0 > 0 \),

\[
q_{m}^{(0)}(0) - q_{m}^{(-\xi_0)} = e^{-(1-\xi_0 \epsilon_m) \lambda ss_m} \left( 1 - e^{-\lambda s \xi_0} \right). \tag{2.47}
\]

**Lemma 2.5.4.** Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( \xi_0 > 0 \),

\[
q_{m}^{(\xi_0)} - q_{m}^{(0)} = e^{-\lambda ss_m} \left( 1 - e^{-\lambda s \xi_0} \right). \tag{2.48}
\]

**Lower Bound**

Define \( \eta(s_m) = e^{\lambda ss_m} \) and \( d_0(\xi_0) = 1 - e^{-\lambda s \xi_0} \). Recall that \( \nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(\beta)}{|a|} \) and \( \epsilon_m = \frac{1}{s_m} \).

- **(B1)** By (2.48), there exists and \( d_0(\xi_0) \) and \( M = M(\xi_0) \) such that for \( m > M \) :

\[
q_{m}^{(\xi_0)} - q_{m}^{(0)} \geq d_0(\xi_0) \frac{1}{\eta(s_m)}.
\]

- **(B2)** \( \lim_{m \to \infty} \left( \frac{(\eta(s_m))^2}{m} \right) = 0 \) since

\[
\log \left( \frac{(\eta(s_m))^2}{m} \right) = \log \left( \frac{e^{2\lambda ss_m}}{m} \right)
\]

\[
= \log (2\lambda ss_m) - \log m
\]

\[
= s_m \left( 2\lambda - \frac{\log m}{s_m} \right)
\]

\[
\to -\infty.
\]
It remains to check (B3) and (B4). Define $\Psi_L(s_m) = e^{\lambda_0 \left( \gamma s_m - \frac{G^{-1}(\bar{\nu})}{|a|} \right)}$. Recall that $\nu(m) = \gamma (1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(P)}{|a|}$, $\epsilon_m = \frac{1}{s_m}$ and $\mathbb{P}(Z > \nu(m)) = e^{-\lambda_0 \nu(m)}$. It is easy to see that (B3) and (B4) are satisfied with $\psi(\xi_0) = e^{-\gamma \xi_0}$.

**Upper Bound**

Define $\Psi_U(s_m) = e^{\lambda_0 \left( \gamma s_m - \frac{G^{-1}(\bar{\nu})}{|a|} \right)}$. Similar to the case for the lower bound, (A1) and (A2) are satisfied with $\psi(\xi_0) = e^{-\lambda_0 \xi_0}$. It remains to verify that

$$
\lim_{m \to \infty} \Psi_U(s_m) e^{-\theta^*_m m \bar{l} u(q_m(0) - q_m(-\xi_0)) + m D_l^2(\theta^*_m)^2} = \lim_{m \to \infty} e^{-\theta^*_m m \bar{l} u(q_m(0) - q_m(-\xi_0)) + m D_l^2(\theta^*_m)^2 + \lambda_0 \left( \gamma s_m - \frac{G^{-1}(\bar{\nu})}{|a|} \right)} = 0.
$$

The logarithm of the quantity in question is

$$
- \theta^*_m m \bar{l} u(q_m(0) - q_m(-\xi_0)) + m D_l^2(\theta^*_m)^2 + \lambda_0 \left( \gamma s_m - \frac{G^{-1}(\bar{\nu})}{|a|} \right) = s_m \left( - \frac{\theta^*_m m \bar{l} u(q_m(0) - q_m(-\xi_0))}{s_m} + m D_l^2(\theta^*_m)^2 + \lambda_0 \gamma + o(1) \right). \quad (2.49)
$$

Define $\theta^*_m = \sqrt{\frac{m}{s_m}}$ so that $\frac{m(\theta^*_m)^2}{s_m} = 1$. By (2.47), there exists $M_1 = M_1(\xi_0) \in \mathbb{N}$ such that for $m > M_1$: $q_m^e(0) - q_m^e(-\xi_0) = e^{-(1 - \xi_0 \epsilon_m) \lambda s_m m} (1 - e^{-\lambda \xi_0})$.

$$
\therefore \frac{\theta^*_m}{s_m} (q_m^e(0) - q_m^e(-\xi_0)) = \frac{\theta^*_m}{s_m} e^{-(1 - \xi_0 \epsilon_m) \lambda s_m m} (1 - e^{-\lambda \xi_0})
$$
\[ \theta_m^* m e^{-(1-\xi_0 \epsilon_m) \lambda s_m} \]

\[ \therefore \frac{\theta_m^* m e^{-(1-\xi_0 \epsilon_m) \lambda s_m}}{s_m} = \frac{m^{1/2}}{s_m} e^{-(1-\xi_0 \epsilon_m) \lambda s_m}. \]

\[ \therefore \log \left( \frac{m^{1/2}}{s_m} e^{-(1-\xi_0 \epsilon_m) \lambda s_m} \right) = \frac{1}{2} \log m - (1 - \xi_0 \epsilon_m) \lambda s_m \]

\[ = s_m \left( \frac{1}{2} \frac{\log m}{s_m} - (1 - \xi_0 \epsilon_m) \lambda s \right) \]

\[ = s_m \left( \frac{1}{2} \frac{\log m}{s_m} - s \lambda \right) \]

\[ \to \infty. \]

Therefore by Theorem 2.4.1, \( \lim_{m \to \infty} \sup P_m \left( L_m > x_m \right) \leq 1. \)

### 2.5.3 Proof of Theorem 2.2.4

We first state two inequalities needed for proof. Let \( \varepsilon_i \sim \text{Str Exp}(k, c_1(\cdot), c_2(\cdot), b(\cdot)) \), and let \( H(\cdot) \) denote the cumulative distribution of the \( \varepsilon_i \). This means that there exists \( k \in (0, 1) \) and slowly varying functions \( b, c_1, c_2 : (0, \infty) \to (0, \infty) \), and \( t_0 > 0 \) such that for \( t > t_0 \)

\[ c_1(t) \exp(-b(t)t^k) \leq 1 - H(t) \leq c_2(t) \exp(-b(t)t^k). \]

We also need the additional assumption that, \( \lim_{m \to \infty} \frac{c_2(s_m)}{c_2(s_m)} = 1 \). Define \( \epsilon_m = \frac{1}{s_m} \).

**Lemma 2.5.5.** Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( \xi_0 > 0 \), there exists \( d_0(\xi_0) \) and \( M = M(\xi_0) \) such that for \( m > M \),
\[ q_m^\epsilon(0) - q_m^\epsilon(-\xi_0) \geq d_0(\xi_0)c_1((1 - \xi_0\epsilon_m)ss_m)e^{-b((1-\xi_0\epsilon_m)ss_m)(1-\xi_0\epsilon_m)ss_m}k. \] (2.50)

**Lemma 2.5.6.** Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( \xi_0 > 0 \), there exists \( d_0(\xi_0) \) and \( M = M(\xi_0) \) such that for \( m > M \),

\[ q_m^\epsilon(\xi_0) - q_m^\epsilon(0) \geq d_0(\xi_0)c_1(ss_m)e^{-b(ss_m)(ss_m)k}. \] (2.51)

**Lower Bound**

We will use Theorem 2.4.2. We need to prove (B1), (B2), (B3) and (B4). By (2.51), there exists \( d_0(\xi_0) \) and \( M = M(\xi_0) \) such that for \( m > M \),

\[ q_m^\epsilon(\xi_0) - q_m^\epsilon(0) \geq d_0(\xi_0)c_1(ss_m)e^{-b(ss_m)(ss_m)k}. \]

Define \( \eta(s_m) = \frac{1}{c_1(ss_m)}e^{b(ss_m)(ss_m)k} \).

- (B1) There exists \( d_0(\xi_0) \) and \( M = M(\xi_0) \) such that for \( m > M \) : \( q_m^\epsilon(\xi_0) - q_m^\epsilon(0) \geq d_0(\xi_0)\frac{1}{\eta(s_m)} \).

- (B2) \( \lim_{m \to \infty} \frac{(\eta(s_m))^2}{m} = 0 \) since

\[
\log \left( \frac{(\eta(s_m))^2}{m} \right) = 2b(ss_m)(ss_m)^k - \log m + o(s_m^k)
= b(s_m)s_m^k \left( \frac{b(ss_m)}{b(s_m)s_m^k} s_m^k - \frac{\log m}{b(s_m)s_m^k} + o(1) \right) \\
\rightarrow -\infty.
\]

Therefore by Theorem 2.4.2, \( \mathbb{P}(L_m > x_m) \geq k_m \mathbb{P}(Z > \nu^{(m)}) \) with \( \lim_{m \to \infty} k_m = 1 \). It remains to check (B3) and (B4). In order to do that we need to consider separate cases for
Recall that \( \nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m)s_m - \frac{G^{-1}(p)}{|a|\gamma s_m} \) and \( \epsilon_m = \frac{1}{s_m} \).

**Case 1:** \( Z \sim \text{Str Exp}(k_0, c_1^0(\cdot), c_2^0(\cdot), b^0(\cdot)) \) with either

1. \( k_0 < k \) or
2. \( k_0 = k \) and \( \lim_{m \to \infty} b^0(s_m) = B < \infty \).

Define \( \Psi_L(s_m) = \frac{1}{c_1^0(s_m)} e^{b^0(s_m)\gamma^0 s_m^{k_0}} \). Using the binomial series

\[
(\nu^{(m)})_{k_0}^{k_0} \\
= \gamma^{k_0} s_m^{k_0} \left( 1 + \left( \frac{1}{s_m} - \frac{G^{-1}(p)}{|a|\gamma s_m} \right) \right)^{k_0} \\
= \gamma^{k_0} s_m^{k_0} \left( 1 + t_1 \left( \frac{1}{s_m} - \frac{G^{-1}(p)}{|a|\gamma s_m} \right) + t_2 \left( \frac{1}{s_m} - \frac{G^{-1}(p)}{|a|\gamma s_m} \right)^2 + \ldots \right),
\]

where \( t_i = \frac{k_0(k_0-1)\ldots(k_0-i+1)}{i!} \) for \( i = 1, 2, 3, 4, \ldots \)

\[
b^0(\nu^{(m)})(\nu^{(m)})_{k_0}^{k_0} \\
\sim b^0(s_m)(\nu^{(m)})_{k_0}^{k_0} \\
= b^0(s_m) \gamma^{k_0} s_m^{k_0} \left( 1 + t_1 \left( \frac{1}{s_m} - \frac{G^{-1}(p)}{|a|\gamma s_m} \right) + t_2 \left( \frac{1}{s_m} - \frac{G^{-1}(p)}{|a|\gamma s_m} \right)^2 + \ldots \right) \\
= b^0(s_m) \gamma^{k_0} s_m^{k_0} + \gamma^{k_0} b(s_m) k_0 \xi_0 \frac{1}{s_m^{k-1}} + o(1).
\]

Define \( \Psi_L(s_m) = \frac{1}{c_1^0(s_m)} e^{b^0(s_m)\gamma^0 s_m^{k_0}} \). Using the binomial series:
If $k > k_0$, then (B3) and (B4) are satisfied with $\Psi_L(s_m)$ and $\psi(\xi_0) = 1$.

If $k = k_0$ and $\lim_{m \to \infty} b^0(s_m) = B < \infty$, then (B3) and (B4) are satisfied with $\Psi_L(s_m)$ and $\psi(\xi_0) = e^{-\gamma k_0 B \xi_0}$.

Therefore by Theorem 2.4.2

$$\liminf_{m \to \infty} \frac{1}{c^1_1(s_m)} e^{\mu_0(s_m) \gamma k_0 s_m^{k_0}} \geq 1.$$ 

**Case 2:** $Z \sim \text{Paretto}(\alpha, \beta)$

Note that $\mathbb{P}(Z > \nu^{(m)}) = \left(\frac{\beta}{\nu^{(m)}}\right)^\alpha$ and $\nu^{(m)} = \gamma(1 + \xi_0 \epsilon_m) s_m - \frac{G^{-1}(P)}{|a|} = \gamma s_m (1 + o(1))$.

Define $\Psi_L(s_m) = \left(\frac{\gamma s_m}{\beta}\right)^\alpha$. (B3) and (B4) are satisfied with $\psi(\xi_0) = 1$. Therefore by Theorem 2.4.2

$$\liminf_{m \to \infty} \left(\frac{\gamma s_m}{\beta}\right)^\alpha \mathbb{P}(L_m > x_m) \geq 1.$$ 

**Upper Bound**

**Case 1:** $Z \sim \text{Str Exp}(k_0, c_1(\cdot), c_2(\cdot), b(\cdot))$ with either

1. $k_0 < k$ or

2. $k_0 = k$ and $\lim_{m \to \infty} b^0(s_m) = B < \infty$

Define $\Psi_U(s_m) = \frac{1}{c^2_1(s_m)} e^{\mu_0(s_m) \gamma k_0 s_m^{k_0}}$. Similar to the lower bound computation:

If $k > k_0$, then (A1) and (A2) are satisfied with $\Psi_U(s_m)$ and $\psi(\xi_0) = 1$.

If $k = k_0$ and $\lim_{m \to \infty} b^0(s_m) = B < \infty$, then (A1) and (A2) are satisfied with $\Psi_U(s_m)$ and
\[ \psi(\xi_0) = e^{k_0 B \xi_0}. \]

It remains to verify that

\[
\lim_{m \to \infty} \Psi_U(s_m) e^{-\theta_m^* m \xi_0 u(q_m^*(0) - q_m^*(-\xi_0)) + m D\tilde{F}(\theta_m^*)^2} = \lim_{m \to 0} \frac{1}{c_m^2(s_m)} e^{-\theta_m^* m \xi_0 u(q_m^*(0) - q_m^*(-\xi_0)) + m D\tilde{F}(\theta_m^*)^2 + b^0(s_m) \gamma k_0 s_m^k} = 0.
\]

The logarithm of the quantity in question is

\[
-\theta_m^* m \xi_0 u(q_m^*(0) - q_m^*(-\xi_0)) + m D\tilde{F}(\theta_m^*)^2 + b^0(s_m) \gamma k_0 s_m^k = b^0(s_m) s_m^k \theta_m^* m \xi_0 u(q_m^*(0) - q_m^*(-\xi_0)) - m D\tilde{F}(\theta_m^*)^2 + b^0(s_m) \gamma k_0 s_m^k + \gamma^k + o(1).
\]

Define \( \theta_m^* = \sqrt{\frac{b^0(s_m) s_m^k}{m}} \) so that \( \frac{m(\theta_m^*)^2}{b^0(s_m) s_m^k} = 1 \).

By Theorem 2.50,

\[
q_m^*(0) - q_m^*(-\xi_0) \geq d_0(\xi_0) c_1((1 - \xi_0 \epsilon_m) s s_m e^{-b((1 - \xi_0 \epsilon_m) s s_m)((1 - \xi_0 \epsilon_m) s s_m)^k}). \tag{2.52}
\]

\[
\frac{\theta_m^* s_m^k}{b^0(s_m) s_m^k} (q_m^*(0) - q_m^*(-\xi_0)) \geq \frac{\theta_m^* m}{b^0(s_m) s_m^k} d_0(\xi_0) c_1((1 - \xi_0 \epsilon_m) s s_m e^{-b((1 - \xi_0 \epsilon_m) s s_m)((1 - \xi_0 \epsilon_m) s s_m)^k}).
\]
\[
\frac{\theta_m^* m e^{-(1-\xi_0 \epsilon_m) \lambda s_m}}{b^0(s_m) s_m^k} = \frac{m^{1/2}}{(b^0(s_m))^{1/2} s_m^{k}} e^{-(1-\xi_0 \epsilon_m) \frac{k}{2} k^k s_m^k}.
\]

\[
\log \left( \frac{m^{1/2}}{k^0 s_m} c_1 ((1 - \xi_0 \epsilon_m) s_m) e^{b((1 - \xi_0 \epsilon_m) s_m)((1 - \xi_0 \epsilon_m) s_m)^k + o(s_m)} \right)
\]

\[
= \frac{1}{2} \log m - b((1 - \xi_0 \epsilon_m) s_m)((1 - \xi_0 \epsilon_m) s_m)^k + o(s_m)
\]

\[
= b(s_m)^k \left( \frac{1}{2} \log m - \frac{b((1 - \xi_0 \epsilon_m) s_m)}{b(s_m)} ((1 - \xi_0 \epsilon_m) s_m)^k + o(1) \right)
\]

\[
\to \infty.
\]

Therefore by Theorem 2.4.1, \( \limsup_{m \to \infty} \frac{1}{c_2(s_m)} e^{b(s_m) \gamma k_0 s_m^k} \leq 1. \)

**Case 2: Z \sim \text{Pareto}(\alpha, \beta)**

Define \( \Psi_U(s_m) = \left( \frac{\gamma s_m}{\beta} \right)^\alpha \). (A1) and (A2) are satisfied with \( \Psi_U(s_m) \) and \( \psi(\xi_0) = 1 \). It remains to verify that

\[
\lim_{m \to \infty} \Psi_U(s_m) e^{-\theta_m^* m L u(q_m'(0) - q_m(-\xi_0)) + m D\bar{l}^2(\theta_m^*)^2}
\]

\[
= \lim_{m \to \infty} \left( \frac{\gamma s_m}{\beta} \right)^\alpha e^{-\theta_m^* m L u(q_m'(0) - q_m(-\xi_0)) + m D\bar{l}^2(\theta_m^*)^2}
\]

\[
= 0.
\]

The logarithm of the quantity in question is
\[- \theta_m^* m \bar{l}_u(q_m^e(0) - q_m^e(-\xi_0)) + m D l^2(\theta_m^*)^2 + o(s_m) \]

\[
s_m \left( - \frac{\theta_m^* m \bar{l}_u(q_m^e(0) - q_m^e(-\xi_0))}{s_m} - \frac{m D l^2(\theta_m^*)^2}{s_m} + o(1) \right).
\]

Define \( \theta_m^* = \sqrt{\frac{c_1}{m}} \). Therefore \( \frac{m(\theta_m^*)^2}{s_m} = 1 \). By (2.50)

\[
q_m^e(0) - q_m^e(-\xi_0) \geq d_0(\xi_0) c_1 ((1 - \xi_0 \epsilon_m) s s_m) e^{-b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k}.
\]

\[
\frac{\theta_m^* m}{s_m} (q_m^e(0) - q_m^e(-\xi_0)) \geq \frac{\theta_m^* m}{s_m} d_0(\xi_0) c_1 ((1 - \xi_0 \epsilon_m) s s_m) e^{-b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k}.
\]

\[
\frac{\theta_m^* m c_1 ((1 - \xi_0 \epsilon_m) s s_m) e^{-b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k}}{s_m} = \frac{m^{1/2} c_1 ((1 - \xi_0 \epsilon_m) s s_m) e^{-b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k}}{s_m}.
\]

\[
\log \left( \frac{m^{1/2} c_1 ((1 - \xi_0 \epsilon_m) s s_m) e^{-b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k}}{s_m} \right) = \frac{1}{2} \log m - b ((1 - \xi_0 \epsilon_m) s s_m) ((1 - \xi_0 \epsilon_m) s s_m)^k + o(s_m^k)
\]

\[
= b(s_m) s_m^k \left( \frac{1}{2} \log m - \frac{b ((1 - \xi_0 \epsilon_m) s s_m)}{b(s_m)} ((1 - \xi_0 \epsilon_m)^k s_m^k + o(1)) \right)
\]

\( \rightarrow \infty \).
Therefore by Theorem 2.4.1 \( \limsup_{m \to \infty} \left( \frac{\gamma s_m}{\beta} \right)^{\alpha} \mathbb{P}(L_m > x_m) \leq 1 \).

2.6 Analysis of the Small Default Probability Regime

Calculation for the small default probability regime is relatively easier than the large loss threshold regime since it is not necessary to verify the inequalities satisfied by \( \varepsilon_k \) (or the distribution function \( H(\cdot) \)). Instead all we require is Assumptions SD2. Recall that Assumptions SD2 states:

1. The density of \( \varepsilon_k, h(\cdot) \) does not vanish at \( H^{-1}(q) : h(H^{-1}(q)) \neq 0 \).

2. \( h(\cdot) \) is continuous on a neighborhood of \( q \).

By observing that \( H \left( H^{-1}(q) \right) - H \left( (1 - \xi_0 \varepsilon_m)H^{-1}(q) \right) = \int_{H^{-1}(q)-(1-\xi_0 \varepsilon_m)}^{H^{-1}(q)} h(x) \, dx \), it follows that for any \( \xi_0 > 0 \) and \( \epsilon_m \to 0 \), there exists \( k_1 = k_1(\xi_0, q), k_2 = k_2(\xi_0, q) > 0 \) such that

\[ k_1 \epsilon_m < H \left( H^{-1}(q) \right) - H \left( (1 - \xi_0 \varepsilon_m)H^{-1}(q) \right) < k_2 \epsilon_m. \]

That is

\[ k_1 \epsilon_m < q - H \left( (1 - \xi_0 \varepsilon_m)H^{-1}(q) \right) < k_2 \epsilon_m. \]  \hfill (2.53)

Once again we introduce some notation. We also reserve the notation \( \theta_m^* \) and \( \epsilon_m \) for two positive sequences such that \( \theta_m^*, \epsilon_m \to 0 \) and \( \xi_0 > 0 \). Recall that we defined, in (2.13)

\[ \gamma_s = \frac{s}{|a|}. \]  \hfill (2.54)
A sequence of real numbers $\mu^{(m)}_*$ and a sequence of sets $G^{(m)}_*$ defined by

$$
\mu^{(m)}_* = \gamma_* s_m + \frac{b}{|a|}(1 - \xi_0 \epsilon_m)H^{-1}(q),
$$

(2.55)

and

$$
G^{(m)}_* = \begin{cases} 
\{ z \in \mathbb{R} : ss_m + b(1 - \xi_0 \epsilon_m)H^{-1}(q) \leq -a.z \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : ss_m + b(1 - \xi_0 \epsilon_m)H^{-1}(q) \geq -a.z - ss_m \} & \text{if } a < 0
\end{cases}
$$

will be used for the upper bound computation. $\mu^{(m)}_*$ and $G^{(m)}_*$ are related in the following manner.

$$
G^{(m)}_* = \begin{cases} 
\{ z \in \mathbb{R} : -s_m \frac{ss_m + b(1 - \xi_0 \epsilon_m)H^{-1}(q)}{|a|} = -\mu^{(m)}_* \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : ss_m + b(1 - \xi_0 \epsilon_m)H^{-1}(q) \leq -a.z - ss_m \} & \text{if } a < 0
\end{cases}
$$

(2.57)

Similarly, for the lower bound computation, we define $\nu^{(m)}_*$ and $H^{(m)}_*$ by

$$
\nu^{(m)}_* = \gamma_* s_m + \frac{b}{|a|} (1 + \xi_0 \epsilon_m)H^{-1}(q)
$$

(2.58)

and

$$
H^{(m)}_* = \begin{cases} 
\{ z \in \mathbb{R} : ss_m + b(1 + \xi_0 \epsilon_m)H^{-1}(q) \leq -a.z \} & \\
\{ z \in \mathbb{R} : b(1 + \xi_0 \epsilon_m)H^{-1}(q) \leq -a.z - ss_m \}
\end{cases}
$$
= \left\{ z \in \mathbb{R} : H((1 + \xi_0 \epsilon_m)H^{-1}(q)) \leq H\left(\frac{-az - ss_m}{b}\right) = p^{(m)}(z) \right\}. \quad (2.59)

\nu^{(m)}_* and \ H^{(m)}_* are related by

\[ H^{(m)}_* = \begin{cases} 
\left\{ z \in \mathbb{R} : z \leq -\left(\frac{ss_m + b(1 + \xi_0 \epsilon_m)H^{-1}(q)}{|a|}\right) = -\nu^{(m)}_* \right\} & \text{if } a > 0 \\
\left\{ z \in \mathbb{R} : z \geq \frac{ss_m + b(1 + \xi_0 \epsilon_m)H^{-1}(q)}{|a|} = \nu^{(m)}_* \right\} & \text{if } a < 0
\end{cases} \quad (2.60)

2.6.1 Upper Limit Computation

Theorem 2.6.1. Suppose \textbf{Assumptions GEN, Assumptions SD1 and Assumptions SD2} hold. Let \( \xi_0 > 0 \). Let \( \gamma_* \) be as in (2.54) and \( \mu^{(m)}_* \) be as in (2.55).

Then, there exists constants \( k_1 > 0 \) and \( D > 0 \) such that for any positive sequence \( \theta^{*}_m \to 0 \)

\[ \mathbb{P}(L_m > x_m) \leq e^{-m\theta_m^* u k_1 \epsilon_m + mD^2(\theta_m^*)^2} + \mathbb{P}(Z > \mu^{(m)}_*) \quad (2.61) \]

Suppose further that there exists functions \( \Psi(\cdot) \) and \( \psi(\cdot) \), and sequences \( \epsilon_m, \theta^{*}_m \to 0 \) such that

\[ \limsup_{m \to \infty} \Psi(s_m)\mathbb{P}(Z > \mu^{(m)}_*) \leq \psi(\xi_0) \quad (C1) \]

and

\[ \lim_{\xi_0 \to 0^+} \psi(\xi_0) = 1. \quad (C2) \]
\[
\lim_{m \to \infty} \Psi(s_m)e^{-m\theta_m^* u_k \epsilon_m + mD^2(\theta_m^*)^2} = 0. \quad (C3)
\]

Then,

\[
\limsup_{m \to \infty} \Psi(s_m)\mathbb{P}(L_m > x_m) \leq 1. \quad (2.62)
\]

**Proof.** First note that if we prove (2.61) then, (2.62) would follow from (C1), (C2) and (C3). Therefore we only need to prove (2.61).

Let \( G^{(m)} \) be defined as in (2.56). By (2.18),

\[
\psi_m(\theta, z) = \frac{1}{m} \sum_{k=1}^{m} \log \left(1 + p^{(m)}(z) \left(e^{\Lambda(\theta l_k)} - 1\right)\right).
\]

Recall that

\[
\theta_m(z) = \arg\min_{\theta \geq 0} \{-\theta x_m + m\psi_m(\theta, z)\},
\]

was defined in (2.16) and that the measure \( \mathbb{P}_m \) was defined in (2.17) by

\[
\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z)L_m + m\psi_m(\theta_m(Z), Z)}.
\]

\[
\mathbb{P}(L_m > x_m)
\]

\[
= \mathbb{E} \left( \mathbb{1} \{ L_m > x_m \} \mathbb{1}_{(G^{(m)}_\epsilon)(Z)} \right) + \mathbb{E} \left( \mathbb{1} \{ L_m > x_m \} \mathbb{1}_{(G^{(m)}_\epsilon)(Z)} \right)
\]

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\[
\begin{align*}
&= \mathbb{E}_m \left( e^{-\theta_m(Z) L_m + m \psi_m(\theta_m(Z), Z)} \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^*_m)^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^*_m)^c}(Z) \right) \\
&\leq \mathbb{E}_m \left( e^{-\theta_m(Z) x_m + m \psi_m(\theta_m(Z), Z)} \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^*_m)^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(G^*_m)^c}(Z) \right) \\
&\leq \mathbb{E}_m \left( e^{-\theta_m(Z) x_m + m \psi_m(\theta_m(Z), Z)} \mathbb{1}_{(G^*_m)^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}_{(G^*_m)^c}(Z) \right) \\
&\leq \mathbb{E}_m \left( e^{-\theta_m^* x_m + m \psi_m(\theta_m^*, Z)} \mathbb{1}_{(G^*_m)^c}(Z) \right) + \mathbb{P}(Z \in G^*_m) \text{ for any } \theta_m^* \geq 0.
\end{align*}
\]  
\(2.63\) follows since \(-\theta_m(z)x_m + m \psi_m(\theta_m, z) \leq -\theta_m^* x_m + m \psi_m(\theta_m^*, z)\) for any \(\theta_m^* \geq 0\).

By \(2.57\),

\[
G^*_m = \begin{cases} 
\{z \in \mathbb{R} : z \leq -\mu^{(m)}_* \} & \text{if } a > 0 \\
\{z \in \mathbb{R} : z \geq \mu^{(m)}_* \} & \text{if } a < 0
\end{cases}
\]

Therefore, by symmetry of \(Z\),

\[
\mathbb{P}(Z \in G^*_m) = \mathbb{P}(Z > \mu^{(m)}_*).
\]

Therefore

\[
\mathbb{P}(L_m > x_m) \leq \mathbb{E}_m \left( e^{-\theta_m^* x_m + m \psi_m(\theta_m^*, Z)} \mathbb{1}_{(G^*_m)^c}(Z) \right) + \mathbb{P}(Z > \mu^{(m)}_*) \text{ for any } \theta_m^* \geq 0.
\]  
\(2.64\)

Now, we will find a bound for the first term of \(2.64\).

Notice that for \(z \in (G^*_m)^c\),
\[ p^{(m)}(z) = H \left( \frac{-az - ss_m}{b} \right) \]

\[ < H \left( (1 - \xi_0 \epsilon_m)H^{-1}(q) \right). \quad \text{(by (2.57))} \]

Recall that by (2.12),

\[ x_m = q \sum_{k=1}^{m} u_k l_k \]

\[ = q u \sum_{k=1}^{m} l_k. \]

Let \( \theta^*_m \) be any sequence such that \( \theta^*_m \rightarrow 0. \)

\[-\theta^*_m x_m + m \psi_m(\theta^*_m, z) \]

\[ = -\theta^*_m q \sum_{k=1}^{m} l_k u_k + \sum_{k=1}^{m} \log \left( 1 + p^{(m)}(z)(e^{\Lambda(\theta^*_m l_k)} - 1) \right) \]

\[ \leq \sum_{k=1}^{m} \left( -\theta^*_m q l_k u_k + p^{(m)}(z) u_k l_k \theta^*_m + D \bar{l}^2 (\theta^*_m)^2 \right) \quad \text{(by (2.3.2))} \]

\[ \leq \sum_{k=1}^{m} \left( -\theta^*_m q l_k u_k + H \left( (1 - \xi_0 \epsilon_m)H^{-1}(q) \right) \theta^*_m l_k + D \bar{l}^2 (\theta^*_m)^2 \right) \quad \text{(by (2.57))} \]

\[ \leq \sum_{k=1}^{m} \left( -\theta^*_m q l_k u_k + H \left( (1 - \xi_0 \epsilon_m)H^{-1}(q) \right) \theta^*_m l_k + D \bar{l}^2 (\theta^*_m)^2 \right) \]
\[
\sum_{k=1}^{m} \left( -\theta^*_m l_k u_k \left( q - H \left( (1 - \xi_0 \epsilon_m) H^{-1}(q) \right) \right) + D\bar{l}^2(\theta^*_m)^2 \right) \\
\leq \sum_{k=1}^{m} \left( -\theta^*_m u_k \left( q - H \left( (1 - \xi_0 \epsilon_m) H^{-1}(q) \right) \right) + D\bar{l}^2(\theta^*_m)^2 \right)
\]

(since \( q - H \left( (1 - \xi_0 \epsilon_m) H^{-1}(q) \right) > 0 \))

\[
= -m\theta^*_m u_l \left( q - H \left( (1 - \xi_0 \epsilon_m) H^{-1}(q) \right) \right) + mD\bar{l}^2(\theta^*_m)^2.
\]

By (2.53) (following from Assumptions SD2), there exists \( k_1 = k_1(\xi_0, q), k_2 = k_2(\xi_0, q) > 0 \) such that

\[
k_1 \epsilon_m < q - H \left( (1 - \xi_0 \epsilon_m) H^{-1}(q) \right) < k_2 \epsilon_m.
\]

Therefore

\[
-\theta^*_m x_m + m\psi_m(\theta^*_m, z) \leq -m\theta^*_m u_l k_1 \epsilon_m + mD\bar{l}^2(\theta^*_m)^2.
\]

Therefore by (2.63),

\[
\mathbb{P}(L_m > x_m) \leq \mathbb{E}_m \left( e^{-\theta x_m + m\psi_m(\theta, z) 1_{(G^*_m)*)c}(Z)} \right) + \mathbb{P}(Z > \mu^*_m) \\
\leq e^{-m\theta^*_m u_l k_1 \epsilon_m + mD\bar{l}^2(\theta^*_m)^2} + \mathbb{P}(Z > \mu^*_m).
\]

\]
Theorem 2.6.2. Suppose Assumptions GEN, Assumptions SD1 and Assumptions SD2 hold. Let

\[ \epsilon_m = \frac{1}{s_m^j}, \]

where \( j > 0 \). Let \( \gamma_* \) be given by (2.54) and \( \nu_*^{(m)} \) be given by (2.58). Let \( \xi_0 > 0 \). Suppose

\[ \lim_{m \to \infty} \frac{(s_m)^{2j}}{m} = 0. \]  
(2.65)

Then,

\[ \mathbb{P}(L_m > x_m) \geq k_m \mathbb{P}
\left(Z > \nu_*^{(m)}\right), \]  
(2.66)

for some sequence \( k_m \to 1 \).

Suppose further that there exists functions \( \Psi(s_m) \) and \( \psi(\xi_0) \) such that

\[ \liminf_{m \to \infty} \Psi(s_m) \mathbb{P}(Z > \nu_*^{(m)}) \geq \psi(\xi_0) \]  
(D1)

\[ \lim_{\xi_0 \to \infty} \psi(\xi_0) = 1. \]  
(D2)

Then,
\[
\lim_{m \to \infty} \Psi(s_m) P(L_m > x_m) \geq 1.
\]

\textit{Proof.} First note that if we prove (2.66), then (2.67) would follow by the assumptions (D1) and (D2). Therefore we only need to prove (2.66).

Let \( H^{(m)}_s \) be defined by (2.59). Define

\[
S_m = \frac{(s_m)^j}{m} (L_m - \mathbb{E}(L_m | Z)).
\]

By Lemma (2.3.4) and the assumption (2.65), we have that for any sequence \( z^{(m)} \) and \( \epsilon > 0 \)

\[
\lim_{m \to \infty} P(|S_m| > \epsilon | Z = z^{(m)}) = 0.
\]

Lemma 2.6.3. Let \( z^{(m)}_s \in H^{(m)}_s \). Then

\[
\lim_{m \to \infty} P \left( L_m > x_m | Z = z^{(m)}_s \right) = 1.
\]

\textit{Proof.} Let \( z^{(m)}_s \in H^{(m)}_s \). By (2.60)

\[
\mathbb{E} \left( L_m | Z = z^{(m)}_s \right) \geq H ((1 + \xi_0 \epsilon_m)H^{-1}(g)) \sum_{k=1}^{m} l_k u_k.
\]

Therefore

\[
\frac{(s_m)^j}{m} \left( x_m - \mathbb{E}(L_m | Z = z^{(m)}_s) \right)
\]
\[
\leq \frac{\nu_m^j}{m} x_m - \frac{\nu_m^j}{m} H ((1 + \xi_0 \epsilon_m)H^{-1}(q)) \sum_{k=1}^{m} l_k u_k
\]
\[
= (q - H ((1 + \xi_0 \epsilon_m)H^{-1}(q))) \frac{\nu_m^j}{m} \sum_{k=1}^{m} l_k u_k \quad \text{(since } x_m = q \sum_{k=1}^{m} l_k u_k) \]
\[
= (H (H^{-1}(q)) - H ((1 + \xi_0 \epsilon_m)H^{-1}(q))) \frac{\nu_m^j}{m} \sum_{k=1}^{m} l_k u_k \]
\[
= - (H ((1 + \xi_0 \epsilon_m)H^{-1}(q)) - H (H^{-1}(q))) \frac{\nu_m^j}{m} \sum_{k=1}^{m} l_k u_k. 
\]

(2.71)

By (2.53), there exists \(k_1 = k_1(\xi_0, q), k_2 = k_2(\xi_0, q) > 0\) such that

\[ k_1 \epsilon_m < H ((1 + \xi_0 \epsilon_m)H^{-1}(q)) - H (H^{-1}(q)) < k_2 \epsilon_m. \]

Therefore, using the fact that \(\epsilon_m = \frac{1}{\nu_m}\) we get

\[
\frac{\nu_m^j}{m} \left( x_m - \mathbb{E}(L_m | Z = z^{(m)}_*) \right) \leq - (H ((1 + \xi_0 \epsilon_m)H^{-1}(q)) - H (H^{-1}(q))) \frac{\nu_m^j}{m} \sum_{k=1}^{m} l_k u_k
\]
\[
= -k_1 \epsilon_m \frac{\nu_m^j}{m} \sum_{k=1}^{m} l_k u_k
\]
\[
= -k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k \quad \text{(2.72)}
\]
\[
\rightarrow -k_1 C.
\]

Let \(0 < \epsilon < C\).

There exists \(M \in \mathbb{N}\) such that for all \(m > M\)
\[
\frac{1}{m} \sum_{k=1}^{m} l_k u_k > \epsilon.
\]

Therefore for \( m > M \)

\[
-k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k < -k_1 \epsilon.
\]  (2.73)

Therefore for \( m > M \)

\[
\frac{(s_m)^j}{m} (x_m - \mathbb{E}(L_m|Z = z_*^{(m)})) \leq -k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k \quad \text{(by (2.72))}
\]

\[
< -k_1 \epsilon. 
\]  (2.74)

\[
\mathbb{P}(L_m > x_m | Z = z_*^{(m)}) = \mathbb{P}\left(L_m - \mathbb{E}[L_m|Z] > x_m - \mathbb{E}[L_m|Z] \big| Z = z_*^{(m)}\right)
\]

\[
= \mathbb{P}\left((s_m)^j \frac{l}{m} (L_m - \mathbb{E}(L_m|Z)) > (s_m)^j \frac{l}{m} (x_m - \mathbb{E}(L_m|Z)) \big| Z = z_*^{(m)}\right)
\]

\[
= \mathbb{P}\left(S_m > \frac{(s_m)^j}{m} (x_m - \mathbb{E}(L_m|Z)) \big| Z = z_*^{(m)}\right)
\]

\[
\geq \mathbb{P}\left(S_m > -k_1 \epsilon \big| Z = z_*^{(m)}\right) \quad \text{(for } m > M \text{ by (2.74)})
\]

\[
\geq \mathbb{P}\left(|S_m| \leq k_1 \epsilon \big| Z = z_*^{(m)}\right) \rightarrow 1 \text{ as } m \rightarrow \infty. 
\]  (2.75)

\[p^{(m)}(z) = H \left( -\frac{az - bsx}{b} \right).\] Therefore \( p^{(m)}(z) \) is increasing in \( z \) if \( a < 0 \). Analogously \( p^{(m)}(z) \)
is decreasing in $z$ if $a > 0$.

If $a < 0$

\[
\mathbb{P}(L_m > x_m) \geq \int_{t \in [0, \infty)} \mathbb{P}\left(L_m > x_m \mid Z = \nu^{(m)}_* + t\right) \mathbb{P}\left(Z = \nu^{(m)}_* + t\right) dt
\]

\[
\geq \mathbb{P}\left(L_m > x_m \mid Z = \nu^{(m)}_*\right) \int_{t \in [0, \infty)} \mathbb{P}\left(Z = \nu^{(m)}_* + t\right) dt
\]

\[
= \mathbb{P}\left(L_m > x_m \mid Z = \nu^{(m)}_*\right) \mathbb{P}\left(Z > \nu^{(m)}_*\right).
\]

Note that $\nu^{(m)}_* \in H^{(m)}_*.$

Similarly if $a > 0$

\[
\mathbb{P}(L_m > x_m) \geq \int_{t \in [0, \infty)} \mathbb{P}\left(L_m > x_m \mid Z = -\nu^{(m)}_* - t\right) \mathbb{P}\left(Z = -\nu^{(m)}_* - t\right) dt
\]

\[
\geq \mathbb{P}\left(L_m > x_m \mid Z = -\nu^{(m)}_*\right) \int_{t \in [0, \infty)} \mathbb{P}\left(Z = -\nu^{(m)}_* - t\right) dt
\]

\[
= \mathbb{P}\left(L_m > x_m \mid Z = -\nu^{(m)}_*\right) \mathbb{P}\left(Z < -\nu^{(m)}_*\right).
\]

Note that $-\nu^{(m)}_* \in H^{(m)}_*.$

Notice also that

\[H^{(m)}_* = \begin{cases} 
\{ z \in \mathbb{R} : z \geq \nu^{(m)}_* \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \leq -\nu^{(m)}_* \} & \text{if } a < 0
\end{cases}
\]

and therefore, by the symmetry of $Z,$
\[ P(Z > \nu_m^*) = P(Z < -\nu_m^*) \]

Therefore, in either case (whether \( a > 0 \) or \( a < 0 \)), we have

\[ P(L_m > x_m) \geq k_m P\left(Z > \nu_m^*\right), \]

for some sequence \( k_m \to 1 \).

\[ \square \]

2.7 Proofs of Main Theorems: Small Default Probability Regime

2.7.1 Proof of Theorem 2.2.5

Recall that \( \mu_m^* = \gamma_s m + \frac{b}{|a|} (1 - \xi_0 \epsilon m) H^{-1}(q) \), and \( \nu_m^* = \gamma_s m + \frac{b}{|a|} H^{-1}(q) + \frac{b}{|a|} \xi_0 \epsilon m H^{-1}(q) \).

1. When \( Z \sim N(0,1) \)

We will show the upper bound,

\[ \limsup_{m \to \infty} \sqrt{2 \pi \gamma_s m e^{\frac{1}{2} \left( \gamma_s m + \frac{b}{|a|} H^{-1}(q) \right)^2}} P(L_m > x_m) \leq 1 \]

holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^2} = 0 \), and the lower bound,

\[ \liminf_{m \to \infty} \sqrt{2 \pi \gamma_s m e^{\frac{1}{2} \left( \gamma_s m + \frac{b}{|a|} H^{-1}(q) \right)^2}} P(L_m > x_m) \geq 1 \]
holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^{1/4}}. \) This would imply that

\[
\lim_{m \to \infty} \sqrt{2\pi \gamma s_m e^{\frac{1}{2} \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)^2}} P(L_m > x_m) = 1
\]

under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^{1/4}} = 0. \)

**Upper Bound: when \( Z \sim \mathcal{N}(0,1) \)**

Suppose \( \lim_{m \to \infty} \frac{s_m}{m^{1/4}} = 0. \) Apply Theorem 2.6.1 with \( \epsilon_m = \frac{1}{s_m}. \) It follows that there exists constants \( k_1, D > 0 \) such that for any positive sequence \( \theta_m \to 0 : \)

\[
P(L_m > x_m) \leq \exp^{-m\theta_m^2 k_1 \epsilon_m + md^2(\theta^{(m)})^2} + P(Z > \mu^{(m)}).
\]

\[
\left( \mu^{(m)} \right)^2 = \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) - \frac{b}{|a|} \xi_0 \epsilon_m H^{-1}(q) \right)^2
\]

\[
= \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 - 2\gamma \frac{b}{|a|} H^{-1}(q) \xi_0 s_m \epsilon_m + o(1)
\]

\[
= \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 - 2\gamma \frac{b}{|a|} H^{-1}(q) \xi_0 + o(1).
\]

Therefore, \( -\frac{1}{2} \left( \mu^{(m)} \right)^2 = -\frac{1}{2} \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 + K\xi_0 + o(1), \) where \( K = \gamma \frac{b}{|a|} H^{-1}(q). \)

Define \( \Psi(s_m) = \sqrt{2\pi \gamma s_m e^{\frac{1}{2} \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)^2}. \) Since \( P(Z > \mu^{(m)}) \sim \frac{1}{\sqrt{2\pi \mu^{(m)}}} e^{-\frac{1}{2}(\mu^{(m)})^2}, \)

(C1) and (C2) are satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = e^K\xi_0. \) It remains to verify that there exists \( \theta_m^* \to 0 \) such that

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lim \limits_{m \to \infty} \Psi(s_m)e^{-m\theta_m^* \tilde{l}_k \epsilon_m + mD \tilde{l}^2(\theta_m^*)^2} = 0.

The logarithm of the above quantity

\[-m\theta_m^* \tilde{l}_k \epsilon_m + mD \tilde{l}^2(\theta_m^*)^2 + \frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 + \log(2\pi \gamma_s s_m)\]

= \[-m\theta_m^* \tilde{l}_k \epsilon_m + mD \tilde{l}^2(\theta_m^*)^2 + \frac{1}{2} \gamma_s^2 s_m^2 + o(s_m^2)\]

= s_m^2 \left( -\frac{m\theta_m^* \tilde{l}_k \epsilon_m}{s_m^2} + \frac{mD \tilde{l}^2(\theta_m^*)^2}{s_m^2} + \frac{1}{2} \gamma_s^2 + o(1) \right).

\[s_m^2 \left( -\frac{m\theta_m^* \tilde{l}_k \epsilon_m}{s_m^2} + \frac{mD \tilde{l}^2(\theta_m^*)^2}{s_m^2} + \frac{1}{2} \gamma_s^2 + o(1) \right)\]

= s_m^2 \left( -\frac{m\theta_m^* \tilde{l}_k \epsilon_m}{s_m^3} + \frac{mD \tilde{l}^2(\theta_m^*)^2}{s_m^2} + \frac{1}{2} \gamma_s^2 + o(1) \right).

Define \( \theta_m^* = \frac{s_m}{\sqrt{m}} \) so \( \frac{(\theta_m^*)^2}{s_m^2} = 1 \) and \( \frac{m\theta_m^*}{s_m} = \frac{\sqrt{m}}{s_m} \to \infty \). Therefore, by Theorem 2.6.1,

\[\limsup \frac{\sqrt{2\pi \gamma_s s_m} e^{\frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2}}{\mathbb{P}(L_m > x_m)} \leq 1.\]

Lower Bound: when \( Z \sim \mathcal{N}(0,1) \)

Apply Theorem 2.6.2 with \( j = 1 \). It follows that we have \( \lim_{m \to \infty} \frac{s_m^2}{m} = 0 \), \( \epsilon_m = \frac{1}{s_m} \), and \( \mathbb{P}(L_m > x_m) \geq k_m \mathbb{P} \left( Z > \nu_{m}^{(m)} \right) \), where \( \lim_{m \to \infty} k_m = 1 \) and \( \nu_{m}^{(m)} = \gamma_s s_m + \frac{b}{|a|} (1 + \xi_0 \epsilon_m) \). By a calculation similar to the upper bound case
\[- \frac{1}{2} (\nu_s^{(m)})^2 = - \frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 - \gamma_s \frac{b}{|a|} H^{-1}(q) \xi_0 s_m \epsilon_m + o(1) \]

\[- = - \frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2 + K \xi_0 + o(1),\]

where \( K = -\gamma_s \frac{b}{|a|} H^{-1}(q). \)

Define \( \Psi(s_m) = \sqrt{2 \pi \gamma_s s_m e \frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2}. \) Since \( \mathbb{P}(Z > \nu_s^{(m)}) \sim \frac{1}{\sqrt{2 \pi \nu_s^{(m)}}} e^{-\frac{1}{2} (\nu_s^{(m)})^2}, \)

(D1) and (D2) is satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = e^{K \xi_0}. \)

Therefore by Theorem 2.6.2 \( \lim \inf \sqrt{2 \pi \gamma_s s_m e \frac{1}{2} \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)^2} \mathbb{P}(L_m > x_m) \geq 1. \)

2. When \( Z \sim \text{Exp}(\lambda) \)

We will show that the upper bound holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{5}} = 0, \) and the lower bound holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{2}{5}} = 0. \) Together, these two bounds will imply

\[ \lim_{m \to \infty} e^{\lambda \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)} \mathbb{P}(L_m > x_m) = 1, \]

under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{3}} = 0. \)

Upper Bound: when \( Z \sim \text{Exp}(\lambda) \)

Suppose \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{3}} = 0. \) Apply Theorem 2.6.1 with \( \epsilon_m = \frac{1}{s_m}. \) It follows that there exists constants \( k_1, D > 0 \) such that for any positive sequence \( \theta_m^* \to 0 \)

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\[ \mathbb{P}(L_m > x_m) \leq e^{-m \theta_m^* k_1 \epsilon_m + mD^2(\theta_m^*)^2} + \mathbb{P}(Z > \mu^*_m). \]

Define \( \Psi(s_m) = e^{\gamma s_m + \frac{b}{|a|} H^{-1}(q)} \). Since \( \mu^m_m = \gamma s_m + \frac{b}{|a|} (1 - \xi_0 \epsilon_m) H^{-1}(q) \) and \( \mathbb{P}(Z > \mu^*_m) = e^{-\lambda \mu^*_m}, \) (C1) and (C2) are satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = 1 \).

It remains to verify that there exists \( \theta^*_m \) such that \( \lim_{m \to \infty} \Psi(s_m) e^{-m \theta^*_m k_1 \epsilon_m + mD^2(\theta^*_m)^2} = 0. \)

The logarithm of the quantity in question

\[
\lambda \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right) - m \theta_m^* k_1 \epsilon_m + mD^2(\theta_m^*)^2 \\
= s_m \left( \gamma \lambda - \frac{m \theta_m^* k_1}{s_m} \epsilon_m + \frac{mD^2(\theta_m^*)^2}{s_m} \right) \\
= s_m \left( \gamma \lambda - \frac{m \theta_m^* k_1}{s_m^2} + \frac{mD^2(\theta_m^*)^2}{s_m} \right). \quad \text{(Since } \epsilon_m = \frac{1}{s_m}\text{)}
\]

Let \( \theta_m^* = \sqrt{\frac{2m}{m}} \), so that \( \frac{m(\theta_m^*)^2}{s_m^2} = 1 \), and \( \frac{m \theta_m^*}{s_m} = \frac{1}{s_m^2} \to \infty \). Therefore by Theorem 2.6.1

\[
\limsup_{m \to \infty} e^{\lambda \left( \gamma s_m + \frac{b}{|a|} H^{-1}(q) \right)} \mathbb{P}(L_m > x_m) \leq 1.
\]

**Lower Bound: \( Z \sim \text{Exp}(\lambda) \)**

Apply Theorem 2.6.2 with \( j = 1 \). It follow that we are left with \( \lim_{m \to \infty} \frac{s_m^2}{m} = 0, \epsilon_m = \frac{1}{s_m}, \)

and \( \mathbb{P}(L_m > x_m) \geq k_m \mathbb{P} \left( Z > \nu^*_m \right), \) where \( \lim_{m \to \infty} k_m = 1 \) and \( \nu^*_m = \gamma s_m + \frac{b}{|a|} (1 + \xi_0 \epsilon_m) H^{-1}(q). \)
Define \( \Psi(s_m) = e^{\lambda(\gamma s_m + \frac{b}{|a|} H^{-1}(q))} \). (D1) and (D2) are satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = 1 \).

Therefore by Theorem 2.6.2

\[
\liminf_{m \to \infty} e^{\lambda(\gamma s_m + \frac{b}{m} H^{-1}(q))} \mathbb{P}(L_m > x_m) \geq 1.
\]

3. When \( Z \sim \text{Gamma}(\alpha, \beta) \) with \( \alpha > 1 \)

We will show that the upper bound holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{3}{2}} = 0 \), and the lower bound holds under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{2}} = 0 \). Together, these two bounds will imply

\[
\lim_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma^\alpha - 1 s_m^{\alpha - 1}} e^{\beta(\gamma s_m + \frac{b}{m} H^{-1}(q))} \mathbb{P}(L_m > x_m) = 1,
\]

under the assumption \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{2}} = 0 \).

**Upper Bound: when \( Z \sim \text{Gamma}(\alpha, \beta) \)**

Suppose \( \lim_{m \to \infty} \frac{s_m}{m^\frac{1}{2}} = 0 \). Apply Theorem 2.6.1 with \( \epsilon_m = \frac{1}{s_m} \). It follows that there exists constants \( k_1, D > 0 \) such that

\[
\mathbb{P}(L_m > x_m) \leq e^{-m \mu_*(m)} k_1 e^{m D^2 (\theta_m)^2} + \mathbb{P}(Z > \mu_*(m)).
\]

Define \( \Psi(s_m) = \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma^\alpha s_m^{\alpha - 1}} e^{\beta(\gamma s_m + \frac{b}{|a|} H^{-1}(q))}. \) Since \( \mathbb{P}(Z > \mu_*(m)) \sim \frac{1}{\beta \Gamma(\alpha)} (\mu_*(m))^{\alpha - 1} e^{-\beta \mu_*(m)} \), and \( \mu_*(m) = \gamma s_m + \frac{b}{|a|} H^{-1}(q) + o(1) \). (C1) and (C2) are satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = 1 \).

It remains to show that
\[
\lim_{m \to 0} \Psi(s_m)e^{-m \theta_m^* k_1 \epsilon_m + m \bar{D}^2(\theta_m^*)^2}.
\]

The logarithm of the quantity in question

\[
\beta \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right) - m \theta_m^* l_1 \epsilon_m + m \bar{D}^2(\theta_m^*)^2 \\
= s_m \left( \beta \gamma_s - \frac{m \theta_m^* l_1 \epsilon_m}{s_m} + \frac{m \bar{D}^2(\theta_m^*)^2}{s_m} + o(1) \right) \\
= s_m \left( \beta \gamma_s - \frac{m \theta_m^* l_1}{s_m^2} + \frac{m \bar{D}^2(\theta_m^*)^2}{s_m} + o(1) \right).
\]

Define \( \theta_m^* = \sqrt{s_m} \) so that \( \frac{m(\theta_m^*)^2}{s_m} = 1 \), and \( \frac{m \theta_m^*}{s_m} = \frac{m}{s_m} \to \infty \). Therefore by Theorem 2.6.1

\[
\lim \sup_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma_s^\alpha - 1} e^{\beta \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)} \mathbb{P}(L_m > x_m) \leq 1.
\]

**Lower Bound: when \( Z \sim \text{Gamma}(\alpha, \beta) \)**

Apply Theorem 2.6.2 with \( j = 1 \). It follows that we are left with \( \lim_{m \to \infty} \frac{s_m^2}{m} = 0 \), \( \epsilon_m = \frac{1}{s_m} \)

and \( \mathbb{P}(L_m > x_m) \geq k_m \mathbb{P}\left( Z > \nu_m^{(m)} \right) \), where \( \lim_{m \to \infty} k_m = 1 \) and \( \nu_m^{(m)} = \gamma_s s_m + \frac{b}{|a|}(1 + \xi_0 \epsilon_m) H^{-1}(q) \). Define \( \Psi(s_m) = \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma_s^\alpha - 1} e^{\beta \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)} \). A similar calculation to the one in the upper bound shows that (D1) and (D2) are satisfied with \( \Psi(s_m) \) and \( \psi(\xi_0) = 1 \).

Therefore by Theorem 2.6.2

\[
\lim \inf_{m \to \infty} \frac{\beta \Gamma(\alpha)}{\beta^\alpha \gamma_s^\alpha - 1} e^{\beta \left( \gamma_s s_m + \frac{b}{|a|} H^{-1}(q) \right)} \mathbb{P}(L_m > x_m) \geq 1.
\]

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4. \( Z \sim \text{Str Exp} (k, c_1(\cdot), c_2(\cdot), b(\cdot)) \)

There exists \( k \in (0, 1) \) and slowly varying functions \( b, c_1, c_2 : (0, \infty) \to (0, \infty) \), and \( t_0 > 0 \) such that for \( t > t_0 \)

\[
c_1(t) \exp(-b(t)t^k) \leq \mathbb{P}(Z > t) \leq c_2(t) \exp(-b(t)t^k).
\]

We will show that the upper bound will hold under the condition \( \lim_{m \to \infty} \frac{s_{mk}^{3k}}{mb(s_m)} = 0 \), and the lower bound will hold under the condition \( \lim_{m \to \infty} \frac{s_{mk}^{2k}}{m} = 0 \). Then the result would follow by the following verification. \( \lim_{m \to \infty} \frac{s_{mk}^{3k}}{mb(s_m)} = 0 \implies \lim_{m \to \infty} \frac{s_{mk}^{2k}}{m} = 0 \), since \( \frac{s_{mk}^{2k}}{m} = \frac{s_{mk}^{3k}}{mb(s_m)} \cdot \frac{b(s_m)}{s_{mk}^{k}} \).

**Upper Bound: when \( Z \sim \text{Str Exp} (k, c_1(\cdot), c_2(\cdot), b(\cdot)) \)**

Suppose \( \lim_{m \to \infty} \frac{s_{mk}^{3k}}{mb(s_m)} = 0 \) and \( \epsilon_m = \frac{1}{(s_m)^2} \). By Theorem 2.6.1 it follows that for any \( \theta^*_m \to 0 \) and for some constants \( k_1, D > 0 \),

\[
\mathbb{P}(L_m > x_m) \leq e^{-m\theta^*_m k_1 \epsilon_m + mD^2(\theta^*_m)^2} + \mathbb{P}(Z > \mu^{(m)}_*) \geq c_2(\mu^{(m)}_*) e^{-b(\mu^{(m)}_*)(\mu^{(m)}_*)^k}.
\]

Note that \( \mathbb{P}(Z > \mu^{(m)}_*) \leq c_2(\mu^{(m)}_*) e^{-b(\mu^{(m)}_*)(\mu^{(m)}_*)^k} \).

\[
(\mu^{(m)}_*)^k = \left( \gamma s_m + \frac{b}{|a|} (1 - \xi_0 \epsilon_m) H^{-1}(q) \right)^k
\]

\[
= \gamma^k s_m^k \left( 1 + \frac{bH^{-1}(q)}{|a|\gamma s_m} \frac{1}{s_m} - \xi_0 \frac{bH^{-1}(q)}{|a|\gamma s_m} \frac{\epsilon_m}{s_m} \right)^k
\]

\[
= \gamma^k s_m^k \left( 1 + d \frac{1}{s_m} - \xi_0 d \frac{1}{s_m} \right)^k,
\]

where \( d = \frac{bH^{-1}(q)}{|a|\gamma} \). Using the binomial series.
\[(\mu^{(m)}_*)^k = \gamma^k s_m^k \left( 1 + d \frac{1}{s_m} - \xi_0 d \frac{1}{k+1} \right)^k \]

\[= \gamma^k s_m^k \left( 1 + t_1 \left( d \frac{1}{s_m} - \xi_0 d \frac{1}{k+1} \right) + t_2 \left( d \frac{1}{s_m} - \xi_0 d \frac{1}{k+1} \right)^2 + \ldots \right),\]

where \(t_i = \frac{k(k-1)\ldots(k-i+1)}{i!}\) for \(i = 1, 2, 3, 4, \ldots\)

\[b(\mu^{(m)}_*) = b(\gamma^k s_m^k \frac{c2(s_m)}{|d|} (1 - \xi_0 \epsilon_m) H^{-1}(q)) \]

\[= b(\gamma^k s_m^k) \sim b(s_m) \]

\[= b(s_m).\]

For any \(\epsilon > 0\), \(\lim_{m \to \infty} \frac{b(s_m)}{(s_m)\epsilon} = 0\). Therefore

\[b(\mu^{(m)}_*)^{(m)}(\mu^{(m)}_*)^k \sim b(s_m)(\mu^{(m)}_*)^k \]

\[= b(s_m)^k \gamma^k s_m^k \left( 1 + t_1 \left( d \frac{1}{s_m} - \xi_0 d \frac{1}{k+1} \right) + t_2 \left( d \frac{1}{s_m} - \xi_0 d \frac{1}{k+1} \right)^2 + \ldots \right) \]

\[= b(s_m)^k \gamma^k s_m^k + o(1).\]

Define \(\Psi(s_m) = c_2(s_m) e^{b(s_m) \gamma^k s_m^k}\). Since \(\mathbb{P}(Z > \mu^{(m)}_*) \leq c_2(\mu^{(m)}_*) e^{-b(\mu^{(m)}_*)(\mu^{(m)}_*)^k}\), (C1) and (C2) are satisfied with \(\psi(\xi_0) = 1\). It remains to show that

\[\lim_{m \to \infty} \Psi(s_m)e^{-m\theta^2_m + m\epsilon_m + mD}(\theta^2_m)^2 = 0.\]
The logarithm of the quantity in question

\[
b(s_m) \gamma^k s_m^k - m \theta^*_m l k_1 \epsilon_m + m D \bar{l}^2 (\theta^*_m)^2 \epsilon_m + m D \bar{l}^2 (\theta^*_m)^2
\]

\[
= b(s_m) s_m^k \left( \gamma^k - \frac{m \theta^*_m l k_1}{b(s_m) s_m^k} \epsilon_m + \frac{m D \bar{l}^2 (\theta^*_m)^2}{b(s_m) s_m^k} \right)
\]

\[
= b(s_m) s_m^k \left( \gamma^k - \frac{m \theta^*_m l k_1}{b(s_m) s_m^k} + \frac{m D \bar{l}^2 (\theta^*_m)^2}{b(s_m) s_m^k} \right).
\]

Choose \( \theta^*_m = \sqrt{\frac{b(s_m) s_m^k}{b}} \) so that \( \frac{m \theta^*_m}{b(s_m) s_m^k} = 1 \). It also follows that

\[
\frac{m \theta^*_m}{b(s_m) s_m^2} = \frac{m \frac{1}{2} (s_m^k)^{\frac{1}{2}} (b(s_m))^{\frac{1}{2}}}{s_m^k} = \frac{m \frac{1}{2} (b(s_m))^{\frac{1}{2}}}{s_m^k} \rightarrow \infty.
\]

Therefore by Theorem 2.6.2,

\[
\frac{1}{c_2(s_m)} e^{b(s_m) \gamma^k s_m^k} \mathbb{P}(L_m > x_m) \leq 1.
\]

**Lower Bound:** \( Z \sim \text{Str Exp} (k, c_1(\cdot), c_2(\cdot), b(\cdot)) \)

Apply Theorem 2.6.2 with \( j = k \). It follow that we have \( \lim_{m \to \infty} \frac{s_m^k}{s_m^k} = 0 \), \( \epsilon_m = \frac{1}{s_m^k} \), and

\[
\mathbb{P}(L_m > x_m) \geq k_m \mathbb{P} \left( Z > \nu^*(m) \right),
\]

where \( \lim_{m \to \infty} k_m = 1 \) and \( \nu^*(m) = \gamma^* s_m + \frac{b}{|a|} (1 + \xi \epsilon_m) H^{-1}(q) \).

Define \( \Psi(s_m) = c_1(s_m) e^{b(s_m) \gamma^k s_m^k} \). A calculation identical to the one in the upper bound shows that \([\text{D1}] \) and \([\text{D2}] \) are satisfied with \( \psi(\xi_0) = 1 \).

Therefore \( \liminf_{m \to \infty} \frac{1}{c_1(s_m)} e^{b(s_m) \gamma^k s_m^k} \mathbb{P}(L_m > x_m) \geq 1 \).
5. When $Z \sim \text{Pareto}(\alpha, \beta)$

We will show that the upper bound holds under the condition, $\lim_{m \to \infty} \frac{(\log s_m)}{m^{\frac{1}{2}}} = 0$, and the lower bound holds under the condition, there exists $j_0$ such that $\lim_{m \to \infty} \frac{s_{j_0}}{s_m} = 0$. Then the full result will follow by noticing that $\frac{(\log s_m)}{m^{\frac{1}{2}}} = \frac{(\log s_m)}{s_m^{\frac{1}{2}}} \text{ and therefore } \lim_{m \to \infty} \frac{s_{j_0}}{s_m} = 0$ implies $\lim_{m \to \infty} \frac{(\log s_m)}{m^{\frac{1}{2}}} = 0$.

**Upper Bound: when $Z \sim \text{Pareto}(\alpha, \beta)$**

Suppose $\lim_{m \to \infty} \frac{(\log s_m)}{m^{\frac{1}{2}}} = 0$. Let $\epsilon_m = \frac{1}{\sqrt{\log s_m}}$. By Theorem 2.6.1 it follows that

$$\mathbb{P}(L_m > x_m) \leq e^{-m\theta^*_{m,m} k_1 \epsilon_m + mD^2(\theta^*_{m,m})^2} + \mathbb{P}(Z > \mu^{(m)}_*)$$

for some constants $k_1, D > 0$, where $\mu^{(m)}_* = \gamma_* s_m + \frac{b}{|a|} (1 - \xi_0 \epsilon_m) H^{-1}(q)$. Define $\Psi(s_m) = \left(\frac{\gamma_* s_m}{\beta}\right)^\alpha$. Since $\mathbb{P}(Z > \mu^{(m)}_*) = \left(\frac{\beta}{\mu^{(m)}_*}\right)^\alpha$, (C1) and (C2) are satisfied with $\psi(\xi_0) = 1$. It remains to show that

$$\lim_{m \to \infty} \left(\frac{\gamma_* s_m}{\beta}\right)^\alpha e^{-m\theta^*_{m,m} k_1 \epsilon_m + mD^2(\theta^*_{m,m})^2} = 0.$$ 

The logarithm of the quantity in question

$$\alpha \log s_m - m\theta^*_{m,m} k_1 \epsilon_m + mD^2(\theta^*_{m,m})^2 + o(\log s_m)$$

$$= \log s_m \left(\alpha - \frac{m\theta^*_{m,m} k_1}{\log s_m} \epsilon_m + \frac{mD^2(\theta^*_{m,m})^2}{\log s_m} + o(1)\right)$$

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Choose $\theta^*_m = \sqrt{\frac{\log s_m}{m}}$ so that $\frac{m(\theta^*_m)^2}{\log s_m} = 1$, and $\frac{\theta^*_m}{\log s_m \sqrt{\log s_m}} \to \infty$. Therefore by Theorem 2.6.1

$$\limsup_{m \to \infty} \left( \frac{\gamma s_m}{\beta} \right)^\alpha \mathbb{P}(L_m > x_m) \leq 1.$$  

Lower Bound: when $Z \sim \text{Pareto}(\alpha, \beta)$

Apply Theorem 2.6.2 with $j = \frac{\log s_m}{2}$. It follow that we are left with $\lim_{m \to \infty} \frac{s_{j0}}{m} = 0$, $\epsilon_m = \frac{1}{s_m}$, and

$$\mathbb{P}(L_m > x_m) \geq k_m \mathbb{P} \left( Z > \nu^{(m)}_{\epsilon} \right)$$

where, $\lim_{m \to \infty} k_m = 1$ and $\nu^{(m)}_{\epsilon} = \gamma_{\epsilon} s_m + \frac{b}{\beta} (1 + \xi_0 \epsilon_m) H^{-1}(q)$. Define $\Psi(s_m) = \left( \frac{\gamma_{\epsilon} s_m}{\beta} \right)^\alpha$.

[D1] and [D2] are satisfied with $\psi(\xi_0) = 1$. Therefore by Theorem 2.6.2

$$\liminf_{m \to \infty} \left( \frac{\gamma s_m}{\beta} \right)^\alpha \mathbb{P}(L_m > x_m) \geq 1.$$  

2.8 Supplementary Proofs for the Chapter

In this section, we prove some inequalities used in section 2.4. As before $H(\cdot)$ denotes the cdf of $\varepsilon_i$, and $q^c_m(\xi) = H((1 + \xi \epsilon_m) s s_m)$. We consider the cases $\varepsilon_i \sim \text{N}(0,1)$, $\text{Exp}(\lambda)$ and $\text{Str Exp}(k, c_1(\cdot), c_2(\cdot), b(\cdot))$ below in section (2.8.1), (2.8.2) and (2.8.3) respectively. We choose $\epsilon_m$ conveniently as follows: $\epsilon_m = \frac{1}{s_m}$ when $\varepsilon_i \sim \text{N}(0,1)$, $\epsilon_m = \frac{1}{s_m}$ when $\varepsilon_i \sim \text{Exp}(\lambda)$, and $\epsilon_m = \frac{1}{s_m}$ when $\varepsilon_i \sim \text{StretchedExp}(k, c_1(\cdot), c_2(\cdot), b(\cdot))$.  

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2.8.1 \( \varepsilon_i \sim N(0,1) \)

**Lemma 2.8.1.** Let \( \epsilon_m = \frac{1}{s_m} \). For any \( 0 \leq \xi_1 < \xi_2 \) there exists \( M_1 = M_1(\xi_1, \xi_2) \in \mathbb{N} \) such that for \( m > M_1 \)

\[
q_m^\epsilon(-\xi_1) - q_m^\epsilon(-\xi_2) \geq \frac{e^{-\frac{1}{2}(1-\xi_2 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi s m}} \left( \frac{1 - e^{-\xi_2 - \xi_1}}{2} \right).
\]

**Proof.** Using the Gaussian tail inequality

\[
\frac{x e^{-\frac{1}{2}x^2}}{\sqrt{2\pi(1 + x^2)}} \leq 1 - \Phi(x) \leq \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi x}},
\]

we get

\[
q_m^\epsilon(-\xi_1) - q_m^\epsilon(-\xi_2) \\
\geq \frac{(1 - \xi_2 \epsilon_m) s m e^{-\frac{1}{2}(1-\xi_2 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi(1 + ((1 - \xi_2 \epsilon_m) s m)^2)}} - \frac{e^{-\frac{1}{2}(1-\xi_1 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi(1 - \xi_1 \epsilon_m) s m}} \\
= e^{\frac{1}{2}(1-\epsilon_2 \epsilon_m)^2 s^2 s_m^2} \left( \frac{(1 - \xi_2 \epsilon_m) s m}{\sqrt{2\pi(1 + ((1 - \xi_2 \epsilon_m) s m)^2)}} - \frac{e^{\frac{1}{2}(1-\xi_1 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi(1 - \xi_1 \epsilon_m) s m}} \right) \\
= e^{\frac{1}{2}(1-\epsilon_2 \epsilon_m)^2 s^2 s_m^2} \left( \frac{(1 - \xi_2 \epsilon_m) s m}{\sqrt{2\pi(1 + ((1 - \xi_2 \epsilon_m) s m)^2)}} - \frac{e^{\frac{1}{2}((1-\xi_1 \epsilon_m)^2 - (1-\xi_2 \epsilon_m)^2) s^2 s_m^2}}{\sqrt{2\pi(1 - \xi_1 \epsilon_m) s m}} \right) \\
= e^{\frac{1}{2}(1-\epsilon_2 \epsilon_m)^2 s^2 s_m^2} \left( \frac{(1 - \xi_2 \epsilon_m) s m}{\sqrt{2\pi(1 + ((1 - \xi_2 \epsilon_m) s m)^2)}} - \frac{e^{\frac{1}{2}(2(\xi_2 - \xi_1) \epsilon_m + (\xi_1^2 - \xi_2^2) \epsilon_m^2) s^2 s_m^2}}{\sqrt{2\pi(1 - \xi_1 \epsilon_m) s m}} \right) \\
= e^{\frac{1}{2}(1-\epsilon_2 \epsilon_m)^2 s^2 s_m^2} \left( \frac{(1 - \xi_2 \epsilon_m) s m}{\sqrt{2\pi s m}}\left( \frac{(1 - \xi_2 \epsilon_m) s m}{\sqrt{2\pi s m}} - \frac{e^{\frac{1}{2}(2(\xi_2 - \xi_1) \epsilon_m + (\xi_1^2 - \xi_2^2) \epsilon_m^2) s^2 s_m^2}}{\sqrt{2\pi(1 - \xi_1 \epsilon_m) s m}} \right) \right).}
\]
Notice that since \( \epsilon_m = \frac{1}{s_m} \),

\[
\lim_{m \to \infty} \frac{(1 - \xi_2 \epsilon_m) ss_m}{\left( \frac{1}{ss_m} + (1 - \xi_2 \epsilon_m)^2 ss_m \right)} = 1,
\]

and

\[
\lim_{m \to \infty} \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_1^2 - \xi_2^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 - \xi_1 \epsilon_m)} = e^{-(\xi_2 - \xi_1)s^2}.
\]

Therefore,

\[
\lim_{m \to \infty} \frac{(1 - \xi_2 \epsilon_m) ss_m}{\left( \frac{1}{ss_m} + (1 - \xi_2 \epsilon_m)^2 ss_m \right)} - \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_1^2 - \xi_2^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 - \xi_1 \epsilon_m)} = 1 - e^{-(\xi_2 - \xi_1)s^2}.
\]

There exists \( M_1 = M_1(\xi_1, \xi_2) \in \mathbb{N} \) such that for \( m > M_1 \)

\[
\frac{(1 - \xi_2 \epsilon_m) ss_m}{\left( \frac{1}{ss_m} + (1 - \xi_2 \epsilon_m)^2 ss_m \right)} - \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_1^2 - \xi_2^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 - \xi_1 \epsilon_m)} > \frac{1 - e^{-(\xi_2 - \xi_1)s^2}}{2}.
\]

Therefore for \( m > M_1 \)

\[
q_m^e(-\xi_1) - q_m^e(-\xi_2) \geq \frac{e^{-\frac{1}{2} (1 - \xi_2 \epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1 - e^{-(\xi_2 - \xi_1)s^2}}{2} \right).
\]

\[\square\]

**Corollary 2.8.2.** Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( 0 < \xi_0 \) there exists \( M_1 = M_1(\xi_0) \in \mathbb{N} \) such that for \( m > M_1 \)
\[ q_m^c(0) - q_m^c(-\xi_0) \geq e^{-\frac{1}{2}(1-\xi_0^2)^2s^2s_m^2} \frac{1 - e^{-\xi_0^2s^2s_m^2}}{2}. \]

**Lemma 2.8.3.** Let \( \epsilon_m = \frac{1}{s_m^2} \). Let \( 0 \leq \xi_1 < \xi_2 \). Then, there exists \( M_1 = M_1(\xi_1, \xi_2) \) such that for \( m > M_1 \),

\[ q_m^c(\xi_2) - q_m^c(\xi_1) \geq e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{1 - e^{-(\xi_2-\xi_1)^2s^2s_m^2}}{2}. \]

**Proof.** Using the Gaussian tail inequality

\[ xe^{-\frac{1}{2}x^2} \leq 1 - \Phi(x) \leq \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}, \]

we get

\[ q_m^c(\xi_2) - q_m^c(\xi_1) \]

\[ \geq (1 + \xi_1\epsilon_m)s_m^2e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{1 - e^{-(\xi_2-\xi_1)^2s^2s_m^2}}{\sqrt{2\pi}(1 + (1 + \xi_1\epsilon_m)s_m^2)^2)} - e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{(1 + \xi_1\epsilon_m)s_m^2}{\sqrt{2\pi}(1 + (1 + \xi_1\epsilon_m)s_m^2)^2}) \]

\[ = e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{(1 + \xi_1\epsilon_m)s_m^2}{\sqrt{2\pi}(1 + (1 + \xi_1\epsilon_m)s_m^2)^2)} - \frac{e^{-\frac{1}{2}(1+\xi_2\epsilon_m)^2s^2s_m^2} + \frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2)}{\sqrt{2\pi}(1 + \xi_2\epsilon_m)s_m^2}} \]

\[ = e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{(1 + \xi_1\epsilon_m)s_m^2}{\sqrt{2\pi}(1 + (1 + \xi_1\epsilon_m)s_m^2)^2)} - \frac{e^{-\frac{1}{2}(1+\xi_2\epsilon_m)^2s^2s_m^2} + \frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2)}{\sqrt{2\pi}(1 + \xi_2\epsilon_m)s_m^2}} \]

\[ = e^{-\frac{1}{2}(1+\xi_1\epsilon_m)^2s^2s_m^2} \frac{(1 + \xi_1\epsilon_m)s_m^2}{\sqrt{2\pi}(1 + (1 + \xi_1\epsilon_m)s_m^2)^2)} - \frac{e^{-\frac{1}{2}(2\xi_2-\xi_1)^2s^2s_m^2} + (\xi_2^2-\xi_1^2)^2s^2s_m^2)}{\sqrt{2\pi}(1 + \xi_2\epsilon_m)s_m^2}} \]
\[
\frac{1}{\sqrt{2\pi s s_m}} e^{-\frac{1}{2} \left( (1 + \xi_1 \epsilon_m) s s_m \right)} \left( \frac{1}{s s_m} + \frac{1}{(1 + \xi_1 \epsilon_m)^2 s s_m} \right) - \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_2^2 - \xi_1^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 + \xi_2 \epsilon_m)}.
\]

Note that
\[
\lim_{m \to \infty} \frac{(1 + \xi_1 \epsilon_m) s s_m}{\left( \frac{1}{s s_m} + (1 + \xi_1 \epsilon_m)^2 s s_m \right)} = 1,
\]
and
\[
\lim_{m \to \infty} \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_2^2 - \xi_1^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 + \xi_2 \epsilon_m)} = e^{-(\xi_2 - \xi_1) s^2}.
\]

Therefore,
\[
\lim_{m \to \infty} \frac{(1 + \xi_1 \epsilon_m) s s_m}{\left( \frac{1}{s s_m} + (1 + \xi_1 \epsilon_m)^2 s s_m \right)} - \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_2^2 - \xi_1^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 + \xi_2 \epsilon_m)} = 1 - e^{-(\xi_2 - \xi_1) s^2}.
\]

There exists \( M_1 = M_1(\xi_1, \xi_2) \in \mathbb{N} \) such that for \( m > M_1 \)
\[
\frac{(1 + \xi_1 \epsilon_m) s s_m}{\left( \frac{1}{s s_m} + (1 + \xi_1 \epsilon_m)^2 s s_m \right)} - \frac{e^{-\frac{1}{2} \left( 2(\xi_2 - \xi_1) \epsilon_m + (\xi_2^2 - \xi_1^2) (\epsilon_m)^2 \right) s^2 s_m^2}}{(1 + \xi_2 \epsilon_m)} > \frac{1 - e^{-(\xi_2 - \xi_1) s^2}}{2}.
\]
Therefore for \( m > M_1 \)
\[
q_m^e(\xi_2) - q_m^e(\xi_1) \geq \frac{e^{-\frac{1}{2} \left( 1 + \xi_1 \epsilon_m \right)^2 s^2 s_m^2}}{\sqrt{2\pi s s_m}} \left( \frac{1 - e^{-(\xi_2 - \xi_1) s^2}}{2} \right).
\]

\( \square \)

**Corollary 2.8.4.** Let \( 0 < \xi_0 \). There exists \( M_1 = M_1(\xi_0) \) such that for \( m > M_1 \),
\[ q_m^\epsilon(\xi_0) - q_m^\epsilon(0) \geq \frac{e^{-\frac{1}{2} s^2 s_m^2}}{\sqrt{2\pi s s_m}} \left( 1 - e^{-s^2 s_m} \right). \]

2.8.2 \( \epsilon_i \sim \text{Exp}(\lambda) \)

Lemma 2.8.5. Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( 0 \leq \xi_1 < \xi_2 \)

\[ q_m^\epsilon(-\xi_1) - q_m^\epsilon(-\xi_2) = e^{-(1-\xi_2 \epsilon_m) \lambda s s_m} \left( 1 - e^{-\lambda(\xi_2-\xi_1)s} \right). \]

Proof.

\[ q_m^\epsilon(-\xi_1) - q_m^\epsilon(-\xi_2) = e^{-(1-\xi_2 \epsilon_m) \lambda s s_m} - (e^{-(1-\xi_1 \epsilon_m) \lambda s s_m}} \]

\[ = e^{-(1-\xi_2 \epsilon_m) \lambda s s_m} \left( 1 - e^{-\lambda(\xi_2-\xi_1)s} \right) \]

\[ = e^{-(1-\xi_2 \epsilon_m) \lambda s s_m} \left( 1 - e^{-\lambda(\xi_2-\xi_1)s} \right). \]

Corollary 2.8.6. Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( 0 \leq \xi_1 < \xi_2 \)

\[ q_m^\epsilon(0) - q_m^\epsilon(-\xi_0) = e^{-(1-\xi_0 \epsilon_m) \lambda s s_m} \left( 1 - e^{-\lambda s} \right). \]

Lemma 2.8.7. Let \( \epsilon_m = \frac{1}{s_m} \). Then, for any \( 0 \leq \xi_1 < \xi_2 \):

\[ q_m^\epsilon(\xi_2) - q_m^\epsilon(\xi_1) = e^{-(1+\xi_1 \epsilon_m) \lambda s s_m} \left( 1 - e^{-\lambda(\xi_2-\xi_1)s} \right). \]
Proof.

\[ q_m^k(\xi_2) - q_m^k(\xi_1) = e^{-(1+\xi_1^s_m)\lambda ss_m} - e^{-(1+\xi_2^s_m)\lambda ss_m} \]

\[ = e^{-(1+\xi_1^s_m)\lambda ss_m} \left( 1 - e^{-\lambda(\xi_2 - \xi_1)s_m^s} \right) \]

\[ = e^{-(1+\xi_1^s_m)\lambda ss_m} \left( 1 - e^{-\lambda(\xi_2 - \xi_1)s} \right). \]

\[ \square \]

Corollary 2.8.8. For any \( \xi_0 > 0 \):

\[ q_m^k(\xi_0) - q_m^k(0) = e^{\lambda ss_m} \left( 1 - e^{-\lambda \xi_0 s} \right). \]

2.8.3 \( \varepsilon_i \sim \text{Str Exp}(k, c_1(\cdot), c_2(\cdot), b(\cdot)) \) with \( \lim_{m \to \infty} \frac{c_1(s_m)}{c_2(s_m)} = 1. \)

There exists \( k \in (0, 1) \) and slowly varying functions \( b, c_1, c_2 : (0, \infty) \to (0, \infty) \) and \( t_0 > 0 \) such that for \( t > t_0 \)

\[ c_1(t) \exp(-b(t)t^k) \leq 1 - H(t) \leq c_2(t) \exp(-b(t)t^k). \]

We also need the additional assumption that \( \lim_{m \to \infty} \frac{c_1(s_m)}{c_2(s_m)} = 1. \). The proofs for this section are similar to the case when \( \varepsilon_i \sim \text{N}(0,1) \). Throughout this section

\[ \varepsilon_m = \frac{1}{s_m^k}. \]

Lemma 2.8.9. Suppose \( 0 \leq \xi_1 < \xi_2 \). There exists \( d_1(\xi_1, \xi_2) \) and \( M = M(\xi_1, \xi_2) \) such that for \( m > M \),
\[ q_m^e(-\xi_1) - q_m^e(-\xi_2) \geq d_0(\xi_1, \xi_2)c_1((1 - \xi_2\epsilon_m)ss_m)e^{-b((1 - \xi_2\epsilon_m)ss_m)((1 - \xi_2\epsilon_m)ss_m)^k}. \]

**Proof.**

\[ q_m^e(-\xi_1) - q_m^e(-\xi_2) \]
\[ \geq c_1((1 - \xi_2\epsilon_m)ss_m)e^{-b((1 - \xi_2\epsilon_m)ss_m)((1 - \xi_2\epsilon_m)ss_m)^k} \]
\[ - c_2((1 - \xi_1\epsilon_m)ss_m)e^{-b((1 - \xi_1\epsilon_m)ss_m)((1 - \xi_1\epsilon_m)ss_m)^k} \]
\[ = c_1((1 - \xi_2\epsilon_m)ss_m)e^{-b((1 - \xi_2\epsilon_m)ss_m)((1 - \xi_2\epsilon_m)ss_m)^k} \left( 1 - \frac{c_2((1 - \xi_1\epsilon_m)ss_m)}{c_1((1 - \xi_2\epsilon_m)ss_m)}e^{A(m)} \right), \]

where \( A(m) = -b((1 - \xi_1\epsilon_m)ss_m)((1 - \xi_1\epsilon_m)ss_m)^k + b((1 - \xi_2\epsilon_m)ss_m)((1 - \xi_2\epsilon_m)ss_m)^k. \)

But \( b((1 - \xi_1\epsilon_m)ss_m) = (1 + t_m)b(s_m) \) and \( b((1 - \xi_2\epsilon_m)ss_m) = (1 + y_m)b(s_m) \) for some sequences \( t_m, y_m \to 0. \)

\[ -(1 + t_m)b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + (1 + y_m)b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k \]
\[ = -(1 + t_m)b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + (1 + y_m)b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k. \]

\[ -(1 + t_m)b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + (1 + y_m)b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k \]
\[ = -b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k \]
\[ + (y_m - t_m)(-b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k). \]
\[-b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k\]

\[+ (y_m - t_m)\left(-b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k\right)\]

\[\sim -b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k.\]

\[-b(s_m)((1 - \xi_1\epsilon_m)ss_m)^k + b(s_m)((1 - \xi_2\epsilon_m)ss_m)^k = b(s_m)k(\xi_1 - \xi_2)s^k + o(1)\]

\[= -b(s_m)k(\xi_2 - \xi_1)s^k + o(1).\]

We assume that \(\lim_{m \to \infty} \frac{c_2(s_m)}{c_1(s_m)} = 1\). Then, depending on whether \(\lim_{m \to \infty} b(s_m)\) is finite or infinite

\[\lim_{m \to \infty} \frac{c_2((1 - \xi_1\epsilon_m)ss_m)}{c_1((1 - \xi_2\epsilon_m)ss_m)}e^{A(m)} = 0 \text{ or } e^{-Bk(\xi_2 - \xi_1)s^k},\]

where \(\lim_{m \to \infty} b(s_m) = B < \infty\).

In any case we have,

There exists \(d_1(\xi_1, \xi_2)\) and \(M = M(\xi_1, \xi_2)\) such that for \(m > M\),

\[q_m^\epsilon(-\xi_1) - q_m^\epsilon(-\xi_2) \geq d_0(\xi_1, \xi_2)c_1((1 - \xi_2\epsilon_m)ss_m)e^{-b((1 - \xi_2\epsilon_m)ss_m)((1 - \xi_2\epsilon_m)ss_m)^k}.\]
**Corollary 2.8.10.** For any $\xi_0 > 0$, there exists $d_0(\xi_0)$ and $M = M(\xi_0)$ such that for $m > M$

$$q_m^\epsilon(0) - q_m^\epsilon(-\xi_0) \geq d_0(\xi_0)c_1((1 - \xi_0 \epsilon_m)s_{ss_m})e^{-b((1 - \xi_0 \epsilon_m)s_{ss_m})((1 - \xi_0 \epsilon_m)s_{ss_m})^k}.$$ 

**Lemma 2.8.11.** There exists $d_0(\xi_1, \xi_2)$ and $M = M(\xi_1, \xi_2)$ such that $m > M$

$$q_m^\epsilon(\xi_2) - q_m^\epsilon(\xi_1) \geq d_0(\xi_1, \xi_2)c_1((1 + \xi_1 \epsilon_m)s_{ss_m})e^{-b((1 + \xi_1 \epsilon_m)s_{ss_m})((1 + \xi_1 \epsilon_m)s_{ss_m})^k}.$$ 

**Proof.** Identical to Lemma 2.8.9.

**Corollary 2.8.12.** There exists $d_0(\xi_0)$ and $M = M(\xi_0)$ such that $m > M$

$$q_m^\epsilon(\xi_0) - q_m^\epsilon(0) \geq d_0(\xi_0)c_1(s_{ss_m})e^{-b(s_{ss_m})(s_{ss_m})^k}. \quad (2.76)$$

### 2.8.4 Gamma Tail Asymptotic

**Lemma 2.8.13.** Let $H(\cdot)$ denote the cdf of the Gamma Distribution, where the pdf $h(\cdot)$ is given by $h(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$. Let $\alpha > 1$. Then,

$$\frac{1}{\beta} \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} \leq 1 - H(x) \leq \frac{x}{x^\beta - \alpha} \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}. \quad (2.77)$$

**Proof.** $U \sim \text{Gamma}(\alpha, \beta)$ satisfies the following standard upper bound (see [7]):

$$1 - H(x) \leq \frac{x}{x^\beta - \alpha} \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}.$$ 

To get a lower bound, we use the assumption $\alpha > 1$. 


1 − H(x) = \int_{x}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} dt
\geq \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \int_{x}^{\infty} e^{-\beta t} dt
= \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \frac{e^{-\beta x}}{\beta}.

Hence the result.

\square

**Corollary 2.8.14.** Suppose \( U \sim \text{Gamma}(\alpha, \beta) \). Then,

\[ P(U > x) \sim \frac{1}{\beta} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}. \quad (2.78) \]

**Proof.** By (2.77). \( \square \)
Chapter 3: Rare Event Simulation for Single Factor Models

3.1 Introduction

Monte Carlo Simulation is a widely used tool in Quantitative Finance. In credit risk management, we are often confronted with estimating the probabilities of rare but high impact events. Monte Carlo methods, although easy to use, has the drawback of being slow, especially when it comes to computing the probabilities of rare events. Therefore importance sampling becomes a very useful tool.

Importance sampling was studied by Glasserman et al. (see [17]) for the single factor case, and Glasserman et al. (see [16]) for the multi factor case, both when the risk factors are Gaussian. In this chapter, we extend their work to non Gaussian risk factors. We develop an importance sampling estimator and prove its logarithmic efficiency. Following [17], we identify two probability regimes: large loss threshold regime and small default probability regime. The results for this chapter bear much similarity to the those of Chapter 2.

Similar to Chapter 2, we consider a factor model given by

\[ X_k = aZ + b\varepsilon_k, \]  

where \( b = \sqrt{1 - a^2} \). \( X_k, Z, \varepsilon_k \) are assumed to have cumulative distribution functions \( G(\cdot), F(\cdot) \) and \( H(\cdot) \). \( \varepsilon_k \) are i.i.d. for \( i = 1, 2, \ldots, m \) where \( m \) is the number of obligors.

However, proving logarithmic efficiency of the importance sampling estimator, as we may
see, turns out to be a little more delicate matter than obtaining large deviations. The logarithmic efficiency results we derive, require that $Z$ satisfies

$$\log P(Z>x) \sim x\theta(x) - \Lambda_f(\theta(x)),$$

where $\theta(x)$ solves $\Lambda'_f(\theta) = x$ for $\theta$.

The rest of this Chapter is organized as follows: In section 3.2 we define the rare event simulation problem and in section 3.3 we develop the importance sampling algorithm. Next we set about proving the logarithmic efficiency of the importance sampling estimator. This is done using two probability regimes. Since we are dealing with two probability regimes, we identify some congruences in the proofs. Therefore we devote section 3.4 for establishing some preliminary results that will be used later. Section 3.5 deals with large loss threshold regime where as section 3.6 is devoted to the small default probability regime. Our approach is to prove two general theorems without considering any particular distributions for the random variables $X_k, Z$ and $\varepsilon_k$. These theorems are given by Theorem 3.5.1 for the large loss threshold regime, and by Theorem 3.6.1 for the small default probability regime. Theorem 3.5.1 and Theorem 3.6.1 hypothesize certain conditions that need to be satisfied by the distributions $X_k, Z$ and $\varepsilon_k$. Then, once we specify particular distributions for $X_k, Z$ and $\varepsilon_k$, the results would follow upon verifying these conditions.

### 3.2 Problem Statement and Notation

The notation we use will be as follows:

$m = \text{Number of obligors}$

$Y_k = 1_{X_k < x_k}$ where $x_k$'s are prespecified deterministic thresholds
\[ p_k = \mathbb{P}(Y_k = 1) \]

\[ L = \sum_{k=1}^{m} l_k U_k Y_k \] where the exposure \( l_k \) is deterministic and the loss rate \( U_k \) is random.

We are interested estimating \( \mathbb{P}(L > x) \) where \( x >> \mathbb{E}(L) \). We will develop an importance sampling algorithm and show it is logarithmically efficient. In order to establish logarithmic efficiency, we will embed the rare event simulation problem inside a large deviation problem in the following manner: replace \( Y_k, p_k, L, x \) above by \( Y_k^{(m)}, p_k^{(m)}, L_m, x_m \) in such a way that \( \mathbb{P}(L_m > x_m) \to 0 \) as \( m \to \infty \). We develop the importance sampling algorithm and prove its logarithmic efficiency using two probability regimes: large loss threshold regime and small default probability regime. We formally state the model assumptions below.

**Assumptions IS-GEN**

1. Latent variable \( X_k \) is defined as follows.

   Let \( Z \) be a random variable with density \( f(\cdot) \) and distribution function \( F(\cdot) \) and let \( \varepsilon_k \) be a random variable with density \( h(\cdot) \) and distribution function \( H(\cdot) \). \( Z \) and \( \varepsilon_k \) are independent for every \( k \). \( \varepsilon_i \) and \( \varepsilon_j \) are independent for \( i \neq j \). Assume \( f(\cdot) \) and \( h(\cdot) \) are symmetric. Let

   \[ X_k = aZ + b\varepsilon_k, \quad (3.2) \]

   where \( 0 < a \neq 1 \) and \( b > 0 \).

2. From (3.2) it follows that

   \[ \mathbb{P}(X_k < x|Z = z) = \mathbb{P}(aZ + b\varepsilon_k < x|Z = z) \]
\[ P\left( \varepsilon_k < \frac{x - az}{b} \right) = H\left(\frac{x - az}{b}\right), \quad (3.3) \]

and

\[ P(X_k < x) = \int H\left(\frac{x - az}{b}\right) f(z) \, dz. \]

We will denote \( G(x) = P(X_k < x) \) and \( g(x) = \frac{d}{dx} G(x) \).

3. \( Y^{(m)}_k = 1_{\{X_k < G^{-1}(p^{(m)}_k)\}} \) so that \( P(Y^{(m)}_k = 1) = p^{(m)}_k \). It also follows from (3.3) that

\[ P(Y^{(m)}_k = 1|Z) = P\left( \varepsilon_k < \frac{G^{-1}(p^{(m)}_k) - az}{b} \right) = H\left(\frac{G^{-1}(p^{(m)}_k) - az}{b}\right). \quad (3.4) \]

4. The total loss from defaults \( L_m \) is given by

\[ L_m = \sum_{k=1}^{m} l_k U_k Y^{(m)}_k. \]

\( l_k \) is deterministic and satisfies \( 0 < l \leq l_k \leq \bar{l} < \infty \). \( U_k \) are i.i.d. random variables that take values in \([u, 1]\) for some \( 0 < u \leq 1 \). \( U_k \) are independent of \( Z \) and \( \varepsilon_k \). The mean of \( U_k \) will be denoted by \( u_k = u \).
\[ \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} u_k l_k = \lim_{m \to \infty} \frac{1}{m} u \sum_{k=1}^{m} l_k = C, \quad (3.5) \]

for some \(0 < C < \infty\).

As stated above we seek to embed our rare event simulation problem inside a large deviation problem. The regimes we consider is the same as we did for our sharp large deviation results in Chapter 2. In the large loss threshold regime, we consider moderate default probabilities with increasing loss levels. In the small default probability regime, we consider moderate loss levels and decreasing default probabilities.

### 3.3 Importance Sampling Procedure

Once we condition on \(Z\), \(L_m\) becomes a sum of independent random variables. Therefore the standard procedure for importance sampling can be applied.

Define

\[ \Lambda(\lambda) = \log \mathbb{E} \left( e^{\lambda U_k} \right). \quad (3.6) \]

By (3.4)

\[ p_k^{(m)}(z) = = H \left( \frac{G^{-1} \left( p_k^{(m)} \right) - az}{b} \right). \]

We change these probabilities to \textit{exponentially twisted} probabilities \(p_{k,\theta}^{(m)}(z)\) given by
\[ P_{k, \theta}^{(m)}(z) = \frac{P_k^{(m)}(z)e^{\Lambda(\theta l_k)}}{1 + P_k^{(m)}(z)e^{\Lambda(\theta l_k) - 1}}, \]  

(3.7)

for some \( \theta \geq 0 \). (See [17] for details.) A direct computation shows that the conditional likelihood ratio associated with this change of measure is given by

\[ \prod_{k=1}^{m} \left( \frac{P_k^{(m)}(z)}{P_k^{(m)}(z)} \right)^{y_k^{(m)}(z)} \left( \frac{1 - P_k^{(m)}(z)}{1 - P_k^{(m)}(z)} \right)^{1 - y_k^{(m)}(z)} = e^{-\sum_{k=1}^{m} y_k^{(m)} \Lambda(\theta l_k) + m \psi_m(\theta, z)}, \]  

(3.8)

where

\[ \psi_m(\theta, z) = \frac{1}{m} \log \mathbb{E} \left( e^{\theta L_m} \left| Z = z \right. \right) \]
\[ = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + P_k^{(m)}(z)e^{\Lambda(\theta l_k) - 1} \right). \]  

(3.9)

Given \( Z \) and \( Y_k^{(m)} \)'s, we apply another conditional importance sampling for the random variables \( l_k U_k \). This step is exponential twisting of \( l_k U_k \) by \( \theta Y_k^{(m)} \). That is, if \( w_k(l) \) denotes the original density of \( l_k U_k \) (of course \( U_k = U_1 \) in distribution!), then the new density function is given by

\[ w_{k, \theta}(l) = w_k(l)e^{\theta Y_k^{(m)}(l - \Lambda(\theta l_k) Y_k^{(m)})}. \]  

(3.10)

(See [16] for details.) Therefore the likelihood ratio given \( Z \) and \( Y_k^{(m)} \)'s is given by
\[
\prod_{k=1}^{m} \frac{w_k(l_k U_k)}{w_k, \theta(l_k U_k)} = \prod_{k=1}^{m} e^{-\theta Y^{(m)}_k l_k U_k + \Lambda(\theta l_k Y^{(m)}_k)}
\]
\[
= e^{-\sum_{k=1}^{m} \theta Y^{(m)}_k l_k U_k + \sum_{k=1}^{m} \Lambda(\theta l_k Y^{(m)}_k)}. \tag{3.11}
\]

The product of the two likelihood ratios (3.8) and (3.11) simplifies to

\[
e^{-\theta L_m + m \psi_m(\theta, z)}. \]

The next task is to choose \( \theta = \theta_m(z) \). Following Glasserman et al. (see [17]) we choose

\[
\theta_m(z) = \text{argmin}_{\theta \geq 0} \{-\theta x_m + m \psi_m(\theta, z)\}. \tag{3.12}
\]

**Shifting the systematic risk factor \( Z \)**

Our objective is to shift the distribution of \( Z \) in such a way that defaults are more likely to occur. By (3.4), it follows that if \( a < 0 \), defaults are more likely for large positive values of \( Z \). Similarly if \( a > 0 \) defaults are more likely for small negative values of \( Z \). Therefore if \( a < 0 \) we wish to sample \( Z \) from a distribution with a large positive mean, and if \( a > 0 \) we would wish to sample \( Z \) from a distribution with small negative mean.

Let \( f(z) \) be the density of \( Z \), and let \( \Lambda_f(\theta) = \log \left( \mathbb{E}(e^{\theta Z}) \right) \).

Let \( \theta(x) \) solve the equation \( \Lambda'_f(\theta) = x \) for \( \theta \). i.e.

\[
\Lambda'_f(\theta(x)) = x. \tag{3.13}
\]
Instead of sampling from \( f(z) \) we sample from \( f_{\theta(\mu^{(m)})}(z) \) where

\[
f_{\theta(\mu^{(m)})}(z) = f(z)e^{\theta(\mu^{(m)})z - \Lambda_f(\theta(\mu^{(m)}))}.
\]  

(3.14)

Under the distribution \( f_{\theta(\mu^{(m)})}(z) \), \( Z \) would have a mean \( \mu^{(m)} \). The associated likelihood ratio is given by

\[
\frac{f(z)}{f_{\theta(\mu^{(m)})}(z)} = e^{-\theta(\mu^{(m)})z + \Lambda_f(\theta(\mu^{(m)}))}.
\]  

(3.15)

The main problem then is finding \( \mu^{(m)} \). Glasserman et al. (see [17], [16]) suggest choosing

\[
\mu^{(m)} = \arg \min_{z \in \mathbb{R}} \left\{ |z| : z \in W^{(m)} \right\},
\]  

(3.16)

where

\[
W^{(m)} = \left\{ z \in \mathbb{R} : \mathbb{E}(L_m | Z = z) > x_m \right\}.
\]  

(3.17)

(We note that (3.16) is motivated by Glasserman et al. [14]). Hence we get the importance sampling identity for computing \( \mathbb{P}(L_m > x_m) \),

\[
\mathbb{P}(L_m > x_m) = \mathbb{E}^{\prime}_{m} \left( \mathbb{1} \{ L_m > x_m \} e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^{(m)})Z + \Lambda_f(\theta(\mu^{(m}))} \right),
\]  

(3.18)

where \( \mathbb{E}^{\prime}_{m} \) denotes the expectation under the probability measure \( \mathbb{P}^{\prime}_{m} \), where
\[
\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z),Z) - \theta(\mu^{(m)})Z + \Lambda_f(\theta(\mu^{(m)}))}.
\]  

(3.19)

General Importance Sampling Procedure

**Main Loop:** Repeat for \(i = 1, 2, ... K\)

1. Find \(\mu^{(m)}\) by solving (3.16). Find \(\theta(\mu^{(m)})\) by solving (3.13). Sample \(Z\) from \(f_{\theta(\mu^{(m)})}(z)\) given in (3.14).

2. Find \(\theta_m(Z)\) by solving (3.12)

3. For \(k = 1, 2..., m\) compute the conditional twisted default probabilities \(p^{(m)}_{k,\theta_m(Z)}\) in (3.7), and then sample \(Y_k^{(m)}\) from Bernoulli distribution with \(p^{(m)}_{k,\theta_m(Z)}\).

4. For \(k\) with \(Y_k^{(m)} = 1\), generate the loss \(l_kU_k\) under the conditional twisted distribution given in (3.10).

5. Calculate \(I^{(i)} = 1 \{ L_m > x_m \} e^{-\theta_m(Z)L_m + \mu_m(\theta_m(Z),Z) - \theta(\mu^{(m)})z + \Lambda_f(\theta(\mu^{(m)}))}\).

**Return the estimate** \(\frac{1}{K} \sum_{i=1}^{K} I^{(i)}\).

The importance sampling procedure is said to be logarithmically efficient if

\[
M_2(\theta_m(Z), x_m, \mu^{(m)}) = \mathbb{E}_m' \left( 1 \{ L_m > x_m \} e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z),Z) - 2\theta(\mu^{(m)})Z + 2\Lambda_f(\theta(\mu^{(m)}))} \right).
\]  

(3.20)

The importance sampling procedure is said to be logarithmically efficient if
\[
\lim_{m \to \infty} \frac{\log M_2(\theta_m(Z), x_m, \mu^{(m)})}{\mathbb{P}(L_m > x_m)} = 2. \tag{3.21}
\]

The rest of this chapter deals with establishing (3.21) for our importance sampling procedure.

### 3.3.1 Main Results: Large Loss Threshold Regime

In this regime, we consider moderate default probabilities and extreme loss levels. Before we provide the main results, we give below the assumptions for the large loss regime.

**Assumptions IS-LL1**

1. The default indicators \( Y_k^{(m)} = Y_k \) and the default probabilities \( p_k^{(m)} = p_k \) so that

\[
Y_k = 1 \{ X_k < G^{-1}(p_k) \}. \tag{3.22}
\]

This implies that default probability of the k-th obligor is

\[
\mathbb{P}(Y_k = 1) = p_k.
\]

2. The default probabilities \( p_k \) satisfy

\[
0 < \underline{p} \leq p_k \leq \bar{p} < 1.
\]

3. By (3.22) above the default probability of the k-th obligor conditioned on \( Z \) is given by

\[
p_k(z) = \mathbb{P} (X_k < G^{-1}(p_k) | Z = z)
\]

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\[
= \mathbb{P} \left( aZ + b\varepsilon_k < G^{-1}(p_k) | Z = z \right)
\]
\[
= \mathbb{P} \left( az + b\varepsilon_k < G^{-1}(p_k) \right)
\]
\[
= \mathbb{P} \left( \varepsilon_k < \frac{G^{-1}(p_k) - az}{b} \right)
\]
\[
= H \left( \frac{G^{-1}(p_k) - az}{b} \right). \quad (3.23)
\]

4. The threshold loss is given by

\[
x_m = H(ss_m) \sum_{k=1}^{m} l_k u_k = H(ss_m) u \sum_{k=1}^{m} l_k. \quad (3.24)
\]

Logarithmic Efficiency: Large Loss Threshold Regime

In order to state our asymptotic results, we first define two important quantities for this section. Let

\[
\gamma = \frac{sb}{|a|}, \quad (3.25)
\]

and for \( \xi \in \mathbb{R} \) and \( \epsilon_m \to 0 \), let

\[
q_m^\xi(\xi) = H(s(1 + \xi\epsilon_m)s_m). \quad (3.26)
\]

We are required to find \( \mu^{(m)} \) by solving \([3.16]\). Note that \( \mathbb{E}(L_m | Z = z) = \sum_{k=1}^{m} p_k(z) u_k l_k \), and \( x_m = H(s s_m) \sum_{k=1}^{m} u_k l_k \). If we assume that \( p_k = \bar{p} \) then,
\[-az > sbm - G^{-1}(\bar{p}) \iff H \left( \frac{-az + G^{-1}(\bar{p})}{b} \right) > H(s \, sm) \]
\[\iff \mathcal{E}(L_m | Z = z) > x_m.\]

Therefore under the assumption \(p_k = \bar{p}\), \(W^{(m)}\) defined in (3.17) becomes

\[W^{(m)} = \{z \in \mathbb{R} : -az > sbm - G^{-1}(\bar{p})\}.\]

In our asymptotic analysis, we fine tune \(\mu^{(m)}\) defined in (3.16), by replacing the set \(W^{(m)}\) defined in (3.17) by

\[G^{(m)}_{LL} = \{z \in \mathbb{R} : -az \geq sb(1 - \epsilon_m)sm - G^{-1}(\bar{p})\} \]
\[= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(\bar{p}) - az}{b} \right) \geq H(s(1 - \epsilon_m)sm) \right\} \]
\[= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(\bar{p}) - az}{b} \right) \geq q'_m(-1) \right\}. \quad (3.27)\]

Define

\[\mu^{(m)}_{LL} = \gamma(1 - \epsilon_m)sm - \frac{G^{-1}(\bar{p})}{|a|}. \quad (3.28)\]

It follows that
\[ G_{LL}^{(m)} = \begin{cases} 
  \{ z \in \mathbb{R} : z \leq -\left( \frac{sb(1-\epsilon_m)s_m-G^{-1}(\hat{p})}{|a|} \right) = -\mu_{LL}^{(m)} \} & \text{if } a > 0, \\
  \{ z \in \mathbb{R} : z \geq \frac{sb(1-\epsilon_m)s_m-G^{-1}(\hat{p})}{|a|} = \mu_{LL}^{(m)} \} & \text{if } a < 0. 
\]  

Now \( \mu^{(m)} \) defined in (3.16) is replaced by \( \mu_{LL}^{*^{(m)}} \), where

\[ \mu_{LL}^{*^{(m)}} = \arg \min_z \{|z| : z \in G_{LL}^{(m)}\} \]

\[ = \begin{cases} 
  -\mu_{LL}^{(m)} & \text{if } a > 0, \\
  \mu_{LL}^{(m)} & \text{if } a < 0. 
\]  

Define the probability measure \( \mathbb{P}_{LL}^m \) by

\[ \frac{d\mathbb{P}}{d\mathbb{P}_{LL}^m} = e^{-\theta_m(Z)\log_m + \psi_m(\theta_m(Z),Z) - \theta(\mu_{LL}^{*^{(m)}})Z + \Lambda_f(\theta(\mu_{LL}^{*^{(m)}}))}. \]  

Then

\[ \mathbb{P}(L_m > x_m) = \mathbb{E}_{LL}^m \left( \mathbb{I}\{L_m > x_m\}e^{-\theta_m(Z)\log_m + \psi_m(\theta_m(Z),Z) - \theta(\mu_{LL}^{*^{(m)})Z + \Lambda_f(\theta(\mu_{LL}^{*^{(m)}))} \right), \]  

and the second moment of the IS estimator is

\[ M_2(\theta_m(Z), x_m, \mu_{LL}^{*^{(m)})} = \mathbb{E}_{LL}^m \left( \mathbb{I}\{L_m > x_m\}e^{-2\theta_m(Z)\log_m + 2\psi_m(\theta_m(Z),Z) - \theta(\mu_{LL}^{*^{(m)})Z + 2\Lambda_f(\theta(\mu_{LL}^{*^{(m)}))} \right), \]

In section 3.5 we prove that the above importance sampling procedure is logarithmically efficient for random variables \( Z \) that satisfy
\[
\log P(Z > x) \sim x\theta(x) - \Lambda_f(\theta(x)),
\]

upon verifying some conditions satisfied by \( \varepsilon_k \). The following results is obtained as a particular instance of the more general Theorem 3.5.1 that we prove in section 3.5.

**Theorem 3.3.1.** Suppose Assumptions IS-GEN and Assumptions IS-LL1 hold. Suppose \( \varepsilon_k \sim N(0,1) \). Suppose either

1. \( \lim_{m \to \infty} \frac{s_m}{\sqrt{\log m}} = 0 \) and \( s > 0 \) or
2. \( s_m = \sqrt{\log m} \) and \( 0 < s < 1 \).

Suppose either

1. \( Z \sim N(0,1) \) or
2. \( Z \sim \text{Double Exp}(\lambda) \).

Then,

\[
\lim_{m \to \infty} \frac{\log M_2(\theta_m(Z), x_m, \mu_{LL}^{(m)})}{\log P(L_m > x_m)} = 2. \tag{3.30}
\]

**3.3.2 Main Results: Small Default Probability Regime**

In this regime we let the default probabilities approach 0 as \( m \to \infty \) and consider moderate losses. Similar to our approach in the previous section, we first state the assumptions for the small default probability regime, and then provide the main results.

**Assumptions IS-SD1**
1. The default indicators $Y_k^{(m)}$ are given by

$$Y_k^{(m)} = \mathbb{1}\{X_k < -s s_m\},$$

(3.31)

for some sequence $s_m \to \infty$ and $0 < s$.

This implies that default probability of the $k^{th}$ obligor is

$$p^{(m)} = \mathbb{P}\left(Y_k^{(m)} = 1\right) = G(-s s_m).$$

(3.32)

2. By (3.31) above, the default probability of the $k$-th obligor conditioned on $Z$ is given by

$$p^{(m)}(z) = \mathbb{P}(Y_k = 1|Z = z)$$

$$= \mathbb{P}(X_k < -s s_m|Z = z)$$

$$= \mathbb{P}(aZ + b\varepsilon_k < -ss_m|Z = z)$$

$$= \mathbb{P}\left(\varepsilon_k \left< \frac{-ss_m - az}{b}\right\right)$$

$$= H\left(\frac{-az - ss_m}{b}\right).$$

3. The threshold loss is given by

$$x_m = q \sum_{k=1}^m l_k u_k = qu \sum_{k=1}^m l_k,$$

(3.33)
where $\frac{1}{2} < q < 1$.

In addition to Assumptions IS-SD1 we need the following assumptions about the distribution of $\varepsilon_k$.

Assumptions IS-SD2

1. The density of $\varepsilon_k$, $h(\cdot)$ does not vanish at $H^{-1}(q) : h(J^{-1}(q)) \neq 0$.

2. $h(\cdot)$ is continuous on a neighborhood of $H^{-1}(q)$.

Logarithmic Efficiency: Small Default Probability Regime

Similar to the large loss regime, we begin by defining the quantity

$$\gamma_* = \frac{s}{|a|}.$$  \hfill (3.34)

We are required to find $\mu^{(m)}$ by solving (3.16). Note that $\mathbb{E}(L_m|Z = z) = p^{(m)}(z) \sum_{k=1}^{m} u_k l_k$, and $x_m = q \sum_{k=1}^{m} u_k l_k$. Therefore,

$$\mathbb{E}(L_m|Z = z) > x_m \iff p^{(m)}(z) > q \iff H\left(\frac{-az - ss_m}{b}\right) > q \iff \frac{-az}{ss_m + bq}.$$  \hfill (3.35)

The set $W^{(m)}$ defined in (3.17) is found to be equal to

$$W^{(m)} = \{z \in \mathbb{R} : -az > ss_m + bH^{-1}(q)\}.$$  \hfill (3.35)
Then, the solution to (3.16), $\mu^{(m)}$ is given by

$$\mu^{(m)} = \arg \min_{z \in \mathbb{R}} W^{(m)}.$$ 

But in our asymptotic analysis, we fine tune the set $W^{(m)}$ in (3.35), and replace it by

$$G_{SD}^{(m)} = \{ z \in \mathbb{R} : ss_m + b(1 - \epsilon_m)H^{-1}(q) \leq -a.z \} \quad \text{(3.36)}$$

$$= \{ z \in \mathbb{R} : b(1 - \epsilon_m)H^{-1}(q) \leq -a.z - ss_m \}$$

$$= \left\{ z \in \mathbb{R} : H((1 - \epsilon_m)H^{-1}(q)) \leq H \left( \frac{-az - ss_m}{b} \right) = \mu^{(m)}(z) \right\}. \quad \text{(3.37)}$$

where $\epsilon_m \to 0$. By letting

$$\mu_{SD}^{(m)} = \gamma_m s_m + \frac{b}{|a|}(1 - \epsilon_m)H^{-1}(q), \quad \text{(3.38)}$$

we see that

$$G_{SD}^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\left( \frac{ss_m + b(1 - \epsilon_m)H^{-1}(q)}{|a|} \right) = -\mu_{SD}^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \frac{ss_m + b(1 - \epsilon_m)H^{-1}(q)}{|a|} = \mu_{SD}^{(m)} \} & \text{if } a < 0
\end{cases}.$$ 

Now the solution to (3.16) is replaced by $\mu_{SD}^{*^{(m)}}$, where

$$\mu_{SD}^{*^{(m)}} = \arg \min_{z \in \mathbb{R}} \{ |z| : z \in G_{SD}^{(m)} \}.$$
Define the measure $\mathbb{P}^{SD}_m$ by

$$
\frac{d\mathbb{P}}{d\mathbb{P}^{SD}_m} = e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^{(m)}_S)Z + \Lambda_f(\theta(\mu^{(m)}_S))}.
$$

$$
\mathbb{P}(L_m > x_m) = \mathbb{E}^{SD}_m \left( 1\{L_m > x_m\} e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^{(m)}_S)Z + \Lambda_f(\theta(\mu^{(m)}_S))} \right).
$$

Define the second moment of the IS estimator

$$
M_2(\theta_m(Z), x_m, \mu^{(m)}_S) = \mathbb{E}^{SD}_m \left( 1\{L_m > x_m\} e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - \theta(\mu^{(m)}_S)Z + 2\Lambda_f(\theta(\mu^{(m)}_S))} \right).
$$

Now, we are ready to provide the main theorem for the small default probability regime.

**Theorem 3.3.2.** Suppose Assumptions IS-GEN, Assumptions IS-SD1 and Assumptions IS-SD2 hold. Suppose either

1. $Z \sim N(0, 1)$ and $\lim_{m \to \infty} \frac{s_m}{m^{\frac{3}{2}}} = 0$ or

2. $Z \sim \text{Double Exp}(\lambda)$ and $\lim_{m \to \infty} \frac{s_m}{m^{\frac{3}{2}}} = 0$.

Then,

$$
\lim_{m \to \infty} \frac{\log M_2(\theta_m(Z), x_m, \mu^{(m)}_S)}{\log \mathbb{P}(L_m > x_m)} = 2.
$$

(3.42)
3.4 Some Probability Estimates

We are dealing with two probability regimes, and it turns out that there are some similarities between our main proofs. In this section we introduce some tools that we will use in common for both the probability regimes. In section 3.4.1 we introduce the tools for the upper bound computation, and in section 3.4.2 we introduce the tools for the lower bound computation.

3.4.1 Tools for the Upper Bound Computation

Recall that the cumulant generating function of $U_k$ was given in (3.6) by

$$\Lambda(\theta) = \log \mathbb{E}(e^{\theta U_k}), \quad (3.43)$$

which obviously holds for both regimes. (Note that $U_k$ are i.i.d random variables taking values in $[u, 1]$).

The cumulant generating function of $L_m$ conditioned on $Z$, was define in (3.9) by

$$\psi_m(\theta, z) = \frac{1}{m} \log \mathbb{E}(e^{\theta L_m}|Z = z).$$

$\theta_m(z)$ was defined in (3.12)

$$\theta_m(z) = \arg\min_{\theta \geq 0} \{-\theta x_m + m\psi_m(\theta, z)\}. \quad (3.44)$$

We next show that the probability measure implicitly defined by the importance sampling estimator (3.20) is indeed a probability measure. Let

$$\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z) L_m + m\psi_m(\theta_m(Z), Z) - \theta(\mu^m)Z + \Lambda_f(\theta(\mu^m))}, \quad (3.45)$$
and let \( \mathbb{E}^*_m \) denote the expectation under \( \mathbb{P}^*_m \).

**Lemma 3.4.1.** \( \mathbb{P}_m(\cdot) \) defined by (3.45) is a probability measure.

**Proof.** It suffices to show that

\[
\mathbb{E} \left( \frac{d\mathbb{P}}{d\mathbb{P}^*_m} \right)^{-1} = \mathbb{E} \left( e^{\theta_m(Z)L_m - m\psi_m(\theta_m(Z), Z) + \theta(\mu^m)Z - \Lambda_f(\theta(\mu^m))} \right) = 1.
\]

\[
\mathbb{E} \left( e^{\theta_m(Z)L_m - m\psi_m(\theta_m(Z), Z) + \theta(\mu^m)Z - \Lambda_f(\theta(\mu^m))} \right)
= \mathbb{E} \left( e^{\theta_m(Z)L_m - \log(\mathbb{E}(e^{\theta_m(Z)L_m})|Z) e^{\theta(\mu^m)Z} - \log(\mathbb{E}(e^{\theta(\mu^m)Z})} \right)
= \left( \mathbb{E}(e^{\theta(\mu^m)Z}) \right)^{-1} \mathbb{E} \left( e^{\theta_m(Z)L_m \left( \mathbb{E}(e^{\theta_m(Z)L_m})|Z \right)^{-1} e^{\theta(\mu^m)Z}} \right)
= \left( \mathbb{E}(e^{\theta(\mu^m)Z}) \right)^{-1} \mathbb{E} \left( e^{\theta_m(Z)L_m} \mathbb{E}(e^{\theta_m(Z)L_m})|Z \right) \left( \mathbb{E}(e^{\theta_m(Z)L_m})|Z \right)^{-1} \left( e^{\theta(\mu^m)Z} \right)
= \left( \mathbb{E}(e^{\theta(\mu^m)Z}) \right)^{-1} \mathbb{E} \left( e^{\theta(\mu^m)Z} \right)
= 1.
\]

It should be noted that \( \psi_m(\theta, z) \) defined above takes different expressions depending on the probability regime we are dealing with. In the large loss threshold regime,

\[
\psi_m(\theta, z) = \frac{1}{m} \log \mathbb{E} \left( e^{\theta L_m} \bigg| Z = z \right)
= \frac{1}{m} \log \mathbb{E} \left( e^{\theta \sum_{k=1}^m I_k Y_k} \bigg| Z = z \right)
\]

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\[
\psi_m(\theta, z) = \begin{cases} 
\frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k(z) \left( e^{\Lambda(\theta_l)} - 1 \right) \right) & \text{for the large loss threshold regime} \\
\frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k^{(m)}(z) \left( e^{\Lambda(\theta_l)} - 1 \right) \right) & \text{for the small default probability regime.}
\end{cases}
\] (3.46)

The following is a restatement of Lemma 4.3 in [15].

**Lemma 3.4.2.** Let \( \Lambda(\cdot) \) is the cumulant generating function of \( U_k \) given by (3.43). Then, there exists a positive constant \( D \) such that

\[
\log \left( 1 + \alpha \left( e^{\Lambda(\theta)} - 1 \right) \right) \leq \alpha u_k \theta + D \theta^2
\] (3.47)
for all $\theta \in [0, 1]$ and $\alpha \in [0, 1]$. ($u_k = u$ is the mean of $U_k$.)

### 3.4.2 Tools for the Lower Bound Computation

The following is a restatement of Lemma 3.10 in [15].

**Lemma 3.4.3.** Suppose a sequence of events $\{A_m\}_{m \in \mathbb{N}}$ and a sequence of positive integers $\{n_m\}_{m \in \mathbb{N}}$ with $\lim_{m \to \infty} n_m = \infty$ are given. Suppose that, given $A_m$, $T_k^{(m)}$, $k = 1, 2, ..., n_m$ are conditionally independent random variables random variables with conditional mean 0 for which,

$$\limsup_{m \to \infty} \frac{1}{(n_m)^2} \sum_{k=1}^{n_m} \text{Var}\left( T_k^{(m)} \mid A_m \right) = 0.$$ 

Let

$$S_m = \frac{1}{m} \sum_{k=1}^{n_m} T_k^{(m)}.$$ 

Then,

$$\lim_{m \to \infty} \mathbb{P}(|S_m| > \epsilon |A_m) = 0.$$ 

(Note that $T_k^{(m)}$ and $T_l^{(m)}$ may have different distributions for $k \neq l$.)

Note that

$$\mathbb{E}(L_m | Z) = \begin{cases} 
\sum_{k=1}^{n_m} l_k u_k p_k(Z) & \text{for the large loss threshold regime} \\
\sum_{k=1}^{n_m} l_k u_k p^{(m)}(Z) & \text{for the small default probability regime}
\end{cases}.$$  
(3.48)
But the following theorem holds for both the probability regimes nonetheless.

**Lemma 3.4.4.** Let \( \eta(s_m) : [0, \infty] \rightarrow [0, \infty] \) be a function such that

\[
\lim_{m \rightarrow \infty} \frac{\eta(s_m)}{\sqrt{m}} = 0.
\]

Let \( z^{(m)} \) be an arbitrary sequence in \( \mathbb{R} \) and let

\[
S_m = \frac{\eta(s_m)}{m} (L_m - \mathbb{E}(L_m|Z)).
\]

Then,

\[
\lim_{m \rightarrow \infty} \mathbb{P} \left( |S_m| > \epsilon \big| Z = z^{(m)} \right) = 0.
\]

**Proof.** We give the proof for the large loss threshold regime. It is easily seen that a similar proof holds for the small default probability regime as well. Apply Lemma 3.4.3 with \( A_m = \{Z = z^{(m)}\} \) and \( n_m = m \). Let

\[
T^{(m)}_k = \eta(s_m) l_k (U_k Y_k - u_k p_k(Z)).
\]

- \( T^{(m)}_k \)'s are conditionally independent given \( Z \).

- \( \mathbb{E}(T^{(m)}_k|Z = z^{(m)}) = 0 \) since \( U_k \) is independent of \( Z \) and \( Y_k \).

- We will show that

\[
\limsup_{m \rightarrow \infty} \frac{1}{m^2} \text{Var}(T^{(m)}_k|Z = z^{(m)}) = 0.
\]
Note that
\[
\text{Var} \left( T_k^{(m)} | Z = z^{(m)} \right) = (\eta(s_m))^2 \mathbb{E}_k \left( (U_k Y_k - u_k p_k(Z))^2 | Z = z^{(m)} \right) \\
\leq 4(\eta(s_m))^2 (\bar{l})^2,
\]
and therefore
\[
\frac{1}{m^2} \sum_{k=1}^{m} \text{Var} \left( T_k^{(m)} | Z = z^{(m)} \right) \leq 4 m (\bar{l})^2 \frac{1}{m^2} 4(\eta(s_m))^2 \\
\leq 4 (\bar{l})^2 \frac{1}{m} 4(\eta(s_m))^2 \\
\to 0.
\]

Therefore
\[
\limsup_{m \to \infty} \frac{1}{m^2} \text{Var}(T_k^{(m)} | Z = z^{(m)}) = 0.
\]

Therefore, for any \( \epsilon > 0 \),
\[
\lim_{m \to \infty} \mathbb{P} \left( |S_m| > \epsilon | Z = z^{(m)} \right) = 0.
\]

Now that we have developed these probabilistic tools, we consider the two probability regimes separately. Section 3.5 deals with the logarithmic efficiency of large loss regime, where as 3.6 is devoted to the small default regime.
3.5 Logarithmic Efficiency: Large Loss Threshold Regime

We reserve the notation $\theta^*_m$ and $\epsilon_m$ for two positive sequences such that $\theta^*_m, \epsilon_m \to 0$. Recall that we defined $q^\epsilon_m(\xi)$ for $\xi \in \mathbb{R}$, in (3.26) by

$$q^\epsilon_m(\xi) = H(s(1 + \xi \epsilon_m)s_m).$$

(3.49)

This turns out to be a useful quantity for this regime. We also defined $\gamma$ in (3.25) by

$$\gamma = \frac{sb}{|a|}.$$

(3.50)

Large losses are more likely to occur for those values of $z$ for which $p_k(z)$ is large. We exploit this fact in the following manner. Recall that a sequence numbers $\mu^{(m)}_{LL}$, and a sequence of sets $G^{(m)}_{LL}$ was defined in (3.28) and (3.27) by

$$\mu^{(m)}_{LL} = \gamma(1 - \epsilon_m)s_m - \frac{G^{-1}(\bar{p})}{|a|},$$

(3.51)

and

$$G^{(m)}_{LL} = \{ z \in \mathbb{R} : -a.z \geq sb(1 - \epsilon_m)s_m - G^{-1}(\bar{p}) \}$$

$$= \left\{ z \in \mathbb{R} : H\left( \frac{G^{-1}(\bar{p}) - az}{b} \right) \geq H\left( s(1 - \epsilon_m)s_m \right) \right\}$$

$$= \left\{ z \in \mathbb{R} : H\left( \frac{G^{-1}(\bar{p}) - az}{b} \right) \geq q^\epsilon_m(-1) \right\}.$$  

(3.52)

$\mu^{(m)}_{LL}$ and $G^{(m)}_{LL}$ will be used in the upper bound computation. They are related in the
following manner.

\[
G_{LL}^{(m)} = \begin{cases} 
  \left\{ z \in \mathbb{R} : z \leq -\left( \frac{sb(1-\epsilon_m)s_m-G^{-1}(p)}{|a|} \right) = -\mu_{LL}^{(m)} \right\} & \text{if } a > 0 \\
  \left\{ z \in \mathbb{R} : z \geq \frac{sb(1-\epsilon_m)s_m-G^{-1}(p)}{|a|} = \mu_{LL}^{(m)} \right\} & \text{if } a < 0 
\end{cases} .
\] (3.53)

Similarly, for the lower bound computation we define \(\nu_{LL}^{(m)}\) and \(H_{LL}^{(m)}\) by

\[
\nu_{LL}^{(m)} = \gamma(1 + \epsilon_m)s_m - \frac{G^{-1}(p)}{|a|},
\] (3.54)

and

\[
H_{LL}^{(m)} = \left\{ z \in \mathbb{R} : -a \cdot z \geq sb(1 + \epsilon_m)s_m - G^{-1}(p) \right\} \\
= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(p) - az}{b} \right) \geq H(s(1 + \epsilon_m)s_m) \right\} \\
= \left\{ z \in \mathbb{R} : H \left( \frac{G^{-1}(p) - az}{b} \right) \geq q_m(1) \right\} .
\]

\(\nu_{LL}^{(m)}\) and \(H_{LL}^{(m)}\) are related by

\[
H_{LL}^{(m)} = \begin{cases} 
  \left\{ z \in \mathbb{R} : z \leq -\left( \frac{sb(1+\epsilon_m)s_m-G^{-1}(p)}{|a|} \right) = -\nu_{LL}^{(m)} \right\} & \text{if } a > 0 \\
  \left\{ z \in \mathbb{R} : z \geq \frac{sb(1+\epsilon_m)s_m-G^{-1}(p)}{|a|} = \nu_{LL}^{(m)} \right\} & \text{if } a < 0 
\end{cases} .
\] (3.55)

The following theorem deals with identifying the conditions under which logarithmic efficiency holds for the large loss regime.

**Theorem 3.5.1.** Suppose Assumptions IS-GEN and Assumptions IS-LL1 hold. Let
$0 < \epsilon_m, \rightarrow 0$. Let $\gamma$ be given by (3.50), and let $q^\epsilon_m(\xi)$ be given by (3.49). Let $\mu^{(m)}_{LL}$ be given by (3.51), and $\nu^{(m)}_{LL}$ be given by (3.54). Suppose

\[ \lim_{m \to \infty} \frac{1}{s_m^0} \log P(Z > \mu^{(m)}_{LL}) = \lim_{m \to \infty} \frac{1}{s_m^0} \log P(Z > \nu^{(m)}_{LL}) = -A \quad \text{(D1) and (D2)} \]

\[ \lim_{m \to \infty} \frac{1}{k_s^0} \left( -\theta(\mu^{(m)}_{LL})\mu^{(m)}_{LL} + \Lambda_f(\theta(\mu^{(m)}_{LL})) \right) \]

\[ = \lim_{m \to \infty} \frac{1}{s_m^k} \left( -\theta(\nu^{(m)}_{LL})\nu^{(m)}_{LL} + \Lambda_f(\theta(\nu^{(m)}_{LL})) \right) = -A \quad \text{(D3) and (D4)} \]

There exists $\theta^* \rightarrow 0$ such that for any $D > 0$

\[ \lim_{m \to \infty} \frac{1}{s_m^k} \left( -\theta^* m L(u(q^\epsilon_m(0) - q^\epsilon_m(-1)) + mD\bar{l}^2(\theta^*)^2 + 2\Lambda_f(\theta(\mu^{(m)}_{LL})) \right) = -\infty \quad \text{(U1)} \]

There exists $\eta : [0, \infty] \rightarrow [0, \infty], d_0$ and $M_1$ such that for $m > M_1$

\[ q^\epsilon_m(1) - q^\epsilon_m(0) \geq d_0 \frac{1}{\eta(s_m)} \quad \text{(L1)} \]

and

\[ \lim_{m \to \infty} \frac{(\eta(s_m))^2}{m} = 0. \quad \text{(L2)} \]

Then,
The proof of Theorem 3.5.1 follows by the Upper Bound Computation (Theorem 3.5.2) and Lower Bound Computation (Theorem 3.5.5) given below.

3.5.1 Upper Bound Computation

Theorem 3.5.2. Suppose Assumptions GEN and Assumptions IS-LL1 hold. Let $0 < \epsilon_m \to 0$. Let $\gamma$ be given by (3.50), and let $q^*_m(\xi)$ be given by (3.49). Let $\mu_{LL}^{(m)}$ be given by (3.51). Suppose

- \[
\lim_{m \to \infty} \frac{1}{s_{m_{k_0}}} \log P\left(Z > \mu_{LL}^{(m)}\right) = -A \tag{D1}
\]

- \[
\lim_{m \to \infty} \frac{1}{s_{m_{k_0}}} \left( -\theta(\mu_{LL}^{(m)}) \mu_{LL}^{(m)} + \Lambda_f(\theta(\mu_{LL}^{(m)})) \right) = -A \tag{D3}
\]

- There exists $\theta^*_m \to 0$ such that for any $D > 0$

  \[
  \lim_{m \to \infty} \frac{1}{s_{m_{k_0}}} \left( -\theta^*_m mL.u(q^*_m(0) - q^*_m(-1)) + mD\bar{l}^2(\theta^*_m)^2 + 2\Lambda_f(\theta(\mu_{LL}^{(m)})) \right) = -\infty \tag{U1}
  \]

Then,

\[
\limsup_{m \to \infty} \frac{1}{s_{m_{k_0}}} \log P(L_m > x_m) \leq -A \tag{3.57}
\]
and
\[ \limsup_{m \to \infty} \frac{1}{s_m} \log M_2(\theta_m(Z), x_m, \mu_{LL}^{(m)}) \leq -2A. \]  
(3.58)

Proof.
\[ G_{LL}^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\mu_{LL}^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \mu_{LL}^{(m)} \} & \text{if } a < 0 
\end{cases} . \]

Therefore, by symmetry of \( Z \),
\[ \mathbb{P}(Z \in G_{LL}^{(m)}) = \mathbb{P}(Z > \mu_{LL}^{(m)}). \]

\[ \mathbb{P}(L_m > x_m) = \mathbb{E} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})}(Z) \right) \]
\[ \leq \mathbb{E} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}_{(G_{LL}^{(m)})}(Z) \right) \]
\[ = \mathbb{E} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})^c}(Z) \right) + \mathbb{P} \left( Z \in G_{LL}^{(m)} \right) \]
\[ = \mathbb{E} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})^c}(Z) \right) + \mathbb{P} \left( Z > \mu_{LL}^{(m)} \right) . \]
(3.59)

\[ M_2(\theta_m(Z), x_m, \mu_{LL}^{(m)}) \]
\[ \leq \mathbb{E}_{LL}^{L_m} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu_{LL}^{(m)} Z) + 2\Lambda_f(\theta(\mu_{LL}^{(m)}))} \right) \]
\[ + \mathbb{E}_{LL}^{L_m} \left( \mathbb{1}_{\{ L_m > x_m \}} \mathbb{1}_{(G_{LL}^{(m)})^c}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu_{LL}^{(m)} Z) + 2\Lambda_f(\theta(\mu_{LL}^{(m)}))} \right) . \]
(3.60)
We recall the following. By (3.46)

\[ \psi_m(\theta, z) = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k(z) \left( e^{\Lambda(\theta_k)} - 1 \right) \right), \]

and by (3.44)

\[ \theta_m(z) = \arg\min_{\theta \geq 0} \{-\theta x_m + m \psi_m(\theta, z)\}. \]

Define the measure \( \mathbb{P}_m \)

\[ \frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z) L_m + m \psi_m(\theta_m(Z), z)}, \]

(3.61)

and let \( \mathbb{E}_m \) denote the expectation under \( \mathbb{P}_m \). Note also that \( -\theta x_m + m \psi_m(\theta, z)|_{\theta=0} = 0 \).

Therefore

\[ -\theta_m(z) x_m + \psi_m(\theta_m(z), z) \leq 0. \]

(3.62)

Also recall that the measure \( \mathbb{P}_m^{LL} \) was defined in (3.29) by

\[ \frac{d\mathbb{P}}{d\mathbb{P}_m^{LL}} = e^{-\theta_m(Z) L_m + \psi_m(\theta_m(Z), z) - \theta(\mu_{LL}^{(m)}) Z + \Lambda_f(\theta(\mu_{LL}^{(m)}))}. \]

(3.63)

**Lemma 3.5.3.**

\[ \mathbb{E}_m^{LL} \left( 1 \{ L_m > x_m \} 1 \{ G_{LL}^{(m)} \} (Z) e^{-2\theta_m(Z) L_m + 2\psi_m(\theta_m(Z), z) - 2\theta(\mu_{LL}^{(m)}) Z + 2\Lambda_f(\theta(\mu_{LL}^{(m)}))} \right) \]

\[ \leq e^{-2\theta(\mu_{LL}^{(m)}) \mu_{LL}^{(m)} + 2\Lambda_f(\theta(\mu_{LL}^{(m)}))}. \]

**Proof.** First assume that \( a < 0 \). Then \( \mu_{LL}^{(m)} = \mu_{LL}^{(m)} > 0 \) and by (3.53), \( z \in G_{LL}^{(m)} \implies z \geq \)
\(\mu^{(m)}_{LL}\). We assume that \(\theta(\mu^{(m)}_{LL}) > 0\).

\[
\mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu^{(m)}_{LL})Z+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
= \mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu^{(m)}_{LL})Z+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
\leq \mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)x_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu^{(m)}_{LL})\mu^{(m)}_{LL}+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
\leq e^{-2\theta(\mu^{(m)}_{LL})\mu^{(m)}_{LL}+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} . \quad \text{(by (3.62))}
\]

Now assume that \(a > 0\). Then \(\mu^{(m)}_{LL} = -\mu^{(m)}_{LL} < 0\) and by (3.53) \(z \in G^{(m)}_{GLL} \implies z \leq -\mu^{(m)}_{LL}\).

Note that \(\Lambda'_f(\theta) = \frac{E[Z e^{\theta Z}]}{E[e^{\theta Z}]}\). By the symmetry of \(Z\), we have \(\Lambda'_f(-\theta) = -\Lambda'_f(\theta)\). Therefore if \(\Lambda'_f(\theta(x)) = x\) then \(\Lambda'_f(-\theta(x)) = -x\). Therefore we have \(\theta(-\mu^{(m)}_{LL}) = -\theta(\mu^{(m)}_{LL})\). Notice also that \(\Lambda_f(-\theta(\mu^{(m)}_{LL})) = \Lambda_f(\theta(\mu^{(m)}_{LL}))\) by the symmetry of \(Z\).

\[
\mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu^{(m)}_{LL})Z+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
= \mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(-\mu^{(m)}_{LL})Z+2\Lambda_f(-\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
= \mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta(-\mu^{(m)}_{LL})Z+2\psi_m(\theta_m(Z),Z)+2\theta(\mu^{(m)}_{LL})\mu^{(m)}_{LL}+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
\[
\leq \mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta(\mu^{(m)}_{LL})\mu^{(m)}_{LL}+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right) . \quad \text{(by (3.62))}
\]

Either way we have

\[
\mathbb{E}^{LL}_{m} \left( \mathbbm{1}_{\{L_m > x_m\}} \mathbbm{1}_{\{G^{(m)}_{GLL}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu^{(m)}_{LL})Z+2\Lambda_f(\theta(\mu^{(m)}_{LL}))} \right)
\]
$$\leq e^{-2\theta(m)\mu_{LL} + 2\Lambda f(\theta(m))}.$$ \hfill \square

Notice that by (3.52), for $z \in (G_{LL})^c$, 

$$p_k(z) = H\left(\frac{G^{-1}(p_k) - az}{b}\right) \leq H\left(\frac{G^{-1}(\bar{p}) - az}{b}\right) = q_m(-1).$$

Recall that 

$$x_m = H(\sum_{k=1}^m l_k u_k).$$

Therefore with our new notation 

$$x_m = q_m(0)\sum_{k=1}^m l_k u_k.$$

**Lemma 3.5.4.** There exists $D > 0$ such that for any $\theta_m^* \geq 0$, 

$$\mathbb{E}_m^{LL} \left( \mathbb{1}_{\{L_m \leq x_m\}} \mathbb{1}_{\{(G_{LL})^c\}}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z),Z) - 2\theta(m)\mu_{LL} + 2\Lambda f(\theta(m))} \right) \leq e^{-2\theta_m^*m L_m u(q_m(0) - q_m(-1)) + 2mD^2(\theta_m^*)^2} \left( \mathbb{E} \left( e^{\theta(m)\mu_{LL}} Z \right) \right)^2$$

(3.64)
By using the change of measure given in (3.61)

$$E \left( \mathbf{1}\{L_m > x_m\}\mathbf{1}\{(G^{(m)}_{LL})^c\}(Z) \right)$$

$$= E_m \left( e^{-\theta_m(Z)L_m} + m\psi_m(\theta_m(Z), Z) \mathbf{1}\{L_m > x_m\}\mathbf{1}\{(G^{(m)}_{LL})^c\}(Z) \right)$$

$$\leq E_m \left( e^{-\theta_m(Z)x_m + m\psi_m(\theta_m(Z), Z)} \mathbf{1}\{L_m > x_m\}\mathbf{1}\{(G^{(m)}_{LL})^c\}(Z) \right)$$

$$\leq E_m \left( e^{-\theta_m(Z)x_m + m\psi_m(\theta_m(Z), Z)} \mathbf{1}\{(G^{(m)}_{LL})^c\}(Z) \right)$$

$$\leq E_m \left( e^{-\theta^*_m x_m + m\psi_m(\theta^*_m, Z)} \mathbf{1}\{(G^{(m)}_{LL})^c\}(Z) \right) \text{ for any } \theta^*_m \geq 0 \text{ by (3.44).}$$

By using the change of measure given in (3.63)

$$E^LL_m \left( \mathbf{1}\{L_m > x_m\}\mathbf{1}\{(G^{(m)}_{LL})^c\}(Z)e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu^*_{LL})Z + 2\Lambda_f(\theta(\mu^*_{LL}))} \right)$$

$$= E \left( \mathbf{1}\{L_m > x_m\}\mathbf{1}\{(G^{(m)}_{LL})^c\}(Z)e^{\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^*_{LL})Z + \Lambda_f(\theta(\mu^*_{LL}))} \right)$$

$$\leq E \left( \mathbf{1}\{(G^{(m)}_{LL})^c\}(Z)e^{-\theta_m(Z)x_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^*_{LL})Z + \Lambda_f(\theta(\mu^*_{LL}))} \right)$$

$$\leq E \left( \mathbf{1}\{(G^{(m)}_{LL})^c\}(Z)e^{-\theta^*_m x_m + \psi_m(\theta^*_m, Z) - \theta(\mu^*_{LL})Z + \Lambda_f(\theta(\mu^*_{LL}))} \right) \text{ for any } \theta^*_m \geq 0. \quad (3.66)$$

For \( z \in (G^{(m)}_{LL})^c \),

$$-\theta^*_m x_m + m\psi_m(\theta^*_m, Z)$$

$$= -\theta^*_m q^*_m(0) \sum_{k=1}^{m} l_k u_k + \sum_{k=1}^{m} \log \left( 1 + p_k(z)(e^{\Lambda(\theta^*_m k)} - 1) \right)$$

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Hence we have proved (3.65). By (3.66) for any \( \theta^* \) we have

\[
= \sum_{k=1}^{m} \left( -\theta^*_m q_m^e(0) l_k u_k + \log \left( 1 + p_k(z) (e^{\Lambda (\theta^*_m l_k)} - 1) \right) \right)
\]

\[
\leq \sum_{k=1}^{m} \left( -\theta^*_m q_m^e(0) l_k u_k + p_k(z) u_k l_k \theta^*_m + Dl^2(\theta^*_m)^2 \right)
\]

(by Lemma (3.4.2))

\[
\leq \sum_{k=1}^{m} \left( -\theta^*_m q_m^e(0) l_k u_k + q_m^e(-1) u_k l_k \theta^*_m + Dl^2(\theta^*_m)^2 \right)
\]

\[
\leq \sum_{k=1}^{m} \left( -\theta^*_m l_k u_k (q_m^e(0) - q_m^e(-1)) + Dl^2(\theta^*_m)^2 \right)
\]

(since \( q_m^e(0) - q_m^e(-1) > 0 \))

\[
= -\theta^*_m m_L u(q_m^e(0) - q_m^e(-1)) + mDl^2(\theta^*_m)^2.
\]

\[
\mathbb{E}_m \left( I \{ L_m > x_m \} \mathbb{I}_{(G^{(m)}_L \subset \mathcal{C})}(Z) \right) \leq \mathbb{E}_m \left( e^{-\theta^*_m x_m + m \psi_m(\theta^*_m, Z) \mathbb{I}_{(G^{(m)}_L \subset \mathcal{C})}(Z)} \right)
\]

\[
\leq \mathbb{E}_m \left( e^{-\theta^*_m q_m^e(0) \sum_{k=1}^{m} l_k u_k + \sum_{k=1}^{m} \log \left( 1 + p_k(z) (e^{\Lambda (\theta^*_m l_k)} - 1) \right)} \right)
\]

\[
\leq e^{-\theta^*_m m_L u(q_m^e(0) - q_m^e(-1)) + mDl^2(\theta^*_m)^2}.
\]

Hence we have proved (3.65). By (3.66) for any \( \theta^*_m \)

\[
\mathbb{E}_m^{LL} \left( I \{ L_m > x_m \} \mathbb{I}_{((G^{(m)}_L \subset \mathcal{C}))}(Z) e^{-2\theta_m(Z) L_m + 2 \psi_m(\theta_m(Z), Z) - 2 \theta(\mu^{*(m)}_{LL}) Z + 2 \Lambda_f(\theta(\mu^{*(m)}_{LL}))} \right)
\]

\[
\leq \mathbb{E} \left( I \{ ((G^{(m)}_L) \subset \mathcal{C}) \}(Z) e^{-\theta^*_m x_m + \psi_m(\theta^*_m, Z) - \theta(\mu^{*(m)}_{LL}) Z + \Lambda_f(\theta(\mu^{*(m)}_{LL}))} \right)
\]

\[
\leq e^{-\theta^*_m m_L u(q_m^e(0) - q_m^e(-1)) + mDl^2(\theta^*_m)^2} \mathbb{E} \left( e^{-\theta(\mu^{*(m)}_{LL}) Z + \Lambda_f(\theta(\mu^{*(m)}_{LL}))} \right)
\]

\[
\leq e^{-2\theta_m(Z) L_m + 2 \psi_m(\theta_m(Z), Z) - 2 \theta(\mu^{*(m)}_{LL}) Z + 2 \Lambda_f(\theta(\mu^{*(m)}_{LL}))} \mathbb{E} \left( e^{-\theta(\mu^{*(m)}_{LL}) Z + \Lambda_f(\theta(\mu^{*(m)}_{LL}))} \right)
\]

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\[
\leq e^{-\theta^*_m mL u(q_m^*(0) - q_m^*(-1)) + mD^2(\theta^*_m)^2} E \left( e^{-\theta(\mu^*_{LL}) Z} \left( E \left( e^{\theta(\mu^*_{LL}) Z} \right) \right) \right)
\]

Hence we have proved (3.64).

It follows that, by Lemma (3.5.3), Lemma (3.5.4), (3.59) and (3.60) that

\[
\mathbb{P}(L_m > x_m) \leq e^{-\theta^*_m mL u(q_m^*(0) - q_m^*(-1)) + mD^2(\theta^*_m)^2} + \mathbb{P}(Z > \mu^*_{LL}),
\]

and

\[
M_2(\theta_m(Z), x_m, \mu^*_{LL}) \leq e^{-2\theta^*_m mL u(q_m^*(0) - q_m^*(-1)) + 2mD^2(\theta^*_m)^2} \left( E \left( e^{\theta(\mu^*_{LL}) Z} \right) \right)^2 \\
+ e^{-2\theta(\mu^*_{LL}) \mu^*_{LL} + 2\Lambda_f(\theta(\mu^*_{LL}))}.
\]  

Now, (3.58) and (3.57) follow by (D1), (D3) and (U1).

### 3.5.2 Lower Bound Computation

**Theorem 3.5.5.** Suppose \textbf{Assumptions GEN} and \textbf{Assumptions IS-LL1} hold. Let \( \epsilon_m > 0 \) such that \( \epsilon_m \to 0 \). Let \( \gamma \) be given by (3.50), and let \( q_m^*(\xi) \) be given by (3.49).

Let \( \nu_{LL}^{(m)} \) be given by (3.54). Suppose
\[ \lim_{m \to \infty} \frac{1}{s_m^{k_0}} \log \mathbb{P} \left( Z > \nu_{LL}^{(m)} \right) = -A \]  \hspace{1cm} (D2)

\[ \lim_{m \to \infty} \frac{1}{s_m^{k_0}} \left( -\theta \nu_{LL}^{(m)} \nu_{LL}^{(m)} + \Lambda f(\nu_{LL}^{(m)}) \right) = -A \]  \hspace{1cm} (D4)

- There exists \( \eta : [0, \infty] \to [0, \infty] \), \( d_0 > 0 \) and \( M_1 \) such that for \( m > M_1 \)

\[ q_m^*(1) - q_m^*(0) \geq d_0 \frac{1}{\eta(s_m)} \]  \hspace{1cm} (L1)

\[ \lim_{m \to \infty} \frac{(\eta(s_m))^2}{m} = 0 \]  \hspace{1cm} (L2)

Then,

\[ \liminf_{m \to \infty} \frac{1}{s_m^{k_0}} \log \mathbb{P}(L_m > x_m) \geq -A \]  \hspace{1cm} (3.68)

and

\[ \liminf_{m \to \infty} \frac{1}{s_m^{k_0}} \log \mathbb{P}(2, \nu_{LL}^{(m)} \nu_{LL}^{(m)} + \Lambda f(\nu_{LL}^{(m)})) = -A. \]  \hspace{1cm} (3.69)

**Proof.** Recall that

\[ \mathbb{P}(L_m > x_m) = \mathbb{E}^{SD}_m \left( \mathbb{1}\{L_m > x_m\} e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^*_{LL}(m))Z + \lambda_f(\theta(\mu^*_{LL}(m)))} \right), \]

and the second moment of the IS estimator

\[ M_2(\theta_m(Z), x_m, \mu^*_{LL}(m)) = \mathbb{E}^{SD}_m \left( \mathbb{1}\{L_m > x_m\} e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu^*_{LL}(m))Z + 2\lambda_f(\theta(\mu^*_{LL}(m)))} \right). \]

By Jensen’s inequality

\[ (\mathbb{P}(L_m > x_m))^2 \leq M_2(\theta_m(Z), x_m, \mu^*_{LL}(m)). \] \hspace{1cm} (3.70)

Therefore it suffices to prove (3.68) since (3.69) would then follow by (3.70). Let

\[ S_m = \frac{\eta(s_m)}{m} (L_m - \mathbb{E}(L_m|Z)), \]

and \( z^{(m)} \) be an arbitrary sequence in \( \mathbb{R} \). Then, by Lemma (3.4.3) and (L2), for any \( \epsilon > 0 \)

\[ \lim_{m \to \infty} \mathbb{P} \left( |S_m| > \epsilon |Z = z^{(m)} \right) = 0. \]

**Lemma 3.5.6.** Let \( z^{(m)}_* \in H_{LL}^{(m)} \). Then,

\[ \lim_{m \to \infty} \mathbb{P}(L_m > x_m|Z = z^{(m)}_*) = 1. \]

**Proof.** Let \( z^{(m)}_* \in H_{LL}^{(m)} \).
\[ \mathbb{E}(L_m | Z = z^{(m)}) \geq q^c_m(1) \sum_{k=1}^{m} l_k u_k. \]

Therefore for \( m > M_1 \)

\[
\frac{\eta(s_m)}{m} \left( x_m - \mathbb{E}(L_m | Z = z^{(m)}) \right) \leq \frac{\eta(s_m)}{m} \left( x_m - q^c_m(1) \sum_{k=1}^{m} l_k u_k \right) = \frac{\eta(s_m)}{m} (q^c_m(0) - q^c_m(1)) \sum_{k=1}^{m} l_k u_k
\]

\[
= -\frac{\eta(s_m)}{m} (q^c_m(1) - q^c_m(0)) \sum_{k=1}^{m} l_k u_k
\]

\[
\leq -\frac{\eta(s_m)}{m} d_0 \frac{1}{\eta(s_m)} \sum_{k=1}^{m} l_k u_k \quad \text{(by } (L1)\text{)}
\]

\[
= -d_0 \frac{1}{m} \sum_{k=1}^{m} l_k u_k \quad \text{(3.71)}
\]

\[
\to -d_0 C, \quad \text{(3.72)}
\]

where

\[
C = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} l_k u_k. \quad \text{(3.73)}
\]

Let \( 0 < \epsilon < d_0 C \). There exists \( M_2 \in \mathbb{N} \) such that for all \( m > M_2 \)

\[
d_0 \frac{1}{m} \sum_{k=1}^{m} l_k u_k > \epsilon,
\]

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or equivalently,

\[-d_0 \frac{1}{m} \sum_{k=1}^{m} l_k u_k < -\epsilon. \tag{3.74}\]

For \(m > \max\{M_1,M_2\}\)

\[
\mathbb{P}(L_m > x_m|Z = z^{(m)}_*) = \mathbb{P}\left(L_m - \mathbb{E}(L_m|Z) > x_m - \mathbb{E}(L_m|Z) \bigg| Z = z^{(m)}_*\right)
\]

\[
= \mathbb{P}\left(\frac{\eta(s_m)}{m} \left(L_m - \mathbb{E}(L_m|Z)\right) > \frac{\eta(s_m)}{m} \left(x_m - \mathbb{E}(L_m|Z)\right) \bigg| Z = z^{(m)}_*\right)
\]

\[
= \mathbb{P}\left(S_m > \frac{\eta(s_m)}{m} \left(x_m - \mathbb{E}(L_m|Z)\right) \bigg| Z = z^{(m)}_*\right)
\]

\[
\geq \mathbb{P}\left(S_m > -d_0 \frac{1}{m} \sum_{k=1}^{m} l_k u_k \bigg| Z = z^{(m)}_*\right) \quad \text{(for } m > M_1 \text{ by (3.71))}
\]

\[
\geq \mathbb{P}\left(S_m > -\epsilon\right) \quad \text{(for } m > M_2 \text{ by (3.74))}
\]

\[
\geq \mathbb{P}\left(|S_m| \leq \epsilon \bigg| Z = z^{(m)}_*\right) \to 1 \text{ as } m \to \infty. \tag{3.75}
\]

\[p_k(z) = H \left(\frac{G^{-1}(p_k)-az}{b}\right). \]

Therefore \(p_k(z)\) is decreasing in \(z\) if \(a > 0\). Analogously \(p_k(z)\) is increasing in \(z\) if \(a < 0\). Notice also that by (3.55)

\[
H_{LL}^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\nu_{LL}^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \nu_{LL}^{(m)} \} & \text{if } a < 0 
\end{cases}.
\]

Assume \(a < 0\).
\[ P(L_m > x_m) \geq \int_{t \in [0, \infty)} P\left(L_m > x_m | Z = \nu_{LL}^{(m)} + t\right) P\left(Z = \nu_{LL}^{(m)} + t\right) dt \]

\[ \geq P\left(L_m > x_m | Z = \nu_{LL}^{(m)}\right) \int_{t \in [0, \infty)} P\left(Z = \nu_{LL}^{(m)} + t\right) dt \]

\[ = P\left(L_m > x_m | Z = \nu_{LL}^{(m)}\right) P\left(Z > \nu_{LL}^{(m)}\right). \]

Note that \( \nu_{LL}^{(m)} \in H_{LL}^{(m)} \). Similarly if \( a > 0 \)

\[ P(L_m > x_m) \geq \int_{t \in [0, \infty)} P\left(L_m > x_m | Z = -\nu_{LL}^{(m)} - t\right) P\left(Z = -\nu_{LL}^{(m)} - t\right) dt \]

\[ \geq P\left(L_m > x_m | Z = -\nu_{LL}^{(m)}\right) \int_{t \in [0, \infty)} P\left(Z = -\nu_{LL}^{(m)} - t\right) dt \]

\[ = P\left(L_m > x_m | Z = -\nu_{LL}^{(m)}\right) P\left(Z < -\nu_{LL}^{(m)}\right). \]

Note that \( -\nu_{LL}^{(m)} \in H_{LL}^{(m)} \). Notice also that by the symmetry of \( Z \)

\[ P(Z > \nu_{LL}^{(m)}) = P(Z < -\nu_{LL}^{(m)}). \]

Therefore in either case (whether \( a > 0 \) or \( a < 0 \)), we have

\[ P(L_m > x_m) \geq k_m P\left(Z > \nu_{LL}^{(m)}\right), \]

for some sequence \( k_m \to 1 \). Now (3.68) follows by (D2).

\[ \square \]
3.5.3 Proof of Main Theorem: Large Loss Threshold Regime

To prove Theorem 3.3.1, we will use Theorem 3.5.1.

Proof of Theorem 3.3.1

We will apply Theorem 3.5.1 as follows. We need to verify (D1) and (D2), (D3) and (D4), (D4), (U1), (L1) and (L2). \( X_k \sim N(0, 1) \). \( Z \) is allowed to take \( N(0, 1) \) and Double Exponential(\( \lambda \)). Define \( \epsilon_m = \frac{1}{s_m} \). We give below two lemmas whose proofs can be found in the Section 2.8.

Lemma 3.5.7. Let \( \epsilon_m = \frac{1}{s_m} \). Then, there exists \( M_1 \in \mathbb{N} \) such that for \( m > M_1 \)

\[
q'_m(0) - q'_m(-1) \geq \frac{e^{-\frac{1}{2}(1-\epsilon_m)^2s^2s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1-e^{-s^2}}{2} \right). \tag{3.76}
\]

Lemma 3.5.8. Let \( \epsilon_m = \frac{1}{s_m} \). Then, there exists \( M_1 \) such that for \( m > M_1 \)

\[
q'_m(1) - q'_m(0) \geq \frac{e^{-\frac{1}{2}s^2s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1-e^{-s^2}}{2} \right). \tag{3.77}
\]

By (3.77), there exists \( M_1 \) such that for \( m > M_1 \), \( q'_m(1) - q'_m(0) \geq \frac{e^{-\frac{1}{2}s^2s_m^2}}{\sqrt{2\pi ss_m}} \left( \frac{1-e^{-s^2}}{2} \right) \). Define \( \eta(s_m) = ss_m e^{\frac{1}{2}s^2s_m^2} \) and \( d_0 = \frac{1}{\sqrt{2\pi}} \left( \frac{1-e^{-s^2}}{2} \right) \).

- (L1) There exists and \( d_0 \) and \( M = \) such that for \( m > M \): \( q'_m(1) - q'_m(0) \geq d_0 \frac{1}{\eta(s_m)} \).

- (L2) \( \lim_{m \to \infty} \left( \frac{(\eta(s_m))^2}{m} \right) = 0 \) since

\[
\log \left( \frac{(\eta(s_m))^2}{m} \right) = \log \left( \frac{(ss_m)^2e^2s^2s_m^2}{m} \right)
\]

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\( \log \left( (s s_m)^2 e^{s^2 s_m^2} \right) - \log m \)
\( = s^2 s_m^2 - \log m + o(s_m) \)
\( = s_m^2 \left( s^2 - \frac{\log m}{s_m^2} + o(1) \right) \)
\( \to -\infty. \)

In order to verify rest of the conditions we consider separate cases.

**Case 1: \( Z \sim N(0,1) \)**

\([D1]\) and \([D2]\) are satisfied with \( k_0 = 2 \) and \( A = \frac{1}{2} \gamma^2 \). Next we will verify \([D3]\) and \([D4]\)
\( Z \sim N(0,1) \). Therefore \( \Lambda_f(\theta) = \frac{1}{2} \theta^2 \) and \( \theta(x) = x \). Therefore

\[ x \theta(x) - \Lambda_f(\theta(x)) = -\frac{1}{2} x^2. \]

Therefore \([D3]\) and \([D4]\) are satisfied with \( k_0 = 2 \) and \( A = \frac{1}{2} \gamma^2 \).

To verify \([U1]\), note that \( \Lambda_f(\mu_{LL}^{(m)}) = \frac{1}{2} (\theta(\mu_{LL}^{(m)}))^2 \) and therefore

\[ \lim_{m \to \infty} \frac{1}{s_m} \Lambda_f(\theta(\mu_{LL}^{(m)})) = \frac{1}{2} \gamma^2. \]

Therefore it suffices to show that

\[ \lim_{m \to \infty} -\frac{\theta_m^* m_1 u(q_m^*(0) - q_m^*(1))}{s_m^2} + \frac{m D \bar{L}^2 (\theta_m^*)^2}{s_m^2} = 0. \quad (3.78) \]

Define \( \theta_m^* = \frac{s_m}{\sqrt{m}} \) so that \( \frac{m \theta_m^*}{s_m^2} = 1 \). By \( [3.76] \), there exists \( M_1 \in \mathbb{N} \) such that for \( m > M_1 \),
\[ q_m^\epsilon(0) - q_m^\epsilon(-1) \geq \frac{e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi s s_m}} \left( \frac{1 - e^{-s^2}}{2} \right). \]

\[ \frac{\theta_m^* m}{s_m^2} (q_m^\epsilon(0) - q_m^\epsilon(-1)) \geq \frac{\theta_m^* m e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{s_m^2 \sqrt{2\pi s s_m}} \left( \frac{1 - e^{-s^2}}{2} \right). \]

\[ \frac{\theta_m^* m e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{s_m^3} = \frac{m^\frac{1}{2}}{s_m^2} e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}. \]

\[ \log \left( \frac{m^\frac{1}{2}}{s_m^2} e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2} \right) = \frac{1}{2} \log m - \frac{1}{2} (1 - \epsilon_m)^2 s^2 s_m^2 + o(s_m) \]

\[ = s_m^2 \left( \frac{1}{2} \log m s_m^2 - \frac{1}{2} (1 - \epsilon_m)^2 s^2 + o(1) \right) \]

\[ = \frac{1}{2} s_m^2 \left( \frac{\log m}{s_m^2} - s^2 + o(1) \right) \]

\[ \to \infty. \]

**Case 2: Z \sim Double Exponential \lambda**

\[ \mathbb{P}(Z > x) = \frac{1}{2} e^{-\lambda x}. \] (D1) and (D2) are satisfied with \( k_0 = 1 \) and \( A = \lambda \gamma. \) Next we will verify (D3) and (D4) \( Z \sim Double Exponential(\lambda). \) Then \( \Lambda_f(\theta) = 2 \log \lambda - \log(\lambda^2 - \theta^2) \) for \( \theta < |\lambda|, \) \( \theta(x) \) solves the equation

\[ x\theta^2 + 2\theta - \lambda^2 x = 0 \] (3.79)

for \( \theta. \) Taking the positive root
\[
\theta(x) = -\frac{1}{x} + \sqrt{\frac{1}{x^2} + \lambda^2}.
\] (3.80)

Therefore

\[
\Lambda_f(\theta(x)) = 2 \log \lambda - \log(\lambda^2 - (\theta(x))^2)
\]
\[= 2 \log \lambda - \log(2 \frac{\theta(x)}{x}). \quad \text{(by (3.79))}
\]

Note that for any \( r > 0 \), \( \lim_{x \to \infty} \frac{1}{x} \log \left( \frac{1}{x^r} \right) = 0 \). Therefore

\[
\lim_{x \to \infty} \frac{1}{x} \Lambda_f(\theta(x)) = 0.
\]

\[
\lim_{x \to \infty} \frac{1}{x} \left( -x \theta(x) + \Lambda_f(\theta(x)) \right) = -\lambda.
\]

Therefore (D3) and (D4) hold with \( k_0 = 1 \) and \( \lambda gamma. \) To verify (U1), note that

\[
\lim_{m \to \infty} \frac{1}{s_m} \Lambda_f(\mu_{\text{LL}}^{(m)}) = 0.
\]

Therefore it suffices to show that

\[
\lim_{m \to \infty} \frac{\theta_m^* mL.u(q_m^*(0) - q_m^*(-1))}{s_m} + \frac{mD\bar{l}^2(\theta_m^*)^2}{s_m} = 0.
\]
Define $\theta^*_m = \sqrt{\frac{s_m}{m}}$ so that $\frac{m \theta^*_m}{s_m} = 1$. By (3.76), there exists $M_1 \in \mathbb{N}$ such that for $m > M_1$,

$$q_m^e(0) - q_m^e(-1) \geq \frac{e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi ss_m}} \left(1 - \frac{e^{-s^2}}{2}\right).$$

Let

$$\frac{\theta^*_m m}{s_m} \left(q_m^e(0) - q_m^e(-1)\right) \geq \frac{\theta^*_m m}{s_m} \frac{e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{\sqrt{2\pi ss_m}} \left(1 - \frac{e^{-s^2}}{2}\right).$$

Then

$$\frac{\theta^*_m m e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}}{s_m^2} = \frac{m^{\frac{1}{2}}}{s_m^2} e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}.$$

Thus

$$\log \left(\frac{m^{\frac{1}{2}}}{s_m^2} e^{-\frac{1}{2}(1-\epsilon_m)^2 s^2 s_m^2}\right) = \frac{1}{2} \log m - \frac{1}{2} (1 - \epsilon_m)^2 s^2 s_m^2 + o(s_m)$$

$$= s_m^2 \left(\frac{1}{2} \log m - \frac{1}{2} (1 - \epsilon_m)^2 s^2 + o(1)\right)$$

$$= \frac{1}{2} s_m^2 \left(\log m - s^2 + o(1)\right)$$

$$\to \infty.$$

### 3.6 Logarithmic Efficiency: Small Default Probability Regime

In this regime, we consider small $p^{(m)}_k$’s and moderate $x_m$’s. We let the default probabilities $p^{(m)}_k = p^{(m)} \to 0$ according to (3.32), and let $x_m = q \sum_{k=1}^m u_k l_k$ according to (3.33). We require two additional assumptions on the cdf $H(\cdot)$ of $\varepsilon_k$. This is given under Assumptions IS-SD2.

**Assumptions IS-SD2**
1. The density of $\varepsilon_k, h(\cdot)$ does not vanish at $H^{-1}(q): h(H^{-1}(q)) \neq 0$.

2. $h(\cdot)$ is continuous on a neighborhood of $H^{-1}(q)$.

By observing that $H \left( H^{-1}(q) \right) - H \left( (1 - \epsilon_m)H^{-1}(q) \right) = \int_{H^{-1}(q) - H^{-1}(q)\epsilon_m}^{H^{-1}(q)} h(x) \, dx$, it follows that for $\epsilon_m \to 0$, there exists $k_1 = k_1(q), k_2 = k_2(q) > 0$ such that

$$k_1\epsilon_m < H \left( H^{-1}(q) \right) - H \left( (1 - \epsilon_m)H^{-1}(q) \right) < k_2\epsilon_m.$$

That is,

$$k_1\epsilon_m < q - H \left( (1 - \epsilon_m)H^{-1}(q) \right) < k_2\epsilon_m. \quad (3.81)$$

Once again we reserve the notation, $\theta^*_m$ and $\epsilon_m$ for two positive sequences such that $\theta^*_m, \epsilon_m \to 0$.

First we recall the definitions of $\gamma^*, \mu^{(m)}_{SD}, \mu^{(m)}_{SD}$ and $G^{(m)}_{SD}$ defined in (3.34), (3.38), (3.39) and (3.36) respectively.

$$\gamma^* = \frac{s}{|a|}. \quad (3.82)$$

$$G^{(m)}_{SD} = \{ z \in \mathbb{R} : ss_m + b(1 - \epsilon_m)H^{-1}(q) \leq -a.z \}$$

$$= \{ z \in \mathbb{R} : b(1 - \epsilon_m)H^{-1}(q) \leq -a.z - ss_m \}$$

$$= \left\{ z \in \mathbb{R} : H((1 - \epsilon_m)H^{-1}(q)) \leq H \left( \frac{-a.z - ss_m}{b} \right) = p^{(m)}(z) \right\}.$$
\[ \mu_{SD}^{(m)} = \gamma s_m + \frac{b}{|a|} (1 - \epsilon_m) H^{-1}(q). \] (3.83)

\[ \mu_{SD}^{\ast(m)} = \arg \min_z \{|z| : z \in G_{SD}^{(m)}\} \] (3.84)

\[ = \begin{cases} -\mu_{SD}^{(m)} & \text{if } a > 0 \\ \mu_{SD}^{(m)} & \text{if } a < 0 \end{cases}. \] (3.85)

\( \mu_{SD}^{(m)} \) and \( G_{SD}^{(m)} \) are related in the following manner.

\[ G_{SD}^{(m)} = \begin{cases} \{ z \in \mathbb{R} : z \leq \left(\frac{ss_m + b(1 - \epsilon_m) H^{-1}(q)}{|a|}\right) = -\mu_{SD}^{(m)} \} & \text{if } a > 0 \\ \{ z \in \mathbb{R} : z \geq \frac{ss_m + b(1 - \epsilon_m) H^{-1}(q)}{|a|} = \mu_{SD}^{(m)} \} & \text{if } a < 0 \end{cases}. \] (3.86)

Similarly, for the lower bound computation we define \( \nu_{SD}^{(m)} \) and \( H_{SD}^{(m)} \) by

\[ \nu_{SD}^{(m)} = \gamma s_m + \frac{b}{|a|} (1 + \epsilon_m) H^{-1}(q), \] (3.87)

and

\[ H_{SD}^{(m)} = \{ z \in \mathbb{R} : ss_m + b(1 + \epsilon_m) H^{-1}(q) \leq -a.z \} \] (3.88)

\[ = \{ z \in \mathbb{R} : b(1 + \epsilon_m) H^{-1}(q) \leq -a.z - ss_m \} \]

\[ = \left\{ z \in \mathbb{R} : H((1 + \epsilon_m) H^{-1}(q)) \leq H \left( \frac{-a.z - ss_m}{b} \right) = p^{(m)}(z) \right\}. \] (3.89)
\( \nu_{SD}^{(m)} \) and \( H_{SD}^{(m)} \) are related by

\[
H_{SD}^{(m)} = \begin{cases} 
    \{ z \in \mathbb{R} : z \leq -\left( \frac{ssm + b(1 + \epsilon_m)H^{-1}(q)}{|a|} \right) = -\nu_{SD}^{(m)} \} & \text{if } a > 0 \\
    \{ z \in \mathbb{R} : z \geq ssm + b(1 + \epsilon_m)H^{-1}(q) = \nu_{SD}^{(m)} \} & \text{if } a < 0 
\end{cases}
\]  

(3.90)

Theorem 3.6.1. Suppose Assumptions IS-GEN, Assumptions IS-SD1 and Assumptions IS-SD2 hold. Let \( \gamma \) be given by (3.82), \( \mu_{SD}^{(m)} \) be given by (3.83), and \( \nu_{SD}^{(m)} \) be given by (3.87). Suppose \( \epsilon_m = \frac{1}{s_m} \) for some \( j \geq 0 \). Suppose

\[
\lim_{m \to \infty} \frac{(s_m)^{2j}}{m} = 0. 
\]

(R1)

Suppose further

\[
\lim_{m \to \infty} \frac{1}{s_m} \log \mathbb{P}(Z > \mu_{SD}^{(m)}) = \lim_{m \to \infty} \frac{1}{s_m} \log \mathbb{P}(Z > \nu_{SD}^{(m)}) = -A \quad (E1) \text{ and } (E2)
\]

\[
\lim_{m \to \infty} \frac{1}{s_m} \left( -\theta(\mu_{SD}^{(m)}) \mu_{SD}^{(m)} + \Lambda_f(\theta(\mu_{SD}^{(m)})) \right)
\]

\[
= \lim_{m \to \infty} \frac{1}{s_m} \left( -\theta(\nu_{SD}^{(m)}) \nu_{SD}^{(m)} + \Lambda_f(\theta(\nu_{SD}^{(m)})) \right) = -A \quad (E3) \text{ and } (E4)
\]

\[
\text{There exists } \theta_m^* \to 0 \text{ such that for any } k_1, D > 0
\]

\[
\lim_{m \to \infty} \frac{1}{s_m} \left( -m \theta_m^* k_1 u \epsilon_m + m D \bar{\epsilon}^2 (\theta_m^*)^2 + 2 \Lambda_f(\theta(\mu_{SD}^{(m)})) \right) = -\infty \quad (R2)
\]

Then,
\[ \lim_{m \to \infty} \frac{\log M_2(\theta_m(Z), x_m, \mu^{(m)}_{SD})}{\log \mathbb{P}(L_m > x_m)} = 2. \] (3.91)

The proof of Theorem 3.6.1 follows from the upper bound computation given in Theorem 3.6.2 and the lower bound computation given in 3.6.5.

### 3.6.1 Upper Bound Computation

**Theorem 3.6.2.** Suppose Assumptions IS-GEN, Assumptions IS-SD1 and Assumptions IS-SD2 hold. Let \( 0 < \epsilon_m \to 0 \). Let \( \gamma_* \) be given by (3.82) and \( \mu^{(m)}_{SD} \) be given by (3.83).

Suppose

- \[ \lim_{m \to \infty} \frac{1}{s_m} \log \mathbb{P} \left( Z > \mu^{(m)}_{SD} \right) = -A \] (E1)

- \[ \lim_{m \to \infty} \frac{1}{s_m} \left( -\theta(\mu^{(m)}_{SD}) \mu^{(m)}_{SD} + \Lambda_f(\theta(\mu^{(m)}_{SD})) \right) = -A \] (E3)

- There exists \( \theta^*_m \to 0 \) such that for any \( k_1, D > 0 \)

\[ \lim_{m \to \infty} \frac{1}{s_m} \left( -m \theta^*_m u k_1 \epsilon_m + m D \bar{P}(\theta^*_m)^2 + 2 \Lambda_f(\theta(\mu^{(m)}_{SD})) \right) = -\infty \] (R2)

Then,

\[ \limsup_{m \to \infty} \frac{1}{s_m} \log \mathbb{P}(L_m > x_m) \leq -A \] (3.92)
\[
\limsup_{m \to \infty} \frac{1}{s_m} \log M_2(\theta_m(Z), x_m, \mu^{*(m)}_{SD}) \leq -2A \quad (3.93)
\]

**Proof.**

\[
G^{(m)}_{SD} = \begin{cases} 
\{ z \in \mathbb{R} : z \leq -\mu^{(m)}_{SD} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \geq \mu^{(m)}_{SD} \} & \text{if } a < 0
\end{cases}
\]

Therefore, by symmetry of \( Z \),

\[
\mathbb{P}(Z \in G^{(m)}_{SD}) = \mathbb{P}(Z > \mu^{(m)}_{SD}).
\]

\[
\mathbb{P}(L_m > x_m) = \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})}(Z) \right)
\]

\[
\leq \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) \right) + \mathbb{E} \left( 1_{(G^{(m)}_{SD})}(Z) \right)
\]

\[
= \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) \right) + \mathbb{P} \left( Z \in G^{(m)}_{SD} \right)
\]

\[
= \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) \right) + \mathbb{P} \left( Z > \mu^{(m)}_{SD} \right). \quad (3.94)
\]

\[
M_2(\theta_m(Z), x_m, \mu^{*(m)}_{SD}) \leq \mathbb{E}_m \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu^{*(m)}_{SD})Z + 2\lambda_f(\theta(\mu^{*(m)}_{SD}))} \right)
\]

\[
+ \mathbb{E}_m \left( \mathbb{1}\{L_m > x_m\} 1_{(G^{(m)}_{SD})^c}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu^{*(m)}_{SD})Z + 2\lambda_f(\theta(\mu^{*(m)}_{SD}))} \right). \quad (3.95)
\]
We recall the following. By (3.46),

$$\psi_m(\theta, z) = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p^{(m)}(z) \left( e^{\Lambda(\theta_k)} - 1 \right) \right),$$

and by (3.12),

$$\theta_m(z) = \arg\min_{\theta \geq 0} \{-\theta x_m + m \psi_m(\theta, z)\}.$$ 

Define a probability measure $P_m$ by

$$\frac{dP}{dP_m} = e^{-\theta_m(Z)L_m + m\psi_m(\theta_m(Z), Z)}. \quad (3.96)$$

Also recall that $P^{SD}_m$ was defined by (3.40) to be

$$\frac{dP}{dP^{SD}_m} = e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - \theta(\mu^{(m)})Z + \Lambda_f(\theta(\mu^{(m)}))}. \quad (3.97)$$

Note also that and $-\theta x_m + m \psi_m(\theta, z)|_{\theta=0} = 0$. Therefore

$$-\theta_m(z)x_m + \psi_m(\theta_m(z), z) \leq 0. \quad (3.98)$$

**Lemma 3.6.3.**

$$\mathbb{E}^{SD}_m \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{G^{(m)}_SD} (Z)e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu^{(m)}_{SD})Z + 2\Lambda_f(\theta(\mu^{(m)}_{SD}))} \right) \leq e^{-2\theta(\mu^{(m)}_{SD})\mu^{(m)}_{SD} + 2\Lambda_f(\theta(\mu^{(m)}_{SD}))}. \quad (3.99)$$

**Proof.** First assume that $a < 0$. By (3.84), $\mu^{(m)}_{SD} = \mu^{(m)}_{SD}$. Notice by (3.86) $z \in G^{(m)}_SD \implies$
$z \geq \mu_{SD}^{(m)}$. We assume $\theta_m(\mu_{SD}^{(m)}) > 0$.

\[
\mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right)
\]

\[
= \mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(\mu_{SD})^{(m)})} \right)
\]

\[
\leq \mathbb{E}_{SD}^{SD} \left( 1\{G_{SD}^{(m)}\}(Z)e^{-2\theta_m(Z)x_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})\mu_{SD}^{(m)}+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right). \quad (3.100)
\]

Now assume that $a > 0$. By (3.84), $\mu_{SD}^{(m)} = -\mu_{SD}^{(m)}$. Notice by (3.86) $z \in G_{SD}^{(m)} \implies z \leq -\mu_{SD}^{(m)}$.

Note that $\Lambda_f'(\theta) = \frac{\mathbb{E}(Ze^{\theta Z})}{\mathbb{E}(e^{\theta Z})}$. By the symmetry of $Z$, we have $\Lambda_f(-\theta) = -\Lambda_f'(\theta)$. Therefore if $\Lambda_f'(\theta(x)) = x$ then $\Lambda_f(-\theta(x)) = -x$. Therefore we have $\theta(-\mu_{LL}^{(m)}) = -\theta(\mu_{LL}^{(m)})$. Notice also that $\Lambda_f(-\theta(\mu_{LL}^{(m)})) = \Lambda_f(\theta(\mu_{LL}^{(m)}))$ by the symmetry of $Z$.

\[
\mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right)
\]

\[
= \mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(-\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(-\mu_{SD}^{(m)}))} \right)
\]

\[
= \mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)+2\theta(\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right)
\]

\[
\leq \mathbb{E}_{SD}^{SD} \left( 1\{G_{SD}^{(m)}\}(Z)e^{-2\theta_m(Z)x_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})\mu_{SD}^{(m)}+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right). \quad (3.101)
\]

Either way we have

\[
\mathbb{E}_{SD}^{SD} \left( 1\{L_m > x_m\} \mathbbm{1}_{\{G_{SD}^{(m)}\}}(Z)e^{-2\theta_m(Z)L_m+2\psi_m(\theta_m(Z),Z)-2\theta(\mu_{SD}^{(m)})Z+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right)
\]

\[
\leq \mathbb{E}_{SD}^{SD} \left( 1\{G_{SD}^{(m)}\}(Z)e^{-2\theta(\mu_{SD}^{(m)})\mu_{SD}^{(m)}+2\Lambda_f(\theta(\mu_{SD}^{(m)}))} \right) \quad \text{(by (3.62))}
\]

\[
\leq e^{-2\theta(\mu_{SD}^{(m)})\mu_{SD}^{(m)}+2\Lambda_f(\theta(\mu_{SD}^{(m)}))}.
\]
Notice that by (3.36), for \( z \in (G_{SD})^c \),
\[
p^{(m)}(z) = H \left( \frac{-az - ss_m}{b} \right)
< H \left( (1 - \epsilon_m)H^{-1}(q) \right).
\]

Recall that
\[
x_m = q \sum_{k=1}^{m} l_k.
\]

Let \( \theta_m^* \) be any sequence such that \( \theta_m^* \to 0 \).

**Lemma 3.6.4.** There exists \( k_1, D > 0 \) such that for any \( \theta_m^* \to 0 \),
\[
\mathbb{E}^{SD}_{m} \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{(G_{SD})^c\}}(Z)e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z),Z) + 2\theta(\mu_{SD}^*)Z + 2\Lambda_f(\theta(\mu_{SD}^*))} \right) \\
\leq e^{-m\theta_m^* u k_1 \epsilon_m + mD^2(\theta_m^*)^2} \tag{3.102}
\]

and
\[
\mathbb{E} \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{(G_{SD})^c\}} \right) \leq e^{-m\theta_m^* u k_1 \epsilon_m + mD^2(\theta_m^*)^2}. \tag{3.103}
\]

**Proof.** By using the measure \( \mathbb{P}_m \) defined in (3.96)
\[
\mathbb{E} \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{(G_{SD})^c\}}(Z) \right) = \mathbb{E}_m \left( e^{-\theta_m(Z)L_m + m\psi_m(\theta_m(Z),Z)} \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{(G_{SD})^c\}}(Z) \right)
\]

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By using the measure $\mathbb{P}^{SD}$ defined given in (3.97)

$$
\mathbb{E}^{SD}_m \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{G^{(m)}_{SD} > \varepsilon\}} (Z) e^{-2\theta_m L_m + 2\psi_m (\theta_m, Z) - 2\theta_m (\mu_{SD}^*) Z + 2\Lambda_f (\theta_m^*)) \right)
$$

$$
= \mathbb{E} \left( \mathbb{1}_{\{L_m > x_m\}} \mathbb{1}_{\{G^{(m)}_{SD} > \varepsilon\}} (Z) e^{-\theta_m L_m + 2\psi_m (\theta_m, Z) - \theta_m (\mu_{SD}^*) Z + \Lambda_f (\theta_m^*)) \right)
$$

$$
\leq \mathbb{E} \left( \mathbb{1}_{\{G^{(m)}_{SD} > \varepsilon\}} (Z) e^{-\theta_m L_m + \psi_m (\theta_m, Z) - \theta_m (\mu_{SD}^*) Z + \Lambda_f (\theta_m^*)) \right)
$$

$$
\leq \mathbb{E} \left( \mathbb{1}_{\{G^{(m)}_{SD} > \varepsilon\}} (Z) e^{-\theta_m L_m + \psi_m (\theta_m, Z) - \theta_m (\mu_{SD}^*) Z + \Lambda_f (\theta_m^*)) \right) \text{ for any } \theta_m^* \geq 0. \tag{3.105}
$$

Let $\theta_m^*$ be any sequence such that $\theta_m^* \to 0$. By $z \in (G^{(m)}_{SD})^c$,

$$
-\theta_m^* x_m + m \psi_m (\theta_m^*, z) = -\theta_m^* q \sum_{k=1}^m l_k u_k + \sum_{k=1}^m \log \left( 1 + p^{(m)}(z) (e^{\Lambda (\theta_m^*) l_k}) - 1 \right)
$$

$$
= \sum_{k=1}^m \left( -\theta_m^* q l_k u_k + \sum_{k=1}^m \log \left( 1 + p^{(m)}(z) (e^{\Lambda (\theta_m^*) l_k}) - 1 \right) \right)
$$

$$
\leq \sum_{k=1}^m \left( -\theta_m^* q l_k u_k + p^{(m)}(z) l_k \theta_m^* + D(\bar{f})^2 (\theta_m^*)^2 \right) \tag{by Lemma 3.4.2}
$$

$$
\leq \sum_{k=1}^m \left( -\theta_m^* q l_k u_k + H \left( (1 - \epsilon_m) H^{-1} (q) \right) \theta_m^* l_k + D(\bar{f})^2 (\theta_m^*)^2 \right) \tag{by (3.36)}
$$
\[\leq \sum_{k=1}^{m} (-\theta^*_m q_k u_k + H ((1 - \epsilon_m) H^{-1}(q)) \theta^*_m l_k + D(l)^2 (\theta^*_m)^2)\]

\[\leq \sum_{k=1}^{m} (-\theta^*_m l_k (q - H ((1 - \epsilon_m) H^{-1}(q))) + D(l)^2 (\theta^*_m)^2)\]

\[\leq \sum_{k=1}^{m} (-\theta^*_m u_k (q - H ((1 - \epsilon_m) H^{-1}(q))) + D(l)^2 (\theta^*_m)^2)\]

\[= -m\theta^*_m u_k (q - H ((1 - \epsilon_m) H^{-1}(q))) + mD(l)^2 (\theta^*_m)^2.\]

By (3.81) under Assumptions IS-SD2 there exists \(k_1 = k_1(q), k_2 = k_2(q) > 0\) such that

\[k_1 \epsilon_m < q - H ((1 - \epsilon_m) H^{-1}(q)) < k_2 \epsilon_m.\]

Therefore

\[-\theta^*_m x_m + m\psi_m(\theta^*_m, z) \leq -m\theta^*_m u_k k_1 \epsilon_m + mD(l)^2 (\theta^*_m)^2.\]

\[\mathbb{E} \left( 1 \{L_m > x_m\} 1\{(G_{mSD})^c\}(Z) \right) = \mathbb{E}_m \left( e^{-\theta^*_m x_m + m\psi_m(\theta^*_m, Z)} 1\{(G_{mSD})^c\}(Z) \right)\]

\[\leq \mathbb{E}_m \left( e^{-\theta^*_m q \sum_{k=1}^{m} l_k u_k + \sum_{k=1}^{m} \log \left( 1 + p(l_m)(z)(e^{\Lambda(\theta^*_m l_k)} - 1) \right) \right)\]

\[\leq e^{-m\theta^*_m u_k k_1 \epsilon_m + mD(l)^2 (\theta^*_m)^2}.\]

We have proved (3.103). By (3.105), for any \(\theta^*_m \geq 0\),

\[\mathbb{E}_m^{SD} \left( 1 \{L_m > x_m\} 1\{(G_{mSD})^c\}(Z) e^{-2\theta_m(Z)L_m + 2\psi_m(\theta_m(Z), Z) - 2\theta(\mu_{SD}^{*(m)})Z + 2\Lambda_f(\theta(\mu_{SD}^{*(m)}))} \right)\]

\[\leq \mathbb{E} \left( 1\{(G_{SD})^c\}(Z) e^{-\theta^*_m x_m + \psi_m(\theta^*_m, Z) - \theta(\mu_{SD}^{*(m)})Z + \Lambda_f(\theta(\mu_{SD}^{*(m)}))} \right)\]

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\[
\begin{align*}
&\leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \mathbb{E} \left( e^{-\theta (\mu_{SD}^{(m)}) Z + \Lambda_f (\theta(\mu_{SD}^{(m)}))} \right) \\
&\leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \mathbb{E} \left( e^{-\theta (\mu_{SD}^{(m)}) Z + \log \left( \mathbb{E} \left( e^{\theta (\mu_{SD}^{(m)}) Z} \right) \right)} \right) \\
&\leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \mathbb{E} \left( e^{-2\theta (\mu_{SD}^{(m)}) Z} \left( \mathbb{E} \left( e^{\theta (\mu_{SD}^{(m)}) Z} \right) \right)^2 \right) \\
&\leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \left( \mathbb{E} \left( e^{\theta (\mu_{SD}^{(m)}) Z} \right) \right)^2. 
\end{align*}
\]

Therefore

\[
\mathbb{E}_m \left( 1 \{ L_m > x_m \} \mathbb{E}_{G_{SD}^{(m)}} (Z) e^{-2 \theta_m (Z) L_m + 2 \psi_m (\theta_m (Z), Z) - 2 \theta (\mu_{SD}^{(m)}) Z + 2 \Lambda_f (\theta (\mu_{SD}^{(m)}))} \right) \\
\leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \left( \mathbb{E} \left( e^{\theta (\mu_{SD}^{(m)}) Z} \right) \right)^2.
\]

\[\square\]

It follows by Lemma (3.6.3), Lemma (3.6.4), (3.95) and (3.94) that

\[
\mathbb{P}(L_m > x_m) \leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} + \mathbb{P}(Z > \mu_{SD}^{(m)}),
\]

and

\[
M_2(\theta_m(Z), x_m, \mu_{SD}^{(m)}) \leq e^{-m\theta_m^* u k_1 \epsilon_m + m D \bar{L}^2 (\theta_m^*)^2} \left( \mathbb{E} \left( e^{\theta (\mu_{SD}^{(m)}) Z} \right) \right)^2 + e^{-2 \theta (\mu_{SD}^{(m)}) \mu_{SD}^{(m)} + 2 \Lambda_f (\theta (\mu_{SD}^{(m)}))}.
\]

Now the result (3.92) and (3.93) follows by the assumptions (E1), (E3) and (R2).

\[\square\]

### 3.6.2 Lower Bound Computation

**Theorem 3.6.5.** Suppose Assumptions IS-GEN, Assumptions IS-SD1 and Assumptions IS-SD2 hold. Let \( \epsilon_m = \frac{1}{s_m} \) for some \( j > 0 \). Let \( \gamma_* \) be given by (3.82), and \( \nu_{SD}^{(m)} \) be
given by (3.87). Suppose

\[ \lim_{m \to \infty} \frac{1}{s_m} \log P \left( Z > \nu_{SD}^{(m)} \right) = -A \]  

(E2)

\[ \lim_{m \to \infty} \frac{1}{s_m} \left( -\theta(\nu_{SD}^{(m)}) \nu_{SD}^{(m)} + \Lambda_f(\nu_{SD}^{(m)}) \right) = -A \]  

(E4)

\[ \lim_{m \to \infty} \frac{(s_m)^{2j}}{m} = 0. \]  

(R1)

Then,

\[ \liminf_{m \to \infty} \frac{1}{s_m} \log P(L_m > x_m) \geq -A \]  

(3.106)

and

\[ \liminf_{m \to \infty} \frac{1}{s_m} \log M_2(\theta_m(Z), x_m, \mu_{SD}^{(m)}) \geq -2A. \]  

(3.107)

Proof. Note that by using the measure \( P_m^{SD} \) defined in (3.40),

\[ P(L_m > x_m) = P_m^{SD} \left( \mathbb{1}_{L_m > x_m} e^{-\theta_m(Z)L_m + \psi_m(\theta_m(Z), Z) - 2\theta(\mu_{SD}^{(m)})Z + \Lambda_f(\nu_{SD}^{(m)})} \right), \]

and we defined the second moment of the IS estimator

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By Jensen’s inequality

\[
\left( \mathbb{P}(L_m > x_m) \right)^2 \leq M_2(\theta_m(Z), x_m, \mu^*_SD).
\]

Therefore it suffices to prove (3.106), since (3.107) would then follow by (3.108).

By Lemma 3.4.3 and the assumption (R1), it follows that for any sequence \( z^{(m)} \) and \( \epsilon > 0 \)

\[
\lim_{m \to \infty} \mathbb{P}(|S_m| > \epsilon | Z = z^{(m)}) = 0.
\]

Lemma 3.6.6. Let \( z^{(m)}_\star \in H^{(m)}_{SD}. \) Then

\[
\lim_{m \to \infty} \mathbb{P} \left( L_m > x_m | Z = z^{(m)}_\star \right) = 1.
\]

Proof. Let \( z^{(m)}_\star \in H^{(m)}_{SD}. \) By (3.88),

\[
\mathbb{E} \left( L_m | Z = z^{(m)}_\star \right) \geq H \left( (1 + \epsilon_m)H^{-1}(q) \right) \sum_{k=1}^{m} l_k u_k.
\]
Therefore

\[
\frac{(s_m)^j}{m} (x_m - \mathbb{E}(L_m|Z = z^{(m)}) \leq \frac{(s_m)^j}{m} x_m - \frac{(s_m)^j}{m} H((1 + \epsilon_m)H^{-1}(q)) \sum_{k=1}^{m} l_k u_k
\]

\[
= (q - H((1 + \epsilon_m)H^{-1}(q))) \frac{(s_m)^j}{m} \sum_{k=1}^{m} l_k u_k
\]

(since \(x_m = q \sum_{k=1}^{m} l_k u_k\))

\[
= (H(H^{-1}(q)) - H((1 + \epsilon_m)H^{-1}(q))) \frac{(s_m)^j}{m} \sum_{k=1}^{m} l_k u_k
\]

\[
= - (H((1 + \epsilon_m)H^{-1}(q)) - H(H^{-1}(q))) \frac{(s_m)^j}{m} \sum_{k=1}^{m} l_k u_k.
\]

By \([3.81]\) under \textbf{IS-SD2} there exists \(k_1 = k_1(q), k_2 = k_2(q) > 0\) such that

\[
k_1 \epsilon_m < H((1 + \epsilon_m)H^{-1}(q)) - H(H^{-1}(q)) < k_2 \epsilon_m.
\]

Therefore

\[
\frac{(s_m)^j}{m} (x_m - \mathbb{E}(L_m|Z = z^{(m)}) \leq - (H((1 + \epsilon_m)H^{-1}(q)) - H(H^{-1}(q))) \frac{(s_m)^j}{m} \sum_{k=1}^{m} l_k u_k
\]

\[
= -k_1 \epsilon_m \frac{(s_m)^j}{m} \sum_{k=1}^{m} l_k u_k
\]

\[
= -k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k
\]

\[
\rightarrow -k_1 C. \quad (3.109)
\]

Let \(0 < \epsilon < C\).
There exists $M \in \mathbb{N}$ such that for all $m > M$

$$\frac{1}{m} \sum_{k=1}^{m} l_k u_k > \epsilon.$$ 

Therefore for $m > M$

$$-k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k < -k_1 \epsilon.$$ 

Therefore for $m > M$

$$\frac{(s_m)^j}{m} \left( x_m - \mathbb{E}(L_m | Z = z^{(m)}_*) \right) \leq -k_1 \frac{1}{m} \sum_{k=1}^{m} l_k u_k \quad \text{(by (3.109))}$$

$$< -k_1 \epsilon. \quad (3.111)$$

$$\mathbb{P}(L_m > x_m | Z = z^{(m)}_*) = \mathbb{P} \left( L_m - \mathbb{E}(L_m | Z) > x_m - \mathbb{E}[L_m | Z] \Big| Z = z^{(m)}_* \right)$$

$$= \mathbb{P} \left( \frac{(s_m)^j}{m} \left( L_m - \mathbb{E}(L_m | Z) \right) > \frac{(s_m)^j}{m} \left( x_m - \mathbb{E}[L_m | Z] \right) \Big| Z = z^{(m)}_* \right)$$

$$= \mathbb{P} \left( S_m > \frac{(s_m)^j}{m} \left( x_m - \mathbb{E}(L_m | Z) \right) \Big| Z = z^{(m)}_* \right)$$

$$\geq \mathbb{P} \left( S_m > -k_1 \epsilon \Big| Z = z^{(m)}_* \right) \quad \text{(for } m > M \text{ by (3.111))}$$

$$\geq \mathbb{P} \left( |S_m| \leq k_1 \epsilon \Big| Z = z^{(m)}_* \right) \to 1 \text{ as } m \to \infty.$$ 

$\square$
\( p^{(m)}(z) = H \left( \frac{-az - ss_m}{b} \right) \). Therefore \( p^{(m)}(z) \) is increasing in \( z \) if \( a < 0 \). Analogously \( p^{(m)}(z) \) is decreasing in \( z \) if \( a > 0 \). If \( a < 0 \),

\[
\mathbb{P}(L_m > x_m) \geq \int_{t \in [0, \infty)} \mathbb{P} \left( L_m > x_m \mid Z = \nu_{SD}^{(m)} + t \right) \mathbb{P} \left( Z = \nu_{SD}^{(m)} + t \right) dt
\]

\[
\geq \mathbb{P} \left( L_m > x_m \mid Z = \nu_{SD}^{(m)} \right) \int_{t \in [0, \infty)} \mathbb{P} \left( Z = \nu_{SD}^{(m)} + t \right) dt
\]

\[
= \mathbb{P} \left( L_m > x_m \mid Z = \nu_{SD}^{(m)} \right) \mathbb{P} \left( Z > \nu_{SD}^{(m)} \right).
\]

Note that \( \nu_{SD}^{(m)} \in H_{SD}^{(m)} \).

Similarly if \( a > 0 \)

\[
\mathbb{P}(L_m > x_m) \geq \int_{t \in [0, \infty)} \mathbb{P} \left( L_m > x_m \mid Z = -\nu_{SD}^{(m)} - t \right) \mathbb{P} \left( Z = -\nu_{SD}^{(m)} - t \right) dt
\]

\[
\geq \mathbb{P} \left( L_m > x_m \mid Z = -\nu_{SD}^{(m)} \right) \int_{t \in [0, \infty)} \mathbb{P} \left( Z = -\nu_{SD}^{(m)} - t \right) dt
\]

\[
= \mathbb{P} \left( L_m > x_m \mid Z = -\nu_{SD}^{(m)} \right) \mathbb{P} \left( Z < -\nu_{SD}^{(m)} \right).
\]

Note that \(-\nu_{SD}^{(m)} \in H_{SD}^{(m)} \).

By (3.90) we have

\[
H_{SD}^{(m)} = \begin{cases} 
\{ z \in \mathbb{R} : z \geq \nu_{SD}^{(m)} \} & \text{if } a > 0 \\
\{ z \in \mathbb{R} : z \leq -\nu_{SD}^{(m)} \} & \text{if } a < 0
\end{cases}
\]

and therefore by the symmetry of \( Z \),

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\[ P(Z > \nu_{SD}^{(m)}) = P(Z < -\nu_{SD}^{(m)}). \]

Therefore in either case (whether \( a > 0 \) or \( a < 0 \)), we have

\[ P(L_m > x_m) \geq k_m P(Z > \nu_{SD}^{(m)}), \]

for some sequence \( k_m \to 1 \). Now (3.106) follows from (E2) and (E4).

\[ \square \]

### 3.6.3 Proof of Main Theorem: Small Default Probability Regime

**Proof of Theorem 3.3.2**

We will apply Theorem 3.6.1. We need to verify (E1), (E2), (E3), (E4), (R1), and (R2). \( Z \) is allowed to take \( N(0,1) \) and Double Exponential (\( \lambda \)).

**Case 1: \( Z \sim N(0,1) \)**

Assume \( \lim_{m \to \infty} \frac{s_m}{m^2} = 0 \). Therefore (R1) is satisfied.

(E1) and (E2) are satisfied with \( k_0 = 2 \) and \( A = \frac{1}{2} \gamma^2 \). Next we will verify (E3) and (E4).

\( Z \sim N(0,1) \). Therefore \( \Lambda_f(\theta) = \frac{1}{2} \theta^2 \) and \( \theta(x) = x \). Therefore

\[ x\theta(x) - \Lambda_f(\theta(x)) = -\frac{1}{2} x^2. \] (3.112)

Therefore (E3) and (E4) are satisfied with \( k_0 = 2 \) and \( A = \frac{1}{2} \gamma^2 \).
Note that \( \Lambda_f(\mu_{SD}^{(m)}) = \frac{1}{2}(\mu_{SD}^{(m)})^2 \). Therefore to verify (R2), it suffices to show that there exists \( \theta_m^* \to 0 \) such that

\[
\lim_{m \to \infty} \frac{-\theta_m^* m \mu_{m}}{s_m^2} + \frac{m D^2(\theta_m^*)^2}{s_m^2} = 0.
\]

That is

\[
\lim_{m \to \infty} \frac{-\theta_m^* m \mu_{m}}{s_m^3} + \frac{m D^2(\theta_m^*)^2}{s_m^2} = -\infty.
\]

Define \( \theta_m^* = \frac{s_m}{\sqrt{m}} \) so that \( \frac{m \theta_m^*}{s_m} = 1 \).

\[
\frac{\theta_m^* m}{s_m^3} = \frac{m^{\frac{1}{2}}}{s_m^4} = m \to \infty.
\]

Case 2: \( Z \sim \text{Double Exponential} (\lambda) \)

Assume \( \lim_{m \to \infty} \frac{s_m}{m^{\frac{3}{2}}} \). Apply (3.6.1) with \( j = 1 \). It follows that \( \epsilon_m = \frac{1}{s_m} \) and \( \lim_{m \to \infty} \frac{s_m}{m^2} = 0 \). Therefore (R1) is satisfied.

\[ P(Z > x) = \frac{1}{2} e^{-\lambda x}. \] (E1) and (E2) are satisfied with \( k_0 = 1 \) and \( A = \lambda \gamma \).

To verify (R1) it suffices to show that,

\[
\lim_{m \to \infty} \frac{-\theta_m^* m \mu_{m} \epsilon_m}{s_m} + \frac{m D^2(\theta_m^*)^2}{s_m} = 0.
\]

That is
\[
\lim_{m \to \infty} \frac{\theta_m^* m u}{s_m^2} + \frac{m D l^2 (\theta_m^*)^2}{s_m} = 0.
\]

Define \( \theta_m^* = \sqrt{\frac{\theta_m^*}{m}} \) so that \( \frac{m \theta_m^*}{s_m} = 1 \).

\[
\frac{\theta_m^* m}{s_m^2} = \frac{m \frac{\theta_m^*}{s_m}}{s_m^2} = \frac{m}{s_m^3} \to \infty.
\]

It remains to verify (E3) and (E4). \( Z \sim \text{Double Exponential}(\lambda) \). (E3) and (E4) follows by the same calculation that we showed under proof of Theorem 3.3.1 in section 3.5.3.
Chapter 4: Large Deviations for Higher Dimensional Multifactor Gaussian Copula

4.1 Introduction

In this chapter, we derive logarithmic large deviations for the widely used multifactor Gaussian Copula (see [19] and [22]). Large deviations for the Gaussian Copula were derived by Glasserman et al. (see [15]), letting the number of obligors $m$ go to infinity but holding the number of factors fixed at $d \in \mathbb{N}$. They also divide the set of obligors into $t$ types where $t \in \mathbb{N}$. In the model they consider, if the $k^{th}$ obligor is of type $j$, its latent variable $X_k$ is given by

$$X_k = (a_j)^T Z + b_j \varepsilon_k, \quad (4.1)$$

where $a_j \in \mathbb{R}^d$ with $0 < \underline{a} < \|a_j\| < \bar{a} < 1$, $Z$ is a $d$ dimensional standard normal random vector, $b_j = \sqrt{1 - (a_j)^T a_j}$, and $\varepsilon_k$ are independent standard normal random variables.

We expand on the results by Glasserman et al. [15], and let the number of factors go to infinity together with the number of obligors. Again we consider two probability regimes, large loss threshold regime and the small default probability regime.

We change the model in (4.1) to accommodate $Z$ to be an $m$ dimensional vector. More specifically, (4.1) is replaced by
\[ X_k^{(m)} = (a_j^{(m)})^T Z^{(m)} + b_j^{(m)} \varepsilon_k, \]

where \( a_j^{(m)} \in \mathbb{R}^m \) with \( 0 < a < \| a_j^{(m)} \| < \bar{a} < 1 \), \( Z^{(m)} \) is a \( m \) dimensional standard normal random vector, \( b_j^{(m)} = \sqrt{1 - (a_j^{(m)})^T a_j^{(m)}} \), and \( \varepsilon_k \)'s are independent standard normal random variables.

This chapter is organized as follows. In section 4.2 we postulate the model assumptions and state the main results. The two main theorems are, Theorem 4.2.1 for the large loss threshold regime, and Theorem 4.2.4 for the small default probability regime. Next we set out to prove these two theorems. Since we are dealing with two probability regimes, there is a congruence of probabilistic tools used in the proofs. These preliminary results are outlined in section 4.3 and they are used to prove Theorem 4.2.1 in section 4.4 and Theorem 4.2.4 in section 4.5.

4.2 Model Assumptions and Main Results

In this section we state the model assumptions and the main results. The model assumptions are given in section 4.2.1 The main result for the large loss threshold regime, Theorem 4.2.1 is given in section 4.2.2. The main result for the small default probability regime, Theorem 4.2.4 is given in section 4.2.3.

4.2.1 Model Assumptions

In this section, we state the assumptions that are universally applicable to both probability regimes, large loss threshold and small default probability. Regime specific assumptions will be given in section 4.2.2 for the large loss threshold regime, and in section 4.2.3 for the small default probability regime. We first describe some of the notation that will be used
throughout this chapter.

\[ m = \text{Number of obligors} \]

\[ Y_k^{(m)} = \text{Default indicator of the k-th obligor. (1 for default and 0 otherwise.)} \]

\[ p_k^{(m)} = P(Y_k^{(m)} = 1) \]

\[ p_k^{(m)}(z^{(m)}) = P(Y_k^{(m)} = 1 | Z^{(m)} = z^{(m)}) \text{ where } Z^{(m)}, z^{(m)} \in \mathbb{R}^m \]

\[ L_m = \sum_{k=1}^{m} l_k U_k Y_k \text{ where } l_k \text{ is deterministic.} \]

\( U_k \) is a random variable taking values in [0,1] in the large loss threshold regime, where as \( U_k \equiv 1 \) in the small default probability regime. The correlations between the default indicators \( Y_k^{(m)} \) are given by an \( m \) dimensional Gaussian Copula.

The following **Assumptions MF-GEN** are universally applicable for both probability regimes. We then postulate regime specific assumptions **Assumptions MF-LL** for the large loss regime, and **Assumptions MF-SD** for the small default regime.

**Assumptions MF-GEN**

There are \( m \) factors and \( t \) types of obligors. \( \{I_1^{(m)}, I_2^{(m)}, \ldots, I_t^{(m)}\} \) is a partition of the set of obligors \{1, 2, ..., \( m \)\} into \( t \) number of types. If \( k \in I_j^{(m)} \), then the \( k^{th} \) obligor is of type \( j \), and its latent variable is given by

\[ X_k^{(m)} = (a_j^{(m)})^T Z^{(m)} + b_j^{(m)} e_k. \]
In the above equation, \(a_j^{(m)} \in \mathbb{R}^m\) with \(0 < \underline{a} < \|a_j^{(m)}\| < \bar{a} < 1\), \(Z^{(m)}\) is a \(m\) dimensional standard normal random vector, \(b_j^{(m)} = \sqrt{1 - (a_j^{(m)})^T a_j^{(m)}}\), and \(\varepsilon_k\) are independent standard normal random variables. It follows that there exists \(\underline{b}, \bar{b}\) such that \(0 < \underline{b} < b_j^{(m)} < \bar{b} < 1\). Let \(n_j^{(m)} = |I_j^{(m)}|\) denote the number of obligors of type \(j\). We assume that for each \(j = 1, 2, ..., t\), 
\[w_j = \lim_{m \to \infty} \frac{1}{m} n_j^{(m)} > 0.\]

### 4.2.2 Main Results for Large Loss Threshold Regime

Before we give the main results, we give below the set of assumptions that are specific to the large loss threshold regime.

**Assumptions MF-LL**

1. The default indicators \(Y_k^{(m)}\) is given by

   \[Y_k^{(m)} = 1 \{X_k^{(m)} > \Phi^{-1}(1 - p_k^{(m)})\}.\]  

2. The default probability of the \(k\)-th obligor, \(p_k^{(m)} = \mathbb{P}(Y_k^{(m)} = 1)\), satisfies

   \[0 < p_k^{(m)} \leq \bar{p} < 1.\]

3. If the \(k^{th}\) obligor is of type \(j\) then its conditional default probability is given by

   \[p_k^{(m)}(Z^{(m)}) := \mathbb{P}(Y_k^{(m)} = 1 \| Z^{(m)}) = \Phi\left(\frac{(a_j^{(m)})^T Z^{(m)} + \Phi^{-1}(p_k^{(m)})}{b_j^{(m)}}\right).\]
\[ P \left( Y_{k}^{(m)} = 1 | Z^{(m)} = z^{(m)} \right) \]
\[ = P \left( X_{k}^{(m)} > \Phi^{-1} \left( 1 - p_{k}^{(m)} \right) | Z^{(m)} = z \right) \]
\[ = P \left( X_{k}^{(m)} > -\Phi^{-1} \left( p_{k}^{(m)} \right) | Z^{(m)} = z \right) \]
\[ = P \left( (a_{j}^{(m)})^{T} z^{(m)} + b_{j}^{(m)} \varepsilon_{k} > -\Phi^{-1} \left( p_{k}^{(m)} \right) \right) \]
\[ = P \left( \varepsilon_{k} > \frac{-(a_{j}^{(m)})^{T} z^{(m)} - \Phi^{-1} \left( p_{k}^{(m)} \right)}{b_{j}^{(m)}} \right) \]
\[ = P \left( -\varepsilon_{k} < \frac{(a_{j}^{(m)})^{T} z^{(m)} + \Phi^{-1} \left( p_{k}^{(m)} \right)}{b_{j}^{(m)}} \right) \]
\[ = \Phi \left( \frac{(a_{j}^{(m)})^{T} z^{(m)} + \Phi^{-1} \left( p_{k}^{(m)} \right)}{b_{j}^{(m)}} \right). \]

4. The maximum loss for obligor \( k \) is \( l_{k} \), and satisfies \( 0 < l \leq l_{k} \leq \bar{l} < \infty \). \( l_{k} \)'s are deterministic. The actual loss upon the default of the obligor \( k \) is \( l_{k} U_{k} \) where \( U_{k} \) is a random variable that take values in \([u, 1]\) for some \( 0 < u \leq 1 \). For each obligor type \( j \), \( \{ U_{k} \}_{k \in \mathcal{I}^{(m)}_{j}} \) is an i.i.d. sequence from a distribution with mean \( u_{j}^{*} \). These random variables are independent of \( Z^{(m)} \) and \( \varepsilon_{k} \). The mean of \( U_{k} \) will be denoted by \( u_{k} \). \( u_{k} = u_{j}^{*} \) if the \( k^{th} \) obligor is of type \( j \).

5. The total loss from the defaults is given by
\[ L_m = \sum_{k=1}^{m} l_k U_k Y_k^{(m)}. \]

6. The portfolio default threshold \( x_m \) is given by

\[ x_m = q_m \sum_{k=1}^{m} l_k u_k, \quad (4.4) \]

where

\[ q_m = \Phi(s \sqrt{\log m}), \]

for some \( 0 < s \).

7.

\[ \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} u_k l_k = H. \quad (4.5) \]

Before we give the main Theorem for the large loss threshold regime, we develop some concepts that are necessary.

Let

\[ A^{(m)}_j = \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} \}, \]

and

\[ A^{(m)} = \bigcap_{j=1}^{t} A^{(m)}_j \]
\[ \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} \text{ for } j = 1, 2, \ldots, t \}. \quad (4.6) \]

We assume \( A^{(m)} \neq \emptyset \) for all \( m \). Let

\[ \gamma^{(m)} = \min \{ \| z \| : z \in A^{(m)} \}, \]

and assume

\[ \lim_{m \to \infty} \gamma^{(m)} = \gamma < \infty. \quad (4.7) \]

Recall that if the \( k^{th} \) obligor is of type \( j \), then its latent variable \( X_k^{(m)} = a_j^{(m)} Z^{(m)} + b_j^{(m)} \varepsilon_k \),

where \( Z^{(m)} \) is a Standard Gaussian vector of dimension \( m \). By writing

\[ N_j^{(m)} = a_j^{(m)} Z^{(m)} \text{ for } j \in \{ 1, 2, \ldots, t \}, \]

and

\[ N^{(m)} = \begin{bmatrix} N_1^{(m)} \\ N_2^{(m)} \\ \vdots \\ N_t^{(m)} \end{bmatrix}, \]

it follows that \( N^{(m)} \) is centered multivariate Gaussian with covariance matrix given by

\[ \Sigma^{(m)} = F^{(m)} (F^{(m)})^T, \]
where

\[ F^{(m)} = \begin{bmatrix} (a_1^{(m)})^T \\ (a_2^{(m)})^T \\ \vdots \\ (a_t^{(m)})^T \end{bmatrix}. \]

**Theorem 4.2.1.** Suppose \textbf{Assumptions MF-GEN} and \textbf{Assumptions MF-LL} hold. Let \( \gamma \) be given by (4.7). Let \( F^{(m)} \) be the \( t \times m \) matrix

\[ F^{(m)} = \begin{bmatrix} (a_1^{(m)})^T \\ (a_2^{(m)})^T \\ \vdots \\ (a_t^{(m)})^T \end{bmatrix}, \]

and \( \Sigma^{(m)} \) be the \( t \times t \) matrix

\[ \Sigma^{(m)} = F^{(m)} (F^{(m)})^T. \]

Let \( H^{(m)} = (\Sigma^{(m)})^{-1} \). Assume

1. \( A^{(m)} \neq \emptyset \) for all \( m \).

2. \( \det(\Sigma^{(m)}) < \log m \).

3. The \( t \times t \) matrices \( H^{(m)} \) are uniformly bounded in the max norm.

Then,
\[
\lim_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(L_m > x_m) = -\frac{1}{2} \gamma^2.
\]

### 4.2.3 Main Results for the Small Default Probability Regime

Before we give the main results, we give below the set of assumptions that are specific to the small default regime.

**MF-SD**

1. If the \( k \text{th} \) obligor is of type \( j \), its default probability \( p_k^{(m)} = \mathbb{P}(Y_k^{(m)} = 1) \), satisfies

\[
p_k^{(m)} = \tilde{p}_j^{(m)} = \Phi(-\tilde{s}_j \sqrt{m}).
\]  

(4.8)

2. If the \( k \)-th obligor is of type \( j \) then its conditional default probability is given by

\[
p_k^{(m)}(Z^{(m)}) = \tilde{p}_j^{(m)}(Z^{(m)}) := \mathbb{P} \left( Y_k^{(m)} = 1 \mid Z^{(m)} \right) = \Phi \left( \frac{(a_j^{(m)})^T Z^{(m)} - \tilde{s}_j \sqrt{m}}{b_j^{(m)}} \right).
\]  

(4.9)

(4.9) follows from (4.8) since

\[
\mathbb{P} \left( Y_k^{(m)} = 1 \mid Z^{(m)} = z \right) = \mathbb{P} \left( X_k^{(m)} > \Phi^{-1} \left( 1 - \tilde{p}_j^{(m)} \right) \mid Z^{(m)} = z \right)
\]

\[
= \mathbb{P} \left( X_k^{(m)} > \Phi^{-1} \left( \tilde{p}_j^{(m)} \right) \mid Z^{(m)} = z \right)
\]

\[
= \mathbb{P} \left( (a_j^{(m)})^T Z^{(m)} + b_j^{(m)} \varepsilon_k > -\Phi^{-1} \left( \Phi \left( -\tilde{s}_j \sqrt{m} \right) \right) \mid Z = z \right)
\]
\[
\begin{align*}
&= \mathbb{P} \left( (a_j^{(m)})^T z^{(m)} + b_j^{(m)} \varepsilon_k > \tilde{s}_j \sqrt{m} \right) \\
&= \mathbb{P} \left( \varepsilon_k > \frac{-(a_j^{(m)})^T z^{(m)} + \tilde{s}_j \sqrt{m}}{b_j^{(m)}} \right) \\
&= \mathbb{P} \left( -\varepsilon_k < \frac{(a_j^{(m)})^T z^{(m)} - \tilde{s}_j \sqrt{m}}{b_j^{(m)}} \right) \\
&= \Phi \left( \frac{(a_j^{(m)})^T z^{(m)} - \tilde{s}_j \sqrt{m}}{b_j^{(m)}} \right).
\end{align*}
\]

3. The exposure for obligor \( k, l_k \) and satisfies \( 0 < l \leq l_k \leq \bar{l} < \infty \). \( l_k \) are deterministic.

4. The total loss from the defaults is given by

\[ L_m = \sum_{k=1}^{m} l_k Y_k^{(m)}. \]

5. The portfolio default threshold loss \( x_m \) is given by

\[ x_m = q \sum_{k=1}^{m} l_k, \]

where \( 0 < q < 1 \).

6. For each \( j = 1, 2, \ldots, t \)

\[ H_j = \lim_{m \to \infty} \frac{1}{m} \sum_{k \in T_j^{(m)}} l_k, \]

and

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\[
H = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} l_k.
\]

It follows that

\[
\sum_{k=1}^{t} H_j = H.
\]

Before we give the main large deviation theorem for this section, we develop some required concepts. We start with some definitions below.

**Definition 4.2.2** (q-minimal index). \( J \subseteq \{1, 2, \ldots, t\} \) is defined to be a q-minimal index set \( m \) if

\[
\max_{J \subseteq J} \sum_{j \in J} H_j < qH \leq \sum_{j \in J} H_j.
\]

Thus, if all the members in the types in \( J \) default (in the limit), the loss exceeds the threshold \( qH \). But no proper subset of \( J \) holds this property.

**Definition 4.2.3.** Define \( \mathcal{M}_q \) to be the set of all q-minimal subsets.

Define

\[
C^{(m)}_j = \left\{ \mathbf{z}^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T \mathbf{z}^{(m)} \geq s_j \right\},
\]

\[
C^{(m)}_J = \bigcap_{j \in J} C^{(m)}_j
\]
\[
\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j \text{ for all } j \in J \subseteq \{1, 2, ..., t\} \}, \quad (4.10)
\]
and
\[
\gamma_J^{(m)} = \min \left\{ \|z^{(m)}\| : z^{(m)} \in C_J^{(m)} \right\}. \quad (4.11)
\]

Define
\[
S_q = \{ J \in \mathcal{M}_q : C_J^{(m)} \neq \emptyset \text{ for all } m \}. \quad (4.12)
\]
Define
\[
C^{(m)} = \bigcup_{J \in S_q} C_J^{(m)}. \quad (4.13)
\]
We assume that \( \gamma_J^{(m)} \) converges for each \( J \in S_q \), and we denote this limit by
\[
\gamma_J = \lim_{m \to \infty} \gamma_J^{(m)}. \quad (4.14)
\]
Define
\[
\gamma_* = \min \{ \|\gamma_J\| : J \in S_q \}. \quad (4.14)
\]

**Theorem 4.2.4.** Suppose Assumptions MF-GEN and Assumptions MF-SD hold. Let \( \gamma_* \) be given by \( (4.14) \). For every \( J = \{j_1, j_2, ..., j_v\} \in S_q \) (\( v \) depends on the set \( J \) but we suppress this dependence for economy), define \( F_J^{(m)} \) to be the \( v \times m \) matrix
\[ F_J^{(m)} = \begin{bmatrix} (a_{j_1})^T \\ (a_{j_2})^T \\ \vdots \\ (a_{j_v})^T \end{bmatrix}, \]

and \( \Sigma_J^{(m)} \) be the \( v \times v \) matrix

\[ \Sigma_J^{(m)} = F_J^{(m)} (F_J^{(m)})^T. \]

Let \( H_J^{(m)} = (\Sigma_J^{(m)})^{-1} \). Assume

1. For every \( J \in S_q \), there exists \( k > 0 : \det(\Sigma_J^{(m)}) < m^k \)

2. The \( v \times v \) matrices \( H_J^{(m)} \) are uniformly bounded in the max norm.

Then,

\[ \lim_{m \to \infty} \frac{1}{m} \log \mathbb{P}(L_m > x_m) = -\frac{1}{2} \gamma^2_s. \]

### 4.3 Some Preliminary Results

We are dealing with two probability regimes, and it turns out there are some similarities between our main proofs. In this section we introduce some tools that we will use in common for both the probability regimes. So far we have introduced the sets \( A^{(m)} \) by (4.6), \( C_J^{(m)} \) by (4.10), and we have seen that these sets are central to our large deviation results. When we get to the proofs, we will introduce the sequences of sets \( A^{(m,\epsilon)}, B^{(m,\epsilon)}, C_J^{(m,\epsilon)} \) and \( D_J^{(m,\epsilon)} \). 
by (4.17), (4.30), (4.42) and (4.54) respectively. We begin by proving two useful lemmas that are concerned with a convergence property of these sets.

**Lemma 4.3.1.** For $j = 1, 2, \ldots, v$, Let $a_j \in \mathbb{R}^m$, $0 < h_j^{(m)} < \tilde{h}$ and $0 < c_j^{(m)} < \tilde{c}$.

Let $0 < s_m \to \infty$ as $m \to \infty$. Let

$$G^{(m)} = \left\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq h_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, v\} \right\},$$

and

$$G^{(m,\epsilon)} = \left\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq h_j^{(m)} s_m + c_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, v\} \right\}.$$  

Then,

$$G^{(m)} \neq \emptyset \iff G^{(m,\epsilon)} \neq \emptyset.$$ 

**Proof.** $\iff$

Let $z_0^{(m)} \in G^{(m,\epsilon)}$. Then $(a_j^{(m)})^T z_0^{(m)} \geq h_j^{(m)} s_m + c_j^{(m)}$ for all $j$. Choose $\lambda_m$ large enough that

$$\lambda_m > \max_j \frac{h_j^{(m)}}{h_j^{(m)} s_m + c_j^{(m)}} \text{ for all } j \in \{1, 2, \ldots, v\}.$$ 

Then,

$$(a_j^{(m)})^T \lambda_m z_0^{(m)} \geq h_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, v\}.$$
Therefore $\lambda_mz_0 \in G^{(m)}$.

" $\implies$ "

Suppose $z_0^{(m)} \in G^{(m)}$.

$$(a_j^{(m)})^T z_0^{(m)} \geq h_j^{(m)} \text{ for } j \in \{1, 2, \ldots, v\}.$$ 

Choose $\lambda_m$ large enough that

$$\lambda_m \geq \max_j \frac{h_j^{(m)} s_m + c_j^{(m)}}{h_j^{(m)}} \text{ for all } j \in \{1, 2, \ldots, v\}.$$ 

Then,

$$(a_j^{(m)})^T (\lambda_m z_0^{(m)}) \geq h_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, v\}.$$ 

Therefore $\lambda_mz_0 \in G^{(m, \epsilon)}$.

Next we summarize another useful convergence property.

**Lemma 4.3.2.** Suppose $a_j^{(m)} \in \mathbb{R}^m$, $0 < h < h_j^{(m)} < \bar{h} < 1$ and $\lim_{m \to 0} v_j^{(m)} = 0$ for $j \in \{1, 2, \ldots, v\}$.

Suppose

$$K_m := \{z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq h_j^{(m)} \text{ for } j \in \{1, 2, \ldots, v\} \} \neq \emptyset,$$
and
\[ \gamma_m := \inf\{ \|z^{(m)}\| : z^{(m)} \in K_m \} \rightarrow \gamma > 0 \text{ as } m \rightarrow \infty. \]

Then,

1. 
\[ J_m := \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq h_j^{(m)} + v_j^{(m)} \text{ for } j \in \{1, 2, \ldots, v\} \} \neq \emptyset \text{ for large } m. \]

2. 
\[ \eta_m := \inf\{ \|z^{(m)}\| : z^{(m)} \in J_m \} \rightarrow \gamma. \]

Proof. Let \( \epsilon = \frac{h}{\gamma} > 0 \). There exists \( M_j : m > M_j \rightarrow -\epsilon < v_j^{(m)} \).

Let \( M = \max_{j \in \{1, 2, \ldots, v\}} \). For \( m > M : 0 < v_j^{(m)} + \epsilon \) for all \( j \in \{1, 2, \ldots, v\} \).

Therefore for \( m > M : 0 < v_j^{(m)} + h_j^{(m)} \) for all \( j \in \{1, 2, \ldots, v\} \).

There exists \( z_0^{(m)} \in K_m \).

Therefore \( (a_j^{(m)})^T z_0^{(m)} \geq h_j^{(m)} \) for all \( j \in \{1, 2, \ldots, v\} \).

Let \( \lambda_m = \max_{j \in \{1, 2, \ldots, v\}} \frac{h_j^{(m)} + v_j^{(m)}}{h_j^{(m)}} \).

Then, for all \( j \in \{1, 2, \ldots, v\} \) and for all \( m > M : (a_j^{(m)})^T (\lambda_m z_0^{(m)}) \geq h_j^{(m)} + v_j^{(m)} \)
Therefore $J_m \neq \emptyset$ for $m > M$.

Let $B_m(R) = \{z^{(m)} \in \mathbb{R}^m : \|z^{(m)}\| < R\}$

By choosing $R_m$ large enough such that $K_m \cap B_m(R_m) \neq \emptyset$ and $J_m \cap B_m(R_m) \neq \emptyset$, and using the compactness of these two sets we see that there exists $x_m \in K_m$ such that $\gamma_m = \|x_m\|$, and $y_m \in J_m$ such that $\eta_m = \|y_m\|$.

$(a_j^{(m)})^T x_m \geq h_j^{(m)}$ for all $j \in \{1, 2, .., v\}$.

Define $\alpha_m = \max_{j \in \{1, 2, .., v\}} \frac{h_j^{(m)} + v_j^{(m)}}{h_j^{(m)}}$.

Then, for all $j \in \{1, 2, .., v\}$ and for all $m > M : (a_j^{(m)})^T \alpha_m x_m \geq h_j^{(m)} + v_j^{(m)}$.

Therefore $\alpha_m x_m \in J_m$. Therefore $\eta_m \leq \|\alpha_m x_m\| = \max_{j \in \{1, 2, .., v\}} \frac{h_j^{(m)} + v_j^{(m)}}{h_j^{(m)}} \gamma_m$.

$$\lim_{m \to \infty} \max_{j \in \{1, 2, .., v\}} \frac{h_j^{(m)} + v_j^{(m)}}{h_j^{(m)}} = 1.$$ 

Therefore $\limsup_{m \to \infty} \eta_m \leq \gamma$.

Similarly,

$(a_j^{(m)})^T y_m \geq h_j^{(m)} + v_j^{(m)}$ for all $j \in \{1, 2, .., v\}$.

Define $\beta_m = \max_{j \in \{1, 2, .., v\}} \frac{h_j^{(m)}}{h_j^{(m)} + v_j^{(m)}}$.

Then, for all $j \in \{1, 2, .., v\}$ and for all $m > M : (a_j^{(m)})^T \beta_m y_m \geq h_j^{(m)}$.

Therefore $\beta_m y_m \in J_m$. Therefore $\gamma_m \leq \|\beta_m y_m\| = \max_{j \in \{1, 2, .., v\}} \frac{h_j^{(m)}}{h_j^{(m)} + v_j^{(m)}} \eta_m$. 

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Therefore \( \max_{j \in \{1, 2, \ldots, v\}} \frac{\gamma_m^{(m)} h_j}{\mu_j^{(m)} + v_j^{(m)}} \leq \eta_m. \)

Therefore \( \liminf_{m \to \infty} \eta_m \geq \gamma. \)

Therefore \( \lim_{m \to \infty} \eta_m = \gamma. \)

The following Lemma deals with change of measure for Multivariate Gaussian Distribution.

**Lemma 4.3.3.** Let \( \mathbf{N} \sim \mathcal{N}(0, \Sigma) \) where \( \Sigma \) is the \( v \times v \) dimensional covariance matrix under the probability measure \( \mathbb{P}. \) \( \mathbf{a} \in \mathbb{R}^v. \) Define a new probability measure \( \mathbb{P}_a \) by

\[
d\mathbb{P}_a = e^{a^T \mathbf{N} - \Lambda(a)} d\mathbb{P}
\]

where \( \Lambda(a) = \log(\mathbb{E}(e^{a^T \mathbf{N}})) \) where \( \mathbb{E} \) denotes the expectation under \( \mathbb{P}. \) Then, \( \mathbf{N} \sim \mathcal{N}(\Sigma \mathbf{a}, \Sigma) \) under the probability measure \( \mathbb{P}_a. \)

**Proof.** We will compute the Moment Generating Function of \( \mathbf{N} \) under \( \mathbb{P}_a. \)

For any \( \theta \in \mathbb{R}^v, \)

\[
\mathbb{E}_a \left( e^{\theta \mathbf{N}} \right) = \mathbb{E} \left( e^{\theta \mathbf{N} - a^T \mathbf{N} + \Lambda(a)} \right) = \frac{\mathbb{E} \left( e^{(\theta + a)^T \mathbf{N}} \right)}{\mathbb{E}(e^{a^T \mathbf{N}})} = \frac{e^{\frac{1}{2} (\theta + a)^T \Sigma (\theta + a)}}{e^{\frac{1}{2} a^T \Sigma a}}
\]
We will use the above Lemma 4.3.3 to prove the following result.

**Lemma 4.3.4.** Let $N^{(m)}$ be a $v$-dimensional multivariate Gaussian vector with covariance matrix $\Sigma^{(m)}$. Let $H^{(m)} = (\Sigma^{(m)})^{-1}$. Suppose $H^{(m)}$ is uniformly bounded in the max-norm. Let

$$\Lambda_m(a) = \log \mathbb{E}(e^{aN^{(m)}}).$$

Define a new measure

$$dP^*_m = e^{k_m N^{(m)} - \Lambda_m(k_m)} dP,$$

where

$$\Sigma^{(m)} k_m = y^{(m)}.$$

Then, there exists $K_1 > 0$ such that for all $m$

$$P^*_m(N^{(m)} \geq y^{(m)}) \geq \frac{1}{\sqrt{2\pi v |\Sigma^{(m)}|}} K_1.$$ 

**Proof.** By Lemma 4.3.3 it follows that under the probability measure $P^*_m$, $N^{(m)}$ is Normally distributed with mean $y^{(m)}$ and covariance matrix $\Sigma^{(m)}$. 

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\[ P^*_m (N^{(m)} \geq y^{(m)}) = \int_{y^{(m)} \leq z} \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} e^{-\frac{1}{2} \left( (z-y^{(m)})^T H^{(m)} (z-y^{(m)}) \right)} dz \]

\[ = \int_{0 \leq z} \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} e^{-\frac{1}{2} z^T H^{(m)} z} dz \]

\[ \geq \int_{h1 \leq z \leq q1} \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} e^{-\frac{1}{2} z^T H^{(m)} z} dz \quad \text{(by picking } 0 < h < q) \]

\[ \geq \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} \inf_{h1 \leq z \leq q1} \left( e^{-\frac{1}{2} z^T H^{(m)} z} \right) \lambda(\{z \in \mathbb{R}^v : h1 \leq z \leq q1\}) \]

\[ = \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} \left( e^{-\frac{1}{2} \sup_{h1 \leq z \leq q1} (z^T H^{(m)} z)} \right) \lambda(\{z \in \mathbb{R}^v : h1 \leq z \leq q1\}) . \]

We assume that the sequence of matrices $H^{(m)}$ is bounded in the max norm. Therefore

\[ \sup_{h1 \leq z \leq q1} \left( z^T H^{(m)} z \right) \]

is bounded above as $m \to \infty$.

Therefore,

\[ P^*_m (N^{(m)} \geq y^{(m)}) \geq \frac{1}{\sqrt{2\pi \mid \Sigma^{(m)} \mid}} K_1 , \]

for some constant $K_1 > 0$.

\[ \square \]

The following is a restatement of Lemma 3.10 in [15].

**Lemma 4.3.5.** Suppose a sequence of events $\{A_m\}_{m \in \mathbb{N}}$ and a sequence of positive integers
\{n_m\}_{m \in \mathbb{N}} \text{ with } \lim_{m \to \infty} n_m = \infty \text{ are given. Suppose that, given } A_m, T^{(m)}_k, k = 1, 2, \ldots, n_m \text{ are conditionally independent random variables random variables with conditional mean } 0 \text{ for which,}

\[
\limsup_{m \to \infty} \frac{1}{(n_m)^2} \sum_{k=1}^{n_m} \text{Var} \left( T^{(m)}_k \mid A_m \right) = 0.
\]

Let

\[
S_m = \frac{1}{m} \sum_{k=1}^{m} T^{(m)}_k.
\]

Then,

\[
\lim_{m \to \infty} \mathbb{P}(|S_m| > \epsilon \mid A_m) = 0.
\]

(Note that \(T^{(m)}_k\) and \(T^{(m)}_l\) may have different distributions for \(k \neq l\).)

We conclude this section by defining a change of measure \(P_m\) that will be used for the upper bound computation of both probability regimes. For \(z^{(m)} \in \mathbb{R}^m\) let

\[
\psi_m(\theta, z^{(m)}) = \frac{1}{m} \log \mathbb{E} \left( e^{\theta L_m} \mid Z^{(m)} = z^{(m)} \right),
\]

and let

\[
\theta_m(z^{(m)}) = \arg\min_{\theta \geq 0} \{-\theta x_m + m\psi_m(\theta, z^{(m)})\}.
\]
Define a new measure \( P_m \) by

\[
\frac{dP}{dP_m} = e^{-\theta(m) L_m + m \psi(m) (\theta(m) Z^{(m)})},
\]

(4.15)

and let \( E_m \) denote the expectation under \( P_m \).

Lemma 4.3.6. \( P_m(\cdot) \) defined by (4.15) is a probability measure.

Proof. It suffices to show that \( E \left( \left( \frac{dP}{dP_m} \right)^{-1} \right) = E \left( e^{\theta_m(Z^{(m)}) L_m - m \psi_m(\theta_m(Z^{(m)}), Z^{(m)})} \right) = 1. \)

\[
E \left( e^{\theta_m(Z^{(m)}) L_m - m \psi_m(\theta_m(Z^{(m)}), Z^{(m)})} \right)
= E \left( e^{\theta_m(Z^{(m)}) L_m - \log \left( E \left( e^{\theta_m(Z^{(m)}) L_m} \right) \right)} \right)
= E \left( e^{\theta_m(Z^{(m)}) L_m - \left( E \left( e^{\theta_m(Z^{(m)}) L_m} \right) \right)^{-1}} \right)
= E \left( \left( E \left( e^{\theta_m(Z^{(m)}) L_m} \right) \right)^{-1} \left( e^{\theta_m(Z^{(m)}) L_m} \right) \right)
= 1.
\]

Having obtained these preliminary results, we now consider the two probability regimes separately. Section 4.4 deals with the large loss threshold regime, where as section 4.5 is devoted to the small default probability regime.
4.4 Analysis of the Large Loss Regime

We give below the proof of Theorem 4.2.1. The proof is based on the upper bound computation given in Theorem 4.4.1, and the lower bound computation given in Theorem 4.4.7.

**Theorem 4.4.1.** Under the assumptions of Theorem 4.2.1,

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(L_m > x_m) \leq -\frac{1}{2} \gamma^2.
\]

**Proof.** Let \( \epsilon_m = \frac{1}{\sqrt{\log m}} \). Define, for \( \xi \in \mathbb{R} \),

\[
q_m^\epsilon(\xi) = \Phi\left(s(1 + \epsilon_m \sqrt{\log m})\right).
\]

Let

\[
A_j^{(m,\epsilon)} = \{z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} (1 - \epsilon_m) \sqrt{\log m} - \Phi^{-1} \tilde{p}\}
\]

\[
= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi\left(\frac{(a_j^{(m)})^T z^{(m)} + \Phi^{-1} \tilde{p}}{b_j^{(m)}}\right) \geq \Phi\left(s(1 - \epsilon_m) \sqrt{\log m}\right)\right\}
\]

\[
= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi\left(\frac{(a_j^{(m)})^T z^{(m)} + \Phi^{-1} \tilde{p}}{b_j^{(m)}}\right) \geq q_m^\epsilon(-1)\right\}, \quad (4.16)
\]

and let

\[
A^{(m,\epsilon)} = \cap_{j=1}^t A_j^{(m,\epsilon)}
\]

\[
= \left\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} (1 - \epsilon_m) \sqrt{\log m} - \Phi^{-1} \tilde{p} \text{ for } j = 1, 2, \ldots, t\right\}. \quad (4.17)
\]
Recall that by (4.6),

\[ A^{(m)} = \left\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} \text{ for } j = 1, 2, ..., t \right\}. \]

The following result is a direct application of Lemma 4.3.1.

**Lemma 4.4.2.**

\[ A^{(m, \epsilon)} \neq \emptyset \iff A^{(m)} \neq \emptyset. \]

Since we assume that \( A^{(m)} \neq \emptyset \), it follows that \( A^{(m, \epsilon)} \neq \emptyset \).

Each \( U_k \) is \([u, 1]\)-valued. Denote the cumulant generating function of \( U_k \) of type \( j \) (whose mean is \( u_k = u_j^* \)) by

\[ \Lambda_j(\lambda) = \log \mathbb{E} \left( e^{\lambda U_k} \right). \quad (4.18) \]

For any \( z^{(m)} \in \mathbb{R}^m \). Let

\[ \psi_m(\theta, z^{(m)}) = \frac{1}{m} \log \mathbb{E} \left( e^{\theta L_m} \left| Z^{(m)} = z^{(m)} \right. \right) \]

\[ = \frac{1}{m} \log \mathbb{E} \left( e^{\theta \sum_{k=1}^{m} l_k Y_k^{(m)}} \left| Z^{(m)} = z^{(m)} \right. \right) \]

\[ = \frac{1}{m} \sum_{k=1}^{m} \log \mathbb{E} \left( e^{\theta l_k Y_k^{(m)}} \left| Z^{(m)} = z^{(m)} \right. \right) \quad \text{(by conditional independence)} \]

\[ = \frac{1}{m} \sum_{k=1}^{m} \log \left( \mathbb{E} \left( e^{\theta l_k Y_k^{(m)}} \left| U_k, Z^{(m)} = z^{(m)} \right. \right) \right) \left| Z^{(m)} = z^{(m)} \right. \]

\[ = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k^{(m)}(z^{(m)}) (e^{\theta l_k U_k} - 1) \left| Z^{(m)} = z^{(m)} \right. \right) \]

\[ = \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k^{(m)}(z^{(m)}) \left( \mathbb{E} \left( e^{\theta l_k U_k} \left| Z^{(m)} = z^{(m)} \right. \right) - 1 \right) \right) \]

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\[
= \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + p_k^{(m)}(z^{(m)}) \left( \mathbb{E}(e^{\theta_k U_k}) - 1 \right) \right) \\
\text{ (by the independence of } U_k \text{ and } Z^{(m)} )
\]
\[
= \frac{1}{m} \sum_{j=1}^{t} \sum_{k \in I_j^{(m)}} \log \left( 1 + p_k^{(m)}(z^{(m)}) \left( e^{\Lambda_j(\theta_k)} - 1 \right) \right).
\]

Recall that by (4.4)

\[
x_m = \Phi \left( s \sqrt{\log m} \right) \sum_{k=1}^{m} l_k u_k.
\]

Therefore with our new notation

\[
x_m = q_m^{(0)}(0) \sum_{k=1}^{m} l_k u_k.
\]

Let

\[
\theta_m(z^{(m)}) = \arg\min_{\theta \geq 0} \{-\theta x_m + m \psi_m(\theta, z^{(m)})\}.
\]

Define a new measure \( \mathbb{P}_m \) by

\[
\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(Z^{(m)}) L_m + m \psi_m(\theta_m(Z^{(m)}), Z^{(m)})},
\]

and let \( \mathbb{E}_m \) denote the expectation under \( \mathbb{P}_m \). (Note that by Lemma 4.15, \( \mathbb{P}_m \) is a probability measure.)

\[
\mathbb{P}(L_m > x_m)
\]
\[
= \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(A_m, \epsilon)(Z^{(m)})} \right) + \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}_{(A_m, \epsilon)(Z^{(m)})} \right)
\]
\[
\mathbb{E}_m \left( e^{-\theta_m(Z^{(m)})} L_m + m \psi_m(\theta_m(Z^{(m)}), Z^{(m)}) \mathbb{1}\{L_m > x_m\} \mathbb{1}(A(m, \epsilon) \cap (Z^{(m)})) \right) \\
+ \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}(A(m, \epsilon)) (Z^{(m)}) \right) \\
\leq \mathbb{E}_m \left( e^{-\theta_m(Z^{(m)})} x_m + m \psi_m(\theta_m(Z^{(m)}), Z^{(m)}) \mathbb{1}\{L_m > x_m\} \mathbb{1}(A(m, \epsilon) \cap (Z^{(m)})) \right) \\
+ \mathbb{E} \left( \mathbb{1}\{L_m > x_m\} \mathbb{1}(A(m, \epsilon)) (Z^{(m)}) \right) \\
\leq \mathbb{E}_m \left( e^{-\theta_m(Z^{(m)})} x_m + m \psi_m(\theta_m(Z^{(m)}), Z^{(m)}) \mathbb{1}(A(m, \epsilon) \cap (Z)) \right) + \mathbb{P} \left( Z^{(m)} \in A(m, \epsilon) \right). 
\] (4.19)

The rest of the analysis is concerned with finding upper bounds for the RHS of (4.19). We first find an upper bound for \( \mathbb{P}(Z^{(m)} \in A(m, \epsilon)) \).

\[
\mathbb{P}\left( Z^{(m)} \in A(m, \epsilon) \right) = \mathbb{P}\left( (a_j^{(m)})^T Z^{(m)} \geq sb_j^{(m)} (1 - \epsilon_m) \sqrt{\log m} - \Phi^{-1}(\bar{p}) \right. \text{ for } j = 1, 2, \ldots t). 
\]

Let

\[
N_j^{(m)} = (a_j^{(m)})^T Z^{(m)},
\]

and

\[
N^{(m)} = \begin{bmatrix}
N_1^{(m)} \\
N_2^{(m)} \\
\vdots \\
N_t^{(m)}
\end{bmatrix}.
\]

Then \( N^{(m)} \) is centered Gaussian with covariance matrix
\[ \Sigma^{(m)} = F^{(m)}(F^{(m)})^T, \]

where

\[ F^{(m)} = \begin{bmatrix} (a_1^{(m)})^T \\ (a_2^{(m)})^T \\ \vdots \\ (a_t^{(m)})^T \end{bmatrix}. \]

Let

\[ r_j^{(m)} = s b_j^{(m)} (1 - \epsilon_m) \sqrt{\log m} - \Phi^{-1}(\bar{p}), \]

and

\[ r^{(m)} = \begin{bmatrix} r_1^{(m)} \\ r_2^{(m)} \\ \vdots \\ r_t^{(m)} \end{bmatrix}. \]

Let

\[ b^{(m)} = \begin{bmatrix} s b_1^{(m)} \\ s b_2^{(m)} \\ \vdots \\ s b_t^{(m)} \end{bmatrix}, \]
and

\[
c = \begin{bmatrix}
\Phi^{-1}(\bar{p}) \\
\Phi^{-1}(\bar{p}) \\
\vdots \\
\Phi^{-1}(\bar{p})
\end{bmatrix}.
\]

Then we have,

\[
r^{(m)} = (1 - \epsilon_m) \sqrt{\log m} b^{(m)} + c.
\]

\[
\mathbb{P}(Z^{(m)} \in A^{(m, \epsilon)}) = \mathbb{P}(F^{(m)} Z^{(m)} \geq r^{(m)})
\]

\[
= \mathbb{P}(N^{(m)} \geq r^{(m)})
\]

\[
= \mathbb{P}(N^{(m)} \geq (1 - \epsilon_m) \sqrt{\log m} b^{(m)} + c). \quad (4.20)
\]

Let

\[
H^{(m)} = (\Sigma^{(m)})^{-1}.
\]

We will use the shorthand notation

\[
\alpha_m = (1 - \epsilon_m) \sqrt{\log m}.
\]
Lemma 4.4.3.

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \left( \mathbb{P}(N^{(m)} \geq \alpha_m b^{(m)} + c) \right) \leq -\frac{1}{2} \gamma^2.
\]

Proof. Let \(0 < \theta < \infty\) and let

\[
\theta = \begin{bmatrix}
\theta \\
\theta \\
\theta \\
\vdots \\
\theta
\end{bmatrix}_{t \times 1}.
\]

Define

\[
T_i^{(m)} = \{ N_i^{(m)} > \alpha_m \theta \},
\]

and

\[
T^{(m)} = \bigcup_{i=1}^t T_i^{(m)}.
\]

Note that \(N_i\) is centered Gaussian with standard deviation \(\|a_i^{(m)}\|\). Therefore

\[
\mathbb{P}(N_i > \alpha_m \theta) \
\leq \frac{\|a_i^{(m)}\|}{\sqrt{2\pi \alpha_m \theta}} e^{-\frac{1}{2} \frac{\|a_i^{(m)}\|^2}{\|a_i^{(m)}\|^2} \alpha_m \theta^2} \\
\leq \frac{1}{\sqrt{2\pi \theta \alpha_m}} e^{-\frac{1}{2} \alpha_m \theta^2} \
\quad \text{(Since } \|a_i^{(m)}\| < 1)\]

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and

$$\Pr(N^{(m)} \in T^{(m)}) \leq t \frac{1}{\sqrt{2\pi}\alpha_m} e^{-\frac{1}{2}\alpha_m^2 \theta^2}.$$  

Note that

$$\left\{ N^{(m)} \in (T^{(m)})^c \right\} = \left\{ N^{(m)} < \alpha_m \theta \right\}.$$

$$\Pr(N^{(m)} \geq \alpha_m b^{(m)} + c)$$

$$\leq \Pr(\alpha_m b^{(m)} + c \leq N^{(m)} \text{ and } N^{(m)} \in (T^{(m)})^c) + \Pr(N^{(m)} \in T^{(m)})$$

$$= \Pr(\alpha_m b^{(m)} + c \leq N^{(m)} \leq \alpha_m \theta) + \Pr(N^{(m)} \in T^{(m)})$$

$$\leq \int_{\alpha_m b^{(m)} + c \leq z \leq \alpha_m \theta} \frac{1}{\sqrt{2\pi}|\Sigma^{(m)}|} e^{-\frac{1}{2}z^T H^{(m)} z} dz + t \frac{1}{\sqrt{2\pi}\alpha_m} e^{-\frac{1}{2}\alpha_m^2 \theta^2}.$$ 

$$y = \frac{z - c}{\alpha_m}.$$ 

Jacobian of the transformation $\text{Jac} = \alpha_m I_{t \times t}$, where $I_{t \times t}$ denoted the identity matrix of dimension $t$. Therefore $|\text{Jac}| = \alpha_m^t$

$$\Pr(N^{(m)} \geq \alpha_m b^{(m)} + c)$$

$$\leq \int_{b^{(m)} \leq y \leq \frac{1}{\alpha_m} (y+c)) \frac{\alpha_m^t}{\sqrt{2\pi}t |\Sigma^{(m)}|} e^{-\frac{1}{2}(\alpha_m y + c)^T H^{(m)} (\alpha_m y + c)} dy + t \frac{1}{\sqrt{2\pi}\alpha_m} e^{-\frac{1}{2}\alpha_m^2 \theta^2}.$$
Recall that $0 < \bar{b} < b_j^{(m)} < \tilde{b} < 1$. Therefore

$$\{ y \in \mathbb{R}^t | b^{(m)} \leq y \leq \theta - \frac{1}{\alpha_m} c \} \subseteq \{ y \in \mathbb{R}^t | s \bar{b} \leq y \leq 2\theta \}$$

for large $m$.

Therefore the set $\{ y \in \mathbb{R}^t | b^{(m)} \leq y \leq \theta - \frac{1}{\alpha_m} c \}$ can be bounded above by a compact set $K$ independent of $m$.

$$\mathbb{P}(N^{(m)} \geq \alpha_m b^{(m)} + c)$$

\[
\leq \int_{K} \frac{\alpha_{m}^t}{\sqrt{2\pi|\Sigma^{(m)}|}} e^{-\frac{1}{2}(\alpha_m y + c)^T H^{(m)} (\alpha_m y + c)} dy + t \frac{1}{\sqrt{2\pi \theta \alpha_m}} e^{-\frac{1}{2} \alpha_m \theta^2} \\
\leq \frac{\alpha_{m}^t}{\sqrt{2\pi|\Sigma^{(m)}|}} \sup_{y \geq b^{(m)}} \left( e^{-\frac{1}{2}(\alpha_m y + c)^T H^{(m)} (\alpha_m y + c)} \right) \lambda(K) + t \frac{1}{\sqrt{2\pi \theta \alpha_m}} e^{-\frac{1}{2} \alpha_m \theta^2} \\
\leq \frac{\alpha_{m}^t}{\sqrt{2\pi|\Sigma^{(m)}|}} \sup_{y \geq b^{(m)}} \left( e^{-\frac{1}{2}(\alpha_m y + c)^T H^{(m)} (\alpha_m y + c)} \right) \lambda(K) + t \frac{1}{\sqrt{2\pi \theta \alpha_m}} e^{-\frac{1}{2} \alpha_m \theta^2} \\
\leq \frac{\alpha_{m}^t}{\sqrt{2\pi|\Sigma^{(m)}|}} e^{-\frac{1}{2} \inf_{y \geq b^{(m)}} (\alpha_m y + c)^T H^{(m)} (\alpha_m y + c)} \lambda(K) + t \frac{1}{\sqrt{2\pi \theta \alpha_m}} e^{-\frac{1}{2} \alpha_m \theta^2}.
\]

We assume that $|\Sigma^{(m)}| < \log m$. Also recall that $\alpha_m = (1 - \epsilon_m)\sqrt{\log m}$.

$$\limsup_{m \to \infty} \frac{1}{\log m} \log \left( \mathbb{P}(N^{(m)} \geq \alpha_m b^{(m)} + c) \right)$$

\[
\leq \max \left\{ -\frac{1}{2} \limsup_{m \to \infty} \frac{1}{\log m} \inf_{y \geq b^{(m)}} (\alpha_m y + c)^T H^{(m)} (\alpha_m y + c), -\frac{1}{2} \theta^2 \right\}.
\]

Assuming that $\limsup_{m \to \infty} \frac{1}{\log m} \inf_{y \geq b^{(m)}} (\alpha_m y + c)^T H^{(m)} (\alpha_m y + c) < \infty$ (which we show to be true), since $\theta$ is arbitrary.
\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \left( P \left( N^{(m)} \geq \alpha_m b^{(m)} + c \right) \right) \\
\leq -\frac{1}{2} \limsup_{m \to \infty} \frac{1}{\log y(\geq b^{(m)})} \inf (\alpha_m y + c)^T H^{(m)} (\alpha_m y + c). \quad (4.21)
\]

Use the change of variable \( \alpha_m y + c = F^{(m)} z \) and recall that \( H^{(m)} = (F^{(m)} (F^{(m)})^T)^{-1} \).

\[
\inf_{y(\geq b^{(m)})} (\alpha_m y + c)^T H^{(m)} (\alpha_m y + c) \\
= \inf_{F^{(m)} z(\geq \alpha_m b^{(m)} + c)} (F^{(m)} z)^T H^{(m)} (F^{(m)} z) \\
= \inf_{F^{(m)} z(\geq \alpha_m b^{(m)} + c)} z^T z \\
= \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq \alpha_m s b_j^{(m)} + \Phi^{-1}(\bar{p}) \text{ for all } j \in \{1, 2, \ldots, t\} \right\} \\
= \alpha_m^2 \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} + \frac{\Phi^{-1}(\bar{p})}{\alpha_m} \text{ for all } j \in \{1, 2, \ldots, t\} \right\}.
\]

By Lemma 4.3.2

\[
\lim m \to \infty \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} + \frac{\Phi^{-1}(\bar{p})}{\alpha_m} \text{ for all } j \in \{1, 2, \ldots, t\} \right\} \\
= \lim m \to \infty \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, t\} \right\} \\
= \lim m \to \infty (\gamma^{(m)})^2 \\
= \gamma^2.
\]

Therefore,
Therefore by (4.21)

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(N^{(m)} > \alpha_m b^{(m)} + c) \leq -\frac{1}{2} \gamma^2.
\]

Therefore by (4.20)

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(Z^{(m)} \in A^{(m,\epsilon)}) \leq -\frac{1}{2} \gamma^2. \tag{4.22}
\]

Our next task is to show that: \( \limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{E}_m \left( e^{-\theta_m x_m + m \psi_m(\theta_m)} \mathbb{1}_{(A^{(m,\epsilon)})^c}(Z) \right) = -\infty \). In order to show this we first state two Lemmas.

Following is a restatement of Lemma 4.3 in [15].

**Lemma 4.4.4.** Let \( \Lambda_j(\cdot) \) is the cumulant generating function of \( U_k \) of type \( j \). Then, there exists a positive constant \( D \) such that for every \( j \in \{1, 2, \ldots, t\} \)

\[
\log \left( 1 + \alpha(e^{\Lambda_j(\theta)} - 1) \right) \leq \alpha u_k \theta + D \theta^2 \tag{4.23}
\]

for all \( \theta \in [0, 1] \) and \( \alpha \in [0, 1] \). ( \( \mathbb{E}(U_k) = u_k = u_j^* \) if \( U_k \) is of type \( j \) for \( j = 1, 2, \ldots, t \).)

The proof of the following lemma is identical to that of Lemma 2.8.1.
Lemma 4.4.5. Let $\epsilon_m = \frac{1}{\sqrt{\log m}}$. Then, for any $0 \leq \xi_1 < \xi_2$ there exists $M_1 = M_1(\xi_1, \xi_2) \in \mathbb{N}$ such that for $m > M_1$

$$
\Phi \left( (1 - \xi_1 \epsilon_m) s \sqrt{\log m} \right) - \Phi \left( (1 - \xi_2 \epsilon_m) s \sqrt{\log m} \right) = q'_m (\xi_1) - q'_m (\xi_2) \geq \frac{1}{2} e^{-\frac{1}{2} (1 - \xi_2 \epsilon_m)^2 s^2 \log m} \sqrt{2 \pi s \sqrt{\log m}}.
$$

With Lemma 4.4.4 and Lemma 4.4.5 at our disposal, we next prove the following.

Lemma 4.4.6. For any $j \in \{1, 2, .., t\}$,

$$
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{E} \left( e^{-\theta_m (z^{(m)})_x + \psi_m (\theta_m (z^{(m)}), z^{(m)} \mathbb{1}_{(A_j^{(m,e)})^c} (z^{(m)}) \right) \leq -\infty.
$$

Proof. For any $\theta^*_m \geq 0,$

$$
\mathbb{E} \left( e^{-\theta_m (z^{(m)})_x + \psi_m (\theta_m (z^{(m)}), z^{(m)}) \mathbb{1}_{(A_j^{(m,e)})^c} (z^{(m)}) \right)
\leq \mathbb{E} \left( e^{-\theta^*_m (z^{(m)})_x + \psi_m (\theta^*_m, z^{(m)}} \mathbb{1}_{(A_j^{(m,e)})^c} (z^{(m)}) \right)
$$

(4.24) follows since $-\theta_m (z^{(m)})_x + m \psi_m (\theta_m, z^{(m)}) \leq -\theta^*_m (z^{(m)})_x + m \psi_m (\theta^*_m, z^{(m)})$ for any $\theta^*_m \geq 0.$

Notice that for $z^{(m)} \in (A_j^{(m,e)})^c \subseteq \mathbb{R}^m$ we have by (4.16)

$$
p_k^{(m)} (z^{(m)}) = \Phi \left( (a_j^{(m)})^T z^{(m)} + \Phi^{-1} (p_k^{(m)}) b_j^{(m)} \right)
\leq \Phi \left( (a_j^{(m)})^T z^{(m)} + \Phi^{-1} (\bar{p}) b_j^{(m)} \right)
$$

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< \Phi \left( s(1 - \epsilon_m) \sqrt{\log m} \right) \\
= q_m^c (-1).

Also recall that by (4.4)

\[ x_m = \Phi \left( s \sqrt{\log m} \right) \sum_{k=1}^{m} l_k u_k \]
\[ = q_m^c(0) \sum_{k=1}^{m} l_k u_k. \]

At this point we will assume \( \theta_m^* \to 0 \) and differ specifying \( \theta_m^* \) for later.

For \( z^{(m)} \in (A_j^{(m,\epsilon)})^c \)

\[ -\theta_m^* q_m^c(0) \sum_{k \in I_j^{(m)}} l_k u_k + \sum_{k \in I_j^{(m)}} \log \left( 1 + p_k^{(m)}(z^{(m)})(e^{\Lambda_j(\theta_m^* l_k)} - 1) \right) \]
\[ = \sum_{k \in I_j^{(m)}} \left( -\theta_m^* q_m^c(0) l_k u_k + \log \left( 1 + p_k^{(m)}(z^{(m)})(e^{\Lambda_j(\theta_m^* l_k)} - 1) \right) \right) \]
\[ \leq \sum_{k \in I_j^{(m)}} \left( -\theta_m^* q_m^c(0) l_k u_k + q_m^c(-1)(e^{\Lambda_j(\theta_m^* l_k)} - 1) \right) \]
\[ \leq \sum_{k \in I_j^{(m)}} \left( -\theta_m^* q_m^c(0) l_k u_k + q_m^c(-1)u_k l_k \theta_m^* + D\bar{l}^2(\theta_m^*)^2 \right) \]
\[ \quad \text{for large } m \text{ by (4.23)} \]
\[ \leq \sum_{k \in I_j^{(m)}} \left( -\theta_m^* q_m^c(0) l_k u_k + q_m^c(-1)u_k l_k \theta_m^* + D\bar{l}^2(\theta_m^*)^2 \right) \]
\[ \leq \sum_{k \in I_j^{(m)}} (-\theta^*_m l_k u_k (q^e_m(0) - q^e_m(-1)) + D \bar{I}^2(\theta^*_m)^2) \]

\[ \leq \sum_{k \in I_j^{(m)}} (-\theta^*_m l_k u_k (q^e_m(0) - q^e_m(-1)) + D \bar{I}^2(\theta^*_m)^2) \]

(since \( q^e_m(0) - q^e_m(-1) > 0 \))

\[ = \sum_{k \in I_j^{(m)}} (-\theta^*_m l_k u_k (q^e_m(0) - q^e_m(-1)) + D \bar{I}^2(\theta^*_m)^2) \]

\[ = \sum_{k \in I_j^{(m)}} (-\theta^*_m l_k u_k (q^e_m(0) - q^e_m(-1)) + mD \bar{I}^2(\theta^*_m)^2) . \] (4.25)

Therefore for \( z^{(m)} \in (A^{(m,e)}_{j^*})^c \),

\[ -\theta^*_m q^e_m(0) \sum_{k \in I_j^{(m)}} l_k u_k + \sum_{k \in I_j^{(m)}} \log \left( 1 + p^{(m)}_k (z^{(m)}) \left( e^{\Lambda_j(\theta^*_m l_k)} - 1 \right) \right) \]

\[ \leq \frac{n_j^{(m)}}{m} (-m\theta^*_m l u (q^e_m(0) - q^e_m(-1)) + mD \bar{I}^2(\theta^*_m)^2) . \] (4.26)

Following in the same lines as how we got (4.26), we get for each type \( j' \) and any \( z^{(m)} \in \mathbb{R}^m \)

\[ -\theta^*_m q^e_m(0) \sum_{k \in I_{j'}^{(m)}} l_k u_k + \sum_{k \in I_{j'}^{(m)}} \log \left( 1 + p^{(m)}_k (z^{(m)}) \left( e^{\Lambda_{j'}(\theta^*_m l_k)} - 1 \right) \right) \]

\[ = \sum_{k \in I_{j'}^{(m)}} \left( -\theta^*_m q^e_m(0) l_k u_k + \log \left( 1 + p^{(m)}_k (z^{(m)}) \left( e^{\Lambda_j(\theta^*_m l_k)} - 1 \right) \right) \right) \]

\[ \leq \sum_{k \in I_{j'}^{(m)}} \left( -\theta^*_m q^e_m(0) l_k u_k + \log \left( 1 + e^{\Lambda_j(\theta^*_m l_k)} - 1 \right) \right) \]
\begin{align*}
\leq & \sum_{k \in I_{j'}} (-\theta^*_m q^c_m(0) l_k u_k + u_k l_k \theta^*_m + D\bar{I}^2(\theta^*_m)^2) \\
& \quad \text{(for large } m \text{ by (4.23))} \\
\leq & \sum_{k \in I_{j'}} (-\theta^*_m l_k u_k (q^c_m(0) - 1) + D\bar{I}^2(\theta^*_m)^2) \\
\leq & \sum_{k \in I_{j'}} (-\theta^*_m l_k u_k (q^c_m(0) - 1) + D\bar{I}^2(\theta^*_m)^2) \\
& \quad = n_{j'}^{(m)} (-\theta^*_m l_k u_k (q^c_m(0) - 1) + D\bar{I}^2(\theta^*_m)^2) \\
& \quad = n_{j'}^{(m)} (\theta^*_m l_k u_k (1 - q^c_m(0)) + D\bar{I}^2(\theta^*_m)^2). \\
\end{align*}

Therefore for any \( z^{(m)} \in \mathbb{R}^m \):

\begin{align*}
-\theta^*_m q^c_m(0) & \sum_{j' \neq j} \sum_{k \in I_{j'}} l_k u_k + \sum_{j' \neq j} \sum_{k \in I_{j'}} \log \left(1 + p_k^{(m)}(z^{(m)}) \left(e^{\Lambda_j(\theta^*_m l_k)} - 1\right)\right) \\
& \quad \leq \sum_{j' \neq j} \left(n_{j'}^{(m)} (\theta^*_m l_k u_k (1 - q^c_m(0)) + D\bar{I}^2(\theta^*_m)^2)\right) \\
& \quad \leq m \left(\theta^*_m l_k u_k (1 - q^c_m(0)) + D\bar{I}^2(\theta^*_m)^2\right). \\
& \quad \text{(4.27)}
\end{align*}

We emphasize that (4.27) holds for any \( z^{(m)} \in \mathbb{R}^m \) where as (4.26) holds only for \( z^{(m)} \in (A_{j}^{(m,c)})^c \). Therefore by (4.26) and (4.27), we get for any \( z^{(m)} \in (A_{j}^{(m,c)})^c \),

\begin{align*}
-\theta^*_m q^c_m(0) & \sum_{t=1}^m \sum_{k \in I_{j}(m)} l_k u_k + \sum_{k=1}^m \log \left(1 + p_k^{(m)}(z^{(m)}) \left(e^{\Lambda_j(\theta^*_m l_k)} - 1\right)\right) \\
& \quad \leq m \left(\theta^*_m l_k u_k (1 - q^c_m(0)) + D\bar{I}^2(\theta^*_m)^2\right) + \frac{n_j^{(m)}}{m} \left(-m\theta^*_m l_k u_k (q^c_m(0) - q^c_m(-1)) + mD\bar{I}^2(\theta^*_m)^2\right).
\end{align*}
That is for any $z^{(m)} \in (A_{j}^{(m, \epsilon)})^{c}$,

$$
-\theta^{*}_{m}x_{m} + \psi_{m}(\theta^{*}_{m}, z^{(m)})
\leq m \left( \theta^{*}_{m} l \cdot w(1 - q_{m}'(0)) + Dl^{2}(\theta^{*}_{m})^{2} \right) + \frac{n^{(m)}_{j}}{m} \left( -m\theta^{*}_{m} l \cdot w(q_{m}'(0) - q_{m}'(-1)) + mDl^{2}(\theta^{*}_{m})^{2} \right).
$$

$$
\mathbb{E}\left( e^{-\theta^{*}_{m}x_{m} + \psi_{m}(\theta^{*}_{m}, Z^{(m)})} 1_{(A_{j}^{(m, \epsilon)})^{c}}(Z^{(m)}) \right)
\leq e^{m\left( \theta^{*}_{m} l \cdot w(1 - q_{m}'(0)) + Dl^{2}(\theta^{*}_{m})^{2} \right) + \frac{n^{(m)}_{j}}{m} \left( -m\theta^{*}_{m} l \cdot w(q_{m}'(0) - q_{m}'(-1)) + mDl^{2}(\theta^{*}_{m})^{2} \right)}.
$$

$$
\frac{1}{\log m} \mathbb{E}\left( e^{-\theta^{*}_{m}x_{m} + \psi_{m}(\theta^{*}_{m}, Z^{(m)})} 1_{(A_{j}^{(m, \epsilon)})}(Z^{(m)}) \right)
\leq \frac{1}{\log m} \left( m \left( \theta^{*}_{m} l \cdot w(1 - q_{m}'(0)) + Dl^{2}(\theta^{*}_{m})^{2} \right) + \frac{n^{(m)}_{j}}{m} \left( -m\theta^{*}_{m} l \cdot w(q_{m}'(0) - q_{m}'(-1)) + mDl^{2}(\theta^{*}_{m})^{2} \right) \right). \quad (4.28)
$$

We will show that the RHS of (4.28) goes to $-\infty$ as $m \to \infty$.

Choose

$$
\theta^{*}_{m} = m^{-1 + \frac{\epsilon}{2}}.
$$

$$
m \left( \theta^{*}_{m} \right)^{2} \frac{1}{\log m} = 0.
$$

Recall that
\[ 1 - q_m^\epsilon(0) = 1 - \Phi(s\sqrt{\log m}) \]
\[ \leq \frac{1}{\sqrt{2\pi s\sqrt{\log m}}} e^{-\frac{1}{2}s^2\log m}. \]

Therefore

\[ \theta_m^* \frac{m(1 - q_m^\epsilon(0))}{\log m} \leq m \frac{s^2}{2} \frac{1}{\sqrt{2\pi s \log m^2}} e^{-\frac{1}{2}s^2\log m}. \]

\[ \log \left( m \frac{s^2}{2} \frac{1}{\sqrt{2\pi s \log m^2}} e^{-\frac{1}{2}s^2\log m} \right) = \frac{s^2}{2} \log m - \frac{s^2}{2} \log m - \frac{3}{2} \log(\log m) + o(1) \]
\[ \to -\infty. \]

Therefore

\[ \frac{m\theta_m^*L_u(1 - q_m^\epsilon(0))}{\log m} + \frac{mD^2(\theta_m^*)^2}{\log m} \to 0. \]

By Lemma 4.4.5

\[ \frac{\theta_m^* m}{\log m} (q_m^\epsilon(0) - q_m^\epsilon(-1)) \geq \frac{\theta_m^* m e^{-\frac{1}{2}(1-\epsilon_m)^2s^2\log m}}{\sqrt{2\pi s\sqrt{\log m}}} \frac{1}{2}. \]

\[ \frac{\theta_m^* me^{-\frac{1}{2}(1-\epsilon_m)s^2\log m}}{(\log m)^{\frac{3}{2}}} = \frac{m^2 e^{-\frac{1}{2}(1-\epsilon_m)s^2\log m}}{(\log m)^{\frac{3}{2}}}. \]
\[
\log \left( \frac{m^{\frac{s^2}{2}} e^{-\frac{1}{2}(1-\epsilon_m)s^2 \log m}}{(\log m)^{\frac{3}{2}}} \right) = \frac{s^2}{2} \log m - \frac{1}{2}(1 - \epsilon_m)^2 s^2 \log m - \frac{3}{2} \log \log m
\]

\[
= s^2 \epsilon_m \log m - \frac{1}{2} s^2 \epsilon_m^2 \log m - \frac{3}{2} \log \log m
\]

\[
= s^2 \sqrt{\log m} - \frac{1}{2} s^2 - \frac{3}{2} \log m \quad \text{(since } \epsilon_m = \frac{1}{\sqrt{\log m}}\text{)}
\]

\[
\rightarrow \infty.
\]

Moreover

\[
\lim_{m \to \infty} \frac{n_j^{(m)}}{m} = w_j.
\]

Therefore

\[
\lim_{m \to \infty} \frac{n_j^{(m)}}{m} \left( -m \frac{\hat{\theta}_m^* l_m(q_m^c(0) - q_m^c(-1))}{\log m} + m D \frac{(\theta_m^*)^2}{\log m} \right) = -\infty.
\]

Therefore, by (4.28) for any \( j \)

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{E} \left( e^{-\theta_m^* x_m + \psi_m(\theta_m^*, Z^{(m)})} 1_{(A_{j}^{(m,c)}, c)(Z^{(m)})} \right) \leq -\infty.
\]

Therefore by (4.24)

\[
\limsup_{m \to \infty} \frac{1}{\log m} \mathbb{E} \left( e^{-\theta_m(\hat{Z}^{(m)}) x_m + \psi_m(\theta_m(\hat{Z}^{(m)}), Z^{(m)})} 1_{(A_{j}^{(m,c)}, c)(Z^{(m)})} \right) \leq -\infty.
\]
Notice that

\[ A^{(m, \epsilon)} = \bigcap_{j=1}^{t} A_j^{(m, \epsilon)} \]

and therefore

\[ (A^{(m, \epsilon)})^c \subseteq \bigcup_{j=1}^{t} (A_j^{(m, \epsilon)})^c. \]

Therefore

\[ 1_{(A^{(m, \epsilon)})^c} \leq \sum_{j=1}^{t} 1_{(A_j^{(m, \epsilon)})^c}. \]

Therefore

\[
\mathbb{E} \left( e^{-\theta_m(Z^{(m)})_{x_m} + \psi_m(Z^{(m)}), Z^{(m)}} 1_{(A^{(m, \epsilon)})^c} (Z^{(m)}) \right) \\
\leq \sum_{j=1}^{t} \mathbb{E} \left( e^{-\theta_m(Z^{(m)})_{x_m} + \psi_m(Z^{(m)}), Z^{(m)}} 1_{(A_j^{(m, \epsilon)})^c} (Z^{(m)}) \right).
\]

Therefore by Lemma 4.4.6

\[
\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{E} \left( e^{-\theta_m(Z^{(m)})_{x_m} + \psi_m(Z^{(m)}), Z^{(m)}} 1_{(A^{(m, \epsilon)})^c} (Z^{(m)}) \right) \\
\leq \max_{j \in \{1, 2, \ldots, t\}} \left\{ \limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{E} \left( e^{-\theta_m(Z^{(m)})_{x_m} + \psi_m(Z^{(m)}), Z^{(m)}} 1_{(A_j^{(m, \epsilon)})^c} (Z^{(m)}) \right) \right\}.
\]

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Now it follows by (4.19), (4.22) and (4.29),

$$\limsup_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(L_m > x_m) \leq \max \left\{ -\infty, -\frac{1}{2} \gamma^2 \right\}$$

$$= -\frac{1}{2} \gamma^2.$$

\[\square\]

**Theorem 4.4.7.** Under the assumptions of Theorem 4.2.1

$$\liminf_{m \to \infty} \frac{1}{\log m} \log \mathbb{P}(L_m > x_m) \geq -\frac{1}{2} \gamma^2.$$  

**Proof.** Let $\epsilon_m = \frac{1}{\sqrt{\log m}}$. Define for $\xi \in \mathbb{R}$

$$q^\epsilon_m(\xi) = \Phi \left( s(1 + \xi \epsilon_m) \sqrt{\log m} \right).$$

Let

$$B_j^{(m,\epsilon)} = \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq sb_j^{(m)} (1 + \epsilon_m) \sqrt{\log m} - \Phi^{-1}(\mathbb{P}) \}$$

$$= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(a_j^{(m)})^T z^{(m)} + \Phi^{-1}(\mathbb{P})}{b_j^{(m)}} \right) \geq \Phi \left( s(1 + \epsilon_m) \sqrt{\log m} \right) \right\}$$

$$= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(a_j^{(m)})^T z^{(m)} + \Phi^{-1}(\mathbb{P})}{b_j^{(m)}} \right) \geq q^\epsilon_m(1) \right\}.$$
and

\[ B^{(m,\epsilon)} = \bigcap_{j=1}^{t} H^{(m)}_j = \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} (1 + \epsilon_m) \sqrt{\log m} - \Phi^{-1}(p) \text{ for all } j \in \{1, 2, \ldots, t\} \}. \] (4.30)

Recall the definition of \( A^{(m)} \) given in (4.6)

\[ A^{(m)} = \{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq s b_j^{(m)} \text{ for } j = 1, 2, \ldots t \}. \] (4.31)

By applying Lemma 4.3.1 we get

**Lemma 4.4.8.**

\[ A^{(m)} \neq \emptyset \iff B^{(m,\epsilon)} \neq \emptyset. \]

Since we assume that \( A^{(m)} \) is nonempty for all \( m \), it follows that \( B^{(m,\epsilon)} \) is nonempty for all \( m \).

Let

\[ T^{(m)}_k = e^{\frac{s^2 \log m}{2}} \sqrt{\log m} l_k(U_k Y^{(m)}_k - u_k p^{(m)}_k(Z^{(m)})) \]

and let

\[ S_m = \frac{1}{m} \sum_{k=1}^{m} T^{(m)}_k \]

\[ = \frac{e^{\frac{s^2 \log m}{2}} \sqrt{\log m}}{m} \sum_{k=1}^{m} l_k(U_k Y^{(m)}_k - u_k p^{(m)}_k(Z^{(m)})) \]

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\[ \frac{e^{s^2 \log m}}{m^2} \sqrt{\log m} \left( L_m - \mathbb{E}(L_m | Z^{(m)}) \right). \]

**Lemma 4.4.9.** Let \( z^{(m)} \) be an arbitrary sequence in \( \mathbb{R}^m \). Then for any \( \epsilon > 0 \)

\[ \lim_{m \to \infty} \mathbb{P} \left( |S_m| > \epsilon |Z^{(m)} = z^{(m)} \right) = 0. \]

(4.32)

**Proof.** Apply Lemma 4.3.5 with \( A_m = \{ Z^{(m)} = z^{(m)} \} \), \( n_m = m \)

\[ T^{(m)} = e^{s^2 \log m} \sqrt{\log m} l_k(U_k Y^{(m)} - u_k p_k^{(m)}(Z^{(m)})). \]

- \( T^{(m)} \)'s are conditionally independent given \( Z \).
- \( \mathbb{E} (T^{(m)} | Z^{(m)} = z^{(m)}) = 0 \) since \( U_k \) is independent of \( Z^{(m)} \) and \( Y^{(m)} \).
- We will show that

\[ \limsup_{m \to \infty} \frac{1}{m^2} \text{Var}(T^{(m)} | Z^{(m)} = z^{(m)}) = 0. \]

Since \( U_k Y^{(m)}_k \leq 1 \),

\[ \text{Var} \left( T^{(m)} | Z^{(m)} = z^{(m)} \right) \]

\[ = e^{s^2 \log m} \log m l_k^2 \mathbb{E} \left( \left( U_k Y^{(m)}_k - u_k p_k^{(m)}(Z^{(m)}) \right)^2 | Z^{(m)} = z^{(m)} \right) \]

\[ \leq 4(t)^2 e^{s^2 \log m} \log m. \]
Therefore
\[
\frac{1}{m^2} \sum_{k=1}^{m} \text{Var} \left( T_k^{(m)} | Z^{(m)} = z^{(m)} \right) \leq 4 \left( \bar{l} \right)^2 \frac{1}{m^2} 4e^{s^2 \log m \log m}
\]
\[
\leq 4 \left( \bar{l} \right)^2 \frac{1}{m} e^{s^2 \log m \log m}.
\]

\[
\log \left( \frac{1}{m} e^{s^2 \log m \log m} \right) = s^2 \log m + \log(\log m) - \log m
\]
\[
= \log m \left( s^2 - \frac{\log m}{\log m} + \frac{\log(\log m)}{\log m} \right) \to -\infty.
\]

Therefore
\[
\limsup_{m \to \infty} \frac{1}{m^2} \text{Var} \left( T_k^{(m)} | Z^{(m)} = z^{(m)} \right) = 0.
\]

Therefore, for any \( \epsilon > 0 \),
\[
\lim_{m \to \infty} \mathbb{P} \left( |S_m| > \epsilon |Z^{(m)} = z^{(m)} \right) = 0.
\]

We give below the following Lemma. The proof is identical to the proof of Lemma 2.8.3

**Lemma 4.4.10.** Let \( 0 \leq \xi_1 < \xi_2 \). There exists \( M_1 = M_1(\xi_1, \xi_2) \) such that for \( m > M_1 \),
\[
\Phi \left( (1 + \xi_2 \epsilon_m) s \sqrt{\log m} \right) - \Phi \left( (1 + \xi_1 \epsilon_m) s \sqrt{\log m} \right) = q_m^\epsilon (\xi_2) - q_m^\epsilon (\xi_1)
\]
\[
\geq e^{-\frac{1}{2} \left( 1 + \xi_1 \epsilon_m \right)^2 s^2 \log m} \frac{1}{\sqrt{2\pi s \sqrt{\log m}}} \frac{1}{2}.
\]
Lemma 4.4.11. Let $z^{(m)}_* \in B^{(m, \epsilon)}$. Then

$$\lim_{m \to \infty} \mathbb{P}(L_m > x_m|Z^{(m)} = z^{(m)}_*) = 1.$$ 

Proof. Let $z^{(m)}_* \in B^{(m, \epsilon)}$. By (4.30)

$$\mathbb{E}(L_m|Z^{(m)} = z^{(m)}_*) \geq q_m^\epsilon(1) \sum_{k=1}^m l_k u_k.$$ 

Therefore for $m > M_1$

$$e^{\frac{s^2 \log m}{2} \sqrt{\log m}} \left( x_m - \mathbb{E}(L_m|Z^{(m)} = z^{(m)}_*) \right) \leq e^{\frac{s^2 \log m}{2} \sqrt{\log m}} \left( x_m - q_m^\epsilon(1) \sum_{k=1}^m l_k u_k \right)$$

$$= e^{\frac{s^2 \log m}{2} \sqrt{\log m}} \left( q_m^\epsilon(0) - q_m^\epsilon(1) \right) \sum_{k=1}^m l_k u_k$$

$$= - e^{\frac{s^2 \log m}{2} \sqrt{\log m}} \left( q_m^\epsilon(1) - q_m^\epsilon(0) \right) \sum_{k=1}^m l_k u_k$$

$$\leq - e^{\frac{s^2 \log m}{2} \sqrt{\log m}} \left( \frac{1}{\sqrt{2\pi} e^{\frac{s^2 \log m}{2} \sqrt{\log m}}} \right) \sum_{k=1}^m l_k u_k$$

(by (4.33))

$$= - \frac{1}{\sqrt{2\pi s}} \sum_{k=1}^m l_k u_k$$

(4.34)

$$\to - \frac{1}{\sqrt{2\pi s}} C,$$

(4.35)

where
\[ C = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} l_k u_k. \]

Let \( 0 < \epsilon < \frac{1}{\sqrt{2\pi s}} C. \)

There exists \( M_2 \in \mathbb{N} \) such that for all \( m > M_2 \)

\[ \frac{1}{\sqrt{2\pi s}} \frac{1}{m} \sum_{k=1}^{m} l_k u_k > \epsilon \]

or equivalently

\[ -\frac{1}{\sqrt{2\pi s}} \frac{1}{m} \sum_{k=1}^{m} l_k u_k < -\epsilon. \] (4.36)

For \( m > \max\{M_1, M_2\} \)

\[ \mathbb{P}(L_m > x_m | Z^{(m)} = z^{(m)}_*) \]

\[ = \mathbb{P} \left( L_m - \mathbb{E}(L_m | Z^{(m)}) > x_m - \mathbb{E}(L_m | Z^{(m)}) | Z = z^{(m)}_* \right) \]

\[ = \mathbb{P} \left( S_m > \frac{\epsilon^2 \log m}{2 \sqrt{\log m}} \left( x_m - \mathbb{E}(L_m | Z^{(m)}) \right) | Z^{(m)} = z^{(m)}_* \right) \]

\[ \geq \mathbb{P} \left( S_m > -\left( \frac{1}{\sqrt{2\pi s}} \right) \frac{1}{m} \sum_{k=1}^{m} l_k u_k | Z^{(m)} = z^{(m)}_* \right) \quad \text{(for large } m \text{ by (4.34))} \]

\[ \geq \mathbb{P} (S_m > -\epsilon | Z^{(m)} = z^{(m)}_*) \quad \text{(for large } m \text{ by (4.36))} \]
\[ P \left( |S_m| \leq \epsilon \left| Z^{(m)} = z^{(m)} \right. \right) \to 1 \text{ as } m \to \infty. \] (By (4.32))

Let

\[ w_m = \inf_{z^{(m)} \in B(m,\epsilon)} P \left( L_m > x_m \left| Z^{(m)} = z^{(m)} \right. \right). \]

Let \( \epsilon > 0 \). There exists \( z_{**}^{(m)} \in B(m,\epsilon) \) such that

\[ w_m + \epsilon > P \left( L_m > x_m \left| Z^{(m)} = z_{**}^{(m)} \right. \right). \] (4.37)

\[ P(L_m > x_m) \geq \int_{z^{(m)} \in B(m,\epsilon)} P \left( L_m > x_m \left| Z^{(m)} = z^{(m)} \right. \right) P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)} \]

\[ \geq w_m \int_{z^{(m)} \in B(m,\epsilon)} P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)} \]

\[ \geq \left( P \left( L_m > x_m \left| Z^{(m)} = z_{**}^{(m)} \right. \right) - \epsilon \right) \int_{z^{(m)} \in B(m,\epsilon)} P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)} \] (by (4.37))

\[ = \left( P \left( L_m > x_m \left| Z^{(m)} = z_{**}^{(m)} \right. \right) - \epsilon \right) P \left( Z^{(m)} \in B(m,\epsilon) \right). \]

\[ P(L_m > x_m) \geq \left( P \left( L_m > x_m \left| Z^{(m)} = z_{**}^{(m)} \right. \right) - \epsilon \right) P \left( Z^{(m)} \in B(m,\epsilon) \right) \] (4.38)

where \( P \left( L_m > x_m \left| Z^{(m)} = z_{**}^{(m)} \right. \right) \to 1 \). Our Next task is to find an expression for \( P(Z^{(m)} \in B(m,\epsilon)) \).
\[
P \left( \mathbf{Z}(m) \in B^{(m, \epsilon)} \right) = P \left( a_j^T \mathbf{Z}(m) \geq s b_j^{(m)} (1 + \epsilon_m) \sqrt{\log m} - \Phi^{-1}(p) \right. \text{ for } j = 1, 2, \ldots, t \).
\]

Let
\[
N_j^{(m)} = (a_j^{(m)})^T \mathbf{Z}(m)
\]

and
\[
\mathbf{N}^{(m)} = \begin{bmatrix}
N_1^{(m)} \\
N_2^{(m)} \\
\vdots \\
N_t^{(m)}
\end{bmatrix}.
\]

Then \( \mathbf{N}^{(m)} \) is centered Gaussian with covariance matrix
\[
\Sigma^{(m)} = \mathbf{F}^{(m)}(\mathbf{F}^{(m)})^T,
\]

where
\[
\mathbf{F}^{(m)} = \begin{bmatrix}
(a_1^{(m)})^T \\
(a_2^{(m)})^T \\
\vdots \\
(a_t^{(m)})^T
\end{bmatrix}.
\]

Let
\[ H^{(m)} = (\Sigma^{(m)})^{-1}. \]

Let

\[ r_j = s \beta_j^{(m)} (1 + \epsilon_m) \sqrt{\log m - \Phi^{-1}(p)}, \]

and

\[ \mathbf{r}^{(m)} = \begin{bmatrix} r_1^{(m)} \\ r_2^{(m)} \\ \vdots \\ r_t^{(m)} \end{bmatrix}. \]

Let

\[ \mathbf{b}^{(m)} = \begin{bmatrix} s b_1 \\ s b_2 \\ \vdots \\ s b_t \end{bmatrix}, \]

and
Then we have

$$r^{(m)} = (1 + \epsilon_m) \sqrt{\log m} b^{(m)} + c.$$ 

Let

$$\beta_m = (1 + \epsilon_m) \sqrt{\log m}.$$ 

Then we have,

$$P(N^{(m)} \in B^{(m, \epsilon)}) = P(N^{(m)} \geq \beta_m b^{(m)} + c).$$ 

Let

$$\Lambda_m(a) = \log \mathbb{E}(e^{aN^{(m)}}).$$

Let

$$y^{(m)} \geq \beta_m b^{(m)} + c.$$ 

Define a new measure
\[ d\mathbb{P}^*_{m} = e^{k_m N^{(m)} - \Lambda_m(k_m)} \, d\mathbb{P}, \]

where

\[ \Sigma^{(m)} k_m = y^{(m)}. \]

By Theorem 4.3.3, under the probability measure \( \mathbb{P}^*_{m} \), \( N^{(m)} \) is Normally distributed with mean \( y^{(m)} \geq \beta_m b^{(m)} + c \) and covariance matrix \( \Sigma^{(m)} \).

\[
\begin{align*}
\mathbb{P}(N^{(m)} \in B^{(m,\epsilon)}) &= \mathbb{P}(N^{(m)} \geq \beta_m b^{(m)} + c) \\
&\geq \mathbb{P}(N^{(m)} \geq y^{(m)}) \\
&= \mathbb{E}^*_m \left( 1_{\{N^{(m)} \geq y^{(m)}\}} e^{-k_m N^{(m)} + \log E(e^{k_m N^{(m)}})} \right) \\
&= e^{-k_m y^{(m)} + \log E(e^{k_m N^{(m)}})} \mathbb{P}^*_{m} \left( 1_{\{N^{(m)} \geq y^{(m)}\}} \right) \\
&= e^{-k_m y^{(m)} + \log E(e^{k_m N^{(m)}})} \mathbb{P}^*_{m} \left( N^{(m)} \geq y^{(m)} \right).
\end{align*}
\]

Therefore

\[
\begin{align*}
\mathbb{P}(N^{(m)} \in B^{(m,\epsilon)}) &\geq e^{-k_m y^{(m)} + \log E(e^{k_m N^{(m)}})} \mathbb{P}^*_{m} \left( N^{(m)} \geq y^{(m)} \right). \\
&= e^{-k_m y^{(m)} + \log E(e^{k_m N^{(m)}})} \mathbb{P}^*_{m} \left( N^{(m)} \geq y^{(m)} \right).
\end{align*}
\]

We assume that the sequence of matrices \( H^{(m)} = (\Sigma^{(m)})^{-1} \) is uniformly bounded in the max norm. Therefore by Theorem 4.3.4, there exists a constant \( K_1 > 0 \) such that for any \( m \),

\[
\begin{align*}
\mathbb{P}(N^{(m)} \in B^{(m,\epsilon)}) &\geq e^{-k_m y^{(m)} + \log E(e^{k_m N^{(m)}})} \mathbb{P}^*_{m} \left( N^{(m)} \geq y^{(m)} \right).
\end{align*}
\]
\[ P^*_m(N^{(m)} \geq y^{(m)}) \geq \frac{1}{\sqrt{2\pi |\Sigma^{(m)}|}} K_1, \]

for some constant \( K_1 > 0 \).

Therefore

\[ \log P^*_m(N^{(m)} \geq y^{(m)}) \geq \log K_1 - \log |\Sigma^{(m)}| - \frac{t}{2} \log 2\pi \]

\[ \geq \log K_1 - \log \log m - \frac{t}{2} \log 2\pi \]

\[ = o(\log m). \quad (4.40) \]

By (4.39)

\[ \log P(N^{(m)} \in B^{(m,\epsilon)}) = -k_m y_m + \log \mathbb{E}(e^{k_m N^{(m)}}) + \log P^*_m(N^{(m)} \geq y^{(m)}) \]

\[ = -k_m y_m + \Lambda_m(k_m) + \log P^*_m(N^{(m)} \geq y^{(m)}) \]

\[ \geq -\Lambda^*_m(y^{(m)}) + \log P^*_m(N^{(m)} \geq y^{(m)}) \]

\[ = -\frac{1}{2}(y^{(m)})^T H^{(m)} y^{(m)} + \log P^*_m(N^{(m)} \geq y^{(m)}) \]

\[ = -\frac{1}{2}(y^{(m)})^T H^{(m)} y^{(m)} + o(\log m) \quad (by \ \text{(4.40)}) \]

\[ \log P(N^{(m)} \in B^{(m,\epsilon)}) \geq \sup_{y^{(m)} \in B^{(m,\epsilon)}} -\frac{1}{2}(y^{(m)})^T H^{(m)} y^{(m)} + o(\log m) \]

\[ = -\frac{1}{2} \inf_{y^{(m)} \in B^{(m,\epsilon)}} (y^{(m)})^T H^{(m)} y^{(m)} + o(\log m) \]
\[
-\frac{1}{2} \inf_{y^{(m)} \geq \beta_m b^{(m)} + c} (y^{(m)})^T H^{(m)} y^{(m)} + o(\log m) \\
= -\frac{1}{2} \inf_{y \geq \beta_m b^{(m)} + c} y^T H^{(m)} y + o(\log m).
\] (4.41)

By using the change of variable \( y = F^{(m)} z \) where \( z \in \mathbb{R}^m \),

\[
\inf_{y \geq \beta_m b^{(m)} + c} y^T H^{(m)} y = \inf_{F^{(m)} z \geq \beta_m b^{(m)} + c} (z)^T z
\]

\[
= \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq \beta_m b_j^{(m)} + \Phi^{-1}(p) \text{ for all } j \in \{1, 2, \ldots, t\} \right\}
\]

\[
= \beta_m^2 \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq s_j^{(m)} + \frac{\Phi^{-1}(p)}{\beta_m} \text{ for all } j \in \{1, 2, \ldots, t\} \right\}.
\]

By Lemma 4.3.2,

\[
\lim_{m \to \infty} \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq s_j^{(m)} + \frac{\Phi^{-1}(p)}{\beta_m} \text{ for all } j \in \{1, 2, \ldots, t\} \right\}
\]

\[
= \lim_{m \to \infty} \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq s_j^{(m)} \text{ for all } j \in \{1, 2, \ldots, t\} \right\}
\]

\[
= \lim_{m \to \infty} (\gamma^{(m)})^2
\]

\[
= \gamma^2.
\]

\[
\lim_{m \to \infty} \frac{1}{\log m} \left( \inf_{y \geq \beta_m b^{(m)} + c} y^T H^{(m)} y \right) = \gamma^2.
\]
Therefore by (4.41),

\[
\liminf_{m \to \infty} \frac{1}{\log m} \log P(N^{(m)} \in B^{(m,\epsilon)}) \geq -\frac{1}{2}\gamma^2.
\]

Therefore by (4.38),

\[
\liminf_{m \to \infty} \frac{1}{\log m} \log P(L_m > x_m) \geq -\frac{1}{2}\gamma^2.
\]

\[\square\]

### 4.5 Analysis of Small Default Regime

In this section we give the proof of Theorem 4.2.4. The upper bound computation is given by Theorem 4.5.1 and the lower bound computation is given by (4.5.11).

**Theorem 4.5.1.** Under the assumptions of Theorem 4.2.4

\[
\limsup_{m \to \infty} \frac{1}{m} \log P(L_m > x_m) \leq -\frac{1}{2}\gamma^2.
\]

**Proof.** Let \( J \in S_q \). Let \( \epsilon_m \to 0 \) such that \( \epsilon_m \sqrt{m} \to \infty \). Define

\[
C_j^{(m,\epsilon)} = \left\{ z^{(m)} \in \mathbb{R}^m : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j (1 - \epsilon_m) \sqrt{m} \right\}
\]

\[
= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(a_j^{(m)})^T z^{(m)} - \tilde{s}_j \sqrt{m}}{b_j^{(m)}} \right) \geq \Phi \left( -\frac{\tilde{s}_j \epsilon_m \sqrt{m}}{b_j^{(m)}} \right) \right\}.
\]

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\begin{align*}
C^{(m,\epsilon)} = \bigcap_{j \in J} C_j^{(m,\epsilon)} \\
= \left\{ \mathbf{z}^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(\mathbf{a}_j^{(m)})^T \mathbf{z}^{(m)} - \bar{s}_j \sqrt{m}}{b_j^{(m)}} \right) \geq \Phi \left( -\frac{\bar{s}_j \epsilon_m \sqrt{m}}{b_j^{(m)}} \right) \text{ for all } j \in J \right\}.
\end{align*}

Define

\begin{equation}
C^{(m,\epsilon)} = \bigcup_{J \in \mathcal{S}_q} C_J^{(m,\epsilon)}, \tag{4.42}
\end{equation}

and

\begin{equation}
\gamma_J^{(m,\epsilon)} = \min \left\{ \| \mathbf{z}^{(m)} \| : \mathbf{z}^{(m)} \in C_J^{(m,\epsilon)} \right\}.
\end{equation}

Recall the definition of \( \mathcal{S}_q \) given in (4.12),

\begin{equation}
\mathcal{S}_q = \left\{ J \in \mathcal{M}_q : C_J^{(m)} \neq \emptyset \text{ for all } m \right\} \tag{4.43}
\end{equation}

and the definition of \( C_J^{(m)} \) given in (4.10),

\begin{equation}
C_J^{(m)} = \left\{ \mathbf{z}^{(m)} \in \mathbb{R}^m : (\mathbf{a}_j^{(m)})^T \mathbf{z}^{(m)} \geq \bar{s}_j \text{ for all } j \in J \subseteq \{1, 2, \ldots, t\} \right\}.
\end{equation}

The following Lemma is a direct application of Lemma 4.3.1.
Lemma 4.5.2.

\[ \mathcal{C}_{\mathcal{J}}^{(m, \epsilon)} \neq \emptyset \iff \mathcal{C}_{\mathcal{J}}^{(m)} \neq \emptyset. \]

Since we assume that \( \mathcal{J} \in \mathcal{S}_q \), it follows that \( \mathcal{C}_{\mathcal{J}}^{(m)} \neq \emptyset \) for all \( m \), and hence \( \mathcal{C}_{\mathcal{J}}^{(m, \epsilon)} \neq \emptyset \) for all \( m \).

\[ \gamma^{(m, \epsilon)}_{\mathcal{J}} = \min \left\{ \|z^{(m)}\| : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j (1 - \epsilon_m) \sqrt{m} \text{ for all } j \in \mathcal{J} \right\} \]

\[ = \min \left\{ (1 - \epsilon_m) \sqrt{m} \|z^{(m)}\| : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j \text{ for all } j \in \mathcal{J} \right\} \]

\[ = (1 - \epsilon_m) \sqrt{m} \min \left\{ \|z^{(m)}\| : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j \text{ for all } j \in \mathcal{J} \right\} \]

\[ = (1 - \epsilon_m) \sqrt{m} \gamma^{(m)}_{\mathcal{J}}. \] (by (4.11))

\[ \lim_{m \to \infty} \frac{1}{m} (\gamma^{(m, \epsilon)}_{\mathcal{J}})^2 = \lim_{m \to \infty} \frac{1}{m} (\gamma^{(m)}_{\mathcal{J}})^2 \]

\[ = (\gamma_{\mathcal{J}})^2. \] (4.44)

For \( z^{(m)} \in \mathbb{R}^m \) let,

\[ \psi_m(\theta, z^{(m)}) = \frac{1}{m} \log \mathbb{E} \left( e^{\theta L_m} \left| Z^{(m)} = z^{(m)} \right. \right) \]

\[ = \frac{1}{m} \log \mathbb{E} \left( e^{\theta \sum_{k=1}^{m} t_k Y_k^{(m)}} \left| Z^{(m)} = z^{(m)} \right. \right) \]

\[ = \frac{1}{m} \sum_{k=1}^{m} \log \mathbb{E} \left( e^{\theta t_k Y_k^{(m)}} \left| Z^{(m)} = z^{(m)} \right. \right) \] (by conditional independence)

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\[= \frac{1}{m} \sum_{k=1}^{m} \log \left( 1 + \tilde{p}_k^{(m)}(z^{(m)}) \left( e^{\theta_k} - 1 \right) \right)\]

\[= \frac{1}{m} \sum_{j=1}^{t} \sum_{k \in I_j} \log \left( 1 + \tilde{p}_j^{(m)}(z^{(m)}) \left( e^{\theta_k} - 1 \right) \right).\]

Let

\[\theta_m(z^{(m)}) = \arg\min_{\theta \geq 0} \{-\theta x_m + m \psi_m(\theta, z^{(m)})\}.\]

Define a new measure \(\mathbb{P}_m\) by

\[
\frac{d\mathbb{P}}{d\mathbb{P}_m} = e^{-\theta_m(z^{(m)}) L_m + m \psi_m(\theta_m(z^{(m)}), z^{(m)})}, \tag{4.45}
\]

and let \(\mathbb{E}_m\) denote the expectation under \(\mathbb{P}_m\). By (4.45), \(\mathbb{P}_m\) is a probability measure.

\[
\mathbb{P}(L_m > x_m)
\]

\[= \mathbb{E} \left( \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon)^c)}(z^{(m)}) \right) + \mathbb{E} \left( \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon))}(z^{(m)}) \right)
\]

\[= \mathbb{E}_m \left( e^{-\theta_m(z^{(m)})} L_m + m \psi_m(\theta_m(z^{(m)}), z^{(m)}) \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon)^c)}(z^{(m)}) \right)
\]

\[+ \mathbb{E} \left( \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon))}(z^{(m)}) \right).
\]

\[\leq \mathbb{E}_m \left( e^{-\theta_m(z^{(m)})} x_m + m \psi_m(\theta_m(z^{(m)}), z^{(m)}) \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon)^c)}(z^{(m)}) \right)
\]

\[+ \mathbb{E} \left( \mathbb{1} \{L_m > x_m\} \mathbb{1}_{(C(m,\epsilon))}(z^{(m)}) \right).
\]

\[\leq \mathbb{E}_m \left( e^{-\theta_m(z^{(m)})} x_m + m \psi_m(\theta_m(z^{(m)}), z^{(m)}) \mathbb{1}_{(C(m,\epsilon)^c)}(z^{(m)}) \right) + \mathbb{P} \left( z^{(m)} \in C(m,\epsilon) \right)
\]

\[\leq \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m(\theta, z^{(m)})} \mathbb{1}_{(C(m,\epsilon)^c)}(z^{(m)}) \right) + \mathbb{P} \left( z^{(m)} \in C(m,\epsilon) \right) \text{ for any } \theta \geq 0. \tag{4.46}
\]
(4.46) follows since $-\theta_m(z^{(m)})x_m + m\psi_m(\theta_m, z^{(m)}) \leq -\theta x_m + m\psi_m(\theta, z^{(m)})$ for any $\theta \geq 0$.

Now we find an expression for $\mathbb{P}(Z^{(m)} \in C_J^{(m, \epsilon)})$.

$$\mathbb{P} \left( Z^{(m)} \in C_J^{(m, \epsilon)} \right) = \mathbb{P} \left( (a_j^{(m)})^T Z^{(m)} \geq \tilde{s}_j (1 - \epsilon_m) \sqrt{m} \text{ for all } j \in J \right).$$

Suppose $J = \{j_1, j_2, \ldots, j_v\}$. Let

$$N_j^{(m)} = (a_j^{(m)})^T Z^{(m)} \text{ for } j \in J,$$

and

$$N_J^{(m)} = \begin{bmatrix} N_{j_1}^{(m)} \\ N_{j_2}^{(m)} \\ \vdots \\ N_{j_v}^{(m)} \end{bmatrix}.$$ 

Then $N_J^{(m)}$ is centered Gaussian with covariance matrix

$$\Sigma_J^{(m)} = F_J^{(m)} (F_J^{(m)})^T,$$
where

\[ F^{(m)}_J = \begin{bmatrix}
(a^{(m)}_{j1})^T \\
(a^{(m)}_{j2})^T \\
\vdots \\
(a^{(m)}_{jv})^T
\end{bmatrix}. \]

Let

\[ r^{(m)}_j = \tilde{s}_j (1 - \epsilon_m) \sqrt{m}, \]

and

\[ r^{(m)} = \begin{bmatrix}
r^{(m)}_{j1} \\
r^{(m)}_{j2} \\
\vdots \\
r^{(m)}_{jv}
\end{bmatrix}. \]

Let

\[ b = \begin{bmatrix}
\tilde{s}_{j1} \\
\tilde{s}_{j2} \\
\vdots \\
\tilde{s}_{jv}
\end{bmatrix}. \]

Then we have
\[ r^{(m)} = (1 - \epsilon_m)\sqrt{m}b. \]

\[ \mathbb{P}(Z^{(m)} \in \mathcal{C}^{(m,\epsilon)}) = \mathbb{P}(F_{\mathcal{J}}^{(m)}Z^{(m)} \geq r^{(m)}) \]

\[ = \mathbb{P}(N^{(m)}_{\mathcal{J}} \geq r^{(m)}) \]

\[ = \mathbb{P}(N^{(m)}_{\mathcal{J}} \geq (1 - \epsilon_m)\sqrt{m}b). \] (4.47)

Define

\[ H^{(m)}_{\mathcal{J}} = (\Sigma^{(m)}_{\mathcal{J}})^{-1}. \]

We will use the shorthand notation

\[ \alpha_m = (1 - \epsilon_m)\sqrt{m}. \]

**Lemma 4.5.3.**

\[ \limsup_{m \to \infty} \frac{1}{m} \log \mathbb{P} \left( N^{(m)}_{\mathcal{J}} \geq \alpha_m b \right) \leq -\frac{1}{2} \gamma_{\mathcal{J}}. \]

**Proof.** Let \( 0 < \theta < \infty \) and let
$$\theta = \begin{bmatrix}
\theta \\
\theta \\
\theta \\
\vdots \\
\theta
\end{bmatrix}_{v \times 1}.$$  

Define

$$T_i^{(m)} = \{ N_i^{(m)} > \alpha_m \theta \},$$

and

$$T^{(m)} = \bigcup_{i=1}^v T_i^{(m)}.$$ 

Note that $N_i^{(m)}$ is centered Gaussian with standard deviation $\|a_i^{(m)}\|$.

$$\mathbb{P}(N_i^{(m)} > \alpha_m \theta) \leq \frac{\|a_i^{(m)}\|}{\sqrt{2\pi \alpha_m \theta}} e^{-\frac{1}{2} \|a_i^{(m)}\|^2 \alpha_m^2 \theta^2}$$

$$\leq \frac{1}{\sqrt{2\pi \alpha_m \theta}} e^{-\frac{1}{2} \alpha_m^2 \theta^2},$$

(Since $\|a_i^{(m)}\| < 1$)

and therefore

$$\mathbb{P}(N_j^{(m)} \in T^{(m)}) \leq v \frac{1}{\sqrt{2\pi \alpha_m \theta}} e^{-\frac{1}{2} \alpha_m^2 \theta^2}.$$
Note that
\[ \left\{ N^{(m)}_J \in (T^{(m)})^c \right\} = \left\{ N^{(m)}_i < \alpha_m \theta \text{ for all } i \in \{1, 2, \ldots, v\} \right\} = \left\{ N^{(m)}_J < \alpha_m \theta \right\}. \]

\[ \mathbb{P}(N^{(m)}_J \geq \alpha_m b) \leq \mathbb{P}(\alpha_m b \leq N^{(m)}_J \leq \alpha_m \theta) + \mathbb{P}(N^{(m)}_J \in T^{(m)}) = \mathbb{P}(\alpha_m b \leq N^{(m)}_J \leq \alpha_m \theta) + \mathbb{P}(N^{(m)}_J \in T^{(m)}) \leq \int_{\alpha_m b \leq z \leq \alpha_m \theta} \frac{1}{\sqrt{2\pi \alpha_m}} e^{-\frac{1}{2} z^2} \mathbf{H}^{(m)}_J \mathbf{z} \, dz + v \frac{1}{\sqrt{2\pi \alpha_m}} e^{-\frac{1}{2} \alpha_m^2 \theta^2}. \]

Use the transformation
\[ y = \frac{z}{\alpha_m}. \]

Jacobian of the transformation \( \text{Jac} = \alpha_m I_{v \times v} \), where \( I_{v \times v} \) denotes the identity matrix of dimension \( v \).

Therefore \( |\text{Jac}| = \alpha_m^v \).

\[ \mathbb{P}(N^{(m)}_J \geq \alpha_m b) \leq \int_{b \leq y \leq \theta} \frac{\alpha_m^v}{\sqrt{2\pi \alpha_m}} e^{-\frac{1}{2} (\alpha_m y)^T \mathbf{H}^{(m)}_J \alpha_m y} \, dy + v \frac{1}{\sqrt{2\pi \alpha_m}} e^{-\frac{1}{2} \alpha_m^2 \theta^2}. \]

Therefore if we let \( 0 < s < \tilde{s}_j \) for all \( j \in J \) then,
\[ \{ y \in \mathbb{R}^v | b \leq y \leq \theta \} \subseteq \{ y \in \mathbb{R}^v | s_1 \leq y \leq 2\theta \} \text{ for large } m. \]
Therefore the set $K = \{ y \in \mathbb{R}^v | b \leq y \leq \theta \}$ is a compact set.

\[
\mathbb{P}(N_J^{(m)} \geq \alpha_m b) \leq \int_K \frac{\alpha_m^v}{\sqrt{2\pi^v}|\Sigma_J^{(m)}|} e^{-\frac{1}{2}(\alpha_m y)^T H_J^{(m)}(\alpha_m y)} dy + \frac{1}{\sqrt{2\pi\theta\alpha_m}} e^{-\frac{1}{2}\alpha_m^2 \theta^2}
\]

\[
\leq \frac{\alpha_m^v}{\sqrt{2\pi^v}|\Sigma_J^{(m)}|} \sup_{y \in K} \left( e^{-\frac{1}{2}(\alpha_m y)^T H_J^{(m)}(\alpha_m y)} \lambda(K) + \frac{1}{\sqrt{2\pi\theta\alpha_m}} e^{-\frac{1}{2}\alpha_m^2 \theta^2} \right)
\]

\[
\leq \frac{\alpha_m^v}{\sqrt{2\pi^v}|\Sigma_J^{(m)}|} \sup_{y \geq b} \left( e^{-\frac{1}{2}(\alpha_m y)^T H_J^{(m)}(\alpha_m y)} \lambda(K) + \frac{1}{\sqrt{2\pi\theta\alpha_m}} e^{-\frac{1}{2}\alpha_m^2 \theta^2} \right)
\]

We assume that $|\Sigma_J^{(m)}| < m^k$ for some $k > 0$. Also recall that $\alpha_m = (1 - \epsilon_m) \sqrt{m}$.

\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \mathbb{P}(N_J^{(m)} \geq \alpha_m b) \right) \leq \max \left\{ -\frac{1}{2} \lim_{m \to \infty} \frac{1}{m} \inf_{y \geq b} (\alpha_m y)^T H_J^{(m)}(\alpha_m y), -\frac{1}{2} \theta^2 \right\}.
\]

Since $\theta$ is arbitrary, assuming that \( \lim_{m \to \infty} \frac{1}{m} \inf_{y \geq b} (\alpha_m y)^T H_J^{(m)}(\alpha_m y) < \infty \),

\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \mathbb{P}(N_J^{(m)} \geq \alpha_m b) \right) \leq -\frac{1}{2} \lim_{m \to \infty} \frac{1}{m} \inf_{y \geq b} (\alpha_m y)^T H_J^{(m)}(\alpha_m y). \tag{4.48}
\]

Use the change of variable $\alpha_m y = F_J^{(m)} z$ and note that $H_J^{(m)} = (F_J^{(m)} (F_J^{(m)})^T)^{-1}$.

\[
\inf_{y \geq b} (\alpha_m y)^T H_J^{(m)}(\alpha_m y) = \inf_{F_J^{(m)} z \geq \alpha_m b} (F_J^{(m)} z)^T H_J^{(m)}(F_J^{(m)} z)
\]

\[
= \inf_{F_J^{(m)} z \geq \alpha_m b} z^T z
\]

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\[= \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq \alpha_m \tilde{s}_j \text{ for all } j \in \{1, 2, \ldots, v\} \right\}\]
\[= \alpha_m^2 \inf \left\{ \|z^{(m)}\|^2 : (a_j^{(m)})^T z^{(m)} \geq \tilde{s}_j \text{ for all } j \in \{1, 2, \ldots, v\} \right\}\]
\[= \alpha_m^2 (\gamma_J^{(m)})^2.\]

Therefore,
\[
\lim \frac{1}{m} \inf_{m \to \infty} (\alpha_m y)^T H_J^{(m)} (\alpha_m y) = \lim_{m \to \infty} (\gamma_J^{(m)})^2
\]
\[= \gamma_J.\]

Therefore by (4.48)
\[
\limsup_{m \to \infty} \frac{1}{m} \log P(N_J^{(m)} > \alpha_m b) \leq -\frac{1}{2} \gamma_J^2.
\]

Therefore by (4.47)
\[
\limsup_{m \to \infty} \frac{1}{m} \log P(Z^{(m)} > C_J^{(m,c)}) \leq -\frac{1}{2} \gamma_J^2.
\]

But by (4.14), \(\gamma = \min_{J \in \mathcal{S}_q} \gamma_J.\) Therefore
\[
\limsup_{m \to \infty} \frac{1}{m} \log P(Z^{(m)} \in C_J^{(m,c)}) \leq -\frac{1}{2} \gamma^2.
\]

Recall that by (4.13)
\[ C^{(m, \epsilon)} = \cup_{J \in S_q} C_J. \]

Therefore

\[ \mathbb{P}(Z^{(m)} \in C^{(m, \epsilon)}) \leq \sum_{J \in \mathcal{M}_q} \mathbb{P}(Z^{(m)} \in C^{(m, \epsilon)}_J). \]

Therefore

\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{P}(Z^{(m)} \in C^{(m, \epsilon)}) \leq \max_{J \in S_q} \limsup_{m \to \infty} \frac{1}{m} \log \mathbb{P}(Z^{(m)} \in C^{(m, \epsilon)}_J) \leq -\frac{1}{2} \gamma^2. \tag{4.49}
\]

With \[4.49\] at our disposal, in order to use \[4.46\] to prove the main result, we should investigate \( \limsup_{m \to \infty} \frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta^*_m x^{(m)}_J + m \psi_m(\theta, Z^{(m)})} \mathbbm{1}_{(C^{(m, \epsilon)}_J)^c}(Z^{(m)}) \right) \). In order to show this we first prove the following Lemma.

**Definition 4.5.4.** \( K \subseteq \{1, 2, ..., t\} \) is said to be a cut-set of \( S_q \) if

1. \( K \cap J \neq \emptyset \) for every \( J \in S_q \).

2. \( K \) is minimal: For any \( K' \subseteq K \), there exists \( J \in S_q \) such that \( K' \cap J = \emptyset \).

Let \( C_q \) denote the set of all cut-sets.

**Lemma 4.5.5.**

\[ (C^{(m, \epsilon)}_J)^c \subseteq \cup_{K \in C_q} \cap_{J \in K} (C^{(m, \epsilon)}_j)^c. \tag{4.50} \]
Proof. Suppose \( z \in (C^{(m,\epsilon)})^c \). \( C^{(m,\epsilon)} = \bigcup_{J \in \mathcal{S}_q} \bigcap_{j \in J} C^{(m,\epsilon)}_j \). Therefore \( C^{(m,\epsilon)} = \bigcap_{J \in \mathcal{S}_q} \bigcup_{j \in J} (C^{(m,\epsilon)}_j)^c \). Therefore for each \( J \in \mathcal{S}_q \) there exists \( j_J \in J \) such that \( z \in (C^{(m,\epsilon)}_{j_J})^c \). \( \{j_J : J \in \mathcal{S}_q\} \) is a cut of \( \mathcal{S}_q \) and it contains a cut-set \( \mathcal{K} \). Therefore

\[
z \in \bigcap_{J \in \mathcal{S}_q} (C^{(m,\epsilon)}_{j_J})^c \subseteq \bigcap_{J \in \mathcal{K}} (C^{(m,\epsilon)}_j)^c \quad \text{(Since } \mathcal{K} \subseteq \{j_J : J \in \mathcal{S}_q\}\text{)}
\]

We have shown there exists a cut-set \( \mathcal{K} \) such that \( z \in \bigcap_{J \in \mathcal{K}} (C^{(m,\epsilon)}_j)^c \).

By (4.50)

\[
1_{(C^{(m,\epsilon)})^c} \leq \sum_{\mathcal{K} \in \mathcal{C}_q} 1_{\bigcap_{J \in \mathcal{K}} (C^{(m,\epsilon)}_j)^c}.
\]

For each \( \mathcal{K} \in \mathcal{C}_q \), and \( z \in \bigcap_{J \in \mathcal{K}} (C^{(m,\epsilon)}_j)^c \)

\[
\tilde{p}_{j,m}(z)(e^{\theta_l} - 1) = \begin{cases} 
\Phi \left(-\frac{\tilde{s}_j}{b_j^{(m)}} \epsilon_m \sqrt{m}\right) \left(e^{\theta_l} - 1\right) & \text{if } j \in \mathcal{K} \\
 e^{\theta_l} - 1 & \text{if } j \notin \mathcal{K}
\end{cases}
\]

\[
e^{m\psi_m(\theta,z^{(m)})} = \prod_{j=1}^{t} \prod_{k \in I_j^{(m)}} \left(1 + \tilde{p}_{j,m}(z)(e^{\theta_l} - 1)\right)
\]

\[
= \prod_{j \in \mathcal{K}} \prod_{k \in I_j^{(m)}} \left(1 + \Phi \left(-\frac{\tilde{s}_j}{b_j^{(m)}} \epsilon_m \sqrt{m}\right) \left(e^{\theta_l} - 1\right)\right) \times \prod_{j \notin \mathcal{K}} \prod_{k \in I_j^{(m)}} e^{\theta_l}.
\]

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\[
\frac{1}{m} \log \left( e^{m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{\cap_{j \in \mathcal{K}} (C_j^{(m, \epsilon)})^c} Z^{(m)} \right)
\]

\[
\leq \frac{1}{m} \sum_{j \in \mathcal{K}} n_j^{(m)} \log \left( 1 + \Phi \left( \frac{-s_j}{b_j^{(m) \epsilon_m \sqrt{m}}} \right) (e^{\theta x_j} - 1) \right) + \theta \sum_{j \not\in \mathcal{K}} \frac{1}{m} \sum_{k \in I_j^{(m)}} l_k
\]

\[
\to \theta \sum_{j \not\in \mathcal{K}} H_j.
\]

Lemma 4.5.6.

\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{(C^{(m, \epsilon)})^c} (Z^{(m)}) \right) \leq \theta \left( \max_{K \in \mathcal{C}_q} \left\{ \sum_{j \not\in \mathcal{K}} H_j \right\} - qH \right).
\]

Proof.

\[
\mathbb{E}_m \left( e^{-\theta x_m + m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{(C^{(m, \epsilon)})^c} (Z^{(m)}) \right) \leq \sum_{K \in \mathcal{C}_q} \left( \mathbb{E}_m e^{-\theta x_m + m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{\cap_{j \in \mathcal{K}} (C_j^{(m, \epsilon)})^c} (Z^{(m)}) \right).
\]

Therefore,

\[
\frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{(C^{(m, \epsilon)})^c} (Z^{(m)}) \right)
\]

\[
\leq \frac{1}{m} \log \sum_{K \in \mathcal{C}_q} \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m(\theta, Z^{(m)})} \mathbb{1}_{\cap_{j \in \mathcal{K}} (C_j^{(m, \epsilon)})^c} (Z^{(m)}) \right).
\]

Therefore,

\[
\limsup_{m \to \infty} \frac{1}{m} \log \left( \mathbb{E}_m e^{-\theta x_m + m \psi_m(\theta, Z)} \mathbb{1}_{(C^{(m, \epsilon)})^c} (Z^{(m)}) \right)
\]

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\[ \leq \max_{K \in C_q} \left( \limsup_{m \to \infty} \frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m} \mathbbm{1}_{\cap_j \in K (C_j^{(m, \epsilon)})_{c} (Z)^{(m)}} \right) \right). \]

For any \( K \in C_q \),

\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m} \mathbbm{1}_{\cap_j \in K (C_j^{(m, \epsilon)})_{c} (Z)^{(m)}} \right)
\leq \limsup_{m \to \infty} \frac{1}{m} \log e^{-\theta x_m} + \limsup_{m \to \infty} \frac{1}{m} \log \left( \mathbb{E}_m e^{m \psi_m} \mathbbm{1}_{\cap_j \in K (C_j^{(m, \epsilon)})_{c} (Z)^{(m)}} \right)
\leq -\theta q H + \theta \sum_{j \notin K} H_j
\leq \theta \left( \sum_{j \notin K} H_j - q H \right).
\]

Therefore,

\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{E}_m \left( e^{-\theta x_m + m \psi_m} \mathbbm{1}_{(C^{(m, \epsilon)})_{c} (Z)^{(m)}} \right)
\leq \max_{K \in C_q} \theta \left( \sum_{j \notin K} H_j - q H \right)
\leq \theta \left( \max_{K \in C_q} \left( \sum_{j \notin K} H_j \right) - q H \right). \quad (4.51)
\]

Next we show that
Lemma 4.5.7.

\[
\max_{K \in C_q} \left( \sum_{j \notin K} H_j \right) < qH. \tag{4.52}
\]

Proof. Suppose for some \( K \in C_q \),

\[
\sum_{j \notin K} H_j = \sum_{j \in K^c} H_j > qH.
\]

Then there exists a \( J \in S_q \) such that \( J \subseteq K^c \). Then \( K \cap J \subseteq K \cap K^c = \emptyset \). This contradicts the assumption that \( K \in C_q \).

Finally we have by (4.46), (4.49) and (4.51) (since \( \theta \) is arbitrary)

\[
\limsup_{m \to \infty} \frac{1}{m} \log \mathbb{P}(L_m > x_m) \leq -\frac{1}{2} \gamma^2_2.
\]

In order to prove the lower bound, we first prove the following result.

Theorem 4.5.8. Suppose the assumption of Theorem 4.2.4 hold. Let \( J \in S_q \). Define the partial portfolio loss

\[
L_J^{(m)} = \sum_{j \in J} \sum_{k \in I_j^{(m)}} l_k Y_k^{(m)},
\]

and the partial portfolio loss threshold

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\[ x^{(m)}_\mathcal{J} = \rho \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}_j} l_k \]

for some \( \rho > 0 \) that could be different from \( q \) used in the full portfolio threshold \( x_m = q \sum_{k=1}^m l_k \).

Then,

\[
\liminf_{m \to \infty} \frac{1}{m} \log \mathbb{P}(L^{(m)}_\mathcal{J} > x^{(m)}_\mathcal{J}) \geq -\frac{1}{2} \gamma_s^2.
\]

Proof. Let \( \mathcal{J} \in \mathcal{S}_q \). Define

\[
D^{(m,\epsilon)}_j = \{ z^{(m)} \in \mathbb{R}^m : (a^{(m)}_j)^T z^{(m)} \geq \tilde{s}_j b^{(m)}_j \sqrt{m} + b^{(m)}_j \Phi^{-1}(\rho + \epsilon) \}
\]

\[
= \left\{ z^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(a^{(m)}_j)^T z^{(m)} - \tilde{s}_j \sqrt{m}}{b^{(m)}_j} \right) \geq \rho + \epsilon \right\}, \quad (4.53)
\]

and

\[
D^{(m,\epsilon)}_\mathcal{J} = \bigcap_{j \in \mathcal{J}} D^{(m,\epsilon)}_j
\]

\[
= \{ z^{(m)} \in \mathbb{R}^m : (a^{(m)}_j)^T z^{(m)} \geq \tilde{s}_j b^{(m)}_j \sqrt{m} + b^{(m)}_j \Phi^{-1}(\rho + \epsilon) \text{ for all } j \in \mathcal{J} \}
\]

\[
= \{ z^{(m)} \in \mathbb{R}^m : \Phi \left( \frac{(a^{(m)}_j)^T z^{(m)} - \tilde{s}_j \sqrt{m}}{b^{(m)}_j} \right) \geq \rho + \epsilon \text{ for all } j \in \mathcal{J} \}, \quad (4.54)
\]

By a direct application of Lemma [4.3.1] we have
\[ D^{(m, \epsilon)} \neq \emptyset \iff C^{(m)} \neq \emptyset. \]

Since \( J \in S_q \) it follows that \( C^{(m)} \neq \emptyset \), and therefore \( D^{(m, \epsilon)} \neq \emptyset \) for large \( m \).

Let
\[ T_k^{(m)} = l_k(Y_k^{(m)} - p_k^{(m)}(Z^{(m)})), \]
and let
\[ S_j^{(m)} = \frac{1}{n_j^{(m)}} \sum_{j \in J} \sum_{k \in I_j^{(m)}} T_k^{(m)} \]
\[ = \frac{1}{n_j^{(m)}} \sum_{j \in J} \sum_{k \in I_j^{(m)}} l_k(Y_k^{(m)} - p_k^{(m)}(Z^{(m)})). \]

**Lemma 4.5.9.** Let \( z^{(m)} \) be an arbitrary sequence in \( \mathbb{R}^m \). Then for any \( \epsilon > 0 \)
\[ \lim_{m \to \infty} \mathbb{P} \left( |S_j^{(m)}| > \epsilon \mid Z^{(m)} = z^{(m)} \right) = 0. \]

**Proof.** Apply Lemma 4.3.5 with \( A_m = \{ Z^{(m)} = z^{(m)} \} \), \( n_m = n_j^{(m)} \).
\[ T_k^{(m)} = l_k(Y_k^{(m)} - p_k^{(m)}(Z^{(m)})). \quad (4.55) \]

- \( T_k^{(m)} \)'s are conditionally independent given \( Z \).
- \( \mathbb{E}(T_k^{(m)} \mid Z^{(m)} = z^{(m)}) = 0 \)
- We will show that
\[
\limsup_{m \to \infty} \frac{1}{(n^{(m)}_J)^2} \sum_{j \in J} \sum_{k \in J_{j}^{(m)}} \text{Var}(T_{k}^{(m)} | Z^{(m)} = z^{(m)}) = 0.
\]

Since \(Y_k^{(m)} \leq 1\),

\[
\text{Var} \left( T_{k}^{(m)} | Z^{(m)} = z^{(m)} \right) = \tilde{l}_k^2 \mathbb{E} \left( (Y_k^{(m)} - \bar{p}_k^{(m)}(Z^{(m)}))^2 | Z^{(m)} = z^{(m)} \right) \leq 4 (\bar{l})^2.
\]

Therefore

\[
\frac{1}{(n^{(m)}_J)^2} \sum_{j \in J} \sum_{k \in J_{j}^{(m)}} \text{Var} \left( T_{k}^{(m)} | Z^{(m)} = z^{(m)} \right) \leq 4 n^{(m)}_J (\bar{l})^2 \frac{1}{(n^{(m)}_J)^2}
\]

\[
\to 0.
\]

Therefore

\[
\limsup_{m \to \infty} \frac{1}{(n^{(m)}_J)^2} \sum_{j \in J} \sum_{k \in J_{j}^{(m)}} \text{Var}(T_{k}^{(m)} | Z^{(m)} = z^{(m)}) = 0.
\]

Therefore, for any \(\epsilon > 0\),

\[
\lim_{m \to \infty} \mathbb{P} \left( \left| S_{J}^{(m)} \right| > \epsilon | Z^{(m)} = z^{(m)} \right) = 0.
\]
Lemma 4.5.10. Let $z_\star^{(m)} \in D_{\mathcal{J}}^{(m,\epsilon)}$. Then

$$\lim_{m \to \infty} \mathbb{P} \left( L_\mathcal{J}^{(m)} > x_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)} \right) = 1.$$ 

Proof. Let $z_\star^{(m)} \in D_{\mathcal{J}}^{(m,\epsilon)}$. By (4.54)

$$\mathbb{E}(L_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)}) \geq (\rho + \epsilon)x_\mathcal{J}^{(m)}$$

or equivalently

$$\mathbb{E}(L_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)}) - \epsilon x_\mathcal{J}^{(m)} \geq \sum_{j \in \mathcal{J}} \sum_{k \in I_j^{(m)}} l_k$$

$$= \rho x_\mathcal{J}^{(m)}.$$ 

$$\mathbb{P} \left( S_\mathcal{J}^{(m)} > \frac{\epsilon x_\mathcal{J}^{(m)}}{n_\mathcal{J}^{(m)}} \mid Z^{(m)} = z_\star^{(m)} \right)$$

$$= \mathbb{P} \left( L_\mathcal{J}^{(m)} - \sum_{j \in \mathcal{J}} \sum_{k \in I_j^{(m)}} l_k p_k^{(m)}(Z^{(m)}) > -\epsilon x_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)} \right)$$

$$= \mathbb{P} \left( L_\mathcal{J}^{(m)} - \mathbb{E}(L_\mathcal{J}^{(m)} \mid Z^{(m)}) > -\epsilon x_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)} \right)$$

$$= \mathbb{P} \left( L_\mathcal{J}^{(m)} > \mathbb{E}(L_\mathcal{J}^{(m)} \mid Z^{(m)}) - \epsilon x_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)} \right)$$

$$\leq \mathbb{P} \left( L_\mathcal{J}^{(m)} > x_\mathcal{J}^{(m)} \mid Z^{(m)} = z_\star^{(m)} \right). \quad (4.56)$$
Let

\[ H^* = \lim_{m \to \infty} \frac{x_J^{(m)}}{n_J^{(m)}} \]

\[ = \lim_{m \to \infty} \frac{\frac{x_J^{(m)}}{m}}{n_J^{(m)} / m} \]

\[ = \frac{\rho \sum_{j \in J} H_j}{\sum_{j \in J} w_j}. \]

For \( m > M_1 \)

\[ \frac{1}{2} H^* < \frac{x_J^{(m)}}{n_J^{(m)}}. \]

Therefore for \( m > M_1 \)

\[ \mathbb{P} \left( L_J^{(m)} > x_J^{(m)} \left| Z^{(m)} = z_*^{(m)} \right. \right) \]

\[ \geq \mathbb{P} \left( S_J^{(m)} > -\epsilon \frac{x_J^{(m)}}{n_J^{(m)}} \left| Z^{(m)} = z_*^{(m)} \right. \right) \]

(by (4.56))

\[ \geq \mathbb{P} \left( S_J^{(m)} > \frac{1}{2} H^* \left| Z^{(m)} = z_*^{(m)} \right. \right) \]

\[ \geq \mathbb{P} \left( |S_J^{(m)}| < \frac{1}{2} \epsilon H^* \left| Z^{(m)} = z_*^{(m)} \right. \right) \]

\[ \to 1. \]
Let

\[ w_*^{(m)} = \inf_{z^{(m)} \in D^{(m,\epsilon)}} P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z^{(m)} \right). \]

Let \( \epsilon > 0 \). There exists \( z_{**}^{(m)} \in D^{(m,\epsilon)} \) such that

\[ w_*^{(m)} + \epsilon > P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z_{**}^{(m)} \right). \]

\[
P(L_J^{(m)} > x_J^{(m)}) \geq \int_{z^{(m)} \in D^{(m,\epsilon)}} P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z^{(m)} \right) P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)}
\]
\[
\geq w_*^{(m)} \int_{z^{(m)} \in D^{(m,\epsilon)}} P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)}
\]
\[
\geq \left( P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z_{**}^{(m)} \right) - \epsilon \right) \int_{z^{(m)} \in D^{(m,\epsilon)}} P \left( Z^{(m)} = z^{(m)} \right) dz^{(m)}
\]
\[
= \left( P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z_{**}^{(m)} \right) - \epsilon \right) P \left( Z^{(m)} \in D^{(m,\epsilon)} \right).
\]

\[
P(L_J^{(m)} > x_J^{(m)}) = \left( P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z_{**}^{(m)} \right) - \epsilon \right) P \left( Z^{(m)} \in D^{(m,\epsilon)} \right) \quad (4.57)
\]

where \( \lim_{m \to \infty} P \left( L_J^{(m)} > x_J^{(m)} | Z^{(m)} = z_{**}^{(m)} \right) = 1. \)

Next we find an expression for \( P \left( Z^{(m)} \in D^{(m,\epsilon)} \right). \)

\[
P \left( Z^{(m)} \in D^{(m,\epsilon)} \right) = P \left( (a_j^{(m)})^T Z^{(m)} \geq \bar{s}_j \sqrt{m} - b_j \Phi^{-1}(\rho + \epsilon) \right. \text{ for all } j \in J.
\]

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Suppose \( \mathcal{J} = \{j_1, j_2, \ldots, j_v\} \subseteq \{1, 2, \ldots, t\} \). Let

\[
N_j^{(m)} = (a_j^{(m)})^T Z^{(m)},
\]

and

\[
N^{(m)} = \begin{bmatrix}
N_1^{(m)} \\
N_{j_2}^{(m)} \\
\vdots \\
N_{j_v}^{(m)}
\end{bmatrix}.
\]

Then \( N_J^{(m)} \) is centered Gaussian with covariance matrix

\[
\Sigma^{(m)}_J = F_J^{(m)} (F_J^{(m)})^T,
\]

where

\[
F_J^{(m)} = \begin{bmatrix}
(a_{j_1}^{(m)})^T \\
(a_{j_2}^{(m)})^T \\
\vdots \\
(a_{j_v}^{(m)})^T
\end{bmatrix}.
\]
\[ r_j^{(m)} = \tilde{s}_j \sqrt{m} - b_j^{(m)} \Phi^{-1}(\rho + \epsilon), \]

and

\[ r^{(m)} = \begin{bmatrix} r_j^{(m)} \\ r_{j_1}^{(m)} \\ r_{j_2}^{(m)} \\ \vdots \\ r_{j_v}^{(m)} \end{bmatrix}. \]

Let

\[ b = \begin{bmatrix} \tilde{s}_{j_1} \\ \tilde{s}_{j_2} \\ \vdots \\ \tilde{s}_{j_v} \end{bmatrix}, \]

and

\[ c^{(m)} = \begin{bmatrix} b_{j_1}^{(m)} \Phi^{-1}(\rho + \epsilon) \\ b_{j_2}^{(m)} \Phi^{-1}(\rho + \epsilon) \\ \vdots \\ b_{j_v}^{(m)} \Phi^{-1}(\rho + \epsilon) \end{bmatrix}. \]

Then we have
\[ r^{(m)} = \sqrt{mb} + c^{(m)}. \]

\[
\mathbb{P}(Z^{(m)} \in D^{(m,\epsilon)}_{\mathcal{J}}) = \mathbb{P}(\mathbb{F}^{(m)}_{\mathcal{J}}Z^{(m)} \geq r^{(m)})
\]

\[
= \mathbb{P}(\mathbb{N}^{(m)}_{\mathcal{J}} \geq r^{(m)})
\]

\[
= \mathbb{P}(\mathbb{N}^{(m)}_{\mathcal{J}} \geq \sqrt{mb} + c^{(m)}).
\] (4.59)

Let

\[ \Lambda_m(a) = \log \mathbb{E}(e^{aN^{(m)}_{\mathcal{J}}}). \]

Let \[ y^{(m)} \geq \sqrt{mb} + c^{(m)}. \]

Define a new measure

\[ d\mathbb{P}^*_{\mathcal{J} m} = e^{k_m N^{(m)}_{\mathcal{J}} - \Lambda_m(k_m)} d\mathbb{P}, \]

where

\[ \Sigma^{(m)}_{\mathcal{J} k_m} = y^{(m)}. \]

By Theorem 4.3.3, under the probability measure \( \mathbb{P}^*_{\mathcal{J} m} \), \( N^{(m)}_{\mathcal{J}} \) is normally distributed with mean \[ y^{(m)} \geq \sqrt{mb} + c^{(m)}, \] and covariance matrix \( \Sigma^{(m)}_{\mathcal{J}} \).
\[ \Pr(N_J^{(m)} \in D_J^{(m, \epsilon)}) = \Pr(N_J^{(m)} \geq \sqrt{mb} + c) \]

\[ \geq \Pr(N_J^{(m)} \geq y^{(m)}) \]

\[ = E^*_m \left( \mathbb{1}_{\{N_J^{(m)} \geq y^{(m)}\}} e^{-k_m N_J^{(m)} + \log E(e^{k_m N_J^{(m)}})} \right) \]

\[ = e^{-k_m y^{(m)} + \log E(e^{k_m N_J^{(m)}})} E^*_m \left( \mathbb{1}_{\{N_J^{(m)} \geq y^{(m)}\}} \right) \]

\[ = e^{-k_m y^{(m)} + \log E(e^{k_m N_J^{(m)}})} P^*_m \left( N_J^{(m)} \geq y^{(m)} \right). \quad (4.60) \]

By Theorem 4.3.4

\[ P^*_m \left( N_J^{(m)} \geq y^{(m)} \right) \geq \frac{1}{\sqrt{2\pi |\Sigma_J^{(m)}|}} K_1 \]

for some constant \( K_1 > 0 \). Therefore

\[ \log P^*_m \left( N_J \geq y^{(m)} \right) \geq \log K_1 - \log |\Sigma_J^{(m)}| - \frac{v}{2} \log 2\pi \]

\[ \geq \log K_1 - \log m^k - \frac{v}{2} \log 2\pi \]

\[ = o(m). \quad (4.61) \]

Therefore by (4.60)

\[ \log \Pr(N_J^{(m)} \in D_J^{(m, \epsilon)}) = -k_m + \log E(e^{k_m N_J^{(m)}}) + \log P^*_m \left( N_J^{(m)} \geq y^{(m)} \right) \]

\[ = -k_m y^{(m)} + A_m(k_m) + \log P^*_m \left( N_J^{(m)} \geq y^{(m)} \right) \]
$$\geq -\Lambda^*_m(y^{(m)}) + \log \mathbb{P}^*_m \left( \mathcal{N}_{\mathcal{F}}^{(m)} \geq y^{(m)} \right)$$

$$= -\frac{1}{2} (y^{(m)})^T \mathbf{H}^{(m)} y^{(m)} + \log \mathbb{P}^*_m \left( \mathcal{N}_{\mathcal{F}}^{(m)} \geq y^{(m)} \right)$$

$$= -\frac{1}{2} (y^{(m)})^T \mathbf{H}^{(m)} y^{(m)} + o(\sqrt{m})$$

$$\log \mathbb{P}(\mathcal{N}_{\mathcal{F}}^{(m)} \in D_{\mathcal{F}}^{(m,c)}) \geq \sup_{y^{(m)} \geq \sqrt{m}b + c^{(m)}} -\frac{1}{2} (y^{(m)})^T \mathbf{H}^{(m)} y^{(m)} + o(m)$$

$$= -\frac{1}{2} \inf_{y^{(m)} \geq \sqrt{m}b + c^{(m)}} (y^{(m)})^T \mathbf{H}^{(m)} y^{(m)} + o(m)$$

$$= -\frac{1}{2} \inf_{y^{(m)} \geq \sqrt{m}b + c^{(m)}} (y^{(m)})^T \mathbf{H}^{(m)} y^{(m)} + o(m)$$

$$= -\frac{1}{2} \inf_{y \geq \sqrt{m}b + c^{(m)}} y^T \mathbf{H}^{(m)} y + o(m).$$

By using the change of variable $y = F_{\mathcal{F}}^{(m)} z$ where $z \in \mathbb{R}^m$,

$$\inf_{y \geq \sqrt{m}b + c^{(m)}} y^T \mathbf{H}^{(m)} y$$

$$= \inf_{F_{\mathcal{F}}^{(m)}z^{(m)} \geq \sqrt{m}b + c^{(m)}} (z^{(m)})^T z^{(m)}$$

$$= \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (\tilde{s}_j^{(m)})^T z \geq \tilde{s}_j \sqrt{m} + b_j^{(m)} \Phi^{-1}(\rho + \epsilon) \text{ for all } j \in \mathcal{F} \right\}$$

$$= m \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq \tilde{s}_j + b_j^{(m)} \Phi^{-1}(\rho + \epsilon) \sqrt{m} \text{ for all } j \in \mathcal{F} \right\}.$$

(4.62)

By Lemma 4.3.2

$$\lim_{m \to \infty} \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq \tilde{s}_j + b_j^{(m)} \Phi^{-1}(\rho + \epsilon) \sqrt{m} \text{ for all } j \in \mathcal{F} \right\}$$

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\[ \lim_{m \to \infty} \inf \left\{ z^T z : z \in \mathbb{R}^m \text{ and } (a_j^{(m)})^T z \geq \tilde{s}_j \text{ for all } j \in \mathcal{J} \right\} = \lim_{m \to \infty} \gamma_{\mathcal{J}}^{(m)} = \gamma_{\mathcal{J}}^2. \]  

Therefore

\[ \lim_{m \to \infty} \frac{1}{m} \left( \inf_{y \geq \sqrt{mb^{(m)}}} y^T H^{(m)} y \right) = \gamma_{\mathcal{J}}^2. \]

Therefore by (4.60)

\[ \liminf_{m \to \infty} \frac{1}{m} \log P(N^{(m)}_{\mathcal{J}} \in D^{(m, \epsilon)}_{\mathcal{J}}) \geq -\frac{1}{2} \gamma_{\mathcal{J}}^2. \]

By (4.57)

\[ \liminf_{m \to \infty} \frac{1}{m} \log P(L^{(m)}_{\mathcal{J}} > x^{(m)}_{\mathcal{J}}) \geq -\frac{1}{2} \gamma_{\mathcal{J}}^2. \]

\[ \square \]

Lower bound for the full portfolio

We now use Theorem 4.5.8 to derive the lower bound for the full portfolio.

**Theorem 4.5.11.** Under the assumptions of Theorem 4.2.4

\[ \liminf_{m \to \infty} \frac{1}{m} \log P(L_m > x_m) \geq -\frac{1}{2} \gamma^2. \]
Proof. Recall that by (4.14)

\[ \gamma_* = \min \{ \| \gamma_J \| : J \in S_q \}. \] (4.64)

Let

\[ \gamma_* = \gamma_{J_*}, \]

where \( J_* \in S_q \). Since \( qC < \sum_{j \in J_*} H_j \), there exists \( \delta > 0 \) such that \( (1 + \delta)qH < \sum_{j \in J_*} H_j \).

Let

\[ \rho = \frac{(1 + \delta)qH}{\sum_{j \in J_*} H_j} < 1. \] (4.65)

Now we have

\[ L_{J_*}^{(m)} = \sum_{j \in J_*} \sum_{k \in \mathcal{I}_j^{(m)}} l_k Y_k^{(m)} \]

and

\[ x_{J_*}^{(m)} = \rho \sum_{j \in J_*} \sum_{k \in \mathcal{I}_j^{(m)}} l_k. \]
Lemma 4.5.12. There exists $M$ such that for $m > M$:

$$\{L_{J*}^{(m)} > x_{J*}^{(m)}\} \subseteq \{L_{J*}^{(m)} > x_m\}.$$  

Proof. Note that

$$\lim_{m \to \infty} \frac{x_{J*}^{(m)}}{m} = \rho \sum_{j \in J*} H_j$$

$$= (1 + \delta) qH \quad \text{(by (4.65))}$$

$$= (1 + \delta) \lim_{m \to \infty} \frac{x_m}{m}.$$  

Let $x^* = (1 + \frac{\delta}{2}) qH$. Then $x^* < \lim_{m \to \infty} \frac{x_{J*}^{(m)}}{m}$ and $\lim_{m \to \infty} \frac{x_m}{m} < x^*$. 

Therefore for $m > M_1$ : $\frac{x_m}{m} < x^*$. Similarly for $m > M_2$ : $x^* < \lim_{m \to \infty} \frac{x_{J*}^{(m)}}{m}$.

Therefore for $m > \max\{M_1, M_2\}$ : $x_m < x_{J*}^{(m)}$.

Therefore for $m > \max\{M_1, M_2\}$ : $\{L_{J*}^{(m)} > x_{J*}^{(m)}\} \subseteq \{L_{J*}^{(m)} > x_m\}$. 

Now,

$$\mathbb{P}(L_m > x_m) \geq \mathbb{P}(L_{J*}^{(m)} > x_m)$$

$$\geq \mathbb{P}(L_{J*}^{(m)} > x_{J*}^{(m)}).$$
Therefore

\[
\liminf_{m \to \infty} \frac{1}{m} \log \mathbb{P}(L_m > x_m) \geq \mathbb{P}(L^{(m)}_{J^*} > x^{(m)}_{J^*}) \geq -\frac{1}{2} \gamma^2_{J^*}.
\]

\[
= -\frac{1}{2} \gamma^2.
\]
Chapter 5: Conclusions

In this thesis, we studied large deviations and rare event simulation algorithms for factor models of portfolio credit risk.

In Chapter 2 and Chapter 3, we studied a single factor model where the latent variable of the \( k^{th} \) obligor is given by

\[
X_k = aZ + b\varepsilon_k, \tag{5.1}
\]

where \( Z \) and \( \varepsilon_k \) are independent random variables, \( \varepsilon_k \) are i.i.d., and \( b = \sqrt{1 - a^2} \) with \( 0 < a < 1 \). All three random variables \( X_k, Z \) and \( \varepsilon_k \) are allowed to take general distributions.

Chapter 2 dealt with sharp large deviation asymptotics. Logarithmic large deviations have been studied for the model in (5.1) by Glasserman et al. in [17], [15], assuming \( X_k, Z \) and \( \varepsilon_k \) are distributed as \( N(0, 1) \). Our work is inspired by Glasserman and his group but differs in the following ways:

1. We consider general probability distributions for \( X_k, Z \) and \( \varepsilon_k \).

2. We derive sharp large deviation asymptotic where as Glasserman et al. (see [17], [15]) derive logarithmic large deviations.

In chapter 2, we considered two probability regimes: large loss threshold regime and small default probability regime. Our approach has been to prove general theorems without assuming particular distributions for \( X_k, Z \) and \( \varepsilon_k \). These results are given in Theorem 2.4.1.
and Theorem 2.4.2 for the large loss threshold regime, and in Theorem 2.6.1 and Theorem 2.6.2 for the small default probability regime. These theorems hypothesize certain conditions that need to be satisfied by the distributions of $X_k, Z$ and $\varepsilon_k$. Then, once we specify particular distributions for $X_k, Z$ and $\varepsilon_k$, precise asymptotics would follow upon verifying these conditions. Some of the probability distributions that we considered are Gaussian, Gamma, Exponential and Stretched Exponential. But other distributional assumptions are also possible.

In Chapter 3 we developed a rare event simulation algorithm and proved its logarithmic efficiency. This algorithm is the same algorithm developed by Glasserman et al. in [17], assuming all random variables are Standard Gaussian. Our contribution is that we prove its logarithmic efficiency for general distribution functions. Logarithmic efficiency is established under two limiting regimes: large loss threshold regime and small default probability regime. Once again, our approach is to prove two general theorems without considering any particular distributions for the random variables $X_k, Z$ and $\varepsilon_k$. These theorems are given in Theorem 3.5.1 and Theorem 3.6.1. These theorems hypothesize certain conditions that need to be satisfied by the distributions $X_k, Z$ and $\varepsilon_k$. Then, once we specify particular distributions for $X_k, Z$ and $\varepsilon_k$ the results would follow upon verifying these conditions.

In Chapter 4 we derived logarithmic large deviations for the multifactor Gaussian copula model with finite number of obligor types, previously considered by Glasserman et al. [15]. Under their model, if the $k^{th}$ obligor is of type $j$, its latent variable is given by

$$X_k = (a_j)^T Z + b_j \varepsilon_k,$$  \hspace{1cm} (5.2)

where $a_j \in \mathbb{R}^d$ with $0 < a < \|a_j\| < \bar{a} < 1$, $Z$ is a $d$ dimensional standard normal random vector, $b_j = \sqrt{1 - (a_j)^T a_j}$, and $\varepsilon_k$'s are independent standard normal random variables.
Glasserman et al. [15] derive logarithmic large deviations for the portfolio loss letting the number of obligors \( m \) go to infinity, but holding the number of factors fixed at \( d \). They also divide the set of obligors into \( t \) types, where \( t \in \mathbb{N} \). We expand on the results by Glasserman et al. [15], and let the number of factors go to infinity together with the number of obligors. To this end we change the model in (5.2) to accommodate \( Z \) to be an \( m \) dimensional vector. More specifically, (5.2) is replaced by

\[
X_k^{(m)} = (a_j^{(m)})^T Z^{(m)} + b_j^{(m)} \varepsilon_k,
\]

where \( a_j^{(m)} \in \mathbb{R}^m \) with \( 0 < a < \|a_j^{(m)}\| < \bar{a} < 1 \), \( Z^{(m)} \) is a \( m \) dimensional standard normal random vector, \( b_j^{(m)} = \sqrt{1 - (a_j^{(m)})^T a_j^{(m)}} \), and \( \varepsilon_k \) are independent standard normal random variables.

We have considered factor models of portfolio credit risk under a very general setting. Our Theorems have the potential to be used for many other distributions than what we have considered in this thesis, upon verifying our distributional assumptions. We next list some of the future work that may lead from this thesis.

5.1 Future Work

1. Rare event simulation for higher dimensional multifactor Gaussian Copula

Glasserman et al. (see [16]) developed an importance sampling algorithm for the multifactor Gaussian Copula. They considered a fixed number of risk factors. Our large deviation results that let the number of factors go to infinity would enable us to prove that the importance sampling algorithm by Glasserman et al. is asymptotically
optimal even as the number of factors go to infinity.

2. Deriving Large Deviations for the t-Copula

Copulas are an indispensable tool when considering linear factor models. Large deviations for the multifactor Gaussian Copula was proved by Glasserman et al (see [15]). Gaussian Copula (see [22]) exhibits tail independence and the t-Copula (see [20] and [2]) has been considered as an alternative. We could extend our large deviation results to the t-Copula. We anticipate that extension to other copulas are also possible in a unified framework.

3. Deriving Sharp Asymptotics for the Expected Shortfall

Our large deviation and rare event simulation algorithms can be used to calculate Value at Risk (VaR). The use of VaR has come under criticism primarily because it is not a coherent risk measure (see [3]). Expected Shortfall has been suggested as an alternative (see [1]). Our results can be extended to deriving sharp large deviations for Expected Shortfall.
Bibliography


Curriculum Vitae

Hasitha de Silva received his Bachelor of Science (Mathematics) from University of Colombo, Sri Lanka, in 2007. He was employed as a Quantitative Analyst at Amba Research (now known as Copal Amba - A Moody’s Analytics Company), from 2007-2009. He received his Master of Science (Mathematics) from George Mason University in 2013.