CYCLE BASES OF DIRECTED GRAPHS

by

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Dedication

It is with genuine gratitude and warm appreciation that I dedicate this thesis to my family, whose continued support helped to make this goal attainable. Their constant encouragement and undying belief in me was a source of inspiration.

Acknowledgments

I am grateful to those with whom I had the pleasure to work during this and related projects. I extend special thanks to Dr. Morris for his patience and enthusiasm in guiding my research and to Dr. Agnarsson and Dr. Lawrence for their interest and participation in my work.

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Abstract

CYCLE BASES OF DIRECTED GRAPHS

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Each cycle of a directed graph can be written as a linear combination of the circuits of a cycle basis for that directed graph. We define two new classes of cycle bases and show how each relates to the known classes of strictly fundamental cycle bases, zero-one cycle bases and integral cycle bases. We provide examples showing the significance of the Möbius band to constructing directed graphs, the bases of which are in some of these classes and not in other classes.

Chapter 1: Introduction

The study of cycles and the characterization of cycle bases is a graph theory topic that continues to grow and develop. Historically, graphs made an early appearance in puzzles and games, yet today studying graphs is of interest to those in fields as varied as mathematics, computer science, management, engineering, economics, information technology and biology. The circuits of a graph are easily recognized as the building blocks of cycles so that cycle bases of directed graphs are important not only to graph drawing but also to network analysis, chemical analysis and periodic scheduling.

1.1 Preliminary Definitions

Central to the study of cycle bases is the understanding of an *undirected graph* G which is a pair of two types of objects denoted by $G = (V, E)$ where V is a finite set and E is a family of unordered pairs of elements of V. The set V contains *vertices* of the graph G, sometimes called *nodes* or *points* of G. The unordered pairs of vertices are the *edges* of the set E. The vertices v and w of an edge $\{v, w\}$ can also be called the *endpoints* of the edge. Notice that a pair of vertices $\{v, w\}$ may occur more than once in E and is, in that case, called a *multiple edge.* Thus distinct edges may be represented in E by the same pair of vertices. A *loop* is an edge of the form $\{v, v\}$ where $v = w$. A graph is *simple* when E contains no multiple edges and no loops. We speak of the edge $\{v, w\}$ being *incident to* the vertices v and w. Likewise the vertices v and w are said to be incident to the edge $\{v, w\}$. A vertex which is not incident to any edge is considered an *isolated vertex.* The *degree* of a vertex v, denoted $deg(v)$, is the number of edges that are incident to the vertex v. Another way to define the degree of a vertex is the number of times the vertex occurs as an endpoint of an edge. Thus, the edge $\{v, w\}$ is incident to the vertex v so it contributes one to the $deg(v)$ while the edge $\{v, v\}$ is a loop joining v to itself and contributes two to the degree of v. The set of edges which are incident to vertex v is denoted by $\delta(v)$.

A graph is an abstract mathematical concept, however it can be given a geometric representation as a diagram in the plane. We represent each vertex with a point or a dot, and we represent each edge by a line segment or curved segment that joins a pair of dots. A graph is called *planar* if it can be represented in the plane so that no two edges meet or cross except at a vertex. Figure 1.1 is an example of an undirected graph that is planar.

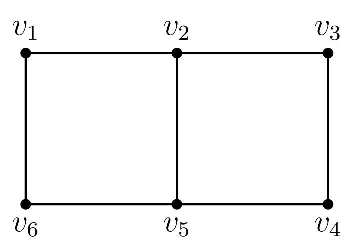

Figure 1.1: Undirected Graph

Suppose now that direction is assigned to the edges of an undirected graph G. A *directed graph* $D = (V, A)$ is a pair of two types of objects where V is a finite set and A is a family of *ordered pairs* of elements of V. A directed graph is often called a *digraph*, and the elements of V are called the *vertices* or *nodes* or *points* of the directed graph D. The elements of A are called the *arcs* or *directed edges* of D. We think of the arc (v, w) as *leaving* v and *entering* w. Consequently, we refer to the vertex v as the *tail* of the arc and w as the *head* of the arc. It is often useful to denote an edge e by vw. We will sometimes use the notation vw to abbreviate both $\{v, w\}$ and (v, w). The notions of *multiple edge*, *simple graph* and *loop* are the same for directed graphs as they are for undirected graphs. A vertex v of a directed graph has two degrees. The *indegree* of a vertex v denoted $indeg(v)$ is the number of arcs that enter the vertex v or equivalently the number of times the vertex v is the head of an arc. The *outdegree* of a vertex v denoted $outdeg(v)$ is the number of arcs that leave the vertex v or equivalently the number of times the vertex v is the tail of an arc. The set of arcs entering v is denoted $\delta^-(v)$ while the set of arcs leaving v is denoted $\delta^+(v)$. For a graph, whether undirected or directed, we will use n to represent the number of vertices or nodes and m to represent the number of edges or arcs. So we can write $n = |V|$ and $m = |E|$ or $m = |A|$.

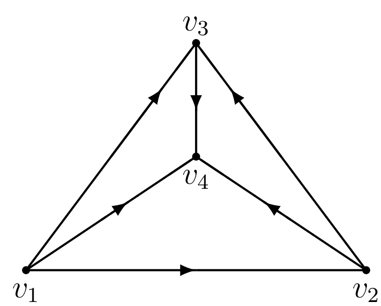

Figure 1.2: Directed Graph

It should be noted that given a directed graph D we can omit the direction of its arcs and thus obtain an undirected graph $G = G(D)$. This graph $G(D)$ is called the *underlying graph of* D, and every digraph has exactly one underlying graph. In contrast, given an undirected graph G we can obtain a directed graph D by arbitrarily assigning direction to each edge $\{v, w\}$ of E and replacing it with either (v, w) or (w, v). If $\{v, w\}$ is a multiple edge then indeed some edges $\{v, w\}$ can be replaced with (v, w) and some can be replaced with (w, v). The resulting digraph D is considered an *orientation* of the graph G. Given this orientation D should we reverse the direction of as few as one of its arcs then the result is another orientation D' of the graph G. An undirected graph can have many different orientations. Figure 1.2 shows an orientation of the *complete graph* K_4 on four vertices, a simple graph with all possible edges.

A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$, $E' \subseteq E$ and the endpoints of an edge $e \in E'$ are the same as its endpoints in G. In a graph, a sequence of edges

$$\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$$

is called a *walk* from vertex x_0 to vertex x_m. If there is a walk from vertex x_0 to vertex x_m then we say the two vertices are *joined.* We call x_0 the *initial* vertex and x_m the *final* vertex. The edges of a walk may repeat, but if the edges of a walk are distinct then the walk is called a *trail.* Furthermore, if a walk has distinct edges and distinct vertices (except for the initial and final vertices), then the walk is called a *path.* If $x_0 \neq x_m$ then the walk is considered *open*, and if $x_0 = x_m$ the walk is *closed.* We define the *length of a walk* to be the number of its edges. A walk that begins with initial vertex x_0 and ends with final vertex x_m may also be denoted as follows

$$x_0 - x_1 - x_2 - \cdots - x_m$$

which would be a walk of length m. The concepts of walk, trail and path as well as length, initial vertex and final vertex are the same for a directed graph. This notation

$$x_0 \to x_1 \to x_2 \to \cdots \to x_m$$

can be used to describe a walk in a directed graph with initial vertex x_0 and final vertex x_m. Let $i = 0, 1, \ldots, m-1$, and notice that while the sequence of vertices in the walk of a digraph may be $\cdots \to x_i \to x_{i+1} \to \cdots$ the arc may appear as either (x_i, x_{i+1}) or (x_{i+1}, x_i) in the sequence of arcs. Thus, the arcs do not have to all be directed forward.

An undirected graph is *connected* if for each pair of vertices v and w there is a walk joining v and w. A directed graph is connected if its underlying undirected graph is connected. A connected graph that contains no closed paths is called a *tree*. Thus, a tree with n vertices has $n-1$ edges. Given a subgraph $G' = (V', E')$ of a connected graph $G = (V, E)$, G' is a *spanning tree* of G if $V' = V$ and G' is a tree.

1.2 Cycle Bases Definitions

Precise definitions in our study as in all fields of mathematics are essential. The notion of cycle for example can have a slightly different meaning in the study of graph theory than in the study of matroids. We define a *cycle in an undirected graph* to be a subgraph such that each vertex has even degree. Notice that the definition of cycle does not require connectivity; however, each connected component of a cycle in an undirected graph can be thought of as a closed trail. A *circuit* is a cycle that is connected and each of its vertices has degree two. We might also think of a circuit in an undirected graph as a closed path. Notice in [1] that the graph theorist may find these definitions of cycle and circuit reversed.

We can represent a cycle of a directed graph with a vector where the entries of the vector are indexed by the arcs of the digraph. Let k be a field. We represent the set of cycles by a set of vectors in k^A indicating that values from k are assigned to the arcs in A. So k^A contains $|A|$-tuples that are indexed by the arcs of the digraph. We will use the convention that the arcs are ordered lexicographically. We define a *k-cycle* C in a digraph D as a vector in k^A such that for any vertex v in the cycle we have

$$\sum_{a \in \delta^+(v)} C(a) = \sum_{a \in \delta^-(v)} C(a)$$

where $C(a)$ denotes the component of cycle C indexed by arc a. This constraint, called *flow conservation*, means that at any vertex in the cycle the total flow entering v is equal to the total flow leaving v. The word *flow* provides a visual for the direction of the arcs of the directed graph together with the components of the cycle indexed by those arcs at the vertices of the digraph. The constraint says that at each vertex of the digraph the sum of the components of the cycle that are indexed by the arcs leaving the vertex is equal to the sum of the components of the cycle that are indexed by the arcs entering the vertex. For example, consider the cycle, using lexicographic order,

$$C = (C(v_1v_2), C(v_1v_3), C(v_1v_4), C(v_2v_3), C(v_2v_4), C(v_3v_4))$$

$$= (3, -1, -2, 2, 1, 1)$$

of the digraph D in Figure 1.2. Look at the vertex v_3 of D. The set of arcs leaving v_3 is $\delta^+(v_3) = \{v_3v_4\}$, and the set of arcs entering v_3 is $\delta^-(v_3) = \{v_1v_3, v_2v_3\}$. To determine the flow leaving v_3 we have $\sum_{a\in\delta^+(v_3)} = C(v_3v_4) = 1$. The flow entering v_3 is $\sum_{a\in\delta^-(v_3)} = C(v_1v_3) + C(v_2v_3) = -1 + 2 = 1$ which satisfies the constraint. A similar check can be made at each vertex in order to determine the desired flow conservation for the cycle C.

The *support* of a cycle is the set of arcs a such that the component $C(a)$ is nonzero. We denote the support of a cycle C by $\underline{C}$. Given a cycle C if $C(a) \in \{-1, 0, +1\}$ for all arcs a then C is a *simple* cycle. A simple cycle is a *circuit* if its support is connected and non-empty, and for any vertex $v \in V$ there are either two arcs in the support incident to v or no arcs in the support incident to v. A circuit C of D uses arcs of A in the *forward* and *backward* direction in that we think of *traversing* the arcs of the digraph with the result being a *directed circuit* in which all arcs point in the same direction. Notice that there are two ways to traverse any circuit. The *incidence vector* of a circuit C is a vector in $\{-1, 0, +1\}^A$ with an entry $+1$ if the arc is used in the forward direction, an entry -1 if the arc is used in the backward direction and an entry 0 if the arc is not used in C at all. As an example, the directed circuit C_1 appearing in Figure 1.3 with sequence of vertices $v_1 \to v_4 \to v_3 \to v_1$ shows all arcs pointing in the same direction. The circuit

$C_1 = (0,-1,1,0,0,-1)$ of the digraph D of Figure 1.2 indicates the circuit uses arcs v_1v_3 and v_3v_4 in the backward direction and arc v_1v_4 in the forward direction and does not use arcs $(v_1,v_2),(v_2,v_3),(v_2,v_4)$ at all.

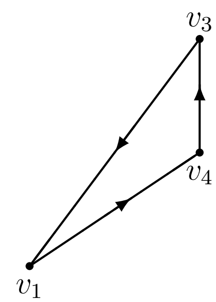

Figure 1.3: Directed Circuit

We can form the *node-arc incidence matrix* for the digraph $D = (V, A)$, a $|V|\text{x}|A|$ matrix with the nodes as labels for the rows and with the arcs as headers for the columns. For each entry of the matrix (v, a) we place a 1 if the node is the tail of a, -1 if the node is the head of a and 0 otherwise. So we can produce the node-arc incidence matrix

$$\begin{array}{c c} & \begin{array}{cccccc} v_1v_2 & v_1v_3 & v_1v_4 & v_2v_3 & v_2v_4 & v_3v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

for the orientation of K_4 in Figure 1.2. The null space of the node-arc incidence matrix is the *cycle space* of D. In the example it is clear that the vector $C_2 = (1,0,-1,1,0,1)$ is in the null space of the matrix. By inspection we see that the sequence of nodes

$v_1 \to v_2 \to v_3 \to v_4 \to v_1$ that the vector represents is a circuit of the digraph and thus is in the cycle space of D. We denote the k-cycle space of a graph by $\mathcal{C}_k(D)$ where

$$\mathcal{C}_k(D) = \{C \mid C \text{ is a } k\text{-cycle of } D\}$$

which forms a vector space over k. We assume the field to be the rational numbers $\mathbb{Q}$ unless stated otherwise. When the associated field is understood we may drop the decoration from the notation and write simply cycle for k-cycle and $\mathcal{C}(D)$ for the cycle space.

Suppose all components of a cycle C are integral. Permitting a slight abuse of notation, we call C a *$\mathbb{Z}$-cycle*, thought of as $C \in \mathcal{C}_{\mathbb{Q}}(D) \cap \mathbb{Z}^A$. Now let $D = (V, A)$ be a directed graph, and let $G = G(D)$ be the underlying undirected graph. For any $\mathbb{Z}$-cycle C of D, the *projection* of C to $\mathbb{Z}_2^A$ is defined to be the $\mathbb{Z}_2$-cycle $\pi(C)$ with $\pi(C(a)) = (C(a) \bmod 2)$ for $a \in A$. A cycle of an undirected graph G may be *lifted* from G to an orientation D of G. Suppose C' is a cycle in G. We call C with $C(a) \in \{-1, 0, +1\}$ for $a \in A$ a *lifting* of C' if C projects to C'.

Notice that for any $\mathbb{Q}$-cycle we can find a $\mathbb{Z}$-cycle that is a scalar multiple of the $\mathbb{Q}$-cycle. Let C be a cycle in $\mathbb{Q}^A$ so that $C = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}\right)$ for $a_i, b_i \in \mathbb{Z}, b_i \neq 0$, $i \in \{1, \ldots, m\}$. Determine the least common multiple of $\{b_1, b_2, \ldots, b_m\}$ and let $\mathrm{LCM}(b_1, b_2, \ldots, b_m) = l$. Then we can write the $\mathbb{Z}$-cycle K here as $K = lC = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ where $\alpha_i = l \cdot \frac{a_i}{b_i} \in \mathbb{Z}$.

Let $C_1 \ldots C_k$ be cycles of an undirected graph G. The *sum of the cycles*, $C_1 + \cdots + C_k$, consists of all edges that are found in an odd number of C_i's. The sum is again a cycle. When we represent these cycles as vectors we notice that when summing the components of the cycles an odd number of ones will sum to one while an even number of ones will sum to zero since we are performing addition in $\mathbb{Z}_2$. As an example we look at the graph in Figure 1.1. We order the edges lexicographically. It is easy to see for circuits $C_1 = (1, 1, 0, 1, 0, 0, 1)$ and $C_2 = (0, 0, 1, 1, 1, 1, 0)$ of this undirected graph that their sum $C_3 = (1, 1, 1, 0, 1, 1, 1)$ is again a cycle. While we work with linear dependences over $\mathbb{Q}$ for directed graphs we work with linear dependences over $\mathbb{Z}_2$ for undirected graphs.

We define an *undirected cycle basis* as a minimal set of circuits such that any cycle can be written as a sum of the circuits in the basis. A *k-cycle basis* is a set of circuits forming a basis of the cycle space. For a connected digraph any cycle basis will consist of $\nu := m - n + 1$ circuits as we will see in Theorem 1.1.

Unifying characteristics of cycle bases have led to the classification of at least seven classes. Liebchen and Rizzi in [5] define a *directed cycle basis* of a directed graph D as a set of circuits whose incidence vectors form a basis over $\mathbb{Q}$ of $\mathcal{C}(D)$. The definitions included here of the other primary classes of cycle bases are based mainly on the definitions found in [4].

Definition 1.1. A directed cycle basis $B = \{C_1, C_2, \ldots, C_\nu\}$ of a graph D is called a(n):

1. *undirected cycle basis* if the projections $\pi(C_i)$ of the basic circuits C_i onto the underlying undirected graph $G(D)$ constitute a cycle basis of $G(D)$;

2. *integral cycle basis* if each $\mathbb{Z}$-cycle C of D can be written as an *integer* linear combination of circuits in B, that is

$$\exists \lambda_i \in \mathbb{Z} : C = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_\nu C_\nu;$$

3. *zero-one cycle basis*, if for every cycle C' of the undirected graph $G(D)$ there exists a simple cycle C of D that projects to C' and can be written as a linear combination of the circuits in B with coefficients in $\{-1, 0, +1\}$, that is

$$\exists C \; (\pi(C) = C' \wedge \exists \lambda_i \in \{-1, 0, +1\} : C = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_\nu C_\nu);$$

4. *weakly fundamental cycle basis* if there exists some permutation σ such that

$$\underline{C}_{\sigma(i)} \setminus (\underline{C}_{\sigma(1)} \cup \cdots \cup \underline{C}_{\sigma(i-1)}) \neq \emptyset, \forall i = 2, \ldots, \nu;$$

5. *strictly fundamental cycle basis* if there exists some spanning forest $T \subseteq A$ such that $B = \{C_{T,a} \mid a \in A \setminus T\}$, where $C_{T,a}$ denotes the unique circuit with support in $T \cup \{a\}$ and with $C_{T,a}(a) = +1$;

6. *planar cycle basis* if each arc is contained in at most two basic circuits and the basis is undirected.

1.3 Preliminary Theorems

As noted earlier, the cycle space of a graph is the null space of the node-arc incidence matrix. Its dimension then indicates the number of circuits in a cycle basis for a graph. Theorem 1.1 which is proven in [4] gives the dimension of the cycle space.

Theorem 1.1. ([4]) The dimension of the k-cycle space of a graph G is given by

$$\nu = m - n + K$$

where K denotes the number of connected components of G.

An important tool in the study of the cycle bases of a directed graph is the cycle matrix. We define the *cycle matrix* that corresponds to a directed basis B of a directed graph D as an $m \times \nu$ matrix with columns that are the incidence vectors of the circuits of the basis and rows that are indexed by the arcs of the graph. The cycle matrix is determined uniquely up to the arrangement of the incidence vectors in the matrix and the arrangement of the arcs in the matrix.

Let Γ be the cycle matrix corresponding to a directed cycle basis B of a directed graph. Choose a spanning forest for the graph. Let Γ' be the $\nu \times \nu$ submatrix of Γ formed by the rows corresponding to the ν non-tree arcs. We will see that this $\nu \times \nu$ submatrix Γ' plays a role in the classification of B. Proofs of Lemma 1.1 and Lemma 1.2 can be found in [4]. These two lemmas lead to the definition of the determinant of a directed cycle basis.

Lemma 1.1. ([4]) Let B be a directed cycle basis of a directed graph and let Γ be the corresponding cycle matrix. A $\nu \times \nu$ submatrix Γ' of Γ is nonsingular if and only if the rows of Γ' correspond to the non-tree arcs of some spanning forest of D.

Lemma 1.2. ([4]) Let B be a directed cycle basis of a directed graph. Let Γ be the cycle matrix corresponding to B. Let A_1 and A_2 be two nonsingular $\nu \times \nu$ submatrices of Γ. Then $\det A_1 = \pm \det A_2$.

Definition 1.2. ([4]) Let B be a directed cycle basis containing ν circuits for a directed graph D. Consider the cycle matrix Γ. Let Γ' be the $\nu \times \nu$ submatrix of Γ that arises when deleting the arcs of some spanning forest of D. We define the determinant of the basis to be:

$$\det B = |\det \Gamma'|.$$

Considerable work has been done to relate one class of cycle bases to another class. Example 1.1 highlights the relationship between the class of directed cycle bases and the class of undirected cycle bases. That is, not all directed cycle bases are undirected. In this example we feature Wagner's graph, Figure 1.4, a digraph that we will explore again in Section 3.2. We present a set of $\mathbb{Z}$-cycles for Wagner's graph that are linearly independent over $\mathbb{Q}$ but whose set of projections is linearly dependent over $\mathbb{Z}_2$. While we don't describe them all we have determined 21 circuits of Wagner's graph which we've numbered $C_1, \ldots, C_{21}$.

Example 1.1.

By Theorem 1.1 we know there are $\nu = 12 - 8 + 1 = 5$ circuits in any cycle basis for Wagner's graph, V_8. Let's construct a cycle matrix Γ_1 for the directed cycle basis B_1 of V_8 which includes the four 4-gons, C_1, C_2, C_3 and C_4 plus the circuit which uses the eight edges "around" the graph which we call C_{17}. We form the $\nu \times \nu$ submatrix Γ_1' from rows

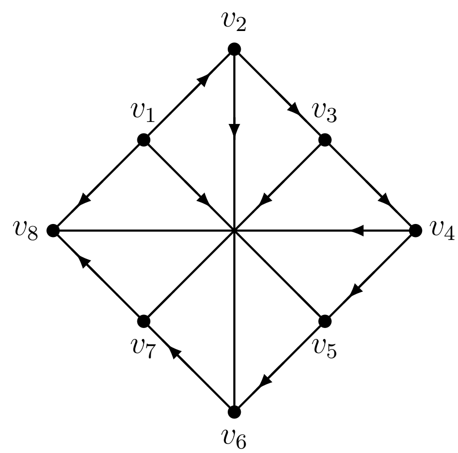

Figure 1.4: Wagner's Graph V_8

$(v_1, v_5), (v_1, v_8), (v_2, v_6), (v_3, v_7)$ and (v_4, v_8) of Γ_1 to find the det $B_1 = |\det \Gamma_1'| = 2$. The circuits of Γ_1 are linearly independent, however, the projections of the circuits are linearly dependent. Notice that $\pi(C_1) + \pi(C_2) + \pi(C_3) + \pi(C_4) = \pi(C_{17})$. Alternately, if we were to display the matrix of the projections of the circuits of Γ_1 we would see that each row of that matrix contains exactly two entries equal to one with the remaining entries equal to zero. This confirms again that the projections of the circuits of B_1 are linearly dependent over $\mathbb{Z}_2$. Thus, B_1 is an example of a directed cycle basis which is not an undirected cycle basis.

$$
\mathbf{\Gamma_1} = \begin{array}{c} \\ (v_1,v_2) \\ (v_1,v_5) \\ (v_1,v_8) \\ (v_2,v_3) \\ (v_2,v_6) \\ (v_3,v_4) \\ (v_3,v_7) \\ (v_4,v_5) \\ (v_4,v_8) \\ (v_5,v_6) \\ (v_6,v_7) \\ (v_7,v_8) \end{array}
\begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_{17} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & -1 & 1 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1
\end{pmatrix} \end{array}
$$

In Proposition 1.1 we show that when a set of linearly dependent $\mathbb{Z}$-cycles projects to $\mathbb{Z}_2^E$ the result is a set of linearly dependent cycles.

Proposition 1.1. Linear dependence of a set of $\mathbb{Z}$-cycles implies linear dependence of the set of their projections.

Proof. Let $\{C_1, C_2, \dots, C_t\}$ be a set of $\mathbb{Z}$-cycles that are linearly dependent over $\mathbb{Q}$. Then there exist coefficients $\lambda_i \in \mathbb{Q}$ not all zero such that $\sum_{i=1}^t \lambda_i C_i = \mathbf{0}$ where $\mathbf{0}$ is the zero vector.

We want to show that the set of the projections $\{\pi(C_1), \pi(C_2), \dots, \pi(C_t)\}$ is linearly dependent over $\mathbb{Z}_2$. Let u be the least common multiple of the denominators of the coefficients $\lambda_i \in \mathbb{Q}$. Then $u \sum_{i=1}^t \lambda_i C_i = u\mathbf{0}$ will result in a linear combination of the $\mathbb{Z}$-cycles which we can write as $\sum_{i=1}^t \gamma_i C_i = \mathbf{0}$ where $\gamma_i = u\lambda_i$ for each i with $\gamma_i \in \mathbb{Z}$ not all zero. If the coefficients γ_i are even for all $i \in \{1, \ldots, t\}$ then we let $v = 2^s$ where s is the smallest positive exponent of the powers of 2 in the prime factorization of each γ_i. Then $\frac{1}{v}\sum_{i=1}^{t} \gamma_i C_i = \frac{1}{v}\mathbf{0}$ can be written as $\sum_{i=1}^{t} \alpha_i C_i = \mathbf{0}$ where $\alpha_i = \frac{1}{v}\gamma_i$ for each i. Thus we can assume there exist coefficients $\alpha_i \in \mathbb{Z}$ not all even such that $\sum_{i=1}^{t} \alpha_i C_i = \mathbf{0}$.

Recall that $\pi(C_i) + \pi(C_i) = \mathbf{0}$. So a linear combination of $\mathbb{Z}_2$-cycles will have coefficients in $\mathbb{Z}_2$. Therefore the linear combination of the projections of the $\mathbb{Z}$-cycles becomes

$$\sum_{i=1}^{t} \alpha_i \pmod 2\ \pi(C_i) = \mathbf{0} \pmod 2$$

where $\alpha_i \pmod 2$ are not all zero. We conclude that the set $\{\pi(C_1), \pi(C_2), \ldots, \pi(C_t)\}$ of $\mathbb{Z}_2$-cycles are linearly dependent. $\square$

Much can be said about the different classes of cycle bases by observing properties of the corresponding cycle matrices. For example, the rows and columns of the cycle matrix of a basis that is strictly fundamental can be permuted so that the last ν rows contain a $\nu \times \nu$ identity matrix.

Throughout our study the conclusions made regarding the classification of a cycle basis rely on Theorem 1.2 proven in [4].

Theorem 1.2. ([4]) Let B be a directed cycle basis with cycle matrix Γ. Then:

1. B is undirected if and only if det B is odd.
2. B is integral if and only if det B is one.
3. B is zero-one if and only if Γ is totally unimodular.
4. B is weakly fundamental if and only if Γ can be permuted so as to have an invertible upper triangular ν x ν matrix in its last ν rows.

5. B is strictly fundamental if and only if Γ can be permuted so as to have the ν x ν identity matrix in its last ν rows.

6. B is a 2-basis if and only if B is an undirected cycle basis and Γ has at most two non-zero entries per row.

Theorem 1.3 proven in [8] provides further characterization of a totally unimodular matrix.

Theorem 1.3. ([8]) Let A be a matrix with entries 0, +1 or -1. Then the following are equivalent:

1. A is totally unimodular, that is, each square submatrix of A has determinant 0 +1 or -1;

2. each collection of columns of A can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries only 0, +1 and -1.

Chapter 2: Results

From a problem found in [4] we explore the class of zero-one cycle bases and how it relates to two new classes of cycle bases. We extend this study so that we can relate the new classes of cycle bases to the classes of strictly fundamental cycle bases and integral cycle bases.

2.1 Zero-one Basis

Open Problem 3 in [4] states

> The definition of zero-one bases may seem strange. It would be equally natural to require that every circuit (every simple cycle) is a linear combination of the basic circuits with coefficients in $\{-1, 0, +1\}$.

As we consider this relationship between the definition of a zero-one basis and the definitions offered in Open Problem 3 we define two new classes of cycle bases. In these new classes of cycle bases the coefficients for the basic circuits of the linear combinations can be found in the box $[-1, 1]^{\nu}$ so we will dub the new classifications *circuit boxed* and *simple cycle boxed*.

Definition 2.1. A directed cycle basis $B = \{C_1, \ldots, C_\nu\}$ of a graph D is called a *circuit boxed cycle basis* if every circuit C of D can be written as a linear combination of the circuits in B with coefficients in $\{-1, 0, +1\}$; that is,

$$\exists \lambda_i \in \{-1, 0, +1\} : C = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_\nu C_\nu.$$

Definition 2.2. A directed cycle basis $B = \{C_1, \dots, C_\nu\}$ of a graph D is called a *simple cycle boxed cycle basis* if every simple cycle C of D can be written as a linear combination of the circuits in B with coefficients in $\{-1, 0, +1\}$; that is,

$$\exists \lambda_i \in \{-1, 0, +1\} \,:\, C = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_\nu C_\nu.$$

We will prove the sequence of implications for the classes of cycle bases that appear in the statement below

$$\text{strictly} \implies \text{simple cycle boxed} \implies \text{zero-one} \implies \text{circuit boxed} \implies \text{integral} \tag{2.1}$$

and decide which implications are certainly not equal. For example, we know that zero-one $\not\Longrightarrow$ simple cycle boxed. We begin with the first implication and show that given a cycle basis which is strictly fundamental we can be sure that it is a simple cycle boxed cycle basis as well.

Theorem 2.1. Every strictly fundamental cycle basis is a simple cycle boxed basis.

Proof. Let $B = \{C_1, \dots, C_\nu\}$ be a directed cycle basis of a directed graph D, and let Γ be the cycle matrix for B. Assume B is a strictly fundamental basis. We know by Theorem 1.2 since B is strictly fundamental we can permute the rows and columns of Γ so as to have the $\nu \times \nu$ identity matrix in its last ν rows. We will call this permutation Γ^*.

Here the $(m-\nu) \times \nu$ submatrix that sits above the identity matrix in Γ^* contains entries c_{11} to $c_{(m-\nu)\nu}$ that are all from the set $\{-1, 0+1\}$.

$$
\mathbf{\Gamma}^* = \begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_{m-\nu} \\ e_{m-\nu+1} \\ e_{m-\nu+2} \\ \vdots \\ e_m \end{array}
\begin{array}{c}
\begin{array}{cccc} C_1 & C_2 & \cdots & C_\nu \end{array} \\
\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1\nu} \\
c_{21} & c_{22} & \cdots & c_{2\nu} \\
\vdots & \vdots & \ddots & \vdots \\
c_{(m-\nu)1} & c_{(m-\nu)2} & \cdots & c_{(m-\nu)\nu} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\end{array}
$$

The matrix Γ^* is again an $m \times \nu$ matrix with m arcs and ν basic circuits of B. We label the rows e_1 to e_m for the m arcs and label the columns C_1 to C_ν for the ν circuits. It is clear that every circuit in the basis contains an arc that is contained in no other circuit of the basis.

Let C be a simple cycle of the directed graph D. By definition $C(e) \in \{-1, 0+1\}$ for all arcs e of C. Consider the m^{th} entry of the $m \times 1$ column vector representing simple cycle C. If $C(e_m) = -1$ then the circuit C_ν will be multiplied by a coefficient of -1 in the linear combination of circuits of B to obtain -1 in the m^{th} entry of the vector for the simple cycle C. Similarly we see that if $C(e_m) = 0$ then the circuit C_ν will carry a coefficient of 0 in the combination of basic circuits to obtain a 0 in the m^{th} entry of the simple cycle C. Lastly, should $C(e_m) = +1$ then the circuit C_ν will carry a coefficient of $+1$ in the linear combination of circuits of B to obtain $+1$ in the m^{th} entry of the vector for the simple cycle C. Thus, the possible coefficients for the circuit C_ν are in the set $\{-1, 0, +1\}$.

Next, we consider the entry that precedes the m^{th} entry (the entry $m-1$) of the $m \times 1$

column vector representing C and apply the above strategy to determine the coefficient of the circuit $C_{\nu-1}$ in the combination of basic circuits to represent the simple cycle C. We continue in this way determining the coefficients of the basic circuits until we have finally determined the coefficient of the circuit C_1. We find that the simple cycle C which is in the cycle space of D can be written as a linear combination of the basic circuits with coefficients in $\{-1, 0, +1\}$. As C was an arbitrary simple cycle we can conclude that B is a simple cycle boxed basis. □

In regard to the second implication in statement (2.1) we show that every simple cycle boxed cycle basis is a zero-one cycle basis. We provide, though, an example which indicates that zero-one does not imply simple cycle boxed.

Theorem 2.2. Every simple cycle boxed cycle basis is a zero-one cycle basis.

Proof. Let D be a directed graph and $B = \{C_1, C_2, \ldots, C_\nu\}$ a directed cycle basis for D. Let $G = G(D)$ be the underlying undirected graph of D. Assume that B is a simple cycle boxed cycle basis and let C' be a cycle of G.

Of course each simple cycle in D is a cycle, but we note that each cycle in G is the projection of a simple cycle in D. For a cycle C' of an undirected graph is by definition a subgraph in which every vertex has even degree. The cycle is a union of connected components. By a well known combinatorics theorem found in [3] we know that each connected component of the subgraph is Eulerian. Thus for each connected component of the cycle there is a closed trail of the edges which uses each edge exactly once. The union of the closed trails determines a simple cycle C of the directed graph D since $C(e) = 0$ if C' does not include the edge e and $C(e) = +1$ if the trail follows edge e in the forward direction and $C(e) = -1$ if the trail follows edge e in the backward direction. The simple cycle C then projects to C' and we note that $C'(e) = +1$ whenever $C(e) \in \{\pm 1\}$ for all edges e in the cycle.

Since, by definition, a cycle in the directed graph is simple if $C(e) \in \{-1, 0, +1\}$ for all e in C then each simple cycle C projects to a cycle C', and each cycle C' lifts to *some* simple cycle C. Thus for each cycle C' of G there exists a simple cycle C of the directed graph that projects to C'. By definition of simple cycle boxed cycle basis then C can be written as a linear combination of the basic circuits with coefficients in $\{-1, 0, +1\}$. So we have for $\lambda_i \in \{-1, 0, +1\}$

$$C = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_\nu C_\nu.$$

Now the cycle C' of G was arbitrary, and we can conclude that for every cycle C' of the undirected graph there exists a simple cycle C of D that projects to C' and is a linear combination of the basic circuits with coefficients in $\{0, \pm 1\}$. Thus the basis B is a zero-one-basis. $\square$

Next, consider the question: is every zero-one cycle basis also a simple cycle boxed cycle basis? We provide an example which shows that we do not have equality between these two classes of bases. Consider the graph V_5 in Figure 2.1 that we created from a complete graph on four vertices K_4, with an additional vertex and two additional edges connecting the fifth vertex to two vertices of K_4. We label the vertices of the graph a through e and direct the edges from the letter in the pair that appears earlier in the alphabet to the letter that appears later. The graph V_5 is clearly planar.

We find $\nu = 8-5+1 = 4$. We create the cycle matrix Γ for a basis consisting of all triangles. We have the circuits $C_1 = a \to b \to e$, $C_2 = a \to b \to c$, $C_3 = a \to c \to e$, $C_4 = c \to d \to e$. The columns of the matrix are the vectors representing circuits C_1 through C_4 and the rows are labeled with the arcs (a, b) through (d, e) in lexicographic order. By Definition 1.1 we know that the basis is planar, and we know from [4] that planar $\implies$ zero-one. Take the simple cycle C represented as $C = (0, -1, 1, -1, 1, -1, -1, -1)$ which follows the vertices in this order $a \to e \to d \to c \to b \to e \to c \to a$. To write the

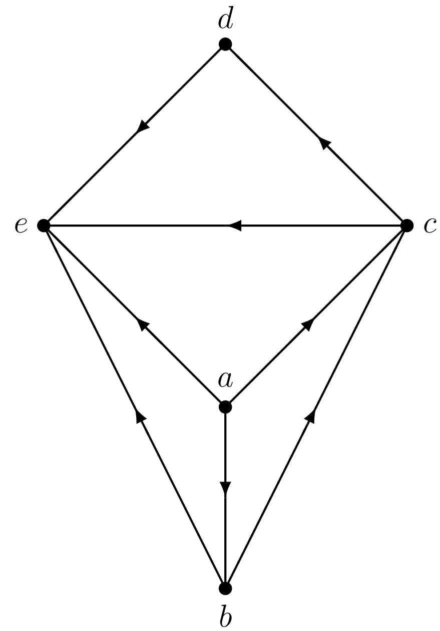

Figure 2.1: Orientation of Directed Graph on Five Vertices V_5

simple cycle as a combination of the basic circuits we find that to obtain a -1 as the last component of C, the coefficient of C_4 must be a -1. Then to obtain a -1 as the seventh component of C, the coefficient of C_3 must be a -2. Thus, we find that the simple cycle C in the cycle space of V_5 can be written uniquely as a linear combination of the basic circuits as follows: $C = (0, -1, 1, -1, 1, -1, -1, -1) = C_1 - C_2 - 2C_3 - C_4$. Since not all simple cycles of V_5 can be written as a linear combination with coefficients in $\{-1, 0, +1\}$ we know that the zero-one basis represented by Γ is not a simple cycle boxed basis. Thus, we can conclude that

$$\text{zero-one} \not\Longrightarrow \text{simple cycle boxed.}$$

$$
\mathbf{\Gamma} = \begin{array}{c} \\ (a,b) \\ (a,c) \\ (a,e) \\ (b,c) \\ (b,e) \\ (c,d) \\ (c,e) \\ (d,e) \end{array}
\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

The next implication in the sequence of implications in statement (2.1) indicates that every zero-one cycle basis is a circuit boxed cycle basis. The proof of Theorem 2.3 shows this result.

Theorem 2.3. Every zero-one cycle basis is a circuit boxed cycle basis.

Proof. Let D be a directed graph and let $G = G(D)$ be the underlying undirected graph. Let B be a directed cycle basis of D and let Γ be the corresponding cycle matrix. Assume B is a *zero-one* basis; $B = \{C_1, \ldots, C_\nu\}$.

By definition of zero-one basis for each circuit C' of G there exists a circuit C of D that projects to C' and can be written as a linear combination with coefficients in $\{-1, 0, 1\}$ of circuits in B,

$$C = \lambda_1 C_1 + \cdots + \lambda_\nu C_\nu$$

for $\lambda_i \in \{-1, 0, 1\}$.

Let $\mathcal{D}$ be a circuit in the directed graph D so that each non-zero component of $\mathcal{D}$ is a non-zero component of C' and each zero component of $\mathcal{D}$ is a zero component of C'. Thus, the support of C' is the support of $\mathcal{D}$ and the components of $\mathcal{D}$ are in the set $\{-1, 0, 1\}$.

We want to show that $C = \pm\mathcal{D}$.

Suppose to the contrary that C is such that it is neither $\mathcal{D}$ nor $-\mathcal{D}$. Recall that since $\mathcal{D}$ is a circuit (and $-\mathcal{D}$ as well) then $\pm\mathcal{D}$ meet the flow conservation constraint

$$\sum_{e\in\delta^+(v)} \mathcal{D}(e) = \sum_{e\in\delta^-(v)} \mathcal{D}(e).$$

We will find a vertex v such that

$$\sum_{e\in\delta^+(v)} C(e) \neq \sum_{e\in\delta^-(v)} C(e)$$

for C.

We let C' be the undirected circuit of finitely many edges. Then for the k edges of the circuit we have edges $e_0, \ldots, e_{k-1}$ where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i < k$ with $v_0 = v_k$; that is for vertices $v_0, \ldots, v_{k-1}$. Since we have assumed that $C \neq \pm\mathcal{D}$ then C differs from $\mathcal{D}$ in as few as one edge or as many as $k-1$ edges. It is easy to see that if C differs from $\mathcal{D}$ in k edges then $C = -\mathcal{D}$.

We traverse the circuit and begin by comparing the directions of edges e_1 and e_2 of circuit $\mathcal{D}$ with those of C. If the directions of e_1 and e_2 in C are the same as those of e_1 and e_2 of the circuit $\mathcal{D}$ or if the directions of e_1 and e_2 in C are both different from the directions of e_1 and e_2 in the circuit $\mathcal{D}$ then we find

$$C(e_1) = \mathcal{D}(e_1) \text{ and } C(e_2) = \mathcal{D}(e_2)$$

or

$$C(e_1) \neq \mathcal{D}(e_1) \text{ and } C(e_2) \neq \mathcal{D}(e_2)$$

respectively. This indicates that the amount of flow entering v_2 is equal to the amount of flow leaving v_2 in C. Thus, for this vertex the flow conservation constraint is met at that vertex in C. We continue in this way comparing the directions of pairs of edges incident to a vertex in circuit $\mathcal{D}$ with the directions of pairs of edges incident to that vertex in C. Since we know at least one edge (but not more than $k-1$ edges) in C differs in direction from that edge in $\mathcal{D}$ we will find a vertex v_i incident to edges e_{i-1} and e_i where

$$C(e_{i-1}) \neq \mathcal{D}(e_{i-1}) \text{ and } C(e_i) = \mathcal{D}(e_i)$$

or where

$$C(e_{i-1}) = \mathcal{D}(e_{i-1}) \text{ and } C(e_i) \neq \mathcal{D}(e_i).$$

In either of these cases we find that the amount of flow entering v_i is not equal to the amount of flow leaving v_i in C. Thus for vertex v_i in C we have

$$\sum_{e \in \delta^+(v_i)} C(e) \neq \sum_{e \in \delta^-(v_i)} C(e)$$

and we have found a vertex in C which violates the flow conservation constraint. Therefore we have a contradiction and we conclude $C = \pm\mathcal{D}$. So every circuit $\mathcal{D}$ of D is a linear combination of the basic circuits with coefficients in $\{-1, 0, +1\}$ and we can finally conclude that every zero-one basis is a circuit boxed basis. $\square$

Lemma 2.1 introduces the concept of *conformal circuits*. The concept of conformal composition of cycles can be found in [2], and we note some key terms here. Let $s(\mathcal{C})$ denote the *signed support* of a cycle $\mathcal{C}$ where the signed set may be written $s(\mathcal{C}) = (s(\mathcal{C})^+, s(\mathcal{C})^-)$. For the m arcs of a directed graph the components of $s(\mathcal{C})$ are defined as follows: $s(\mathcal{C})^+ := \{i \colon \mathcal{C}(e_i) > 0\}$ - the positive elements of the signed set $s(\mathcal{C})$, and $s(\mathcal{C})^- := \{i \colon \mathcal{C}(e_i) < 0\}$ - the negative elements of the signed set $s(\mathcal{C})$, for $1 \leq i \leq m$. For example, we can use the signed set notation for a cycle $\mathcal{C} = (1, -1, 0, 0, 1, 1)$ consisting of six arcs and write $s(\mathcal{C}) = (\{e_1, e_5, e_6\}, \{e_2\})$ which indicates arcs e_1, e_5 and e_6 are the positive elements of the cycle, and arc e_2 is the negative element. The same signed support could be written using a signed incidence vector in which case the signed set of the cycle $\mathcal{C}$ could be represented by the vector $s(\mathcal{C}) = (+, -, 0, 0, +, +)$. The composition of two signed sets results in a signed set. The *composition* of two signed sets $s(\mathcal{C}_1)$ and $s(\mathcal{C}_2)$ is by definition

$$s(\mathcal{C}_1) \circ s(\mathcal{C}_2) = (s(\mathcal{C}_1)^+ \cup (s(\mathcal{C}_2)^+ \setminus s(\mathcal{C}_1)^-), s(\mathcal{C}_1)^- \cup (s(\mathcal{C}_2)^- \setminus s(\mathcal{C}_1)^+)).$$

In vector notation this becomes

$$(s(\mathcal{C}_1) \circ s(\mathcal{C}_2))(e) = \begin{cases} s(\mathcal{C}_1)(e), & \text{if } \mathcal{C}_1(e) \neq 0; \\ s(\mathcal{C}_2)(e), & \text{otherwise.} \end{cases}$$

Repeated compositions of the signed sets for k circuits produces the signed set $s(\mathcal{C})$:

$$s(\mathcal{C}) = s(\mathcal{C}_1) \circ s(\mathcal{C}_2) \circ s(\mathcal{C}_3) \circ \cdots \circ s(\mathcal{C}_k)$$

a union of the supports of the circuits $\mathcal{C}_1, \ldots, \mathcal{C}_k$ to give the support of $\mathcal{C}$. If the composition is *conformal* then for each edge e of $\mathcal{C}$ we have: $\mathcal{C}_h(e)\mathcal{C}_j(e) \geq 0$ for all h, j. That is, the sign of component $\mathcal{C}_j(e)$ is the same as the sign of component $\mathcal{C}(e)$ for each circuit $\mathcal{C}_j$ of arc e or else the component $\mathcal{C}_j(e)$ is zero.

Lemma 2.1. For any cycle of a directed graph there exists a circuit of the digraph that conforms to the cycle.

Proof. Let D be a directed graph, and let $\mathcal{C}$ be a cycle of D. As we have seen we can assume that $\mathcal{C}$ is a $\mathbb{Z}$-cycle. We want to show that we can always find a circuit that conforms to $\mathcal{C}$. Select a vertex v of the digraph that is a vertex of $\mathcal{C}$. Assume v is not an isolated vertex.

Let $v = x_0$, and construct a walk W from x_0. Continue the walk by selecting an arc of $\mathcal{C}$ incident to x_0 which we call a_0. Let x_1 be the other endpoint of a_0. Now either x_0 is the head of a_0 or x_1 is the head of a_0 and the component of $\mathcal{C}$ indexed by a_0 is either positive or negative. Thus we have the following cases for this arc a_0 of W

1. (x_0, x_1) and $\mathcal{C}(a_0) > 0$;

2. (x_0, x_1) and $\mathcal{C}(a_0) < 0$;

3. (x_1, x_0) and $\mathcal{C}(a_0) > 0$;

4. (x_1, x_0) and $\mathcal{C}(a_0) < 0$.

Suppose Case 1 first, that is x_0 is the tail of a_0, the head is x_1, and the component of $\mathcal{C}$ indexed by a_0 is positive. Since each vertex of $\mathcal{C}$ agrees with the flow conservation constraint, $\sum_{a\in\delta^+(v)} C(a) = \sum_{a\in\delta^-(v)} C(a)$, we know that there is an arc of $\mathcal{C}$ incident to x_1 that we will label a_1 and another vertex incident to a_1 that we will name x_2 such that either of the following is true

a. (x_1, x_2) and $\mathcal{C}(a_1) > 0$;

b. (x_2, x_1) and $\mathcal{C}(a_1) < 0$.

In either case we can continue the walk along such an arc a_1. As we construct W in this way the walk becomes a sequence of vertices and arcs

$$\{x_0, a_0, x_1, a_1, x_2, a_2, \ldots, x_r\}. \tag{2.2}$$

Recall that each component of $\mathcal{C}$ is connected so we will conclude the walk once the sequence (2.2) repeats a vertex. We use the resulting closed path to construct the circuit C. For the arcs a of the closed path let $C(a) = 1$ if $\mathcal{C}(a) > 0$, $C(a) = -1$ if $\mathcal{C}(a) < 0$, and let all other components of C be 0. Then the arcs of the closed path provide the support of C. The flow conservation constraint is satisfied at each vertex of the circuit C, and the circuit conforms to the cycle $\mathcal{C}$. As v was an arbitrary vertex of a cycle of D we can conclude that we can always find a circuit that conforms to the given cycle.

Cases 2, 3 and 4 are shown in an analogous way. We highlight some details for those proofs here. Suppose Case 2 next where x_0 is the tail of a_0, the head is x_1, and the component of $\mathcal{C}$ indexed by a_0 is negative. We know there is an arc a_1 with endpoints x_1 and x_2 such that either of the following is true: (x_1, x_2) and $\mathcal{C}(a_1) < 0$ or (x_2, x_1) and $\mathcal{C}(a_1) > 0$. Now suppose for Case 3 where x_1 is the tail of a_0, x_0 is the head, and the component of $\mathcal{C}$ indexed by a_0 is positive. We know there is an arc a_1 with endpoints x_1 and x_2 such that either (x_1, x_2) and $\mathcal{C}(a_1) < 0$ is true or (x_2, x_1) and $\mathcal{C}(a_1) > 0$ is true. Lastly, for Case 4 where x_1 is the tail of a_0, the head is x_0 and the component of $\mathcal{C}$ indexed by a_0 is negative. We know there is an arc a_1 with endpoints x_1 and x_2 such that either of the following is true (x_1, x_2) and $\mathcal{C}(a_1) > 0$ or (x_2, x_1) and $\mathcal{C}(a_1) < 0$. $\square$

Theorem 2.4. Every circuit boxed cycle basis is an integral cycle basis.

Proof. Let $B = \{C_1, \ldots, C_\nu\}$ be a directed basis for a directed graph D where $G = G(D)$ is the underlying undirected graph of D. Let C be a cycle of D and assume that B is a circuit boxed cycle basis. We can assume that C is a $\mathbb{Z}$-cycle, that is, all of the entries of the vector C are integral. We want to show that C can be written as an integer linear combination of circuits in B.

Denote the set of nonnegative integers by $\mathbb{Z}^* = \{0\} \cup \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of positive integers. We will use the second principle of mathematical induction to show that C is a linear combination of circuits $C_1, \ldots, C_k$, not necessarily basic circuits, with coefficients in $\mathbb{Z}^*$. We will induct on the ℓ_1 norm

$$\|C\|_1 = \sum_{i=1}^{m} |C(a_i)|$$

of the vector C where m is the number of arcs of D, and the components of the cycle are indexed by the arcs a_i. Let $\|C\|_1 = x$ for $x \geq 0$. For the basic step, we note that if $\|C\|_1 = 0$ then $C = 0 \cdot C_j = \mathbf{0}$ for any circuit C_j, $1 \leq j \leq k$.

Let y be in the set of all possible values of the ℓ_1 norm such that $y < x$. For the inductive step, we assume for a cycle C' with $\|C'\|_1 = y$ that C' can be written as a linear combination of circuits with coefficients in $\mathbb{Z}^*$ that is

$$\exists \lambda_i \in \mathbb{Z}^* \colon C' = \lambda_1 C_1 + \cdots + \lambda_k C_k$$

for circuits $C_1, \ldots, C_k$, not necessarily basic circuits. By Lemma 2.1 we know that there is a circuit that conforms to cycle C. Let C_h be that circuit. Now let $C^* = C - C_h$ be the cycle found when circuit C_h is subtracted from cycle C.. Recall that the nonzero entries of each circuit C are from the set $\{\pm 1\}$ and are the support of each circuit. Thus, the ℓ_1 norm of C_h is strictly greater than one so we can conclude that $\|C^*\|_1 < x$. Thus, by the inductive assumption, C^* can be written as a linear combination of circuits with coefficients in $\mathbb{Z}^*$.

Now, if we consider the combination $C^* + C_h$ this must be a linear combination of circuits with coefficients in $\mathbb{Z}^*$ since C^* is such a cycle. It follows then that C can be written as a linear combination of circuits $C_1, \ldots, C_k$, that is

$$\exists \lambda_i \in \mathbb{Z}^* \colon C = \lambda_1 C_1 + \cdots + \lambda_k C_k.$$

Since B is circuit boxed each circuit C_i for $1 \leq i \leq k$ can be written as a linear combination with coefficients in $\{-1, 0, +1\}$. Therefore, cycle C can be written as an integer linear combination of the circuits of B. $\square$

Here, we might ask if we have equality for the last two classes of cycle bases in statement (2.1); that is: can we conclude circuit boxed $\iff$ integral? We return to Wagner's graph in Figure 1.4 for an answer. We choose another basis B_2 and create a cycle matrix Γ_2. The $\nu = 5$ circuits include the first four circuits in our previous cycle matrix Γ_1 together with the circuit $C_5 = v_6 \to v_5 \to v_4 \to v_3 \to v_2$. We form the $\nu \times \nu$ submatrix Γ_2' with rows $(v_1, v_5), (v_1, v_8), (v_2, v_6), (v_3, v_7)$ and (v_4, v_8). We find that det $B_2 = |\det \Gamma_2'| = |-1| = 1$. By Theorem 1.2 we know that B_2 is an integral basis for V_8. Consider, again, the circuit C_{17} which uses the eight edges "around" the digraph. It is written uniquely as a linear combination of the circuits of B_2 as

$$C_{17} = -(C_1 + C_2 + C_3 + C_4 + 2C_5).$$

Thus, we must conclude that the basis B_2 is not circuit boxed. We assert then that not every integral basis is circuit boxed; that is

$$\text{integral} \;\not\Rightarrow\; \text{circuit boxed.}$$

$$\mathbf{\Gamma_2} = \begin{array}{c} \\ (v_1,v_2) \\ (v_1,v_5) \\ (v_1,v_8) \\ (v_2,v_3) \\ (v_2,v_6) \\ (v_3,v_4) \\ (v_3,v_7) \\ (v_4,v_5) \\ (v_4,v_8) \\ (v_5,v_6) \\ (v_6,v_7) \\ (v_7,v_8) \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2.2 Remarks

We close this chapter with suggested open problems. The first two questions follow immediately from the work in Section 2.1. We want to know more about the equality or inequality of the classes of cycle bases in the sequence of implications in statement 2.1. The last three questions are considered when we extend this study of relationships among the classes of cycle bases to include the class of weakly fundamental cycle bases.

1. Can every simple cycle boxed cycle basis be made strictly fundamental by multiplying some of its vectors by -1?

2. Is every circuit boxed cycle basis a zero-one basis?

3. Is every simple cycle boxed cycle basis weakly fundamental?

4. Is every zero-one cycle basis weakly fundamental?

5. Is every circuit boxed cycle basis weakly fundamental?

Our search for instances where a cycle basis can be classified as circuit boxed but not zero-one led to the discovery of a number of examples of cycle matrices that are not totally unimodular. Section 3.1 includes results on the relationship of some of these non-totally unimodular cycle matrices and their corresponding bases of circuits to the Möbius band.

Chapter 3: Examples

3.1 Möbius Band

The *Möbius band*, a topological surface, can be obtained by taking a rectangular strip of paper, giving one end of the paper a $180°$ twist and then gluing the ends of shorter length together. This gluing is indicated in Figure 3.1 by the direction of the arrows on side e. As a practicality the construction works more easily if the unlabeled side is about three to four times the length of the labeled side.

Figure 3.1: Paper Strip for Möbius Band

We find that there exist bases of digraphs containing circuits which correspond to disks that when glued together in a particular way form a Möbius band. Yet another circuit of the basis corresponding to a disk turns out to be the boundary of the Möbius band. We obtain the *real projective plane* with the gluing of this last disk along the boundary of the Möbius band we constructed from the first disks. Example 3.1 demonstrates this construction.

Example 3.1. A directed graph appears in [6] with a basis whose cycle matrix is not totally unimodular. This digraph on nine vertices V_9 is included in Figure 3.2. The cycle matrix for a basis Γ_{V_9} is presented as well. We use the circuits of this basis to construct a Möbius Band with boundary.

The dimension of the cycle space of the digraph V_9 is $\nu = 12-9+1 = 4$ so a cycle matrix will include four circuits. Notice that the submatrix Γ'_{V_9} in rows $(v_1, v_2), (v_2, v_9), (v_3, v_4)$, (v_4, v_5) has determinant -2. This is a submatrix that arises after deleting the arcs of a spanning forest of V_9. So the determinant of the basis is det $B_{V_9} = |\det\ \Gamma'| = 2$. Thus, the matrix is not totally unimodular, and the basis is not zero-one. Additionally, this directed basis is not an undirected basis.

The disks that correspond to the circuits of B_{V_9} and which will form the band with boundary appear in Figure 3.3. We glue disks C_1 and C_2 along the path $v_2 \to v_3 \to v_4$. When gluing arcs we respect the direction and identify the arcs to be glued according to the direction of the arcs. Next, glue this union $C_1 \cup C_2$ and this disk C_3 along arc (v_1, v_2). Finally, give a $180°$ twist to the end of the union of these three disks and glue arc (v_4, v_5) in C_3 to arc (v_4, v_5) in C_2. The result is a Möbius band. By carefully following the arcs of C_4 we see that it bounds this band. That is, the sequence of arcs $(v_9, v_5), (v_2, v_9), (v_2, v_6), (v_6, v_4), (v_8, v_4), (v_1, v_8), (v_1, v_7), (v_7, v_5)$ bounds the union $C_1 \cup C_2 \cup C_3$.

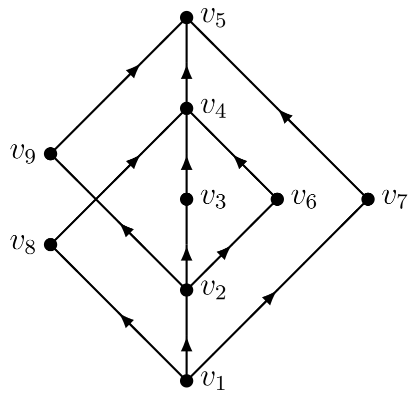

Figure 3.2: Directed Graph on Nine Vertices V_9

$$\mathbf{\Gamma_{V_9}} = \begin{array}{c} \\ (v_1,v_2) \\ (v_1,v_7) \\ (v_1,v_8) \\ (v_2,v_3) \\ (v_2,v_6) \\ (v_2,v_9) \\ (v_3,v_4) \\ (v_4,v_5) \\ (v_6,v_4) \\ (v_7,v_5) \\ (v_8,v_4) \\ (v_9,v_5) \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \end{array}$$

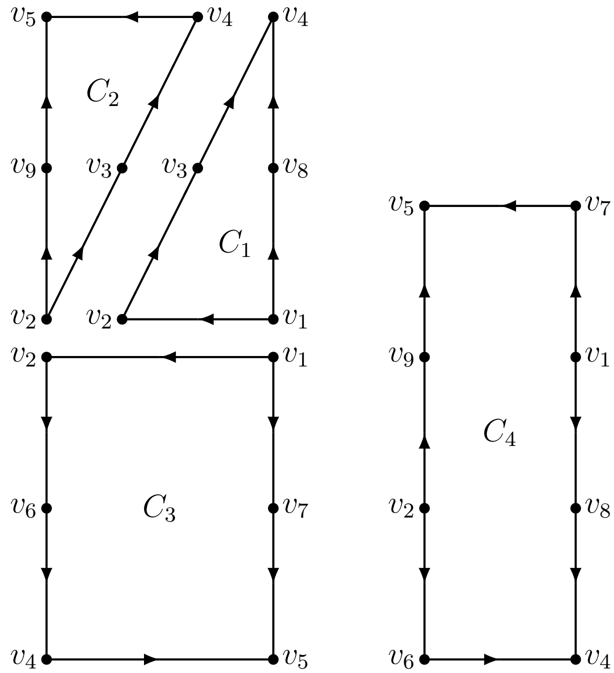

Figure 3.3: Disks C_1, C_1, C_3 and Boundary Disk C_4 for Digraph V_9

The Möbius band can be given a graph representation known as the *Möbius Ladder*. In [7] Oxley describes this family of graphs. The usual form of the Möbius Ladder appears in Figure 3.8 where we have given the arcs direction. Example 3.2 includes a description of the smallest of these directed graphs.

Example 3.2. This Möbius Ladder on four vertices L_4 is presented in Figure 3.4. The dimension of the cycle space of the digraph L_4 is $\nu = 6 - 4 + 1 = 3$, so a cycle matrix will include three circuits. The cycle matrix Γ_{L_4} corresponds to a basis of three 4-gons. We will use the circuits of the basis to construct a Möbius Band with boundary.

The 3×3 submatrix Γ'_{L_4} in rows $(u_1, v_1), (u_2, v_1), (u_2, v_2)$ has determinant -2. Thus, the determinant of the basis is det $B_{L_4} = |\det\ \Gamma'_{L_4}| = 2$. By definition the matrix is not totally unimodular, and by Theorem 1.2 the basis is not zero-one. Furthermore, the directed basis is not an undirected basis.

Figure 3.5 shows the circuits of B_{L_4} that correspond to the disks of the band and the boundary. Disks C_1 and C_2 are glued along arc (u_2, v_2) according to the direction of the arc. These two disks are also glued along arc (u_1, v_1) according to the direction of the arc. Careful attention to the direction of the arcs when gluing will produce the desired twist. The union of the disks form a Möbius band with C_3 being the bound of the surface. Observe that $(u_1, u_2), (u_2, v_1), (v_1, v_2), (v_2, u_1)$ bounds the union $C_1 \cup C_2$.

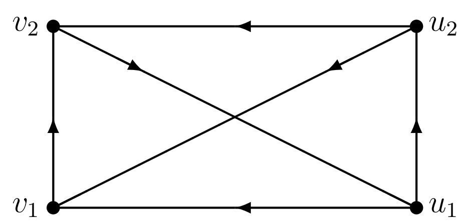

Figure 3.4: Möbius Ladder L_4, $t = 3$

$$\mathbf{\Gamma}_{L_4} = \begin{array}{c} \\ (u_1,u_2) \\ (u_1,v_1) \\ (u_2,v_1) \\ (u_2,v_2) \\ (v_1,v_2) \\ (v_2,u_1) \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

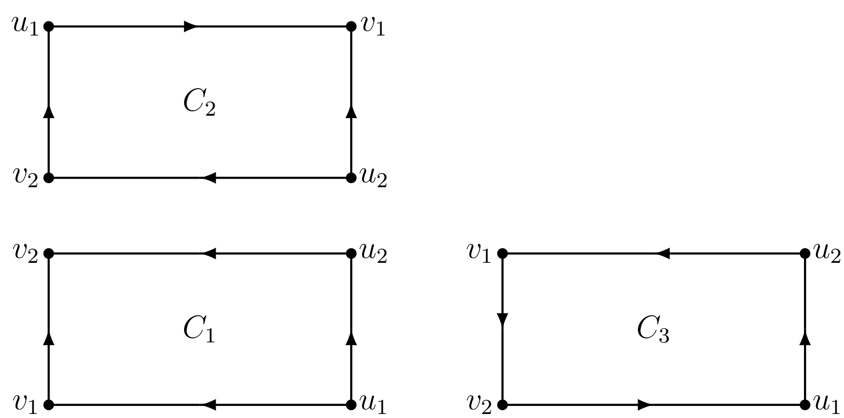

Figure 3.5: Disks C_1, C_2 and Boundary Disk C_3 for Digraph L_4

Example 3.3. We present an additional Möbius Ladder before we discuss the general graph. The Möbius Ladder on six vertices L_6 appears in Figure 3.6. The dimension of the cycle space of digraph L_6 is $\nu = 9 - 6 + 1 = 4$ so a cycle matrix will contain four circuits. The cycle matrix Γ_{L_6} contains the circuits of the basis using three 4-gons plus a circuit of six arcs.

The 4×4 submatrix, Γ'_{L_6}, in rows $(u_1, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_3)$ has determinant 2. By definition the determinant of the basis is det $B_{L_6} = |\det \Gamma_{L_6}| = 2$. Therefore the matrix is not totally unimodular, and the basis is not zero-one. Additionally, the directed basis B_{L_6} is not an undirected basis.

Figure 3.7 displays the circuits of cycle matrix Γ_{L_6} corresponding to the disks that form the band and boundary of the band. We form the union of the first three disks by gluing C_1 and C_2 along arc (u_2, v_2) then gluing C_2 and C_3 along arc (u_3, v_3). The Möbius Band is completed by gluing C_1 to C_3 on arc (u_1, v_1) while paying attention to the direction of the arc (u_1, v_1). The arcs $(u_1, u_2), (u_2, u_3), (u_3, v_1), (v_1, v_2), (v_2, v_3), (v_3, u_1)$ bound the union $C_1 \cup C_2 \cup C_3$ so that C_4 is the boundary of the band.

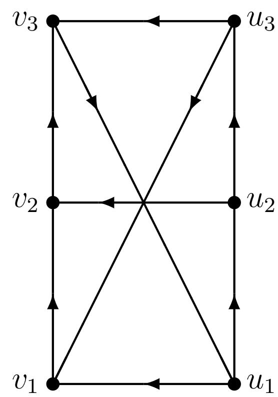

Figure 3.6: Möbius Ladder L_6, $t = 4$

$$\mathbf{\Gamma}_{L_6} = \begin{array}{c} \\ (u_1,u_2) \\ (u_1,v_1) \\ (u_2,u_3) \\ (u_2,v_2) \\ (u_3,v_1) \\ (u_3,v_3) \\ (v_1,v_2) \\ (v_2,v_3) \\ (v_3,u_1) \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

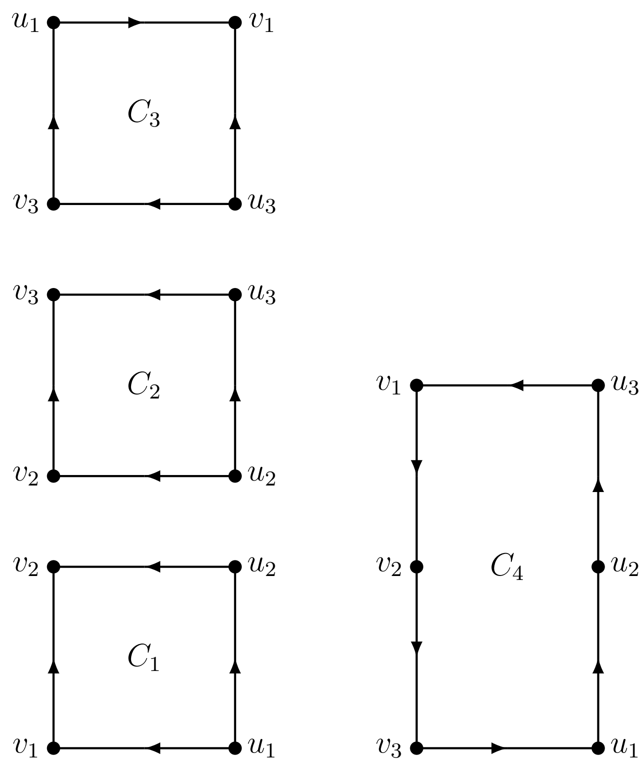

Figure 3.7: Disks C_1, C_1, C_3 and Boundary Disk C_4 for Digraph L_6

The directed graph of the Möbius Ladder is usually drawn as it is in Figure 3.8. The arcs (u_i, v_i) are considered the rungs of the ladder and the arcs (u_i, u_{i+1}) and (v_i, v_{i+1}) as well as arcs (u_{t-1}, v_1) and (v_{t-1}, u_1) make up the side rails. Notice that the arc (u_1, v_1) is drawn twice. When these two copies of (u_1, v_1) are glued according the direction of the arc then the desired twist is achieved and the Möbius Band is constructed. There are $t-1$ circuits that correspond to the $t-1$ four-gons of the ladder. The digraph consists of $3t-3$ arcs and $2t-2$ vertices. A basis consists of $\nu = (3t-3) - (2t-2) + 1 = t$ circuits. Thus, one more circuit is needed to make a cycle basis with these $t-1$ circuits.

The Möbius Band is bound by the circuit C_t that appears in (3.1)

$$u_1 \to u_2 \to u_3 \to \cdots \to u_{t-2} \to u_{t-1} \to v_1 \to v_2 \to v_3 \to \cdots \to v_{t-2} \to v_{t-1} \to u_1, \quad (3.1)$$

the t^{th} circuit of the basis. We present a cycle matrix $\Gamma^1_{L_{2t-2}}$ for this basis of circuits $B^1_{L_{2t-2}}$ just described.

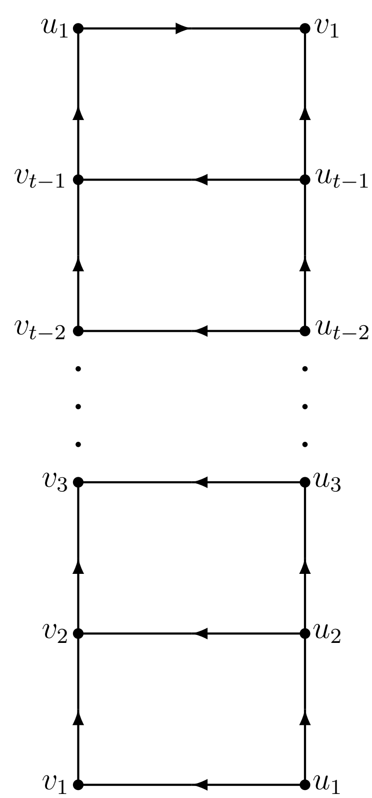

Figure 3.8: Möbius Ladder L_{2t-2}

$$
\mathbf{\Gamma}^1_{L_{2t-2}} = \begin{array}{c} \\ (u_1,u_2) \\ (u_1,v_1) \\ (u_2,u_3) \\ (u_2,v_2) \\ (u_3,u_4) \\ (u_3,v_3) \\ (u_4,u_5) \\ (u_4,v_4) \\ (u_5,u_6) \\ (u_5,v_5) \\ \vdots \\ (u_{t-1},v_1) \\ (u_{t-1},v_{t-1}) \\ (v_1,v_2) \\ (v_2,v_3) \\ (v_3,v_4) \\ (v_4,v_5) \\ \vdots \\ (v_{t-1},u_1) \end{array}
\begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & \cdots & C_{t-1} & C_t \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-1 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 1 \\
0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 1 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & -1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & -1 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & -1 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix} \end{array}
$$

Proposition 3.1. The basis $B^1_{L_{2t-2}} = \{C_1, C_2, \ldots, C_t\}$ for the Möbius Ladder L_{2t-2} is not zero-one.

Proof. Let L_{2t-2} be the usual directed graph of the Möbius Ladder with the orientation as it appears in Figure 3.8. Let $B^1_{L_{2t-2}} = \{C_1, C_2, \ldots, C_t\}$ be the set of circuits of L_{2t-2} where each C_i for $1 \le i \le t-2$ is the circuit $u_i \to u_{i+1} \to v_{i+1} \to v_i \to u_i$, and C_{t-1} is the circuit $u_{t-1} \to v_1 \to u_1 \to v_{t-1} \to u_{t-1}$, and the circuit C_t is as described in (3.1). Let $\Gamma^1_{L_{2t-2}}$ be the cycle matrix containing the t circuits. Notice that the column vector for C_t contains a 0 for each rung and a $+1$ for each arc of the side rails. Each column vector C_i for $1 \le i \le t-1$ has entries $C_i(a) \in \{+1, -1\}$ for exactly four arcs a and 0 otherwise.

We choose a spanning tree consisting of arcs (u_i, u_{i+1}) and (v_i, v_{i+1}) for $1 \le i \le t-2$ of the two side rails together with the side rail arc (v_{t-1}, u_1). Thus we have a spanning tree of $2(t-2)+1 = 2t-3$ arcs. Now the submatrix $\Gamma^{1'}_{L_{2t-2}}$ will be a $(3t-3)-(2t-3) \times t = t \times t$ matrix as desired. The non-tree arcs are the arcs which are the rungs of L_{2t-2}; that is, (u_i, v_i) for $1 \le i \le t-1$, plus the arc (u_{t-1}, v_1). These arcs index the rows of $\Gamma^{1'}_{L_{2t-2}}$.

$$
\mathbf{\Gamma^{1'}_{L_{2t-2}}} = \begin{array}{c} \\ (u_1,v_1) \\ (u_2,v_2) \\ (u_3,v_3) \\ (u_4,v_4) \\ (u_5,v_5) \\ \vdots \\ (u_{t-2},v_{t-2}) \\ (u_{t-1},v_{t-1}) \\ (u_{t-1},v_1) \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \cdots & C_{t-2} & C_{t-1} & C_t \end{array} \\
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$

Notice that the $(t-1) \times (t-1)$ submatrix formed by removing the first column and the first row of $\Gamma^{1'}_{L_{2t-2}}$ is lower triangular. Also notice that the $(t-1) \times (t-1)$ submatrix formed by removing the $(t-1)$st column and first row of $\Gamma^{1'}_{L_{2t-2}}$ is upper triangular. Then when t is odd the determinant becomes

$$\det(\Gamma') = (-1)(-1) - (-1)(1) = 2$$

and when t is even the determinant becomes

$$\det(\Gamma') = (-1)(1) + (-1)(1) = -2.$$

Thus det $B^{1'}_{L_{2t-2}} = |\pm 2| = 2$ and we conclude by definition that the cycle matrix is not totally unimodular. Consequently by Theorem 1.2 the basis is not zero-one. $\square$

We consider, next, the circuit of the Möbius Ladder L_{2t-2} that includes all of the arcs of the left side rail; that is, arcs of the form (v_i, v_{i+1}) for $1 \le i \le t-2$ and arc (v_{t-1}, u_1) together with the arc (u_1, v_1). Call this circuit C_{t+1}. Similar statements can be made about the circuit that includes all of the arcs of the right side rail together with the arc (u_1, v_1). Given the orientation of L_{2t-2} as in Figure 3.8 we will use the circuit C_{t+1} with all nonzero components equal to +1.

$$
\mathbf{\Gamma}^1_{L_{2t-2}} = \begin{array}{c} \\ (u_1,u_2) \\ (u_1,v_1) \\ (u_2,u_3) \\ (u_2,v_2) \\ (u_3,u_4) \\ (u_3,v_3) \\ (u_4,u_5) \\ (u_4,v_4) \\ (u_5,u_6) \\ (u_5,v_5) \\ \vdots \\ (u_{t-1},v_1) \\ (u_{t-1},v_{t-1}) \\ (v_1,v_2) \\ (v_2,v_3) \\ (v_3,v_4) \\ (v_4,v_5) \\ \vdots \\ (v_{t-1},u_1) \end{array}
\begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & \cdots & C_{t-1} & C_t \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-1 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 1 \\
0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 1 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & -1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & -1 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & -1 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix} \end{array}
, \quad
\begin{array}{c} C_{t+1} \\
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{array}
$$

Proposition 3.2. The basis $B^1_{L_{2t-2}} = \{C_1, C_2, \ldots, C_t\}$ for the Möbius Ladder L_{2t-2} is not circuit boxed.

Proof. To show that the basis is not circuit boxed we will show that circuit C_{t+1}, described above, can be uniquely expressed in terms of the basic circuits of cycle matrix $\Gamma^1_{L_{2t-2}}$ as

$$C_{t+1} = \frac{-1}{2} \cdot C_1 + \frac{-1}{2} \cdot C_2 + \cdots + \frac{-1}{2} \cdot C_{t-1} + \frac{1}{2} \cdot C_t \tag{3.2}$$

that is, the coefficients of the circuits of the Möbius band equal $\frac{-1}{2}$ and the coefficient of the circuit of the boundary is $\frac{1}{2}$.

Since $\Gamma^1_{L_{2t-2}}$ is a matrix of linearly independent circuits of the digraph L_{2t-2} we know that the circuit C_{t+1} of the cycle space of L_{2t-2} can be written uniquely as a linear combination of the basic circuits. The circuit C_{t+1} appears beside the cycle matrix for reference. Notice that each rung appears in exactly two basic circuits and each basic circuit contains exactly two rungs. The circuit corresponding to the boundary of the Möbius Band contains a zero for each rung. The circuit C_{t+1} includes only one rung which is arc (u_1, v_1). For all rungs except (u_1, v_1) the two nonzero components are of opposite sign. Thus the coefficients of the basic circuits C_i for $1 \leq i \leq t-1$ must be equal as seen in the following

$$\lambda_i C_i(u_{i+1}, v_{i+1}) + \lambda_{i+1} C_{i+1}(u_{i+1}, v_{i+1}) = 0$$

$$\lambda_i \cdot 1 + \lambda_{i+1} \cdot (-1) = 0$$

$$\lambda_i = \lambda_{i+1}$$

for $1 \leq i \leq t-2$. Now the component for the rung (u_1, v_1) in C_1 equals the component for that rung in C_{t-1}, and the component of (u_1, v_1) in C_{t+1} is opposite in sign so that we find

$$\lambda_1 C_1(u_1, v_1) + \lambda_{t-1} C_{t-1}(u_1, v_1) = C_{t+1}(u_1, v_1)$$

$$\lambda_1 \cdot (-1) + \lambda_1 \cdot (-1) = 1$$

$$\lambda_1 = \frac{-1}{2}.$$

Thus the coefficients of the circuits $C_1, C_2, \ldots, C_{t-1}$ are all equal to $\frac{-1}{2}$.

Next, consider the arcs of the right side rail which are of the form (u_i, u_{i+1}) and (u_{t-1}, v_1). Each of these arcs appear in exactly one basic circuit of the Möbius band, circuits C_i for $1 \leq i \leq t-1$, and each of these basic circuits contain exactly one of these arcs. Each arc of the right side rail appears in the circuit of the boundary of the Möbius band, C_t, but not in the circuit C_{t+1}. The nonzero components indexed by these side rail arcs are of equal sign. So we have the following

$$\lambda_i C_i(u_i, u_{i+1}) + \lambda_t C_t(u_i, u_{i+1}) = C_{t+1}(u_i, u_{i+1})$$

$$\frac{-1}{2} \cdot (1) + \lambda_t \cdot (1) = 0$$

$$\lambda_t = \frac{1}{2}.$$

Thus the coefficient of the circuit C_t is $\frac{1}{2}$. Therefore the expression for the circuit C_{t+1} in terms of the basic circuits becomes

$$C_{t+1} = \frac{-1}{2} \cdot C_1 + \frac{-1}{2} \cdot C_2 + \cdots + \frac{-1}{2} \cdot C_{t-1} + \frac{1}{2} \cdot C_t$$

as desired. Since there exists a circuit of the cycle space that cannot be expressed as a linear combination of basic circuits with coefficients only in $\{-1, 0, +1\}$ then we know by definition that $B^1_{L_{2t-2}}$ is not a circuit boxed cycle basis. $\square$

Now we will create another basis for digraph L_{2t-2}. Begin with the cycle matrix $\Gamma^1_{L_{2t-2}}$ and remove the circuit that corresponds to the boundary of the Möbius Band. Replace C_t with circuit C_{t+1}. Call this cycle matrix for the new basis $\Gamma^2_{L_{2t-2}}$. Let $B^2_{L_{2t-2}}$ be the basis that includes the circuits $C_1, C_2, \ldots, C_{t-1}$ together with the circuit C_{t+1}.

Proposition 3.3. The basis $B^2_{L_{2t-2}} = \{C_1, C_2, \ldots C_{t-2}, C_{t-1}, C_{t+1}\}$ for the Möbius Ladder L_{2t-2} is weakly fundamental.

Proof. Use the spanning tree for the digraph L_{2t-2} just as in the proof for Proposition 3.1. The submatrix $\Gamma^{2'}_{L_{2t-2}}$ found by using the non-tree arcs will be the desired $t \times t$ matrix.

$$\mathbf{\Gamma^{2'}_{L_{2t-2}}} = \begin{array}{c} \\ (u_1,v_1) \\ (u_2,v_2) \\ (u_3,v_3) \\ (u_4,v_4) \\ (u_5,v_5) \\ \vdots \\ (u_{t-2},v_{t-2}) \\ (u_{t-1},v_{t-1}) \\ (u_{t-1},v_1) \end{array} \begin{array}{c} \begin{array}{ccccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \cdots & C_{t-2} & C_{t-1} & C_{t+1} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \end{array}$$

A permutation of the columns of $\Gamma^{2'}_{L_{2t-2}}$ results in matrix $\sigma_1\left(\Gamma^{2'}_{L_{2t-2}}\right)$ which is an upper triangular matrix with ones on its main diagonal. By Theorem 1.2 we know that the basis

$B^2_{L_{2t-2}}$ is weakly fundamental.

$$
\sigma_1\left(\mathbf{\Gamma^{2'}_{L_{2t-2}}}\right) = \begin{array}{c} \\ (u_1,v_1) \\ (u_2,v_2) \\ (u_3,v_3) \\ (u_4,v_4) \\ (u_5,v_5) \\ \vdots \\ (u_{t-2},v_{t-2}) \\ (u_{t-1},v_{t-1}) \\ (u_{t-1},v_1) \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} C_{t+1} & C_1 & C_2 & C_3 & C_4 & C_5 & \cdots & C_{t-2} & C_{t-1} \end{array} \\
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\end{array}
$$

□

Proposition 3.4. The basis $B^2_{L_{2t-2}} = \{C_1, C_2, \ldots, C_{t-2}, C_{t-1}, C_{t+1}\}$ for the Möbius Ladder L_{2t-2} is not circuit boxed.

Proof. To show that $B^2_{L_{2t-2}}$ is not circuit boxed we will show that the boundary of the Möbius Band, the circuit C_t, can be uniquely expressed in terms of the basic circuits of cycle matrix $\Gamma^2_{L_{2t-2}}$ as

$$C_t = C_1 + C_2 + \cdots + C_{t-1} + 2 \cdot C_{t+1} \tag{3.3}$$

that is the coefficients of the 4-gon circuits of the Möbius Ladder equal 1 and the coefficient of the circuit C_{t+1} is 2. Begin with the expression for circuit C_{t+1} found in the proof of Proposition 3.2; that is, equation (3.2), and multiply the equation by 2 so that the coefficient of C_t becomes 1 as seen in the following

$$2 \cdot C_{t+1} = 2 \cdot \left(\frac{-1}{2} \cdot C_1 + \frac{-1}{2} \cdot C_2 + \cdots + \frac{-1}{2} \cdot C_{t-1} + \frac{1}{2} \cdot C_t \right).$$

We solve for C_t to get the desired linear combination of basic circuits

$$C_t = C_1 + C_2 + \cdots + C_{t-1} + 2 \cdot C_{t+1}$$

for C_t. Since there exists a circuit of the cycle space that cannot be expressed as a linear combination of basic circuits with coefficients only in $\{-1, 0, +1\}$ then we know by definition that $B^2_{L_{2t-2}}$ is not a circuit boxed cycle basis. $\square$

Since we have zero-one $\implies$ circuit boxed from Theorem 2.3 it follows immediately from Proposition 3.4 that the basis $B^2_{L_{2t-2}}$ for the Möbius ladder is not a zero-one cycle basis.

3.2 Wagner's Graph

We consider Wagner's Graph again, a graph containing eight vertices and twelve arcs where each vertex of the underlying undirected graph has degree three. In Figure 3.9 we feature the Möbius ladder on eight vertices and Wagner's graph. We have provided a labeling of the vertices to show a one-to-one correspondence between the vertex sets of each graph such that if two vertices are joined by an edge in one graph, then the corresponding vertices are joined by an edge in the other graph. Thus, the Möbius ladder L_8 is isomorphic to Wagner's graph V_8. Notice that the four 4-gon circuits of the Möbius ladder L_8 appear in Wagner's graph as the figure eight-like circuits that cross through the center of the graph.

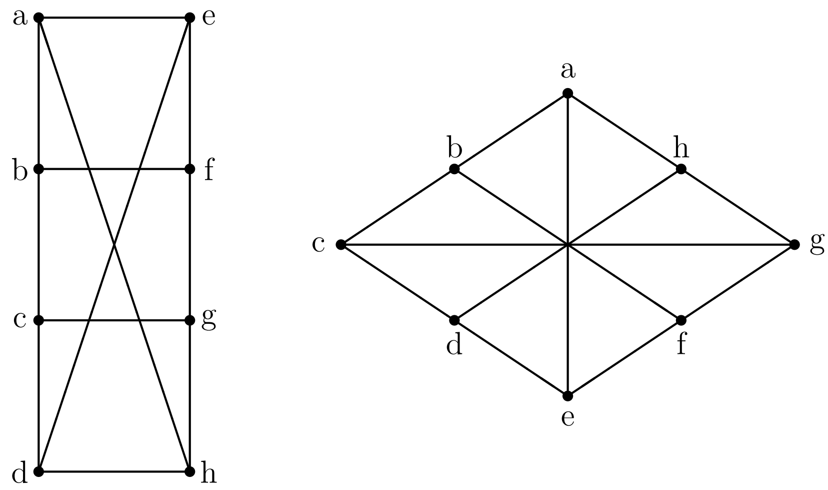

Figure 3.9: Möbius Ladder L_8 and Wagner's Graph V_8

We assign direction to the edges of Wagner's graph in Figure 3.10. Recall that we have $\nu = 5$. The circuits $C_1 = v_2 \to v_3 \to v_7 \to v_6$, $C_2 = v_3 \to v_4 \to v_8 \to v_7$, $C_3 = v_4 \to v_5 \to v_1 \to v_8$, $C_4 = v_5 \to v_6 \to v_2 \to v_1$, $C_5 = v_6 \to v_5 \to v_4 \to v_3 \to v_2$ form a cycle basis B_2 for V_8. We let Γ_2 be the cycle matrix associated with the basis B_2. The five circuits are clearly independent.

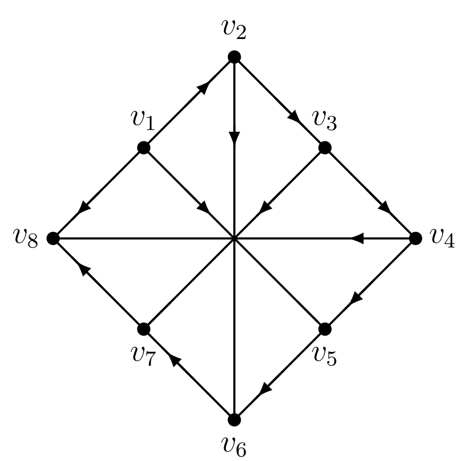

Figure 3.10: Wagner's Graph V_8

$$
\mathbf{\Gamma_2} = \begin{array}{c} \\ (v_1,v_2) \\ (v_1,v_5) \\ (v_1,v_8) \\ (v_2,v_3) \\ (v_2,v_6) \\ (v_3,v_4) \\ (v_3,v_7) \\ (v_4,v_5) \\ (v_4,v_8) \\ (v_5,v_6) \\ (v_6,v_7) \\ (v_7,v_8) \end{array}
\begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

The permutation of the rows and columns of Γ_2 produces a matrix Γ_2^P where the last five rows form a 5 x 5 upper triangular submatrix. Thus by Theorem 1.2 we know that B_2 is weakly fundamental.

$$
\Gamma_2^P = \begin{array}{c} \\ (v_1,v_5) \\ (v_2,v_3) \\ (v_2,v_6) \\ (v_3,v_4) \\ (v_3,v_7) \\ (v_4,v_5) \\ (v_4,v_8) \\ (v_6,v_7) \\ (v_7,v_8) \\ (v_1,v_8) \\ (v_5,v_6) \\ (v_1,v_2) \end{array}
\begin{array}{c}
\begin{array}{ccccc} C_1 & C_2 & C_3 & C_5 & C_4 \end{array} \\
\begin{pmatrix}
0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\end{array}
$$

Liebchen and Rizzi take six copies of V_8 to create a star-like graph as seen in [5]. They determine a basis for the star-like graph. They use the five circuits of basis B_2 for each one of the six copies of V_8. They assert the independence of the 30 circuits and note that this cycle basis is not weakly fundamental since each arc is contained in at least two circuits. Furthermore they offer a proof to show that the cycle matrix of the basis of the star-like graph is totally unimodular. However, we have just seen that B_2 is weakly fundamental, and we see in the following that the cycle matrix of the basis B_2 for one copy of Wagner's graph is not totally unimodular. It is not the case that each collection of columns of Γ_2 can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries only $0, +1,$ and -1. Take the collection of columns C_1, C_2, C_3, C_4. By inspection of Γ_2 it is clear that there is no way to split this collection into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries only $0, +1,$ and -1. Notice that the entries in rows $(v_1, v_5), (v_3, v_7)$ and (v_4, v_8) will force columns C_1, C_2, C_3, C_4 to all be in one part. However, the sum of the entries in row (v_2, v_6) for these four columns is -2. Thus, by Theorem 1.3 we can conclude that the matrix Γ_2 is not totally unimodular. Alternatively, we can form the square submatrix using these same four rows $(v_1, v_5), (v_2, v_6), (v_3, v_7)$ and (v_4, v_8) from columns C_1, C_2, C_3, C_4, and we find that the determinant of this square submatrix is two. Again, we are assured that Γ_2 is not totally unimodular.

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Curriculum Vitae

For as long as she can remember, Barbara Brown has loved math and helping children learn. Combining these passions, she earned a degree in secondary education with a concentration in mathematics at the University of Virginia in 1982. After teaching for a short time in Alexandria City Public Schools, she and her husband started a family. While Barbara stayed home to raise their five wonderful children, she maintained her teaching license and fostered a love of learning in their children.

In 2006, she returned to school part time at the University of Mary Washington. While there, she prepared an honors thesis entitled Generalized Dihedral Groups of Small Order under the direction of Dr. Randall Helmstutler and completed a degree in mathematics. Beginning in 2013 when their youngest child started his college career, Barbara enrolled in the graduate mathematics program at George Mason University. Upon completion of her degree, she plans to continue teaching mathematics.