CALCULATING THE MINIMAL FREE RESOLUTION OF THE STANLEY-REISNER IDEAL OF A SELF-DUAL SIMPLICIAL COMPLEX

by

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at George Mason University

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Abstract

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Combinatorial Commutative Algebra is a popular subject for investigation and for application. It has various types of benefits widely acknowledged in some other branches of Mathematics and other Sciences. One of those is calculating the minimal free resolution of a Stanley-Reisner ring, the main purpose for this project. To be convenient for readers, this thesis is divided into two parts, the theory part and the example part. In the theory, we concentrate building the process in two ways, the Gröbner basis way and the Hochster's theorem way. However, to understand both of them, we need to cover the basic knowledge about abstract simplicial complex, minimal non-faces, Stanley-Reisner ring and etc. After defining these, the two ways can be accessed. For the Gröbner basis way, we assume that the readers know the concepts of an ideal over a polynomial ring, especially in this case the Noetherian domain (because, the polynomial ring is over a field), and have some intuition for divisibility from Number theory. Then we have a foundation to comprehend the definitions of syzygy, Gröbner basis, S-polynomial etc. From those, the algorithm is displayed effectively. Schreyer's theorem and the Macaulay matrix help a lot for doing the computation.

However, the second way is more direct. Observing instead directly the bases of the modules, we can go through the computation of Betti numbers by using the reduced homology. This is acomplished through several concepts of upper Koszul complex and link to determine the Betti number of the grade that we are looking at. Then, gradually, both are enough to detect the resolution. Gaining these theories is sufficient for the second part. In this, we process some problems for self-dual simplicial complexes to identify their minimal free resolutions, using the two ways above. With the solution I for the Gröbner basis way and solution II for Hochster's theorem way, we can distinguish and analyze the interaction between them. This is a good thing to better understand and obtain a good orientation for the new application in computer science and maybe in the real life. In conclusion, this topic is very useful and realistic for several new approaches. Hence, this thesis is written to clarify the computation and serve as material for advanced investigations. I hope through this work, we can obtain enough ingredients to continue on the research and to develop the skills for interesting areas like this, especially Combinatorial Commutative Algebra and Computational Algebra.

Chapter 1: Introduction and Theory

1.1 Introduction

Combinatorial commutative algebra is a branch of Mathematics, popularized by Richard Stanley with various types of application such as the upper bound theorem for simplicial spheres, g-theorem, enumeration of magic squares, etc. This subject consumes many results of algebra by mathematicians such as Reisner and Hochster. Especially, "Computing the minimal free resolution of the Stanley-Reisner ring of a self-dual simplicial complex" is presented by this thesis below.

1.2 Definitions

In this section, the definitions from 1 to 8 are taken from the book of Miller and Sturmfels [4].

Definition 1. *(Definition of a simplicial complex) Let* $V = \{v_1, v_2, \ldots, v_n\}$ *be a finite set of vertices. Let* Δ *be a collection of subsets of* V *satisfying that* $\forall \alpha \in \Delta$ *then* $\forall \beta \subseteq \alpha \implies \beta \in \Delta$*. Hence,* Δ *is called a simplicial complex of* V*. Every subset of* V *that belongs to* Δ *is called a face. Every subset of* V *that does not belong to* Δ *is called a non-face. (We denote* Π *to be the collection of those non-faces).*

Notice that it is not necessary that a vertex in a simplicial complex is a face. Let's denote Π to be the collection of all non-faces of V.

Proposition 1. *(Properties of* Δ *and* Π*) Let* $V = \{v_1, v_2, \ldots, v_n\}$ *be a finite set of vertices. Let* Δ *and* Π *be the collections of faces and non-faces respectively. Then,*

- *A subset of a face is a face.*

- *A superset of a non-face is a non-face.*
- Δ *is bounded above* $\implies$ Δ *has maximal elements, are called maximal faces.*
- Π *is bounded below* $\implies$ Π *has minimal elements, are called minimal non-faces.*

Notice that the dimension of a face or a non-face X is defined by $|X|-1$.

Definition 2. *(Definition of a self-dual simplicial complex) Let* $V=\{v_1,v_2,\dots,v_n\}$ *be a finite set of vertices. Let* Δ *be a simplicial complex on* V*. Let* $\Delta^*=\{\overline{\tau}|\forall\tau\notin\Delta\}$ *where* $\overline{\tau}=V\backslash\tau$ *then* Δ^* *is called the duality of* Δ*. If* $\Delta=\Delta^*$ *then* Δ *is said to be self-dual.*

Definition 3. *(Definition of the monomial corresponding to a face or a non-face) Let* Δ *be a self-dual simplicial complex of* $V=\{v_1,v_2,\dots,v_n\}$*. Let* $F[v_1,v_2,\dots,v_n]$ *be the polynomial ring over a field* F *corresponding to* V*. The monomial* m_γ *corresponding to a face (or a non-face)* γ *where* $\gamma=\{v_{i_1},v_{i_2},\dots,v_{i_k}\}$*, is defined by* $m_\gamma=\prod_{j=1}^{k}v_{i_j}=v_{i_1}v_{i_2}\dots v_{i_k}$*.*

Notice that the monomial corresponding to a face (or a non-face) is square-free.

Definition 4. *(Definition of the Stanley-Reisner ring (or Stanley-Reisner module)) Let* Δ *be a self-dual simplicial complex on* $V=\{v_1,v_2,\dots,v_n\}$*. Let* $F[v_1,v_2,\dots,v_n]$ *be the polynomial ring over a field* F *corresponding to* V*. Let* I_Δ *be an ideal generated by the monomials corresponding to the minimal non-faces. Then,* $F[v_1,v_2,\dots,v_n]/I_\Delta$ *is called the Stanley-Reisner ring (or Stanley-Reisner module).*

Assume that we already know the definition of module over a commutative ring. In particular, in our case, the ring is regarded to be a polynomial ring over a field or a division ring, which is suitable for our succeeding arguments. Next, the definition of a free module is described in more detail, which are taken from the book of "Using Algebraic Geometry" [2] and "An introduction to Homological Algebra" [5].

Definition 5. *(Definition of a free module) Let* R *be a ring. Let* M *be a module over* R*. Let* E *be a generating set of* M*. If* E *is linearly independent then* M *is said to be free. Furthermore,* $Rank(M)=|E|$*.*

Definition 6. *(Definition of a minimal free resolution) Let* Δ *be a self-dual simplicial complex on* $V = \{v_1, v_2, \dots, v_n\}$*. Let* $M = F[v_1, v_2, \dots, v_n]/I_\Delta$*. Given a sequence of modules of* M

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Then it is called the minimal free resolution of M *if*

- *This sequence is exact i.e* $Im(d_i) = Ker(d_{i-1})\, \forall i = 0, 1, 2, \dots$.
- *Every* T_i *is a free module over* $F[v_1, v_2, \dots, v_n]$*.*
- $Rank(T_i) = |S_i|\, \forall i = 0, 1, 2, \dots$ *where* S_i *is the minimal generating set of* $Im(d_i)$*.*

Before comprehending Schreyer's Theorem which applies directly to the technique which we plan to use, let's look at the definitions of lexicographic ordering and the leading term of a polynomial. Beginning with a set of monomials, we form an ordering on it.

Definition 7. *(Definition of lexicographic order) Let* S *be a set, containing monomials. A lexicographic order or dictionary order* $\leq$ *is the order relation, identified by when we consider all the monomials of* S *by their vectors of exponents. i.e. if* $m \in S$ *such that* $m = \prod_{i=1}^{n} {x_i}^{a_i}$*, then we write* $vec(m) = (a_1, a_2, a_3, \dots)$*. From that, the relation is certified that if* m_1 *and* $m_2 \in S$ *where* $vec(m_1) = (b_1, b_2, b_3, \dots)$ *and* $vec(m_2) = (c_1, c_2, c_3, \dots)$*, we compare from the first position gradually to later until identifying* i *is the smallest index that* $b_j = c_j\, \forall j < i$ *and* $b_i \neq c_i$*, if* $b_i < c_i$ *then* $m_1 \leq m_2$*.*

Note: There exists a monomial term order that is the graded reverse lexicographic order defined by: $x^A > x^B$ if either $degree(x^A) > degree(x^B)$ or $degree(x^A) = degree(x^B)$ and the last non-zero entry of the vector of integers $A - B$ is negative. This is the default order in Macaulay2 [3], in large part because it is often the most efficient order for use with Gröbner bases. By giving GRevLex a list of integers, one may change the definition of the order: $degree(x^A)$ is the dot product of A with the argument of GRevLex.

Definition 8. *(Definition of the leading term of a polynomial) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let f be a polynomial in $F[v_1, v_2, \ldots, v_n]$. The leading term of f (denoted by $lt(f)$) is the term of f, whose monomial is the biggest monomial in f.*

Definition 9. *(Definition of Gröbner basis) Let Δ be a self-dual simplicial complex on a finite set $V = \{v_1, v_2, \ldots, v_n\}$. Let $F[v_1, v_2, \ldots, v_n]$ be a polynomial ring over a field F. Assume we have the minimal resolution.*

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $G_i = \{g_1, g_2, \ldots, g_m\}$ be a generating set of $Im(d_i)$ then G_i is called a Gröbner basis of $Im(d_i)$ if $\langle \{lt(f) : f \in Im(d_i)\} \rangle = \langle lt(g_1), lt(g_2), \ldots, lt(g_m) \rangle$.

Notice that we can form S_i from G_i by taking some elements from G_i which is generated by others.

Theorem 1. *(Schreyer's theorem) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let $M = F[v_1, v_2, \ldots, v_n]/I_\Delta$. Assume we have the minimal free resolution*

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $S_i = \{g_1, g_2, \ldots, g_n\}$ be a minimal generating set, contained in a Gröbner basis of $Im(d_i)$. Let $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ be the set of basis of T_i. Then $\{\sigma_{ij} | \sigma_{ij} = \frac{M(g_i,g_j)}{lt(g_i)} \epsilon_i - \frac{M(g_i,g_j)}{lt(g_j)} \epsilon_j - (\frac{M(g_i,g_j)}{lt(g_i)} g_i - \frac{M(g_i,g_j)}{lt(g_j)} g_j)$ where $M(g_i, g_j) = LCM(lt(g_i), lt(g_j))$ forms a Gröbner basis of $Im(d_{i+1})$.

Definition 10. *(Definition of Macaulay matrix) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let $M = F[v_1, v_2, \ldots, v_n]/I_\Delta$. Assume we have the minimal free* *resolution*

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ be the set of basis of T_i. Let $S_{i+1} = \{s_1, s_2, \ldots, s_m\}$ be a minimal generating set of $Im(d_{i+1}$. Then, the matrix A_{nm} for which its ij-entry a_{ij} is the coefficient of ϵ_i in s_j is called a Macauley matrix of T_i.

Notice that all of the entries of the Macauley matrix A_{nm} is square-free monomials. So, it is really a monomial matrix.

In addition, to compute the minimal resolution, we also have another way to be quicker and more direct to the observation for each term of free modules. "Betti number" and "Hochster's theorem," are two necessary ingredients in our second approach. To see what they say, let's observe the statements from [4], [1] below. We assume the readers know how to calculate reduced simplicial homology.

Definition 11. *(Definition of Betti-number) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let Π be the collection of all non-faces, which does not belong to Δ. Let $M = F[v_1, v_2, \ldots, v_n]/I_\Delta$. Assume we have the minimal free resolution*

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $\mathbf{b} \in \Pi$. Then, the Betti-number $\beta_i(\mathbf{b}, M)$ is the number of basis in T_i with the same degree $\mathbf{b}$.

Definition 12. *(Definition of the upper Koszul complex) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let Π be the collection of all non-faces, which does not belong to Δ. Let $\mathbf{b} \in \Pi$. Then, the upper Koszul complex of $\mathbf{b}$ is defined by $K^{\mathbf{b}} = \{\tau | \mathbf{b} \backslash \tau \in \Pi\}$.*

Definition 13. *(Definition of the link) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. The link of a face $\sigma \in \Delta$ is denoted $link(\sigma) = \{\tau | \tau \cup \sigma \in \Delta \, and \, \tau \cap \sigma = \emptyset\}$.*

Notice that when Δ is self-dual, then $link_{\Delta^*}(\overline{\sigma}) = K^{\sigma}$.

Theorem 2. *(Hochster's theorem) Let Δ be a self-dual simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$. Let Π be the collection of all non-faces, which does not belong to Δ. Let $M = F[v_1, v_2, \ldots, v_n]/I_{\Delta}$. Assume we have the minimal free resolution*

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $\mathbf{b} \in \Pi$. Then, $\beta_{i+1}(\mathbf{b}, M) = |\tilde{H}_{i-1}(K^{\mathbf{b}}, F)| = |\tilde{H}_{i-1}(link_{\Delta^*}(\overline{\mathbf{b}}); F)|$.

(Notice that $|\tilde{H}_{i-1}(K^{\mathbf{b}}; F)|$ denotes $Dim(\tilde{H}_{i-1}(K^{\mathbf{b}}; F))$). And it is the same for $|\tilde{H}_{i-1}(link_{\Delta^*}(\overline{\mathbf{b}}); F)|$.

1.3 Properties of the self-dual simplicial complex

Proposition 2. *(Properties of self-dual simplicial complex) According to the definition, we can see that when Δ is self-dual then*

1. *Δ does not have any empty minimal non-faces unless Δ is the void complex.*
2. *If τ and σ are two non-faces of Δ then $\tau \cap \sigma \neq \emptyset$*
3. *If τ and σ are two faces of Δ then $\tau \cup \sigma \neq \{1, 2, \ldots, n\}$ where $\{1, 2, \ldots, n\}$ is the vertex set of Δ.*
4. *There is at most one vertex that is not a face of Δ.*
5. *If $\tau \subseteq \{1, 2, \ldots, n\}$ then τ or $\overline{\tau}$ is in Δ.*

Proof :

1. If Δ is non-void then if $\emptyset$ is a non-face of Δ then $\emptyset \neq \overline{\emptyset} \in \Delta$, but $\emptyset \in \Delta \implies \overline{\emptyset}$ is a non-face of Δ (contradiction).

2. If $\tau \cap \sigma = \emptyset$ then if τ is a non-face of $\Delta \implies \sigma \subseteq \overline{\tau} \implies \sigma \in \Delta$ (contradicts the hypothesis).

3. Using De Morgan's law, we have if $\tau \cup \sigma = 1, 2, \ldots, n \implies \overline{\tau} \cap \overline{\sigma} = \emptyset \implies \overline{\tau}$ or $\overline{\sigma} \in \Delta$ (contradiction) .

4. Assume we have two distinct vertices v and w that are non-faces of Δ then $\{v\} \cap \{w\} = \emptyset$ (contradicts the second property)

5. If $\tau \subseteq \{1, 2, \ldots, n\}$ then if $\tau \in \Delta \implies \overline{\tau}$ is a non-face of Δ and conversely, if τ is a non-face of $\Delta \implies \overline{\tau} \in \Delta$.

Now, we have gained enough ingredients for our work. However, to be easy to observe and access, this writing is divided into two parts to make the readers able to compare and punctually consume theories to get used to it. Those are the applications, written under the examples in the following part.

Chapter 2: Examples

In this part, we will use sets and square-free monomials interchangably.

Example 1 :

A triangle wxy and a vertex z where Δ is the collection of all subsets of $\{w, x\}$, $\{x, y\}$, $\{w, y\}$ and $\{z\}$.

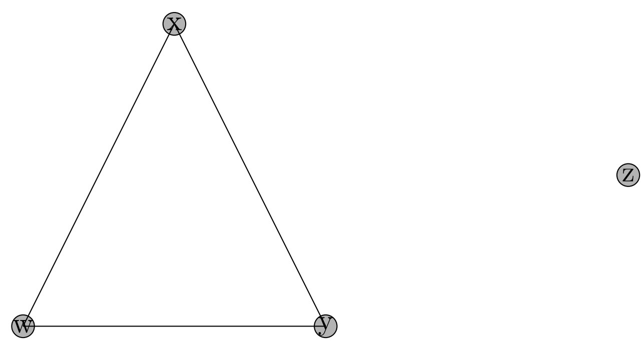

Figure 2.1: Example 1

Solution I:

Let F be a field. Let $S = F[w, x, y, z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ to be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{wz, xz, yz, wxy\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle wz, xz, yz, wxy \rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing

$S \xrightarrow{d_0} S/I_\Delta$, we can immediately see that $Ker(d_0) = I_\Delta = \langle wz, xz, yz, wxy\rangle$.

Denoting $(g_1, g_2, g_3, g_4) = (wz, xz, yz, wxy)$ respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. Define $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, \ldots, 4$. Then, $T_1 = \bigoplus_{i=1}^4 S\epsilon_i \cong S^4$ and we see that $T_1 \xrightarrow{d_1} T_0$ to be converted to $S^4 \xrightarrow{d_1} S$. We define the degree ϵ_i to be the same degree of g_i $\forall i = 1, 2, 3, 4$.

Let $p_1, p_2, p_3, p_4 \in T_1$ be such that

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix} \begin{bmatrix} -x & -y & 0 & 0 \\ w & 0 & -y & 0 \\ 0 & w & x & -wx \\ 0 & 0 & 0 & z \end{bmatrix}.$$

Clearly, we can see that $d_1(p_i) = 0 \, \forall i = 1, \ldots, 4$ and that the syzygies $p_5 = -wy\epsilon_2 + z\epsilon_4$ and $p_6 = -xy\epsilon_1 + z\epsilon_4$ are in $\langle p_1, p_2, p_3, p_4\rangle$. By Schreyer's Theorem, $Ker(d_1) = \langle p_1, p_2, p_3, p_4\rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we define T_2 to be the next free module with the basis $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_2(\sigma_i) = p_i \forall i = 1, \ldots, 4$. Then $T_2 = \bigoplus_{i=1}^4 S\sigma_i \cong S^4$. Thus, $T_2 \xrightarrow{d_2} T_1$ is converted to $S^4 \xrightarrow{d_2} S^4$. We define the degree of σ_i is the same degree with p_i $\forall i = 1, 2, 3, 4$.

Let $q \in T_2$ be such that

$$\begin{bmatrix} q \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{bmatrix} \begin{bmatrix} y \\ -x \\ w \\ 0 \end{bmatrix}.$$

Clearly, $q \in Ker(d_2)$ and by Schreyer's theorem, then $Ker(d_2) = \langle q\rangle$. With $T_3 \xrightarrow{d_3} T_2$, we define T_3 to be the next free module with the basis γ. Defining $d_3 : T_3 \longrightarrow T_2$ such that $d_3(\gamma) = q$. Then $T_3 = S\gamma \cong S$. Thus $T_3 \xrightarrow{d_3} T_2$ is converted to $S \xrightarrow{d_3} S^4$. We define the degree of γ is the same degree with q.

After having these, we can detect that $Ker(d_3) = 0$ and there is no reason that $T_4 \neq 0$ while observing

$$T_4 \xrightarrow{d_4} T_3.$$

So, $T_4 \xrightarrow{d_4} T_3$ is rewritten as $0 \xrightarrow{d_4} S$. Moving along to later terms T_n where $n \geq 4$, we also obtain the result that $T_n = 0$.

So, we obtain the minimal resolution

$$0 \longrightarrow S \xrightarrow{d_3} S^4 \xrightarrow{d_2} S^4 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[w, x, y, z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{wz, xz, yz, wxy\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle wz, xz, yz, wxy \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2$ where λ_i is defined to be a i-th level set of I_Δ.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We see

$\lambda_1 = \mu_\Delta = \{wz, xz, yz, wxy\}$.

$\lambda_2 = \{wzx, wzy, xzy, wxyz\}$.

- For $\lambda_1 = \mu_\Delta = \{wz, xz, yz, wxy\}$, we can see that $\forall \mathbf{b} \in \lambda_1$. Because, $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \forall n \geq 2$.
- For $\lambda_2 = \{wzx, wzy, xzy, wxyz\}$, we can obtain that $\forall \mathbf{b} \in \lambda_2 \backslash \{wxyz\}$ then

$\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \forall n \geq 3$ because, at this time, $K^{\mathbf{b}}$ has three elements are the empty set and two 0-faces. Further, especially for $wxyz$, $K^{wxyz} = \{\emptyset, w, x, y, z, wx, wy, xy\} = \Delta$ then $\beta_2(wxyz, S/I_\Delta) =$
$|\tilde{H}_0(K^{wxyz}; F)| = 1$; $\beta_3(wxyz, S/I_\Delta =$
$|\tilde{H}_1(K^{wxyz}; F)| = 1$ and $\beta_n(wxyz, S/I_\Delta) = |\tilde{H}_{n-2}(K^{wxyz}; F)| = 0 \forall n \geq 2$.

So, we can see that

$M = S/I_\Delta$

$T_0 = S$

Since $Rank(T_1) = \sum_{\mathbf{b} \in I_\Delta} \beta_1(\mathbf{b}; S/I_\Delta) = 4$. Then $T_1 = S^4$.

Since $Rank(T_2) = \sum_{\mathbf{b} \in I_\Delta} \beta_2(\mathbf{b}; S/I_\Delta) = 4$. Then $T_2 = S^4$.

Since $Rank(T_3) = \sum_{\mathbf{b} \in I_\Delta} \beta_3(\mathbf{b}; S/I_\Delta) = 1$. Then $T_3 = S$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S \xrightarrow{d_3} S^4 \xrightarrow{d_2} S^4 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0$$

Example 2

An empty tetrahedron $wxyz$ and a vertex v outside. Δ is the collection of all subsets of $\{x, y, z\}$, $\{w, y, z\}$, $\{w, x, z\}$, $\{w, x, y\}$ and $\{v\}$.

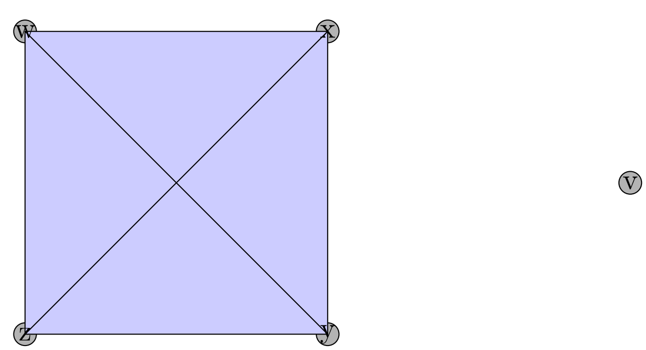

Figure 2.2: Example 2

Solution I:

Let F be a field. Let $S = F[v,w,x,y,z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ to be a set of monomials corresponding to of minimal non-faces of Δ. Clearly, $\mu_\Delta = \{vw, vx, vy, vz, wxyz\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle vw, vx, vy, vz, wxyz\rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that $Ker(d_0) = I_\Delta = \langle vw, vx, vy, vz, wxyz\rangle$.

Denoting $(g_1, g_2, g_3, g_4, g_5) = (vw, vx, vy, vz, wxyz)$ respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5$. Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, \ldots, 5$. Then, $T_1 = \bigoplus_{i=1}^{5} S\epsilon_i \cong S^5$ and we see that $T_1 \xrightarrow{d_1} T_0$ to be converted to $S^5 \xrightarrow{d_1} S$.We define the degree of ϵ_i is the same degree with the degree of $g_i \; \forall i = 1, \ldots, 5$.

Let $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in T_1$ be such that $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{bmatrix} =$

$$\begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 \end{bmatrix} \begin{bmatrix} -x & -y & 0 & -z & 0 & 0 & 0 \\ w & 0 & -y & 0 & -z & 0 & 0 \\ 0 & w & x & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & w & x & y & -wxy \\ 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix}.$$

Clearly, $d_1(p_i) = 0 \, \forall i = 1, \ldots, 7$ and by Schreyer's theorem, $Ker(d_1) = \langle p_1, p_2, p_3, p_4, p_5,$

$p_6, p_7\rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we can define T_2 to be the next free module with the basis $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_2(\sigma_i) = p_i \, \forall i = 1, \ldots, 7$. Then $T_2 = \bigoplus_{i=1}^{7} S\sigma_i \cong S^7$. Thus, $T_2 \xrightarrow{d_2} T_1$ is converted to $S^7 \xrightarrow{d_2} S^5$. We define the degree of σ_i is the same degree with the degree of $p_i \; \forall i = 1, \ldots, 7$.

Let $q_1, q_2, q_3, q_4 \in T_2$ be such that

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 \end{bmatrix} \begin{bmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ w & 0 & 0 & z \\ 0 & -x & -y & 0 \\ 0 & w & 0 & -y \\ 0 & 0 & w & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $d_2(q_i) = 0 \, \forall i = 1, \ldots, 4$ and by Schreyer's theorem, $Ker(d_2) = \langle q_1, q_2, q_3, q_4 \rangle$.

With $T_3 \xrightarrow{d_3} T_2$, we define T_3 to be the next free module with the basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Defining $d_3 : T_3 \longrightarrow T_2$ such that $d_3(\gamma_i) = q_i \, \forall i = 1, \ldots, 4$. Then $T_3 = \bigoplus_{i=1}^{4} S\gamma_i \cong S^4$. Thus, $T_3 \xrightarrow{d_3} T_4$ is converted to $S^4 \xrightarrow{d_3} S^7$. We define the degree of γ_i to be the same degree with the degree of $q_i \; \forall i = 1, \ldots, 4$.

Let $r \in T_3$ be such that

$$\begin{bmatrix} r \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \begin{bmatrix} -z \\ y \\ -x \\ w \end{bmatrix}.$$

Clearly, $d_3(r) = 0$ and by Schreyer's theorem, $Ker(d_3) = \langle r \rangle$.

With $T_4 \xrightarrow{d_4} T_3$, we define T_4 to be the next free module with the basis β. Defining $d_4 : T_4 \longrightarrow T_3$ such that $d_3(\beta) = r$. Then $T_4 = S\beta \cong S$. Thus, $T_4 \xrightarrow{d_4} T_3$ is converted to $S \xrightarrow{d_4} S^4$. We define the degree of β to be the same degree with the degree of r.

We can easily see that $Ker(d_4) = 0$ then certainly $T_i = 0\, \forall i \geq 5$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S \xrightarrow{d_4} S^4 \xrightarrow{d_3} S^7 \xrightarrow{d_2} S^5 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II :

Let F be a field. Let $S = F[v, w, x, y, z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{vw, vx, vy, vz, wxyz\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle vw, vx, vy, vz, wxyz \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3$ where λ_i is defined to be a i-th level set of I_Δ.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We can see that

$\lambda_1 = \mu_\Delta = \{vw, vx, vy, vz, wxyz\}$.

$\lambda_2 = \{vwx, vwy, vwz, vxy, vxz, vyz, vwxyz\}$.

$\lambda_3 = \{vwxy, vwyz, vwxz, vxyz\}$

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0\, \forall n \geq 1$.

- $\forall \mathbf{b} \in \lambda_2 \backslash \{wxyzv\}$then

  $\beta_2(\mathbf{b}; S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0\, \forall n \geq 3$ because

$K^{\mathbf{b}}$ contains the empty and two 1-faces. But it will be different for $vwxyz$ when we detect that $K^{vwxyz} = \Delta = \{\emptyset, v, w, x, y, z, wx, wy, wz, xy, xz, yz, wxy, wyz, wxz, xyz\}$ then $\beta_2(vwxyz, S/I_\Delta) = |\tilde{H}_0(K^{vwxyz}; F)| = 1$; $\beta_3(vwxyz, S/I_\Delta) = |\tilde{H}_1(K^{vwxyz}; F)| = 0$; $\beta_4(vwxy, S/I_\Delta) = |\tilde{H}_2(K^{vwxyz}; F)| = 1$ and further $\beta_n(vwxyz, S/I_\Delta) = |\tilde{H}_{n-2}(K^{vwxyz}; F)| = 0\, \forall n \geq 5$.

- $\forall \mathbf{b} \in \lambda_2$ then $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 0; \beta_3(\mathbf{b}, S/I_\Delta) = |\tilde{H}_1(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \forall n \geq 4$.

With these results, we see

$M = S/I_\Delta)$

$T_0 = S$

Since, $Rank(T_1) = 5 \times 1 = 5$ then $T_1 = S^5$.

Since $Rank(T_2) = \binom{4}{2} \times 1 + 1 = 7$ then $T_2 = S^7$.

Since $Rank(T_3) = \binom{4}{3} \times 1 = 4$ then $T_3 = S^4$.

Since $Rank(T_4) = 1$ then $T_4 = S$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S \xrightarrow{d_4} S^4 \xrightarrow{d_3} S^7 \xrightarrow{d_2} S^5 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Example 3 :

There are three vertices x,y,z. Δ is collection of subsets $\{x\}$, $\{y\}$ and $\{z\}$.

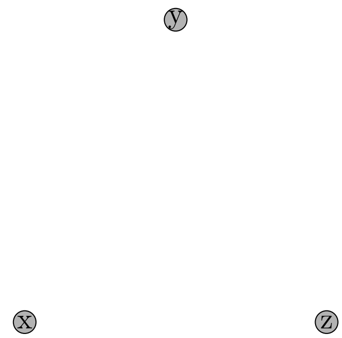

Figure 2.3: Example 3

Solution I :

Let F be a field. Let $S = F[x, y, z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ to be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{xy, xz, yz\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle xy, xz, yz \rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that $Ker(d_0) = I_\Delta = \langle xy, xz, yz \rangle$.

Denoting $(g_1, g_2, g_3) = (xy, xz, yz)$ respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3$. Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, 2, 3$. Then, $T_1 = \bigoplus_{i=1}^3 S\epsilon_i \cong S^3$ and we see that $T_1 \xrightarrow{d_1} T_0$ to be converted to $S^3 \xrightarrow{d_1} S$. We define the degree of ϵ_i to be the same degree with the degree of $p_i \ \forall i = 1, 2, 3$.

Let $p_1, p_2 \in T_1$ such that

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix} \begin{bmatrix} -z & 0 \\ y & -y \\ 0 & x \end{bmatrix}.$$

Clearly, $d_1(p_i) = 0 \, \forall i = 1, \dots, 3$ and by Schreyer's Theorem, $Ker(d_1) = \langle p_1, p_2 \rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we can define T_2 to be the next free module with basis σ_1, σ_2. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_2(\sigma_i) = p_i \, \forall i = 1, 2$. Then $T_2 = \bigoplus_{i=1}^{2} S\sigma_i \cong S^2$. Thus, $T_2 \xrightarrow{d_2} T_1$ is converted to $S^2 \xrightarrow{d_2} S^3$. We define the degree of σ_i to be the same degree with the degree of p_i $\forall i = 1, 2$.

By calculating, we detect that $Ker(d_2) = 0$ and certainly, $T_n = 0 \, \forall n \geq 3$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^2 \xrightarrow{d_2} S^3 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[x, y, z]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{xy, xz, yz\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle xy, xz, yz \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2$ where λ_i is defined to be a i-th level set of I_Δ.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We see that

$\lambda_1 = \{xy, yz, xz\}$

$\lambda_2 = \{xyz\}$

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/i_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 2$.

- $K^{xyz} = \Delta = \{\emptyset, x, y, z\}$ then $\beta_2(xyz, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 2$ and $\beta_n(xyz, S/I_\Delta) = |\tilde{H}_{n-2}(K^{xyz}; F)| = 0 \, \forall n \geq 3$

Therefore,

$M = S/I_\Delta$

$T_0 = S$

Since, $Rank(T_1) = 3$ then $T_1 = S^3$.

Since, $Rank(T_2) = 2$ then $T_2 = S^2$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^2 \xrightarrow{d_2} S^3 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Example 4 :

A solid triangle abc with a vertex d but Δ is the collection of all subsets of $\{a, b, c\}$

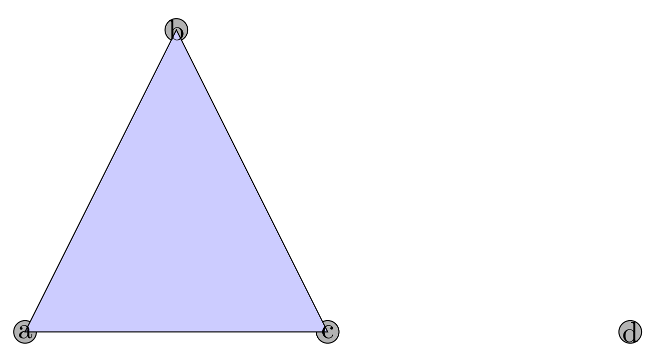

Figure 2.4: Example 4

Solution I :

Let F be a field. Let $S = F[a, b, c, d]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{d\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle d \rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, is the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that $Ker(d_0) = I_\Delta = \langle d \rangle$.

Denoting $g = d$ and observing $T_1 \xrightarrow{d_1} T_0$, we can define T_1 to be the free module with basis ϵ. Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon) = g$. Then $T_1 = S$. So, $T_1 \xrightarrow{d_1} T_0$ is converted to $S \xrightarrow{d_1} S$. We define the degree of ϵ to be the same degree with the degree of g.

We can see that $Ker(d_1) = 0$ so, $T_n = 0 \, \forall n \geq 2$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[a, b, c, d]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{d\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle d \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3 \cup \lambda_4$ where λ_i is i-level set.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We identify:

$\lambda_1 = \mu_\Delta = \{d\}$

$\lambda_2 = \{da, db, dc\}$

$\lambda_3 = \{dab, dbc, dac\}$

$\lambda_4 = \{abcd\}$

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F) = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \,\forall n \geq 2$

- $\forall \mathbf{b} \in \lambda_2$ then $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; f)| = 0 \,\forall n$

- $\forall \mathbf{b} \in \lambda_3$ then $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; f)| = 0 \,\forall n$

- $\forall \mathbf{b} \in \lambda_4$ then $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; f)| = 0 \,\forall n$

Therefore,

$M = S/I_\Delta$

$T_0 = S$

$T_1 = S$

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Example 5:

A picture with five vertices x,y,a,b,c with Δ is the collection of subsets of $\{a, x, y\}$, $\{b, x, y\}$, $\{a, b\}$,$\{c, x\}$ and $\{c, y\}$.

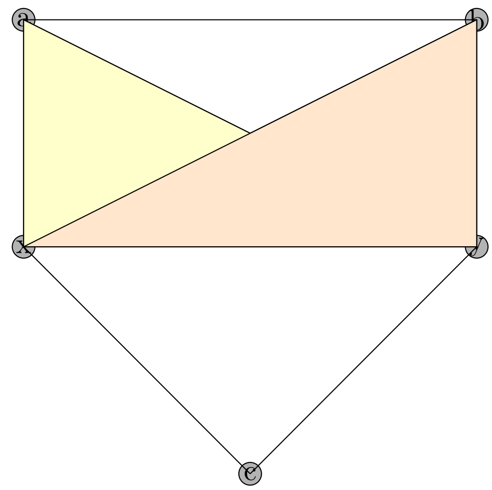

Figure 2.5: Example 5

Solution I:

Let F be a field. Let $S = F[x, y, a, b, c]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{ac, bc, xab, yab, xyc\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle ac, bc, xab, yab, xyc \rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, is the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that $Ker(d_0) = I_\Delta = \langle ac, bc, xab, yab, xyc, wxy \rangle$.

Denoting $(g_1, g_2, g_3, g_4, g_5) = (ac, bc, xab, yab, xyc)$ respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5$. Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, \ldots, 5$. Then, $T_1 = \bigoplus_{i=1}^{5} S\epsilon_i \cong S^5$ and we see that $T_1 \xrightarrow{d_1} T_0$ to be converted to $S^5 \xrightarrow{d_1} S$. Hence, $Ker(d_1)$ is identified through its generators. We define the degree of ϵ_i to be the same with the degree of p_i $\forall i = 1, \ldots, 5$.

Let $p_1, p_2, p_3, p_4, p_5, p_6 \in T_1$ such that $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix} =$

$$\begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 \end{bmatrix} \begin{bmatrix} -b & 0 & -xy & 0 & 0 & 0 \\ a & 0 & 0 & -xy & -xa & -ya \\ 0 & -y & 0 & 0 & c & 0 \\ 0 & x & 0 & 0 & 0 & c \\ 0 & 0 & a & b & 0 & 0 \end{bmatrix}.$$

Clearly $d_1(p_i) = 0 \, \forall i = 1, \ldots, 6$ and by Schreyer's Theorem, $Ker(d_1) = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we can define T_2 to be the next free module with basis $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_1(\sigma_i) = p_i \, \forall i = \overline{1, 6}$. Then $T_2 = \bigoplus_{i=1}^{6} S\sigma_i \cong S^6$. Thus, $T_2 \xrightarrow{d_2} T_1$ is converted to $S^6 \xrightarrow{d_2} S^5$. We define the degree of σ_i to be the same with the degree of p_i $\forall i = \overline{1, 6}$.

Let $q_1, q_2 \in T_2$ be such that

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \end{bmatrix} \begin{bmatrix} 0 & xy \\ -c & 0 \\ 0 & -b \\ 0 & a \\ -y & 0 \\ x & 0 \end{bmatrix}.$$

Clearly, $d_2(q_i) = 0 \, \forall i = 1, 2$ and by Schreyer's Theorem, $Ker(d_2) = \langle q_1, q_2 \rangle$.

With $T_3 \xrightarrow{d_3} T_2$, we can define T_3 to be the next free module with the basis γ_1, γ_2. Definining $d_3 : T_3 \longrightarrow T_2$ such that $d_3(\gamma_i) = q_i \, \forall i = 1, 2$. Then, $T_3 = \bigoplus_{i=1}^{2} S\gamma_i \cong S^2$. Thus,

$T_3 \xrightarrow{d_3} T_2$ is converted to $S^2 \xrightarrow{d_3} S^6$. We define the degree of γ_i to be the same with the degree of q_i $\forall i = 1, 2$.

By calculating, we can detect that $Ker(d_3) = 0$ hence, $T_n = 0 \, \forall n \geq 4$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^2 \xrightarrow{d_3} S^6 \xrightarrow{d_2} S^5 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[x, y, a, b, c]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{ac, bc, xab, yab, xyc\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle ac, bc, xab, yab, xyc \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3 \cup \lambda_4$ where λ_i is i-level set.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We identify:

$\lambda_1 = \mu_\Delta = \{ac, bc, xab, yab, xyc\}$

$\lambda_2 = \{acb, acx, acy, bcx, bcy, xzbc, xaby, yabc, cxya, cxyb\}$

$\lambda_3 = \{abcxy\}$

Therefore,

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 2$.

- $\forall \mathbf{b} \in \{acx, acy, bcx, bcy\}$ then $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n$ because, $K^{\mathbf{b}}$ contains a single vertex. Further, $\beta_2(abc, S/I_\Delta) = |\tilde{H}_0(K^{abc}; F)| = 1$ and $\beta_n(abc, S/I_\Delta) = |\tilde{H}_{n-2}(K^{abc}; F)| = 0 \, \forall n \geq 3$ because $K^{abc} = \{\emptyset, a, b\}$. Further, $\forall \mathbf{b} \in \{abcx, abcy, acxy, bcxy\}$ then $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| =$

$0\,\forall n \geq 3$. Further, $\beta_n(abxy, S/I_\Delta) = |\tilde{H}_{n-2}(K^{abxy}; F)| = 0\,\forall n$ because, $K^{abxy} = \{\emptyset, x, y, xy\}$. Further, $\beta_2(abc, S/I_\Delta) = |\tilde{H}_0(K^{abc}; F)| = 1$ and $\beta_n(abc, S/I_\Delta) = |\tilde{H}_{n-2}(K^{abc}; F)| = 0\,\forall n \geq 3$ because $K^{abc} = \{\emptyset, a, b\}$.

- $\beta_2(xyabc, S/I_\Delta) = |\tilde{H}_0(K^{xyabc}; F)| = 1$; $\beta_3(xyabc, S/I_\Delta) = |\tilde{H}_1(K^{xyabc}; F)| = 2$ and $\beta_n(xyabc, S/I_\Delta) = |\tilde{H}_{n-2}(K^{xyabc}; F)| = 0\,\forall n > 4$.

Therefore,

$M = S/I_\Delta$

$T_0 = S$

Since, $Rank(T_1) = 5$ then $T_1 = S^5$.

Since, $Rank(T_2) = 4 + 1 + 1 = 6$ then $T_2 = S^6$

Since, $Rank(T_3) = 2$ then $T_3 = S^2$.

Therefore,

$$0 \longrightarrow S^2 \xrightarrow{d_3} S^6 \xrightarrow{d_2} S^5 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Example 6 :

A picture with five vertices a, b, c, d, e with Δ is the collection of all subsets of $\{a,b\}$,$\{a,c\}$,$\{a,d\}$,$\{a,e\}$,$\{b,c\}$,$\{b,d\}$,$\{b,e\}$,$\{c,d\}$,$\{c,e\}$.

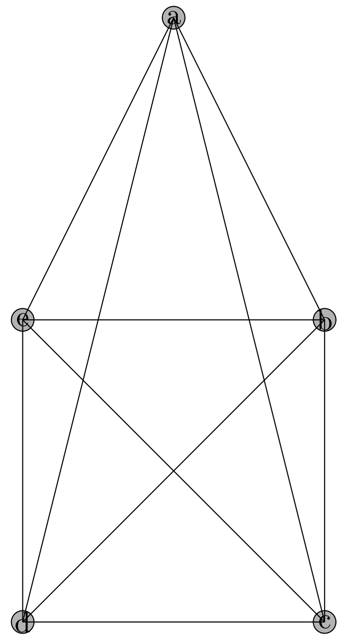

Figure 2.6: Example 6

Solution I:

Let F be a field. Let $S = F[a, b, c, d, e]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde\rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, is the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that

$$Ker(d_0) = I_\Delta = \langle abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde\rangle.$$

For the following situation, we denote

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}) = (abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde)$$

respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9, \epsilon_{10}$. Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, \ldots, 10$. Then, $T_1 = \bigoplus_{i=1}^{10} S\epsilon_i \cong S^{10}$ and we see that $T_1 \xrightarrow{d_1} T_0$ to be converted to $S^{10} \xrightarrow{d_1} S$. We define the degree of ϵ_i to be the same with the degree of p_i $\forall i = 1, \ldots, 10$.

Let $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15} \in T_1$ such that $p = \epsilon M$ where

$$p = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \end{bmatrix},$$

$$\epsilon = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 & \epsilon_9 & \epsilon_{10} \end{bmatrix},$$

$$M = \begin{bmatrix} -d & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & -c & 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & -b & 0 & 0 & 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e & 0 & 0 \\ 0 & 0 & 0 & c & -c & 0 & d & -d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & -b & 0 & 0 & 0 & d & -d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & d & -d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & -b & 0 & c & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & b \end{bmatrix}.$$

Clearly, $d_1(p_i) = 0 \, \forall i = 1, \ldots, 15$ and by Schreyer's Theorem, $Ker(d_1) = \langle p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15} \rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we can define T_2 to be the next free module with the basis $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_2(\sigma_i) = p_i \, \forall i = 1, \ldots, 15$. Then $T_2 = \bigoplus_{i=1}^{15} S\sigma_i \cong S^{15}$. Thus, $T_2 \xrightarrow{d_2} T_1$ is converted to $S^{15} \xrightarrow{d_2} S^{10}$. We define the degree of σ_i to be the same with the degree of p_i $\forall i = 1, \ldots, 15$.

Let $q_1, q_2, q_3, q_4, q_5, q_6 \in T_2$ be such that $q = \sigma M'$, where

$$q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \end{bmatrix},$$

$$M' = \begin{bmatrix} e & e & e & 0 & 0 & 0 \\ 0 & e & e & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \\ -d & -d & -d & 0 & 0 & 0 \\ 0 & -d & -d & d & d & 0 \\ 0 & 0 & -d & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & -c & 0 \\ 0 & 0 & 0 & 0 & -c & c \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

Clearly, $d_2(q_i) = 0 \, \forall i = 1, \ldots, 6$ and by Schreyer's theorem, $Ker(d_2) = \langle q_1, q_2, q_3, q_4, q_5,$

$q_6\rangle$.

With $T_3 \xrightarrow{d_3} T_2$, we can define T_3 to be the next free module with the basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_3(\gamma_i) = q_i \, \forall i = 1, \ldots, 6$. Then, $T_3 = \bigoplus_{i=1}^{6} S\gamma_i \cong S^6$. Thus, $T_3 \xrightarrow{d_3} T_2$ is converted to $S^6 \xrightarrow{d_3} S^{15}$. We define the degree of γ_i to be the same with the degree of q_i $\forall i = 1, \ldots, 6$.

By calculating, we can detect that $Ker(d_3) = 0$ and so, $T_n = 0 \forall n \geq 4$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^6 \xrightarrow{d_3} S^{15} \xrightarrow{d_2} S^{10} \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[a, b, c, d, e]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle abc, abd, acd, bcd, abe, ace, bce, ade, bde, cde \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3$ where λ_i is i-level set.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We identify:

λ_1 contains all 2-faces.

λ_2 contains all 3-faces.

λ_3 contains all 4-faces. (it means $\lambda_3 = \{abcde\}$.)

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 2$.

- $\forall \mathbf{b} \in \lambda_2$, $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 3$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)|$

$= 0\,\forall n \geq 3$ because, $K^{\mathbf{b}}$ only contains empty set and four vertices.

- We can see that K^{abcde} contains the empty set, all five vertices and all 2-faces. So, we can calculate that $\beta_2(abcde, S/I_\Delta) = |\tilde{H}_0(K^{abcde}; F)| = 0$; $\beta_3(abcde, S/I_\Delta) = |\tilde{H}_1(K^{abcde}; F)| = 10 - 4 = 6$ and $\beta_n(abcde; S/I_\Delta) = |\tilde{H}_{n-2}(K^{abcde}; F)| = 0\,\forall n \geq 4$.

Therefore,

$M = S/I_\Delta$.

$T_0 = S$.

Since, $Rank(T_1) = \binom{5}{3} \times 1 = 10$ then $T_1 = S^{10}$.

Since, $Rank(T_2) = \binom{5}{4} \times 3 = 15$ then $T_2 = S^{15}$.

Since, $Rank(T_3) = 6$ then $T_3 = S^6$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^6 \xrightarrow{d_3} S^{15} \xrightarrow{d_2} S^{10} \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Example 7:

There are five vertices and they are a,b,c,d,e and the simplicial complex Δ contains all subsets of the sets $\{c, d, e\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$,$\{b, c\}$, $\{b, d\}$, $\{b, e\}$.

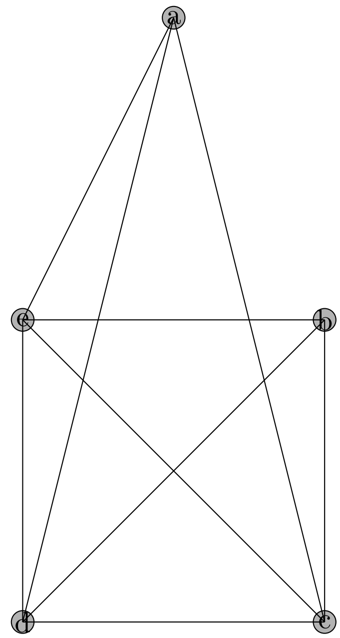

Figure 2.7: Example 7

Solution I:

Let F be a field. Let $S = F[a, b, c, d, e]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{ab, acd, bcd, ace, bce, ade, bde\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle ab, acd, bcd, ace, bce, ade, bde \rangle$ is the Stanley Reisner ideal. So, we can see that the module is $M = S/I_\Delta$, is the Stanley Reisner ring that we are accessing. Hence, we can obtain the general description for the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Starting from $T_0 \xrightarrow{d_0} M$, we define $T_0 = S$. Since, d_0 is the quotient map then observing $S \xrightarrow{d_0} S/I_\Delta$, we can immediately show that $Ker(d_0) = I_\Delta = \langle ab, acd, bcd, ace, bce, ade, bde \rangle$.

Denoting $(g_1, g_2, g_3, g_4, g_5, g_6, g_7) = (ab, acd, bcd, ace, bce, ade, bde)$ respectively and continuing with $T_1 \xrightarrow{d_1} T_0$, we define T_1 to be a free module with the basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$.

Defining $d_1 : T_1 \longrightarrow T_0$ such that $d_1(\epsilon_i) = g_i \, \forall i = 1, \ldots, 7$. Then, $T_1 = \bigoplus_{i=1}^{7} S\epsilon_i \cong S^7$ and we see that $T_1 \xrightarrow{d_1} T_0$ is converted to $S^7 \xrightarrow{d_1} S$. We define the degree of ϵ_i to be the same with the degree of g_i $\forall i = 1, \ldots, 7$.

Let $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10} \in T_1$ such that $p = \epsilon M$, where

$$p = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{bmatrix},$$

$$\epsilon = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_7 \end{bmatrix},$$

$$M = \begin{bmatrix} -cd & 0 & -ce & 0 & -de & 0 & 0 & 0 & 0 & 0 \\ b & -b & 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & -e & 0 \\ 0 & 0 & b & -b & 0 & 0 & d & -d & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & d & -d \\ 0 & 0 & 0 & 0 & b & -b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & c \end{bmatrix}.$$

Clearly, $d_1(p_i) = 0 \, \forall i = 1, \ldots, 10$ and by Schreyer's Theorem, $Ker(d_1) = \langle p_1, p_2, p_3, p_4 \rangle$.

With $T_2 \xrightarrow{d_2} T_1$, we can define T_2 to be a next free module with the basis $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$. Defining $d_2 : T_2 \longrightarrow T_1$ such that $d_2(\sigma_i) = p_i \, \forall i = 1, \ldots, 10$. Then $T_2 = \bigoplus_{i=1}^{10} S\sigma_i \cong S^{10}$. Then $T_2 \xrightarrow{d_2} T_1$ is converted to $S^{10} \xrightarrow{d_2} S^7$. We define the degree of σ_i to be the same with the degree of p_i $\forall i = 1, \ldots, 10$.

Let $q_1, q_2, q_3, q_4 \in T_2$ be such that $q = \sigma M'$, where

$$q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} \end{bmatrix},$$

$$M' = \begin{bmatrix} e & e & 0 & 0 \\ 0 & e & 0 & 0 \\ -d & -d & d & d \\ 0 & -d & 0 & d \\ 0 & 0 & -c & -c \\ 0 & 0 & 0 & -c \\ b & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & a \end{bmatrix}.$$

Clearly, $d_2(q_i) = 0 \, \forall i = 1, \ldots, 4$ and by Schreyer's theorem, $Ker(d_2) = \langle q_1, q_2, q_3, q_4 \rangle$.

With $T_3 \xrightarrow{d_3} T_2$, we can define T_3 to be the next free module with the basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Defining $d_3 : T_3 \longrightarrow T_2$ such that $d_3(\gamma_i) = q_i \, \forall i = 1, \ldots, 4$. Then $T_3 = \bigoplus_{i=1}^{4} S\gamma_i \cong S^4$. Thus, $T_3 \xrightarrow{d_3} T_2$ is converted to $S^4 \xrightarrow{d_3} S^{10}$. We define the degree of γ_i to be the same with the degree of q_i $\forall i = 1, \ldots, 4$.

By calculating, we can detect that $Ker(d_3) = 0$ and so, $T_n = 0 \, \forall n \geq 4$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^4 \xrightarrow{d_3} S^{10} \xrightarrow{d_2} S^7 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_\Delta \longrightarrow 0.$$

Solution II:

Let F be a field. Let $S = F[a, b, c, d, e]$ be the polynomial ring that we are using. With this problem, we denote μ_Δ be a set of monomials corresponding to minimal non-faces of Δ. Clearly, $\mu_\Delta = \{ab, acd, bcd, ace, bce, ade, bde\}$ implies that $I_\Delta = \langle \mu_\Delta \rangle = \langle ab, acd, bcd, ace, bce, ade, bde \rangle$ is the Stanley Reisner ideal. So, $I_\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3 \cup \lambda_4$ where

λ_i is i-level set.

Before working with these, we recall the description of the resolution.

$$\cdots \xrightarrow{d_{n+1}} T_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We identify:

$\lambda_1 = \{ab, acd, bcd, ace, bce, ade, bde\}$.

$\lambda_2 = \{abc, abd, abe\}$.

λ_3 contains all 3-faces.

$\lambda_4 = \{abcde\}$.

So, we have :

- $\forall \mathbf{b} \in \lambda_1$. Because $K^{\mathbf{b}} = \{\emptyset\}$ then $\beta_1(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{-1}(K^{\mathbf{b}}; F)| = 1$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 2$.

- $\forall \mathbf{b} \in \lambda_2$ then $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n$ because, $K^{\mathbf{b}}$ contains only the empty set and one vertex.

- $\forall \mathbf{b} \in abcd, abce, abde$ then $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 2$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 3$ because, $K^{\mathbf{b}}$ contains the empty set, four vertices and one 1-face. Furthermore, $\forall \mathbf{b} \in acde, bcde$ then $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 2$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; F)| = 0 \, \forall n \geq 3$ because, at this time, $K^{\mathbf{b}}$ contains the empty set and three vertices.

- $\forall \mathbf{b} \in \lambda_4$ then $\beta_2(\mathbf{b}, S/I_\Delta) = |\tilde{H}_0(K^{\mathbf{b}}; F)| = 0; \beta_3(\mathbf{b}, S/I_\Delta) = |\tilde{H}_1(K^{\mathbf{b}}; F)| = 4$ and $\beta_n(\mathbf{b}, S/I_\Delta) = |\tilde{H}_{n-2}(K^{\mathbf{b}}; f)| = 0 \, \forall n \geq 4$ because

$$K^{\mathbf{b}} = K^{abcde} = \{\emptyset, a, b, c, d, e, ac, ad, ae, bc, bd, be, cd, ce, de, cde\}.$$

Therefore,

$M = S/I_{\Delta}$.

$T_0 = S$.

Since $Rank(T_1) = 7$ then $T_1 = S^7$.

Since $Rank(T_2) = 2 \times 3 + 2 \times 2 = 10$ then $T_2 = S^{10}$.

Since $Rank(T_3) = 4$ then $T_3 = S^4$.

Therefore, we obtain the minimal resolution.

$$0 \longrightarrow S^4 \xrightarrow{d_3} S^{10} \xrightarrow{d_2} S^7 \xrightarrow{d_1} S \xrightarrow{d_0} S/I_{\Delta} \longrightarrow 0.$$

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Curriculum Vitae

My name is Phuong Dang, a Math student of George Mason University. During my Bachelor degree in the University of Pedagogy in Ho Chi Minh, and two masters in Kent State and George Mason, I have learnt a lot of knowledge and skills to prepare for my future. I can speak English and access the computer quite well. With these, I approach Mathematics more conveniently and gain an orientation for myself. I am very interested in Algebra, an area that helps me a lot for visualizing and systemizing concretely the knowledge which is suitable with my circumstance. Especially for Algebraic Geometry and Commutative Algebra, I have chosen a good way to apply and investigate them, that is ”Combinatorial Commutative Algebra.” As I continue with Algebra, I believe that I will obtain more new research for myself and later. In addition, I also studied some programs which are very useful for my typing and exploring, especially Latex and programming language, two things that are beneficial for my contribution to society. To do that, I have experienced two years for teaching internships and one year for research. Those trained me to be confident when I stand in front of many people and present my consequences and even answer their questions proficiently. Those are what I am doing and my belief. So, I always continue thinking and do it until being able to contribute a part of my life for the community and the society.