A Class of Operators with Symbol on the Bloch Space of a Bounded Homogeneous Domain

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Dedication

This dissertation is dedicated to my mother and father.

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Abstract

A CLASS OF OPERATORS WITH SYMBOL ON THE BLOCH SPACE OF A BOUNDED HOMOGENEOUS DOMAIN

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Let X be a Banach space of holomorphic functions on a domain D in $\mathbb{C}^n$. If ψ is a holomorphic function on D, and φ is a holomorphic self-map of D, we define the weighted composition operator on X with symbols ψ and φ by $W_{\psi,\varphi}f = \psi(f \circ \varphi)$. This operator is a generalization of the multiplication operator $M_\psi f = \psi f$ and the composition operator $C_\varphi f = f \circ \varphi$, which are known as degenerate weighted composition operators.

The weighted composition operators have been an object of interest since the early 30's with their connection to the isometries of various spaces of analytic functions on the unit disk. The Bloch space has been of interest since the early 70's to function theorists and to operator theorists. However, these two concepts did not meet until 2001.

Classical operator theory on spaces of holomorphic functions in several complex variables is typically carried out on the unit ball and the unit polydisk. The respective function theories are very different. In this dissertation, we attempt to unify the operator theory on the Bloch space on these domains and extend it further to bounded homogeneous domains in $\mathbb{C}^n$.

In this unified manner, we study the fundamental properties of the weighted composition operators:

1. For what symbols ψ and φ is $W_{\psi,\varphi}$ bounded?
2. For what symbols ψ and φ is $W_{\psi,\varphi}$ compact?
3. What is an expression for $||W_{\psi,\varphi}||$?
4. For what symbols ψ and φ is $W_{\psi,\varphi}$ an isometry?
5. What is the spectrum of $W_{\psi,\varphi}$?

It is our hope that this work will mark the beginning of a paradigm shift in operator theory research in several complex variables. This will bring in new fields of study such as differential geometry into the study of operators, thus enriching the field.

Chapter 1: Introduction

The goal of this dissertation is to study the multiplication, composition, and weighted composition operators on the Bloch space of bounded homogeneous domains in $\mathbb{C}^n$. We study these operators on a bounded homogeneous domain in a first attempt at unifying operator theory research in several complex variables. Historically, when studying operators on function spaces in several variables, either the unit ball or the unit polydisk is chosen as the ambient space. Results from this research apply to both of these domains, as well as to a large class of domains not typically considered.

- In Chapter 2, we recall the notion of a Bloch function, and provide historical perspective. We then give an overview of the linear structure of the space of Bloch functions in one and higher dimensions.

- In Chapter 3, we collect pertinent definitions and results from function theory and operator theory which are used throughout the dissertation. We then give a formal definition of the operators studied in this dissertation.

- In Chapter 4, we study the multiplication operator on the Bloch space of the unit disk. We extend what is known about this operator to include operator norm estimates, the determination of the spectrum, and a characterization of the isometries amongst the multiplication operators.

- In Chapter 5, we present the results from Chapter 4 extended to the Bloch space and $*$-little Bloch space of a bounded homogeneous domain. We characterize the bounded and the compact multiplication operators, determine operator norm estimates and the spectrum, and characterize the isometric multiplication operators on a large class of bounded symmetric domains.

- In Chapter 6, we study the spectrum of a composition operator on the Bloch space of the unit disk, unit ball, and unit polydisk. We determine the spectrum of the isometric composition operators for the unit disk, and a large class of isometric composition operators on the unit polydisk. In the case of the unit ball, we determine the spectrum for the isometric composition operator induced by an automorphism.

- In Chapter 7, we study the weighted composition operators on the unit disk. We determine operator norm estimates, provide an alternative characterization for boundedness and compactness to what is currently known, as well as determine the spectrum of a large class of isometric weighted composition operators.

- In Chapter 8, we study the weighted composition operators from the Bloch space and $*$-little Bloch space of a bounded homogeneous domain into the space of bounded holomorphic functions. We characterize boundedness, determine the operator norm, and provide a sufficient condition for compactness. We also consider what these properties become when restricting to the unit ball and the unit polydisk. In addition, we analyze what these results say about the multiplication and composition operators. Finally, we show that in the environment of the unit polydisk, there are no isometric weighted composition operators from the Bloch space to the space of bounded holomorphic functions.

- In Chapter 9, we analyze the weighted composition operators on the Bloch space and $*$-little Bloch space of a general bounded homogeneous domain. From the results in Chapter 7, we determine a necessary and a sufficient condition for a weighted composition operator to be bounded. We conjecture these two conditions to be a characterization for boundedness. We also determine operator norm estimates. Finally, we establish a sufficient condition for compactness of the weighted composition operator. We conjecture this condition to be necessary as well.

- In Chapter 10, we prove the conjectures presented in Chapter 9 in the case of the unit ball and the unit polydisk.

- In Chapter 11, we discuss topics which require further research, and possible strategies for studying them.

Chapter 2: Bloch Functions and the Bloch Space

2.1 The Bloch Constant

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in $\mathbb{C}$. For a domain Ω in $\mathbb{C}$, we define $H(\Omega)$ to be the set of analytic functions on Ω and $H^\infty(\Omega)$ to be the Banach algebra of bounded analytic functions on Ω, with the *supremum norm*

$$||f||_\infty = \sup_{z\in\Omega} |f(z)|.$$

Consider the functions $f \in H(\overline{\mathbb{D}})$, by which we mean f is a function analytic on some domain containing $\overline{\mathbb{D}}$, satisfying $f(0) = 0$ and $f'(0) = 1$. All such functions are non-constant since $f'(0) \neq 0$, and thus are open by the Open Mapping Theorem (c.f. [31]). So $f(\mathbb{D})$ must contain a disk of positive radius. Is there a positive number r such that the image of each such function contains a disk of radius r?

In 1925, André Bloch [14] answered the above question in the affirmative under more general assumptions.

Definition 2.1.1. Let f be analytic on $\mathbb{D}$. A *schlicht disk* in the image $f(\mathbb{D})$ is an open disk $\Delta \subset f(\mathbb{D})$ such that there exists a domain $\Omega \subset \mathbb{D}$ with f mapping Ω bijectively onto Δ. We denote the radius of the largest schlicht disk centered at $f(z_0)$ associated to f by $d_f(z_0)$.

Definition 2.1.2. For a function f analytic on $\mathbb{D}$, we define the *Bloch number* of f by

$$\beta(f) = \sup_{z\in\mathbb{D}} d_f(z).$$

Theorem 2.1.3. (Bloch's Theorem) *Let $\mathcal{F}$ denote the family of functions f analytic on $\mathbb{D}$ such that $f'(0) = 1$. Then the* Bloch constant

$$B := \inf_{f \in \mathcal{F}} \beta(f)$$

is strictly positive.

Bloch's Theorem says that every function in $\mathcal{F}$ contains a schlicht disk of positive radius. In fact, the proof of Bloch's Theorem in [50] shows that $B \geq 0.21$. Thus, we see that in the image of $\mathbb{D}$ under any f in $\mathcal{F}$, there exists a schlicht disk of radius at least 0.21.

Shortly after Bloch's Theorem appeared, several papers were published establishing bounds on the Bloch constant. In 1929, Landau [59] showed that $B > .397$. Then Ahlfors [1] obtained the sharper bound $B \geq \sqrt{3}/4$. Subsequently, Heins [50] proved that the Bloch number is strictly greater than $\sqrt{3}/4$. In 1990, Bonk showed that $B > \sqrt{3}/4 + 10^{-14}$ [15] and in 1996, Chen and Gauthier obtained $B > \sqrt{3}/4 + 2 \times 10^{-4} \approx 0.4332127$ [18]. As for an upper bound on B, Ahlfors and Grunsky [2] established the inequality

$$B < \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} \approx 0.471862$$

where Γ is the gamma function. They also conjectured that the value of B is this upper limit. To this day, the exact value of B remains an open question.

2.2 Bloch Function on the Unit Disk

Definition 2.2.1. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is called *Bloch* if

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|f'(z)\right| < \infty. \tag{2.1}$$

The mapping $f \mapsto \beta_f$ is called the *Bloch semi-norm* of f. The set of Bloch functions on $\mathbb{D}$ is denoted by $\mathcal{B}(\mathbb{D})$, or simply $\mathcal{B}$ when the domain is understood.

From the definition of the Bloch semi-norm, we see that if f and g are Bloch functions, then so are $f+g$ and αf, where $\alpha \in \mathbb{C}$. Thus, $\mathcal{B}(\mathbb{D})$ is a complex vector space called the *Bloch space*.

As suggested by its name, the Bloch semi-norm is a semi-norm on $\mathcal{B}(\mathbb{D})$. It fails to be a norm because it cannot distinguish between constant functions; $\beta_f = 0$ for any constant function f. Define the equivalence relation $\sim$ on $\mathcal{B}(\mathbb{D})$ by $f \sim g$ if and only if $f - g$ is constant. Then the Bloch semi-norm is a norm on $\mathcal{B}(\mathbb{D})/\sim$ that is, the Bloch semi-norm is a norm on the Bloch functions modulo the constants.

We wish to define a norm on the Bloch space itself. For $f \in \mathcal{B}(\mathbb{D})$, define

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

Then the map $f \mapsto ||f||_{\mathcal{B}}$ is a norm called the *Bloch norm*. Thus, the Bloch space is a normed linear space. Under this norm, the Bloch space is complete, and so is a Banach space [8].

As we now show, the complex polynomials are also Bloch functions.

Proposition 2.2.2. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$. Then f is Bloch with*

$$\beta_f \leq |a_1| + \sum_{k=2}^{n} \frac{2k\,|a_k|}{k+1} \left(\frac{k-1}{k+1}\right)^{\frac{k-1}{2}}.$$

Proof. First note that $\beta_{\alpha f} = |\alpha|\,\beta_f$ and $\beta_{f+g} \leq \beta_f + \beta_g$ for any complex number α and functions $f, g \in H(\mathbb{D})$. So, we must compute the Bloch semi-norm for the monomials

$p_k(z) = z^k$. Clearly, $\beta_{p_1} = 1$. Let $k \geq 2$. By direct calculation,

$$\beta_{p_k} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| kz^{k-1} \right| = k \sup_{z \in \mathbb{D}} (1 - |z|^2) |z|^{k-1}.$$

We can define the function $\tilde{f}_k : [0,1] \to \mathbb{R}$ as $\tilde{f}_k(x) = (1 - x^2)x^{k-1}$, and rewrite β_{p_k} as

$$\beta_{p_k} = k \max_{x \in [0,1]} \tilde{f}_k(x).$$

By elementary calculus, we see that $\tilde{f}_k(x)$ has an absolute maximum in $[0, 1]$ at $\left(\frac{k-1}{k+1}\right)^{1/2}$. Thus, we have

$$\beta_f \leq \sum_{k=0}^{n} |a_k| \, \beta_{p_k} \leq |a_1| + \sum_{k=2}^{n} \frac{2k \, |a_k|}{k+1} \left(\frac{k-1}{k+1} \right)^{\frac{k-1}{2}},$$

as desired. □

A common tool used in analyzing Bloch functions is the Schwarz-Pick Lemma, which we recall shortly. First, we define the automorphism group of $\mathbb{D}$.

Definition 2.2.3. The set $\mathrm{Aut}(\mathbb{D})$ of bijective analytic self-maps of $\mathbb{D}$ is a group under composition called the *automorphism group of* $\mathbb{D}$.

Theorem 2.2.4. [31] *Every element in* $\mathrm{Aut}(\mathbb{D})$ *can be written as* $\varphi(z) = \lambda L_a(z)$, *for* $z \in \mathbb{D}$, *where* $|\lambda| = 1$, $a \in \mathbb{D}$ *and*

$$L_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Note that L_a in an involution which interchanges 0 and a.

Theorem 2.2.5. (Schwarz-Pick Lemma) *Let* f *be an analytic function from* $\mathbb{D}$ *into* $\overline{\mathbb{D}}$. *Then*

$$(1 - |z|^2) \left| f'(z) \right| \leq 1 - |f(z)|^2$$

for all $z \in \mathbb{D}$. Furthermore, if $f \in \mathrm{Aut}(\mathbb{D})$, then equality holds for each $z \in \mathbb{D}$; otherwise the inequality is strict for all $z \in \mathbb{D}$.

An application of the Schwarz-Pick Lemma illustrates the richness of the set of Bloch functions; the bounded analytic functions on $\mathbb{D}$ are all Bloch functions.

Proposition 2.2.6. *If f is an element of $H^\infty(\mathbb{D})$, then f is Bloch. Moreover, $\beta_f \leq ||f||_\infty$.*

Proof. If $||f||_\infty = 0$ then f is identically 0. Thus $\beta_f = 0$. Now suppose $0 < ||f||_\infty < \infty$. Define $g(z) = \frac{1}{||f||_\infty} f(z)$. It is clear that $||g||_\infty = 1$, and so g maps $\mathbb{D}$ into $\overline{\mathbb{D}}$. By linearity and the Schwarz-Pick Lemma, we have

$$\frac{1}{||f||_\infty}\beta_f = \beta_g = \sup_{z\in\mathbb{D}} (1-|z|^2)\left|g'(z)\right| \leq \sup_{z\in\mathbb{D}} (1-|g(z)|^2) \leq 1.$$

Therefore, we have $\beta_f \leq ||f||_\infty$, as desired. □

The containment of $H^\infty(\mathbb{D})$ in $\mathcal{B}(\mathbb{D})$ is proper. Indeed, consider the function $f(z) = \frac{1}{2}\mathrm{Log}(1-z)$ where Log denotes the principal branch of the logarithm. Then f is unbounded, yet $\beta_f \leq 1$.

We use the Schwarz-Pick Lemma to show that the set of Bloch functions is *Möbius invariant*, that is, invariant under the action of $\mathrm{Aut}(\mathbb{D})$.

Proposition 2.2.7. *Let f be a Bloch function and φ an analytic self-map of $\mathbb{D}$. Then $f\circ\varphi$ is Bloch. Moreover, $\beta_{f\circ\varphi} \leq \beta_f$ and equality holds when $\varphi \in \mathrm{Aut}(\mathbb{D})$.*

Proof. By direct calculation, and appealing to the Schwarz-Pick Lemma, we have

$$\begin{aligned}\beta_{f\circ\varphi} &= \sup_{z\in\mathbb{D}} (1-|z|^2)\left|(f\circ\varphi)'(z)\right| = \sup_{z\in\mathbb{D}} (1-|z|^2)\left|\varphi'(z)\right|\left|f'(\varphi(z))\right| \\ &\leq \sup_{z\in\mathbb{D}} (1-|\varphi(z)|^2)\left|f'(\varphi(z))\right| = \sup_{w\in\varphi(\mathbb{D})} (1-|w|^2)\left|f'(w)\right| \\ &\leq \sup_{w\in\mathbb{D}} (1-|w|^2)\left|f'(w)\right| = \beta_f.\end{aligned}$$

If $\varphi \in \text{Aut}(\mathbb{D})$, then $(1-|z|^2)|\varphi'(z)| = 1-|\varphi(z)|^2$ for all $z \in \mathbb{D}$ and $\varphi(\mathbb{D}) = \mathbb{D}$. Thus the above inequalities are all equalities. $\square$

As we shall see, Bloch functions have several characterizations some of which are geometric in nature, whereas some are purely function-theoretic. This is one reason why Bloch functions are widely studied. The first characterization relates the Bloch semi-norm to the radii of schlicht disks in the image of f.

Theorem 2.2.8. [85] *Let f be analytic on $\mathbb{D}$. Then the following are equivalent:*

(a) *The function f is Bloch.*

(b) *The radius of the largest schlicht disk in $f(\mathbb{D})$ is bounded above, i.e.,* $\sup_{z\in\mathbb{D}} d_f(z) < \infty$.

The unit disk in $\mathbb{C}$ can be made into a metric space in several ways. Specifically, we can make $\mathbb{D}$ into a metric space via the hyperbolic metric. The *hyperbolic metric* (also known as the *Poincaré metric* and *Bergman distance*), between points z, w in $\mathbb{D}$ is defined as

$$\rho(z,w) = \frac{1}{2}\log\frac{1+|L_z(w)|}{1-|L_z(w)|} = \frac{1}{2}\log\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}. \tag{2.2}$$

Theorem 2.2.9. [8] *Let f be analytic in $\mathbb{D}$. Then f is Bloch if and only if f is uniformly continuous as a map from the metric space $(\mathbb{D}, \rho)$ to the metric space $(\mathbb{C}, d)$, where d is the Euclidean distance $d(z,w) = |z-w|$, for $z, w \in \mathbb{C}$.*

Theorem 2.2.10. [27] *Let f be analytic in $\mathbb{D}$. Then f is Bloch if and only if it is Lipschitz with respect to the hyperbolic metric of $\mathbb{D}$ and the Euclidean metric in $\mathbb{C}$, that is, there exists $M > 0$ such that*

$$|f(z) - f(w)| \leq M\rho(z,w)$$

for all $z, w \in \mathbb{D}$. Furthermore the Bloch semi-norm is precisely the Lipschitz number

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

The Riemann sphere $\widehat{\mathbb{C}}$ is the one-point compactification of the complex plane, that is $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Riemann sphere is compact under the chordal metric, defined by

$$\chi(u, v) = \frac{2\,|u - v|}{\sqrt{(1 + |u|^2)(1 + |v|^2)}}, \text{ for } u, v \in \mathbb{C}$$

and

$$\chi(u, \infty) = \frac{2}{\sqrt{1 + |u|^2}}, \text{ for } u \in \mathbb{C}.$$

Definition 2.2.11. Let D be a domain in $\mathbb{C}$. A function $f : D \to \widehat{\mathbb{C}}$ is *meromorphic* on D if f is analytic on D except possibly at isolated singularities, each of which is a pole.

Definition 2.2.12. [31] A family $\mathcal{F}$ of meromorphic functions on domain D is *normal* on D if every sequence in $\mathcal{F}$ has either a subsequence that converges uniformly on compact subsets, with respect to the chordal metric, of D to an analytic function or a subsequence that converges uniformly on compact subsets, with respect to the chordal metric, of D to ∞.

The next characterization of Bloch functions is a purely function-theoretic result.

Theorem 2.2.13. [74] *Let f be analytic on $\mathbb{D}$. Then the following are equivalent:*

(a) *The function f is Bloch.*

(b) *The family $\{(f \circ \varphi)(z) - (f \circ \varphi)(0) : \varphi \in \mathrm{Aut}(\mathbb{D})\}$ is normal on $\mathbb{D}$.*

(c) *There exists a constant* $\alpha > 0$ *and a univalent (one-one) analytic function* g *on* $\mathbb{D}$ *such that* $f(z) = \alpha \log g'(z)$.

The last characterization of Bloch functions on the unit disk that we present is both geometric and function-theoretic in nature.

Theorem 2.2.14. [9] *A function* f *analytic on* D *is Bloch if and only if the family*

$$\left\{ \sum_{j=1}^{n} a_j (f \circ \varphi_j)(z) : \varphi_j \in \mathrm{Aut}(\mathbb{D}),\ a_j \in \mathbb{C},\ \sum_{j=1}^{n} |a_j| \leq 1, n \in \mathbb{N} \right\}$$

is normal on $\mathbb{D}$.

Definition 2.2.15. Let V be a complex vector space and S a subset of V.

(a) S is said to be *balanced* if $\alpha S \subseteq S$ for all scalars α such that $|\alpha| \leq 1$, where $\alpha S = \{\alpha s : s \in S\}$.

(b) S is said to be *convex* if for all points $u, v \in S$, the line segment $\{tu + (1-t)v : t \in (0,1)\}$ is contained in S.

(c) S is said to be *absolutely convex* if it is convex and balanced and the *absolute convex hull* of S is the intersection of all absolutely convex sets containing S.

It turns out the family of functions in Theorem 2.2.14 is the absolute convex hull of the orbit of f under $\mathrm{Aut}(\mathbb{D})$, which is defined as $\{f \circ \varphi : \varphi \in \mathrm{Aut}(\mathbb{D})\}$.

So, in summary, Bloch functions have several characterizations, which we compile here.

Theorem 2.2.16. *Let* f *be analytic on* $\mathbb{D}$. *Then the following are equivalent:*

(a) *The function* f *is Bloch.*

(b) *The radii of the schlicht disks in* $f(\mathbb{D})$ *are bounded above.*

(c) *The function f is uniformly continuous as a map from the metric space $(\mathbb{D}, \rho)$ to the metric space $(\mathbb{C}, d)$, where ρ is the hyperbolic distance on $\mathbb{D}$ and d is the Euclidean distance in $\mathbb{C}$.*

(d) *The function f is Lipschitz as a function between the metric spaces $(\mathbb{D}, \rho)$ and $(\mathbb{C}, d)$.*

(e) *The family $\{(f \circ \varphi)(z) - (f \circ \varphi)(0) : \varphi \in \mathrm{Aut}(\mathbb{D})\}$ is normal on $\mathbb{D}$.*

(f) *There exists a constant $\alpha > 0$ and a univalent analytic function g on $\mathbb{D}$ such that $f(z) = \alpha \log g'(z)$.*

(g) *The absolute convex hull of the orbit of f under $\mathrm{Aut}(\mathbb{D})$ is a normal family on $\mathbb{D}$.*

2.3 Bloch Functions in Higher Dimensions

Definition 2.3.1. [58] Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* if f is analytic in each variable separately, that is, for each $j \in \{1, \ldots, n\}$ and each fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ the function

$$\eta \mapsto f(z_1, \ldots, z_{j-1}, \eta, z_{j+1}, \ldots, z_n)$$

is analytic on the set

$$\{\eta \in \mathbb{C} : (z_1, \ldots, z_{j-1}, \eta, z_{j+1}, \ldots, z_n) \in \Omega\}.$$

It is natural to wonder how one might generalize the notion of a Bloch function to higher dimensions. A consequence of the Riemann Mapping Theorem is that every proper, simply-connected domain Ω in $\mathbb{C}$ is biholomorphic to $\mathbb{D}$, that is, there is a bijective holomorphic map $\Phi : \Omega \to \mathbb{D}$. This is one reason why we study Bloch functions primarily on the unit disk. However, in higher dimensions, there is no analog to the Riemann Mapping Theorem.

This complicates how the unit disk is generalized in $\mathbb{C}^n$. There are two very natural generalizations of $\mathbb{D}$ in $\mathbb{C}^n$:

(1) The unit ball

$$\mathbb{B}_n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 1 \right\}.$$

(2) The unit polydisk

$$\mathbb{D}^n = \underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{\text{n-times}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n\}.$$

Poincaré's Theorem states that for $n \geq 2$, $\mathbb{B}_n$ and $\mathbb{D}^n$ are not biholomorphic. In fact, the set of biholomorphic equivalence classes of domains close to the ball in any reasonable sense is uncountable [58]. Thus, we look for a type of domain which includes $\mathbb{B}_n$ and $\mathbb{D}^n$, but gives more choices which do not have restrictions to $\mathbb{D}$ in the complex plane.

If we choose a bounded domain D in $\mathbb{C}^n$, it must have a geometry which resembles the hyperbolic geometry on $\mathbb{D}$ induced by the Poincaré metric ρ. This leads us to the Bergman metric on bounded subsets of $\mathbb{C}^n$.

Definition 2.3.2. A *Hermitian form* on a complex vector space V is a function $f : V \times V \to \mathbb{C}$ which satisfies the following properties for all $u, v, w \in V$, and $\alpha, \beta \in \mathbb{C}$:

(a) $f(\alpha u + \beta v, w) = \alpha f(u, w) + \beta f(v, w)$,

(b) $f(u, v) = \overline{f(v, u)}$.

A *Hermitian metric* on $D \subset \mathbb{C}^n$ is given by assigning to each point $z \in \mathbb{C}$ a positive-definite Hermitian form $H_z(u, \overline{v})$ on $\mathbb{C}^n$ in such a way that the entries of the matrix of H_z are infinitely differentiable functions of z.

Let D be a bounded domain in $\mathbb{C}^n$ and $\{\phi_k\}$ be an arbitrary orthonormal basis for the

Hilbert space

$$\mathcal{H} = \left\{ f \in H(D) : \int_D |f(z)|^2 \, dV(z) < \infty \right\},$$

where dV denotes Lebesgue measure on $\mathbb{C}^n$. For all $z \in D$, the *Bergman kernel* on D at z is defined as

$$K(z, \overline{z}) = \sum_{k=1}^{\infty} \phi_k(z) \overline{\phi_k(\overline{z})}.$$

The Bergman metric is the Hermitian metric whose matrix, with respect to the usual basis for $\mathbb{C}^n$, has (j, k)-th entry

$$\frac{1}{2} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K(z, \overline{z}).$$

Remark 2.3.3. On initial inspection, it may not be apparent how the Bergman kernel is dependent on $\overline{z}$. Since φ_k is a holomorphic function on D, then it has a power series representation

$$\phi_k(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Thus, we can write the conjugate as

$$\overline{\phi_k(z)} = \sum_{j=0}^{\infty} \overline{a}_j \overline{z}^j,$$

and thus we see the dependence on $\overline{z}$.

Every bounded domain D of $\mathbb{C}^n$ has an associated Bergman metric H_z [51].

Example 2.3.4. The Bergman metric on $\mathbb{B}_n$ is

$$H_z(u,\overline{v}) = \frac{n+1}{2} \cdot \frac{(1-||z||^2)\langle u,v\rangle + \langle u,z\rangle\langle z,v\rangle}{(1-||z||^2)^2},$$

where $u, v \in \mathbb{C}^n, z \in \mathbb{B}_n$ and $\langle u,v\rangle = \sum_{k=1}^{n} u_k\overline{v_k}$ is the standard inner product in $\mathbb{C}^n$.

Example 2.3.5. The Bergman metric on $\mathbb{D}^n$ is

$$H_z(u,\overline{v}) = \sum_{j=1}^{n} \frac{u_j\overline{v_j}}{(1-|z_j|^2)^2}, \tag{2.3}$$

where $u, v \in \mathbb{C}^n$ and $z \in \mathbb{D}^n$.

Example 2.3.6. For $n = 1$, the expressions of the Bergman metric on $\mathbb{B}_n$ and $\mathbb{D}^n$ reduce to

$$H_z(u,\overline{u}) = \frac{|u|^2}{(1-|z|^2)^2} \tag{2.4}$$

for $u \in \mathbb{C}$, which is the Poincaré metric on $\mathbb{D}$.

If $\gamma : [0,1] \to D$ is a piecewise smooth curve, then the length of γ in the Bergman metric is given by

$$\ell(\gamma) = \int_0^1 H_{\gamma(t)}(\gamma'(t), \overline{\gamma'(t)})^{1/2}\, dt.$$

The Bergman distance function on D is defined for $z, w \in D$ by

$$\rho(z,w) = \inf\{\ell(\gamma) : \gamma : [0,1] \to D, \gamma \text{ piecewise smooth}, \gamma(0) = z, \gamma(1) = w\}.$$

Definition 2.3.7. Let D be a domain in $\mathbb{C}^n$. The *automorphism group of* D is defined as

$$\mathrm{Aut}(D) = \{\varphi : D \to D \mid \varphi \text{ biholomorphic}\}.$$

Example 2.3.8. [82] The automorphism group of $\mathbb{B}_n$ is

$$\mathrm{Aut}(\mathbb{B}_n) = \{U \circ \varphi_a : U \text{ unitary}, a \in \mathbb{B}_n\},$$

where for $z \in \mathbb{B}_n$, $\varphi_0(z) = z$ and for $a \in \mathbb{B}_n \setminus \{0\}$

$$\varphi_a(z) = \frac{a - P_a(z) - (1 - ||a||^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle},$$

with $P_a(z) = \dfrac{\langle z, a \rangle}{||a||^2} \cdot a$ and $Q_a(z) = z - P_a(z)$.

Example 2.3.9. [81] The automorphism group of $\mathbb{D}^n$ is

$$\mathrm{Aut}(\mathbb{D}^n) = \{(T_1(z_{\tau(1)}), \dots, T_n(z_{\tau(n)})) : T_k \in \mathrm{Aut}(\mathbb{D}), \tau \in S_n\},$$

where S_n is the group of permutations of the set $\{1, 2, \dots, n\}$.

Definition 2.3.10. For any function $\varphi = (\varphi_1, \dots, \varphi_n)$ mapping a domain $D \subset \mathbb{C}^n$ into $\mathbb{C}^n$ and for $z \in D$, we define the *Jacobian matrix of* φ *at* z, to be the $n \times n$ matrix $J\varphi(z)$ whose (j, k)-entry is $\dfrac{\partial \varphi_j}{\partial z_k}(z)$.

Definition 2.3.11. A domain D in $\mathbb{C}^n$ is called *homogeneous* if $\mathrm{Aut}(D)$ acts transitively on D, that is, for every z and w in D there exists an automorphism φ of D such that $\varphi(z) = w$.

We note that the spaces $\mathbb{B}_n$ and $\mathbb{D}^n$ are homogeneous since for each of their points a there exists an automorphism φ_a that interchanges a and 0, and hence for each pair of points z and w in their respective domains, the automorphism $\varphi_w \circ \varphi_z$ maps z to w.

Much work has been done in defining Bloch functions on domains in $\mathbb{C}^n$. The notion of Bloch functions on bounded homogeneous domains was introduced in 1975 by Hahn [45] using terminology and notation from differential geometry. In 1980, Timoney (cf. [88] and [89]) defined Bloch functions on bounded homogeneous domains using an approach more in line with the definition of Bloch function on the unit disk.

Bloch functions have been defined on more general domains in $\mathbb{C}^n$. Krantz and Ma [57] defined Bloch functions on strongly pseudoconvex domains. Such domains, however, have automorphism groups which can be sparse or even trivial and more geometric tools are needed for the study of Bloch functions. For this reason, we restrict our attention to bounded homogeneous domains. We will follow the approach of Timoney, which we outline now.

Definition 2.3.12. Let $f : D \to \mathbb{C}$ be a holomorphic function on a bounded homogeneous domain D in $\mathbb{C}^n$. Then for $z \in D$, define

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|\nabla(f)(z)u|}{H_z(u, \overline{u})^{1/2}}$$

where $\nabla(f)(z)u = \displaystyle\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)u_j$. The function f is called *Bloch* if

$$\beta_f = \sup_{z \in D} Q_f(z) < \infty.$$

The mapping $f \mapsto \beta_f$ is called the Bloch semi-norm. With the Bloch semi-norm being defined in terms of the linear operator ∇, it is easily verified that for f and g Bloch functions on D, $\beta_{f+g} \leq \beta_f + \beta_g$ and $\beta_{\alpha f} = |\alpha| \beta_f$ where $\alpha \in \mathbb{C}$. Thus, the space $\mathcal{B}(D)$ is a complex vector space. The mapping $f \mapsto \beta_f$ is a semi-norm on $\mathcal{B}(D)$. Timoney [88] showed that $\mathcal{B}(D)$ is a Banach space under β_f, modulo the constants.

Recalling the one-dimensional case, to make a norm which distinguishes constant functions, the term $|f(0)|$ is added. If D is a bounded homogeneous domain, it is not necessarily the case that $0 \in D$. Instead, we fix a point $z_0 \in D$ and define $||f||_{\mathcal{B}} = |f(z_0)| + \beta_f$. Under this norm, the set of Bloch functions is a Banach space. For convenience, we shall assume throughout that $0 \in \mathbb{D}$ and take $z_0 = 0$.

If we consider the Bergman metric on $\mathbb{D}$ from (2.4), we see that for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$,

$$Q_f(z) = \sup_{u \in \mathbb{C}\setminus\{0\}} \frac{|f'(z)|\,|u|}{\frac{|u|}{1-|z|^2}} = (1 - |z|^2)\left|f'(z)\right|,$$

and f is a Bloch function if and only if $\sup\limits_{z\in\mathbb{D}} (1 - |z|^2)\left|f'(z)\right| < \infty$, which is the familiar condition (2.1).

Definition 2.3.13. Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. The *Bergman constant* B_φ of φ is defined by $B_\varphi = \sup\limits_{z\in D} B_\varphi(z)$, where

$$B_\varphi(z) = \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})^{1/2}}{H_z(u, \overline{u})^{1/2}}.$$

Theorem 2.3.14. [88] *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. Then there exists $c > 0$, depending only on D, such that*

$$H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u}) \leq cH_z(u, \overline{u}),$$

for all $z \in D$ and $u \in \mathbb{C}^n$. Furthermore, if $\varphi \in \mathrm{Aut}(D)$, then

$$H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u}) = H_z(u, \overline{u}),$$

for all $z \in D$ and $u \in \mathbb{C}^n$.

The following is a consequence of Definition 2.3.13 and Theorem 2.3.14.

Corollary 2.3.15. *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. Then the Bergman constant B_φ of D is bounded above by a constant depending only on D. Furthermore, if $\varphi \in \mathrm{Aut}(D)$, then $B_\varphi = 1$.*

The following result shows that the set of Bloch functions is invariant under composition with holomorphic self-maps of the domain.

Proposition 2.3.16. *Let D be a bounded homogeneous domain, φ a holomorphic self-map of D, and f a Bloch function on D. Then $f \circ \varphi$ is Bloch.*

Proof. Let $z \in D$ and $u \in \mathbb{C}^n \setminus \{0\}$. Then for $J\varphi(z)u \neq 0$,

$$\begin{aligned}
\frac{|\nabla(f \circ \varphi)(z)u|}{H_z(u, \overline{u})^{1/2}} &= \frac{|\nabla(f)(\varphi(z))J\varphi(z)u|}{H_z(u, \overline{u})^{1/2}} \\
&= \left(\frac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})^{1/2}}{H_z(u, \overline{u})^{1/2}} \right) \frac{|\nabla(f)(\varphi(z))J\varphi(z)u|}{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})^{1/2}} \\
&\leq B_\varphi(z) \frac{|\nabla(f)(\varphi(z))J\varphi(z)u|}{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})^{1/2}} \\
&\leq B_\varphi(z) Q_f(\varphi(z)).
\end{aligned}$$

Taking the supremum over all $u \in \mathbb{C}^n \setminus \{0\}$, we obtain

$$Q_{f \circ \varphi}(z) \leq B_\varphi(z) \beta_f. \tag{2.5}$$

Taking the supremum over all $z \in D$, we have $Q_{f\circ\varphi} \leq B_\varphi \beta_f$. By Corollary 2.3.15, B_φ is bounded above by a constant independent of f and φ, so $f \circ \varphi$ is Bloch. □

Let D be a bounded homogeneous domain in $\mathbb{C}^n$. Given two holomorphic functions f and g on D, and $z \in D$, we wish to have an upper bound on $Q_{fg}(z)$. Using the product rule, we see that for $z \in D$

$$\begin{aligned}
Q_{fg}(z) &= \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\nabla(fg)(z)u|}{H_z(u,\overline{u})^{1/2}} \\
&= \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|f(z)\nabla(g)(z)u + g(z)\nabla(f)(z)u|}{H_z(u,\overline{u})^{1/2}} \\
&\leq \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|f(z)\nabla(g)(z)u|}{H_z(u,\overline{u})^{1/2}} + \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|g(z)\nabla(f)(z)u|}{H_z(u,\overline{u})^{1/2}} \\
&= |f(z)| \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\nabla(g)(z)u|}{H_z(u,\overline{u})^{1/2}} + |g(z)| \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\nabla(f)(z)u|}{H_z(u,\overline{u})^{1/2}}.
\end{aligned}$$

Thus,

$$Q_{fg}(z) \leq |f(z)|\, Q_g(z) + |g(z)|\, Q_f(z). \tag{2.6}$$

If we restrict our attention to the unit ball $\mathbb{B}_n$, we have a useful alternative definition of $Q_f(z)$ and characterization of Bloch functions..

Theorem 2.3.17. [100] *Let $f : \mathbb{B}_n \to \mathbb{C}$ be holomorphic. For all $z \in \mathbb{B}_n$*

$$Q_f(z) = \left[(1-||z||^2)\left(||\nabla f(z)||^2 - |\langle\nabla(f)(z),\overline{z}\rangle|^2\right)\right]^{1/2}.$$

Theorem 2.3.18. [100] *Let f be holomorphic on $\mathbb{B}_n$. Then the following conditions are equivalent:*

(a) *f is Bloch.*

(b) *$(1-||z||^2)\,||\nabla(f)(z)||$ is bounded in $\mathbb{B}_n$.*

(c) *$(1-||z||^2)\,|\langle\nabla(f)(z),\overline{z}\rangle|$ is bounded in $\mathbb{B}_n$.*

This formulation is useful when trying to verify certain holomorphic functions are Bloch. Its main advantage is the elimination of the supremum over all $u \in \mathbb{C}^n \setminus \{0\}$, which appears in both the numerator and denominator of the original definition of $Q_f(z)$. We will use Theorem 2.3.17 to show that a particular function that will be very useful in later chapters is Bloch.

Lemma 2.3.19. *Let* φ *be a holomorphic self-map of* $\mathbb{B}_n$ *and* $\lambda \in \mathbb{B}_n$*. Then the function*

$$f(z) = \operatorname{Log}\frac{1}{1 - \langle z, \varphi(\lambda)\rangle},\ z \in \mathbb{B}_n,$$

is Bloch and $||f||_{\mathcal{B}} \leq 2$.

Proof. We note that if $\varphi(\lambda) = 0$, then f is the constant function 0, and thus Bloch with Bloch norm 0. So we assume $\varphi(\lambda) \neq 0$. For convenience, we will calculate

$$Q_f(z)^2 = (1 - ||z||^2)\left[||\nabla f(z)||^2 - |\langle \nabla(f)(z), \overline{z}\rangle|^2\right].$$

Since

$$\frac{\partial f}{\partial z_k}(z) = \frac{\overline{\varphi_k(\lambda)}}{1 - \langle z, \varphi(\lambda)\rangle}$$

for all $k \in \{1, \ldots, n\}$, we obtain

$$\nabla(f)(z) = \frac{\overline{\varphi(\lambda)}}{1 - \langle z, \varphi(\lambda)\rangle}$$

and

$$||\nabla f(z)||^2 = \frac{||\varphi(\lambda)||^2}{|1 - \langle z, \varphi(\lambda)\rangle|^2}.$$

Furthermore,

$$\langle \nabla(f)(z), \overline{z} \rangle = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z) = \sum_{k=1}^{n} \frac{z_k \overline{\varphi_k(\lambda)}}{1 - \langle z, \varphi(\lambda) \rangle} = \frac{\langle z, \varphi(\lambda) \rangle}{1 - \langle z, \varphi(\lambda) \rangle}.$$

Finally, we have

$$Q_f(z)^2 = (1 - ||z||^2) \left(\frac{||\varphi(\lambda)||^2 - |\langle z, \varphi(\lambda) \rangle|^2}{|1 - \langle z, \varphi(\lambda) \rangle|^2} \right). \tag{2.7}$$

By the Cauchy-Schwarz inequality, $|\langle z, \varphi(\lambda) \rangle| \leq ||z|| \, ||\varphi(\lambda)||$. So

$$|1 - \langle z, \varphi(\lambda) \rangle| \geq 1 - |\langle z, \varphi(\lambda) \rangle| \geq 1 - ||z|| \, ||\varphi(\lambda)|| \, . \tag{2.8}$$

Since z and $\varphi(\lambda)$ are both elements of the unit ball, (2.8) yields

$$|1 - \langle z, \varphi(\lambda) \rangle| \geq 1 - ||z|| \, . \tag{2.9}$$

In applying (2.9), we get

$$\frac{1 - ||z||^2}{|1 - \langle z, \varphi(\lambda) \rangle|} \leq \frac{(1 - ||z||)(1 + ||z||)}{1 - ||z||} = 1 + ||z|| \leq 2. \tag{2.10}$$

Furthermore, using (2.9), the inequality $||\varphi(\lambda)|| - |\langle z, \varphi(\lambda) \rangle| \leq 1 - |\langle z, \varphi(\lambda) \rangle|$, and (2.10), from (2.7) we obtain

$$
\begin{aligned}
Q_f(z)^2 &\le \frac{1-||z||^2}{|1-\langle z,\varphi(\lambda)\rangle|}\frac{||\varphi(\lambda)||^2-|\langle z,\varphi(\lambda)\rangle|^2}{1-|\langle z,\varphi(\lambda)\rangle|} \\
&\le \frac{2\left(||\varphi(\lambda)||^2-|\langle z,\varphi(\lambda)\rangle|^2\right)}{1-|\langle z,\varphi(\lambda)\rangle|} \\
&= \frac{2\left(||\varphi(\lambda)||+|\langle z,\varphi(\lambda)\rangle|\right)\left(||\varphi(\lambda)||-|\langle z,\varphi(\lambda)\rangle|\right)}{1-|\langle z,\varphi(\lambda)\rangle|} \\
&\le 2\left(||\varphi(\lambda)||+|\langle z,\varphi(\lambda)\rangle|\right) \\
&\le 4.
\end{aligned}
$$

Hence $||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{B}_n} Q_f(z) \le 2$. □

Definition 2.3.20. Let D be a domain in $\mathbb{C}^n$ and for $j \in \{1,\dots,n\}$ define the projection map $p_j : \mathbb{C}^n \to \mathbb{C}$ to be $p_j(z_1,\dots,z_n) = z_j$. A function $g : \mathbb{D} \to D$ is *holomorphic* if for each $j \in \{1,\dots,n\}$, $p_j \circ g : \mathbb{D} \to \mathbb{C}$ is analytic.

Definition 2.3.21. Given a domain D in $\mathbb{C}^n$, let $f \in H(D)$, $r > 0$ and $w_0 \in \mathbb{C}$. The set

$$\Delta_{w_0}(r) = \{w \in \mathbb{C} : |w - w_0| < r\}$$

is called a *schlicht disk in the range of f* if there exists a holomorphic function $g : \mathbb{D} \to D$ such that $f \circ g$ maps $\mathbb{D}$ bijectively onto $\Delta_{w_0}(r)$.

Definition 2.3.22. [58] Let $\Omega \subseteq \mathbb{C}^n$ be an open set. A family of holomorphic functions $\mathcal{F}$ from Ω to $\mathbb{D}$ is *normal* if every sequence in $\mathcal{F}$ contains either a subsequence that converges uniformly on compact subsets to a holomorphic function from Ω to $\mathbb{D}$ or a subsequence $\{f_j\}$ such that for every K_1 relatively compact in Ω and K_2 relatively compact in $\mathbb{D}$, there is a $J > 0$ that satisfies $f_j(K_1) \cap K_2 = \emptyset$ when $j \ge J$.

We have the following characterizations of Bloch functions on bounded homogeneous domains.

Theorem 2.3.23. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and let $f : D \to \mathbb{C}$ be a holomorphic function. Then the following conditions on f are equivalent:*

(a) *The function f is Bloch.*

(b) *The radii of the schlicht disks in the range of f are bounded above.*

(c) *As a function from the metric space (D, ρ) (where ρ is the distance function associated with the Bergman metric on D) to the metric space $(\mathbb{C}, d)$ (where d is Euclidean distance), the function f is uniformly continuous.*

(d) *The function f is a Lipschitz map as a function from D under the Bergman metric to $\mathbb{C}$ under the Euclidean metric. Furthermore*

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

(e) *The family*

$$\{(f \circ \varphi)(z) - (f \circ \varphi)(z) : \varphi \in \mathrm{Aut}(D)\}$$

is normal for every $z \in D$.

(f) *The supremum*

$$\sup \{||\nabla(f \circ \varphi)(z)|| : \varphi \in \mathrm{Aut}(D)\}$$

is finite for every $z \in D$.

(g) *The set $\{f \circ g \mid g : \mathbb{D} \to D, g \text{ holomorphic}\}$ is a family of Bloch functions with uniformly bounded Bloch norm.*

(h) *The family $\{(f \circ g)(z) - (f \circ g)(0) \mid g : \mathbb{D} \to D, g \text{ holomorphic}\}$ is normal.*

Note that this theorem is an analog to Theorem 2.2.16, for the one-dimensional case. Characterization (d) is due to Allen and Colonna [4], and the others are due to Timoney [88].

The study of Bloch functions on general bounded homogeneous domains is complicated by the fact that these domains and their automorphism groups do not have a canonical representation. To overcome this complication, when more structure is needed, we restrict our study to Bloch functions on bounded symmetric domains in $\mathbb{C}^n$.

Definition 2.3.24. A domain $D \subseteq \mathbb{C}^n$ is called *symmetric at* $z_0 \in D$ if there exists an automorphism φ of D such that $\varphi \circ \varphi$ is the identity map on D and z_0 is an isolated fixed point of φ. The domain D is called *symmetric* if it is symmetric at each $z \in D$.

Any bounded symmetric domain in $\mathbb{C}^n$ is homogeneous [51]. Conversely, any bounded homogeneous domain which is symmetric at a point is symmetric. Thus $\mathbb{B}_n$ and $\mathbb{D}^n$ are symmetric since they are homogeneous and symmetric at the origin via the automorphism $\varphi(z) = -z$. While bounded homogeneous domains in dimensions 2 and 3 are symmetric, there are examples of bounded homogeneous domains in dimensions greater than 3 which are not symmetric [73].

Definition 2.3.25. A bounded symmetric domain D is *irreducible* if it can not be written as the direct product of other bounded symmetric domains.

Cartan [17] showed that every bounded symmetric domain in $\mathbb{C}^n$ is biholomorphic to a finite product of irreducible bounded symmetric domains, unique up to order. Moreover, Cartan classified the irreducible bounded symmetric domains into six classes. Four of the classes are referred to as *Cartan classical domains*, whereas the other two, each consisting of a single domain of dimension 16 and 27, respectively, are referred to as the *exceptional domains*.

Let $M_{m,n}(\mathbb{C})$ denote the set of $m \times n$ matrices with entries in $\mathbb{C}$, and let $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$. The Cartan classical domains, and their corresponding Bergman metrics are defined as follows:

$$R_1 = \{Z \in M_{m,n}(\mathbb{C}) : I_m - ZZ^* > 0\}, \text{ for } m \geq n \geq 1,$$

$$H_Z(U, \overline{V}) = \frac{m+n}{2} \text{Trace} \left[(I_m - ZZ^*)^{-1} U (I_n - Z^* Z)^{-1} V^* \right].$$

$$R_2 = \{Z \in M_n(\mathbb{C}) : Z = Z^T, I_n - ZZ^* > 0\}, \text{ for } n \geq 2,$$

$$H_Z(U, \overline{V}) = \frac{n+1}{2} \text{Trace} \left[(I_n - ZZ^*)^{-1} U (I_n - Z^* Z)^{-1} V^* \right].$$

$$R_3 = \{Z \in M_n(\mathbb{C}) : Z = -Z^T, I_n - ZZ^* > 0\}, \text{ for } n \geq 5,$$

$$H_Z(U, \overline{V}) = \frac{n-1}{2} \text{Trace} \left[(I_n - ZZ^*)^{-1} U (I_n - Z^* Z)^{-1} V^* \right].$$

$$R_4 = \left\{ z \in \mathbb{C}^n : A > 0, \left| \sum z_j^2 \right|^2 < 1, n \geq 5 \right\},$$

$$H_z(u, \overline{v}) = nAu[A(I_n - z^T \overline{z}) + (I_n - z^T \overline{z}) Z^* z (I_n - z^T \overline{z})] v^*,$$

where the superscript ^{T} indicates the transpose, the superscript * is the conjugate transpose, and $A = \left| \sum z_j^2 \right|^2 + 1 - 2\,||z||^2$. Here we use Kobayashi's definitions, except that the Bergman metrics have been divided by 4 [55]. Also, we require $n \geq 2$ for domains in R_2 and $n \geq 5$ for domains in R_3 and R_4. These restrictions guarantee that the irreducible domains belong to a single class of domains.

The descriptions of the exceptional domains involves the construction of certain non-associative algebras, and thus we will not describe them here. The reader is referred to [36] for a description of the exceptional domains. We denote by R_5 the exceptional domain of dimension 16, and by R_6 the exceptional domain of dimension 27.

A bounded symmetric domain D is said to be in *standard form* if it can be written as $D = D_1 \times D_2 \times \cdots \times D_k$ where each factor D_j is a Cartan classical domain or an exceptional domain.

Definition 2.3.26. If D is a bounded symmetric domain, we define the *Bloch constant* of D as

$$c_D = \sup\{\beta_f : f \in H(D), f(D) \subseteq \mathbb{D}\}$$

and the *inner radius* of D as

$$r_D = \inf\{H_0(u, \overline{u})^{1/2} : u \in \partial D'\}$$

where D' is a bounded symmetric domain in standard form biholomorphically equivalent to D and $\partial D'$ is the boundary of D'.

Cohen and Colonna in [24] showed that for any bounded symmetric domain D,

$$c_D \leq \frac{1}{r_D}$$

and if D is a Cartan classical domain, then

$$c_D = \frac{1}{r_D} = \begin{cases} \sqrt{2/(m+n)}, & \text{if } D \in R_1 \\ \sqrt{2/(n+1)}, & \text{if } D \in R_2 \\ \sqrt{1/(n-1)}, & \text{if } D \in R_3 \\ \sqrt{2/n}, & \text{if } D \in R_4. \end{cases} \tag{2.11}$$

In [93] Zhang extended this result to the exceptional domains, for which he calculated the Bloch constants to be

$$c_D = \frac{1}{r_D} = \begin{cases} 1/\sqrt{6}, & \text{if } D = R_5 \\ 1/3, & \text{if } D = R_6. \end{cases} \tag{2.12}$$

Theorem 2.3.27. *If $D = D_1 \times \cdots \times D_k$ is a bounded symmetric domain in standard form, then*

$$c_D = \max_{1 \le j \le k} c_{D_j},$$

where $D_1, \ldots, D_k$ are the irreducible factors of D.

The statement and the proof of Theorem 2.3.27 under the hypothesis that D has no exceptional factors was given in [24]. However, the proof can be easily extended to include the exceptional factors.

2.4 Important Subspace of the Bloch Space

In this section, we discuss important subspaces of the Bloch space, both on the unit disk and on bounded homogeneous domains.

2.4.1 The Little Bloch Space

Definition 2.4.1. The *little Bloch space* is the subspace $\mathcal{B}_0(\mathbb{D})$ of $\mathcal{B}(\mathbb{D})$ consisting of all functions f satisfying

$$\lim_{|z| \to 1} (1 - |z|^2) \left| f'(z) \right| = 0.$$

It is an immediate consequence of the Schwarz-Pick Lemma that $\mathcal{B}_0(\mathbb{D})$ is Möbius invariant, i.e., if f is a function in the little Bloch space and φ is an automorphism of $\mathbb{D}$ then $f \circ \varphi$ is in the little Bloch space. From the definition, it is immediate that $\mathcal{B}_0(\mathbb{D})$ contains all Bloch functions whose derivatives are bounded on $\mathbb{D}$. In particular, the set of polynomials is contained in $\mathcal{B}_0(\mathbb{D})$. In fact, $\mathcal{B}_0(\mathbb{D})$ is the closure of the polynomials in $\mathcal{B}(\mathbb{D})$ and the little Bloch functions can be approximated by *dilations* of Bloch functions.

Theorem 2.4.2. [8] *$\mathcal{B}_0(\mathbb{D})$ is a separable (strongly) closed nowhere dense subspace of $\mathcal{B}(\mathbb{D})$ and is equal to the closure of the polynomials in the Bloch norm. Furthermore, $f \in \mathcal{B}_0(\mathbb{D})$*

if and only if

$$||f(z) - f(z\zeta)||_{\mathcal{B}} \to 0 \qquad \textit{as } \zeta \to 1, |\zeta| \le 1.$$

There is a relation between the Bloch space and the little Bloch space in terms of dual spaces. For a complex vector space V, the *dual space* V^* of V is the vector space of linear functionals on V.

Theorem 2.4.3. [80] *The double dual space of the little Bloch space is isomorphic to the Bloch space, that is,* $\mathcal{B}_0^{**}(\mathbb{D}) \cong \mathcal{B}(\mathbb{D})$.

Definition 2.4.4. A bounded linear operator $T : X \to Y$ between normed linear spaces is called an *isometry* if

$$||Tx||_Y = ||x||_X$$

for all $x \in X$.

The isometries on the subspace $\widetilde{\mathcal{B}_0}$ of the little Bloch space consisting of the functions which fix the origin are characterized in the following result.

Theorem 2.4.5. [22] *If* $S : \widetilde{\mathcal{B}_0} \to \widetilde{\mathcal{B}_0}$ *is an isometry, then there exists a conformal automorphism* φ *of* $\mathbb{D}$ *and* $\lambda \in \partial\mathbb{D}$ *such that* $Sf = \lambda(f \circ \varphi - f(\varphi(0)))$ *for all* $f \in \mathcal{B}_0(\mathbb{D})$.

2.4.2 The Besov Space

Definition 2.4.6. For $1 < p < \infty$, the *analytic Besov space* B_p is defined as the set of analytic functions f on $\mathbb{D}$ such that

$$\begin{aligned} ||f||_p^p &= \int_{\mathbb{D}} \left|f'(z)\right|^p (1 - |z|^2)^{p-2} \, dA(z) \\ &= \int_{\mathbb{D}} \left|f'(z)\right|^p (1 - |z|^2)^p \, d\mu(z) < \infty, \end{aligned}$$

where dA denotes the normalized 2-dimensional Lebesgue measure and

$$d\mu(z) = \frac{1}{(1-|z|^2)^2} dA(z).$$

In the limiting cases of $p = 1$ and $p = \infty$, we define the Besov spaces as

$$B_1 = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} \left| f''(z) \right| \, dA(z) < \infty \right\}, \text{ and}$$

$$B_\infty = \mathcal{B}(\mathbb{D}).$$

Since the function $f(z) = \dfrac{1}{1-|z|^2}$ is the density for the hyperbolic metric on $\mathbb{D}$, $d\mu$ can be considered as the hyperbolic area density. The Besov space B_p is a Banach space under the norm

$$||f||_{B_p} = \begin{cases} \int_{\mathbb{D}} \left| f''(z) \right| \, dA(z), & \text{for } p = 1 \\ |f(0)| + ||f||_p \,, & \text{for } 1 < p < \infty \\ |f(0)| + \beta_f, & \text{for } p = \infty. \end{cases}$$

The Besov space B_2 is a Hilbert space, called the *Dirichlet space* $\mathcal{D}$ under the inner product

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \, dA$$

for $f, g \in \mathcal{D}$. It is the only Besov space that is a Hilbert space.

We have the following subspace relationship amongst the Besov spaces and the Bloch space.

Proposition 2.4.7. [90] For $1 < p < q$, $B_p \subset B_q \subset \mathcal{B}$, and for $f \in B_p$,

$$\beta_f \leq c_1 \left|\left|f\right|\right|_q \leq c_2 \left|\left|f\right|\right|_p.$$

The $*$-Little Bloch Space

Definition 2.4.8. [89] Let D be a bounded symmetric domain in $\mathbb{C}^n$. Define the *little Bloch space* $\mathcal{B}_0(D)$ of D to be the closure of the polynomials in $\mathcal{B}(D)$.

In the special case of the unit ball, the little Bloch space can be characterized in an analogous manner to the case of the unit disk. In fact, Timoney used the following characterization as the definition of the little Bloch space on $\mathbb{B}_n$.

Theorem 2.4.9. [89] *A function $f \in H(\mathbb{B}_n)$ is in $\mathcal{B}_0(\mathbb{B}_n)$ if and only if*

$$\lim_{||z|| \to 1} Q_f(z) = 0.$$

In analogy to the little Bloch space of $\mathbb{D}$, $\mathcal{B}_0(\mathbb{B}_n)$ is invariant under the action of $\mathrm{Aut}(\mathbb{B}_n)$ and is a separable Banach space. Furthermore, $\mathcal{B}_0^{**}(\mathbb{B}_n) \cong \mathcal{B}(\mathbb{B}_n)$. In the case of a bounded symmetric domain $D \neq \mathbb{B}_n$, the little Bloch space $\mathcal{B}_0(D)$ is invariant under $\mathrm{Aut}(D)$ and separable, but the second dual of the little Bloch space is not isomorphic to the Bloch space [89].

Theorem 2.4.10. [89] *If D is a bounded symmetric domain in $\mathbb{C}^n$ other than $\mathbb{B}_n$, then*

$$\left\{ f \in \mathcal{B}(D) : \lim_{z \to \partial D} Q_f(z) = 0 \right\}$$

is the set of constant functions on D.

The above theorem illustrates the need for defining the little Bloch space on a bounded symmetric domain as the closure of the polynomials in $\mathcal{B}(D)$. However, a limit condition is easier to test than checking whether a function is a limit of polynomials in the Bloch norm. So we wish to define a space which is analogous to the little Bloch space on $\mathbb{B}_n$.

Definition 2.4.11. Let D be a bounded domain in $\mathbb{C}^n$. The *distinguished boundary* $\partial^* D$ is the smallest closed subset of ∂D such that

$$\sup_{z\in\overline{D}} |f(z)| = \sup_{z\in\partial^* D} |f(z)|$$

for each function f continuous on $\overline{D}$ and holomorphic on D.

Definition 2.4.12. For a bounded homogeneous domain D, the $*$-*little Bloch space* $\mathcal{B}_{0^*}(D)$ is defined as

$$\mathcal{B}_{0^*}(D) = \left\{ f \in \mathcal{B}(D) : \lim_{z\to\partial^* D} Q_f(z) = 0 \right\}.$$

If D is the unit ball, then $\partial D = \partial^* D$ and thus $\mathcal{B}_0(D) = \mathcal{B}_{0^*}(D)$, while when $D \neq \mathbb{B}_n$, $\mathcal{B}_0(D)$ is a proper subspace of $\mathcal{B}_{0^*}(D)$, and $\mathcal{B}_{0^*}(D)$ is a non-separable subspace of $\mathcal{B}(D)$ [89].

Chapter 3: Bounded Operators on Banach Spaces

In this chapter, we collect relevant definitions and results from functional analysis and operator theory. This is by no means a complete list of topics, and the reader is directed to [31] and [61] for complete treatments of the subject. We focus on the idea of bounded operators, compact operators, isometric operators, and the spectrum of bounded operators, all on Banach spaces of holomorphic functions on domains in $\mathbb{C}^n$. We conclude the chapter with a discussion of the operators of interest in this dissertation, the weighted composition operators and their component operators: the multiplication and the composition operators.

3.1 Bounded Linear Operators and the Operator Norm

The object of study in this dissertation is the bounded linear operator.

Definition 3.1.1. Let X and Y be complex-Banach spaces and T a function from X to Y.

(a) T is a *linear* operator if $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{C}$.

(b) The linear operator T is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $||T(x) - T(x_0)|| < \varepsilon$ whenever $||x - x_0|| < \delta$. The operator T is *continuous* if it is continuous at every point $x_0 \in X$.

(c) The linear operator T is *bounded* if there exists a positive constant C such that $||Tx|| \leq C\,||x||$ for all $x \in X$. A bounded operator on X is a bounded operator from X to itself.

It is an immediate consequence of linearity that $T(0) = T(x - x) = T(x) - T(x) = 0$. Also, the boundedness of the linear operators is equivalent to continuity, as we see in the following theorem.

Theorem 3.1.2. *If $T : X \to Y$ is a linear operator between complex Banach spaces, the following are equivalent:*

(a) *T is bounded.*

(b) *T is continuous.*

(c) *T is continuous at 0.*

Definition 3.1.3. If $T : X \to Y$ is a bounded linear operator between complex Banach spaces, the *operator norm* of T is defined as

$$||T|| = \sup_{||x||=1} ||T(x)|| \, .$$

The operator norm of a bounded linear operator can be formulated in the following different ways [31]:

$$||T|| = \begin{cases} \sup\limits_{||x|| \leq 1} ||T(x)|| \\ \sup\limits_{x \neq 0} \left|\left| T\left(\dfrac{x}{||x||}\right) \right|\right| \\ \inf \{C : ||T(x)|| \leq C \, ||x|| \text{ for all } x \in X\}. \end{cases}$$

Definition 3.1.4. Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. We say T is *invertible* if T is bijective.

As we will see from Theorem 3.2.4, if T is bijective, then T^{-1} is a bounded linear operator from Y to X. So, we assume in the definition of invertibility that if T is invertible, then T^{-1} is a bounded operator. The condition of being a bijection can be weakened to the operator T being bounded below and having dense range.

Definition 3.1.5. Let $T : X \to Y$ be a bounded linear operator between Banach spaces. We say T is *bounded below* if there exists $\delta > 0$ such that $||Tx|| \geq \delta \, ||x||$ for all $x \in X$.

As an immediate consequence of the definition, if T is bounded below then T is injective. The converse is false. For example, if $\mathscr{H}$ is a Hilbert space with orthonormal basis $\{e_n\}$, the diagonal operator $T : \mathscr{H} \to \mathscr{H}$ defined by $Te_n = \frac{1}{n}e_n$ and extended linearly, is injective but not bounded below.

Definition 3.1.6. Let $T : X \to Y$ be a bounded linear operator between Banach spaces. We say T has *dense range* if $T(X)$ is dense in Y.

Theorem 3.1.7. [61] *Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Then T is invertible if and only if T is bounded below and has dense range.*

3.2 The Big Three

The results known as the Hahn-Banach theorem, the Principle of Uniform Boundedness, and the Open Mapping Theorem are quite often referred to as the "Big Three" theorems in functional analysis.

Theorem 3.2.1. (The Hahn-Banach Theorem) *Let X be a complex-Banach space and Y a proper subspace of X. If $\lambda : Y \to \mathbb{C}$ is a bounded linear functional, then there exists a bounded linear functional $\Lambda : X \to \mathbb{C}$ with $\Lambda\big|_Y = \lambda$ and $||\Lambda|| = ||\lambda||$.*

Theorem 3.2.2. (Principle of Uniform Boundedness) *Suppose X is a Banach space and $\mathcal{F}$ is a family of bounded linear operators from X to some Banach space Y. If for every $x \in X$,*

$$\sup_{T\in\mathcal{F}} ||Tx|| < \infty,$$

then

$$\sup_{T\in\mathcal{F}} ||T|| < \infty.$$

Theorem 3.2.3. (Open Mapping Theorem) *Suppose X and Y are Banach spaces and T is a bounded linear operator from X to Y. If T maps X onto Y, then $T(G)$ is open in Y whenever G is open in X.*

The following two results are consequences of the Open Mapping Theorem.

Theorem 3.2.4. (Bounded Inverse Theorem) *Suppose X and Y are Banach spaces and T is a bounded linear operator from X to Y. If T is bijective, then T^{-1} is a bounded linear operator from Y to X.*

Definition 3.2.5. Let X and Y be Banach spaces and $T : X \to Y$ a linear map. The *graph of T* is defined as

$$\text{gra}(T) = \{(x, Tx) : x \in X\}.$$

Theorem 3.2.6. (Closed Graph Theorem) *Suppose X and Y are Banach spaces and $T : X \to Y$ is a linear map. Then T is bounded if and only if* gra(T) *is closed in $X \times Y$.*

3.3 Spectrum of a Bounded Operator

Definition 3.3.1. Let T be a bounded linear operator on a Banach space X. The *spectrum* of T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

where I is the identity operator on X. The *resolvent* of T is defined as $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Theorem 3.3.2. [61] *If T is a bounded linear operator on a Banach space X, then its spectrum is a nonempty, compact subset of $\mathbb{C}$, which is contained in the closed disk $\{z : |z| \leq ||T||\}$.*

Theorem 3.3.3. (Spectral Mapping Theorem) *Let T be a bounded linear operator on a Banach space X, $\mathscr{B}(X)$ the space of bounded linear operators on X, $C(\sigma(T))$ the space of complex-valued functions analytic on a neighborhood of $\sigma(T)$, and $f \in C(\sigma(T))$. Then there exists a unique map $\Phi : C(\sigma(T)) \to \mathscr{B}(X)$ such that*

$$\sigma[\Phi(f)] = \{f(\lambda) : \lambda \in \sigma(T)\}.$$

For ease of notation, we denote $\Phi(f) = f \circ T$.

Remark 3.3.4. The map Φ has many properties, and the reader is referred to [30] for further treatment. As a consequence, if T is a bounded linear operator and $0 \notin \sigma(T)$, then the function $f(z) = z^{-1}$ is an element of $C(\sigma(T))$. Thus

$$\sigma(f \circ T) = \sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}.$$

From Theorem 3.1.7, λ is an element of the spectrum of the operator T either if $T - \lambda I$ is not bounded below or does not have dense range. This results in a decomposition of the spectrum.

Definition 3.3.5. Let T be a bounded linear operator on a Banach space X. Then the *point spectrum* of T is defined as

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}.$$

The elements of $\sigma_p(T)$ are called *eigenvalues*. For $\lambda \in \sigma_p(T)$, the non-zero vectors in $\ker(T - \lambda I)$ are called *eigenvectors* associated with the eigenvalue λ.

Definition 3.3.6. Let T be a bounded linear operator on a Banach space X. The *approximate point spectrum* of T is defined as

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

Proposition 3.3.7. [30] *If T is a bounded linear operator on a Banach space X, then $\lambda \in \sigma_{ap}(T)$ if and only if there exists a sequence $\{x_n\}$ in X such that $||x_n|| = 1$ for all $n \in \mathbb{N}$ and $||(T - \lambda I)x_n|| \to 0$ as $n \to \infty$.*

If λ is an eigenvalue of T and x is an eigenvector associated with λ, then the constant sequence $x_n = \frac{x}{||x||}$ is such that $||x_n|| = 1$ for all n and $||(T - \lambda I)x_n|| = 0$. Thus $\sigma_p(T) \subseteq \sigma_{ap}(T)$.

Proposition 3.3.8. [30] *If T is a bounded linear operator on a Banach space X, then $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.*

Definition 3.3.9. Let $T : X \to X$ be a bounded linear operator on a Banach space X. The *residual spectrum* of T is defined as

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have dense range}\}.$$

Since the eigenvalues are contained in the approximate point spectrum, the spectrum of a bounded linear operator is $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$.

3.4 Isometries

An immediate consequence of Definition 2.4.4 is that isometries are injective, since their kernel is trivial.

Theorem 3.4.1. *Let X be a complex Banach space and suppose $T : X \to X$ is an isometry. If T is invertible, then $\sigma(T) \subseteq \partial\mathbb{D}$. If T is not invertible, then $\sigma(T) = \overline{\mathbb{D}}$.*

Proof. Suppose T is an invertible isometry on X. Then $0 \notin \sigma(T)$, and so the function $z \mapsto z^{-1}$ is analytic in some neighborhood of $\sigma(T)$. By the Spectral Mapping Theorem (Theorem 3.3.3) we have $\sigma(f \circ T) = f(\sigma(T))$, and so

$$\sigma(T^{-1}) = \sigma(T)^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}.$$

Since T^{-1} exists and is an isometry, we have $\sigma(T^{-1}) \subseteq \overline{\mathbb{D}}$. Therefore $\sigma(T) \subseteq \partial\mathbb{D}$.

Next, suppose T is not invertible. In order to prove that $\sigma(T) = \overline{\mathbb{D}}$, it suffices to show that $\overline{\mathbb{D}} \subseteq \sigma(T)$. For $\lambda \in \mathbb{D}$, $T - \lambda I$ is bounded below by $1 - |\lambda|$. Thus, $\lambda \notin \sigma_{ap}(T)$. By Proposition 3.3.8, we deduce $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \partial\mathbb{D}$.

Since T is not invertible, $0 \in \sigma(T)$. Assume $\lambda \in \overline{\mathbb{D}} \cap \rho(T)$. Note that $\lambda \notin \partial\sigma(T)$ since $\partial\sigma(T) = \sigma(T) \cap \overline{\rho(T)}$. Consider $\Gamma = \{t\lambda : t \in [0, \infty)\}$, the radial line through λ. Since

$\sigma(T)$ is closed, there exists $t \in [0,1)$ such that $t\lambda \in \partial\sigma(T)$. This contradicts the fact that $\partial\sigma(T) \subseteq \partial\mathbb{D}$. Consequently, $\overline{\mathbb{D}} \cap \rho(T) = \emptyset$, whence $\overline{\mathbb{D}} \subseteq \sigma(T)$. □

3.5 Compact Operators and Essential Norm

An important class of bounded operators are known as the compact operators.

Definition 3.5.1. The *unit ball* of a Banach space X is the set $\{x \in X : ||x|| \leq 1\}$.

Definition 3.5.2. A linear operator $T : X \to Y$ between Banach spaces is *compact* if the image of the unit ball of X under T has compact closure in Y.

For X and Y complex Banach spaces, $\mathscr{B}(X,Y)$ denotes the algebra of bounded linear operators from X to Y, and $\mathscr{K}(X,Y)$ denotes the subspace of compact linear operators.

Theorem 3.5.3. [30] *Let X, Y, and L be Banach spaces.*

(a) *$\mathscr{K}(X,Y)$ is a closed linear subspace of $\mathscr{B}(X,Y)$.*

(b) *If $K \in \mathscr{K}(X,Y)$ and $T \in \mathscr{B}(Y,L)$, then $TK \in \mathscr{K}(X,L)$.*

(c) *If $K \in \mathscr{K}(X,Y)$ and $T \in \mathscr{B}(L,X)$, then $KT \in \mathscr{K}(L,Y)$.*

Definition 3.5.4. Let $T \in \mathscr{B}(X,Y)$. The *essential norm* of T is the distance of T from $\mathscr{K}(X,Y)$, that is

$$||T||_e = \inf_{K \in \mathscr{K}(X,Y)} ||T - K||.$$

As an immediate corollary of Theorem 3.5.3, $\mathscr{K}(X,X)$ is a two-sided ideal in $\mathscr{B}(X,X)$. If we let $\mathscr{B} = \mathscr{B}(X,X)$ and $\mathscr{K} = \mathscr{K}(X,X)$, then we can form the quotient algebra $\mathscr{C} = \mathscr{B}/\mathscr{K}$, called the *Calkin algebra*. By definition, if $T \in \mathscr{K}$, then $||T||_e = 0$.

We conclude this section with a spectral theorem for compact operators due to F. Riesz.

Theorem 3.5.5. [30] *If X is an infinite-dimensional Banach space and $T \in \mathscr{K}$, then one and only one of the following possibilities occurs.*

(a) $\sigma(T) = \{0\}$.

(b) $\sigma(T) = \{0, \lambda_1, \ldots, \lambda_n\}$, *where for* $1 \leq k \leq n$, $\lambda_k \neq 0$, *each* λ_k *is an eigenvalue of* T *and* $\dim \ker(T - \lambda_k I) < \infty$.

(c) $\sigma(T) = \{0, \lambda_1, \lambda_2, \ldots\}$, *where for each* $k \geq 1$, λ_k *is an eigenvalue of* T, $\dim \ker(T - \lambda_k I) < \infty$, *and* $\lim_{k \to \infty} \lambda_k = 0$.

Corollary 3.5.6. *If* X *is an infinite-dimensional Banach space and* $T \in \mathscr{K}$, *then* $\sigma(T)$ *is at most countably infinite. Moreover, if* $\sigma(T)$ *is a singleton, then* $\sigma(T) = \{0\}$.

3.6 Weighted Composition Operators

In this section, we formalize the notion of weighted composition operators on a Banach space. Also, we show that the multiplication and composition operators are weighted composition operators, which may be considered as degenerate.

The study of weighted composition operators first began with Banach himself. In [13], Banach proved that the surjective isometries on the space of continuous real-valued functions on a compact metric space are certain weighted composition operators. For $p \in \mathbb{R}, 0 < p < \infty$, the *Hardy space* H^p on the unit disk is defined as

$$H^p = \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta < \infty \right\},$$

with norm

$$||f||_{H^p} = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p}.$$

The *Bergman space* A^p on the unit disk is defined as

$$A^p = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \, dA < \infty \right\},$$

with norm

$$||f||_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA \right)^{1/p}.$$

In [43], Forelli proved that for $p \neq 2$, the isometries on H^p are weighted composition operators. Kolaski in [56] proved that for $p \neq 2$, the surjective isometries on A^p are also weighted composition operators. For further treatment on the Hardy and Bergman spaces, the reader is referred to [38] and [39].

This is not to say that the importance of weighted composition operators is restricted to the study of isometries on spaces of analytic functions. In some sense, the study of weighted composition operators is the natural progression of the field of composition operators, which was begun by Nordgren in his doctoral dissertation, see [70]. Weighted composition operators are found in applied fields such as dynamical systems and evolution equations. The weighted composition operator is tied to the classification of dichotomies in certain dynamical systems, see [19].

Let Ω be a bounded domain in $\mathbb{C}^n$ and let X be a Banach space of holomorphic functions $f : \Omega \to \mathbb{C}$. For a fixed holomorphic function ψ on Ω and φ any fixed holomorphic self-map of Ω, we define the *weighted composition operator* $W_{\psi,\varphi}$ on X as

$$W_{\psi,\varphi} f = \psi(f \circ \varphi),$$

for all f in X. The function ψ is called the *multiplication symbol* of $W_{\psi,\varphi}$, and the map φ is called the *composition symbol*.

From the definition of the weighted composition operator, we can formulate the multiplication and composition operators on the space X. For ψ a fixed holomorphic function from Ω into $\mathbb{C}$, we define the *multiplication operator* M_ψ as

$$M_\psi f = \psi f,$$

for all $f \in X$. The multiplication operator can be thought of as a weighted composition operator whose composition symbol is the identity function, since in this case $f \circ \varphi = f$.

Definition 3.6.1. A *functional Banach space* X is a Banach space of complex-valued functions on a set Ω such that point evaluation is a bounded linear functional and there is no point in Ω at which all functions in X vanish.

Proposition 3.6.2. [40] *Let X be a functional Banach space on the set Ω and let ψ be a complex-valued function on Ω such that $\psi X \subset X$. Then the operator M_ψ is a bounded operator on X, and $|\psi(s)| \leq ||M_\psi||$ for all $s \in \Omega$. In particular, $\psi \in H^\infty(\Omega)$.*

For a fixed holomorphic self-map φ of Ω, we define the *composition operator* C_φ on X as

$$C_\varphi f = f \circ \varphi,$$

for all $f \in X$. The composition operator can be viewed as a weighted composition operator by taking the multiplication symbol to be the constant function 1. For further treatment of the composition operator on spaces of analytic functions, the reader is referred to [86] and [33].

Chapter 4: Multiplication Operators on the Bloch Space of the Unit Disk

In this chapter, we analyze the multiplication operator on the Bloch space on the unit disk. The boundedness of the multiplication operators was first characterized by Arazy in [10].

Theorem 4.0.1. [10] *If $\psi \in H(\mathbb{D})$, then M_ψ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $\psi \in H^\infty(\mathbb{D})$ and*

$$\sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log\frac{1+|z|}{1-|z|} < \infty.$$

Independently, Brown and Shields characterized the bounded multiplication operators in the following theorem.

Theorem 4.0.2. [16] *If $\psi \in H(\mathbb{D})$, then the following are equivalent:*

(a) *M_ψ is bounded on $\mathcal{B}(\mathbb{D})$;*

(b) *M_ψ is bounded on $\mathcal{B}_0(\mathbb{D})$;*

(c) *$\psi \in H^\infty(\mathbb{D})$ and*

$$\left|\psi'(z)\right| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

In the literature there are no further results concerning the multiplication operators on the Bloch space of the unit disk until 2001, when Ohno and Zhao characterized the compact multiplication operators on the Bloch space [72].

Proposition 4.0.3. [72] *Let $\psi \in H(\mathbb{D})$. Then the following are equivalent:*

(a) *M_ψ is compact on $\mathcal{B}(\mathbb{D})$.*

(b) M_ψ *is compact on* $\mathcal{B}_0(\mathbb{D})$.

(c) $\psi = 0$.

We first wish to expand the boundedness conditions to include estimates on the norm of the multiplication operator. We use these estimates to characterize the isometric multiplication operators, and determine the spectra.

4.1 Operator Norm Estimates

For an analytic function $\psi : \mathbb{D} \to \mathbb{C}$, we define the quantity

$$\sigma_\psi = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log \frac{1+|z|}{1-|z|}. \tag{4.1}$$

A key ingredient to obtaining estimates on the norm of M_ψ is the following lemma.

Lemma 4.1.1. *If* $f \in \mathcal{B}(\mathbb{D})$*, then for all* $z \in \mathbb{D}$

$$|f(z)| \le |f(0)| + \frac{1}{2}\beta_f \log \frac{1+|z|}{1-|z|}. \tag{4.2}$$

Proof. For $z = 0$, the inequality is satisfied trivially. So assume $z \neq 0$. From Theorem 2.2.10, we have that

$$\frac{|f(z) - f(0)|}{\rho(z,0)} \le \beta_f.$$

Thus

$$|f(z) - f(0)| \le \beta_f \rho(z,0),$$

whence

$$|f(z)| \le |f(0)| + |f(z) - f(0)| \le |f(0)| + \beta_f \rho(z,0).$$

From (2.2), the Bergman distance from z to 0 is given by

$$\rho(z,0) = \frac{1}{2}\log\frac{1+|z|}{1-|z|}, \tag{4.3}$$

and we obtain (4.2). □

We are now ready to prove the following estimates on the norm of the multiplication operator.

Theorem 4.1.2. *Suppose ψ is the symbol of a bounded multiplication operator M_ψ on $\mathcal{B}(\mathbb{D})$. Then*

$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \le ||M_\psi|| \le \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\}.$$

In particular, if $\psi(0) = 0$, then

$$||\psi||_\infty \le ||M_\psi|| \le ||\psi||_\infty + \sigma_\psi.$$

Proof. Let $f \in \mathcal{B}(\mathbb{D})$ such that $||f||_{\mathcal{B}} = 1$. Then by Lemma 4.1.1, we have

$$\begin{aligned}
||M_\psi f||_{\mathcal{B}} &= |\psi(0)|\,|f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|(\psi f)(z)'\right| \\
&= |\psi(0)|\,|f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f(z)\psi'(z) + \psi(z)f'(z)\right| \\
&\le |\psi(0)|\,|f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f(z)|\left|\psi'(z)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\psi(z)|\left|f'(z)\right| \\
&\le |\psi(0)|\,|f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left(|f(0)| + \frac{1}{2}\beta_f\log\frac{1+|z|}{1-|z|}\right)\left|\psi'(z)\right| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\psi(z)|\left|f'(z)\right| \\
&\le |\psi(0)|\,|f(0)| + |f(0)|\,\beta_\psi + \sigma_\psi\beta_f + ||\psi||_\infty\,\beta_f \\
&= |f(0)|\,||\psi||_{\mathcal{B}} + \sigma_\psi\beta_f + ||\psi||_\infty\,\beta_f. \qquad (4.4)
\end{aligned}$$

Since $||f||_{\mathcal{B}} = 1$, we have $|f(0)| = 1 - \beta_f$. Applying this to (4.4), we deduce

$$\begin{aligned} ||M_\psi f||_{\mathcal{B}} &\leq (1-\beta_f)\,||\psi||_{\mathcal{B}} + \sigma_\psi \beta_f + ||\psi||_\infty \,\beta_f \\ &= ||\psi||_{\mathcal{B}} + (\sigma_\psi + ||\psi||_\infty - ||\psi||_{\mathcal{B}})\,\beta_f \end{aligned} \tag{4.5}$$

We have two cases to analyze. If $\sigma_\psi + ||\psi||_\infty - ||\psi||_{\mathcal{B}} \leq 0$, then (4.5) yields

$$||M_\psi f||_{\mathcal{B}} \leq ||\psi||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}}\,.$$

Therefore $||M_\psi|| \leq ||\psi||_{\mathcal{B}}$. On the other hand, if $\sigma_\psi + ||\psi||_\infty - ||\psi||_{\mathcal{B}} \geq 0$, then

$$\begin{aligned} ||M_\psi f||_{\mathcal{B}} &\leq ||\psi||_{\mathcal{B}} + (\sigma_\psi + ||\psi||_\infty - ||\psi||_{\mathcal{B}})\,\beta_f \\ &\leq ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}} + (\sigma_\psi + ||\psi||_\infty - ||\psi||_{\mathcal{B}})\,||f||_{\mathcal{B}} \\ &= (\sigma_\psi + ||\psi||_\infty)\,||f||_{\mathcal{B}}\,. \end{aligned}$$

Thus $||M_\psi|| \leq ||\psi||_\infty + \sigma_\psi$. Therefore

$$||M_\psi|| \leq \max\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty + \sigma_\psi\},$$

as desired.

By Theorem 3.6.2, $||\psi||_\infty \leq ||M_\psi||$. To show that $||\psi||_{\mathcal{B}} \leq ||M_\psi||$, we use the test function $f(z) = 1$ for all $z \in \mathbb{D}$. Since f is Bloch, we have $||M_\psi f||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}$. Thus $||M_\psi|| \geq ||\psi||_{\mathcal{B}}$. Therefore $||M_\psi|| \geq \max\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty\}$. If $\psi(0) = 0$, then by Proposition 2.2.6, $||\psi||_{\mathcal{B}} = \beta_\psi \leq ||\psi||_\infty$. □

4.2 Spectrum

In this section, we determine the spectrum of the multiplication operators on the Bloch space and little Bloch space of the unit disk.

Theorem 4.2.1. *Let ψ be the symbol of a bounded multiplication operator M_ψ on $\mathcal{B}(\mathbb{D})$ or $\mathcal{B}_0(\mathbb{D})$. Then $\sigma(M_\psi) = \overline{\psi(\mathbb{D})}$.*

Proof. For $\lambda \in \mathbb{C}$, the operator $M_\psi - \lambda I$ can be rewritten as $M_{\psi-\lambda}$. Thus $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible. Clearly, if $M_{\psi-\lambda}^{-1}$ exists, it is the multiplication operator $M_{(\psi-\lambda)^{-1}}$.

Let $\lambda \in \psi(\mathbb{D})$. Then there exists $z_0 \in \mathbb{D}$ such that $\psi(z_0) = \lambda$. So $(\psi - \lambda)^{-1}$ has a pole at z_0, which means $M_{(\psi-\lambda)^{-1}}$ is not a well-defined operator. Thus $M_{\psi-\lambda}$ is not invertible. Since the spectrum is closed, this implies that $\overline{\psi(\mathbb{D})} \subseteq \sigma(M_\psi)$.

Suppose $\lambda \notin \overline{\psi(\mathbb{D})}$. Then $|\psi - \lambda|$ is bounded away from 0 by some positive constant c. Thus the function $g(z) = \frac{1}{\psi(z)-\lambda}$ is bounded analytic on $\mathbb{D}$. In addition, by the boundedness of M_ψ and Theorem 4.0.2, we obtain

$$\left|g'(z)\right| = \frac{|\psi'(z)|}{|\psi(z)-\lambda|^2} \leq \frac{1}{c^2}\left|\psi'(z)\right| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

So $M_g = M_{(\psi-\lambda)^{-1}}$ is a bounded operator on $\mathcal{B}(\mathbb{D})$. Thus $\lambda \notin \sigma(M_\psi)$. Therefore $\sigma(M_\psi) = \overline{\psi(\mathbb{D})}$. □

4.3 Isometries

We end this chapter by characterizing the isometric multiplication operators on the Bloch space on the unit disk. We show that the only isometric multiplication operators on the Bloch space are those induced by constant functions of modulus 1. In a certain sense, the set of isometric multiplication operators on the Bloch space is small. By this, we mean that there exist no non-trivial isometries on the Bloch space amongst the multiplication operators.

In order to prove the characterization of the isometric multiplication operators, we need the following lemmas.

Lemma 4.3.1. *Let* ψ *be the symbol of an isometric multiplication operator on* $\mathcal{B}(\mathbb{D})$*. Then* $||\psi||_\infty \leq 1$ *and* $\left|\left|\psi^k\right|\right|_{\mathcal{B}} = 1$ *for all* $k \in \mathbb{N}$.

Proof. By the lower estimate in Theorem 4.1.2, we obtain $||\psi||_\infty \leq ||M_\psi|| = 1$. Since M_ψ is an isometry, we have $||\psi||_{\mathcal{B}} = ||M_\psi 1||_{\mathcal{B}} = ||1||_{\mathcal{B}} = 1$ and $\left|\left|\psi^2\right|\right|_{\mathcal{B}} = ||M_\psi(\psi)||_{\mathcal{B}} = ||\psi||_{\mathcal{B}} = 1$. By induction, it follows that $\left|\left|\psi^k\right|\right|_{\mathcal{B}} = 1$ for all $k \in \mathbb{N}$. □

Lemma 4.3.2. *If* $\psi \in H^\infty(\mathbb{D})$ *such that* $||\psi||_\infty \leq 1$ *and* $\psi(0) = 0$*, then* $\left|\left|\psi^k\right|\right|_{\mathcal{B}} < 1$ *for all* $k \geq 2$.

Proof. By the Schwarz-Pick Lemma, for all $k \in \mathbb{N}$, $k \geq 2$, we have

$$\begin{aligned}
\beta_{\psi^k} &= \sup_{z\in\mathbb{D}} (1-|z|^2)k\,|\psi(z)|^{k-1}\left|\psi'(z)\right| \leq \sup_{z\in\mathbb{D}} k(1-|\psi(z)|^2)\,|\psi(z)|^{k-1} \\
&\leq k \max_{x\in[0,1]} (x^{k-1}-x^{k+1}) = \frac{2k}{k+1}\left(\frac{k-1}{k+1}\right)^{\frac{k-1}{2}} \\
&= \frac{2k(k-1)^{\frac{k-1}{2}}}{(k+1)^{\frac{k+1}{2}}}. \qquad (4.6)
\end{aligned}$$

For x and a positive real numbers and m a real number greater than 1, we have $(x+a)^m > x^m + amx^{m-1}$. Thus for $k \geq 2$,

$$(k+1)^{\frac{k+1}{2}} > (k-1)^{\frac{k+1}{2}} + 2\left(\frac{k+1}{2}\right)(k-1)^{\frac{k-1}{2}} = 2k(k-1)^{\frac{k-1}{2}}. \qquad (4.7)$$

From (4.6) and (4.7), we deduce

$$\beta_{\psi^k} \leq \frac{2k(k-1)^{\frac{k-1}{2}}}{(k+1)^{\frac{k+1}{2}}} < 1.$$

Therefore, $\left|\left|\psi^k\right|\right|_{\mathcal{B}} = \beta_{\psi^k} < 1$ for $k \geq 2$. □

Corollary 4.3.3. *If ψ is the symbol of an isometric multiplication operator on $\mathcal{B}$, then ψ does not fix the origin.*

Proof. Arguing by contradiction, assume $\psi(0) = 0$. By Lemma 4.3.1, $||\psi||_\infty \leq 1$ and $\left|\left|\psi^2\right|\right|_{\mathcal{B}} = 1$. However, $\left|\left|\psi^2\right|\right|_{\mathcal{B}} < 1$ by Lemma 4.3.2. □

Theorem 4.3.4. [29] *Let f be an analytic self-map of $\mathbb{D}$ such that $\beta_f = 1$. Then either f is a conformal automorphism of $\mathbb{D}$, or the zeros of f form an infinite sequence $\{a_k\}$ such that*

$$\limsup_{k\to\infty}(1 - |a_k|^2)\left|f'(a_k)\right| = 1.$$

Lemma 4.3.5. *Suppose $\psi \in H^\infty(\mathbb{D})$ such that $||\psi||_\infty \leq 1$ and the map $g(z) = z\psi(z)$ has Bloch norm 1. Then either ψ is a constant of modulus 1, or ψ has infinitely many zeros $\{a_k\}$ in $\mathbb{D}$ such that*

$$\beta_\psi = \limsup_{k\to\infty}\,(1 - |a_k|^2)\left|\psi'(a_k)\right| = 1.$$

If ψ is not a constant of modulus one and $||\psi||_{\mathcal{B}} = 1$, then $\psi(0) = 0$.

Proof. Assume ψ is not a constant of modulus 1. Note that the function g maps $\mathbb{D}$ into itself, fixes the origin, and by assumption has Bloch norm 1. By Theorem 4.3.4, there exists an infinite sequence $\{a_k\}$ in $\mathbb{D}$ such that $g(a_k) = 0$ and

$$\limsup_{k\to\infty}\,(1 - |a_k|^2)\left|g'(a_k)\right| = 1. \tag{4.8}$$

Since a non-constant analytic function cannot have a zero set with an accumulation point inside the domain, either $|a_k| \to 1$ as $k \to \infty$ or else ψ is identically zero. The latter case cannot occur because $\beta_g = 1$. Since the non-zero zeros of g are also zeros of ψ, and $g'(z) = \psi(z) + z\psi'(z)$ for $z \in \mathbb{D}$, evaluation at the zeros a_k yields $g'(a_k) = a_k\psi'(a_k)$. Thus (4.8) yields

$$\beta_\psi \geq \limsup_{k\to\infty} (1 - |a_k|^2) \left|\psi'(a_k)\right| = 1.$$

Since $\beta_\psi \leq ||\psi||_\infty \leq 1$, we obtain

$$\beta_\psi = \limsup_{k\to\infty} (1 - |a_k|^2) \left|\psi'(a_k)\right| = 1.$$

The conclusion for the case $||\psi||_{\mathcal{B}} = 1$ follows at once. □

We now prove the main theorem of this section.

Theorem 4.3.6. *The multiplication operator M_ψ is an isometry on $\mathcal{B}$ if and only if ψ is a constant function of modulus 1.*

Proof. Clearly, if ψ is a constant function of modulus 1 then M_ψ is an isometry on $\mathcal{B}$. Conversely, suppose M_ψ is an isometry on $\mathcal{B}$ and assume ψ is not a constant function of modulus 1. Then by Lemma 4.3.1, $||\psi||_\infty \leq 1$ and $||\psi||_{\mathcal{B}} = 1$. Also, for $g(z) = z\psi(z)$, $||g||_{\mathcal{B}} = ||M_\psi(\text{id})||_{\mathcal{B}} = ||\text{id}||_{\mathcal{B}} = 1$, where id is the identity map of $\mathbb{D}$. Then by Lemma 4.3.5, $\psi(0) = 0$, contradicting Corollary 4.3.3. Therefore, if M_ψ is an isometry on $\mathcal{B}$, then ψ must be a constant function of modulus 1. □

As an immediate consequence of Theorems 4.2.1 and 4.3.6, we obtain the following.

Corollary 4.3.7. *Let M_ψ be an isometric multiplication operator on the Bloch space. Then $\sigma(M_\psi) = \{\eta\}$, where η is the unimodular constant value of ψ.*

Chapter 5: Multiplication Operators on the Bloch Space of a Bounded Homogeneous Domain

In Chapter 4, we discussed the properties of the multiplication operator on the Bloch space of the unit disk. Zhu studied the multiplication operators on the Bloch space of the unit ball under

$$||f|| = |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - ||z||^2) \, ||\nabla(f)(z)|| \, ,$$

which is equivalent to the Bloch norm [88].

Theorem 5.0.1. [100] *For a holomorphic function ψ in $\mathbb{B}_n$, the following are equivalent:*

(a) *M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$.*

(b) *M_ψ is bounded on $\mathcal{B}_0(\mathbb{B}_n)$.*

(c) *$\psi \in H^\infty(\mathbb{B}_n)$ and* $\sup_{z \in \mathbb{B}_n} (1 - ||z||^2) \, ||\nabla(f)(z)|| \log \dfrac{1}{1 - ||z||^2} < \infty.$

We extend these results, and those in Chapter 4, to the Bloch space and $*$-little Bloch space of a bounded homogeneous domain in $\mathbb{C}^n$. We characterize the bounded and the compact multiplication operators, establish operator norm estimates, determine the spectrum of the multiplication operators, and characterize the isometric operators on the Bloch space and $*$-little Bloch space of a bounded symmetric domain.

5.1 Boundedness

Let D be a bounded homogeneous domain in $\mathbb{C}^n$. In [6], it was shown that for $f \in \mathcal{B}(D)$, the Bloch semi-norm of f is precisely the Lipschitz number of f, that is,

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

Thus, for any function $f \in \mathcal{B}(D)$ and $z, w \in D$, we have

$$|f(z) - f(w)| \leq \rho(z, w)\beta_f. \tag{5.1}$$

In particular, if f fixes the origin and $||f||_{\mathcal{B}} \leq 1$, then

$$|f(z)| \leq \rho(z, 0) \tag{5.2}$$

by taking $w = 0$ in (5.1). This also holds in the case that $f \in \mathcal{B}_{0^*}(D)$ with $f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1$.

For $z \in D$, define

$$\begin{aligned} \omega(z) &= \sup\{|f(z)| : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\}, \\ \omega_0(z) &= \sup\{|f(z)| : f \in \mathcal{B}_{0^*}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\}. \end{aligned} \tag{5.3}$$

Since $\mathcal{B}_{0^*}(D) \subseteq \mathcal{B}(D)$, we have $\omega_0(z) \leq \omega(z)$ for all $z \in D$. From (5.2), we also have $\omega(z) \leq \rho(z, 0)$. Thus, $\omega_0(z)$ and $\omega(z)$ are finite for each $z \in D$.

Lemma 5.1.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $z \in D$.*

(a) *If $f \in \mathcal{B}(D)$, then*

$$|f(z)| \leq |f(0)| + \omega(z)\beta_f.$$

(b) *If* $f \in \mathcal{B}_{0^*}(D)$, *then*

$$|f(z)| \leq |f(0)| + \omega_0(z)\beta_f.$$

Proof. To prove (a), let $f \in \mathcal{B}(D)$. The result is immediate if f is constant. So assume f is not constant, and for $z \in D$, define $g(z) = \frac{1}{\beta_f}(f(z) - f(0))$. Then g is Bloch, $g(0) = 0$, and $||g||_{\mathcal{B}} = \beta_g = 1$. Thus $|g(z)| \leq \omega(z)$ for all $z \in D$. Consequently

$$|f(z)| \leq |f(0)| + |f(z) - f(0)| = |f(0)| + |g(z)|\,\beta_f \leq |f(0)| + \omega(z)\beta_f.$$

The proof of (b) is analogous. □

Theorem 5.1.2. [100] *Let* $z, w \in \mathbb{B}_n$. *Then*

$$\begin{aligned}\rho(z,w) &= \sup\{|f(z) - f(w)| : f \in \mathcal{B}(\mathbb{B}_n), ||f||_{\mathcal{B}} \leq 1\} \\ &= \sup\{|f(z) - f(w)| : f \in \mathcal{B}_0(\mathbb{B}_n), ||f||_{\mathcal{B}} \leq 1\}.\end{aligned}$$

In the case when $D = \mathbb{B}_n$, from Theorem 5.1.2, it follows that $\omega(z)$ and $\omega_0(z)$ coincide with the Bergman distance from z to 0, that is

$$\omega_0(z) = \omega(z) = \rho(z, 0) = \frac{1}{2}\log\frac{1 + ||z||}{1 - ||z||}. \tag{5.4}$$

In the case when $D = \mathbb{D}^n$, we do not have an explicit formulation of $\omega(z)$ or $\omega_0(z)$. However, the following lemma provides useful estimates involving $\omega(z)$ and $\rho(z, 0)$ for the unit polydisk.

Theorem 5.1.3. [25] *Let* $f \in \mathcal{B}(\mathbb{D}^n)$. *Then*

$$\beta_f = \sup_{z \in \mathbb{D}^n}\left|\left|\left((1 - |z_1|^2)\frac{\partial f}{\partial z_1}(z), \ldots, (1 - |z_n|^2)\frac{\partial f}{\partial z_n}(z)\right)\right|\right|.$$

Lemma 5.1.4. *For $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$ and $k = 1, \ldots, n$,*

(a) $\dfrac{1}{2} \log \dfrac{1 + |z_k|}{1 - |z_k|} \leq \omega(z)$.

(b) $\rho(z, 0) \leq \dfrac{1}{2} \displaystyle\sum_{k=1}^{n} \log \dfrac{1 + |z_k|}{1 - |z_k|}$.

Proof. To prove (a), fix $z \in \mathbb{D}^n$, $k \in \{1, \ldots, n\}$, and define the function

$$h(w) = \frac{1}{2}\text{Log}\frac{|z_k| + w_k\overline{z_k}}{|z_k| - w_k\overline{z_k}},$$

for $w \in \mathbb{D}^n$, where Log denotes the principal branch of the logarithm. Thus $h \in H(\mathbb{D}^n)$ and $h(0) = 0$. Also, $\frac{\partial h}{\partial w_j}(w) = 0$ for $j \neq k$ and $\frac{\partial h}{\partial w_k}(w) = \frac{\overline{z_k}|z_k|}{|z_k|^2 - w_k^2\overline{z_k}^2}$. By Theorem 5.1.3,

$$||h||_{\mathcal{B}} = \beta_h \leq \sup_{w \in \mathbb{D}^n} (1 - |w_k|^2) \frac{|z_k|^2}{|z_k|^2 - |w_k|^2 |z_k|^2} = 1.$$

By the definition of $\omega(z)$, it follows that

$$\frac{1}{2} \log \frac{1 + |z_k|}{1 - |z_k|} = |h(z)| \leq \omega(z).$$

To prove (b), recall that the Bergman metric for the unit polydisk (2.3) is defined for $z \in \mathbb{D}^n$ and $u \in \mathbb{C}^n$ by

$$H_z(u, \overline{u}) = \sum_{k=1}^{n} \frac{|u_k|^2}{(1 - |z_k|^2)^2}.$$

If $\gamma : [0, 1] \to \mathbb{D}^n$ is the geodesic from w to z in $\mathbb{D}^n$, then

$$\rho(z, w) = \int_0^1 H_{\gamma(t)}(\gamma'(t), \overline{\gamma'(t)})^{1/2}\, dt.$$

Since the geodesic from z to 0 in $\mathbb{D}^n$ is parameterized by $\gamma(t) = tz$, we obtain

$$\begin{aligned}\rho(z,0) &= \int_0^1 \left(\sum_{k=1}^{n} \frac{|z_k|^2}{(1-|z_k|^2 t^2)^2} \right)^{1/2} dt \\ &\leq \int_0^1 \sum_{k=1}^{n} \frac{|z_k|}{1-|z_k|^2 t^2}\, dt \\ &= \frac{1}{2}\sum_{k=1}^{n} \log\frac{1+|z_k|}{1-|z_k|}. \quad \square\end{aligned}$$

Lemma 5.1.5. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $z \in D$.*

(a) *If $f \in \mathcal{B}(D)$, then*

$$|f(z)| \leq |f(0)| + \omega(z)\beta_f.$$

(b) *If $f \in \mathcal{B}_{0^*}(D)$, then*

$$|f(z)| \leq |f(0)| + \omega_0(z)\beta_f.$$

Proof. Let $f \in \mathcal{B}(D)$. If f is constant, then $\beta_f = 0$, $|f(z)| = |f(0)|$, and we are done. If f is not constant, $\beta_f \neq 0$ and we may define function $g(z) = \frac{1}{\beta_f}(f(z) - f(0))$, which is holomorphic on D, $g(0) = 0$ and $||g||_{\mathcal{B}} = 1$. So $|g(z)| \leq \omega(z)$ for all $z \in D$ and

$$|f(z)| \leq |f(0)| + |f(z) - f(0)| = |f(0)| + g(z)\beta_f \leq |f(0)| + \omega(z)\beta_f.$$

The proof of (b) is analogous. $\square$

Since $\omega_0(z)$ and $\omega(z)$ are bounded above by $\rho(z,0)$ for all $z \in D$, we obtain the following result.

Corollary 5.1.6. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $z \in D$. If f is in $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$, then*

$$|f(z)| \leq |f(0)| + \rho(z,0)\beta_f.$$

We wish to consider how to generalize the conditions for boundedness of the multiplication operator on the Bloch space of $\mathbb{D}$ to a general bounded homogeneous domain. On the Bloch space of the unit disk, the characterizing properties of the bounded multiplication operator M_ψ are $\psi \in H^\infty(\mathbb{D})$ and $\sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\,|\psi'(z)|\log\frac{1+|z|}{1-|z|} < \infty$. In trying to generalize this quantity to bounded homogeneous domains, we replace $(1-|z|^2)\,|\psi'(z)|$ with $Q_\psi(z)$ and $\frac{1}{2}\log\frac{1+|z|}{1-|z|}$ with $\rho(z,0)$, which equals $\omega(z)$, on $\mathbb{D}$. However, in domains other than the unit ball, we will replace $\rho(z,0)$ with $\omega(z)$.

For a bounded homogeneous domain D and $\psi \in H(D)$, define

$$\sigma_\psi = \sup_{z\in D} \omega(z) Q_\psi(z),$$

$$\sigma_{0,\psi} = \sup_{z\in D} \omega_0(z) Q_\psi(z).$$

Theorem 5.1.7. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$. Then*

(a) *M_ψ is bounded on $\mathcal{B}(D)$ if and only if $\psi \in H^\infty(D)$ and $\sigma_\psi < \infty$.*

(b) *M_ψ is bounded on $\mathcal{B}_{0^*}(D)$ if and only if $\psi \in H^\infty(D) \cap \mathcal{B}_{0^*}(D)$ and $\sigma_{0,\psi} < \infty$.*

Proof. To prove (a), assume $\psi \in H^\infty(D)$ such that $\sigma_\psi < \infty$ and let $f \in \mathcal{B}(D)$. From (2.6), we have

$$Q_{\psi f}(z) \le |\psi(z)|\, Q_f(z) + |f(z)|\, Q_\psi(z)$$

for all $z \in D$. We need to show that M_ψ maps $\mathcal{B}(D)$ into $\mathcal{B}(D)$ and there exists $c > 0$ such that $||M_\psi f||_{\mathcal{B}} \le c\,||f||_{\mathcal{B}}$ for all $f \in \mathcal{B}(D)$. By using Lemma 5.1.5(a) and the fact that ψ is bounded, we deduce

$$\begin{aligned}
\beta_{\psi f} &= \sup_{z\in D} Q_{\psi f}(z) \\
&\leq \sup_{z\in D} \left[|\psi(z)|\, Q_f(z) + |f(z)|\, Q_\psi(z) \right] \\
&\leq \sup_{z\in D} |\psi(z)|\, Q_f(z) + \sup_{z\in D} |f(z)|\, Q_\psi(z) \\
&\leq ||\psi||_\infty \, \beta_f + \sup_{z\in D} (|f(0)| + \omega(z)\beta_f) Q_\psi(z) \\
&\leq ||\psi||_\infty \, \beta_f + \sup_{z\in D} |f(0)|\, Q_\psi(z) + \sup_{z\in D} \omega(z) Q_\psi(z)\beta_f \\
&= |f(0)|\, \beta_\psi + (||\psi||_\infty + \sigma_\psi)\beta_f .
\end{aligned}$$

Thus $M_\psi f \in \mathcal{B}(D)$. Furthermore,

$$\begin{aligned}
||M_\psi f||_{\mathcal{B}} &= |\psi(0)|\, |f(0)| + \beta_{\psi f} \\
&\leq |f(0)|\, ||\psi||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi)\beta_f \qquad (5.5) \\
&\leq (||\psi||_{\mathcal{B}} + ||\psi||_\infty + \sigma_\psi)\, ||f||_{\mathcal{B}} .
\end{aligned}$$

Taking the supremum over all $f \in \mathcal{B}(D)$ such that $||f||_{\mathcal{B}} \leq 1$, we have

$$||M_\psi|| \leq ||\psi||_{\mathcal{B}} + ||\psi||_\infty + \sigma_\psi < \infty.$$

Therefore, M_ψ is bounded on $\mathcal{B}(D)$.

Conversely, assume M_ψ is bounded on $\mathcal{B}(D)$. By Theorem 3.6.2, $\psi \in H^\infty(D)$. So, it suffices to show that σ_ψ is finite. Let $f \in \mathcal{B}(D)$, $z \in D$, and $u \in \mathbb{C}^n \setminus \{0\}$. By the product rule, we have

$$\nabla(\psi f)(z)u = \psi(z)\nabla(f)(z)u + f(z)\nabla(\psi)(z)u,$$

and

$$|f(z)|\,|\nabla(\psi)(z)u| = |\nabla(\psi f)(z)u - \psi(z)\nabla(f)(z)u|$$

$$\leq |\nabla(\psi f)(z)u| + |\psi(z)|\,|\nabla(f)(z)u|\,.$$

By dividing by $H_z(u,\overline{u})^{1/2}$ and taking the supremum over all $u \in \mathbb{C}^n \setminus \{0\}$, we obtain

$$|f(z)|\,Q_\psi(z) \leq Q_{\psi f}(z) + |\psi(z)|\,Q_f(z)$$

$$\leq ||M_\psi f||_{\mathcal{B}} + |\psi(z)|\,||f||_{\mathcal{B}}$$

$$\leq (||M_\psi|| + |\psi(z)|)\,||f||_{\mathcal{B}}\,.$$

Taking the supremum over all $f \in \mathcal{B}(D)$ such that $f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1$, we deduce $\omega(z)Q_\psi(z) \leq ||M_\psi|| + |\psi(z)|$. Finally, taking the supremum over all $z \in D$, we have $\sigma_\psi \leq ||M_\psi|| + ||\psi||_\infty$, which is finite by assumption.

To prove (b), we assume $\psi \in H^\infty(D) \cap \mathcal{B}_{0^*}(D)$ and $\sigma_{0,\psi} < \infty$. Then $\omega_{0,\psi}(z)Q_\psi(z)$ is bounded on D and $\lim_{z\to\partial^*D} Q_\psi(z) = 0$, which implies

$$\lim_{z\to\partial^*D} \omega_{0,\psi}(z)Q_\psi(z) = 0.$$

If $f \in \mathcal{B}_{0^*}(D)$, then by Lemma 5.1.5, we get

$$\lim_{z\to\partial^*D} Q_{\psi f}(z) \leq \lim_{z\to\partial^*D} (|\psi(z)|\,Q_f(z) + |f(z)|\,Q_\psi(z))$$

$$\leq \lim_{z\to\partial^*D} |\psi(z)|\,Q_f(z) + \lim_{z\to\partial^*D} (|f(0)| + \omega_0(z)\beta_f)Q_\psi(z)$$

$$\leq ||\psi||_\infty \lim_{z\to\partial^*D} Q_f(z) + |f(0)| \lim_{z\to\partial^*D} Q_\psi(z) + \beta_f \lim_{z\to\partial^*D} \omega_0(z)Q_\psi(z)$$

$$= 0.$$

Therefore, $M_\psi f \in \mathcal{B}_{0^*}(D)$. The proof of the boundedness of M_ψ on $\mathcal{B}_{0^*}(D)$ is similar to that of part (a).

Conversely, we assume M_ψ is bounded on $\mathcal{B}_{0^*}(D)$. Then $\psi = M_\psi 1 \in \mathcal{B}_{0^*}(D)$ since $1 \in \mathcal{B}_{0^*}(D)$. By Theorem 3.6.2, $||\psi||_\infty \leq ||M_\psi||$, and so $\psi \in H^\infty(D)$. Arguing as in part (a), we obtain $\sigma_{0,\psi} \leq ||M_\psi|| + ||\psi||_\infty$, completing the proof. □

The characterizations of the bounded multiplication operators on the Bloch space of the unit disk and unit ball due to Brown and Shields, and Zhu, respectively, show an equivalence between the boundedness of the operator on the Bloch space and the little Bloch space. In what follows, we give an alternative proof of the equivalence when the ambient space is the unit ball and show that this equivalence also holds for the polydisk.

Theorem 5.1.8. *Let $\psi \in H(\mathbb{B}_n)$. Then the following are equivalent:*

(a) *M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$.*

(b) *M_ψ is bounded on $\mathcal{B}_0(\mathbb{B}_n)$.*

(c) *$\psi \in H^\infty(\mathbb{B}_n)$ and $\sup_{z\in\mathbb{B}_n} Q_\psi(z) \log \frac{1+||z||}{1-||z||}$ is finite.*

Proof. By (5.4), we have that $\omega_0(z) = \omega(z)$ for all $z \in \mathbb{B}_n$. Thus, we have $\sigma_{0,\psi} = \sigma_\psi$ as well. By Theorem 5.1.7, M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if $\psi \in H^\infty(\mathbb{B}_n)$ and

$$\sup_{z\in\mathbb{B}_n} \frac{1}{2} Q_\psi(z) \log \frac{1+||z||}{1-||z||} = \sup_{z\in\mathbb{B}_n} \omega(z) Q_\psi(z) < \infty,$$

and hence $(a) \Longleftrightarrow (c)$.

Likewise, if M_ψ is bounded on $\mathcal{B}_0(\mathbb{B}_n)$, then $\psi \in H^\infty(\mathbb{B}_n)$ and

$$\sup_{z\in\mathbb{B}_n} \frac{1}{2} Q_\psi(z) \log \frac{1+||z||}{1-||z||} = \sup_{z\in\mathbb{B}_n} \omega_0(z) Q_\psi(z) < \infty.$$

So $(b) \implies (c)$. To show $(c) \implies (b)$, it suffices to show $\psi \in \mathcal{B}_0(\mathbb{B}_n)$. Note that as $||z|| \to 1$, the function $\log \dfrac{1+||z||}{1-||z||}$ goes to ∞. So, the finiteness of $\sup_{z\in\mathbb{B}_n} Q_\psi(z) \log \dfrac{1+||z||}{1-||z||}$ implies that $Q_\psi(z) \to 0$ as $||z|| \to 1$. Hence $\psi \in \mathcal{B}_0(\mathbb{B}_n)$ and $(c) \implies (b)$, as desired. □

Next, we prove the analogous result for the Bloch space of the unit polydisk.

Theorem 5.1.9. *Let $\psi \in H(\mathbb{D}^n)$. Then the following are equivalent:*

(a) *M_ψ is bounded on $\mathcal{B}(\mathbb{D}^n)$.*

(b) *M_ψ is bounded on $\mathcal{B}_{0^*}(\mathbb{D}^n)$.*

(c) *$\psi \in H^\infty(\mathbb{D}^n)$ and $\sup_{z\in\mathbb{D}^n} Q_\psi(z) \sum_{k=1}^n \log \dfrac{1+|z_k|}{1-|z_k|}$ is finite.*

Proof. Assume M_ψ is bounded on $\mathcal{B}(\mathbb{D}^n)$. From Theorem 5.1.7, it follows that $\psi \in H^\infty(\mathbb{D}^n)$ and $\sup_{z\in\mathbb{D}^n} \omega(z) Q_\psi(z) < \infty$. By Lemma 5.1.4(a), we have

$$\frac{1}{2} \sup_{z\in\mathbb{D}^n} \sum_{k=1}^n Q_\psi(z) \log \frac{1+|z_k|}{1-|z_k|} \le \frac{1}{2} \sum_{k=1}^n \sup_{z\in\mathbb{D}^n} Q_\psi(z) \log \frac{1+|z_k|}{1-|z_k|} \le n \sup_{z\in\mathbb{D}^n} \omega(z) Q_\psi(z) < \infty.$$

So $(a) \implies (c)$. Conversely, assume $\psi \in H^\infty(\mathbb{D}^n)$ and $\sup_{z\in\mathbb{D}^n} Q_\psi(z) \sum_{k=1}^n \log \dfrac{1+|z_k|}{1-|z_k|} < \infty$. Then by Lemma 5.1.4(b), we deduce

$$\sup_{z\in\mathbb{D}^n} \omega_0(z) Q_\psi(z) \le \sup_{z\in\mathbb{D}^n} \omega(z) Q_\psi(z) \le \sup_{z\in\mathbb{D}^n} \rho(z,0) Q_\psi(z) \le \sup_{z\in\mathbb{D}^n} Q_\psi(z) \sum_{k=1}^n \log \frac{1+|z_k|}{1-|z_k|}.$$

Thus $\sigma_{0,\psi}$ and σ_ψ are finite. So by Theorem 5.1.7, we have $(c) \implies (a)$. Furthermore, as $z \to \partial^*\mathbb{D}^n$, $\sum_{k=1}^n \log \dfrac{1+|z_k|}{1-|z_k|} \to \infty$, and the boundedness of $\sup_{z\in\mathbb{D}^n} Q_\psi(z) \sum_{k=1}^n \log \dfrac{1+|z_k|}{1-|z_k|}$ implies that $Q_\psi(z) \to 0$. Thus $\psi \in \mathcal{B}_{0^*}(D)$, and $(c) \implies (b)$.

It remains to show that $(b) \implies (c)$. Assume M_ψ is bounded on $\mathcal{B}_{0^*}(D^n)$. Then by Theorem 5.1.7, $\psi \in H^\infty(\mathbb{D}^n) \cap \mathcal{B}_{0^*}(\mathbb{D}^n)$ and $\sigma_{0,\psi} < \infty$. If for each $k \in \{1,\ldots,n\}$ $\sup_{z\in\mathbb{D}^n} Q_\psi(z)\log\frac{1+|z_k|}{1-|z_k|} < \infty$, then

$$\sup_{z\in\mathbb{D}^n} Q_\psi(z)\sum_{k=1}^{n}\log\frac{1+|z_k|}{1-|z_k|} \leq \sum_{k=1}^{n}\sup_{z\in\mathbb{D}^n} Q_\psi(z)\log\frac{1+|z_k|}{1-|z_k|} < \infty.$$

So it suffices to show that $\sup_{z\in\mathbb{D}^n} Q_\psi(z)\log\frac{1+|z_k|}{1-|z_k|} < \infty$ for each $k \in \{1,\ldots,n\}$.

Fix $k \in \{1,\ldots,n\}$ and $w \in \mathbb{D}^n \setminus \{0\}$, and for $z \in \mathbb{D}^n$ define

$$f_w(z) = \frac{1}{2}\mathrm{Log}\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}.$$

It follows that $f_w(0) = 0$, $\frac{\partial f_w}{\partial z_j}(z) = 0$ for all $j \neq k$, and $\frac{\partial f_w}{\partial z_k}(z) = \frac{\overline{w_k}}{1-\overline{w_k}^2 z_k^2}$. From Theorem 5.1.3, we have

$$\begin{aligned} Q_{f_w}(z) &= \left|\left|\left((1-|z_1|^2)\frac{\partial f_w}{\partial z_1}(z),\ldots,(1-|z_n|^2)\frac{\partial f_w}{\partial z_n}(z)\right)\right|\right| \\ &= (1-|z_k|^2)\left|\frac{\partial f_w}{\partial z_k}(z)\right| \\ &\leq |w_k|\,\frac{1-|z_k|^2}{1-|w_k|^2\,|z_k|^2}. \end{aligned}$$

Define the real-valued function $g : [0,1] \to \mathbb{R}$ by $g(x) = \frac{1-x^2}{1-\alpha^2x^2}$ for $0 < \alpha < 1$. By elementary calculus, g attains it's absolute maximum on $[0,1]$ at $x = 0$. Thus

$$||f_w||_{\mathcal{B}} = |f_w(0)| + \sup_{z\in\mathbb{D}^n} Q_{f_w}(z) \leq |w_k| < 1,$$

and $f_w \in \mathcal{B}_{0^*}(\mathbb{D}^n)$ since

$$\lim_{z\to\partial^*D} Q_{f_w}(z) \leq |w_k| \lim_{|z_k|\to 1} \frac{1-|z_k|^2}{1-|w_k|^2|z_k|^2} = 0.$$

Thus, for $z \in \mathbb{D}^n$,

$$|f_w(z)|\, Q_\psi(z) \leq \omega_{0,\psi}(z) Q_\psi(z) \leq \sigma_{0,\psi}. \tag{5.6}$$

Observe that

$$\begin{aligned} |f_w(z)| &= \frac{1}{2}\left|\mathrm{Log}\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}\right| \geq \frac{1}{2}\left(\log\left|\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}\right| - \mathrm{Arg}\left(\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}\right)\right) \\ &\geq \frac{1}{2}\left(\log\left|\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}\right| - \frac{\pi}{2}\right), \end{aligned}$$

where Arg denotes the principal value of the argument. Thus

$$\frac{1}{2}\log\left|\frac{1+\overline{w_k}z_k}{1-\overline{w_k}z_k}\right| \leq |f_w(z)| + \frac{\pi}{4}.$$

Let $w_k = |w_k|\, e^{i\theta_k}$ and choose z_k so that $\arg(z_k) = \theta_k$. From (5.6), we obtain

$$\frac{1}{2}Q_\psi(z)\log\frac{1+|w_k|\,|z_k|}{1-|w_k|\,|z_k|} \leq \sigma_{0,\psi} + \frac{\pi}{4}\beta_\psi.$$

By letting $|w_k| \to 1$, we obtain $\sup_{z\in\mathbb{D}^n} Q_\psi(z)\log\frac{1+|z_k|}{1-|z_k|} \leq \sigma_{0,\psi} + \frac{\pi}{4}\beta_\psi < \infty$, as desired. □

Corollary 5.1.10. *Let $n \geq 2$ and $\psi \in H(\mathbb{D}^n)$. Then M_ψ is bounded on $\mathcal{B}(\mathbb{D}^n)$ if and only if ψ is a constant function.*

Proof. If ψ is a constant function, then it is immediate that M_ψ is bounded on $\mathcal{B}(\mathbb{D}^n)$.

Conversely, suppose M_ψ is bounded on $\mathcal{B}(\mathbb{D}^n)$. By Theorem 5.1.9,

$$\sup_{z\in\mathbb{D}^n} Q_\psi(z) \sum_{k=1}^{n} \log \frac{1+|z_k|}{1-|z_k|} < \infty.$$

As $z \to \partial\mathbb{D}^n$, $\log \frac{1+|z_k|}{1-|z_k|} \to \infty$ for some $k \in \{1, \ldots, n\}$. Thus,

$$\lim_{z\to\partial\mathbb{D}^n} Q_\psi(z) = 0.$$

Since $n \geq 2$, the unit polydisk is not biholomorphically equivalent to the unit ball, and so by Theorem 2.4.10, ψ is a constant function. □

Remark 5.1.11. An analogous result is not true for the unit ball; there exist non-constant symbols which induce bounded multiplication operators on the Bloch space of $\mathbb{B}_n$. For $j \in \{1, \ldots, n\}$, the projection map $p_j(z_1, \ldots, z_n) = z_j$ is a bounded holomorphic function on $\mathbb{B}_n$ with $||\nabla(p_j)(z)|| = 1$. Since $1-||z||^2 \to 0$ as $z \to \partial\mathbb{B}_n$ faster than $\log(1-||z||^2) \to -\infty$ as $z \to \partial\mathbb{B}_n$, we have

$$\sup_{z\in\mathbb{B}_n} (1-||z||^2)\, ||\nabla(p_j)(z)|| \log \frac{1}{1-||z||^2} < \infty.$$

So ψ is bounded and non-constant, and by Theorem 5.0.1, M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$.

We end this section with a sufficient condition for the equivalence of the boundedness of M_ψ as an operator on the Bloch space and on the $*$-little Bloch space of a bounded homogeneous domain. If A and B are positive constants, then we denote $A \asymp B$ to mean there exist $c, C > 0$ such that $cA \leq B \leq CA$.

Theorem 5.1.12. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$. If $\sigma_\psi \asymp \sigma_{0,\psi}$ and $\lim_{z\to\partial^* D} \omega_0(z) = \infty$, then M_ψ is bounded on $\mathcal{B}(D)$ if and only if it is bounded on $\mathcal{B}_{0^*}(D)$.*

Proof. Let $\psi \in H(D)$ such that $\sigma_\psi \asymp \sigma_{0,\psi}$ and $\omega_0(z) \to \infty$ as $z \to \partial^* D$. We want to show that the boundedness of M_ψ on $\mathcal{B}(D)$ is equivalent to the boundedness on $\mathcal{B}_{0^*}(D)$. Assume M_ψ is bounded on $\mathcal{B}(D)$. Then by Theorem 5.1.7, $\psi \in H^\infty(D)$ and $\sigma_\psi < \infty$, and in particular $\sigma_{0,\psi} < \infty$. Since $\sigma_{0,\psi} = \sup_{z \in \mathbb{D}^n} \omega_0(z) Q_\psi(z)$ is finite, and $\omega_0(z) \to \infty$ as $z \to \partial^* D$, it must be the case that $Q_\psi(z) \to 0$. Thus $\psi \in \mathcal{B}_{0^*}(D)$, proving the M_ψ is bounded on $\mathcal{B}_{0^*}(D)$.

Next, assume M_ψ is bounded on $\mathcal{B}_{0^*}(D)$. Then $\psi \in H^\infty(D)$ and $\sigma_{0,\psi} < \infty$. Since $\sigma_\psi \asymp \sigma_{0,\psi}$, it follows immediately that $\sigma_\psi < \infty$. Thus M_ψ is bounded on $\mathcal{B}(D)$. □

5.2 Operator Norm Estimates

In this section, we provide estimates on the norm of the bounded multiplication operators acting on the Bloch space and the $*$-little Bloch space of a bounded homogeneous domain. These norm estimates, when applied to the case of the Bloch space and the little Bloch space of the unit disk correspond to those of Theorem 4.1.2 in Chapter 4.

Theorem 5.2.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$.*

(a) *If M_ψ is bounded on $\mathcal{B}(D)$, then*

$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \le ||M_\psi|| \le \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\}.$$

(b) *If M_ψ is bounded on $\mathcal{B}_{0^*}(D)$, then*

$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \le ||M_\psi|| \le \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_{0,\psi}\}.$$

Proof. To prove (a), assume $\psi \in H(D)$ induces a bounded multiplication operator M_ψ on $\mathcal{B}(D)$. By Theorem 3.6.2, we have $||\psi||_\infty \le ||M_\psi||$. By considering the constant function 1,

$||\psi||_{\mathcal{B}} = ||M_\psi 1||_{\mathcal{B}} \le ||M_\psi||$, and thus

$$\max\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty\} \le ||M_\psi||\,.$$

For $f \in \mathcal{B}(D)$, inequality (5.5) shows that

$$||M_\psi f||_{\mathcal{B}} \le |f(0)|\,||\psi||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi)\beta_f.$$

Since $|f(0)| = ||f||_{\mathcal{B}} - \beta_f$, we obtain

$$\begin{aligned} ||M_\psi f||_{\mathcal{B}} &\le (||f||_{\mathcal{B}} - \beta_f)\,||\psi||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi)\beta_f \\ &\le ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi - ||\psi||_{\mathcal{B}})\beta_f. \end{aligned}$$

If $||\psi||_\infty + \sigma_\psi \le ||\psi||_{\mathcal{B}}$, then $||M_\psi f||_{\mathcal{B}} \le ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}}$. This implies that $||M_\psi|| \le ||\psi||_{\mathcal{B}}$. On the other hand, if $||\psi||_\infty + \sigma_\psi \ge ||\psi||_{\mathcal{B}}$, then

$$\begin{aligned} ||M_\psi f||_{\mathcal{B}} &\le ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi - ||\psi||_{\mathcal{B}})\beta_f \\ &\le ||\psi||_{\mathcal{B}}\,||f||_{\mathcal{B}} + (||\psi||_\infty + \sigma_\psi - ||\psi||_{\mathcal{B}})\,||f||_{\mathcal{B}} \\ &= (||\psi||_\infty + \sigma_\psi)\,||f||_{\mathcal{B}}\,. \end{aligned}$$

Thus $||M_\psi|| \le ||\psi||_\infty + \sigma_\psi$, hence

$$||M_\psi|| \le \max\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty + \sigma_\psi\}.$$

The proof of (b) is similar. $\square$

5.3 Spectrum

In this section, we determine the spectrum of the bounded multiplication operators on the Bloch space and the $*$-little Bloch space of a bounded homogeneous domain in $\mathbb{C}^n$. Thus, we extend Theorem 4.2.1 to the bounded homogeneous domain setting.

Remark 5.3.1. If M_ψ is bounded on some Banach space of holomorphic functions, $\lambda \in \sigma(M_\psi)$ if and only if $M_\psi - \lambda I$ is not invertible where I is the identity operator. Since $\lambda I = M_\lambda$, and $M_\psi - M_\lambda = M_{\psi-\lambda}$, we see that $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible. Since $M_{\psi-\lambda}^{-1} = M_{(\psi-\lambda)^{-1}}$, we finally arrive at $\lambda \in \sigma(M_\psi)$ if and only if $M_{(\psi-\lambda)^{-1}}$ is undefined.

Theorem 5.3.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$ such that M_ψ is bounded on either $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$. Then $\sigma(M_\psi) = \overline{\psi(D)}$.*

Proof. First, assume $\psi \in H(D)$ induces a bounded multiplication operator on $\mathcal{B}(D)$. Let $\lambda \in \psi(D)$. Then there exists $z_0 \in D$ such that $\psi(z_0) = \lambda$, and thus the function $(\psi - \lambda)^{-1}$ is singular at z_0. So $M_{(\psi-\lambda)^{-1}}$ is undefined as an operator on $\mathcal{B}(D)$. Thus $\psi(D) \subseteq \sigma(M_\psi)$, and since the spectrum is closed, we have $\overline{\psi(D)} \subseteq \sigma(M_\psi)$.

Now, suppose $\lambda \notin \overline{\psi(D)}$. Then the function $\psi - \lambda$ is bounded away from zero, that is there exists $c > 0$ such that $|\psi(z) - \lambda| \geq c$ for all $z \in D$. Thus the function g defined by

$$g(z) = \frac{1}{\psi(z) - \lambda}, \text{ for } z \in D, \tag{5.7}$$

is bounded and holomorphic with $|g(z)| \leq \frac{1}{c}$, for all $z \in D$. Observe that for $z \in D$,

$$\nabla(g)(z) = \frac{-1}{(\psi(z) - \lambda)^2} \nabla(\psi)(z),$$

so

$$\sigma_g = \sup_{z\in D} \omega(z)Q_g(z) \le \sup_{z\in D} \frac{1}{c^2}\omega(z)Q_\psi(z) = \frac{1}{c^2}\sigma_\psi < \infty.$$

By Theorem 5.1.7, M_g is bounded on $\mathcal{B}(D)$. Thus $\lambda \notin \sigma(M_\psi)$.

We will now show that if M_ψ is bounded on $\mathcal{B}_{0^*}(D)$, then $\sigma(M_\psi) = \overline{\psi(D)}$. Arguing as in the case of $\mathcal{B}(D)$, it suffices to show that the bounded holomorphic function g defined in (5.7) is in the $*$-little Bloch space for $\lambda \notin \overline{\psi(D)}$. Since M_ψ is bounded on $\mathcal{B}_{0^*}(D)$, $\psi = M_\psi 1 \in \mathcal{B}_{0^*}(D)$. So

$$\lim_{z\to\partial^* D} Q_g(z) \le \lim_{z\to\partial^* D} \frac{1}{c^2} Q_\psi(z) = 0,$$

as desired. □

5.4 Compactness

In this section, we characterize the compact multiplication operators acting on the Bloch space or the $*$-little Bloch space of a bounded homogeneous domain in $\mathbb{C}^n$.

Theorem 5.4.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$. Then M_ψ is compact on $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ if and only if ψ is identically zero.*

Proof. If ψ is identically zero, then M_ψ is compact. Suppose M_ψ is compact on $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$. By Corollary 3.5.6, the spectrum of M_ψ is at most countable and by Theorem 5.3.2, the spectrum of M_ψ is $\overline{\psi(D)}$, thus $\overline{\psi(D)}$ is at most countable. If $\overline{\psi(D)}$ contains two distinct points, it must contain a continuum, a contradiction. So $\overline{\psi(D)}$ is a singleton, and thus by Corollary 3.5.6, $\psi(D) = \{0\}$. Therefore ψ is identically zero. □

5.5 Isometries

In this section, we characterize the isometric multiplication operations on the Bloch space and on the $*$-little Bloch space of a class of bounded symmetric domains in $\mathbb{C}^n$. Recall the computed values of the Bloch constant c_D for an irreducible bounded symmetric domain were given in (2.11) and (2.12).

Lemma 5.5.1. *Let $D = D_1 \times \cdots \times D_k$ be a bounded symmetric domain in standard form. Then $c_D \leq 1$ and $c_D = 1$ if and only if $D_j = \mathbb{D}$ for some $j \in \{1, \ldots, k\}$.*

Proof. Fix $j \in \{1, \ldots, k\}$. If D_j is an exceptional domain, then $c_{D_j} < 1$ from (2.12). If D_j is a classical Cartan domain of type R_2, R_3 or R_4, recalling the dimensional restrictions $n \geq 2$ for the domains in R_2 and $n \geq 5$ for the domains in R_3 and R_4, we see that $c_{D_j} < 1$. If $D_j \in R_1$, then $c_{D_j} \leq 1$ and $c_{D_j} = 1$ if and only if $m = n = 1$, that is, if and only if $D_j = \mathbb{D}$. Therefore, by Theorem 2.3.27, $c_D \leq 1$ and $c_D = 1$ if and only if there exists $j \in \{1, \ldots, k\}$ such that $D_j = \mathbb{D}$. □

We denote by $\mathfrak{D}$ the set of bounded symmetric domains $D = D_1 \times \cdots \times D_k$ in $\mathbb{C}^n$ for which $D_j \neq \mathbb{D}$ for all $j = 1, \ldots, k$. By Lemma 5.5.1, $D \in \mathfrak{D}$ if and only if $c_D < 1$.

Lemma 5.5.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$. If M_ψ is an isometry on $\mathcal{B}(D)$ $(\mathcal{B}_{0^*}(D))$, then M_{ψ^k} is an isometry on $\mathcal{B}(D)$ $(\mathcal{B}_{0^*}(D))$ for all $k \in \mathbb{N}$. In particular, $\left|\left|\psi^k\right|\right|_{\mathcal{B}} = 1$ for all $k \in \mathbb{N}$.*

Proof. We will prove the result for M_ψ an isometry on $\mathcal{B}(D)$. The proof for M_ψ an isometry on $\mathcal{B}_{0^*}(D)$ is exactly the same. If $f \in \mathcal{B}(D)$, then

$$\left|\left|M_{\psi^2} f\right|\right|_{\mathcal{B}} = \left|\left|\psi^2 f\right|\right|_{\mathcal{B}} = \left|\left|M_\psi(\psi f)\right|\right|_{\mathcal{B}} = \left|\left|\psi f\right|\right|_{\mathcal{B}} = \left|\left|M_\psi f\right|\right|_{\mathcal{B}} = \left|\left|f\right|\right|_{\mathcal{B}}.$$

Thus M_{ψ^2} is an isometry on $\mathcal{B}(D)$. Thus by induction, M_{ψ^k} is an isometry on $\mathcal{B}(D)$ for all $k \in \mathbb{N}$. Also, $1 = ||1||_{\mathcal{B}} = ||M_\psi 1||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}$ and $1 = ||\psi||_{\mathcal{B}} = ||M_\psi \psi||_{\mathcal{B}} = \left|\left|\psi^2\right|\right|_{\mathcal{B}}$. Thus, by induction, $\left|\left|\psi^k\right|\right|_{\mathcal{B}} = 1$ for all $k \in \mathbb{N}$. □

Lemma 5.5.3. *Let D be a bounded symmetric domain in $\mathbb{C}^n$ and $\psi \in H(D)$. If M_ψ is an isometry on $\mathcal{B}(D)$ $(\mathcal{B}_{0^*}(D))$, then $\beta_{\psi^k} \leq c_D$ for all $k \in \mathbb{N}$. In particular, if $D \in \mathfrak{D}$ then*

$\beta_{\psi^k} < 1$ *for all* $k \in \mathbb{N}$.

Proof. Since M_ψ is an isometry, then M_{ψ^k} is an isometry for all $k \in \mathbb{N}$ by Lemma 5.5.2. From Theorem 3.6.2, we have $||\psi^k||_\infty \leq ||M_{\psi^k}|| = 1$. Thus ψ is either a constant function of modulus one or a bounded holomorphic function mapping into $\mathbb{D}$. If ψ^k is a constant function of modulus one, then $\beta_{\psi^k} = 0$. On the other hand, if ψ^k maps into $\mathbb{D}$, then by the definition of the Bloch constant of D, $\beta_{\psi^k} \leq c_D$ for all $k \in \mathbb{N}$. In either case, $\beta_{\psi^k} \leq c_D$ for all $k \in \mathbb{N}$. If $D \in \mathfrak{D}$, then $\beta_{\psi^k} \leq c_D < 1$. □

Theorem 5.5.4. *Let* $D \in \mathfrak{D}$ *and* $\psi \in H(D)$. *Then* M_ψ *is an isometry on* $\mathcal{B}(D)$ *or* $\mathcal{B}_{0^*}(D)$ *if and only if* ψ *is a constant function of modulus one.*

Proof. If ψ is a constant function of modulus one, then M_ψ is an isometry on $\mathcal{B}(D)$. Conversely, assume M_ψ is an isometry on $\mathcal{B}(D)$ and ψ is not a constant function of modulus one. Then $\psi(0) = a$ for some $|a| < 1$. Since $||\psi^k||_{\mathcal{B}} = 1$ by Lemma 5.5.2, we have

$$|a|^k = 1 - \beta_{\psi^k} \geq 1 - c_D > 0.$$

So $|a|^k$ is bounded away from zero. However, as $k \to \infty$, $|a|^k \to 0$, a contradiction. Thus, if M_ψ is an isometry, then ψ must be a constant function of modulus one. An analogous argument holds for M_ψ acting on $\mathcal{B}_{0^*}(D)$. □

We are also able to characterize the isometric multiplication operators on the Bloch space of the unit polydisk for all $n \in \mathbb{N}$. The $n = 1$ case is Theorem 4.3.6, and the $n > 1$ case is a result of Corollary 5.1.10.

Corollary 5.5.5. *Let* $\psi \in H(\mathbb{D}^n)$. *Then* M_ψ *is an isometry on* $\mathcal{B}(\mathbb{D}^n)$ *if and only if* ψ *is a constant function of modulus one.*

Remark 5.5.6. Recall that M_ψ is an isometry on $\mathcal{B}(\mathbb{D})$ or $\mathcal{B}_0(\mathbb{D})$ if and only if ψ is a constant function of modulus one (Theorem 4.3.6). The key to proofs of Theorem 4.3.6

and Theorem 5.5.4 was to find a means by which to make $\beta_{\psi^k} < 1$ for the symbols which induce an isometric multiplication operator. In the higher dimensional case, this required the restriction to a class of bounded symmetric domains which excludes the unit disk. At this time, we are unable to connect these two characterizations unless the domain is the unit polydisk.

Chapter 6: Spectrum of an Isometric Composition Operator on the Bloch Space of the Unit Disk, Unit Ball, and Unit Polydisk

In this chapter, we consider the determination of the spectrum of the composition operators on the Bloch space of the unit disk, unit ball, and unit polydisk induced by various symbols. In the case of the unit disk, we determine the spectrum of the isometric composition operators using Theorem 3.4.1. For the unit ball, the spectra of composition operators induced by automorphisms fixing an interior point of the ball is considered. Finally, in the case of the polydisk, we determine the spectrum for the isometric composition operators induced by automorphisms that fix the origin, as well as isometric composition operators induced by a surjective, non-automorphic symbol.

6.1 Isometric Composition Operators

In [91], Xiong showed that if φ is a rotation of the unit disk, then the induced composition operator C_φ is an isometry on $\mathcal{B}(\mathbb{D})$, but was unsuccessful in characterizing the symbol of the isometries amongst the composition operators. In [29], Colonna characterized the isometric composition operators as those induced by the maps φ for which $\varphi(0) = 0$ and $\beta_\varphi = 1$. Furthermore, Colonna provided a means of constructing the symbols that induce an isometry.

In [6], we collect several equivalent conditions for the symbols to induce an isometric composition operator on $\mathcal{B}(\mathbb{D})$.

Definition 6.1.1. A *Blaschke product* is an analytic self-map of $\mathbb{D}$ of the form

$$B(z) = z^m \prod \frac{\overline{z_n}}{|z_n|} \left(\frac{z_n - z}{1 - \overline{z_n} z} \right), \; z \in \mathbb{D}$$

where the product is taken over the $z_n \neq 0$ and m is the multiplicity at 0. If the zero set is finite, B is said to be a finite Blaschke product and the number of zeros of B counted according to multiplicity is called the *degree* of B. If the zero set is infinite, B is said to be an infinite Blaschke product.

Theorem 6.1.2. [4] *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is an isometry on $\mathcal{B}(\mathbb{D})$ if and only if $\varphi(0) = 0$ and any of the following equivalent conditions holds:*

(a) $\beta_\varphi = 1$.

(b) $B_\varphi := \sup_{z \in \mathbb{D}} \dfrac{(1 - |z|^2)\,|\varphi'(z)|}{1 - |\varphi(z)|^2} = 1$.

(c) *Either $\varphi \in \mathrm{Aut}(\mathbb{D})$ or for every $a \in \mathbb{D}$ there exists a sequence $\{z_k\}$ in $\mathbb{D}$ such that $|z_k| \to 1$, $\varphi(z_k) = a$, and*

$$\lim_{k \to \infty} \frac{(1 - |z_k|^2)\,|\varphi'(z_k)|}{1 - |\varphi(z_k)|^2} = 1.$$

(d) *Either $\varphi \in \mathrm{Aut}(\mathbb{D})$ or for every $a \in \mathbb{D}$ there exists a sequence $\{z_k\}$ in $\mathbb{D}$ such that $|z_k| \to 1$, $\varphi(z_k) \to a$, and*

$$\lim_{k \to \infty} \frac{(1 - |z_k|^2)\,|\varphi'(z_k)|}{1 - |\varphi(z_k)|^2} = 1.$$

(e) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or the zeros of* φ *form an infinite sequence* $\{z_k\}$ *in* $\mathbb{D}$ *such that*

$$\limsup_{k\to\infty} (1 - |z_k|^2) \left|\varphi'(z_k)\right| = 1.$$

(f) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or* $\varphi = gB$ *where* $g : \mathbb{D} \to \overline{\mathbb{D}}$ *is non-vanishing, analytic and* B *is an infinite Blaschke product whose zeros form a sequence* $\{z_k\}$ *containing 0 and an infinite subsequence* $\{z_{k_j}\}$ *such that* $\left|g(z_{k_j})\right| \to 1$ *and*

$$\lim_{j\to\infty} \prod_{\ell \neq k_j} \left| \frac{z_{k_j} - z_\ell}{1 - \overline{z}_{k_j} z_\ell} \right| = 1.$$

(g) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or there exists a sequence* $\{S_k\}$ *in* $\mathrm{Aut}(\mathbb{D})$ *such that* $|S_k(0)| \to 1$ *and* $\{\varphi \circ S_k\}$ *approaches the identity locally uniformly in* $\mathbb{D}$.

In addition, we consider the problem of characterizing the isometries in higher dimensions. The following sufficient condition was established for the Bloch space of a bounded homogeneous domain.

Theorem 6.1.3. [6] *Let* D *be a bounded homogeneous domain,* $\varphi \in H(D)$ *such that* $\varphi(0) = 0$ *and suppose there exists a sequence* $\{S_k\}$ *in* $\mathrm{Aut}(D)$ *such that* $\{\varphi \circ S_k\}$ *converges to the identity locally uniformly in* D*. If the Bergman constant of* φ *does not exceed 1, then* C_φ *is an isometry on* $\mathcal{B}(D)$.

The following necessary conditions were established for the Bloch space of a bounded symmetric domain.

Theorem 6.1.4. [6] *Let* $D = D_1 \times \cdots \times D_k$ *be a bounded symmetric domain in standard form. Let* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *be a holomorphic self-map of* D *such that* C_φ *is an isometry on* $\mathcal{B}(D)$*. Then:*

(a) *The components* $\varphi_1, \ldots, \varphi_n$ *are linearly independent.*

(b) *If D does not contain any exceptional factors, then $\varphi(0) = 0$.*

(c) *If none of the factors of D is in R_4, then the components of φ have semi-norms equal to $c_{D_1}, \ldots, c_{D_k}$, repeated according to the dimension of each factor.*

(d) *If D has factors in R_4, then for each $D_\ell \in R_4$ and each pair r, s of distinct indices, with $\sum_{i=1}^{\ell-1} dim(D_i) < r, s \leq \sum_{i=1}^{\ell} dim(D_i)$, the modified components $\varphi_r + i\varphi_s, \varphi_r - i\varphi_s$ are holomorphic from D to $\mathbb{D}$ and have semi-norm equal to c_{D_ℓ}.*

6.2 The Unit Disk

In this section, we determine the spectrum for the isometric composition operators on $\mathcal{B}(D)$. In [29], Colonna characterized the isometries amongst the composition operators on the Bloch space of $\mathbb{D}$ as those being induced by symbol φ such that $\varphi(0) = 0$ and $\beta_\varphi = 1$. With Theorems 3.4.1 and 6.1.2, we can determine the spectrum of the isometric composition operators on $\mathcal{B}(\mathbb{D})$.

Definition 6.2.1. For an element $\lambda \in \partial\mathbb{D}$, the *order* of λ, denoted $\text{ord}(\lambda)$, is defined as the smallest natural number m such that $\lambda^m = 1$. If no such natural number exists, we say λ has infinite order, and write $\text{ord}(\lambda) = \infty$.

Theorem 6.2.2. *Let φ be an analytic self-map of $\mathbb{D}$ which induces an isometric composition operator C_φ on $\mathcal{B}(\mathbb{D})$. If φ is not a rotation, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$. If there exists $\zeta \in \partial\mathbb{D}$ such that $\varphi(z) = \zeta z$, then*

$$\sigma(C_\varphi) = \begin{cases} \partial\mathbb{D} & \textit{if } \text{ord}(\zeta) = \infty \\ \langle\zeta\rangle & \textit{if } \text{ord}(\zeta) < \infty, \end{cases}$$

where $\langle\zeta\rangle = \{\zeta^k : k \in \{1, \ldots, \text{ord}(\zeta)\}\}$, the cyclic group generated by ζ.

Proof. Assume φ is not a rotation. Then by Theorem 3.4.1, is suffices to show that $0 \in \sigma(C_\varphi)$, that is, C_φ is not invertible. Since C_φ is an isometry, and thus injective, we show that C_φ is not surjective. Arguing by contradiction, assume C_φ is surjective. Since φ is not a rotation, by Theorem 6.1.2(e) φ has infinitely many zeros. Let a and a' be two distinct zeros of φ.

The function defined by $h(z) = z - a$ is Bloch, and since C_φ is surjective, there exists a Bloch function f such that $f \circ \varphi = h$. On the one hand, $f(0) = f(\varphi(a)) = h(a) = 0$. On the other hand, $f(0) = f(\varphi(a')) = h(a') \neq 0$, a contradiction. Therefore C_φ is not surjective, and thus $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

Now suppose φ is a rotation, that is, there exists $\zeta \in \partial\mathbb{D}$ such that $\varphi(z) = \zeta z$. Thus φ is invertible with inverse $\varphi^{-1}(z) = \overline{\zeta} z$, and since φ^{-1} is analytic on $\mathbb{D}$, $C_\varphi^{-1} = C_{\varphi^{-1}}$ is bounded on $\mathcal{B}(\mathbb{D})$. Therefore by Theorem 3.4.1, $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$.

Let $G = \langle \zeta \rangle = \{\zeta^k : k \in \mathbb{N} \cup \{0\}\}$, which is a subset of $\partial\mathbb{D}$. Consider the Bloch function $f_k(z) = z^k$ for $k \in \mathbb{N} \cup \{0\}$. Then

$$(C_\varphi f_k)(z) = f_k(\zeta z) = \zeta^k z^k = \zeta^k f_k(z).$$

Thus ζ^k is an eigenvalue of C_φ corresponding to the eigenfunction f_k. So $G \subseteq \sigma(C_\varphi)$.

If the order of ζ is infinite, then G is dense in $\partial\mathbb{D}$. Since the spectrum is closed, we have $\partial\mathbb{D} = \overline{G} \subseteq \sigma(C_\varphi)$. Thus $\sigma(C_\varphi) = \partial\mathbb{D}$.

Now suppose $\operatorname{ord}(\zeta) = m < \infty$. Then $G = \{\zeta^k : k = 1, \ldots, m\}$. We now wish to show that $\sigma(C_\varphi) \subseteq G$. Let $\mu \in \partial\mathbb{D} \setminus G$. We will show that $C_\varphi - \mu I$ is invertible by proving that for every $g \in \mathcal{B}(\mathbb{D})$, there exists a unique $f \in \mathcal{B}(\mathbb{D})$ such that $f \circ \varphi - \mu f = g$. Since $\operatorname{ord}(\zeta) = m$, then $\varphi^{(m)}(z) := (\underbrace{\varphi \circ \cdots \circ \varphi}_{m\text{-times}})(z) = \zeta^m z = z$. By repeated application of φ, we can form the system of linear equations:

$$\begin{array}{lclcl} f(\varphi(z)) & - & \mu f(z) & = & g(z) \\ f(\varphi^{(2)}(z)) & - & \mu f(\varphi(z)) & = & g(\varphi(z)) \\ & \vdots & & \vdots & \\ f(z) & - & \mu f(\varphi^{(m-1)}(z)) & = & g(\varphi^{(m-1)}(z)). \end{array} \tag{6.1}$$

Equivalently, (6.1) can be posed as the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} -\mu & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 & -\mu \end{bmatrix}, x = \begin{bmatrix} f(z) \\ f(\varphi(z)) \\ \vdots \\ \vdots \\ f(\varphi^{(m-2)}(z)) \\ f(\varphi^{(m-1)}(z)) \end{bmatrix}, \text{ and } b = \begin{bmatrix} g(z) \\ g(\varphi(z)) \\ \vdots \\ \vdots \\ g(\varphi^{(m-2)}(z)) \\ g(\varphi^{(m-1)}(z)) \end{bmatrix}.$$

The determinant of A is $(-1)^m(\mu^m - 1)$, which is not zero since $\mu \notin G$. Thus $C_\varphi - \mu I$ is invertible. For $\mu \notin G$, the unique solution f of (6.1) is a finite linear combination of Bloch functions $g \circ \varphi^{(j-1)}$ for $j = 1, \ldots, n$, and thus is Bloch. Therefore $\sigma(C_\varphi) = G$. □

6.3 The Unit Ball

The automorphisms of the unit ball are characterized by their fixed points. In fact, $\varphi \in \text{Aut}(\mathbb{B}_n)$ falls into three classes (see Remark 2.4.5 of [82]):

- Type I: φ fixes at least one point in $\mathbb{B}_n$.
- Type II: φ fixes two points on $\partial\mathbb{B}_n$, and no points in $\mathbb{B}_n$.
- Type III: φ fixes one point on $\partial\mathbb{B}_n$, and no points in $\mathbb{B}_n$.

By Theorem 6.1.4(b), if $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ induces an isometric composition operator C_φ, then φ fixes the origin. Thus, isometric composition operators induced by automorphisms are only of Type I.

Lemma 6.3.1. [100] *An automorphism U of $\mathbb{B}_n$ is unitary if and only if $U(0) = 0$.*

Lemma 6.3.2. *If U is a unitary map of $\mathbb{B}_n$, then C_U is an isometry on $\mathcal{B}(\mathbb{B}_n)$ and $\sigma(C_U)$ is contained in the unit circle.*

Proof. By the Möbius invariance of the Bloch space, we have $\beta_{f\circ U} = \beta_f$ for all Bloch functions f. Thus, for any Bloch function f, we have

$$||C_U f||_{\mathcal{B}} = |f(U(0))| + \beta_{f\circ U} = |f(0)| + \beta_f = ||f||_{\mathcal{B}}.$$

Therefore C_U is an isometry on $\mathcal{B}(\mathbb{B}_n)$. Since $U \in \mathrm{Aut}(\mathbb{B}_n)$, $U^{-1} \in \mathrm{Aut}(\mathbb{B}_n)$ and $C_U^{-1} = C_{U^{-1}}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$. Therefore, by Theorem 3.4.1, $\sigma(C_U) \subseteq \partial\mathbb{D}$. □

Now, we compute the spectrum of the composition operator C_φ on the Bloch space of the unit ball when φ is an automorphism of $\mathbb{B}_n$ that fixes a point in $\mathbb{B}_n$.

Definition 6.3.3. Operators S and T on a Banach space X are *similar* if there exists a bounded operator V with bounded inverse such that $S = V^{-1}TV$.

Lemma 6.3.4. [33] *If C_φ is similar to C_ϕ, then $\sigma(C_\varphi) = \sigma(C_\phi)$.*

Lemma 6.3.5. [6] *Let D be a bounded symmetric domain in standard form. For each $a \in D$, there exists an involution $\phi_a \in \mathrm{Aut}(D)$ such that $\phi_a(a) = 0$.*

Theorem 6.3.6. *Let $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ fix at least one point in $\mathbb{B}_n$ and induce a composition operator C_φ on $\mathcal{B}(\mathbb{B}_n)$. Then the spectrum of C_φ is the closure of all possible products of the eigenvalues of $J\varphi(a)$, where a is any interior fixed point. This closure is either the entire unit circle, or a finite subgroup of the circle.*

Proof. Let ϕ_a be as in Lemma 6.3.5. Then $\phi_a \circ \varphi \circ \phi_a$ is an automorphism of $\mathbb{B}_n$ that fixes the origin, and hence is unitary by Lemma 6.3.1. Furthermore, $\phi_a \circ \varphi \circ \phi_a$ is unitarily similar to a diagonal matrix. So there exists a unitary transformation V of $\mathbb{B}_n$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$ such that

$$U(z) = T(\varphi(T^{-1}(z))) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n),$$

where $z = (z_1, \ldots, z_n)$ and $T = V \circ \phi_a$.

Taking the Jacobian at 0, we arrive at the following matrix equality

$$JU(0) = JT(a)J\varphi(a)JT^{-1}(0) = JT(a)J\varphi(a)\left[JT(a)\right]^{-1}.$$

For all $z \in \mathbb{B}_n$,

$$JU(z) = \begin{bmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_n} \end{bmatrix}.$$

Thus the eigenvalues of $JU(0)$, and by similarity $J\varphi(a)$, are $e^{i\theta_1}, \ldots, e^{i\theta_n}$.

Consider the function

$$f(z_1, \ldots, z_n) = \prod_{j=1}^{n} z_j^{m_j},$$

where $m_j \geq 0$ are integers. Since f is a polynomial, it is a Bloch function. Also, we see that

$$(f \circ U)(z_1, \ldots, z_n) = \prod_{j=1}^{n} (e^{i\theta_j} z_j)^{m_j} = \left(\prod_{j=1}^{n} e^{im_j\theta_j} \right) f(z_1, \ldots, z_n).$$

Thus, $\lambda = \prod_{j=1}^{n} e^{im_j\theta_j}$ is an eigenvalue of C_U, and the spectrum of C_U contains the closure of all possible products of eigenvalues of $J\varphi(a)$. Since C_φ and C_U are similar, by Lemma 6.3.4, the spectra of C_φ and C_U are the same. We denote the closure of all possible products of eigenvalues of $J\varphi(a)$ by $\mathcal{E}$.

By Lemma 6.3.2, the spectrum of C_U, and thus C_φ, is contained in the unit circle. Suppose there exists $k \in \{1, \ldots, n\}$ such that $\operatorname{ord}(e^{i\theta_k}) = \infty$. Then the powers of $e^{i\theta_k}$ form a dense subset of $\partial\mathbb{D}$, and thus $\mathcal{E} = \partial\mathbb{D}$. On the other hand, if $\operatorname{ord}(e^{i\theta_k}) = \nu_k < \infty$ for all $k \in \{1, \ldots, n\}$, then $\mathcal{E}$ is a finite subgroup of $\partial\mathbb{D}$. Thus $\mathcal{E}$ consists of all the m^{th} roots of unity where $m = \operatorname{lcm}(\nu_1, \ldots, \nu_n)$. By letting $U^{(m)} = \underbrace{U \circ \cdots \circ U}_{m\text{-times}}$, we deduce that $U^{(m)}$ is the identity since

$$U^{(m)}(z_1, \ldots, z_n) = (e^{im\theta_1} z_1, \ldots, e^{im\theta_n} z_n) = (z_1, \ldots, z_n).$$

Suppose $\mu \in \partial\mathbb{D} \setminus \mathcal{E}$. We will show $C_\varphi - \mu I$ is invertible by proving that for every $g \in \mathcal{B}(\mathbb{B}_n)$, there exists a unique $f \in \mathcal{B}(\mathbb{B}_n)$ such that $C_\varphi f - \mu f = g$. Arguing as in the proof of Theorem 6.2.2, we may form a system of linear equations

$$\begin{array}{lcllcl} f(U(z)) & - & \mu f(z) & = & g(z) \\ f(U^{(2)}(z)) & - & \mu f(U(z)) & = & g(U(z)) \\ & \vdots & & \vdots & \\ f(z) & - & \mu f(U^{(m-1)}(z)) & = & g(U^{(m-1)}(z)). \end{array}$$

The determinant of the coefficient matrix is $(-1)^m(\mu^m - 1) \neq 0$. Thus $C_U - \mu I$ is invertible, so $\mu \notin \sigma(C_\varphi)$, and thus $\sigma(C_\varphi) \subseteq \mathcal{E}$. Therefore $\sigma(C_\varphi) = \mathcal{E}$. □

Corollary 6.3.7. *Let $\varphi \in \operatorname{Aut}(\mathbb{B}_n)$ which induces an isometric composition operator on $\mathcal{B}(\mathbb{B}_n)$. Then the spectrum of C_φ is the closure of all possible products of the eigenvalues of $J\varphi(0)$, which is either the entire unit circle, or a finite subgroup of the circle.*

6.4 The Unit Polydisk

In the case of the Bloch space of the polydisk, we consider the spectrum of a large class of isometric composition operators. The spectrum we determine on the polydisk reduces to Theorem 6.2.2 in the case of the isometric composition operators on $\mathcal{B}(\mathbb{D})$.

Theorem 6.4.1. *If C_φ is an isometry on $\mathcal{B}(\mathbb{D}^n)$ with φ a surjective function, but not an automorphism, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*

Proof. By Theorem 3.4.1, it suffices to show that $0 \in \sigma(C_\varphi)$, that is, C_φ is not invertible. Since C_φ is an isometry, it is necessarily injective. Thus, it suffices to show C_φ is not surjective. Arguing by contradiction, assume C_φ is surjective. Since φ is onto but not an automorphism, φ is not injective. Thus, there exist $\zeta = (\zeta_1, \dots, \zeta_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ in $\mathbb{D}^n$ such that $\zeta \neq \eta$ but $\varphi(\zeta) = \varphi(\eta)$. In particular, there exists $k \in \{1, \dots, n\}$ such that $\zeta_k \neq \eta_k$.

Define $p_k : \mathbb{D}^n \to \mathbb{D}$ to be the projection map onto the k^{th} coordinate of $\mathbb{D}^n$, that is $p_k(z_1, \dots, z_n) = z_k$. Since $p_k(0) = 0$, $\frac{\partial p_k}{\partial z_j}(z) = 0$ for $j \neq k$, $\frac{\partial p_k}{\partial z_k}(z) = 1$, by Theorem 5.1.3, p_k is Bloch with

$$||p_k||_{\mathcal{B}} = \beta_{p_k} = \sup_{z \in \mathbb{D}^n} (1 - |z_k|^2) \left| \frac{\partial p_k}{\partial z_k}(z) \right| = 1.$$

The function defined by $g(z) = p_k(z) - \eta_k$ is, therefore, Bloch. Note that $g(\zeta) = p_k(\zeta) - \eta_k = \zeta_k - \eta_k \neq 0$. Since C_φ is surjective, there exists a Bloch function f such that $f \circ \varphi = g$. In particular, $f(\varphi(\zeta)) = g(\zeta) \neq 0$. On the other hand, $f(\varphi(\zeta)) = f(\varphi(\eta)) = g(\eta) = 0$, a contradiction. Thus C_φ is not surjective. Therefore, $\sigma(C_\varphi) = \overline{\mathbb{D}}$. □

Next, we consider the isometric composition operators induced by automorphisms which act as rotations in each coordinate.

Theorem 6.4.2. *Let* $\varphi(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$ *for* $\lambda_j \in \partial\mathbb{D}$, $j = 1, \ldots, n$. *Then*

$$\sigma(C_\varphi) = \begin{cases} \partial\mathbb{D} & \textit{if } \operatorname{ord}(\lambda_j) = \infty \textit{ for some } j \in \{1, \ldots, n\} \\ G & \textit{if } \operatorname{ord}(\lambda_j) < \infty \textit{ for all } j \in \{1, \ldots, n\}, \end{cases}$$

where G *is the finite cyclic group generated by* $\{\lambda_1, \ldots, \lambda_n\}$.

Proof. Since φ is invertible with $\varphi^{-1}(z) = (\lambda_1^{-1} z_1, \ldots, \lambda_n^{-1} z_n)$ a holomorphic self-map of $\mathbb{D}^n$, C_φ is invertible with $C_\varphi^{-1} = C_{\varphi^{-1}}$ bounded on $\mathcal{B}(\mathbb{D}^n)$. So by Theorem 3.4.1, $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$.

Observe that the group G generated by $\{\lambda_1, \ldots, \lambda_n\}$ is

$$\left\{\lambda_1^{k_1} \cdots \lambda_n^{k_n} : k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}\right\} \subseteq \partial\mathbb{D}.$$

For $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$, the function $f(z) = z_1^{k_1} \cdots z_n^{k_n}$ is Bloch and

$$(C_\varphi f)(z) = f(\lambda_1 z_1, \ldots, \lambda_n z_n) = \lambda_1^{k_1} \ldots \lambda_n^{k_n} (z_1^{k_1} \ldots z_n^{k_n}) = \lambda_1^{k_1} \ldots \lambda_n^{k_n} f(z).$$

Thus, $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$ is an eigenvalue of C_φ corresponding to the eigenfunction f, and so $\overline{G} \subseteq \sigma(C_\varphi)$.

If there exists $j \in \{1, \ldots, n\}$ such that $\operatorname{ord}(\lambda_j) = \infty$, then G is a dense subset of $\partial\mathbb{D}$. Thus $\partial\mathbb{D} = \overline{G} \subseteq \sigma(C_\varphi)$, and hence $\sigma(C_\varphi) = \partial\mathbb{D}$. If $\operatorname{ord}(\lambda_j) < \infty$ for all $j \in \{1, \ldots, n\}$, then G is a finite cyclic group of order equal to the least common multiple of $\operatorname{ord}(\lambda_1), \ldots, \operatorname{ord}(\lambda_n)$. We wish to show that $\sigma(C_\varphi) \subseteq G$. If we let $\mu \in \partial\mathbb{D} \backslash G$, then the argument used in Theorems 6.2.2 and 6.3.6 follows through to show $\sigma(C_\varphi) = G$. □

We now determine the spectrum of the composition operator induced by an automorphism that permutes the coordinates.

Theorem 6.4.3. *Let $\varphi(z) = (z_{\tau(1)}, \dots, z_{\tau(n)})$ where $\tau \in S_n$ is decomposed into a product of disjoint cycles $c_1, \dots, c_\ell$ of order $\alpha_1, \dots, \alpha_\ell$, respectively.*

(a) *If λ is an α_j^{th} root of unity for some $j \in \{1, \dots, \ell\}$, then λ is an eigenvalue of C_φ.*

(b) *The spectrum of C_φ is the group of the α^{th} roots of unity, where α is the order of τ.*

Proof. To prove (a), assume λ is an α_j^{th} root of unit for some $j \in \{1, \dots, \ell\}$. To prove that λ is an eigenvalue, we need to show there exists a non-zero function $f \in \mathcal{B}(\mathbb{D}^n)$ such that

$$f(\varphi(z)) = \lambda(f(z)), \tag{6.2}$$

for all $z \in \mathbb{D}^n$. We are going to show that there exists a linear function f satisfying (6.2). Let

$$f(z) = \sum_{k=1}^{n} x_k z_k,$$

where the coefficients x_k are to be determined.

Equation (6.2) can be thought of as a matrix equation $Bx = 0$, where x is the column vector $x = (x_1, \dots, x_n)^T$, and B is a matrix with all diagonal entries $-\lambda$ and whose rows and columns contain an entry 1 and all other off-diagonal entries 0. The rows and columns of B can be permuted to yield the block diagonal matrix

$$A = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & & \ddots & \vdots \\ O & O & \cdots & A_\ell \end{bmatrix}$$

where each matrix A_k has order $\alpha_k \times \alpha_k$ and contains all diagonal entries 1, and each row contains one entry of $-\lambda$ and all other off-diagonal entries 0. The matrix A_k corresponds to the cycle c_k. Since λ is an α_j^{th} root of unity, $\det(A_j) = 1 - \lambda^{\alpha_j} = 0$. Thus $\det(B) = \det(A) = \prod_{k=1}^{\ell} \det(A_k) = 0$, and so $Bx = 0$ has non-trivial solutions. Therefore, λ is an eigenvalue.

To prove (b), assume μ is not an α^{th} root of unity. We will show that $C_\varphi - \mu I$ is invertible, thus $\mu \notin \sigma(C_\varphi)$. Arguing as in the proofs of Theorems 6.2.2 and 6.3.6, given a function $g \in \mathcal{B}(\mathbb{D}^n)$, there exists a unique solution f to the system of linear equations

$$\begin{array}{lcll} f(\varphi(z)) & - & \mu f(z) & = & g(z) \\ f(\varphi^{(2)}(z)) & - & \mu f(\varphi(z)) & = & g(\varphi(z)) \\ & \vdots & & \vdots & \\ f(z) & - & \mu f(\varphi^{(\alpha-1)}(z)) & = & g(\varphi^{(\alpha-1)}(z)) \end{array} \tag{6.3}$$

whose coefficient matrix has non-zero determinant. Thus $C_\varphi - \mu I$ is invertible.

On the other hand, suppose μ is an α^{th} root of unity. By forming the augmented matrix from system (6.3) and reducing the last row, we obtain

$$\left[\begin{array}{cccccc|c} -\mu & 1 & 0 & 0 & \cdots & 0 & g(z) \\ 0 & -\mu & 1 & 0 & \cdots & 0 & g(\varphi(z)) \\ \vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & \ddots & 1 & g(\varphi^{\alpha-2}(z)) \\ 0 & 0 & \cdots & \cdots & 0 & -\mu^\alpha + 1 & h_g(z) \end{array}\right]$$

where

$$h_g(z) = g(\varphi^{\alpha-2}(z)) + \mu g(\varphi^{\alpha-3}(z)) + \mu^2 g(\varphi^{\alpha-4}(z)) + \dots$$
$$\dots + \mu^{\alpha-3} g(\varphi(z)) + \mu^{\alpha-2} g(z) + \mu^{\alpha-1} g(\varphi^{\alpha-1}(z)).$$

Since μ is an α^{th} root of unity, for system (6.3) to have a solution, it must be the case that $h_g(z) = 0$ for all $z \in \mathbb{D}^n$. If $j_1, \ldots, j_k$ are any indices in the cycles $c_1, \ldots, c_j$, respectively, and define $g(z) = z_{j_1} \cdots z_{j_k}$. Then h_g is a linear combination of distinct monomials in k variables with non-zero coefficients. Therefore, h_g is not identically zero. Thus $C_\varphi - \mu I$ is not invertible, and so $\mu \in \sigma(C_\varphi)$. □

By Example 2.3.9, the automorphisms of the unit polydisk that fix the origin are of the form $\varphi(z) = (\lambda_1 z_{\tau(1)}, \ldots, \lambda_n z_{\tau(n)})$, where $\lambda_j \in \partial\mathbb{D}$ for all $j \in \{1, \ldots, n\}$ and $\tau \in S_n$. The arguments used in the proofs of Theorems 6.4.2 and 6.4.3 carry over to the general case, where $\alpha = \operatorname{lcm}(\operatorname{ord}(\tau), \operatorname{ord}(\lambda_1), \ldots, \operatorname{ord}(\lambda_n))$.

Theorem 6.4.4. *Let φ be the symbol of an isometric composition operator on $\mathcal{B}(\mathbb{D}^n)$.*

(a) *If $\varphi \notin \operatorname{Aut}(\mathbb{D}^n)$ and φ is onto, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*

(b) *If $\varphi \in \operatorname{Aut}(\mathbb{D}^n)$, let $\lambda_1, \ldots, \lambda_n \in \partial\mathbb{D}$ and $\tau \in S_n$ be such that $\varphi(z) = (\lambda_1 z_{\tau(1)}, \ldots, \lambda_n z_{\tau(n)})$.*

  (i) *If $\operatorname{ord}(\lambda_j) = \infty$ for some $j \in \{1, \ldots, n\}$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*

  (ii) *If $\operatorname{ord}(\lambda_j) < \infty$ for all $j \in \{1, \ldots, n\}$, then $\sigma(C_\varphi)$ is the cyclic group G generated by $\lambda_1, \ldots, \lambda_n$ and the m^{th} roots of unity, where $m = \operatorname{ord}(\tau)$. Furthermore, each element of the group G generated by $\lambda_1, \ldots, \lambda_n$ is an eigenvalue.*

Remark 6.4.5. By Theorem 6.1.2(c), the symbols which induce isometric composition operators on $\mathcal{B}(\mathbb{D})$ are onto. If we apply Theorem 6.4.4 to the case of $\mathcal{B}(\mathbb{D})$, then we arrive at Theorem 6.2.2.

Remark 6.4.6. It is unknown whether the symbols of an isometric composition operator on $\mathcal{B}(\mathbb{D}^n)$ are necessarily onto for $n \geq 2$. If they are onto, then Theorem 6.4.4 provides a complete description of the spectriom of the isometric composition operators.

Chapter 7: Weighted Composition Operators on the Bloch Space of the Unit Disk

Ohno and Zhao characterized the bounded and the compact weighted composition operators on the Bloch space and little Bloch space of the unit disk [72] in terms of the following quantities: $s_{\psi,\varphi} = \sup_{z\in\mathbb{D}} s_{\psi,\varphi}(z)$ and $\tau_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \tau_{\psi,\varphi}(z)$ where

$$s_{\psi,\varphi}(z) = \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log\frac{2}{1-|\varphi(z)|^2},$$

$$\tau_{\psi,\varphi}(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\varphi'(z)\right| |\psi(z)|.$$

Theorem 7.0.1. [72] *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

(a) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $s_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} s_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

(b) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}_0(\mathbb{D})$ if and only if $\psi \in \mathcal{B}_0(\mathbb{D})$, $s_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and*

$$\lim_{|z|\to 0}(1-|z|^2)\,|\psi(z)|\left|\varphi'(z)\right| = 0.$$

Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact if and only if

$$\lim_{|z|\to 1} s_{\psi,\varphi}(z) = \lim_{|z|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

In this chapter, we extend the results of Ohno and Zhao to include operator norm estimates, an alternative characterization of boundedness and compactness, and the determination of the spectrum for a particular class of isometric weighted composition operators on the Bloch space of the polydisk. The underlying goal of this chapter is to define quantities which can be generalized to bounded homogeneous domains in $\mathbb{C}^n$.

7.1 Operator Norm Estimates

Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Define $\sigma_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \sigma_{\psi,\varphi}(z)$ where

$$\sigma_{\psi,\varphi}(z) = \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|}.$$

Since $\dfrac{1}{2}\log\dfrac{1+|\varphi(z)|}{1-|\varphi(z)|} \le \log\dfrac{2}{1-|\varphi(z)|^2}$ for all $z \in \mathbb{D}$, we have $\sigma_{\psi,\varphi} \le s_{\psi,\varphi}$.

Theorem 7.1.1. *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ which induce a bounded weighted composition operator on $\mathcal{B}(\mathbb{D})$. Then*

(a) $||W_{\psi,\varphi}|| \ge \max\left\{||\psi||_{\mathcal{B}}\,, \frac{1}{2}\,|\psi(0)| \log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}$.

(b) $||W_{\psi,\varphi}|| \le \max\left\{||\psi||_{\mathcal{B}}\,, \frac{1}{2}\,|\psi(0)| \log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\right\}$.

Proof. To prove (a), taking the constant function 1, we have

$$||W_{\psi,\varphi}|| \ge ||W_{\psi,\varphi}1||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}\,.$$

If $\varphi(0) = 0$, then the inequality (a) holds trivially. If $\varphi(0) \ne 0$, then write $\varphi(0) = |\varphi(0)|\, e^{i\theta}$, for $\theta \in \mathbb{R}$. Define

$$f(z) = \frac{1}{2}\mathrm{Log}\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z},$$

where Log denotes the principal branch of the logarithm. Since $f(0) = 0$ and

$$\beta_f = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right| = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{|1-e^{-2i\theta}z^2|} = 1,$$

the function f is Bloch with $||f||_{\mathcal{B}} = 1$. Thus

$$||W_{\psi,\varphi}|| \geq ||W_{\psi,\varphi}f||_{\mathcal{B}} \geq |\psi(0)|\,|f(\varphi(0))| = \frac{1}{2}\,|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

Therefore

$$||W_{\psi,\varphi}|| \geq \max\left\{||\psi||_{\mathcal{B}}\,, \frac{1}{2}\,|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}.$$

We now prove (b). Let $f \in \mathcal{B}(\mathbb{D})$ such that $||f||_{\mathcal{B}} = 1$. Then

$$\begin{aligned}
||W_{\psi,\varphi}f||_{\mathcal{B}} &= |\psi(0)|\,|f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi(z)f'(\varphi(z))\varphi'(z) + \psi'(z)f(\varphi(z))\right| \\
&\leq |\psi(0)|\,|f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\psi(z)|\left|f'(\varphi(z))\right|\left|\varphi'(z)\right| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\,|f(\varphi(z))| \\
&= |\psi(0)|\,|f(\varphi(0))| + \sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}\,|\psi(z)|\left|\varphi'(z)\right|(1-|\varphi(z)|^2)\left|f'(\varphi(z))\right| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\,|f(\varphi(z))| \\
&\leq |\psi(0)|\,|f(\varphi(0))| + \tau_{\psi,\varphi}\beta_f + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi'(z)\right|\,|f(\varphi(z))|\,.
\end{aligned}$$

By applying Corollary 5.1.6 and (4.3), we obtain

$$
\begin{aligned}
||W_{\psi,\varphi}f||_{\mathcal{B}} &\leq |\psi(0)| \left(|f(0)| + \frac{1}{2}\beta_f \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right) + \tau_{\psi,\varphi}\beta_f \\
&\quad + \sup_{z\in\mathbb{D}} (1-|z|^2) \left|\psi'(z)\right| \left(|f(0)| + \frac{1}{2}\beta_f \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|} \right) \\
&\leq |\psi(0)| \left(|f(0)| + \frac{1}{2}\beta_f \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right) + \tau_{\psi,\varphi}\beta_f + |f(0)|\,\beta_\psi + \sigma_{\psi,\varphi}\beta_f \\
&\leq |\psi(0)| \left(|f(0)| + \frac{1}{2}\beta_f \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right) + |f(0)|\,\beta_\psi \\
&\quad + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\beta_f \\
&= ||\psi||_{\mathcal{B}}\,|f(0)| + \left(\frac{1}{2}\,|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} \right) \beta_f.
\end{aligned}
$$

Since $|f(0)| = 1 - \beta_f$, we deduce

$$
\begin{aligned}
||W_{\psi,\varphi}f||_{\mathcal{B}} &\leq ||\psi||_{\mathcal{B}}\,(1-\beta_f) + \left(\frac{1}{2}\,|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} \right) \beta_f \\
&= ||\psi||_{\mathcal{B}} + \left(\frac{1}{2}\,|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} - ||\psi||_{\mathcal{B}} \right) \beta_f.
\end{aligned}
$$

If $\frac{1}{2}\,|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} \leq ||\psi||_{\mathcal{B}}$, then

$$
||W_{\psi,\varphi}f||_{\mathcal{B}} \leq ||\psi||_{\mathcal{B}}\,.
$$

On the other hand, if $\frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}+\tau_{\psi,\varphi}+\sigma_{\psi,\varphi}\geq ||\psi||_{\mathcal{B}}$, then

$$||W_{\psi,\varphi}f||_{\mathcal{B}} \leq ||\psi||_{\mathcal{B}} + \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} - ||\psi||_{\mathcal{B}}$$

$$= \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}.$$

Recalling that $||W_{\psi,\varphi}|| = \sup\limits_{||f||_{\mathcal{B}}=1} ||W_{\psi,\varphi}f||_{\mathcal{B}}$, we obtain

$$||W_{\psi,\varphi}|| \leq \max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\right\}. \quad \square$$

Remark 7.1.2. We now consider the application of the operator norm estimates to the degenerate weighted composition operators, namely the multiplication and composition operators on $\mathcal{B}(\mathbb{D})$. If ψ is the constant function 1, then $W_{\psi,\varphi} = C_\varphi$, $||\psi||_{\mathcal{B}} = 1$, $\sigma_{\psi,\varphi} = 0$, and

$$\tau_{\psi,\varphi} = \tau_\varphi = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\varphi'(z)\right|.$$

Thus, the norm estimates become

$$\max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \leq ||C_\varphi|| \leq \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi\right\}.$$

These norm estimates match those obtained by Xiong in [91].

If φ is the identity map of $\mathbb{D}$, then $W_{\psi,\varphi} = M_\psi$, $\sigma_{\psi,\varphi} = \sigma_\psi$ (see Section 4.1, equation (4.1)), and $\tau_{\psi,\varphi} = ||\psi||_\infty$. So

$$||\psi||_{\mathcal{B}} \leq ||M_\psi|| \leq \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\}.$$

These estimates were observed in Theorem 4.1.2; however, the operator norm in Theorem 4.1.2 has a sharper lower bound.

7.2 Boundedness and Compactness

At a cursory glance, the need for $\sigma_{\psi,\varphi}$ may not be apparent. The ultimate goal is to extend the characterizations of the bounded and the compact weighted composition operators to the Bloch space of a bounded homogeneous domain. Thus, we need to define terms which are amenable to the geometry induced by the Bergman metric of the domain. Recall from (4.3) the Bergman distance between the points $\varphi(z)$ and 0 is given by

$$\rho(\varphi(z), 0) = \frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}.$$

The goal of this section is to show that, indeed, the quantities $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ can be used to characterize the bounded and the compact weighted composition operators on the Bloch space and little Bloch space of $\mathbb{D}$.

Theorem 7.2.1. *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

(a) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $\psi \in \mathcal{B}(\mathbb{D})$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \sigma_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

(b) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}_0(\mathbb{D})$ if and only if $\psi \in \mathcal{B}_0(\mathbb{D})$, $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and*

$$\lim_{|z|\to 1} (1-|z|^2)\,|\psi(z)|\,\left|\varphi'(z)\right| = 0.$$

Furthermore, the bounded operator $W_{\psi,\varphi}$ *is compact if and only if*

$$\lim_{|z|\to 1} \sigma_{\psi,\varphi}(z) = \lim_{|z|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

Proof. We will prove part (a). The proof of part (b) is analogous. Assume $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$. Considering the constant function 1, we see that $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(\mathbb{D})$. Since $\sigma_{\psi,\varphi} \le s_{\psi,\varphi}$, by Theorem 7.0.1(a), $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Conversely, assume $\psi \in \mathcal{B}(\mathbb{D})$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. By Theorem 7.1.1, $W_{\psi,\varphi}$ is bounded.

Assume $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D})$. Since $\sigma_{\psi,\varphi}(z) \le s_{\psi,\varphi}(z)$ for all $z \in \mathbb{D}$, by Theorem 7.0.1(b) we obtain

$$\lim_{|\varphi(z)|\to 1} \sigma_{\psi,\varphi}(z) \le \lim_{|\varphi(z)|\to 1} s_{\phi,\varphi}(z) = 0.$$

Conversely, suppose $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ and

$$\lim_{|\varphi(z)|\to 1} \sigma_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

Since $s_{\psi,\varphi}(z) \le 2\sigma_{\psi,\varphi}(z)$ for $|\varphi(z)| \ge \frac{1}{2}$, we have

$$\lim_{|\varphi(z)|\to 1} s_{\psi,\varphi}(z) \le 2 \lim_{|\varphi(z)|\to 1} \sigma_{\psi,\varphi}(z) = 0.$$

By Theorem 7.0.1(b), $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D})$. □

7.3 Spectrum

In this section, we determine the spectrum of the isometric weighted composition operator $W_{\psi,\varphi}$ for which the associated operators M_ψ and C_φ are isometries.

Theorem 7.3.1. *Let* $\psi \in H(\mathbb{D})$ *induce an isometric multiplication operator and* φ *be an analytic self-map of* $\mathbb{D}$ *which induces an isometric composition operator on* $\mathcal{B}(\mathbb{D})$*. If* φ *is* *not a rotation, then* $\sigma(W_{\psi,\varphi}) = \overline{\mathbb{D}}$. *If* $\varphi(z) = \zeta z$ *for* $z \in \mathbb{D}$ *and* $\zeta \in \partial\mathbb{D}$, *then*

$$\sigma(W_{\psi,\varphi}) = \begin{cases} \partial\mathbb{D} & \textit{if } \operatorname{ord}(\zeta) = \infty \\ \psi(0)\langle\zeta\rangle & \textit{if } \operatorname{ord}(\zeta) < \infty, \end{cases}$$

where $\psi(0)\langle\zeta\rangle = \{\psi(0)\zeta^k : k = 1, \ldots, n\}$.

Proof. Since ψ induces an isometric multiplication operator on $\mathcal{B}(\mathbb{D})$, then by Theorem 4.3.6, ψ is a constant function of modulus one. Observe that $W_{\psi,\varphi} = \psi(0)C_\varphi$ and for $\lambda \in \mathbb{C}$,

$$W_{\psi,\varphi} - \lambda I = \psi(0)C_\varphi - \lambda I = \psi(0)(C_\varphi - \lambda\overline{\psi(0)}I),$$

where I denotes the identity operator on $\mathcal{B}(\mathbb{D})$. Thus, $W_{\psi,\varphi} - \lambda I$ is not invertible if and only if $C_\varphi - \lambda\overline{\psi(0)}I$ is not invertible. Thus, $\lambda \in \sigma(W_{\psi,\varphi})$ if and only if $\lambda\overline{\psi(0)} \in \sigma(C_\varphi)$. The result follows immediately from Theorem 6.2.2. $\square$

Chapter 8: Weighted Composition Operators from the Bloch Space to H^∞ of a Bounded Homogeneous Domain

Before we investigate the weighted composition operators mapping the Bloch space of a bounded homogeneous domain D into itself, we will study a related problem. In this chapter, we will study the weighted composition operators which map the Bloch space into the Hardy space H^∞ of a bounded homogeneous domain. In [53], Hosokawa, Izuchi, and Ohno characterized the bounded and the compact weighted composition operators from $\mathcal{B}(\mathbb{D})$ and $\mathcal{B}_0(\mathbb{D})$ into $H^\infty(\mathbb{D})$.

Theorem 8.0.1. [53] *Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is bounded.*

(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is bounded.*

(c) $\psi \in H^\infty(\mathbb{D})$ *and* $\sup_{z\in\mathbb{D}} |\psi(z)| \log \dfrac{1}{1-|\varphi(z)|} < \infty.$

Theorem 8.0.2. [53] *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is compact.*

(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is compact.*

(c) $\psi \in H^\infty(\mathbb{D})$ *and* $\lim_{\varphi(z)\to\partial\mathbb{D}} |\psi(z)| \log \dfrac{1}{1-|\varphi(z)|} = 0.$

8.1 Boundedness

In this section, we characterize the bounded weighted composition operators from $\mathcal{B}(D)$ and $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$ for a bounded homogeneous domain D. In the instances where D is the unit ball or the unit polydisk, we extend Theorem 8.0.1.

Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. Define

$$\eta_{\psi,\varphi} = \sup_{z\in D} |\psi(z)|\, \omega(\varphi(z)),$$

$$\eta_{0,\psi,\varphi} = \sup_{z\in D} |\psi(z)|\, \omega_0(\varphi(z)).$$

Lemma 8.1.1. *Let D be a bounded homogeneous domain, $\psi \in H(D)$, and φ a holomorphic self-map of D. If $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is bounded, then*

$$\eta_{0,\psi,\varphi} \leq \eta_{\psi,\varphi} \leq ||W_{\psi,\varphi}||\,.$$

Proof. Since $\eta_{0,\psi,\varphi} \leq \eta_{\psi,\varphi}$, it suffices to show that $\eta_{\psi,\varphi} \leq ||W_{\psi,\varphi}||$. Let $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}} \leq 1$. For every $z \in D$,

$$\begin{aligned} ||W_{\psi,\varphi}|| &= \sup\{||\psi(g\circ\varphi)||_\infty : g \in \mathcal{B}(D), ||g||_{\mathcal{B}} \leq 1\} \\ &\geq ||\psi(f\circ\varphi)||_\infty \\ &\geq |\psi(z)|\,|f(\varphi(z))|\,. \end{aligned}$$

Taking the supremum over all such $f \in \mathcal{B}(D)$ such that $f(0) = 0$, we obtain

$$|\psi(z)|\,\omega(\varphi(z)) \leq ||W_{\psi,\varphi}||\,.$$

Finally, taking the supremum over all $z \in D$, we have

$$\eta_{\psi,\varphi} \le ||W_{\psi,\varphi}|| .$$

$\square$

Theorem 8.1.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is bounded if and only if $\psi \in H^\infty(D)$ and $\eta_{\psi,\varphi} < \infty$.*

Proof. First assume $W_{\psi,\varphi}$ is bounded. Then $\psi = W_{\psi,\varphi}1 \in H^\infty(D)$, and by Lemma 8.1.1 $\eta_{\psi,\varphi}$ is finite.

Next, assume $\psi \in H^\infty(D)$ and $\eta_{\psi,\varphi} < \infty$. By Lemma 5.1.1(a), for $f \in \mathcal{B}(D)$ and $z \in D$,

$$|f(\varphi(z))| \le |f(0)| + \omega(\varphi(z))\beta_f.$$

From this, we deduce

$$\begin{aligned} ||W_{\psi,\varphi}f||_\infty &= \sup_{z\in D} |\psi(z)|\,|f(\varphi(z))| \\ &\le \sup_{z\in D} \left(|\psi(z)|\,|f(0)| + |\psi(z)|\,\omega(\varphi(z))\beta_f\right) \\ &\le \sup_{z\in D} \left(|\psi(z)| + |\psi(z)|\,\omega(\varphi(z))\right) ||f||_{\mathcal{B}} \\ &\le \left(||\psi||_\infty + \eta_{\psi,\varphi}\right) ||f||_{\mathcal{B}} . \end{aligned}$$

Thus $W_{\psi,\varphi}$ is a bounded operator mapping $\mathcal{B}(D)$ into $H^\infty(D)$. Furthermore,

$$||W_{\psi,\varphi}|| \le ||\psi||_\infty + \eta_{\psi,\varphi}.$$

$\square$

The proof of the following result is analogous.

Theorem 8.1.3. *Let D be a bounded homogeneous domain, $\psi \in H(D)$, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi} : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ if and only if $\psi \in H^\infty(D)$ and $\eta_{0,\psi,\varphi} < \infty$.*

8.1.1 The Unit Ball

In [60] Li and Stević studied the weighted composition operators from the α-Bloch space into H^∞ of the unit ball. The α-Bloch space on the unit ball, for $\alpha \in (0, \infty)$, is defined as the space of functions $f \in H(\mathbb{B}_n)$ such that

$$b_\alpha(f) = \sup_{z \in \mathbb{B}_n} (1 - ||z||^2)^\alpha \, |\langle \nabla(f)(z), \overline{z} \rangle| < \infty$$

under the norm

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f).$$

The Bloch space coincides with the α-Bloch space with $\alpha = 1$ by Theorem 2.3.18.

Theorem 8.1.4. [60] *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then the following are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(c) $\psi \in H^\infty(\mathbb{B}_n)$ *and* $\displaystyle\sup_{z \in \mathbb{B}_n} |\psi(z)| \log \frac{2}{1 - ||\varphi(z)||^2} < \infty.$

Recall from (5.4) that if φ is a holomorphic self-map of $\mathbb{B}_n$ and $z \in \mathbb{B}_n$, then

$$\omega_0(\varphi(z)) = \omega(\varphi(z)) = \frac{1}{2} \log \frac{1 + ||\varphi(z)||}{1 - ||\varphi(z)||}.$$

Thus, for the unit ball $\eta_{0,\psi,\varphi} = \eta_{\psi,\varphi}$ for all $\psi \in H(D)$. Thus we deduce the following characterization of the bounded weighted composition operators from $\mathcal{B}(\mathbb{B}_n)$ into $H^\infty(\mathbb{B}_n)$. This equivalence extends Theorem 8.0.1 to the unit ball.

Corollary 8.1.5. *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then the following are equivalent.*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(c) $\psi \in H^\infty(\mathbb{B}_n)$ *and* $\sup_{z\in\mathbb{B}_n} |\psi(z)| \log \dfrac{1 + ||\varphi(z)||}{1 - ||\varphi(z)||} < \infty.$

The benefit of this characterization over that of Theorem 8.1.4, is that this is formulated in terms of the Bergman metric and lends itself to a generalization to bounded homogeneous domains.

8.1.2 The Unit Polydisk

We now extend Theorem 8.0.1 to the unit polydisk.

Lemma 8.1.6. *Let φ be a holomorphic self-map of $\mathbb{D}^n$, $\lambda \in \mathbb{D}^n$, and fix $j \in \{1, \ldots, n\}$. Then the function defined by*

$$f(z) = \mathrm{Log}\frac{4}{1 - z_j\overline{\overline{\varphi_j(\lambda)}}}$$

is in the $*$-little Bloch space and $||f||_{\mathcal{B}} \leq 2 \log 2 + 2$.

Proof. Since $\frac{\partial f}{\partial z_k}(z) = 0$ for $k \neq j$ and $\frac{\partial f}{\partial z_j}(z) = \frac{\overline{\varphi_j(\lambda)}}{1 - z_j\overline{\varphi_j(\lambda)}}$, Theorem 5.1.3 yields

$$\begin{aligned}
\beta_f &= \sup_{z \in \mathbb{D}^n} Q_f(z) \\
&= \sup_{z \in \mathbb{D}^n} \left|\left| \left((1-|z_1|^2)\frac{\partial f}{\partial z_1}, \dots, (1-|z_n|^2)\frac{\partial f}{\partial z_n} \right) \right|\right| \\
&= \sup_{z_j \in \mathbb{D}} \frac{(1-|z_j|^2)\,|\varphi_j(\lambda)|}{\left|1 - z_j\overline{\varphi_j(\lambda)}\right|} \\
&\leq \sup_{z_j \in \mathbb{D}} \frac{(1-|z_j|^2)\,|\varphi_j(\lambda)|}{1-|z_j|} \\
&= \sup_{z_j \in \mathbb{D}} (1+|z_j|)\,|\varphi_j(\lambda)| \\
&\leq 2.
\end{aligned}$$

Thus $f \in \mathcal{B}(\mathbb{D}^n)$ and $||f||_{\mathcal{B}} \leq 2\log 2 + 2$. Furthermore,

$$\lim_{z \to \partial^* \mathbb{D}^n} Q_f(z) = \lim_{z_j \to \partial \mathbb{D}} \frac{(1-|z_j|^2)\,|\varphi_j(\lambda)|}{\left|1 - z_j\overline{\varphi_j(\lambda)}\right|} \leq \lim_{z_j \to \partial \mathbb{D}} \frac{(1-|z_j|^2)\,|\varphi_j(\lambda)|}{1-|\varphi_j(\lambda)|} = 0.$$

Thus, $f \in \mathcal{B}_{0^*}(D)$. ☐

Theorem 8.1.7. *Let $\psi \in H(\mathbb{D}^n)$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then the following are equivalent.*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is bounded.*

(b) $W_{\psi,\varphi} : \mathcal{B}_{0^*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is bounded.*

(c) $\psi \in H^\infty(\mathbb{D}^n)$ *and* $\sup_{z \in \mathbb{D}^n} |\psi(z)| \sum_{j=1}^n \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} < \infty$.

Proof. If $W_{\psi,\varphi}$ is bounded from $\mathcal{B}(\mathbb{D}^n)$ to $H^\infty(\mathbb{D}^n)$, then there exists $M > 0$ such that, for each $f \in \mathcal{B}(\mathbb{D}^n)$, $||\psi(f \circ \varphi)||_\infty \leq M\,||f||_{\mathcal{B}}$. So this is true for each $f \in \mathcal{B}_{0^*}(\mathbb{D}^n)$, and thus $W_{\psi,\varphi}$ is bounded from $\mathcal{B}_{0^*}(\mathbb{D}^n)$ into $H^\infty(\mathbb{D}^n)$. Thus $(a) \Longrightarrow (b)$.

Suppose $W_{\psi,\varphi}$ is bounded from $\mathcal{B}_{0^*}(\mathbb{D}^n)$ into $H^\infty(\mathbb{D}^n)$. Then $\psi = W_{\psi,\varphi}1 \in H^\infty(\mathbb{D}^n)$. Fix $j \in \{1, \ldots, n\}$ and $\lambda \in \mathbb{D}^n$. Then by Lemma 8.1.6, the function defined by

$$h_j(z) = \operatorname{Log}\frac{4}{1 - z_j\overline{\varphi_j(\lambda)}}$$

is an element of $\mathcal{B}_{0^*}(\mathbb{D}^n)$. Define the function

$$f(z) = \sum_{j=1}^{n} h_j(z) = \sum_{j=1}^{n} \operatorname{Log}\frac{4}{1 - z_j\overline{\varphi_j(\lambda)}}.$$

Then $f \in \mathcal{B}_{0^*}(\mathbb{D}^n)$ and $||f||_{\mathcal{B}} \leq \sum_{j=1}^{n} ||h_j||_{\mathcal{B}} = 2n(\log 2 + 1)$. Since $W_{\psi,\varphi}$ is bounded, we have $||W_{\psi,\varphi}g||_\infty \leq ||W_{\psi,\varphi}||\,||g||_{\mathcal{B}}$ for all $g \in \mathcal{B}_{0^*}(\mathbb{D}^n)$. Thus

$$\begin{aligned} 2n(\log 2 + 1)\,||W_{\psi,\varphi}|| &\geq ||W_{\psi,\varphi}f||_\infty \\ &\geq |\psi(\lambda)|\,|f(\varphi(\lambda))| \\ &= |\psi(\lambda)| \sum_{j=1}^{n} \log\frac{4}{1 - |\varphi(\lambda)|^2} \\ &\geq |\psi(\lambda)| \sum_{j=1}^{n} \log\frac{1 + |\varphi_j(\lambda)|}{1 - |\varphi_j(\lambda)|}. \end{aligned}$$

By taking the supremum over all $\lambda \in \mathbb{D}^n$,

$$\sup_{\lambda \in \mathbb{D}^n} |\psi(\lambda)| \sum_{j=1}^{n} \log \frac{1 + |\varphi_j(\lambda)|}{1 - |\varphi_j(\lambda)|} \leq 2n(\log 2 + 1) \, ||W_{\psi,\varphi}|| < \infty,$$

completing the proof of $(b) \implies (c)$.

Finally, suppose $\psi \in H^\infty(\mathbb{D}^n)$ and $\sup_{z \in \mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} < \infty$. By Lemma 5.1.4(b), we have

$$\omega(\varphi(z)) \leq \rho(\varphi(z), 0) \leq \frac{1}{2} \sum_{j=1}^{n} \log \frac{1 + |\varphi_j(z)|}{1 - |\varphi_j(z)|},$$

and thus

$$\sup_{z \in \mathbb{D}^n} |\psi(z)| \, \omega(\varphi(z)) \leq \sup_{z \in \mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log \frac{1 + |\varphi_j(z)|}{1 - |\varphi_j(z)|},$$

which is finite by assumption. Thus, by Theorem 8.1.2, $W_{\psi,\varphi}$ is bounded from $\mathcal{B}(\mathbb{D}^n)$ to $H^\infty(\mathbb{D}^n)$, and so $(c) \implies (a)$. □

8.2 Operator Norm

In this section, we determine the norm of $W_{\psi,\varphi}$ acting from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ to $H^\infty(D)$ for D a bounded homogeneous domain.

Theorem 8.2.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. If $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is bounded, then*

$$||W_{\psi,\varphi}|| = \max\{||\psi||_\infty \, , \eta_{\psi,\varphi}\}.$$

Proof. From the boundedness of $W_{\psi,\varphi}$, we have $||\psi||_\infty = ||W_{\psi,\varphi} 1||_\infty \leq ||W_{\psi,\varphi}||$, and by

Lemma 8.1.1, $\eta_{\psi,\varphi} \leq ||W_{\psi,\varphi}||$. Thus

$$\max\{||\psi||_\infty, \eta_{\psi,\varphi}\} \leq ||W_{\psi,\varphi}||.$$

From Lemma 5.1.1(a), for $f \in \mathcal{B}(D)$ we have

$$\begin{aligned} ||W_{\psi,\varphi} f||_\infty &= \sup_{z\in D} |\psi(z)|\,|f(\varphi(z))| \\ &\leq \sup_{z\in D} |\psi(z)|\,(|f(0)| + \omega(\varphi(z))\beta_f) \\ &\leq ||\psi||_\infty\,|f(0)| + \eta_{\psi,\varphi}\beta_f \\ &\leq ||\psi||_\infty\,(||f||_{\mathcal{B}} - \beta_f) + \eta_{\psi,\varphi}\beta_f \\ &= ||\psi||_\infty\,||f||_{\mathcal{B}} + (\eta_{\psi,\varphi} - ||\psi||_\infty)\beta_f \quad \leq \max\{||\psi||_\infty, \eta_{\psi,\varphi}\}\,||f||_{\mathcal{B}}. \end{aligned}$$

Taking the supremum over all $f \in \mathcal{B}(D)$ such that $||f||_{\mathcal{B}} = 1$, we obtain $||W_{\psi,\varphi}|| \leq \max\{||\psi||_\infty, \eta_{\psi,\varphi}\}$. Therefore

$$||W_{\psi,\varphi}|| = \max\{||\psi||_\infty, \eta_{\psi,\varphi}\}. \quad \square$$

By using Lemma 5.1.1(b), we obtain the analogous norm for $W_{\psi,\varphi}$ between $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Theorem 8.2.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. If $W_{\psi,\varphi} : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is bounded, then*

$$||W_{\psi,\varphi}|| = \max\{||\psi||_\infty, \eta_{0,\psi,\varphi}\}.$$

As a corollary, we obtain the following result when the ambient space is taken to be the unit ball.

Corollary 8.2.3. *Let $\psi \in H(\mathbb{B}_n)$, and φ a holomorphic self-map of $\mathbb{B}_n$. If $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ is bounded, then*

$$||W_{\psi,\varphi}|| = \max\left\{||\psi||_\infty\, , \sup_{z\in\mathbb{B}_n} \frac{1}{2}\,|\psi(z)| \log \frac{1+||\varphi(z)||}{1-||\varphi(z)||}\right\}.$$

8.3 Compactness

In this section, we provide sufficient conditions for the weighted composition operator from $\mathcal{B}(D)$ to $H^\infty(D)$ to be compact when D is a bounded homogeneous domain.

Theorem 8.3.1. (Montel's Theorem) *Let D be a domain in $\mathbb{C}^n$ and $\{f_n\}$ a sequence of holomorphic functions on D. Then $\{f_n\}$ is locally bounded if and only if $\{f_n\}$ has a subsequence which converges locally uniformly in D to some function $f \in H(D)$.*

Theorem 8.3.2. [4] *Let $\{f_n\}$ be a sequence of Bloch functions on a bounded homogenous domain D which converges locally uniformly in D to some holomorphic function f. If the sequence $\{\beta_{f_n}\}$ is bounded, then f is Bloch and*

$$\beta_f \le \liminf_{n\to\infty} \beta_{f_n}.$$

That is, the function $f \mapsto \beta_f$ is lower semi-continuous on $\mathcal{B}(D)$ under the topology of uniform convergence on compact subsets.

Lemma 8.3.3. *Let D be a bounded homogeneous domain, $\psi \in H(D)$, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is compact if and only if for every bounded sequence $\{f_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D, $||\psi(f_k \circ \varphi)||_\infty \to 0$ as $k \to \infty$.*

Proof. Assume $W_{\psi,\varphi}$ is compact. Let $\{f_k\}$ be a bounded sequence in $\mathcal{B}(D)$ which converges to 0 locally uniformly in D. We need to show that $||\psi(f_k \circ \varphi)||_\infty \to 0$ as $k \to \infty$. Since $W_{\psi,\varphi}$ is compact, $\{\psi(f_k \circ \varphi)\}$ contains a subsequence which converges to some function

$f \in H^\infty(D)$. For $z \in D$, $\psi(z)(f_k(\varphi(z))) \to 0$ as $k \to \infty$, and hence f must be identically 0. Thus $||\psi(f_k \circ \varphi)||_\infty \to 0$ as $k \to \infty$.

Conversely, assume $||\psi(g_k \circ \varphi)||_\infty \to 0$ as $k \to \infty$ for each bounded sequence $\{g_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D. To prove the compactness of $W_{\psi,\varphi}$, it suffices to show that if $\{f_k\}$ is a sequence in $\mathcal{B}(D)$ with $||f_k||_\mathcal{B} \leq 1$ for all $k \in \mathbb{N}$, then there exists a subsequence $\{f_{k_j}\}$ such that $\psi(f_{k_j} \circ \varphi)$ converges in $H^\infty(D)$. Fix $z_0 \in D$, and without loss of generality assume $f_k(z_0) = 0$ (otherwise, replace f_k by $f_k - f_k(z_0)$). By (5.2), $|f_k(z)| \leq \rho(z, z_0)$, and thus $\{f_k\}$ is uniformly bounded on every closed disk centered at z_0 in the Bergman metric. Thus $\{f_k\}$ is uniformly bounded on every compact subset of D. By Theorem 8.3.1, there exists a subsequence $\{f_{k_j}\}$ which converges locally uniformly to some $f \in H(D)$. By Theorem 8.3.2, $f \in \mathcal{B}(D)$ with $||f||_\mathcal{B} \leq 1$. By defining $g_{k_j} = f_{k_j} - f$, we have a bounded sequence $\{g_{k_j}\}$ in $\mathcal{B}(D)$ with $\left|\left|g_{k_j}\right|\right|_\mathcal{B} \leq 2$ converging to 0 locally uniformly in D. Thus $\left|\left|\psi(g_{k_j} \circ \varphi)\right|\right|_\infty \to 0$ as $k \to \infty$. Therefore, $\psi(f_{k_j} \circ \varphi)$ converges to $\psi(f \circ \varphi)$ in $H^\infty(D)$. □

Theorem 8.3.4. *Let D be a bounded homogeneous domain, $\psi \in H(D)$, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is compact if $\psi \in H^\infty(D)$ and* $\lim_{\varphi(z) \to \partial D} |\psi(z)|\, \omega(\varphi(z)) = 0$.

Proof. Assume $\psi \in H^\infty(D)$ and $\lim_{\varphi(z)\to\partial D} |\psi(z)|\, \omega(\varphi(z)) = 0$. By Lemma 8.3.3, to prove that $W_{\psi,\varphi}$ is compact, it suffices to show that for any bounded sequence $\{f_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D, $||\psi(f_k \circ \varphi)||_\infty \to 0$ as $k \to \infty$. Let $\{f_k\}$ be such a sequence. Without loss of generality, we may assume $f_k(0) = 0$ for all $k \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there exists $r > 0$ such that $|\psi(z)|\, \omega(\varphi(z)) < \varepsilon$ whenever $\rho(\varphi(z), \partial D) \geq r$. Thus, if $\rho(\varphi(z), \partial D) \geq r$, then

$$|\psi(z)|\, |f_k(\varphi(z))| \leq |\psi(z)|\, \omega(\varphi(z)) < \varepsilon.$$

On the other hand, since $f_k \to 0$ locally uniformly on D, $|f_k(\varphi(z))| \to 0$ uniformly on the set $\{z \in D : \rho(\varphi(z), \partial D) \leq r\}$. So $|f_k(\varphi(z))| < \frac{\varepsilon}{||\psi||_\infty}$ whenever $\rho(\varphi(z), \partial D) \leq r$, and thus

$$|\psi(z)|\,|f_k(\varphi(z))| \leq ||\psi||_\infty\,|f_k(\varphi(z))| < \varepsilon.$$

So, for k large enough, $|\psi(z)|\,|f_k(\varphi(z))| < \varepsilon$ for all $z \in D$. Therefore, $||\psi(f_k \circ \varphi)||_\infty \to 0$, completing the proof. □

When the ambient space is the unit ball, Li and Stević [60] characterized the compact weighted composition operators from $\mathcal{B}(\mathbb{B}_n)$ or $\mathcal{B}_0(\mathbb{B}_n)$ into $H^\infty(\mathbb{B}_n)$. This is an extension of Theorem 8.0.1 to the unit ball.

Theorem 8.3.5. [60] *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then the following are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*

(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*

(c) $\psi \in H^\infty(\mathbb{B}_n)$ *and* $\displaystyle\lim_{||\varphi(z)||\to 1} |\psi(z)| \log \frac{2}{1 - ||\varphi(z)||^2} = 0.$

In the following theorem, we extend Theorem 8.0.1 to the unit polydisk. First, we collect the following lemma.

Lemma 8.3.6. *Fix $j \in \{1, \ldots, n\}$ and let $\lambda \in \mathbb{D}^n$. Then the function*

$$f(z) = \frac{\left(\operatorname{Log} \frac{4}{1 - z_j \overline{\varphi_j(\lambda)}}\right)^2}{\log \frac{4}{1 - |\varphi_j(\lambda)|^2}}$$

is Bloch with $||f||_{\mathcal{B}} \leq \log 4 + 4\left(1 + \frac{\pi}{2 \log 4}\right)$.

Proof. For $z \in \mathbb{D}^n$, $\frac{\partial f}{\partial z_k}(z) = 0$ for $k \neq j$ and

$$\frac{\partial f}{\partial z_j}(z) = \frac{2\left(\operatorname{Log}\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right)\frac{\overline{\varphi_j(\lambda)}}{1-z_j\overline{\varphi_j(\lambda)}}}{\log\frac{4}{1-|\varphi_j(\lambda)|}}. \tag{8.1}$$

Since $\operatorname{Re}\left(\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right) > 0$, we have $\left|\operatorname{Arg}\left(\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right)\right| < \frac{\pi}{2}$, where Arg denotes the principal value of the argument. Also $\left|\operatorname{Log}\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right|^2 = \left(\log\left|\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right|\right)^2 + \left(\operatorname{Arg}\left(\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right)\right)^2$. Thus,

$$\begin{aligned}
\left|\operatorname{Log}\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right| &= \left[\left(\log\left|\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right|\right)^2 + \left(\operatorname{Arg}\left(\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right)\right)^2\right]^{1/2} \\
&\leq \log\left|\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right| + \left|\operatorname{Arg}\left(\frac{4}{1-z_j\overline{\varphi_j(\lambda)}}\right|\right) \\
&\leq \log\frac{4}{\left|1-z_j\overline{\varphi_j(\lambda)}\right|} + \frac{\pi}{2}. \qquad (8.2)
\end{aligned}$$

By Theorem 5.1.3, (8.1), and (8.2), we have

$$\begin{aligned}
Q_f(z) &= \left\|\left((1-|z_1|^2)\frac{\partial f}{\partial z_1}, \ldots, (1-|z_n|^2)\frac{\partial f}{\partial z_n}\right)\right\| \\
&\leq \frac{2\left(\log\frac{4}{\left|1-z_j\overline{\varphi_j(\lambda)}\right|} + \frac{\pi}{2}\right)\frac{|\varphi_j(\lambda)|}{\left|1-z_j\overline{\varphi_j(\lambda)}\right|}(1-|z_j|^2)}{\log\frac{4}{1-|\varphi_j(\lambda)|^2}}. \qquad (8.3)
\end{aligned}$$

Since $|\varphi_j(\lambda)| < 1$, we have $\left|1 - z_j\overline{\varphi_j(\lambda)}\right| \geq 1 - |z_j|\,|\varphi_j(\lambda)| \geq 1 - |z_j|$. Thus

$$\frac{|\varphi_j(\lambda)|\,(1-|z_j|^2)}{\left|1 - z_j\overline{\varphi_j(\lambda)}\right|} \leq |\varphi_j(\lambda)|\,(1+|z_j|) \leq 2. \tag{8.4}$$

By combining (8.3) and (8.4), we obtain

$$\begin{aligned} Q_f(z) &\leq \frac{4\left(\log\frac{4}{1-|\varphi_j(\lambda)|^2} + \frac{\pi}{2}\right)}{\log\frac{4}{1-|\varphi_j(\lambda)|^2}} \\ &\leq 4\left(1 + \frac{\pi}{2\log 4}\right). \end{aligned}$$

Therefore $\beta_f = \sup_{z\in\mathbb{D}^n} Q_f(z) < \infty$, and so f is Bloch with

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f \leq \log 4 + 4\left(1 + \frac{\pi}{2\log 4}\right). \quad \square$$

Theorem 8.3.7. *Let $\psi \in H(\mathbb{D}^n)$ and φ be a holomorphic self-map of $\mathbb{D}^n$. Then the following are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is compact.*

(b) $W_{\psi,\varphi} : \mathcal{B}_{0^*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is compact.*

(c) $\psi \in H^\infty(\mathbb{D}^n)$ *and* $\displaystyle\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log\frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} = 0.$

Proof. The implication $(a) \implies (b)$ is obvious. Suppose $W_{\psi,\varphi} : \mathcal{B}_{0^*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ is compact. Then $W_{\psi,\varphi}$ is bounded, and by Theorem 8.1.7, $\psi \in H^\infty(\mathbb{D}^n)$ and

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log\frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} < \infty. \tag{8.5}$$

Let $\{z^{(k)}\}$ be a sequence in $\mathbb{D}^n$ such that $\varphi(z^{(k)}) \to \partial\mathbb{D}^n$ as $k \to \infty$. Then there is an index $m \in \{1,\ldots,n\}$ such that $\left|\varphi_m(z^{(k)})\right| \to 1$ as $k \to \infty$. It follows that $\sum_{j=1}^{n} \log \frac{1+\left|\varphi_j(z^{(k)})\right|}{1-\left|\varphi_j(z^{(k)})\right|} \to \infty$, whence by (8.5)

$$\lim_{k\to\infty} \psi(z^{(k)}) = 0.$$

Fix $j = 1,\ldots,n$. Then by Lemma 8.3.6, the sequence $\{f_k\}$ defined by

$$f_k(z) = \frac{\left(\operatorname{Log}\frac{4}{1-z_j\overline{\varphi_j(z^{(k)})}}\right)^2}{\log\frac{4}{1-\left|\varphi_j(z^{(k)})\right|^2}}$$

is bounded in $\mathcal{B}(\mathbb{D}^n)$. In fact, $f_k \in \mathcal{B}_{0^*}(\mathbb{D}^n)$ since it is holomorphic on the closure of $\mathbb{D}^n$. Also $\{f_k\}$ converges to 0 locally uniformly in $\mathbb{D}^n$. By the compactness of $W_{\psi,\varphi}$, we obtain

$$\begin{aligned}
\left|\psi(z^{(k)})\right| \log\frac{1+\left|\varphi_j(z^{(k)})\right|}{1-\left|\varphi_j(z^{(k)})\right|} &\leq \left|\psi(z^{(k)})\right| \log\frac{4}{1-\left|\varphi_j(z^{(k)})\right|^2} \\
&= \left|\psi(z^{(k)}) f_k(\varphi(z^{(k)}))\right| \\
&\leq \|\psi(f_k\circ\varphi)\|_\infty \to 0 \qquad (8.6)
\end{aligned}$$

as $k \to \infty$.

Next, suppose there exists $\ell \in \{1,\ldots,n\}$ such that $\left|\varphi_\ell(z^{(k)})\right| \not\to 1$ as $k \to \infty$. So there exists $r \in (0,1)$ such that $\left|\varphi_\ell(z^{(k)})\right| \leq r$ for all $k \in \mathbb{N}$. Since $\psi(z^{(k)}) \to 0$ as $k \to \infty$, we obtain

$$\left|\psi(z^{(k)})\right| \log\frac{1+\left|\varphi_\ell(z^{(k)})\right|}{1-\left|\varphi_\ell(z^{(k)})\right|} \leq \left|\psi(z^{(k)})\right| \log\frac{1+r}{1-r} \to 0 \qquad (8.7)$$

as $k \to \infty$. By combining the two cases (8.6) and (8.7), we have

$$\lim_{k\to\infty} \left|\psi(z^{(k)})\right| \sum_{j=1}^{n} \log \frac{1+\left|\varphi_j(z^{(k)})\right|}{1-\left|\varphi_j(z^{(k)})\right|} = 0,$$

showing $(b) \Longrightarrow (c)$.

If we suppose $\psi \in H^\infty(\mathbb{D}^n)$ and $\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} = 0$, then from Lemma 5.1.4(b) we have,

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)|\, \omega(\varphi(z)) \leq \lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|} = 0.$$

Therefore, by Theorem 8.3.4, $(c) \Longrightarrow (a)$, completing the proof. □

8.4 Isometries

In this section, we study the isometries of weighted composition operators from the Bloch space into H^∞. We show that when the ambient space is the unit polydisk, there are no isometric weighted composition operators.

Fix $j \in \{1, \ldots, n\}$ and let $p_j : \mathbb{D}^n \to \mathbb{D}$ be the projection map onto the j^{th} coordinate, that is, $p_j(z_1, \ldots, z_n) = z_j$.

Lemma 8.4.1. *Let $j \in \{1, \ldots, n\}$. Then p_j is Bloch and $||p_j||_{\mathcal{B}} = 1$.*

Proof. For $z \in \mathbb{D}^n$, we have $\frac{\partial p_j}{\partial z_k}(z) = 0$ for $k \neq j$ and $\frac{\partial p_j}{\partial z_j} = 1$. Since $p_j(0) = 0$, from Theorem 5.1.3 we obtain

$$||p_j||_{\mathcal{B}} = \sup_{z\in\mathbb{D}^n} \left|\left| \left((1-|z_1|^2)\frac{\partial p_j}{\partial z_1}(z), \cdots, (1-|z_n|^2)\frac{\partial p_j}{\partial z_n} \right) \right|\right| = \sup_{z_j\in\mathbb{D}} (1-|z_j|^2) = 1. \quad \square$$

Theorem 8.4.2. *Let $\psi \in H(\mathbb{D}^n)$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ is not an isometry.*

Proof. Assume $W_{\psi,\varphi}$ is an isometry from $\mathcal{B}(\mathbb{D}^n)$ to $H^\infty(\mathbb{D}^n)$. Then $||\psi||_\infty = ||W_{\psi,\varphi}1||_\infty = 1$ and, fixing $j \in \{1, \dots, n\}$,

$$||\psi\varphi_j||_\infty = ||W_{\psi,\varphi}p_j||_\infty = ||p_j||_{\mathcal{B}} = 1.$$

Thus, there exists a sequence $\{z^{(m)}\}$ in $\mathbb{D}^n$ such that $\left|\psi(z^{(m)})\varphi_j(z^{(m)})\right| \to 1$ as $m \to \infty$. Since ψ and φ_j map $\mathbb{D}^n$ into $\overline{\mathbb{D}}$, it follows that $\left|\psi(z^{(m)})\right| \to 1$ and $\left|\varphi_j(z^{(m)})\right| \to 1$ as $m \to \infty$.

On the other hand, by Theorem 8.1.7,

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{k=1}^{n} \log \frac{1 + |\varphi_k(z)|}{1 - |\varphi_k(z)|} < \infty. \tag{8.8}$$

If $\left|\varphi_j(z^{(m)})\right| \to 1$ as $m \to \infty$, then $\log \frac{1+\left|\varphi_j(z^{(m)})\right|}{1-\left|\varphi_j(z^{(m)})\right|} \to \infty$ as $m \to \infty$. Thus, in order for (8.8) to hold, we must have $\left|\psi(z^{(m)})\right| \to 0$, a contradiction. □

8.5 Component Operators

In this section, we collect the results on boundedness and compactness of multiplication and composition operators acting from the Bloch space or the $*$-little Bloch space into H^∞ of a bounded homogeneous domain. These results are found by looking at special cases of the results from the previous sections in this chapter.

8.5.1 Multiplication Operators

Let D be a bounded homogeneous domain and $\psi \in H(D)$. If φ is the identity map id of D, then $W_{\psi,\varphi}$ is the multiplication operator M_ψ. For $z \in D$, we define

$$\eta_\psi = \eta_{\psi,\mathrm{id}} = \sup_{z\in D} |\psi(z)|\, \omega(z),$$

$$\eta_{0,\psi} = \eta_{0,\psi,\mathrm{id}} = \sup_{z\in D} |\psi(z)|\, \omega_0(z).$$

As a corollary of Theorems 8.1.2 and 8.1.3, we obtain the following characterization of the bounded multiplication operators from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Corollary 8.5.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$. Then*

(a) *$M_\psi : \mathcal{B}(D) \to H^\infty(D)$ is bounded if and only if $\psi \in H^\infty(D)$ and $\eta_\psi < \infty$.*

(b) *$M_\psi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is bounded if and only if $\psi \in H^\infty(D)$ and $\eta_{0,\psi} < \infty$.*

As a corollary to Theorems 8.2.1 and 8.2.2, we obtain the norm of a bounded multiplication operator from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Corollary 8.5.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$.*

(a) *If $M_\psi : \mathcal{B}(D) \to H^\infty(D)$ is bounded, then*

$$||M_\psi|| = \max\{||\psi||_\infty, \eta_\psi\}.$$

(b) *If $M_\psi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is bounded, then*

$$||M_\psi|| = \max\{||\psi||_\infty, \eta_{0,\psi}\}.$$

As a special case of Theorem 8.3.4, the following is a sufficient condition for the multiplication operator to be compact from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into H^∞(D).

Corollary 8.5.3. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H^\infty(D)$.*

(a) *If $\lim\limits_{z\to\partial D} |\psi(z)|\,\omega(z) = 0$, then $M_\psi : \mathcal{B}(D) \to H^\infty(D)$ is compact.*

(b) *If $\lim\limits_{z\to\partial D} |\psi(z)|\,\omega_0(z) = 0$, then $M_\psi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is compact.*

In Chapter 5, we proved that the only compact multiplication operators on $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ are precisely those induced by the constant function 0. The next result provides a condition on the domain for which this result is true in the present setting.

Proposition 8.5.4. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$. If $\lim_{z\to\partial D} \omega_0(z) = \infty$, then the following are equivalent:*

(a) *$M_\psi : \mathcal{B}(D) \to H^\infty(D)$ is compact.*

(b) *$M_\psi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is compact.*

(c) *ψ is identically 0.*

Proof. Assume $\lim_{z\to\partial D} \omega_0(z) = \infty$. It is clear that $(a) \implies (b)$. Now suppose $M_\psi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is compact. In particular, M_ψ is bounded, and so $\eta_{0,\psi}$ is finite by Lemma 8.1.1. The finiteness of $\eta_{0,\psi}$ implies $\psi(z) \to 0$ as $z \to \partial D$. Thus, ψ is identically 0, proving $(b) \implies (c)$. If ψ is identically 0, then M_ψ is compact from $\mathcal{B}(D)$ to $H^\infty(D)$, and thus $(c) \implies (a)$. □

Since for $z \in \mathbb{B}_n$, $\omega_0(z) = \omega(z) = \dfrac{1}{2}\log\dfrac{1+||z||}{1-||z||}$, we have the following characterization of the compact multiplication operators on the Bloch space of $\mathbb{B}_n$.

Corollary 8.5.5. *Let $\psi \in H(\mathbb{B}_n)$. Then the following are equivalent.*

(a) *$M_\psi : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ is compact.*

(b) *$M_\psi : \mathcal{B}_{0^*}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ is compact.*

(c) *ψ is identically 0.*

8.5.2 Composition Operators

Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and φ a holomorphic self-map of D. Then the weighted composition operator $W_{1,\varphi}$ is the composition operator C_φ. For $z \in D$, define

$$\eta_\varphi = \eta_{1,\varphi} = \sup_{z\in D} \omega(\varphi(z)),$$

$$\eta_{0,\varphi} = \eta_{0,1,\varphi} = \sup_{z\in D} \omega_0(\varphi(z)).$$

As a corollary of Theorems 8.1.2 and 8.1.3, we obtain the following characterization of the bounded composition operators from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Corollary 8.5.6. *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. Then*

(a) $C_\varphi : \mathcal{B}(D) \to H^\infty(D)$ *is bounded if and only if* $\eta_\varphi < \infty$.

(b) $C_\varphi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ *is bounded if and only if* $\eta_{0,\varphi} < \infty$.

As an immediate consequence, we obtain the following characterization on $\mathcal{B}(\mathbb{B}_n)$.

Corollary 8.5.7. *Let φ a holomorphic self-map of $\mathbb{B}_n$. Then the following are equivalent.*

(a) $C_\varphi : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(b) $C_\varphi : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*

(c) $\displaystyle \sup_{z\in\mathbb{B}_n} \log \frac{1+||\varphi(z)||}{1-||\varphi(z)||} < \infty.$

(d) *The range of φ has compact closure in $\mathbb{B}_n$.*

From this characterization, we can determine examples of holomorphic self-maps on $\mathbb{B}_n$ which do not induce bounded composition operators from $\mathcal{B}(\mathbb{B}_n)$ to $H^\infty(\mathbb{B}_n)$. This is interesting, since any holomorphic self-map of $\mathbb{B}_n$ induces a bounded composition operator from $\mathcal{B}(\mathbb{B}_n)$ to $\mathcal{B}(\mathbb{B}_n)$.

As a corollary to Theorems 8.2.1 and 8.2.2, we obtain the norm of a bounded composition operator from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Corollary 8.5.8. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and φ a holomorphic self-map of D.*

(a) *If $C_\varphi : \mathcal{B}(D) \to H^\infty(D)$ is bounded, then*

$$||C_\varphi|| = \max\{1, \eta_\varphi\}.$$

(b) *If $C_\varphi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is bounded, then*

$$||C_\varphi|| = \max\{1, \eta_{0,\varphi}\}.$$

Lastly, the following is a sufficient condition for the composition operator to be compact from $\mathcal{B}(D)$ or $\mathcal{B}_{0^*}(D)$ into $H^\infty(D)$.

Corollary 8.5.9. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and φ be a holomorphic self-map of D.*

(a) *If $\lim_{\varphi(z)\to\partial D} \omega(\varphi(z)) = 0$, then $C_\varphi : \mathcal{B}(D) \to H^\infty(D)$ is compact.*

(b) *If $\lim_{\varphi(z)\to\partial D} \omega_0(\varphi(z)) = 0$, then $C_\varphi : \mathcal{B}_{0^*}(D) \to H^\infty(D)$ is compact.*

Chapter 9: Weighted Composition Operators on the Bloch Space of a Bounded Homogeneous Domain

In this chapter, we extend the results pertaining to boundedness and compactness of the previous chapter to the Bloch space of a bounded homogeneous domain. We provide necessary and sufficient conditions for the boundedness and compactness of the weighted composition operators on the Bloch space and $*$-little Bloch space. In addition, norm estimates are established.

9.1 Boundedness

Recall the definition of ω and ω_0 (5.3) on a bounded homogeneous domain D. For φ a holomorphic self-map of D, and $z \in D$, define

$$T_\varphi(z) = \sup\{Q_{f\circ\varphi}(z) : f \in \mathcal{B}(D), \beta_f \leq 1\}$$

$$T_{0,\varphi}(z) = \sup\{Q_{f\circ\varphi}(z) : f \in \mathcal{B}_{0^*}(D), \beta_f \leq 1\}.$$

Lemma 9.1.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, φ a holomorphic self-map of D, and $z \in D$.*

(a) *If $f \in \mathcal{B}(D)$, then*

$$Q_{f\circ\varphi}(z) \leq T_\varphi(z)\beta_f.$$

(b) *If $f \in \mathcal{B}_{0^*}(D)$, then*

$$Q_{f\circ\varphi}(z) \leq T_{0,\varphi}(z)\beta_f.$$

Proof. The result is trivially satisfied if f is constant. So let $f \in \mathcal{B}(D)$ be non-constant and define the function $g = \frac{1}{\beta_f}(f \circ \varphi)$. Then g is Bloch and $\beta_g \leq 1$. So

$$\frac{1}{\beta_f} Q_{f\circ\varphi}(z) = Q_g(z) \leq T_\varphi(z),$$

and (a) follows immediately. The proof of (b) is analogous. □

Lemma 9.1.2. *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. Then*

$$T_{0,\varphi}(z) \leq T_\varphi(z) \leq B_\varphi(z) \tag{9.1}$$

for all $z \in D$.

Proof. Since $\mathcal{B}_{0^*}(D) \subset \mathcal{B}(D)$, we have $T_{0,\varphi}(z) \leq T_\varphi(z)$ for all $z \in D$. By (2.5), we have for $f \in \mathcal{B}(D)$ and $z \in D$,

$$Q_{f\circ\varphi}(z) \leq B_\varphi(z)\beta_f.$$

Taking the supremum over all functions f with $\beta_f \leq 1$, we deduce $T_\varphi(z) \leq B_\varphi(z)$, as desired. □

For D a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D, define

$$\sigma_{\psi,\varphi} = \sup_{z\in D} \omega(\varphi(z)) Q_\psi(z),$$

$$\tau_{\psi,\varphi} = \sup_{z\in D} |\psi(z)|\, T_\varphi(z),$$

$$\sigma_{0,\psi,\varphi} = \sup_{z\in D} \omega_0(\varphi(z)) Q_\psi(z),$$

$$\tau_{0,\psi,\varphi} = \sup_{z\in D} |\psi(z)|\, T_{0,\varphi}(z).$$

These quantities are direct generalizations of the quantities defined in Chapter 7. We determine separate necessary and sufficient conditions for the weighted composition operator to be bounded on the Bloch space and the $*$-Bloch space of a bounded homogeneous domain. We conjecture that these conditions are both necessary and sufficient.

Theorem 9.1.3. *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. If $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$.*

Proof. We need to show that $W_{\psi,\varphi}$ maps $\mathcal{B}(D)$ into $\mathcal{B}(D)$ and there exists $M > 0$ such that $||W_{\psi,\varphi} f||_{\mathcal{B}} \leq M\,||f||_{\mathcal{B}}$ for all $f \in \mathcal{B}(D)$. Let $f \in \mathcal{B}(D)$. From (2.6), Lemma 5.1.1(a), and Lemma 9.1.1(a), for $z \in D$, we deduce

$$
\begin{aligned}
Q_{\psi(f\circ\varphi)}(z) &\leq |\psi(z)|\, Q_{f\circ\varphi}(z) + |f(\varphi(z))|\, Q_\psi(z) \\
&\leq |\psi(z)|\, T_\varphi(z)\beta_f + (|f(0)| + \omega(\varphi(z))\beta_f)\, Q_\psi(z) \\
&\leq |f(0)|\, Q_\psi(z) + (|\psi(z)|\, T_\varphi(z) + \omega(\varphi(z)) Q_\psi(z))\, \beta_f.
\end{aligned}
$$

Taking the supremum over all $z \in D$, we have

$$
\beta_{\psi(f\circ\varphi)} \leq |f(0)|\, \beta_\psi + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\beta_f, \tag{9.2}
$$

which is finite by assumption. Thus $W_{\psi,\varphi}$ maps $\mathcal{B}(D)$ into itself.

From (9.2) and Corollary 5.1.6, for $f \in \mathcal{B}(D)$, we have

$$
\begin{aligned}
||W_{\psi,\varphi} f||_{\mathcal{B}} &= |\psi(0)|\, |f(\varphi(0))| + \beta_{\psi(f\circ\varphi)} \\
&\leq |\psi(0)|\, (|f(0)| + \rho(\varphi(0), 0)\beta_f) + |f(0)|\, \beta_\psi + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\beta_f \\
&= |f(0)|\, ||\psi||_{\mathcal{B}} + (|\psi(0)|\, \rho(\varphi(0), 0) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\, \beta_f \\
&\leq (||\psi||_{\mathcal{B}} + |\psi(0)|\, \rho(\varphi(0), 0) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\, ||f||_{\mathcal{B}}.
\end{aligned}
$$

Since $||\psi||_{\mathcal{B}} + |\psi(0)|\, \rho(\varphi(0), 0) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}$ is finite by assumption, and is independent of f, we conclude that $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$. □

Theorem 9.1.4. *Let D be a bounded homogeneous domain and φ a holomorphic self-map of D. If $\psi \in \mathcal{B}_{0^*}(D)$, $\sigma_{0,\psi,\varphi}$ and $\tau_{0,\psi,\varphi}$ are finite and*

$$\lim_{z \to \partial^* D} |\psi(z)|\, T_{0,\varphi}(z) = \lim_{z \to \partial^* D} \omega_0(\varphi(z)) Q_\psi(z) = 0,$$

then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{0^*}(D)$.

Proof. We will first show that $W_{\psi,\varphi}$ maps $\mathcal{B}_{0^*}(D)$ into $\mathcal{B}_{0^*}(D)$. Let $f \in \mathcal{B}_{0^*}(D)$. From (2.6), Lemma 5.1.1(b), and Lemma 9.1.1(b), and arguing as in the proof of Theorem 9.1.3, for $z \in D$, we have

$$Q_{\psi(f\circ\varphi)}(z) \leq |f(0)|\, Q_\psi(z) + \left(|\psi(z)|\, T_{0,\varphi}(z) + \omega_0(\varphi(z)) Q_\psi(z)\right) \beta_f, \tag{9.3}$$

and

$$\beta_{\psi(f\circ\varphi)} \leq |f(0)|\, \beta_\psi + \left(\tau_{0,\psi,\varphi} + \sigma_{0,\psi,\varphi}\right) \beta_f.$$

So $W_{\psi,\varphi} f$ is Bloch. In particular, $W_{\psi,\varphi} f \in \mathcal{B}_{0^*}(D)$ since the right-hand side of (9.3) goes to 0 as $z \to \partial^* D$ by assumption. So $W_{\psi,\varphi}$ maps $\mathcal{B}_{0^*}(D)$ into itself. Likewise, we have

$$||W_{\psi,\varphi} f||_{\mathcal{B}} \leq \left(||\psi||_{\mathcal{B}} + |\psi(0)|\, \rho(\varphi(0), 0) + \tau_{0,\psi,\varphi} + \sigma_{0,\psi,\varphi}\right) ||f||_{\mathcal{B}},$$

and thus $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{0^*}(D)$. □

Theorem 9.1.5. *Let D be a bounded homogeneous domain, $\psi \in H(D)$ and φ a holomorphic self-map of D. If $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$, then $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ is finite if and only if $\tau_{\psi,\varphi}$ is finite.*

Proof. Since $W_{\psi,\varphi}$ is bounded, we have $\psi \in \mathcal{B}(D)$ since $\psi = W_{\psi,\varphi} 1$. Let $f \in \mathcal{B}(D)$, $z \in D$

and $u \in \mathbb{C}^n$. Then

$$\nabla(\psi(f \circ \varphi))(z)u = \psi(z)\nabla(f \circ \varphi)(z)u + f(\varphi(z))\nabla(\psi)(z)u.$$

Thus for $u \neq 0$ we have

$$\begin{aligned}\frac{|f(\varphi(z))|\,|\nabla(\psi)(z)u|}{H_z(u,\overline{u})^{1/2}} &= \frac{|\nabla(\psi(f \circ \varphi))(z)u - \psi(z)\nabla(f \circ \varphi)(z)u|}{H_z(u,\overline{u})^{1/2}} \\ &\leq \frac{|\nabla(\psi(f \circ \varphi))(z)u|}{H_z(u,\overline{u})^{1/2}} + \frac{|\psi(z)|\,|\nabla(f \circ \varphi)(z)u|}{H_z(u,\overline{u})^{1/2}}.\end{aligned}$$

Taking the supremum over all $u \in \mathbb{C}^n \setminus \{0\}$, and applying Lemma 9.1.1(a), we obtain

$$\begin{aligned}|f(\varphi(z))|\,Q_\psi(z) &\leq Q_{\psi(f\circ\varphi)}(z) + |\psi(z)|\,Q_{f\circ\varphi}(z) \\ &\leq \beta_{\psi(f\circ\varphi)} + |\psi(z)|\,T_\varphi(z)\beta_f \\ &\leq ||W_{\psi,\varphi}f||_{\mathcal{B}} + |\psi(z)|\,T_\varphi(z)\,||f||_{\mathcal{B}} \\ &\leq (||W_{\psi,\varphi}|| + |\psi(z)|\,T_\varphi(z))\,||f||_{\mathcal{B}}\,.\end{aligned}$$

Finally, taking the supremum over all $f \in \mathcal{B}(D)$ with $f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1$ yields

$$\omega(\varphi(z))Q_\psi(z) \leq ||W_{\psi,\varphi}|| + |\psi(z)|\,T_\varphi(z).$$

Thus

$$\sigma_{\psi,\varphi} \leq ||W_{\psi,\varphi}|| + \tau_{\psi,\varphi}. \tag{9.4}$$

On the other hand, let $g \in \mathcal{B}(D)$ such that $g(0) = 0$ and $||g||_{\mathcal{B}} \leq 1$. Then

$$\begin{aligned} |\psi(z)|\, Q_{g\circ\varphi}(z) &\leq Q_{\psi(g\circ\varphi)}(z) + |g(\varphi(z))|\, Q_\psi(z) \\ &\leq ||W_{\psi,\varphi} g||_{\mathcal{B}} + \omega(\varphi(z)) Q_\psi(z) \\ &\leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}. \end{aligned}$$

For any non-constant Bloch function f with $\beta_f \leq 1$, we define the function $g(z) = \frac{1}{\beta_f}(f(z) - f(0))$. The function g is Bloch, $g(0) = 0$, and $\beta_g = 1$. Applying the previous case to this function g yields

$$|\psi(z)|\, Q_{f\circ\varphi}(z) = |\psi(z)|\, Q_{g\circ\varphi}(z)\beta_f \leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}.$$

Taking the supremum over all such Bloch functions f, we deduce

$$\tau_{\psi,\varphi} \leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}. \tag{9.5}$$

From (9.4) and (9.5), we obtain $\sigma_{\psi,\varphi}$ is finite if and only if $\tau_{\psi,\varphi}$ is finite. □

The proof of the following result is analogous.

Theorem 9.1.6. *Let D be a bounded homogeneous domain, $\psi \in H(D)$ and φ a holomorphic self-map of D. If $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{0^*}(D)$, then $\psi \in \mathcal{B}_{0^*}(D)$, and $\sigma_{0,\psi,\varphi}$ is finite if and only if $\tau_{0,\psi,\varphi}$ is finite.*

Although we are unable to connect the necessary and sufficient conditions, we conjecture they do indeed characterize the bounded weighted composition operators. In an attempt to lend credence to this conjecture, we prove the conjecture, in Chapter 10, in the case when the ambient space is taken to be either the unit ball or the unit polydisk.

Conjecture 9.1.7. Let D be a bounded homogeneous domain, $\psi \in H(D)$ and φ a holomorphic self-map of D. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$ if and only if $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.

9.2 Operator Norm Estimates

In this section, we establish estimates on the operator norm for the bounded weighted composition operators on the Bloch space of a bounded homogeneous domain. These estimates, when restricted to the multiplication and composition operators, agree with established estimates discussed in Chapter 7.

Theorem 9.2.1. *Let D be a bounded homogeneous domain. If $\psi \in H(D)$ and φ a holomorphic self-map of D induce a bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{B}(D)$, then*

$$\max\{||\psi||_{\mathcal{B}}, |\psi(0)|\,\omega(\varphi(0))\} \leq ||W_{\psi,\varphi}|| \leq \max\{||\psi||_{\mathcal{B}}, |\psi(0)|\,\omega(\varphi(z)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\}.$$

Proof. We first prove the lower estimate. By considering as test function the constant function 1, we have $||W_{\psi,\varphi}1||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}$, and so $||W_{\psi,\varphi}|| \geq ||\psi||_{\mathcal{B}}$. Furthermore,

$$\begin{aligned} ||W_{\psi,\varphi}|| &= \sup\{||W_{\psi,\varphi}f||_{\mathcal{B}} : f \in \mathcal{B}(D) \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &\geq \sup\{||W_{\psi,\varphi}f||_{\mathcal{B}} : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &\geq \sup\{|\psi(0)|\,|f(\varphi(0))| : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &= |\psi(0)|\,\omega(\varphi(0)). \end{aligned}$$

Thus, $||W_{\psi,\varphi}|| \geq \max\{||\psi||_{\mathcal{B}}, |\psi(0)|\,\omega(\varphi(0))\}$.

Let $f \in \mathcal{B}(D)$. From Lemma 5.1.1(a) and (9.2), we obtain

$$\begin{aligned}
\|W_{\psi,\varphi}f\|_{\mathcal{B}} &= |\psi(0)|\,|f(\varphi(0))| + \beta_{\psi(f\circ\varphi)} \\
&\le |\psi(0)|\,(|f(0)| + \omega(\varphi(0))\beta_f) + |f(0)|\,\beta_\psi + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\,\beta_f \\
&= \|\psi\|_{\mathcal{B}}\,(\|f\|_{\mathcal{B}} - \beta_f) + (|\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\,\beta_f \\
&= \|\psi\|_{\mathcal{B}}\,\|f\|_{\mathcal{B}} + (|\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} - \|\psi\|_{\mathcal{B}})\,\beta_f.
\end{aligned}$$

If $|\psi(0)|\,\omega(\varphi(0))+\tau_{\psi,\varphi}+\sigma_{\psi,\varphi} \le \|\psi\|_{\mathcal{B}}$, then $\|W_{\psi,\varphi}f\|_{\mathcal{B}} \le \|\psi\|_{\mathcal{B}}\,\|f\|_{\mathcal{B}}$. Taking the supremum over all $f \in \mathcal{B}(D)$ such that $\|f\|_{\mathcal{B}} \le 1$, we have $\|W_{\psi,\varphi}\| \le \|\psi\|_{\mathcal{B}}$. On the other hand, if $|\psi(0)|\,\omega(\varphi(0))+\tau_{\psi,\varphi}+\sigma_{\psi,\varphi} \ge \|\psi\|_{\mathcal{B}}$, then $\|W_{\psi,\varphi}f\|_{\mathcal{B}} \le (|\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\,\|f\|_{\mathcal{B}}$. So $\|W_{\psi,\varphi}\| \le |\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}$, and

$$\|W_{\psi,\varphi}\| \le \max\{\|\psi\|_{\mathcal{B}}\,, |\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\}. \quad \Box$$

The proof of the next result on $\mathcal{B}_{0^*}(D)$ is analogous.

Theorem 9.2.2. *Let D be a bounded homogeneous domain. If $\psi \in H(D)$ and φ a holomorphic self-map of D induce a bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{B}_{0^*}(D)$, then*

$$\max\{\|\psi\|_{\mathcal{B}}\,, |\psi(0)|\,\omega_0(\varphi(0))\} \le \|W_{\psi,\varphi}\| \le \max\{\|\psi\|_{\mathcal{B}}\,, |\psi(0)|\,\omega_0(\varphi(z)) + \tau_{0,\psi,\varphi} + \sigma_{0,\psi,\varphi}\}.$$

9.3 Compactness

In this section, we determine sufficient conditions for the bounded weighted composition operator $W_{\psi,\varphi}$ to be compact on $\mathcal{B}(D)$.

Lemma 9.3.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a* *holomorphic self-map of D. Then $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ if and only if for each bounded sequence $\{f_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D, $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$, as $k \to \infty$.*

Proof. Assume $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$. Let $\{f_k\}$ be a bounded sequence in $\mathcal{B}(D)$ which converges to 0 locally uniformly in D. By rescaling the sequence, we can assume $||f_k||_{\mathcal{B}} \leq 1$ for all $k \in \mathbb{N}$. We need to show that $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Since $W_{\psi,\varphi}$ is compact, the sequence $\{\psi(f_k \circ \varphi)\}$ has a subsequence which converges in the Bloch norm to some function $f \in \mathcal{B}(D)$. For convenience, we re-index this convergent subsequence as $\{f_k\}$.

We are going to show that f is identically 0. Fix $z_0 \in D$ and, without loss of generality, assume $f(z_0) = 0$. For $z \in D$, by (5.1), we obtain

$$\begin{aligned} |\psi(z)f_k(\varphi(z)) - f(z)| &\leq |\psi(z)f_k(\varphi(z)) - f(z) - (\psi(z_0)f_k(\varphi(z_0)) - f(z_0))| \\ &\quad + |\psi(z_0)f_k(\varphi(z_0)) - f(z_0)| \\ &\leq ||\psi(f_k \circ \varphi) - f||_{\mathcal{B}}\, \rho(z, z_0) + |\psi(z_0)f_k(\varphi(z_0))| \,. \end{aligned}$$

So $|\psi(f_k \circ \varphi) - f| \to 0$ locally uniformly as $k \to \infty$, since $||\psi(f_k \circ \varphi) - f||_{\mathcal{B}} \to 0$ and $\psi(z_0)f_k(\varphi(z_0)) \to 0$ as $k \to \infty$. On the other hand, $\psi(f_k \circ \varphi) \to 0$ locally uniformly, so f must be identically 0.

Conversely, assume $||\psi(g_n \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$ for each bounded sequence $\{g_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D. To prove the compactness of $W_{\psi,\varphi}$ it suffices to show that given any sequence $\{f_k\}$ in the unit ball of $B(D)$, there exists a subsequence $\{f_{k_j}\}$ such that $\psi(f_{k_j} \circ \varphi)$ converges in $\mathcal{B}(D)$. Fix $z_0 \in D$. Replacing f_k with $f_k - f_k(z_0)$, we can assume $f_k(z_0) = 0$ for all $k \in \mathbb{N}$. By (5.1), $|f(z)| = |f(z) - f(z_0)| \leq \rho(z, z_0)$, for every $z \in D$. Thus, on each disk centered at z_0 with respect to the Bergman distance, the sequence $\{f_k\}$ is uniformly bounded, and hence on each compact subset of D.

By Theorem 8.3.1, some subsequence $\{f_{k_j}\}$ converges locally uniformly to some function f holomorphic in D. By Theorem 8.3.2, f is Bloch and $||f||_{\mathcal{B}} \leq 1$. Letting $g_{k_j} = f_{k_j} - f$, we obtain a bounded sequence in $\mathcal{B}(D)$ converging to 0 locally uniformly in D. Thus, by the hypothesis, $\left|\left|\psi(g_{k_j} \circ \varphi)\right|\right|_{\mathcal{B}} \to 0$ as $k \to \infty$. Therefore, $\psi(f_{k_j} \circ \varphi)$ converges in norm to $\psi(f \circ \varphi)$. □

Theorem 9.3.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. If*

$$\lim_{\varphi(z)\to\partial D} \omega(\varphi(z))Q_\psi(z) = \lim_{\varphi(z)\to\partial D} |\psi(z)|\, T_\varphi(z) = 0,$$

then $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$.

Proof. By Lemma 9.3.1, to prove that $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ it suffices to show that for any sequence $\{f_k\}$ in the unit ball of $\mathcal{B}(D)$ converging to 0 locally uniformly in D, $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Let $\{f_k\}$ be such a sequence, and fix $\varepsilon > 0$. Then $|f_k(0)| < \frac{\varepsilon}{3||\psi||_{\mathcal{B}}}$ for all k sufficiently large, and there exists $r > 0$ such that for all $k \in \mathbb{N}$, $|\psi(z)|\, Q_{f_k\circ\varphi}(z) < \frac{\varepsilon}{3}$ and $\omega(\varphi(z))Q_\psi(z) < \frac{\varepsilon}{3}$ whenever $\rho(\varphi(z), \partial D) \geq r$. Thus, by Lemma 5.1.1(a), if $\rho(\varphi(z), \partial D) \geq r$, then

$$\begin{aligned} Q_{\psi(f_k\circ\varphi)}(z) &\leq |\psi(z)|\, Q_{f_k\circ\varphi}(z) + |f_k(\varphi(z))|\, Q_\psi(z) \\ &\leq \frac{\varepsilon}{3} + (|f_k(0)| + \omega(\varphi(z)))\, Q_\psi(z) \\ &< \varepsilon. \end{aligned}$$

On the other hand, since $f_k \to 0$ locally uniformly in D, $|f_k(\varphi(z))| \to 0$ and $Q_{f_k\circ\varphi}(z) \to 0$ uniformly on the set $\{z \in D : \rho(\varphi(z), \partial D) \leq r\}$. Consequently, for all k sufficiently large, $Q_{\psi(f_k\circ\varphi)}(z) < \varepsilon$ for all $z \in D$. Furthermore, $|\psi(0)f_k(\varphi(0))| \to 0$ as $k \to \infty$, so $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$. □

Although we have not been able to prove the sufficient condition for compactness to be necessary, we conjecture the necessity of it. In the next chapter, we will prove the conjecture true when the ambient space is taken to be the unit ball and the unit polydisk.

Conjecture 9.3.3. Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $\psi \in H(D)$, and φ a holomorphic self-map of D. Then the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ if and only if

$$\lim_{\varphi(z)\to\partial D} \omega(\varphi(z))Q_\psi(z) = \lim_{\varphi(z)\to\partial D} |\psi(z)|\, T_\varphi(z) = 0.$$

Chapter 10: Weighted Composition Operators on the Bloch Space of the Unit Ball and Unit Polydisk

In the previous chapter, conjectures on the characterizations of boundedness and compactness were made for the weighted composition operators acting on the Bloch space of a general bounded homogeneous domain. In this chapter, we prove these conjectures when the domain is the unit ball or the unit polydisk. Thus, the quantities used in the conjectures do unify the typical operator theory on the Bloch space in several complex variables. However, more study of such general domains and their corresponding function theory is needed in order to prove the conjectures in complete generality. We end this chapter with examples of weighted composition operators on $\mathcal{B}(\mathbb{B}_n)$ and $\mathcal{B}(\mathbb{D}^n)$ that illustrate that the weighted composition operators are more than the sum of their parts.

10.1 The Unit Ball

In this section, we prove Conjectures 9.1.7 and 9.3.3 on the Bloch space of $\mathbb{B}_n$. Let $p, q, s \in \mathbb{R}$ such that $0 < p, s < \infty$, $-n-1 < q < \infty$ and $q + s > -1$. A function $f \in H(\mathbb{B}_n)$ is in the $F(p, q, s)$ space if

$$||f||_{F(p,q,s)} = |f(0)| + \left(\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} ||\nabla(f)(z)||^p \,(1 - ||z||^2)^q (G(z, a))^s \, d\nu(z)\right)^{1/p} < \infty,$$

where $d\nu$ denotes normalized Lebesgue measure on $\mathbb{B}_n$, $G(z, a) = \log|\phi_a(z)|^{-1}$ is the Green's function on $\mathbb{B}_n$ with logarithmic singularity at a, and ϕ_a is the involutive automorphism which maps 0 to a. In [95], Zhou and Chen characterized the bounded and the compact weighted composition operators on the $F(p, q, s)$ spaces. If $p \geq 1$, $s > n$, and $\frac{q+n+1}{p} = 1$, then $F(p,q,s) = \mathcal{B}(\mathbb{B}_n)$, and so the results in [95] can be formulated, as a special case, on the Bloch space of $\mathbb{B}_n$ since the $F(p,q,s)$ norm, under these conditions, is equivalent to the Bloch norm.

Theorem 10.1.1. [95] *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if the following two conditions are satisfied:*

(a) $\sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z) < \infty$;

(b) $\sup_{z\in\mathbb{B}_n} (1 - \|z\|^2)\, \|\nabla(\psi)(z)\| \log \dfrac{2}{1 - \|\varphi(z)\|^2} < \infty$.

Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact if and only if the following two conditions are satisfied:

(c) $\lim_{\|\varphi(z)\|\to 1} |\psi(z)|\, B_\varphi(z) = 0$;

(d) $\lim_{\|\varphi(z)\|\to 1} (1 - \|z\|^2)\, \|\nabla(\psi)(z)\| \log \dfrac{2}{1 - \|\varphi(z)\|^2} = 0$.

10.1.1 Boundedness

To prove Conjecture 9.1.7, we will show the conditions $\psi \in \mathcal{B}(\mathbb{B}_n)$, $\sigma_{\psi,\varphi} < \infty$ and $\tau_{\psi,\varphi} < \infty$ are equivalent to the conditions (a) and (b) in Theorem 10.1.1. However, as we will see later, we cannot use this same technique to prove Conjecture 9.3.3.

Theorem 10.1.2. *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if $\psi \in \mathcal{B}(\mathbb{B}_n)$ and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.*

Proof. If $\psi \in \mathcal{B}(\mathbb{B}_n)$ and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then by Theorem 9.1.3, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$. Conversely, suppose $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$. We need to show $\psi \in \mathcal{B}(\mathbb{B}_n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Since $W_{\psi,\varphi}$ maps $\mathcal{B}(\mathbb{B}_n)$ into itself, $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(\mathbb{B}_n)$.

Furthermore, by (9.1), $T_\varphi(z) \le B_\varphi(z)$ for all $z \in \mathbb{B}_n$. Thus

$$\tau_{\psi,\varphi} = \sup_{z\in\mathbb{B}_n} |\psi(z)|\, T_\varphi(z) \le \sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z),$$

which is finite by Theorem 10.1.1(a). By Theorem 9.1.5, we deduce that $\sigma_{\psi,\varphi}$ is finite. $\square$

10.1.2 Compactness

To prove Conjecture 9.3.3, we need the following lemmas.

Lemma 10.1.3. *Let φ be a holomorphic self-map of $\mathbb{B}_n$ and $\lambda \in \mathbb{B}_n$. Then the function defined by*

$$f(z) = \frac{\left(\operatorname{Log}\dfrac{2}{1-\langle z,\varphi(\lambda)\rangle}\right)^2}{\log\dfrac{2}{1-\|\varphi(\lambda)\|^2}},$$

is in $\mathcal{B}(\mathbb{D})$ with $\|f\|_{\mathcal{B}} \le \log 2 + 4\left(2+\frac{\pi}{\log 4}\right)$.

Proof. From Lemma 2.3.19, the function defined by

$$g(z) = \operatorname{Log}\frac{2}{1-\langle z,\varphi(\lambda)\rangle} = \log 2 + \operatorname{Log}\frac{1}{1-\langle z,\varphi(\lambda)\rangle}$$

is Bloch. By the Cauchy-Schwarz Inequality, $|\langle z,\varphi(\lambda)\rangle| \le \|z\|\,\|\varphi(\lambda)\|$ for all z and λ in $\mathbb{B}_n$. So we have the following inequalities:

$$|1-\langle z,\varphi(\lambda)\rangle| \;\ge\; 1-|\langle z,\varphi(\lambda)\rangle| \ge 1-\|z\| \tag{10.1}$$

$$|1-\langle z,\varphi(\lambda)\rangle| \;\ge\; 1-\|\varphi(\lambda)\|. \tag{10.2}$$

For all $z \in \mathbb{B}_n$, $\operatorname{Re}\left(\frac{2}{1-\langle z,\varphi(\lambda)\rangle}\right) > 0$ and so $\operatorname{Arg}\left(\frac{2}{1-\langle z,\varphi(\lambda)\rangle}\right) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. From this, and

(10.2), we have

$$
\begin{aligned}
|g(z)| &= \left| \text{Log} \frac{2}{1 - \langle z, \varphi(\lambda) \rangle} \right| \\
&\leq \log \frac{2}{|1 - \langle z, \varphi(\lambda) \rangle|} + \left| \text{Arg} \left(\frac{2}{1 - \langle z, \varphi(\lambda) \rangle} \right) \right| \\
&\leq \log \frac{2}{1 - ||\varphi(\lambda)||} + \frac{\pi}{2} \\
&= \log \frac{2(1 + ||\varphi(\lambda)||)}{1 - ||\varphi(\lambda)||^2} + \frac{\pi}{2} \\
&\leq \log \frac{4}{1 - ||\varphi(\lambda)||^2} + \frac{\pi}{2}.
\end{aligned}
$$

Since $f(z) = \dfrac{(g(z))^2}{\log \frac{2}{1-||\varphi(\lambda)||^2}}$ and by Lemma 2.3.19 $Q_g(z) \leq 2$ for all $z \in \mathbb{B}_n$, we have

$$
\begin{aligned}
Q_f(z) &= \frac{2\,|g(z)|}{\log \frac{2}{1-||\varphi(\lambda)||^2}} Q_g(z) \\
&\leq \frac{4}{\log \frac{2}{1-||\varphi(\lambda)||^2}} \left(\log \frac{4}{1 - ||\varphi(\lambda)||^2} + \frac{\pi}{2} \right) \\
&\leq 4 \left(2 + \frac{\pi}{\log 4} \right).
\end{aligned}
$$

Thus $||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{B}_n} Q_f(z) \leq \log 2 + 4 \left(2 + \frac{\pi}{\log 4} \right)$. □

Lemma 10.1.4. *Let φ be a holomorphic self-map of $\mathbb{B}_n$ and $\lambda \in \mathbb{B}_n$. For $z \in \mathbb{B}_n$, the function defined by*

$$
h(z) = \frac{1 - ||\varphi(\lambda)||^2}{1 - \langle z, \varphi(\lambda) \rangle}
$$

is Bloch with $||h||_{\mathcal{B}} \leq 5$.

Proof. For $j \in \{1, \ldots, n\}$, we have

$$\frac{\partial h}{\partial z_j}(z) = \frac{(1 - ||\varphi(\lambda)||^2)\overline{\varphi_j(\lambda)}}{(1 - \langle z, \varphi(\lambda)\rangle)^2}.$$

So

$$\nabla(h)(z) = \frac{(1 - ||\varphi(\lambda)||^2)\overline{\varphi(\lambda)}}{(1 - \langle z, \varphi(\lambda)\rangle)^2}, \tag{10.3}$$

and

$$\langle \nabla(h)(z), \overline{z}\rangle = \frac{(1 - ||\varphi(\lambda)||^2)\,\langle z, \varphi(\lambda)\rangle}{(1 - \langle z, \varphi(\lambda)\rangle)^2}.$$

Thus by Theorem 2.3.17, we obtain

$$\begin{aligned} Q_h(z) &= (1 - ||z||^2)^{1/2}\left[||\nabla(h)(z)||^2 - |\langle \nabla(h)(z), \overline{z}\rangle|^2\right]^{1/2} \\ &= \frac{(1 - ||z||^2)^{1/2}(1 - ||\varphi(\lambda)||^2)(||\varphi(\lambda)||^2 - |\langle z, \varphi(\lambda)\rangle|^2)^{1/2}}{|1 - \langle z, \varphi(\lambda)\rangle|^2}. \end{aligned} \tag{10.4}$$

By combining (10.4), (10.1), and (10.2), we deduce

$$\begin{aligned} Q_h(z) &\leq \frac{(1 - ||z||^2)^{1/2}(1 - ||\varphi(\lambda)||^2)(1 + |\langle z, \varphi(\lambda)\rangle|)^{1/2}(1 - |\langle z, \varphi(\lambda)\rangle|)^{1/2}}{(1 - ||z||)^{1/2}(1 - ||\varphi(\lambda)||)(1 - |\langle z, \varphi(\lambda)\rangle|)^{1/2}} \\ &= (1 + ||z||)^{1/2}(1 + ||\varphi(\lambda)||)(1 + \langle z, \varphi(\lambda)\rangle)^{1/2} \\ &\leq 4. \end{aligned}$$

Therefore, h is Bloch and $||h||_{\mathcal{B}} = |h(0)| + \sup_{z \in \mathbb{B}_n} Q_h(z) \leq 5$. □

Lemma 10.1.5. *Let ψ and φ be the symbols of a bounded weighted composition operator*

$W_{\psi,\varphi}$ on $\mathcal{B}(\mathbb{B}_n)$. Then

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z) = 0.$$

Proof. By Theorem 10.1.2, $\sigma_{\psi,\varphi}$ is finite, that is

$$\sup_{z\in\mathbb{B}_n} Q_\psi(z) \log \frac{1+||\varphi(z)||}{1-||\varphi(z)||} < \infty.$$

Since $\log \frac{1+||\varphi(z)||}{1-||\varphi(z)||} \to \infty$ as $||\varphi(z)|| \to 1$, the boundedness of $\sigma_{\psi,\varphi}$ implies that $Q_\psi(z) \to 0$ as $||\varphi(z)|| \to 1$. □

Lemma 10.1.6. *Suppose ψ and φ are the symbols of a compact weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{B}(\mathbb{B}_n)$, and $\{z^{(k)}\}$ is a sequence in $\mathbb{B}_n$ such that $\left|\left|\varphi(z^{(k)})\right|\right| \to 1$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \frac{\left|\psi(z^{(k)})\right|}{1-\left|\left|\varphi(z^{(k)})\right|\right|^2} \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{H_{z^{(k)}}(u,\overline{u})^{1/2}} = 0.$$

Proof. By Lemma 10.1.4, the sequence of functions defined by

$$h_k(z) = \frac{1-\left|\left|\varphi(z^{(k)})\right|\right|^2}{1-\left\langle z, \varphi(z^{(k)})\right\rangle}, z \in \mathbb{B}_n,$$

is bounded in $\mathcal{B}(\mathbb{B}_n)$ with $||h_k||_{\mathcal{B}} \le 5$ for all $k \in \mathbb{N}$. Also, $h_k(\varphi(z^{(k)})) = 1$ and $\{h_k\}$ converges to 0 locally uniformly in $\mathbb{B}_n$. By the compactness of $W_{\psi,\varphi}$, we have $||\psi(h_k \circ \varphi)||_{\mathcal{B}} \to 0$ as

$k \to \infty$. Moreover, by (10.3), we find

$$\begin{aligned}
||\psi(h_k \circ \varphi)||_{\mathcal{B}} &\geq Q_{\psi(h_k\circ\varphi)}(z^{(k)}) \\
&= \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|h_k(\varphi(z^{(k)}))\nabla\psi(z^{(k)})u + \psi(z^{(k)})\nabla(h_k \circ \varphi)(z^{(k)})u\right|}{H_{z^{(k)}}(u,\overline{u})^{1/2}} \\
&\geq \left| Q_\psi(z^{(k)}) - \left|\psi(z^{(k)})\right| \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\nabla(h_k)(\varphi(z^{(k)}))J\varphi(z^{(k)})u\right|}{H_{z^{(k)}}(u,\overline{u})^{1/2}} \right| \\
&= \left| Q_\psi(z^{(k)}) - \left|\psi(z^{(k)})\right| \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\overline{\varphi(z^{(k)})}J\varphi(z^{(k)})u\right|}{(1 - \left|\left|\varphi(z^{(k)})\right|\right|^2)H_{z^{(k)}}(u,\overline{u})^{1/2}} \right| \\
&= \left| Q_\psi(z^{(k)}) - \frac{\left|\psi(z^{(k)})\right|}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{H_{z^{(k)}}(u,\overline{u})^{1/2}} \right|.
\end{aligned}$$

Since $||\psi(h_k \circ \varphi)||_{\mathcal{B}} \to 0$ and $Q_\psi(z^{(k)}) \to 0$ as $k \to \infty$, it must be the case that

$$\lim_{k\to\infty} \frac{\left|\psi(z^{(k)})\right|}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{H_{z^{(k)}}(u,\overline{u})^{1/2}} = 0. \quad \square$$

We now prove Conjecture 9.3.3 for the unit ball.

Theorem 10.1.7. *Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$ if and only if*

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, T_\varphi(z) = 0 \textit{ and } \lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \frac{1 + ||\varphi(z)||}{1 - ||\varphi(z)||} = 0.$$

Proof. Recall $\omega(z) = \frac{1}{2} \log \frac{1+||z||}{1-||z||}$ for all $z \in \mathbb{B}_n$. By Theorem 9.3.2, if

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, T_\varphi(z) = 0 \text{ and } \lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \frac{1 + ||\varphi(z)||}{1 - ||\varphi(z)||} = 0,$$

then $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$. Conversely, assume $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$, and from Theorem 10.1.1(c), we deduce

$$\lim_{\varphi(z)\to 1} |\psi(z)|\, T_\varphi(z) \leq \lim_{\varphi(z)\to 1} |\psi(z)|\, B_\varphi(z) = 0.$$

Let $\{z^{(k)}\}$ be a sequence in $\mathbb{B}_n$ such that $\left|\left|\varphi(z^{(k)})\right|\right| \to 1$ as $k \to \infty$. For each $k \in \mathbb{N}$ and $z \in \mathbb{B}_n$, define the function

$$f_k(z) = \frac{\left(\operatorname{Log}\frac{2}{1-\left\langle z,\varphi(z^{(k)})\right\rangle}\right)^2}{\log\frac{2}{1-\left|\left|\varphi(z^{(k)})\right|\right|^2}}.$$

By Lemma 10.1.3, $\{f_k\}$ is a bounded sequence in $\mathcal{B}(\mathbb{B}_n)$. Also, $\{f_k\}$ converges to 0 locally uniformly on $\mathbb{B}_n$. By the compactness of $W_{\psi,\varphi}$, $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Observe that

$$\nabla(f_k)(z) = \frac{2\operatorname{Log}\frac{2}{1-\left\langle z,\varphi(z^{(k)})\right\rangle}}{\log\frac{2}{1-\left|\left|\varphi(z^{(k)})\right|\right|^2}} \frac{\overline{\varphi(z^{(k)})}}{1-\left\langle z,\varphi(z^{(k)})\right\rangle},$$

and for $u \in \mathbb{C}^n \setminus \{0\}$

$$\left|\nabla(f_k)(\varphi(z^{(k)}))J\varphi(z^{(k)})u\right| = \frac{2\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{1-\left|\left|\varphi(z^{(k)})\right|\right|^2},$$

and hence,

$$
\begin{aligned}
||\psi(f_k \circ \varphi)||_{\mathcal{B}} &\geq \sup_{z \in \mathbb{B}_n} Q_{\psi(f_k \circ \varphi)}(z) \\
&\geq Q_{\psi(f_k \circ \varphi)}(z^{(k)}) \\
&= \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left| f_k(\varphi(z^{(k)}))\nabla(\psi)(z^{(k)})u + \psi(z^{(k)})\nabla(f_k \circ \varphi)(z^{(k)})u \right|}{H_{z^{(k)}}(u, \overline{u})^{1/2}} \\
&\geq \left| Q_\psi(z^{(k)}) f_k(\varphi(z^{(k)})) - \left|\psi(z^{(k)})\right| Q_{f_k \circ \varphi}(z^{(k)}) \right| \\
&= \left| Q_\psi(z^{(k)}) \log \frac{2}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} - \frac{2\left|\psi(z^{(k)})\right|}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{H_{z^{(k)}}(u, \overline{u})^{1/2}} \right|.
\end{aligned}
$$

By Lemma 10.1.6, we have

$$
\lim_{k \to \infty} \frac{\left|\psi(z^{(k)})\right|}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\left\langle J\varphi(z^{(k)})u, \varphi(z^{(k)})\right\rangle\right|}{H_{z^{(k)}}(u, \overline{u})^{1/2}} = 0.
$$

Since $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$, it must be the case that

$$
\lim_{||\varphi(z)|| \to 1} Q_\psi(z^{(k)}) \log \frac{2}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} = 0.
$$

Using Lemma 10.1.5, and the fact that

$$
\begin{aligned}
Q_\psi(z^{(k)}) \log \frac{1 + \left|\left|\varphi(z^{(k)})\right|\right|}{1 - \left|\left|\varphi(z^{(k)})\right|\right|} &\leq Q_\psi(z^{(k)}) \log \frac{4}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} \\
&= Q_\psi(z^{(k)}) \log \frac{2}{1 - \left|\left|\varphi(z^{(k)})\right|\right|^2} + Q_\psi(z^{(k)}) \log 2,
\end{aligned}
$$

we deduce

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \frac{1+||z||}{1-||z||} = 0,$$

thus completing the proof. □

If we take ψ to be the constant function 1, then $\sigma_{\psi,\varphi} = 0$ and we have the following characterization of boundedness and compactness of C_φ on the Bloch space of the unit ball.

Corollary 10.1.8. *Let φ be a holomorphic self-map of B_n. Then C_φ is compact on $\mathcal{B}(\mathbb{B}_n)$ if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{B}_n} T_\varphi(z) = 0.$$

10.2 The Unit Polydisk

In this section, we prove Conjectures 9.1.7 and 9.3.3 on the Bloch space of $\mathbb{D}^n$. In [94], Zhou and Chen characterized the bounded and the compact weighted composition operators on $\mathcal{B}(\mathbb{D}^n)$ under the norm

$$||f||_* = |f(0)| + \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} (1-|z_k|^2) \left| \frac{\partial f}{\partial z_k}(z) \right|.$$

By Theorem 5.1.3, we have

$$Q_f(z) = \left|\left| \left((1-|z_1|^2) \frac{\partial f}{\partial z_1}(z), \ldots, (1-|z_n|^2) \frac{\partial f}{\partial z_n}(z) \right) \right|\right|,$$

and thus

$$\frac{1}{n} \sum_{k=1}^{n} (1-|z_k|^2) \left| \frac{\partial f}{\partial z_k}(z) \right| \le Q_f(z) \le \sum_{k=1}^{n} (1-|z_k|^2) \left| \frac{\partial f}{\partial z_k}(z) \right|. \tag{10.5}$$

So $||\cdot||_*$ is equivalent to the Bloch norm.

Theorem 10.2.1. [94] *Let $\psi \in H(\mathbb{D}^n)$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\sup_{z\in\mathbb{D}^n} \sum_{j,k=1}^{n} (1-|z_j|^2) \left|\frac{\partial\psi}{\partial z_j}(z)\right| \log\frac{4}{1-|\varphi_k(z)|^2} < \infty,$$

and

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^{n} \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} < \infty.$$

Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$ if and only if

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{j,k=1}^{n} (1-|z_j|^2) \left|\frac{\partial\psi}{\partial z_j}(z)\right| \log\frac{4}{1-|\varphi_k(z)|^2} = 0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^{n} \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} = 0.$$

Corollary 10.2.2. *Let φ be a holomorphic self-map of $\mathbb{D}^n$. Then C_φ is compact on $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{j,k=1}^{n} \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} = 0.$$

To prove the conjectures, we will show the conditions in the conjectures to be equivalent to the conditions in the above theorem. To this end, we will need the following lemmas.

Lemma 10.2.3. [24] *Let φ be a holomorphic self-map of $\mathbb{D}^n$. Then for $z \in \mathbb{D}^n$,*

$$B_\varphi(z) = \max_{||w||=1} \left(\sum_{k=1}^{n} \left| \sum_{j=1}^{n} \frac{\partial\varphi_k}{\partial z_j}(z) \frac{(1-|z_j|^2)w_j}{1-|\varphi_k(z)|^2} \right|^2 \right)^{1/2}.$$

Lemma 10.2.4. *Let $\psi \in H(\mathbb{D}^n)$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then for $z \in \mathbb{D}^n$, the following inequalities hold:*

(a) $\omega(\varphi(z))Q_\psi(z) \le \left(\sum_{j=1}^{n}(1-|z_j|^2)\left|\frac{\partial\psi}{\partial z_j}(z)\right|\right)\sum_{k=1}^{n}\log\frac{4}{1-|\varphi_k(z)|^2}$;

(b) $|\psi(z)|\,T_\varphi(z) \le |\psi(z)|\sum_{j,k=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(z)\right|\frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}$.

Proof. Let $z \in \mathbb{D}^n$. To prove (a), we use Theorem 5.1.4(b) to obtain

$$\omega(\varphi(z)) \le \rho(\varphi(z),0) \le \frac{1}{2}\sum_{k=1}^{n}\log\frac{1+|\varphi_k(z)|}{1-|\varphi_k(z)|} \le \sum_{k=1}^{n}\log\frac{(1+|\varphi_k(z)|)^2}{1-|\varphi_k(z)|^2} \le \sum_{k=1}^{n}\log\frac{4}{1-|\varphi_k(z)|^2}.$$

By the upper estimate of (10.5) we deduce for all $z \in \mathbb{D}^n$,

$$\omega(\varphi(z))Q_\psi(z) \le \left(\sum_{j=1}^{n}(1-|z_j|^2)\left|\frac{\partial\psi}{\partial z_j}(z)\right|\right)\sum_{k=1}^{n}\log\frac{4}{1-|\varphi_k(z)|^2}.$$

To prove (b), observe that by Lemmas 9.1.2 and 10.2.3,

$$\begin{aligned}
T_\varphi(z) \le B_\varphi(z) &= \max_{||w||=1}\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n}\frac{\partial\varphi_k}{\partial z_j}(z)\frac{(1-|z_j|^2)w_j}{1-|\varphi_k(z)|^2}\right|^2\right)^{1/2}\\
&\le \max_{||w||=1}\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(z)\right|\frac{(1-|z_j|^2)\,|w_j|}{1-|\varphi_k(z)|^2}\right)^2\right)^{1/2}\\
&\le \sum_{j,k=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(z)\right|\frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}.
\end{aligned}$$

Thus for all $z \in \mathbb{D}^n$,

$$|\psi(z)|\, T_\varphi(z) \leq |\psi(z)| \sum_{j,k=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}. \quad \square$$

Theorem 10.2.5. *Let $\psi \in H(\mathbb{D}^n)$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$ if and only if $\psi \in \mathcal{B}(\mathbb{D}^n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.*

Proof. First suppose $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$. Then $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(\mathbb{D}^n)$. By Lemma 10.2.4(b), we have

$$\tau_{\psi,\varphi} = \sup_{z\in\mathbb{D}^n} |\psi(z)|\, T_\varphi(z) \leq \sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2},$$

which is finite by Theorem 10.2.1. By Theorem 9.1.5, $\sigma_{\psi,\varphi}$ is finite as well.

Conversely, if $\psi \in \mathcal{B}(\mathbb{D}^n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then by Theorem 9.1.3, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$. $\square$

Theorem 10.2.6. *Let $\psi \in H(\mathbb{D}^n)$, and φ a holomorphic self-map of $\mathbb{D}^n$. Then the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \omega(\varphi(z))Q_\psi(z) = 0 \ \textit{and} \ \lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)|\, T_\varphi(z) = 0.$$

Proof. First suppose $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$. Then, from Lemma 10.2.4(a) and Theorem 10.2.1, we obtain

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \omega(\varphi(z))Q_\psi(z) \leq \lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{j,k=1}^{n} (1-|z_j|^2) \left|\frac{\partial \psi}{\partial z_j}(z)\right| \log \frac{4}{1-|\varphi_k(z)|^2} = 0.$$

By Lemma 10.2.4(b) and Theorem 10.2.1, we have

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)|\, T_\varphi(z) \leq \lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^{n} \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} = 0.$$

Conversely, suppose that

$$\lim_{\varphi(z)\to\partial D} \omega(\varphi(z))Q_\psi(z) = 0 \text{ and } \lim_{\varphi(z)\to\partial D} |\psi(z)|\, T_\varphi(z) = 0.$$

Then by Theorem 9.3.2, $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$. □

10.3 Examples

We end this chapter with examples which illustrate that weighted composition operators are more than the sum of their parts. These examples are inspired by the examples of Ohno and Zhao in the one-dimensional setting.

Example 10.3.1. [72] For $z \in \mathbb{D}$, let $\psi(z) = \log \frac{2}{1-z}$ and $\varphi(z) = \frac{1-z}{2}$. Then M_ψ is not bounded on $\mathcal{B}(\mathbb{D})$, but $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$.

Example 10.3.2. [72] For $z \in \mathbb{D}$, let $\psi(z) = 1 - z$ and $\varphi(z) = \frac{1+z}{2}$. Then neither M_ψ nor C_φ is compact on $\mathcal{B}(\mathbb{D})$, but $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D})$.

10.3.1 The Unit Ball

The following examples pertain to the Bloch space of the unit ball, analogous to Examples 10.3.1 and 10.3.2.

Example 10.3.3. For $\lambda \in \partial\mathbb{B}_n$ and $z \in \mathbb{B}_n$, let $\psi(z) = \frac{1}{2}\mathrm{Log}(1-\langle z, \lambda\rangle)$ and $\varphi(z) = \frac{1}{2}(\lambda - z)$. Since $\psi \notin H^\infty(\mathbb{B}_n)$, by Theorem 5.1.7, the associated multiplication operator M_ψ is not bounded.

Observe that for $z \in \mathbb{B}_n$, $\nabla(\psi)(z) = -\frac{1}{2}\frac{\overline{\lambda}}{1-\langle z,\lambda\rangle}$, so that $||\nabla(\psi)(z)|| = \frac{1}{2}\frac{1}{|1-\langle z,\lambda\rangle|}$, and so

$$(1-||z||^2)\,||\nabla(\psi)(z)||\log\frac{2}{1-||\varphi(z)||^2} = \frac{1-||z||^2}{2\,|1-\langle z,\lambda\rangle|}\log\frac{2}{1-||\frac{\lambda-z}{2}||^2}. \tag{10.6}$$

The boundedness of (10.6) is immediate for all z bounded away from $\pm\lambda$. If $z \to \lambda$, then $1-||z||^2$ approaches 0 faster than $|1-\langle z,\lambda\rangle|$ approaches 0, while the logarithmic term approaches $\log 2$, and so (10.6) is bounded. If $z \to -\lambda$, then $1-||z||^2$ approaches 0 faster than $\log(1-||\frac{\lambda-z}{2}||^2)$ goes to $-\infty$, while $|1-\langle z,\lambda\rangle|$ is bounded away from zero. Thus,

$$\sup_{z\in\mathbb{B}_n}(1-||z||^2)\,||\nabla(\psi)(z)||\log\frac{2}{1-||\varphi(z)||^2} < \infty. \tag{10.7}$$

Also, since $J\varphi(z) = -\frac{1}{2}I_n$, where I_n is the $n\times n$ identity matrix, for $u \in \mathbb{C}^n\setminus\{0\}$, we have

$$\begin{aligned}
B_\varphi(z)^2 &= \frac{H_{\varphi(z)}(J\varphi(z)u,\overline{J\varphi(z)u})}{H_z(u,\overline{u})}\\
&= \frac{(1-||\varphi(z)||^2)\,||J\varphi(z)u||^2+|\langle J\varphi(z),\varphi(z)\rangle|^2}{(1-||z||^2)\,||u||^2+|\langle u,z\rangle|^2}\frac{(1-||z||^2)^2}{(1-||\varphi(z)||^2)^2}\\
&= \frac{1}{4}\frac{(1-||\frac{\lambda-z}{2}||^2)\,||u||^2+\left|\left\langle u,\frac{\lambda-z}{2}\right\rangle\right|^2}{(1-||z||^2)\,||u||^2+|\langle u,z\rangle|^2}\frac{\left(1-||z||^2\right)^2}{\left(1-||\frac{\lambda-z}{2}||^2\right)^2}.
\end{aligned} \tag{10.8}$$

As z approaches λ, (10.8) yields

$$|\psi(z)|\,B_\varphi(z)^2 = |\psi(z)|\,\frac{H_{\varphi(z)}(J\varphi(z)u,\overline{J\varphi(z)u})}{H_z(u,\overline{u})} = O\left(|\mathrm{Log}(1-\langle z,\lambda\rangle)|\,(1-||z||^2)\right).$$

Since $1-||z||^2$ goes to 0 as $z\to\lambda$ faster than $\mathrm{Log}(1-\langle z,\lambda\rangle)$ goes to ∞, $|\psi(z)|\,(B_\varphi(z))^2 < \infty$

for all $z \in \mathbb{B}_n$. Thus

$$\sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z) < \infty.$$

Therefore, by (10.7) and Theorem 10.1.1, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$.

Example 10.3.4. For $z \in \mathbb{B}_n$, let $\psi(z) = 1 - z_1$ and $\varphi(z) = \left(\frac{1+z_1}{2}, \frac{z_2}{2}, \ldots, \frac{z_n}{2}\right)$. Since ψ is not identically zero, by Theorem 5.4.1 the associated multiplication operator M_ψ is not compact on $\mathcal{B}(\mathbb{B}_n)$. Observe that $||\varphi(z)|| \to 1$ only when $z \to (1, 0, \ldots, 0)$, i.e., $z_1 \to 1$ and $z_j \to 0$ for $j \neq 1$. We now show that C_φ is not compact by proving that there exists $u \in \mathbb{C}^n \setminus \{0\}$ such that $\dfrac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})}{H_z(u,\overline{u})}$ is bounded away from zero as $||\varphi(z)|| \to 1$ (see Theorem 10.2.2). Observe that $J\varphi(z) = \frac{1}{2}I_n$ for each $z \in \mathbb{B}_n$. Set $u = (1, 0, \ldots, 0)$. Then

$$\frac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})}{H_z(u,\overline{u})} = \frac{\frac{1}{4}(1 - ||\varphi(z)||^2) + \frac{1}{4}\left|\frac{1+z_1}{2}\right|^2}{1 - ||z||^2 + |z_1|^2} \frac{(1-||z||^2)^2}{(1-||\varphi(z)||^2)^2}.$$

Thus, as $z_1 \to 1$ and $z_j \to 0$ for $j \neq 1$, we have

$$\begin{aligned}
\lim_{z\to(1,0,\ldots,0)} \frac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})}{H_z(u,\overline{u})} &= \frac{1}{4} \lim_{z\to(1,0,\ldots,0)} \frac{(1-||z||^2)^2}{(1-||\varphi(z)||^2)^2} \\
&= \frac{1}{4} \lim_{z_1\to 1} \left(\frac{1-|z_1|^2}{1-\left|\frac{1+z_1}{2}\right|^2}\right)^2 \\
&= \lim_{\substack{x\to 1\\ y\to 0}} \frac{4(1-x^2-y^2)^2}{(3-2x-x^2-y^2)^2} = 1.
\end{aligned}$$

Therefore, the associated composition operator C_φ is not compact on $\mathcal{B}(\mathbb{B}_n)$.

Furthermore, since $\psi(z) \to 0$, when $||\varphi(z)|| \to 1$ and $B_\varphi(z)$ is bounded above by a constant independent of φ, we deduce

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, B_\varphi(z) = 0.$$

Also, since $\nabla(\psi)(z) = -1$, we have

$$(1-||z||^2)\,||\nabla(\psi)(z)||\log\frac{2}{1-||\varphi(z)||^2} = (1-||z||^2)\log\frac{2}{1-||\varphi(z)||^2}.$$

Since $1-||z||^2$ approaches 0 faster than $\log(1-||\varphi(z)||^2)$ approaches $-\infty$, we have

$$\lim_{||\varphi(z)||\to 1}(1-||z||^2)\,||\nabla(\psi)(z)||\log\frac{2}{1-||\varphi(z)||^2} = 0.$$

Therefore, by Theorem 10.1.1, $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$.

10.3.2 The Unit Polydisk

The following examples pertain to the Bloch space of the unit polydisk, analogous to Examples 10.3.1 and 10.3.2.

Example 10.3.5. Fix an index $j \in \{1, \ldots, n\}$ and, for $z \in \mathbb{D}^n$, define $\psi(z) = \mathrm{Log}\frac{2}{1-z_j}$ and $\varphi(z)$ be the vector with k^{th} component 0 for $k \neq j$ and j^{th} component $\frac{1-z_j}{2}$. Since $\psi \notin H^\infty(\mathbb{D}^n)$, by Theorem 5.1.7, the associated multiplication operator M_ψ is not bounded on $\mathcal{B}(\mathbb{D}^n)$.

Since $\frac{\partial\psi}{\partial z_k}(z) = 0$ for $k \neq j$ and $\frac{\partial\psi}{\partial z_j}(z) = \frac{1}{1-z_j}$, then

$$\sum_{k,\ell=1}^{n}(1-|z_\ell|^2)\left|\frac{\partial\psi}{\partial z_\ell}(z)\right|\log\frac{4}{1-|\varphi_k(z)|^2} = \frac{1-|z_j|^2}{|1-z_j|}\log\frac{4}{1-\left|\frac{1-z_j}{2}\right|^2}.$$

The boundedness of the above quantity is immediate for z_j bounded away from ± 1. As $z_j \to -1$, $1-|z_j|^2$ goes to 0 faster than $\log \frac{4}{1-\left|\frac{1-z_j}{2}\right|^2}$ goes to ∞, while $|1-z_j| \to 2$. On the other hand, as $z_j \to 1$,

$$\frac{1-|z_j|^2}{|1-z_j|} \leq 1+|z_j| \leq 2,$$

while $\log \frac{4}{1-\left|\frac{1-z_j}{2}\right|^2} \to \log 4$. Therefore

$$\sup_{z\in\mathbb{D}^n} \frac{1-|z_j|^2}{|1-z_j|} \log \frac{4}{1-\left|\frac{1-z_j}{2}\right|^2} < \infty.$$

Since $\frac{\partial \varphi_k}{\partial z_\ell}(z) = 0$ for k or ℓ unequal to j and $\frac{\partial \varphi_j}{\partial z_j}(z) = -\frac{1}{2}$, we obtain

$$|\psi(z)| \sum_{k,\ell=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_\ell}(z)\right| \frac{1-|z_\ell|^2}{1-|\varphi_k(z)|^2} = \frac{1}{2}\left|\text{Log}\frac{2}{1-z_j}\right| \frac{1-|z_j|^2}{1-\left|\frac{1-z_j}{2}\right|^2}$$

$$\leq \frac{1}{2}\frac{1-|z_j|^2}{1-\left|\frac{1-z_j}{2}\right|^2}\left(\log\frac{2}{|1-z_j|}+\frac{\pi}{2}\right).$$

The boundedness of the above quantity is immediate for z_j bounded away from ± 1. By the same argument as before, the above quantity is bounded also as z approaches ± 1. Therefore

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{k,\ell=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_\ell}(z)\right| \frac{1-|z_\ell|^2}{1-|\varphi_k(z)|^2} < \infty.$$

Therefore, by Theorem 10.2.1, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$.

Example 10.3.6. Fix an index $j \in \{1,\ldots,n\}$, and define $\psi(z) = 1 - z_j$ and let $\varphi(z)$ be the vector with k^{th} component 0 for $k \neq j$ and j^{th} component $\frac{1+z_j}{2}$. Since $\frac{\partial \varphi_k}{\partial z_\ell}(z) = 0$ for

$k, \ell \neq j$, $\frac{\partial \varphi_j}{\partial z_j}(z) = \frac{1}{2}$, and $|\varphi_j(z)| = \left|\frac{1+z_j}{2}\right|$, we have

$$\sum_{k,\ell=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_\ell}(z)\right| \frac{1-|z_\ell|^2}{1-|\varphi_k(z)|^2} = \frac{1}{2}\left(\frac{1-|z_j|^2}{1-\left|\frac{1+z_j}{2}\right|^2}\right). \tag{10.9}$$

Arguing as in Example 10.3.4 we see that the right hand side of (10.9) is bounded away from 0 as $z_j \to 1$. Thus, by Corollary 10.2.2, C_φ is not compact on $\mathcal{B}(\mathbb{D}^n)$.

Since $\frac{\partial \psi}{\partial z_k}(z) = 0$ for $k \neq j$ and $\frac{\partial \psi}{\partial z_j}(z) = -1$,

$$\sum_{k,\ell=1}^{n} (1-|z_\ell|^2) \left|\frac{\partial \psi}{\partial z_\ell}(z)\right| \log \frac{4}{1-|\varphi_k(z)|^2} = (1-|z_j|^2) \log \frac{4}{1-\left|\frac{1+z_j}{2}\right|^2}.$$

As $z_j \to 1$, $1-|z_j|^2$ approaches 0 faster than $\log(1-\left|\frac{1+z_j}{2}\right|^2)$ approaches $-\infty$. So

$$\lim_{z_j \to 1} \sum_{k,\ell=1}^{n} (1-|z_\ell|^2) \left|\frac{\partial \psi}{\partial z_\ell}(z)\right| \log \frac{4}{1-|\varphi_k(z)|^2} = 0.$$

Since $\frac{\partial \varphi_k}{\partial \varphi_\ell}(z) = 0$ for k or ℓ unequal to j and $\varphi(z) \to \partial\mathbb{D}^n$ precisely when $z_j \to 1$, we see that $\frac{\partial \varphi_j}{\partial z_j}(z) = \frac{1}{2}$, then

$$|\psi(z)| \sum_{k,\ell=1}^{n} \left|\frac{\partial \varphi_k}{\partial z_\ell}(z)\right| \frac{1-|z_\ell|^2}{1-|\varphi_k(z)|^2} = \frac{|1-z_j|}{2}\left(\frac{1-|z_j|^2}{1-\left|\frac{1+z_j}{2}\right|^2}\right) \to 0$$

as $\varphi(z) \to \partial\mathbb{D}^n$. Therefore, by Theorem 10.2.1, $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$.

Chapter 11: Further Questions

Although the work in this dissertation answers many questions on operators acting on the Bloch space of a bounded homogeneous domain, along the way many questions have been raised. In this chapter, we will discuss some questions which deserve further investigation, as well as new questions which have not been considered yet.

11.1 Further Developments on the Bloch Space

In this section, we will discuss the results concerning the isometric multiplication operators and the bounded weighted composition operators on bounded homogeneous domains.

11.1.1 Isometries

The results from Theorem 4.3.6 and Theorem 5.5.4 characterize the isometric multiplication operators on the Bloch space of the unit disk and the Bloch space of a bounded symmetric domain which does not contain the unit disk as a factor. At first glance, these two results seem to be at opposite ends of the spectrum. However, the proofs of both theorems rely on a similar concept.

To prove there are no non-trivial isometric multiplication operator M_ψ on the Bloch space of the unit disk, we showed that $\beta_{\psi^k} < 1$ for all k. For the Bloch space of a bounded symmetric domain, we showed that $\beta_{\psi^k} \leq c_D$ where c_D is the Bloch constant of the domain. In order to ensure $\beta_{\psi^k} < 1$, we must remove the unit disk as a factor, which would make $c_D = 1$.

Thus, it seems to be the idea to determine a quantity strictly less than one which bounds β_ψ in the case of the Bloch space on a bounded homogeneous domain. This, however, does not seem to be an easy task. Although the Bloch constant is defined for a bounded homogeneous domain, the exact value, or even a usable upper bound, has not been determined. So, it is not clear whether the arguments in the proofs of Theorems 4.3.6 and 5.5.4 can be directly extended to the bounded homogeneous domain case.

11.1.2 Characterization of Bounded Weighted Composition Operators on Bounded Homogeneous Domains

The task of characterizing the bounded weighted composition operators on the Bloch space of a bounded homogeneous domain resulted in good news and bad news. The good news, necessary conditions (Theorem 9.1.5) and sufficient conditions (Theorem 9.1.3) were established. The bad news, the conditions have not been proven to be necessary and sufficient. Since we can show the conditions are necessary and sufficient in the case of the unit ball and the unit polydisk, we conjecture these conditions to be necessary and sufficient in general.

The underlying obstacle in showing these conditions are in fact a characterization is the inability to choose test functions. In the case of the unit ball and unit polydisk, we can select appropriate test functions involving logarithms. On a general bounded homogeneous domain, the only Bloch functions which we have available for use are polynomials and compositions of logarithmic functions and projection maps. Although the bounded symmetric domains have more structure, we are not aware of other test functions which may be helpful in our quest. Also, since the bounded symmetric domains are defined in terms of matrices, functions on such domains are described in terms of matrices as well, and thus are very difficult to work with.

In an attempt to alleviate some of the above problems, it may be beneficial to consider a different type of bounded homogeneous domain, called a Siegel domain. The Siegel domains have a more developed function theory [92], and the hope is that having a larger set of functions to choose from may assist in the proof that the necessary and the sufficient conditions of boundedness are, in fact, necessary and sufficient.

11.2 New Developments on Other Spaces

The work in this dissertation focused on operators acting on the Bloch space. However, this is not the only space of interest to operator theorists. As with the research on the Bloch space, when working in higher dimensions, either the unit ball or the unit polydisk is considered. It will be advantageous to define other spaces on bounded homogeneous or bounded symmetric domains, and consider the same issues raised in this dissertation for the Bloch space.

Several spaces have been defined on bounded symmetric domains, including the Hardy space [46], and the Besov space [98] and [99]. With the definitions of such spaces on bounded symmetric domains, a unification of the operator theory of such spaces is an interesting area of research.

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Curriculum Vitae

Robert F. Allen graduated from T.C. Williams High School, Alexandria, Virginia in 1990. He received his Bachelor of Science in Computer Science from the University of Virginia in 1994. He won the Computer Science Undergraduate Education Award at graduation. He was employed as a software engineer for 9 years, working for government contractors in the Washington D.C. area.

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