EXTREMAL COMBINATORICS IN GEOMETRY AND GRAPH THEORY

by

Jonathan E. Beagley
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Committee:

_______________________ Dr. Walter Morris, Dissertation Director

_______________________ Dr. Valeriu Soltan, Committee Member

_______________________ Dr. Geir Agnarsson, Committee Member

_______________________ Dr. Dana Richards, Committee Member

_______________________ Dr. Stephen Saperstone, Department Chair

_______________________ Dr. Timothy L. Born, Associate Dean for Student and Academic Affairs, College of Science

_______________________ Dr. Vikas Chandhoke, Dean, College of Science

Date: _______________________ Spring Semester 2013
George Mason University
Fairfax, VA

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at George Mason University

By

Jonathan E. Beagley
Master of Science
George Mason University, 2011
Bachelor of Science
Illinois Institute of Technology, 2004

Director: Dr. Walter Morris, Professor
Department of Mathematical Sciences

Spring Semester 2013
George Mason University
Fairfax, VA

Dedication

I dedicate this dissertation to my wife Callie. Her patience and understanding while completing this research made it much easier. I also dedicate this work to my parents Janice and the late David, and to my late grandfather Gerald who encouraged my curiosity in mathematics from a very young age.

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Abstract

EXTREMAL COMBINATORICS IN GEOMETRY AND GRAPH THEORY

Jonathan E. Beagley, PhD

George Mason University, 2013

Dissertation Director: Dr. Walter Morris

We study a problem in extremal geometry posed by Paul Erdős and George Szekeres in 1935. This problem is to find the smallest positive integer $N(n)$ such that every point set in general position (no three on a line) of $N(n)$ points contains the vertex set of a convex n-gon. Erdős and Szekeres showed that $N(n)$ exists and conjectured that $N(n) = 2^{n-2} + 1$. In 2006, Walter Morris introduced a graph on the copoints of a planar point set in general position, where cliques in the graph correspond to subsets of points in convex position, and showed that the chromatic number of the copoint graph was n if the point set contained at least $2^{n-2} + 1$ points. We extend this copoint graph to abstract convex geometries studied by Edelman and Jamison, where the cliques of this graph are convexly independent sets. A major goal of this dissertation is to study the clique and chromatic numbers for the copoint graph of convex geometries. Much of this dissertation would be trivial if every graph were a copoint graph, so we provide a family of graphs that are not copoint graphs for any convex geometries.

We begin by relating this copoint graph of Morris to the strict graph of critical pairs from the theory of posets. The chromatic number of the strict graph of critical pairs was shown by Felsner and Trotter to be a lower bound for the order dimension of the poset.

After establishing this relationship between these two graphs, we prove that the chromatic number of the copoint graph is non-increasing under minors. We introduce an operation for constructing a new convex geometry from two smaller convex geometries and use this construction to show that the difference between clique number and chromatic number of the copoint graph can become arbitrarily large as for any graph.

We construct a family of copoint graphs for which the ratio of the chromatic number to the clique number can be arbitrarily large. For any natural numbers $1 < d < k$, we study the existence of a number $K_d(k)$ so that the chromatic number of the copoint graph of a convex geometry on a set of at least $K_d(k)$ elements, with every d-element subset closed, has chromatic number at least k. This problem of studying $K_d(k)$ is related to the order dimension of the complete graph.

Finally, we conclude by stating several properties of the copoint graph of a convex geometry for convex geometries realized by planar point sets in general position. We prove an Erdős-Szekeres type result in which for each n we describe a planar point set in general position with order dimension n and show that every point set of larger size has order dimension $n + 1$. We provide computational results detailing the difference of chromatic number and clique number for the copoint graph of planar point sets in general position for sets of size 10 or less. Lastly, we show results for characterizing the hyperedges of the strict hypergraph of critical pairs for a convex geometry.

Chapter 1: Introduction

1.1 A Problem of Erdős and Szekeres

The motivation of this dissertation comes from a problem of Erdős and Szekeres posed in 1935: for any $n \geq 3$, to determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position (no three on a line) in the plane contains n points that are the vertices of a convex n-gon. Morris and Soltan [MS00] surveyed results related to this problem. It was established by Erdős and Szekeres [ES35],[ES61] that $N(n)$ exists and $N(n) > 2^{n-2}$ and conjectured that $N(n) = 2^{n-2} + 1$. It is currently known that $N(n) \leq \binom{2n-5}{n-2} + 1$ when $n \geq 5$, due to Tóth and Valtr [TV04]. It is worth noting that exact values for $N(n)$ are only known for $n = 3, 4, 5$, and 6 [MS00], [SP06] and are equal to $2^{n-2} + 1$ is these cases.

The problem of Erdős and Szekeres is sometimes referred as the Happy Ending Problem. The name "Happy Ending Problem" is due to the marriage of Esther Klein and George Szekeres after Erdős and Szekeres' solution was published. The problem was initiated by Esther Klein, who observed that any set of five points in general position in the plane contains four points that are the vertices of a convex 4-gon. There are three distinct types of placement of five points in the plane, with no three on a line, as shown in the Figure 1.1. In each of these cases, there is at least one convex 4-gon determined by the points.

Klein suggested the more general problem on the existence of a finite number $N(n)$ such that from any set containing at least $N(n)$ points in general position in the plane, there are n points forming a convex polygon. There have been numerous generalizations of this problem, related to higher dimensions, families of convex bodies, and an abstract convex setting. We explore the abstract convex setting of the Erdős-Szekeres problem to

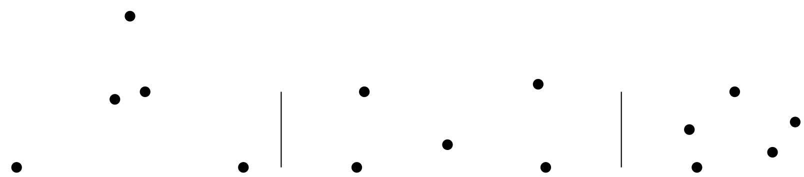

Figure 1.1: Three Distinct Order Types of Five Points in General Position

gain insight into the original problem.

The rest of this chapter is devoted to introducing the concepts and tools used throughout this dissertation. We introduce abstract convex geometries and a graph on specific closed sets originally developed by Morris [Mor06] referred to as the "copoint graph". The copoint graph will be central to the investigations in Chapters 2, 3, and 4.

Chapter 2 starts by completely describing which closed sets of a convex geometry are critical to understanding the poset structure of the collection of closed sets, partially ordered by inclusion. We also describe an inequality relating the chromatic number of the copoint graph and the "order dimension" of the lattice of closed sets for a convex geometry in Corollary 2.6, which appears in [Beaar]. It is also shown in Chapter 2 that the chromatic number of the copoint graph is monotone non-increasing under minors. We conclude by describing a "direct sum" of convex geometries. The copoint graph of a direct sum of convex geometries can be completely described in terms of the copoint graphs of its component convex geometries, which appears in [Bea].

In Chapter 3, we answer a question posed by Beagley [Beaar] by showing that there is a convex geometry where the ratio of the chromatic number to the clique number of the copoint graph is larger than any constant in Theorem 3.6. We continue by proving a result in the spirit of Esther Klein's for abstract convex geometries, and use this result to propose a new problem, related to the Erdős-Szekeres problem. We loosen the restriction of the convex geometry being "realized" by a set of planar points, but keep an abstraction of general position for Corollary 3.15. This appears in [BM].

Chapter 4 shows results to enhance the study of the copoint graph and some computational results related to planar point sets in general position. We prove in Theorem 4.7 that the planar point sets of Erdős and Szekeres of 2^{n-2} points in general position containing no vertex set of a convex n-gon have order dimension $n-1$ and any larger point set has order dimension at least n. We conclude by investigating the possible difference between the order dimension of the lattice of closed sets and the chromatic number of the copoint graph.

1.2 Convex Geometries

Let X be a finite set and $\mathscr{L}$ be a collection of subsets of X with the properties: $\emptyset \in \mathscr{L}$, $X \in \mathscr{L}$, and $A \cap B \in \mathscr{L}$ whenever $A, B \in \mathscr{L}$. Then $\mathscr{L}$ is called an *alignment* on X. Following the example of Edelman and Jamison [EJ85], we also view $\mathscr{L}$ as a *closure operator*. For any subset A of X we define the closure of A, $\mathscr{L}(A)$, to be the intersection of all $C \in \mathscr{L}$ such that $A \subseteq C$. The subsets in $\mathscr{L}$, or equivalently those subsets of X of the form $\mathscr{L}(A)$ for some subset A of X, are called *closed* or *convex*. We say that $\mathscr{L}$ is *anti-exchange* if, given any set $C \in \mathscr{L}$ and two distinct points p and q in X, neither in C, then $q \in \mathscr{L}(C \cup p)$ implies that $p \notin \mathscr{L}(C \cup q)$.

Definition 1.1. *Let X be a finite set. A pair $(X, \mathscr{L})$ is a convex geometry if:*

1. *$\mathscr{L}$ is an alignment on X, and*
2. *$\mathscr{L}$ is anti-exchange.*

Edelman and Jamison [EJ85] presented several equivalent definitions of convex geometry. We use the equivalent statement Theorem 1.2 in the proof of Proposition 2.2 and we use Theorem 1.3 throughout Chapters 3 and 4.

Theorem 1.2 ([EJ85], 2.2). *A pair $(X, \mathscr{L})$ is a convex geometry if and only if $\mathscr{L}$ is an alignment on X and every maximal chain of convex sets of $\emptyset \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq X$ has the same length $|X|$.*

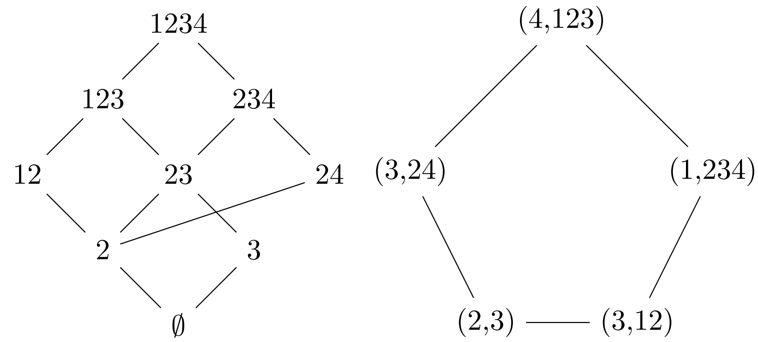

Figure 1.2: The lattice of closed sets for a convex geometry with its copoint graph

Theorem 1.3 ([EJ85], 2.1)**.** *A pair $(X, \mathscr{L})$ is a convex geometry if and only if $\mathscr{L}$ is an alignment on X and for every $C \in \mathscr{L}$, $C \neq X$, there exists a $p \in X$ such that $C \cup p \in \mathscr{L}$.*

For an alignment $(X, \mathscr{L})$ we denote by $L_{\mathscr{L}} = (\mathscr{L}, \subseteq)$ the partial order on $\mathscr{L}$ by containment. This partial order is a lattice, where $A \wedge B = A \cap B$ and $A \vee B = \mathscr{L}(A \cup B)$. A set $C \in \mathscr{L}$ is a *copoint* if it is maximal in $X - p$ for some $p \in X$. If C is a copoint, there is exactly one set in $\mathscr{L}$ of the form $C \cup p$ for $p \notin C$. The unique p is denoted $\alpha(C)$, and we say that the copoint C is *attached* to $\alpha(C)$. We will sometimes refer to a copoint C by the pair $(\alpha(C), C)$. The set of copoints, partially ordered by inclusion, is denoted $M(X)$. We say $B \subseteq X$ is *independent* if for all $p \in B$, $p \notin \mathscr{L}(B - p)$. We denote the size of the largest independent set by $b(L_{\mathscr{L}})$. A convex geometry is said to *d-free* if every d-element subset of X is closed. For an element p of X, we write $\mathscr{L}(p)$ instead of $\mathscr{L}(\{p\})$. We note that $\mathscr{L}(p) \neq p$ in general, as we see in Figure 1.2.

Definition 1.4. *Let X be a finite set of points in $\mathbb{R}^d$. The convex geometry $(X, \mathscr{L})$ realized by X is defined by $\mathscr{L}(A) = conv(A) \cap X$ for all $A \subseteq X$.*

Morris [Mor06] showed that an independent set of size k in the convex geometry realized by a set of points in general position in $\mathbb{R}^2$ corresponds to the vertex set of a convex k-gon. The Erdős-Szekeres Conjecture, that $2^{n-2} + 1$ points in the plane in general position contain the vertex set of a convex n-gon, equivalently stated is: let convex geometry $(X, \mathscr{L})$ be realized by a planar point set in general position. Then $|X| > 2^{n-2}$ implies $b(L_{\mathscr{L}}) \geq n$.

1.2.1 Graph of Copoints

We define $\mathcal{G}(X, \mathscr{L})$ to be the *graph of copoints* or *copoint graph* with vertex set $M(X)$ and $\{A, B\} \in E(\mathcal{G}(X, \mathscr{L}))$, the edge set of $\mathcal{G}(X, \mathscr{L})$, if and only if $\alpha(A) \in B$ and $\alpha(B) \in A$. $\mathcal{G}(X, \mathscr{L})$ was described and studied by Morris in [Mor06] for convex geometries realizable by planar point sets in general position. This graph can be created for any convex geometry, not merely those realized by planar point sets. We see an example of a convex geometry with its copoint graph in Figure 1.2. A *k-clique* is a graph G which has an edge between any two distinct vertices of G with $|V(G)| = k$, this is also called a complete graph on k vertices. The *clique number* of a graph G, denoted $\omega(G)$, is the size of the largest complete graph contained in G. It follows directly from Proposition 1.5 that $\omega(\mathcal{G}(X, \mathscr{L})) = b(L_{\mathscr{L}})$ for the copoint graph.

Proposition 1.5. *If Y is a k-clique in $\mathcal{G}(X, \mathscr{L})$, then $\{\alpha(A) : A \in Y\}$ is an independent set of size k in $(X, \mathscr{L})$. If P is an independent set of size k in $(X, \mathscr{L})$, then there is a k-clique Y in $\mathcal{G}(X, \mathscr{L})$ so that $P = \{\alpha(A) : A \in Y\}$.*

Proof. Suppose Y is a k-clique of $\mathcal{G}(X, \mathscr{L})$. If $A \in Y$, then $\{\alpha(B) : B \in Y \backslash \{A\}\} \subseteq A$, because Y is a clique. Thus, since $\mathscr{L}(A) = A$ and $\alpha(A) \notin A$ we see that $\alpha(A)$ is not in $\mathscr{L}(\{\alpha(B) : B \in Y\} \backslash \alpha(A))$. So, $\{\alpha(B) : B \in Y\}$ is an independent set of size k. Next, suppose P is an independent set of size k in $(X, \mathscr{L})$. For each $x \in P$, let $A(x)$ be a copoint attached to x containing $P \backslash \{x\}$. Then $\{A(x) : x \in P\}$ is a k-clique in $\mathcal{G}(X, \mathscr{L})$. □

Let G be a graph. An *induced* subgraph H of G, has the properties that $V(H) \subseteq V(G)$ and for all $v, u \in V(H)$, one has $\{v, u\} \in E(H)$ if and only if $\{v, u\} \in E(G)$. A function $f : V(G) \to [n]$ is said to be a *proper coloring of* G if f has the property that $f(A) \neq f(B)$ whenever $\{A, B\} \in E(G)$. The smallest n for which there is a proper coloring of G is called the *chromatic number* of G, and denoted $\chi(G)$. As a complete subgraph on n vertices requires n colors, it follows that $\omega(G) \leq \chi(G)$ for all graphs. Much of graph theory has been devoted to understanding the difference between $\omega(G)$ and $\chi(G)$. One famous recent example of this is the Strong Perfect Graph Theorem of Chudnovsky et. al [CRST06] that says $\omega(H) = \chi(H)$ for every induced subgraph H of a graph G if and only if G contains no odd hole or odd-antihole. A hole is an induced cycle of size 4 or more, while an antihole is an induced subgraph that is the complement of a cycle of size 4 or more. Morris also showed that if X is a planar point set in general position with $\chi(\mathcal{G}(X, \mathscr{L})) = k$, then $|X| \leq 2^{k-1}$. This result is stated and applied in Chapter 2 as Theorem 4.6. If we are able to fully understand the difference between the $\omega(\mathcal{G}(X, \mathscr{L}))$ and $\chi(\mathcal{G}(X, \mathscr{L}))$ for planar point sets in general position, this theorem of Morris would be important for resolving the Erdős-Szekeres Conjecture.

Much of the discussion in this dissertation would just be restating many results long known in graph theory if not for one fact: there are graphs that are not copoint graphs. We establish this by showing that the cycle on 6 or more vertices is not a copoint graph for any convex geometry.

Theorem 1.6. *Let G be a cycle on 6 or more vertices. There is no convex geometry $(X, \mathscr{L})$ such that G is isomorphic to $\mathcal{G}(X, \mathscr{L})$.*

Proof. Suppose there is a convex geometry with $\mathcal{G}(X, \mathscr{L})$ a cycle of length 6 or more. The cycle has clique number 2, so the size of the largest independent set of $(X, \mathscr{L})$ is 2. $(X, \mathscr{L})$ cannot have only one copoint of size $n-1$, and still have $\mathcal{G}(X, \mathscr{L})$ be connected, so it has two copoints of size $n-1$ which form a 2-clique. Let these copoints of size $n-1$ be A and B, where $A = \{1, 2, ..., n-1\}$ and $B = \{1, 2, ..., n-2, n\}$. In the graph $\mathcal{G}(X, \mathscr{L})$, A and B are adjacent, so there are two more copoints A_1 and B_1 where A_1 is adjacent to A (and not B) and B_1 is adjacent to B (and not A). We know that A_1 contains n but not $n-1$. Every maximal chain in $L_{\mathscr{L}}$ containing A_1 must contain B and an $(n-2)$-element subset of B. This $(n-2)$-element subset must be a copoint adjacent to A, so it must be A_1. Similarly, B_1

must be an $(n-2)$-element subset of A. Suppose that A_1 and B_1 are attached to different points, this means that $\alpha(A_1) \in B_1$ and $\alpha(B_1) \in A_1$, and we have a cycle of size 4 in $\mathcal{G}(X, \mathscr{L})$. Thus, A_1 and B_1 must be attached to the same point; without loss of generality, let this point be $n-2$. We know that $A_1 = \{1, 2, ...n-3, n\}$ and $B_1 = \{1, 2, ...n-3, n-1\}$. Because $\mathcal{G}(X, \mathscr{L})$ is a cycle of length 6 or more, there are at least 2 more copoints, A_2 and B_2, such that A_2 is adjacent to A_1 (but not A, B, B_1) and B_2 is adjacent to B_1 (but not A, B, A_1). Thus, A_2 and B_2 cannot contain $n-1$ or n, or they would be adjacent to A or B in $\mathcal{G}(X, \mathscr{L})$. So, A_2 and B_2 are subsets of $\{1, 2, ..., n-2\}$ that must contain $n-2$ as they are adjacent to A_1 and B_1 respectively. Consider A_2; this copoint must be incomparable with A_1 and B_1 as it contains $n-2$. So there is a closed set $C \supseteq A_2$ such that C is of size $n-3$ with $C \subset \{1, 2, \ldots, n-2\}$ and without loss of generality, let $C = \{2, 3, ...n-2\}$. C is a copoint attached to 1, that contains $n-2$, so C is adjacent to both A_1 and B_1, forming a cycle of size 5 in the $\mathcal{G}(X, \mathscr{L})$. Thus, there is no convex geometry for which $\mathcal{G}(X, \mathscr{L})$ is a cycle of length 6 or higher. □

1.3 Order Dimension of Partially Ordered Sets

Let $P = (X, \leq)$ be a partially ordered set or *poset* where X is a set and $\leq$ is a reflexive, antisymmetric, and transitive binary relation on X. Two elements $x, y \in X$ are *incomparable* if neither $x \leq y$ nor $y \leq x$. A poset P is called a *chain* if every pair of distinct elements from X is comparable. Similarly, if every pair of distinct elements from X is incomparable in P, then P is called an *antichain*. For Y, a nonempty subset of X, the restriction of P to Y, denoted $P(Y)$, is a partial order on Y and we call $(Y, \leq)$ a *subposet* of $(X, \leq)$. A nonempty subset $Y \subseteq X$ is called a *chain* or *antichain* if the subposet $(Y, \leq)$ is a chain or antichain respectively. When P and Q are posets on the same set X, Q is called an *extension* of P if $x \leq y$ in P implies $x \leq y$ in Q. The maximal extensions of P are linear orders on X and are called *linear extensions*.

When $P = (X, \leq)$ is a poset, the *order dimension* of P, denoted $\dim(P)$, is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P such that $P = \cap\mathcal{R} = \bigcap_{i=1}^{t} L_i$. Any family of linear extensions $\mathcal{R}$ where $P = \cap\mathcal{R}$ is called a *realizer* of P. Many applications of order dimension are discussed by Trotter [Tro92].

An ordered pair, $(x, y) \in X \times X$, of incomparable elements of P is called a *critical pair* if for all $a, b \in X$, $a \leq x$ implies that $a \leq y$ and $y \leq b$ implies $x \leq b$. It was shown by Rabinovitch and Rival [RR79] that $\mathcal{R}$ is a realizer of P if and only if for every critical pair (x, y), there is some $L \in \mathcal{R}$ for which $y \leq x$ in L.

Chapter 2: Order Dimension of Convex Geometries

2.1 Copoints and Critical Pairs

We are able to conclude the following connection between copoints of a convex geometry and critical pairs of its lattice of closed sets.

Theorem 2.1. *Let $(X, \mathscr{L})$ be a convex geometry. For $A, B \in \mathscr{L}$, (A, B) is a critical pair of $L_{\mathscr{L}}$ if and only if B is a copoint, $A = \mathscr{L}(\alpha(B))$ and A is incomparable with B.*

Proof. Let (A, B) be a critical pair of the lattice $L_{\mathscr{L}}$. Suppose that B is not a copoint. Then there are distinct $p, q \in X - B$ such that $B \cup p$ and $B \cup q$ are in $\mathscr{L}$. Since $B \cup p$ and $B \cup q$ both properly contain B, they must also properly contain A. The assumption that p and q are distinct implies $A \subseteq (B \cup p) \cap (B \cup q) = B$, which contradicts A and B being incomparable. Thus, B must be a copoint. The set $B \cup \alpha(B)$ properly contains B, so it properly contains A. Because A is incomparable with B it follows that $\alpha(B) \in A$. Because $\mathscr{L}(\alpha(B)) \subseteq A$ and $\mathscr{L}(\alpha(B)) \nsubseteq B$, we have $\mathscr{L}(\alpha(B)) = A$.

Suppose B is a copoint attached to $\alpha(B) \in X$, $A = \mathscr{L}(\alpha(B))$, and A is incomparable with B. Any $q \in A$, $q \neq \alpha(B)$ must be in B because $A = \mathscr{L}(\alpha(B)) \subseteq B \cup \alpha(B)$. Therefore, $D \subset A$ means that $D \subset B$. Since B is a copoint, $U \supset B$ implies that $U \supseteq B \cup \alpha(B)$. This means that $U \supseteq \mathscr{L}(\alpha(B)) = A$ and $A \neq B \cup \alpha(B)$ because A and B are incomparable. Thus, (A, B) is a critical pair of $L_{\mathscr{L}}$. □

In order to ensure that $(\mathscr{L}(\alpha(B)), B)$ is a critical pair, it is necessary for $\mathscr{L}(\alpha(B))$ to be incomparable with B to eliminate the possibility $\mathscr{L}(\alpha(B)) = B \cup \alpha(B)$. In Proposition 2.2, we describe the lattice of closed sets for those convex geometries with copoints B such that B is comparable with $\mathscr{L}(\alpha(B))$.

Proposition 2.2. *Let $(X, \mathscr{L})$ be a convex geometry with copoint B. If B is comparable with $\mathscr{L}(\alpha(B))$, then $\mathscr{L}(\alpha(B)) = B \cup \alpha(B)$. Furthermore, every maximal chain from $\emptyset$ to X in $L_{\mathscr{L}}$ contains B and $B \cup \alpha(B)$.*

Proof. Suppose B is a copoint of $(X, \mathscr{L})$ comparable to $\mathscr{L}(\alpha(B))$. We know that $\mathscr{L}(\alpha(B)) \subseteq B \cup \alpha(B)$ for any copoint B. Since B is comparable with $\mathscr{L}(\alpha(B))$ and B does not contain $\alpha(B)$, $\mathscr{L}(\alpha(B)) = B \cup \alpha(B)$. Let $\emptyset = C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_k = X$ be a maximal chain in $L_{\mathscr{L}}$. We know from Theorem 1.2 that this maximal chain has length $|X|$. So there is some $C_i \subset C_{i+1}$ where $C_i \cup \alpha(B) = C_{i+1}$. This means that $\mathscr{L}(\alpha(B)) \subseteq C_{i+1}$ and $B \subseteq C_i$. If B is properly contained in C_i, then $\mathscr{L}(\alpha(B)) = B \cup \alpha(B) \subseteq C_i$ because B is a copoint attached to $\alpha(B)$. So, $C_i \cup \alpha(B) = C_i$ and $C_i = C_{i+1}$ but $C_i \subset C_{i+1}$ by assumption. Thus, every maximal chain in $L_{\mathscr{L}}$ contains both B and $B \cup \alpha(B)$. □

We can guarantee that $(X, \mathscr{L})$ has no copoint B such that B is comparable with $\mathscr{L}(\alpha(B))$ when $|X| > 1$ and $(X, \mathscr{L})$ is *atomic*, that is for all $p \in X$, $\mathscr{L}(p) = p$. This yields the following corollary.

Corollary 2.3. *Let $(X, \mathscr{L})$ be an atomic convex geometry with $|X| > 1$. If B is a copoint, then B is incomparable with $\mathscr{L}(\alpha(B))$.*

Proof. Let B be a copoint of $(X, \mathscr{L})$ attached to $\alpha(B)$. If $B \subset \mathscr{L}(\alpha(B)) = \alpha(B)$, then B must be $\emptyset$. Since there is more than one element in X, $\emptyset$ cannot be a copoint. Therefore, $\mathscr{L}(\alpha(B))$ is incomparable with B. □

A convex geometry need not be atomic to be have every copoint B incomparable with $\mathscr{L}(\alpha(B))$ as we see in Figure 1.2.

2.2 Order Dimension of Convex Geometries

We consider the critical digraph, $\mathcal{D}(P)$ for the poset $P = (X, \leq)$ as defined by Trotter [Tro92], and used by Reading [Rea02]. This digraph has vertex set equal to the set of critical pairs of P and there is a directed edge $(A, B) \to (C, D)$ when $C \leq B$. In the case where $P = L_{\mathscr{L}}$, the lattice of closed sets for a convex geometry $(X, \mathscr{L})$, this is simply where $C \subseteq B$ (and if $(X, \mathscr{L})$ is atomic, $\alpha(D) \in B$). The minimal cycles of this digraph induce a hypergraph, $\mathcal{H}_P^C$ with vertices from the set of critical pairs and hyperedges being minimal cycles of $\mathcal{D}(P)$. For any cycle of $\mathcal{D}(P)$ there is no linear extension of P that reverses all critical pairs belonging to that cycle. Felsner and Trotter [FT00] showed that the chromatic number of this hypergraph is the order dimension of P and greater than or equal to the chromatic number of the graph, G_P^C, induced from $\mathcal{H}_P^C$ by only considering the edges of size 2. We state this as a lemma.

Lemma 2.4 ([FT00], 3.3)**.** *For every poset P, $\dim(P) = \chi(\mathcal{H}_P^C) \geq \chi(G_P^C)$*

For $(X, \mathscr{L})$ a convex geometry we define $\mathcal{H}_{L_{\mathscr{L}}}^C$ as described above. We make note of the relationship between the graphs $\mathcal{G}(X, \mathscr{L})$ and $G_{L_{\mathscr{L}}}^C$ for all convex geometries.

Theorem 2.5. *If $(X, \mathscr{L})$ is a convex geometry, then the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the vertices B for which B is incomparable with $\mathscr{L}(\alpha(B))$ is isomorphic to the graph $G_{L_{\mathscr{L}}}^C$.*

Proof. Theorem 2.1 proves that the function ϕ which maps the copoint A to the pair $(\mathscr{L}(\alpha(A)), A)$ is a bijection from those copoints A that are incomparable with $\mathscr{L}(\alpha(A))$ to the critical pairs $(\mathscr{L}(\alpha(A), A)$. Copoint A is adjacent to copoint B in $\mathcal{G}(X, \mathscr{L})$ if and only if $\alpha(B) \in A$ and $\alpha(A) \in B$. This occurs if and only if $(\mathscr{L}(\alpha(A)), A) \to (\mathscr{L}(\alpha(B)), B)$ and $(\mathscr{L}(\alpha(B)), B) \to (\mathscr{L}(\alpha(A)), A)$ are directed edges in $\mathcal{D}(L_{\mathscr{L}})$. Therefore, the edges of $\mathcal{G}(X, \mathscr{L})$ are exactly the two-cycles of $\mathcal{D}(L_{\mathscr{L}})$, which are the edges of $G_{L_{\mathscr{L}}}^C$. □

We make use of Theorem 2.5 and Lemma 2.4 to yield a lower bound for the order dimension of convex geometries.

Corollary 2.6. *For any convex geometry $(X, \mathscr{L})$, $\dim(L_{\mathscr{L}}) = \chi(\mathcal{H}_{L_{\mathscr{L}}}^C) \geq \chi(G_{L_{\mathscr{L}}}^C) = \chi(\mathcal{G}(X, \mathscr{L}))$.*

Proof. From Theorem 2.5, we know $\chi(G^C_{L_{\mathscr{L}}}) \geq \chi(\mathcal{G}(X, \mathscr{L}))$. The only vertices that were removed from $\mathcal{G}(X, \mathscr{L})$ were those corresponding to copoints B such that $B \cup \alpha(B) = \mathscr{L}(\alpha(B))$. Suppose that there is some copoint A with $\alpha(B) \in A$, then because $\mathscr{L}(\alpha(B)) = B \cup \alpha(B) \subset A$ we have that $\alpha(A) \notin B$. Thus, B is an isolated vertex of $\mathcal{G}(X, \mathscr{L})$. Therefore $\chi(\mathcal{G}(X, \mathscr{L}))$ equals the chromatic number of the the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the vertices B for which B is incomparable with $\mathscr{L}(\alpha(B))$. This implies that $\chi(G^C_{L_{\mathscr{L}}}) = \chi(\mathcal{G}(X, \mathscr{L}))$. ☐

There are examples of posets P where $\chi(\mathcal{H}^C_P) > \chi(G^C_P)$ [Tro92], [FT00]. We would like to point out that for a convex geometry $(X, \mathscr{L})$, the hypergraph $\mathcal{H}^C_{L_{\mathscr{L}}}$ may contain hyperedges of size greater than 2. Consider the convex geometry realized by the point set in Figure 4.1 with its poset of copoints. One can quickly verify that both $(z, uwv), (v, xyu), (y, xwz)$ and $(z, xyw), (y, xvu), (v, uwz)$ are minimal cycles of length 3 in the critical digraph of this convex geometry. However, $\chi(\mathcal{G}(X, \mathscr{L})) = \chi(\mathcal{H}^C_{L_{\mathscr{L}}}) = 4$, so the order dimension is 4.

We define a cycle of copoints of length l, to be an ordered collection $\mathcal{A}$ of l copoints $(A_1, \dots, A_l)$ such that $\alpha(A_1) \in A_2, \alpha(A_2) \in A_3, \dots, \alpha(A_{l-1}) \in A_l, \alpha(A_l) \in A_1$ and define $\alpha(\mathcal{A}) = \{\alpha(A_i) : i = 1, 2, \dots, l\}$. We will use Proposition 2.7 to help prove Corollary 4.5.

Proposition 2.7. *If $\mathcal{A}$ is a minimal cycle of length $l > 2$, then the points $\alpha(A_1), \alpha(A_2), \dots, \alpha(A_l)$ are distinct.*

Proof. Suppose that $\alpha(A_i) = \alpha(A_j)$ for $j > i$. Then, $\alpha(A_j) \in A_{i-1}$ and the cycle of copoints can be shortened, contradicting the minimality of $\mathcal{A}$. ☐

We pose a question based on these results. Is there some convex geometry $(X, \mathscr{L})$ for which $\chi(\mathcal{H}^C_{L_{\mathscr{L}}}) > \chi(\mathcal{G}(X, \mathscr{L}))$? That is, does there exist some convex geometry with $\dim(L_{\mathscr{L}}) > \chi(\mathcal{G}(X, \mathscr{L}))$? The construction of Felsner and Trotter [FT00] uses posets that are not lattices of closed sets for convex geometries. We characterize the hyperedges of $\mathcal{H}^C_{L_{\mathscr{L}}}$ when $(X, \mathscr{L})$ is a convex geometry realized by planar point sets in general position in Section 4.3.

2.3 Monotonicity of the Chromatic Number of $\mathcal{G}(X, \mathscr{L})$

Let $(X, \mathscr{L})$ be a convex geometry and $Y \subseteq X$. The *relative alignment* on Y, or the restriction of $\mathscr{L}$ to Y, is the alignment $\mathscr{L}|_Y = \{C \cap Y | C \in \mathscr{L}\}$. We note the result of Edelman and Jamison on the relative alignment of a convex geometry.

Theorem 2.8 ([EJ85], 5.9)**.** *If $(X, \mathscr{L})$ is a convex geometry and $Y \subseteq X$, then $(Y, \mathscr{L}|_Y)$ is also a convex geometry.*

Let $(X, \mathscr{L})$ be a convex geometry, and $Y \subseteq X$ have the property that $Y \in \mathscr{L}$. The *contraction of $\mathscr{L}$ with respect to Y*, $\mathscr{L}/Y$, is the alignment on $X - Y$ defined by $\mathscr{L}/Y = \{C \subseteq X - Y | C = \mathscr{L}(D \cup Y) - Y \text{ for some } D \subseteq X - Y\}$.

Theorem 2.9 ([EJ85], 5.10)**.** *If $(X, \mathscr{L})$ is a convex geometry and $Y \subseteq X$, $Y \in \mathscr{L}$, then $(X - Y, \mathscr{L}/Y)$ is also a convex geometry.*

A minor of a convex geometry $(X, \mathscr{L})$ is any convex geometry of a subset $Y \subseteq X$ obtained by a sequence of restrictions and contractions. Therefore, with the previous two results, every minor of a convex geometry is a convex geometry. In this section, we aim to show that for any convex geometry $(X, \mathscr{L})$, and its graph of copoints, $\mathcal{G}(X, \mathscr{L})$, that the chromatic number of the copoint graph is monotone under minors. This result is by no means obvious at first glance, whereas the analogous result for clique number is obvious from the fact that the cliques of $\mathcal{G}(X, \mathscr{L})$ correspond to convexly independent sets in $L_{\mathscr{L}}$.

To prove the result for the for restrictions, we employ graph homomorphisms. For graphs G and H, a graph homomorphism $\phi : G \to H$ is a map from $V(G)$ to $V(H)$ such that $\{u, v\} \in E(G)$ implies that $\{\phi(u), \phi(v)\} \in E(H)$ or $\phi(u) = \phi(v)$. Let $\mathcal{G}(X, \mathscr{L})|_{X-p}$ be the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the copoints attached to elements of $X - p$. We define a map between the vertex sets of $\mathcal{G}(X, \mathscr{L})|_{X-p}$ and $\mathcal{G}(X - p, \mathscr{L}|_{X-p})$. Let ψ :

$V(\mathcal{G}(X-p,\mathscr{L}|_{X-p}))\to V(\mathcal{G}(X,\mathscr{L})|_{X-p})$ where $\psi(A)=A$ if A is a copoint in $(X,\mathscr{L})$ not attached to p, otherwise $\psi(A)=A\cup p$.

Lemma 2.10. *Let $(X,\mathscr{L})$ be a convex geometry and $p\in X$. Then, $\psi : V(\mathcal{G}(X-p,\mathscr{L}_{X-p}))\to V(\mathcal{G}(X,\mathscr{L})|_{X-p})$ is a graph homomorphism.*

Proof. We first prove that the range of ψ is contained in $V(\mathcal{G}(X,\mathscr{L}))|_{X-p}$, that is, $\psi(A)$ is always a copoint of $\mathscr{L}$ not attached to p. Let A be a copoint of $(X-p,\mathscr{L}|_{X-p})$. That is, $A\in\mathscr{L}|_{X-p}$ is maximal in $(X-p)-q$ for some $q\in X-p$. If $A\in\mathscr{L}$ is also maximal in $X-q$, then A is not a copoint attached to p so $\psi(A)=A\in V(\mathcal{G}(X,\mathscr{L})|_{X-p})$. Suppose A is a copoint in $(X,\mathscr{L})$ not attached to q, then A must be attached to p. Further $q\notin A\cup p$, and $A\cup q$ is closed in $(X-p,\mathscr{L}|_{X-p})$, so $A\cup p\cup q$ is closed in $(X,\mathscr{L})$. Moreover, let $r\in X-(A\cup p\cup q)$, then $A\cup p\cup r$ is not closed in $(X,\mathscr{L})$ or A would not be a copoint attached to q in $(X-p,\mathscr{L}|_{X-p})$. Thus, $\psi(A)=A\cup p$ is a copoint of $(X,\mathscr{L})$ attached to q. Next, suppose A is not a copoint in $(X,\mathscr{L})$. This means that there is some $A\cup r$ that is closed in $(X,\mathscr{L})$ for some $r\neq q$. If $r\neq p$, this would imply that A is not a copoint of $(X-p,\mathscr{L}|_{X-p})$. Again, let $r\in X-(A\cup p\cup q)$, then $A\cup p\cup r$ is not closed in $(X,\mathscr{L})$ or A would not be a copoint attached to q in $(X-p,\mathscr{L}|_{X-p})$. Thus, $\psi(A)=A\cup p$ is a copoint of $(X,\mathscr{L})$ attached to q. Lastly, if $A\notin\mathscr{L}$, then $A\cup p\in\mathscr{L}$ and by the previous argument, $\psi(A)=A\cup p$ is a copoint of $(X,\mathscr{L})$ attached to q. Therefore, $\psi(A)$ is always in $V(\mathcal{G}(X,\mathscr{L}))|_{X-p}$. We note that by definition of ψ, $\alpha(A)=\alpha(\psi(A))$ for each copoint A of $(X,\mathscr{L})$.

Let $\{A,B\}\in E(\mathcal{G}(X-p,\mathscr{L}|_{X-p}))$, this means that $\alpha(A)\in B$ and $\alpha(B)\in A$. Since $\alpha(A),\alpha(B)\neq p$, $A\subseteq\psi(A)$ and $B\subseteq\psi(B)$, so $\alpha(\psi(A))\in\psi(B)$ and $\alpha(\psi(B))\in\psi(A)$. This means that $\{\psi(A),\psi(B)\}\in E(\mathcal{G}(X,\mathscr{L})|_{X-p})$ and ψ is a graph homomorphism. $\square$

We define another map $\phi : V(\mathcal{G}(X,\mathscr{L})|_{X-p})\to V(\mathcal{G}(X-p,\mathscr{L}|_{X-p}))$ and note that for $q\in X-p$, there may be fewer copoints attached to q in $(X-p,\mathscr{L}_{X-p})$ than there are in $(X,\mathscr{L})$. Let A be a copoint attached to q in $(X,\mathscr{L})$. We define $\phi(A)$ to be a copoint of $(X - p, \mathscr{L}|_{X-p})$ attached to q containing $A - p$. If more than one such copoint exists, choose one arbitrarily. At least one such copoint must exist as a result of the definition of convex geometry, because $A - p \in \mathscr{L}|_{X-p}$. Further, if $B, C \in \phi^{-1}(A)$, then $\alpha(B) = \alpha(C)$ which implies that B and C are not adjacent in $\mathcal{G}(X, \mathscr{L})$.

Lemma 2.11. *Let $(X, \mathscr{L})$ be a convex geometry and $p \in X$. Then $\phi : V(\mathcal{G}(X)|_{X-p}) \to V(\mathcal{G}(X - p))$ is a graph homomorphism.*

Proof. Let $\{A, B\} \in E(\mathcal{G}(X, \mathscr{L})|_{X-p})$, this means that $\alpha(A) \in B$ and $\alpha(B) \in A$. Since $\alpha(A), \alpha(B)$ are not p, $A - p \subseteq \phi(A)$ and $B - p \subseteq \phi(B)$, so $\alpha(\phi(A)) \in \phi(B)$ and $\alpha(\phi(B)) \in \phi(A)$. This means that $\{\phi(A), \phi(B)\} \in E(\mathcal{G}(X - p, \mathscr{L}|_{X-p}))$ and ϕ is a graph homomorphism. □

We use a theorem from Hell and Nešetřil [HN04], that if there is a graph homomorphism $\phi : G \to H$ then $\chi(G) \le \chi(H)$, to prove Theorem 2.12.

Theorem 2.12. *Let $(X, \mathscr{L})$ be a convex geometry with copoint graph $\mathcal{G}(X, \mathscr{L})$. For all $p \in X$, $\chi(\mathcal{G}(X - p, \mathscr{L}|_{X-p})) + 1 \ge \chi(\mathcal{G}(X, \mathscr{L})) \ge \chi(\mathcal{G}(X - p, \mathscr{L}|_{X-p}))$.*

Proof. We use the fact that if H is a subgraph of G then, $\chi(G) \ge \chi(H)$. Further, we know from Lemmas 2.10 and 2.11 that there are graph homomorphisms between $\mathcal{G}(X, \mathscr{L})|_{X-p}$ and $\mathcal{G}(X - p, \mathscr{L}|_{X-p})$. Therefore, $\chi(\mathcal{G}(X - p, \mathscr{L}|_{X-p})) = \chi(\mathcal{G}(X, \mathscr{L})|_{X-p}) \le \chi(\mathcal{G}(X, \mathscr{L}))$.

Moreover, we can use a proper coloring of $\mathcal{G}(X, \mathscr{L})|_{X-p})$ to properly color $\mathcal{G}(X, \mathscr{L})|_{X-p}$ together with one additional color for all copoints of $(X, \mathscr{L})$ attached to p. The set of copoints attached to p form an independent set in $\mathcal{G}(X, \mathscr{L})$, so there is a proper coloring of $\mathcal{G}(X, \mathscr{L})$ with $\chi(\mathcal{G}(X, \mathscr{L})|_{X-p}) + 1$ colors. □

To prove the monotonicity for the chromatic number under contractions, let $M(X, \mathscr{L})|_p$ be the set of copoints containing p. We define a map $\mu : M(X, \mathscr{L})|_p \to V((X - p, \mathscr{L}/p))$, where the copoint A in $M(X, \mathscr{L})|_p$ maps to $\mu(A) = A - p$. $A - p$ is a copoint in $(X - p, \mathscr{L}/p)$ attached to $\alpha(A)$, because if there were some some $q \in X - p$ not equal to $\alpha(A)$ such that

$(A-p)\cup q \in \mathscr{L}/p$ this would mean that $A \cup q \in \mathscr{L}$. However, since A is a copoint attached to $\alpha(A)$, this means that $q = \alpha(A)$ and $(A-p)$ is a copoint of $(X-p, \mathscr{L}/p)$ attached to $\alpha(A)$. We can describe the graph $\mathcal{G}(X-p, \mathscr{L}/p)$ as a subgraph of $\mathcal{G}(X, \mathscr{L})$.

Theorem 2.13. *Let $(X, \mathscr{L})$ be a convex geometry with $p \in X$, $p \in \mathscr{L}$. $\mathcal{G}(X-p, \mathscr{L}/p)$ is isomorphic to the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the set copoints containing p. Also, $\chi(\mathcal{G}(X, \mathscr{L})) \geq \chi(\mathcal{G}(X-p, \mathscr{L}/p))$.*

Proof. The map μ is a bijection of vertex sets. To see that adjacent vertices remain adjacent under the map μ, we note that only p is removed by the map μ from the copoint A containing p, which means that all $q \in (X-p) \cap A$ are contained in $\mu(A)$. Thus, if $\{A, B\}$ are adjacent vertices in $\mathcal{G}(X, \mathscr{L})$ with $p \in A \cap B$, then $\alpha(A) \in B$ and $\alpha(B) \in A$. This means that $\alpha(A) = \alpha(\mu(A)) \in \mu(B)$ and $\alpha(B) = \alpha(\mu(B)) \in \mu(A)$. Therefore, $\mathcal{G}(X-p, \mathscr{L}/p)$ is isomorphic to the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the set of copoints containing p.

As $\mathcal{G}(X-p, \mathscr{L}/p)$ is an induced subgraph of $\mathcal{G}(X, \mathscr{L})$, the fact that $\chi(\mathcal{G}(X, \mathscr{L})) \geq \chi(\mathcal{G}(X-p, \mathscr{L}/p))$ follows immediately. □

We now obtain the final result of this section as a direct corollary of Theorems 2.12 and 2.13.

Corollary 2.14. *Let $(X, \mathscr{L})$ be a convex geometry and $(Y, \mathscr{L}')$ a minor of $(X, \mathscr{L})$. Then $\chi(\mathcal{G}(X, \mathscr{L})) \geq \chi(\mathcal{G}(Y, \mathscr{L}'))$.*

2.4 Direct Sum of Convex Geometries

Definition 2.15. *Let $(X_1, \mathscr{L}_1)$ and $(X_2, \mathscr{L}_2)$ be convex geometries and let $X = X_1 \sqcup X_2$, be the disjoint union of X_1 and X_2 with $\mathscr{L}(C) = \mathscr{L}_1(C_{X_1}) \sqcup \mathscr{L}_2(C_{X_2})$ where $C_{X_i} = C \cap X_i$. We define $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) = (X, \mathscr{L})$ to be the* direct sum *of convex geometries.*

We now show that that $(X, \mathscr{L})$ defined as a direct sum of convex geometries is itself a convex geometry. We also classify the copoints of $(X, \mathscr{L})$ in terms of the summands.

Proposition 2.16. *Let* $(X_1, \mathscr{L}_1), (X_2, \mathscr{L}_2), \ldots (X_n, \mathscr{L}_n)$ *be convex geometries. Then*

1. $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2)$ *is a convex geometry.*

2. $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) \oplus \cdots \oplus (X_n, \mathscr{L}_n)$ *is a convex geometry.*

3. *The copoints of* $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2)$ *have the form* $A_1 \sqcup X_2$ *or* $X_1 \sqcup A_2$ *where* A_i *is a copoint of* $(X_i, \mathscr{L}_i)$.

Proof. 1. Let $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) = (X, \mathscr{L})$ as in the definition. It is obvious that $\mathscr{L}$ is a closure operator on the subsets of $X_1 \sqcup X_2$. Let $A \sqcup B$ be a closed set of $(X, \mathscr{L})$, then there is a $p \in X_1 \sqcup X_2 - A \sqcup B$ such that either $A \cup p \in \mathscr{L}_1$ or $B \cup p \in \mathscr{L}_2$. Thus, $(A \sqcup B) \cup p \in \mathscr{L}$. So, $(X, \mathscr{L})$ is a convex geometry.
2. This follows by induction on n.
3. Let $A = A_1 \sqcup A_2$ be a copoint of $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) = (X, \mathscr{L})$ attached to p. We know that either $p \in X_1$ or $p \in X_2$. Suppose $p \in X_1$ and $A_2 \neq X_2$, then there is a $q \in X_2 - A_2$ such that $A_1 \sqcup A_2 = A \subset \mathscr{L}(A \cup p) = (\mathscr{L}_1(A_1 \cup p) \sqcup \mathscr{L}_2(A_2)) \subset (\mathscr{L}_1(A_1 \cup p) \sqcup \mathscr{L}_2(A_2 \cup q))$ and $(A_1 \sqcup A_2) = A \subset \mathscr{L}(A \cup q) = (\mathscr{L}_1(A_1) \sqcup \mathscr{L}_2(A_2 \cup q)) \subset (\mathscr{L}_1(A_1 \cup p) \sqcup \mathscr{L}_2(A_2 \cup q))$ which contradicts that fact that A is a copoint. We have a similar argument if $p \in X_2$ and $A_1 \neq X_1$. Thus, if $p \in X_1$, then we have the case $(A_1 \sqcup X_2)$ where A_1 is a copoint of $(X_1, \mathscr{L}_1)$ attached to p. If $q \in X_2$, then we have the case $(X_1 \sqcup A_2)$ where A_2 is a copoint of $(X_2, \mathscr{L}_2)$. □

2.4.1 Graph of Copoints under Direct Sum

Definition 2.17. *The* join *of simple graphs* G *and* H*, written* $G \vee H$*, is the graph obtained from the disjoint union* $G + H$ *by adding the edges* $\{xy : x \in V(G), y \in V(H)\}$

We prove the following proposition about the graph of copoints of a direct sum of convex geometries using the definition of the join of graphs.

Proposition 2.18. *Let* $(X_1, \mathscr{L}_1), (X_2, \mathscr{L}_2)$ *be convex geometries and* $(X, \mathscr{L}) = (X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2)$*. Then,* $\mathcal{G}(X, \mathscr{L}) = \mathcal{G}(X_1, \mathscr{L}_1) \vee \mathcal{G}(X_2, \mathscr{L}_2)$

Proof. Part 3) of Proposition 2.16 shows us that the copoints of $(X, \mathscr{L})$ are either of the form $(A_1 \sqcup X_2)$ or $(X_1 \sqcup A_2)$ where A_i is a copoint of $(X_i, \mathscr{L}_i)$. Directly from this statement we see that the disjoint union of the graphs $\mathcal{G}(X_1, \mathscr{L}_1) + \mathcal{G}(X_2, \mathscr{L}_2)$ is contained in $\mathcal{G}(X, \mathscr{L})$. We also have the edge $\{(A_1 \sqcup X_2), (X_1 \sqcup A_2)\} \in E(\mathcal{G}(X, \mathscr{L}))$ as $\alpha(A_2) \in (A_1 \sqcup X_2)$ and $\alpha(A_1) \in (X_1 \sqcup A_2)$. □

The following lemma is taken from an exercise in the book by West [Wes01].

Lemma 2.19 ([Wes01], Exercise 5.1.5)**.** *$\omega(G \vee H) = \omega(G) + \omega(H)$ and $\chi(G \vee H) = \chi(G) + \chi(H)$*

We make note of the of the convex geometry in Figure 1.2. This is a convex geometry where $\mathcal{G}(X, \mathscr{L}) = C_5$, the cycle on 5 vertices. The copoint graph of this convex geometry has chromatic number 3 and clique number 2. We use this prove the next proposition.

Theorem 2.20. *For all integers $m \geq 0$, there exists a convex geometry $(X, \mathscr{L})$ with lattice of closed sets $L_{\mathscr{L}}$ such that $\chi(\mathcal{G}(X, \mathscr{L})) - \omega(\mathcal{G}(X, \mathscr{L})) > m$.*

Proof. Let $(X_i, \mathscr{L}_i)$ be the convex geometry in Figure 1.2 for $i = 1, 2, \ldots, m+1$. Then consider the convex geometry $(X, \mathscr{L}) = \oplus_{i=1}^{m+1}(X_i, \mathscr{L}_i)$. With Lemma 2.19, we see that $\chi(\mathcal{G}(X, \mathscr{L})) = 3(m+1)$ and $\omega(\mathcal{G}(X, \mathscr{L})) = 2(m+1)$. We see that $\chi(\mathcal{G}(X, \mathscr{L})) - \omega(\mathcal{G}(X, \mathscr{L})) = m + 1 > m$. □

Lastly, in the construction described in Theorem 2.20, we note that while the difference between the chromatic number and the clique number can be larger than any integer m but the ratio $\frac{\chi(\mathcal{G}(X, \mathscr{L}))}{\omega(\mathcal{G}(X, \mathscr{L}))}$ is a constant $\frac{3}{2}$. We compare this to the construction of Chapter 3 for a convex geometry $(X, \mathscr{L})$ where the ratio $\frac{\chi(\mathcal{G}(X, \mathscr{L}))}{\omega(\mathcal{G}(X, \mathscr{L}))} > c$ for each constant c. That construction gives a convex geometry where $\omega(\mathcal{G}([n], \mathscr{L})) = 2$, while $\chi(\mathcal{G}([n], \mathscr{L})) = \lceil \log_2(n+1) \rceil$ so $\chi(\mathcal{G}([n], \mathscr{L})) - \omega(\mathcal{G}([n], \mathscr{L})) = \lceil \log_2(n+1) \rceil - 2$. Meanwhile, the construction given in Theorem 2.20 for $n = 4m$ gives us a difference of $\chi(\mathcal{G}([n], \mathscr{L})) - \omega(\mathcal{G}([n], \mathscr{L})) = \frac{n}{4}$. We also note that the operation of the direct sum applied to convex geometries generated from planar point sets in general position does not produce a convex geometry generated from a planar point set in general position.

Chapter 3: Chromatic Numbers of Copoint Graphs of Convex Geometries

3.1 Construction of A Convex Geometry

Beagley [Beaar] asked the following question: Is $\frac{\chi(\mathcal{G}(X,\mathscr{L}))}{\omega(\mathcal{G}(X,\mathscr{L}))} \leq c$ for some constant c? We construct a family of convex geometries indexed by integers m, n with clique number of $m+1$ and chromatic number at least $\lceil \log_2(n+1) \rceil$.

Let n be a positive integer and $\{1, 2, \ldots, n\} = [n]$. When $n = 0$, then $[n] = \emptyset$. Let m be a positive integer, $m < n$, and define $\mathscr{L}_{m,n} = \{([i] \cup J) | 0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq m\}$. Figure 3.1 is the lattice of $\mathscr{L}_{2,5}$.

Proposition 3.1. *For n, m positive integers with $m < n$, the pair $([n], \mathscr{L}_{m,n})$ is an m-free convex geometry.*

Proof. It is easy to see that $\mathscr{L}_{m,n}$ is closed under intersection and $\emptyset, [n] \in \mathscr{L}_{m,n}$. Let C be in $\mathscr{L}_{m,n}, C \neq [n]$. If $C = [i] \cup J$ with $0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq m$, then $C \cup \{i+1\} \in \mathscr{L}_{m,n}$, so $([n], \mathscr{L}_{m,n})$ is a convex geometry. To see that $([n], \mathscr{L}_{m,n})$ is m-free, note that if $|J| \leq m$ and i is the smallest element of $[n] \backslash J$, then $J = [i-1] \cup J'$ where $|J'| \leq m$. □

For each $i \in \{1, 2, \ldots, n-m\}$, define $A_i = \{[i-1] \cup J | J \subseteq \{i+1, i+2, \ldots, n\}, |J| = m\}$ and for each $i \in \{n-m+1, n-m+2, \ldots, n\}$ let $A_i = \{[i-1] \cup \{i+1, i+2, \ldots, n\}\}$.

Proposition 3.2. *For $i = 1, 2, \ldots, n$, A_i is the set of copoints of $([n], \mathscr{L}_{m,n})$ attached to i.*

Proof. If $C = [i-1] \cup J \neq [n]$ for $1 \leq i \leq n$, $J \subseteq \{i+1, \ldots, n\}$, $|J| \leq m$, then $C \cup \{i\}$ is in $\mathscr{L}_{m,n}$. If C is not in A_i, then there is an element $p \neq i$ of $[n]$ such that $C \cup \{p\}$ is in $\mathscr{L}_{m,n}$. □

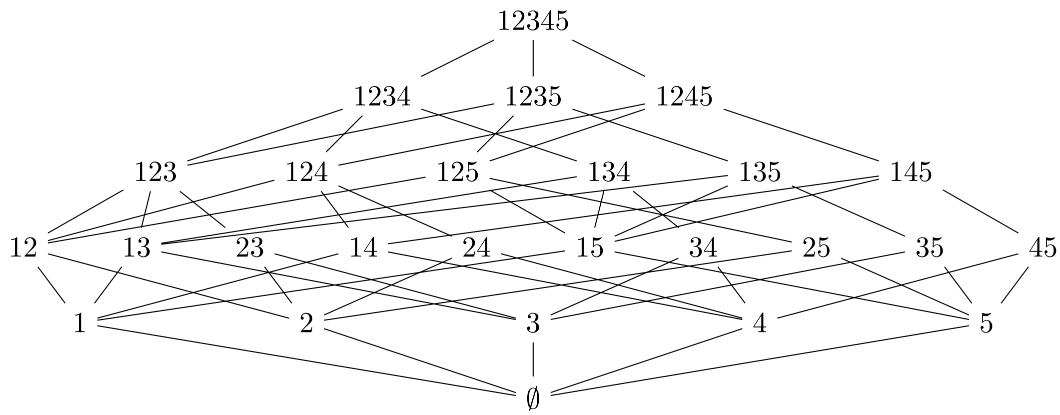

Figure 3.1: Lattice of Closed Sets for $\mathscr{L}_{2,5}$

The size of the maximum clique in $\mathcal{G}([n], \mathscr{L}_{m,n})$ can be found using the size of the largest independent set.

Lemma 3.3. *The clique number of* $\mathcal{G}([n], \mathscr{L}_{m,n})$ *is* $m+1$.

Proof. Let $C \in \mathscr{L}_{m,n}$. If $|C| \leq m$, then $\mathscr{L}_{m,n}(C) = C$. So, let $|C| > m$. We can write $C = [i] \cup J$ where $0 \leq i \leq n-m$, $J \subseteq \{i+1, \ldots, n\}$, $|J| = m$. Thus, $C = \mathscr{L}_{m,n}(\{i\} \cup J)$ and $|\{i\} \cup J| = m+1$. Further, $\mathcal{G}([n], \mathscr{L}_{m,n})$ contains a $m+1$-clique consisting of the copoints of the form $[n] \backslash \{i\}$ for $i = n-m, \ldots, n$. Since every closed set C can be written as the closure of at most $m+1$ elements of $[n]$, there is no independent set of size $m+2$ and $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m+1$. □

To bound the chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ we make note of the following property of the set A_i.

Proposition 3.4. *Suppose that* $B \subseteq [n]$, $|B| \leq m$, *and* $i < b$ *for all* $b \in B$. *Then there exists* $C \in A_i$ *so that* C *contains every element of* B.

Proof. Choose a copoint $C = [i-1] \cup J$ in A_i with $B \subseteq J$, and the result is immediate. □

Corollary 3.5. *Suppose that $B \subseteq [n]$, $|B| \leq m$, and that $i < b$ for all $b \in B$. Then there exists $C \in A_i$ so that C is adjacent in $\mathcal{G}([n], \mathscr{L}_{m,n})$ to every copoint D in $\bigcup_{b \in B} A_b$.*

Proof. By the previous proposition, $b \in C$ for every $b \in B$, and because $b > i$, we have $i \in [b-1] \subseteq D$ for all $D \in A_b$. □

We shall answer Beagley's question, using $([n], \mathscr{L}_{m,n})$ to show that the ratio $\frac{\chi(\mathcal{G}(X \mathscr{L}))}{\omega(\mathcal{G}(X, \mathscr{L}))}$ is bounded by no constant c.

Theorem 3.6. *The convex geometry $([n], \mathscr{L}_{m,n})$ has $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m+1$ and $\lceil \log_2(n+1) \rceil \leq \chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$.*

Proof. $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m + 1$ by Lemma 3.3.

For any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ with c colors, let S_i be the set of colors used to color the copoints of A_i, $i = 1, 2, \ldots, n$. For $1 \leq i < j \leq n$, the fact that there is a copoint of A_i adjacent to every copoint of A_j means that the S_i are distinct and nonempty. Therefore, $n \leq 2^c - 1$, and any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ requires at least $\lceil \log_2(n+1) \rceil$ colors. □

The graph for the convex geometry $([n], \mathscr{L}_{m,n})$ has clique number that is a function of m and independent of n, while the chromatic number is at least $\lceil \log_2(n+1) \rceil$. Therefore the ratio $\frac{\chi(\mathcal{G}([n], \mathscr{L}_{m,n}))}{\omega(\mathcal{G}([n], \mathscr{L}_{m,n}))}$ can be bigger than any fixed constant c, provided n is large enough.

The precise determination of $\chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$ for $m \geq 2$ is an interesting question in its own right. Let S be a finite set. An m-nondecreasing sequence of subsets of S is a sequence $S_1, S_2, \ldots, S_t$ so that for any set $B \subseteq [t]$, $|B| \leq m$, and for $j \in [t]$ with $j > b$ for all $b \in B$, we have $S_j \not\subseteq \bigcup_{b \in B} S_b$.

Lemma 3.7. *The chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ is the smallest integer s for which there is an m-nondecreasing sequence of length n of subsets of an s-element set S.*

Proof. For any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ with s colors, let S_i be the set of colors used to color the copoints of A_i, $i = 1, 2, \ldots, n$. It follows from Corollary 3.5 and the definition of m-nondecreasing sequence of subsets of $[s]$, that $S_1, S_2, \ldots, S_n$ is an m-sequence of length n. Then, it is possible to color the vertices in levels $n, n-1, \ldots, 1$ successively where for any $D = [i-1] \cup J$ in A_i that is adjacent to the copoints in A_j for $j \in J$, there is a color in S_i that does not appear in the label of S_j for $j \in J$. This color can be used for the copoint D. Therefore, there is a proper coloring with s colors of $\mathcal{G}([n], \mathscr{L}_{m,n})$. □

We are confronted with the problem of determining the smallest integer s for which there is an m-nondecreasing sequence of length n of subsets of an s-element set S.

A *binary covering array of strength* $m+1$ is an $s \times n$ matrix A with entries in $\{0, 1\}$ so that for every $s \times m+1$ submatrix B of A, every possible $0-1$ vector of length $m+1$ appears as a row of B.

Lemma 3.8. *If A is an $s \times n$ binary covering array of strength $m+1$, then the columns of A are the characteristic vectors of an m-nondecreasing sequence of length n of subsets of an s-element set.*

Proof. If A is an $s \times n$ binary covering array of strength $m+1$ and B is an $s \times m+1$ submatrix of A, then there is a row of B which consists of m zeroes followed by a 1. This implies that the set with characteristic vector equal to column $m+1$ of B is not contained in the union of the sets whose characteristic vectors are the first m columns of B. □

The survey paper of Lawrence et. al. [LKL$^+$11] on covering arrays gives the result of Kleitman and Spencer [KS73] that there exist c_{m+1} and d_{m+1} such that the smallest integer s for which there exists an $s \times n$ binary covering array of strength $m+1$ is bounded below by $(c_{m+1} - o(1)) \log n$ and above by $(d_{m+1} + o(1)) \log n$.

Corollary 3.9. *There exists a constant d_{m+1} so that the chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ is at most $(d_{m+1} + o(1)) \log n$. This means that $\chi(\mathcal{G}([n], \mathscr{L}_{m,n})) \in \Theta(\log(n))$.*

3.1.1 Order Dimension

We are able to make the following remarks about $\dim(L_{\mathscr{L}_{m,n}})$ due to the results of Chapter 2.

Proposition 3.10. $\dim(L_{\mathscr{L}_{m,n}}) = \chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$

Proof. We show that $\mathcal{H}([n], \mathscr{L}_{m,n}) \cong \mathcal{G}([n], \mathscr{L}_{m,n})$ from which we get the result from Corollary 2.6. Suppose that there is a hyperedge of size strictly more than 2, $\{(\alpha(B_1), B_1), (\alpha(B_2), B_2), \ldots, (\alpha(B_k), B_k)\}$ where $k > 2$. There is some $i \in [k]$ such that $\alpha(B_i) < \alpha(B_{i+1})$ in $[n]$, so $\alpha(B_i) \in B_{i+1}$. Also by definition of $\mathcal{H}([n], \mathscr{L}_{m,n})$, we have that $\alpha(B_{i+1}) \in B_i$. Thus, there is an edge in $\mathcal{G}([n], \mathscr{L}_{m,n})$ between B_{i+1} and B_i. So $\{(\alpha(B_1), B_1), (\alpha(B_2), B_2), \ldots, (\alpha(B_k), B_k)\}$ was not a hyperedge with $k > 2$. $\square$

Let $f : (V(\mathcal{G}([n], \mathscr{L}_{1,n})) \backslash \{1, 2, \ldots, n-1\}) \rightarrow \binom{[n]}{2}$, where $f([i-1] \cup \{j\}) = \{i, j\}$. Then f is a graph isomorphism from the subgraph of $\mathcal{G}([n], \mathscr{L}_{1,n})$ induced by the vertices other than $\{1, 2, \ldots, n-1\}$ to the shift graph of K_n (see [Tro92],Chapter 8). The *shift graph* of K_n is known to have clique number 2 and chromatic number $\lceil \log_2(n) \rceil$.

In this construction, we have shown that the ratio between the chromatic number and the clique number of the graph $\mathcal{G}([n], \mathscr{L}_{m,n})$ can get arbitrarily large. There is a related result about posets in a book of Trotter [Tro92]. The *standard example*, S_n for $n \geq 3$, is a partial order on $X = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ with the relations $a_i < b_j$ if and only if $i \neq j$, for $i, j = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$, a_i is a minimal element and b_i is a maximal element of the partial order. Figure 3.2 is the Hasse diagram of the standard example S_5. It is known that the order dimension of S_n is n. However, posets with large order dimension do not require S_n as a subposet. Further, [Tro92] gave examples where the ratio between the order dimension of a poset and the largest standard example becomes arbitrarily large. Proposition 3.11 shows that the independent sets of $(X, \mathscr{L})$ in $L_{\mathscr{L}}$ act in much the same manner as S_n in posets.

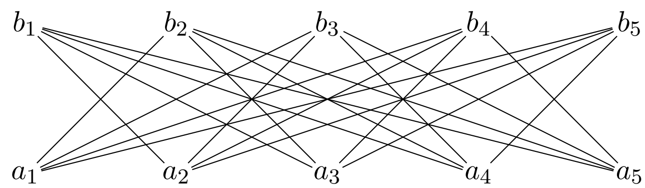

Figure 3.2: The Standard Example, S_5

Proposition 3.11. *Let $(X, \mathscr{L})$ be a convex geometry. $L_{\mathscr{L}}$ contains a subposet isomorphic to S_k, the standard example, if and only if $\mathcal{G}(X, \mathscr{L})$ contains a k-clique.*

Proof. We label the standard example S_k contained in $L_{\mathscr{L}}$ in the usual way. Let p_i be a point in X such that $p_i \in (a_i - b_i)$ for $i = 1, 2, \ldots, k$. As $p_i \in a_i$, this means that for all $j \neq i$, $p_i \in b_j$. We now construct the copoint C_i to be a maximal subset of $X - p_i$ containing b_i. Consider the copoints C_i and C_j for $j \neq i$. C_j is a copoint attached to p_j and C_i is a copoint attached to p_i. By definition, $p_i \in b_j \subseteq C_j$ and $p_j \in b_i \subseteq C_i$. This means that C_i and C_j are adjacent in the graph $\mathcal{G}(X, \mathscr{L})$. Since C_i and C_j are adjacent for all $i \neq j$, we have a clique of size k in $\mathcal{G}(X, \mathscr{L})$.

Conversely, let $\mathcal{G}(X, \mathscr{L})$ contain a k-clique composed of copoints $C_1, C_2, \ldots, C_k$ attached to $p_1, p_2, \ldots, p_k$ respectively. By definition of $\mathcal{G}(X, \mathscr{L})$, this means that $p_i \in C_j$ when $i \neq j$. Thus, we let $a_i = \{p_i\}$ and $b_i = C_i$ for $i = 1, 2, \ldots, k$ and we have that $L_{\mathscr{L}}$ contains a subposet isomorphic to S_k. □

Theorem 3.6 and Proposition 3.11 together show that the convex geometry $([n], \mathscr{L}_{1,n})$ and its lattice of closed sets, $L_{\mathscr{L}_{1,n}}$, is an example of a poset that has order dimension that becomes arbitrarily large but does not contain a poset isomorphic to S_3. The lattice $L_{\mathscr{L}_{1,n}}$ is of order dimension k when $|X| = 2^{k-1}$, which means that $|\mathscr{L}_{1,n}| = 2^{2k-3} + 2^{k-1} + 2^{k-2} + 1$. The example given by Trotter (Example 5.3, [Tro92]) requires a poset of size $R_3(k, 4)$ to have the order dimension equal to k, where $R_3(k, 4)$ is the Ramsey number on 3-regular hypergraphs. It is known that $R_3(k,4)$ is at least $2^{ck\log(k)}$ for some constant c [CFS10]. Thus the posets $L_{\mathscr{L}_{1,n}}$ perform the function of making the order dimension high at a greater economy than do the examples of [Tro92].

3.1.2 Remarks

Convex geometries isomorphic to $([4], \mathscr{L}_{1,4})$ are in the references [EJ85] and [ES88]. The copoint graph for $([4], \mathscr{L}_{1,4})$ contains an induced 5-cycle. The convex geometry $([5], \mathscr{L}_{2,5})$, for which the copoint graph has clique number 3, shows that 5 elements do not force a 4-clique for general convex geometries even when every 2-element subset is closed. Thus one would need more restrictions for combinatorial analogues of Esther Klein's result that 5 point sets in general position in the plane must contain vertex sets of convex 4-gons. The chromatic number of $\mathcal{G}([5], \mathscr{L}_{2,5})$, however, is 4. This will be implied by Theorem 3.17 that we prove in the next section.

One can compute that for any m, n the total number of copoints of the convex geometry $([n], \mathscr{L}_{m,n})$ is:

$$\sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n-m} \binom{n-i}{m} + \sum_{i=n-m+1}^{n} 1 = \binom{n}{m+1} + m$$

For the case $m = \lfloor \frac{n-1}{2} \rfloor$, we get that the total number of copoints is $\binom{n}{\lfloor \frac{n}{2} \rfloor} + \lfloor \frac{n-1}{2} \rfloor$. In the paper [Jam80] it is stated that no examples of convex geometries with total number of copoints greater than the middle binomial coefficient for the number of elements are known.

3.2 Consequences of Freeness

We now introduce a new problem analogous to the Erdős - Szekeres problem: for any integer $k \geq d \geq 2$, determine the smallest positive integer $K_d(k)$ such that for any d-free convex geometry with $|X| \geq K_d(k)$ it follows that $\chi(\mathcal{G}(X, \mathscr{L})) \geq k$. There are two questions of interest related to the study of $K_d(k)$:

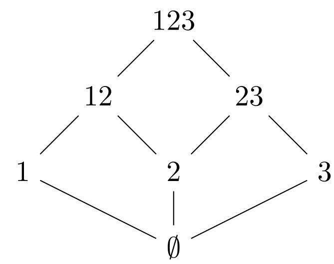

Figure 3.3: Lattice of Closed Sets for a Convex Geometry

1. Does the number $K_d(k)$ exist?

2. If so, how is $K_d(k)$ determined as a function of k?

We specify $d \geq 2$ because of the following 1-free convex geometry. Let $X = [k]$, and for $S \subseteq [k]$ let $\mathscr{L}(S) = [\min(S), \max(S)] \cap X$. Figure 3.3 shows this convex geometry for $k = 3$. It is clear that there are two chains of copoints for $(X, \mathscr{L})$, those containing 1 and those containing k. The graph $\mathcal{G}(X, \mathscr{L})$ has chromatic number 2, for all k for this convex geometry, as each of the chains of copoints is an independent set in $\mathcal{G}(X, \mathscr{L})$. This convex geometry has every 1-element subset closed. Thus 1-freeness alone does not force the chromatic number of the copoint graph to increase with $|X|$.

To show that the number $K_d(k)$ exists for $d > 1$, we focus on $K_2(k)$. It is sufficient to show $K_2(k)$ is finite, because d-freeness for $d > 2$ implies that every 2-element subset is also closed, because $\mathscr{L}$ is an alignment.

Let $(X, \mathscr{L})$ be a 2-free convex geometry and $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$ be a partition of $\mathcal{G}(X, \mathscr{L})$ into independent sets. For $x, y \in X, x \neq y$, define $S_{xy} = \{j \in [t]$: there is a copoint C with $\alpha(C) = y, x \in C, C \in I_j\}$. For each $x \in X$, let $D_x = \{S_{yx} : y \neq x\}$.

A family of subsets of $[t]$ is called *intersecting* if $A \cap B \neq \emptyset$ whenever $A, B \in [t]$. An intersecting family of subsets is *maximal* if it is contained in no other intersecting family.

Lemma 3.12. *For each $x \in X$, D_x is an intersecting family.*

Proof. A copoint C attached to x is a maximal closed subset in $X - x$. For any $\{y, z\}$ with x, y, and z distinct $\{y, z\}$ is closed, so there is a copoint containing $\{y, z\}$ attached to x. This copoint must be in one of the independent sets I_j. Therefore, $j \in S_{yx} \cap S_{zx}$ and $S_{yx} \cap S_{zx} \neq \emptyset$. □

Corollary 3.13. *No two families D_x for $x \in X$ are contained in the same maximal intersecting family of $[t]$.*

Proof. For $x \neq y$, S_{yx} is contained in the complement of S_{xy} in $[t]$, because $\mathcal{I}$ is a proper coloring of $\mathcal{G}(X, \mathscr{L})$. □

Results similar to Lemma 3.12 and Corollary 3.13 also appear in [HM99] and [Mor01]. Moreover, in [Mor01], Morris noted that the number $\gamma(n)$ of maximal intersecting families of subsets of an n-element set is at least $2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$, which is a result of Spencer [Spe71].

Theorem 3.14. $K_2(k) = \gamma(k)$

Proof. Corollary 3.13 shows that $K_2(k) \leq \gamma(k)$. The construction of Hoşten and Morris [HM99] gives a 2-free convex geometry of convex dimension k with $\gamma(k)$ elements, for any k. Edelman and Jamison [EJ85] proved that the convex dimension is bounded below by the order dimension and Beagley [Beaar] proved that the order dimension is bounded below by $\chi(\mathcal{G}(X, \mathscr{L}))$. So, $K_2(k) \geq \gamma(k)$. Therefore, $K_2(k) = \gamma(k)$. □

Corollary 3.15. *$K_d(k)$ exists for $d \geq 2$, and $K_d(k) \leq \gamma(k)$*

The computation of the numbers $K_d(k)$ for $d > 2$ appears to be difficult, in general. We will calculate $K_d(d + 2)$. Before we do this, we recall a result of Morris and Soltan [MS00] to indicate the kind of combinatorial restrictions that lead to analogous results for the clique number. The *Carathéodory number* of a convex geometry $(X, \mathscr{L})$ is the least positive integer c such that $\mathscr{L}(Y) = \cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq c\}$ for any $Y \subseteq X$.

Let c be the Carathéodory number of a convex geometry $(X, \mathscr{L})$, and suppose that every

$c-1$-element subset of X is closed. We say that $(X, \mathscr{L})$ satisfies the *simplex partition property* if for any set $\{z_1, z_2, \ldots, z_{c+2}\}$ of $c+2$ elements of X with $z_{c+1}, z_{c+2} \in \mathscr{L}(z_1, z_2, \ldots, z_c)$, the point z_{c+2} belongs to exactly one of the sets $\mathscr{L}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_c, z_{c+1})$, $i = 1, \ldots, c$. We state a result of Morris and Soltan [MS00].

Proposition 3.16 ([MS00], 5.6)**.** *Let $(X, \mathscr{L})$ be a $(c-1)$-free convex geometry. If $(X, \mathscr{L})$ has Carathéodory number c, the simplex partition property, and $|X| = c+2$, then X contains $c+1$ convexly independent points.*

The analogous result for chromatic number does not require the simplex partition property or any condition on the Carathéodory number, only that every $c-1$ element subset be closed.

Theorem 3.17. *$K_{c-1}(c+1) = c+2$ for $c \geq 2$.*

Proof. The example from Section 3.1, $([c+1], \mathscr{L}_{c-1,c+1})$, is realizable by a $(c-1)$-simplex with a point in the interior. The copoints of the form $[c+1]\backslash\{i\}$ for $i = 2, 3, \ldots, c+1$ form a c-clique. The remaining copoints are $[c+1]\backslash\{1, i\}$ for $i = 2, 3, \ldots, c+1$. For $i = 2, 3, \ldots, c+1$, the copoint $[c+1]\backslash\{1, i\}$ can be colored with the same color as $[c+1]\backslash\{i\}$, so $\chi(\mathcal{G}([c+1], \mathscr{L}_{c-1,c+1})) = c$. Thus $K_{c-1}(c+1) \geq c+2$.

Let $X = \{q_1, q_2, p_1, \ldots, p_c\}$ and assume that $q_1, q_2 \in \mathscr{L}(p_1, \ldots, p_c)$. Further, consider the copoints of $(X, \mathscr{L})$. If there is a convexly independent set of size $c+1$, we have the conclusion. So we may assume that there is no convexly independent set of size $c+1$. There are copoints $(p_i, q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$ for $i = 1, \ldots, c$. Also, there are copoints of the form $(q_{j_i}, q_{k_i} p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$, because the set $\{q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c\}$ is closed and the set $\{q_{k_i} p_1 \ldots p_i \ldots p_c\}$ is not closed. We see that for $i_1, i_2 \in \{1, \ldots c\}, i_1 \neq i_2$, $p_{i_1} \in \{q_{k_{i_1}} p_1 \ldots p_{i_2-1} p_{i_2+1} \ldots p_c\}$ and $q_{j_{i_2}} \in \{q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c\}$, so these two copoints are adjacent. Suppose that $\chi(\mathcal{G}(X, \mathscr{L})) = c$, then the following copoints must be colored with the same color: $(p_i, q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c), (q_{j_i}, q_{k_i} p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$ for $i = 1, \ldots, c$. Consider the closed set $q_{j_1} p_2 \ldots p_c$, which is a copoint attached to q_{k_1} because $\{q_{j_1} p_1 p_2 \ldots p_c\}$

is not a closed set. The copoint $(q_{k_1}, q_{j_1}p_2 \ldots p_c)$ is adjacent to some copoint in each color class because $q_{k_1} \in \{q_1q_2p_1 \ldots p_{i-1}p_{i+1} \ldots p_c\}$ and $p_i \in \{q_{j_1}p_2 \ldots p_c\}$ for $i = 2, \ldots c$. In addition, $q_{k_1} \in \{q_{k_1}p_2 \ldots p_c\}$ and $q_{j_1} \in \{q_{j_1}p_2 \ldots p_c\}$. Thus, $\chi(\mathcal{G}(X, \mathscr{L})) \geq c + 1$. □

Consider the convex geometry $([n], \mathscr{L}_{m,n})$. We show that the following:

Proposition 3.18. *The convex geometry $([n], \mathscr{L}_{m,n})$ has Carathéodory number $m + 1$ and never has simplex partition property.*

Proof. As noted in Lemma 3.3, every set in $\mathscr{L}_{m,n}$ can be written as the closure of at most $m+1$ elements of $[n]$. Therefore the Carathéodory number is at most $m+1$. In particular $[n] = \mathscr{L}(Y)$ where $Y = [n]-[n-m-1]$. Consider $Z \subseteq Y$ where $|Z| \leq m$, since $\mathscr{L}_{m,n}$ is m-free $\mathscr{L}(Z) = Z$. So $\cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq m\} = \cup\{Z : Z \subseteq Y, |Z| \leq m\} = Y \neq \mathscr{L}(Y) = [n]$. Thus, the Carathéodory number of $([n], \mathscr{L}_{m,n})$ is $m + 1$.

If $n < m + 3$, then $([n], \mathscr{L}_{m,n})$ does not satisfy the hypothesis of the simplex partition property. Let $n \geq m + 3$. To see that $([n], \mathscr{L}_{m,n})$ does not have the simplex partition property, we look at the set $[m+3]$. This convex geometry has Carathéodory number $m+1$ with every m element subset closed. Both 1 and 2 are contained in $\mathscr{L}([m + 3] - \{1, 2\})$, however 1 is in every set $\mathscr{L}([m + 3] - \{1, i\})$ for $3 \leq i \leq m + 3$. This violates the simplex partition property that 1 belong to only one of the sets $\mathscr{L}([m + 3] - \{1, i\})$ for $3 \leq i \leq m + 3$. □

3.2.1 Necessary Conditions for Realizable Convex Geometries in General Position

There are several known necessary conditions for a convex geometry to be realizable by a set of points X in $\mathbb{R}^2$ in general position.

Proposition 3.19. *Let X be a planar point set in general position and $(X, \mathscr{L})$ the realizable convex geometry of X. Then $(X, \mathscr{L})$ has Carathéodory number 3 and the simplex partition property.*

Proof. Let $Y \subseteq X$, then $\mathscr{L}(Y) = \text{conv}(Y) \cap X$. $\text{conv}(Y)$ can be rewritten by Carathéodory's Theorem as $\cup\{\text{conv}(Z) : Z \subseteq Y, |Z| \leq 3\}$. So, $\mathscr{L}(Y) = \text{conv}(Y) \cap X = (\cup\{\text{conv}(Z) : Z \subseteq Y, |Z| \leq 3\}) \cap X = \cup\{\text{conv}(Z) \cap X : Z \subseteq Y, |Z| \leq 3\} = \cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq 3\}$ and $(X, \mathscr{L})$ has Carathéodory number 3.

To see that $(X, \mathscr{L})$ has the simplex partition property, we note that every planar point set in general position of 5 points with 2 interior points, must look like the point set in Figure 1.1. The result becomes obvious. □

Another property that planar point sets in general position have, is that complements of copoints are closed.

Proposition 3.20. *Let X be a planar point set in general position and $(X, \mathscr{L})$ the realizable convex geometry of X. Then the complement of every copoint is closed.*

Proof. Copoints of planar point sets in general position are intersections of X with specific open halfspaces that have bounding lines through the point to which the copoint is attached. The complement of an open halfspace is a closed halfspace. A closed halfspace is a convex set and that part of X contained in that closed set must be in $\mathscr{L}$. □

Let $n > 2m + 1$, then $A = \{2, 3, \ldots, m + 1\}$ is a copoint of $\mathscr{L}_{m,n}$ attached to 1. The complement of A is $\{1, m + 2, m + 3, \ldots, n\}$ and $\{m + 2, m + 3, \ldots, n\}$ has size at least $m + 1$. Every closed set in $\mathscr{L}_{m,n}$ can be written as a closure of an initial segment together with a set of size no more than m other elements. The initial segment of the complement of A is 1 together with a set of size at least $m + 1$. So, the complement of A is not in $\mathscr{L}_{m,n}$. If $n \leq 2m + 1$, the complement of every copoint is closed, because every complement of a copoint is of size m or less and $\mathscr{L}_{m,n}$ is an m-free convex geometry.

Chapter 4: Results for Realizable Convex Geometries

4.1 Coloring of Planar Point Sets

Let X be a set of n points in $\mathbb{R}^2$ and for $A \subseteq X$ let $\mathscr{L}(A) = \text{conv}(A) \cap X$. The convex geometry $(X, \mathscr{L})$ is atomic as $\mathscr{L}(p) = \text{conv}(p) \cap X = p$ for every $p \in X$. In this chapter we abbreviate the realizable convex geometry $(X, \mathscr{L})$ as X. We also denote $\mathcal{G}(X, \mathscr{L})$ by $\mathcal{G}(X)$ and $\mathcal{H}^C_{L_{\mathscr{L}}}$ by $\mathcal{H}(X)$. Figure 4.1 is a planar point set in general position with its poset of copoints.

Morris [Mor06] describes an algorithm to compute all the copoints of a planar point set in general position. Start with a directed vertical line through $p \in X$. Call the part of the line above p the *head* and the part of the line below p the *tail*. Rotate the line clockwise around p, noting the order in which the points of $X - p$ are met by the line. If a point q is met by the head of the line, write q, and if q is met by the tail of the line, write $-q$. The sequence of $2n - 2$ symbols written as the line makes a complete revolution around p, viewed as a circular sequence, is called the *circular local sequence* of p. At one or more places in the circular local sequence of p there will be an element q followed by an element

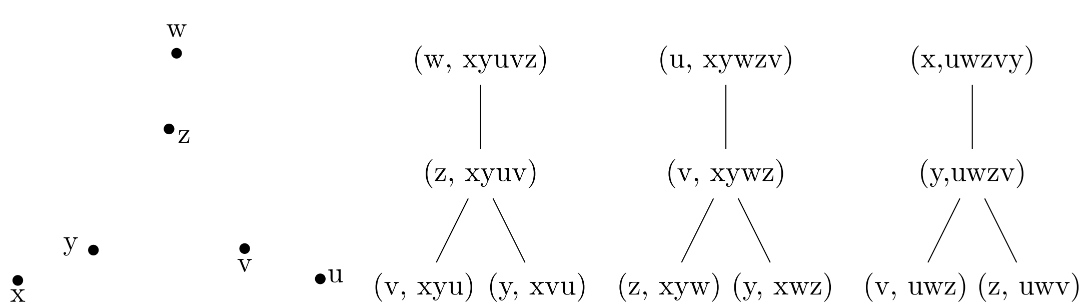

Figure 4.1: A six point set and its poset of copoints

$-r$. At such a place we can find a copoint. Let m be a line through p of which q and r are on the same side. Let H be the open halfspace defined by m that contains q and r. Then $H \cap X$ is a copoint attached to p.

Let B be a consecutive subsequence in the circular local sequence of $p \in X$. We say B is a *block* if all $b \in B$ are met by the same part of the line ℓ rotated around p.

Given two disjoint planar point sets L and M, we define a *composition* of L and M to be a point set of L together with a translation of M in which

1. every point of M has greater first coordinate than the first coordinates of points of L,
2. the slope of any line connecting a point of L to a point of M is greater than the slope of any line connecting two points of L or two points of M.

We easily see as a result of the second condition, that the circular local sequence of each $l \in L$ contains $-M$ and M as blocks and that the circular local sequence of each $m \in M$ contains $-L$ and L as blocks. We use the composition of point sets to construct point sets of particular interest following Corollary 4.4.

Lemma 4.1. *Let X be a planar point set such that X is a composition of L and M. If A is a copoint of X attached to $\alpha(A) \in L$ that contains $m \in M$, then $M \subseteq A$. Also, if B is a copoint of X attached to $\alpha(B) \in M$ that contains $l \in L$, then $L \subseteq B$.*

Proof. We show that if A is a copoint of X attached to $\alpha(A) \in L$ that contains $m \in M$, then $M \subseteq A$. The proof of the other statement is similar. A is the intersection of X with an open halfplane H defined by a line through $\alpha(A)$ that contains both $q, r \in X$, where q and $-r$ are consecutive symbols in the circular local sequence of $\alpha(A)$. M is a block in the circular local sequence of $\alpha(A)$ and there exists $m \in M \cap A$, so $M \subseteq A$. □

For X a point set in general position, we say X is a *composition of point sets* $L_1, L_2, \ldots, L_n$ if there exists a full rooted binary tree, $T(X)$ for which

1. the leaves of the tree are denoted $N_{L_1}, N_{L_2}, \ldots, N_{L_n}$ corresponding to $L_1, L_2, \ldots, L_n$ respectively,

2. every node N_M of $T(X)$ that is not a leaf corresponds to a point set M which is a composition of two point sets M_1 and M_2, where N_{M_1} and N_{M_2} are descendants of N_M in $T(X)$,

3. the root of the tree is N_X corresponding to X.

For a node M of $T(X)$ let P_M be the unique path from N_M to N_X in $T(X)$.

Let $p \in M \subseteq X$, with M being a composition of M_1 and M_2. Suppose that $p \in M_1$. We know that the circular local sequence of p in M contains M_2 as a block. The requirement that the the slope of any line connecting p to a point outside of M must have greater slope than any line connecting two points of M means that M_2 is a block in the circular local sequence for p in X. Similarly, if $p \in M_2$, the circular local sequence of p in X contains M_1 as a block.

Lemma 4.2. *Let X be a composition of point sets $L_1, L_2, \ldots, L_n$. If A is a copoint of X attached to $\alpha(A) \in L_i$ and $A \cap L_j \neq \emptyset$ for some $j \neq i$, then $L_j \subseteq A$.*

Proof. There is at least one node common to the sequences P_{L_i} and P_{L_j}, namely N_X. So there is a first node, N_M, common to both paths. M is a composition of M_1 and M_2 where N_{M_1} is an ancestor of N_{L_i} but not N_{L_j}, and N_{M_2} is an ancestor of N_{L_j} but not N_{L_i}. M_2 is a block in the circular local sequence of $\alpha(A)$ in M, so M_2 is a block in the circular local sequence of $\alpha(A)$ in X. Since A is a copoint of X attached to $\alpha(A) \in M_1$ that contains $p \in M_2$, it follows from Lemma 4.1 that $M_2 \subseteq A$ and $L_j \subseteq A$. □

We are now able to show a result on an arbitrary composition of point sets and the hyperedges of their associated hypergraph, $\mathcal{H}(X)$.

Theorem 4.3. *Let X be a composition of point sets $L_1, L_2, \ldots, L_n$. If $\mathcal{H}(X)$ contains a hyperedge $\mathcal{E}$ such that $|\mathcal{E}| > 2$, then $\alpha(\mathcal{E}) \subseteq L_i$ for some $i = 1, 2, \ldots, n$.*

Proof. Suppose that $(A_1, A_2, \ldots, A_l)$ is a cycle of copoints corresponding to the hyperedge $\{A_1, \ldots, A_l\} = \mathcal{E}$ of $\mathcal{H}(X)$ such that $|\mathcal{E}| > 2$. Let $I = \{i \in \{1, 2, \ldots, n\} : L_i \cap \alpha(\mathcal{E}) \neq \emptyset\}$. If

$|I| = 1$, then the conclusion is satisfied.

Suppose that $|I| > 1$. Consider all paths P_{L_i} in $T(X)$ for $i \in I$. These paths have at least one node in common, N_X, so there is a first common node to all paths P_{L_i}, call this node N_M. There exist point sets M_1 and M_2 and subsets I_1 and I_2 so that M is the composition of M_1 and M_2 with N_{L_i} a descendant of N_{M_1} if $i \in I_1$ and N_{L_i} a descendant of N_{M_2} if $i \in I_2$. The sets I_1 and I_2 must be disjoint and non-empty, because N_M is the first node in common to all paths. By the proof of Lemma 4.2, if any copoint A is attached to $\alpha(A) \in M_1$ and contains $p \in M_2$, then $M_2 \subseteq A$. So, let $\alpha(A_1) \in M_1$ and $\alpha(A_2) \in M_2$. Then, there is some $k \geq 2$ such that $\alpha(A_k) \in M_2$ and $\alpha(A_{k+1}) \in M_1$ so $M_2 \subseteq A_{k+1}$. This means that $\alpha(A_{k+1}) \in A_2$ and $\alpha(A_2) \in A_{k+1}$. Thus, $\alpha(\mathcal{E}) \subseteq M_1$ or $\alpha(\mathcal{E}) \subseteq M_2$. If $\alpha(\mathcal{E}) \subseteq M_1$, then $\alpha(\mathcal{E}) \cap M_2 = \emptyset$, so there is no $i \in I$ with $L_i \subseteq M_2$. This contradicts the minimality of M. □

From Theorem 4.3 we get the following corollary.

Corollary 4.4. *If the point set X is the composition of singletons, then $\mathcal{H}(X) \cong \mathcal{G}(X)$.*

Proof. Since X is the composition of singletons, Theorem 4.3 shows that if there is a hyperedge of size greater than 2 in $\mathcal{H}(X)$, every copoint of the hyperedge must be attached to the same point. This violates Proposition 2.7 on minimal cycles. Thus, $\mathcal{H}(X) \cong \mathcal{G}(X)$. □

Erdős and Szekeres [ES61] describe a construction of large point sets without large subsets in convex position. We describe these point sets with the notation of [Mor06]. For all positive integers, k, we first define $ES(0, k)$ and $ES(k, 0)$ to be singletons. For $i \geq 1, j \geq 1$ define $ES(i, j)$ to be a composition of $ES(i - 1, j)$ and $ES(i, j - 1)$.

The extended Erdős-Szekeres point set $XES(k)$ is a composition of $ES(0, k), ES(1, k - 1), \cdots, ES(k, 0)$ where the compositions are performed in order from left to right. The number of points in $XES(k)$ is 2^k. The size of the largest independent set in $XES(k)$ is $k + 1$ ([Mor06]). We apply Corollary 4.4 to the point set $XES(k)$ and conclude that

$\dim(XES(k)) = k + 1$.

Corollary 4.5. *$\mathcal{H}(XES(k)) \cong \mathcal{G}(XES(k))$. Further, $\dim(XES(k)) = k + 1$ for positive integers k.*

Proof. Since $XES(k)$ is the composition of 2^k singletons, Corollary 4.4 implies that $\mathcal{H}(XES(k)) \cong \mathcal{G}(XES(k))$.

It was shown that $\chi(\mathcal{G}(XES(k))) = k+1$ by Morris [Mor06]. Therefore, $\chi(\mathcal{H}(XES(k))) = k + 1$ and the order dimension of $XES(k)$ is $k + 1$ by Corollary 2.6. □

The Erdős-Szekeres conjecture is an upper bound on the size of the planar point set in general position for a given size of the largest convex polygon in the set. We use a theorem of Morris [Mor06] to come to a conclusion on the maximum size of a planar point set in general position with fixed order dimension. Morris studied pseudoline arrangements, however planar point sets in general position are equivalent to stretchable pseudoline arrangements.

Theorem 4.6 ([Mor06], 4.5)**.** *If L is a pseudoline arrangement and $\chi(\mathcal{G}(L)) = k$, then $|L| \leq 2^{k-1}$.*

As a direct corollary to this theorem, we are able to bound the size of a planar point set in general position by a function of its order dimension. This implies that any planar point set in general position of size greater than 2^{n-2} must have order dimension at least n.

Corollary 4.7. *If X is a planar point set in general position and $\dim(X) = k$, then $|X| \leq 2^{k-1}$.*

We have shown in Proposition 1.5 that k-cliques of $\mathcal{G}(X, \mathscr{L})$ correspond to independent sets of size k, so the size of the largest independent set of $(X, \mathscr{L})$ is $\omega(\mathcal{G}(X, \mathscr{L}))$. We have shown in Chapter 3 that there is no constant c such that $\frac{\chi(\mathcal{G}(X,\mathscr{L}))}{\omega(\mathcal{G}(X,\mathscr{L}))} \leq c$ for a convex geometry, but we do not know this result for convex geometries realized by planar point sets.

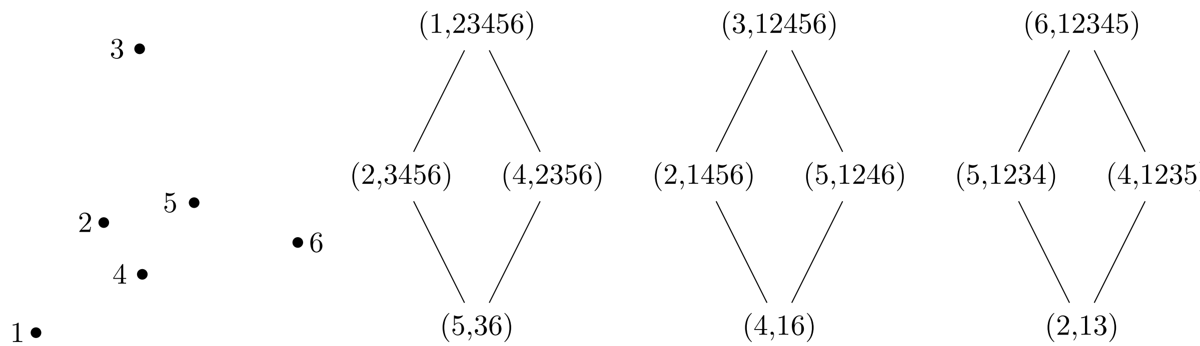

Figure 4.2: Smallest point set with different clique and chromatic number and its poset of copoints

4.2 Inequality of Clique and Chromatic Number of the Copoint Graph for Planar Point Sets

As has been noted on multiple occasions in [Mor06] and [Beaar], there are examples of planar point sets in general position where $\omega(\mathcal{G}(X,\mathscr{L})) < \chi(\mathcal{G}(X,\mathscr{L}))$. Of the 16 order types of 6 planar points in general position [AAK02], there is only one with this property. It is given in Figure 4.2. The copoints are shown to the right of the point set, in the form $(\alpha(C), C)$ where $\alpha(C)$ is the point to which copoint C is attached. The copoints are partially ordered by set containment. The subgraph of $\mathcal{G}(X,\mathscr{L})$ induced by the copoints of $(X,\mathscr{L})$ of size bigger than 3 form the complement of a 9-cycle. This graph has chromatic number 5 and clique number 4. To see that the chromatic number must be 5, note that the size of the largest independent set in the graph is of size 2. Thus, $\chi(\mathcal{G}(X)) \geq \lceil 9/2 \rceil = 5$.

We have used the database of Aichholzer et. al. [AAK02], to compute the number of distinct point sets in general position for which the chromatic number and clique number of the copoint graph are not equal. The results are in Table 4.1. It should also be noted that for all planar point sets in general position the difference between the chromatic number and clique number of $\mathcal{G}(X,\mathscr{L})$ is at most 1.

Of the 8 point sets of size 7, 5 of these contain the point set in Figure 4.2. We provide

n	# of Order Types	$\chi = 5, \omega = 4$	$\chi = 6, \omega = 5$	$\chi = 7, \omega = 6$	$\chi = 8, \omega = 7$
3	1	0	0	0	0
4	2	0	0	0	0
5	3	0	0	0	0
6	16	1	0	0	0
7	135	8	0	0	0
8	3315	51	34	0	0
9	158817	0	7949	0	0
10	14309547	0	1206402	20258	1

Table 4.1: Number of Order Types with distinct Chromatic and Clique Number

the other 3 point sets along with an induced subgraph of copoints that have chromatic number 5 and clique number 4 in Figures 4.3, 4.4, and 4.5. To see that the chromatic number must be 5, note that the size of the largest independent set in the graph is of size 2. Thus, $\chi(\mathcal{G}(X)) \geq \lceil 9/2 \rceil = 5$.

4.3 Describing Hyperedges of $\mathcal{H}(X, \mathscr{L})$ for a Realizable Convex Geometry

Let $(X, \mathscr{L})$ be a convex geometry realized by a set in $\mathbb{R}^d$. If A is a copoint of $(X, \mathscr{L})$ attached to $\alpha(A)$, then $\alpha(A) \notin \text{conv}(A)$, so $\text{conv}(A)$ and $\alpha(A)$ can be properly seperated by a hyperplane H_A in $\mathbb{R}^n$ (Theorem 2.4.10, [Web94]) with $\alpha(A) \in H_A$ and $\text{conv}(A)$ a subset of an open halfspace defined by H_A. Since open halfspaces with one $p \in X$ on its boundary are maximal convex sets not containing p, all copoints of $(X, \mathscr{L})$ can be represented by such H_A. We use this description of copoints to study $\alpha(\mathcal{A})$ where $\mathcal{A}$ is minimal cycle of $(X, \mathscr{L})$.

Proposition 4.8. *Let $(X, \mathscr{L})$ be the convex geometry of points in $\mathbb{R}^d$. If $\mathcal{A}$ is a minimal cycle of copoints with length l, that is $\mathcal{A} \in E(\mathcal{H}^C_{L_\mathscr{L}})$ with $|\mathcal{A}| = l$, then $\alpha(\mathcal{A})$ is the vertex set of a polytope with l vertices.*

Proof. The result is trivial if $l = 2$, so let $l > 2$. Suppose $\alpha(\mathcal{A})$ is not the vertex set of a

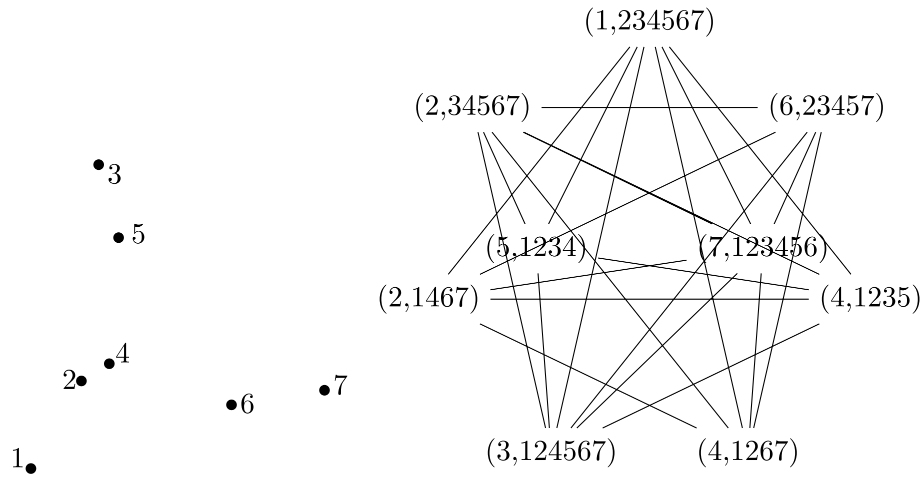

Figure 4.3: 7 point set in general position with an induced subgraph of copoints

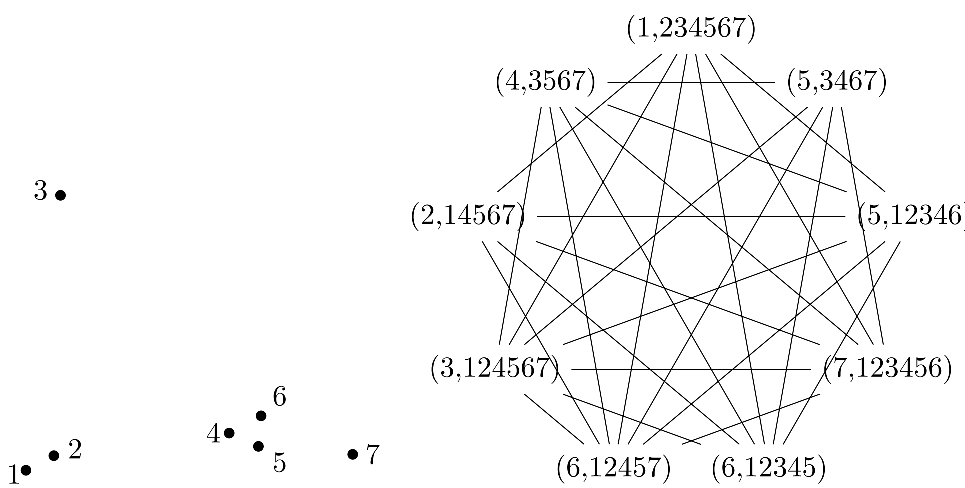

Figure 4.4: 7 point set in general position with an induced subgraph of copoints

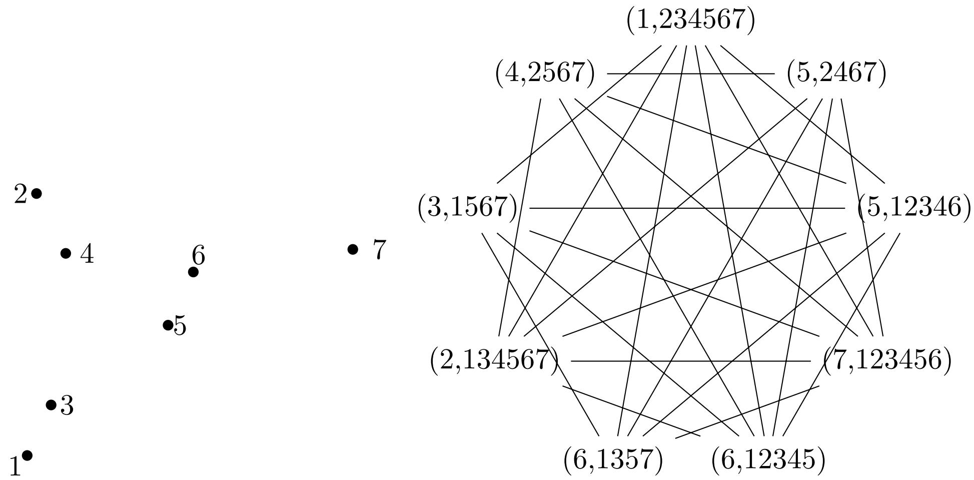

Figure 4.5: 7 point set in general position with an induced subgraph of copoints

polytope with l vertices, then there is at least one $\alpha(A_i)$ such that $\alpha(A_i) \in \mathscr{L}(\{\alpha(A_j) : j \neq i\}) = C$. By assumption, $\alpha(A_i) \in A_{i+1}$ so there is some $\lambda > 0$ such that $\lambda\alpha(A_i) + (1-\lambda)\alpha(A_{i+1}) = x \in \text{rbd}(\text{conv}(C))$, the *relative boundary* of conv(C). So, $x \in F$ a face of conv(C), with $W \subset C$, $F = \text{conv}(W)$. Then, either $x \in W$ or $x \in \text{rint}(F)$, the *relative interior* of F. It is clear that any open halfspace with $\alpha(A_{i+1})$ on its boundary that contains $\alpha(A_i)$ must also contain x. In particular, there is a hyperplane $H_{A_{i+1}}$ with $\alpha(A_{i+1})$ on its boundary with an open halfspace containing $\alpha(A_i)$ and hence x. Thus, $x \in A_{i+1}$, so $\mathcal{A}$ was not a minimal cycle.

If $x \in \text{rint}(F)$, then we write $x = \sum_{j=1}^{k} \lambda_j w_j$ for $w_j \in W$, $\lambda_j > 0$, and $\sum_{j=1}^{k} \lambda_j = 1$. Let $H = \{y \in \mathbb{R}^n : y \cdot a = \alpha_0\}$ be a hyperplane of $\mathbb{R}^n$ containing $\alpha(A_{i+1})$ and x (and thus $\alpha(A_i)$). So, $x \cdot a = \alpha_0 = (\sum_{j=1}^{k} \lambda_j w_j) \cdot a = \sum_{j=1}^{k} \lambda_j (w_j \cdot a)$. Since $\lambda_j > 0$, if there is some w_{j_1} such that $w_{j_1} \cdot a > \alpha_0$, then there is some and $w_{j_2} \cdot a < \alpha_0$. In particular, there is a hyperplane $H_{A_{i+1}} = \{y \in \mathbb{R}^n : y \cdot a_{i+1} = \alpha_{i+1}\}$ with $\alpha(A_i) \cdot a_{i+1} < \alpha_{i+1}$ and hence

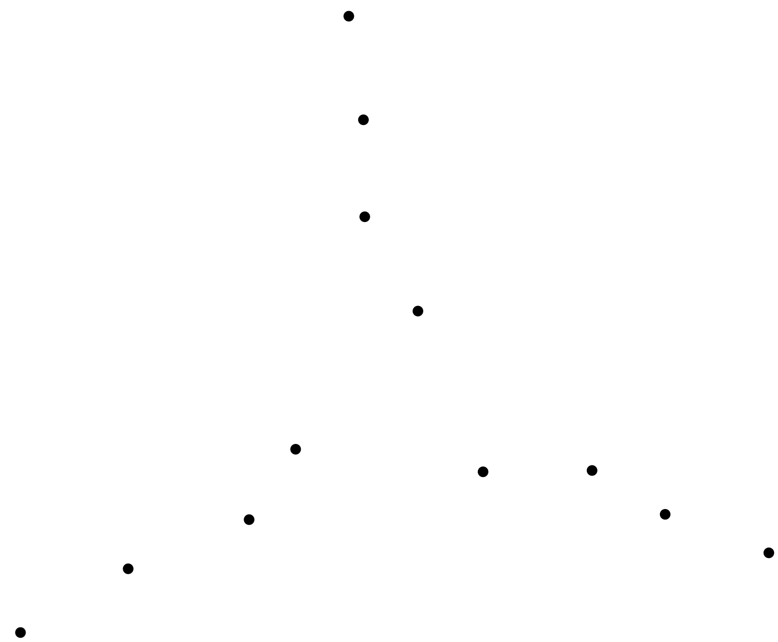

Figure 4.6: Welzl's Little Devil

$x \cdot a_{i+1} = (\sum_{j=1}^{k} \lambda_j w_j) \cdot a_{i+1} = \sum_{j=1}^{k} \lambda_j (w_j \cdot a_{i+1}) < \alpha_{i+1}$. Thus, there is a $w_j \in W$ such that $w_j \cdot a_{i+1} < \alpha_{i+1}$. Therefore, the open halfspace that contains $\alpha(A_i)$ also contains $\alpha(A_k) = w_j \in W \subset C$ with k not equal to i or $i+1$. This implies that $\mathcal{A}$ was not a minimal cycle, which is a contradiction. □

We would also note that there are many well-known examples of planar point sets in general position where the hypergraph $\mathcal{H}(X)$ contains many hyperedges of size larger than 2. These examples are known as Welzl's Little Devils [GPP08] and are known to contain many lines through two points that evenly divide the remaining points. We provide an example on 12 points in Figure 4.6, this contains 63 hyperedges of size 3 with $\chi(\mathcal{G}(X)) = \chi(\mathcal{H}(X) = 6$. Every example of a convex geometry we have studied satisfies $\chi(\mathcal{G}(X, \mathscr{L})) = \chi(\mathcal{H}(X, \mathscr{L}))$.

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Curriculum Vitae

Research Interests

- Discrete Geometry (Combinatorial Geometry)
- Graph Theory
- Partially Ordered Sets

Education

George Mason University, Fairfax, VA USA

Ph.D, Mathematics, expected May, 2013

- Dissertation Title: "Extremal Combinatorics in Geometry and Graph Theory"
- Advisor: Dr. Walter D. Morris

M.S. Mathematics, January, 2012

Illinois Institute of Technology, Chicago, IL USA

B.S. Applied Mathematics, May, 2008

Experience

George Mason University, Fairfax, VA USA

Graduate Teaching Assistant & Adjunct Faculty **July, 2011 - January, 2013**

Institute for Defense Analyses, Alexandria, VA USA

Summer Intern VI **May, 2010 - August, 2010**

System Planning and Analysis, Inc., Alexandria, VA USA

Operations Research Intern **May, 2009 - August, 2009**

Publications

- "Properties of the Copoint Graph of Convex Geometries," Submitted 2013
- "Chromatic Numbers of Copoint Graphs of Convex Geometries," Submitted 2012, (with W. Morris).

- "On the Order Dimension of Convex Geometries," Accepted 2012, *Order*.
- "Bounds for Minimum Semidefinite Rank from Superpositions and Cut Sets," *Linear Algebra and its Applications*, 438:4041-4061, 2013 (with L. Mitchell, S. Narayan, E. Radzwion, S. Rimer, R. Tomasino, J. Wolfe, A. Zimmer).
- "Work in Progress - Improving Interrater Agreement Used to Measure Learning Outcomes," *Conference Proceedings of IEEE Frontiers in Education*, 2008 (with H. Shi, D. Ferguson, and M. Huyck).

Talks

- "The Order Dimension of Convex Geometries," CAGS Seminar, George Mason University. November, 2011.
- "The Order Dimension and Coloring of Planar Point Sets," Joint Mathematics Meetings, Boston, MA, USA. January, 2012.
- "The Order Dimension and Coloring of Planar Point Sets," SIAM Conference on Discrete Math, Halifax, NS, Canada. June, 2012.
- "A Happy Coloring Theorem," CAGS Seminar, George Mason University. October, 2012.
- "An Esther Klein Type Coloring Theorem," Joint Mathematics Meetings, San Diego, CA, USA. January, 2013.
- "Properties of the Copoint Graph of Convex Geometries", SEICCGTC 2013, Boca Raton, FL, USA. March, 2013.
- "Properties of the Copoint Graph of Convex Geometries", CAGS Seminar, George Mason University, March, 2013.
- "Convex Geometries and the Copoint Graph", Mathematics Graduate Student Seminar, George Washington University, March 2013.
- "Convex Geometries and the Copoint Graph", Discrete Mathematics Seminar, Virginia Commonwealth University, April 2013.

Professional and Volunteer Activities

- Member, Mathematical Association of America
- Member, American Mathematical Society
- Volunteer, Fairfax County Animal Shelter
- Assistant to the Financial Secretary, Living Savior Lutheran Church

Computer Skills

Languages: LaTeX, VBA, Java
Software: Maple

Honors/Awards

- Camras-Heald Scholar, Illinois Institute of Technology
- NSF MASS Fellowship, Pennsylvania State University
- Penn State MASS Fellowship, Pennsylvania State University
- Presidential Scholarship, George Mason University
- Graduate Teaching Assistantship, George Mason University
- Provost's Dissertation Completion Grant, George Mason University
- Winner 'Best Student Paper Competition' at the Forty-Fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing