ZERO SUM PROPERTIES IN GROUPS

by

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Abstract

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A conjecture by Erdős and Lemke in elementary number theory goes as follows: If d is a divisor of n and we have d divisors of n, say $a_1, \ldots, a_d$, not necessarily distinct, can we always find a subsequence among them such that their sum is (i) divisible by d, and (ii) at most n? – This was proved by Lemke and Kleitman to be indeed the case. They also noted that an equivalent version of their theorem, stated in terms of the additive cyclic group $G = \mathbb{Z}_n$ is as follows: Every sequence of n elements of G, not necessarily distinct, contains a subsequence $g_1, \ldots, g_k$ such that $g_1 + \cdots + g_k = 0$ and $\sum_{i=1}^{k} 1/|g_i| \leq 1$. This has been shown to be correct for every finite abelian group G. Hence a natural question is therefore if this holds true for any finite group G. By the aid of a computer this has been verified for all solvable groups of order 21 or less, but it is still not known whether it holds for all finite groups. – This paper proves that some well-known non-abelian groups have this property, for example the alternating groups A_n and symmetric groups S_n for $n = 3, 4, 5, 6$, the dihedral group D_n for every n and the dicyclic group of every order. Some speculations on possible plan of attack for S_n for larger n are finally discussed.

Chapter 1: Erdős-Lemke Conjecture and Zero-Sum

The original addition conjecture on the integers modulo n was framed by Paul Lemke and Paul Erdős in May 1987. It goes as follows:

> If d is a divisor of n, and suppose we have d divisors of n, not necessarily distinct, $a_1, a_2, \ldots, a_d$, can we always find a subsequence among these divisors so that (i) the sum is a multiple of d, and (ii) they sum to at most n?

In July 1988, Paul Lemke and Daniel Kleitman proved in [1] that for the case of $d = n$ this is indeed the case by proving the theorem below:

Theorem 1.0.1. *Given a positive integer d and integers a_1, a_2, $\ldots$, a_d, there exists a non-empty set $Q \subseteq \{1, 2, \ldots, d\}$ such that d divides $\sum_{i \in Q} a_i$ and $\sum_{i \in Q} \gcd(a_i, d) \leq d$.*

From the above theorem, we can obtain as the corollary, a positive answer to the original Erdős-Lemke conjecture as demonstrated in the Lemke-Kleitman paper [1]:

Corollary 1.0.2. *If d is a divisor of n, and $a_1, a_2, \ldots, a_d$ divide n, there is a non-empty subset S of $\{1, 2, \ldots, d\}$ such that d divides $\sum_{i \in S} a_i$ and $\sum_{i \in S} a_i \leq n$.*

Their proof goes as follows, where we have expanded it slightly:

Proof. From Theorem 1.0.1, there is a non-empty subset of $S = \{1, 2, \ldots, d\}$ with $d | \sum_{i \in S} a_i$ and $\sum_{i \in S} \gcd(a_i, d) \leq d$. Since a_i and d both divide n for all i, then $a_i = \gcd(a_i, n)$. Using the distributive property of gcd, we have $\gcd(a_i, n) \leq \frac{n}{d} \gcd(a_i, d)$ for each $i \in S$. Hence we have the following:

$$\sum_{i\in S} a_i = \sum_{i\in S} \gcd(a_i, n) \le \frac{n}{d} \sum_{i\in S} \gcd(a_i, d) \le \frac{n}{d} \cdot d = n.$$

$\square$

Lemke and Kleitman further conjectured, in the same paper, a more generalised version of the original theorem, the following, which is still open to this day for finite non-Abelian groups.

Conjecture 1.0.3. *Any sequence of $|G|$ elements (not necessarily distinct) of the finite group G contains a non-empty subsequence g_1, g_2, $\ldots$, g_k such that $g_1 g_2 \ldots g_k = e$ and $\sum_{i=1}^{k} \frac{1}{|g_i|} \le 1$.*

Please note that a subsequence need not have distinct elements, while a subset does.

We now follow the outline in [2] and point out that Corollary 1.0.2 is a special case of Conjecture 1.0.3 where G is a finite cyclic group. In short, for $G = (\mathbb{Z}_n, +)$, the two are equivalent. We can break this into two parts:

i. by proving that, given $g_k = \overline{a_k}$, if $\sum_{k\in Q} g_k = 0_G$, then $\sum_{k\in Q} a_k \equiv 0 \pmod{n}$ and vice versa;

ii. and by proving that if $\sum_{k\in Q} 1/|g_k| \le 1$, then $\sum_{k\in Q} \gcd(a_k, n) \le n$ and vice versa.

Before we start the proof, the following Lemma might come in handy:

Lemma 1.0.4. *For $\overline{a} \in \mathbb{Z}_n$, we have $|\overline{a}| = \dfrac{n}{\gcd(a, n)}$.*

Proof. Clearly, $\frac{n}{\gcd(a,n)} \cdot \overline{a} = \frac{a}{\gcd(a,n)} \cdot \overline{n} = \overline{0}$ in $\mathbb{Z}_n$.

Now, suppose $h \cdot \overline{a} = \overline{0}$ in $\mathbb{Z}_n$. Then $h \cdot a$ is divisible by n. Dividing both $h \cdot a$ and n by $\gcd(a,n)$, we see that $h \cdot \frac{a}{\gcd(a,n)}$ is divisible by $\frac{n}{\gcd(a,n)}$. Since $\frac{a}{\gcd(a,n)}$ and $\frac{n}{\gcd(a,n)}$ are relatively prime, then h has to be divisible by $\frac{n}{\gcd(a,n)}$.

□

Using the above lemma, we now prove the previous statement:

Proof. The first part of this is self-evident: let $G = (\mathbb{Z}_n, +)$. $0_G = \sum_{k \in Q} \overline{a_k} = \overline{\sum_{k \in Q} a_k}$, which means that $\sum_{k \in Q} a_k \equiv 0 \pmod{n}$.

The second part of this is slightly more involved. By Lemma 1.0.4, we have:

$$\begin{aligned} \sum_{k \in Q} \frac{1}{|g_k|} \leq 1 \quad &\Leftrightarrow \quad \sum_{k \in Q} \frac{1}{|\overline{a_k}|} \leq 1 \\ &\Leftrightarrow \quad \sum_{k \in Q} \frac{\gcd(a_k, n)}{n} \leq 1 \\ &\Leftrightarrow \quad \sum_{k \in Q} \gcd(a_k, n) \leq n \end{aligned}$$

□

Hurlbert, in [2], pointed out that $\sum_{k \in Q} \frac{1}{|g_k|} \leq 1$ implies that $|Q| \leq |G|$, with equality if and only if the order of every element in the subsequence is $|G|$.

Hurlbert further mentioned that using computer, Conjecture 1.0.3 has been shown to hold for non-abelian solvable groups with 21 elements. It is also noted that considering solvable group in general might be interesting, since the method in [2] relies on the abelian property. It is commented that if one were to consider subsets instead of subsequences, the solution presented might still hold.

It is the purpose of the rest of this article to explore the zero sum property for certain well known non-abelian finite groups, most of which with order larger than 21.

To simplify description of groups where Conjecture 1.0.3 holds, the following folklore definition shall be used:

Definition 1.0.5. *A finite group G of n elements is said to have the* zero-sum property *if Conjecture 1.0.3 holds true for G.*

Chapter 2: Reminders and Preliminaries for Zero Sum and Non-Abelian Groups

In what follows, we shall assume the results from Hurlbert and Chung, in [2] and [3] respectively, that the zero sum property holds for all finite abelian groups.

The proofs in [2] and [3] are interesting from the graph theoretic perspective, in that they are based on graph pebbling to prove the zero sum property.

In this section, we shall remind ourselves of concepts that we might need to show that the zero sum property holds for some symmetric group, S_n.

First, we shall remind ourselves of the cyclic group. We are introduced to the concept of *cyclic group* in Algebra I. A cyclic group is a group that is generated by a single element, operated with itself multiple times. The single element being operated with itself multiple times is called the *generator* of the cyclic group. The order of the cyclic group is the order of the generator. Any finite cyclic group is isomorphic to the additive quotient group $\mathbb{Z}/n\mathbb{Z}$ where n is the order of the cyclic group, and $\mathbb{Z}$ is the group of integers.

We also know that every group has a cyclic subgroup (not necessarily a proper subgroup). We can form this group by simply taking an element and keep operating with itself until we get back to identity. There might be cases where the cyclic group is infinite (i.e. the group $\mathbb{Z}$ with the generator 1), but in the case of symmetric groups, only the finite ones will occur. Please also note that cyclic groups are abelian.

A "helper" concept we shall refresh ourselves in is *conjugacy class*. Also in Algebra I, we are introduced to the concept of *conjugates*, that is for a and b in a group G, a and b are *conjugates* if there exists an element g in G where $gag^{-1} = b$. Also mentioned was that conjugacy is an equivalence relation, $a \sim b$, which means that conjugacy classes are equivalence classes. This means that every single element belongs to only one conjugacy class, and that conjugacy classes are disjoint from each other.

It is also worth noting that if two elements are in the same conjugacy class, they have the same order. And that if two elements are in the same conjugacy class, so are their powers: that is, if $a \sim b$, then $a^k \sim b^k$. Related to conjugacy class is the *partition.*

Before going into partition, we shall review *group covering.*

Definition 2.0.6. *A* group cover *of the group G is a collection of proper subgroups, $\{H_1, H_2, \dots, H_k\}$, with $H_i \subset G$ and $H_i \neq G$ for $i \in \{1, 2, \dots, k\}$ and $\bigcup_{i=1}^{k} H_i = G$. A group is* coverable *if a covering exists for said group.*

A few notes on coverable group:

Theorem 2.0.7. *A group G is coverable if and only if it is not cyclic.*

Proof. Let the group G be coverable, and $g \in G$ be an element of a subgroup, H, in the cover. To be a cover, then $H \neq G$, which means that g can not be a generator of G since if g is a generator, the whole of G will be in H. Therefore, if G is coverable, G is not cyclic.

On the other hand, if G is not cyclic, then for any $g \in G$, $\langle g \rangle \subset G$, and $\langle g \rangle \neq G$, which means that it is a proper subgroup. And if we do this for all $g \in G$, then $G = \bigcup_{g \in G} \langle g \rangle$.

Therefore, if G is not cyclic, G is coverable.

□

We now define *group partitioning*. Originally alluded to by Miller in [4] in 1906, in [5], Young defined *partition* to be:

Definition 2.0.8. *A* partition *of any group G is a set $\{H_1, H_2, \ldots, H_k\}$ of subgroups of G such that every element other than the identity of G is contained in one and only one H_i. In other words:*
A group G is partitioned *into subgroups H_1, H_2, $\ldots$, H_k if $G^* = \bigcup_{i=1}^{k} H_i^*$ and $H_i^* \cap H_j^* = \varnothing$ for $i \neq j$, where $G^* = G \setminus \{e\}$, and $H_i^* = H_i \setminus \{e\}$ for all $i = 1, 2, \ldots, k$.*

We can refer to [6] for more results in partitionable groups, and how to determine if a group is partitionable. Quoting Zappa in [6] without any justification, a group G is partitionable if and only if it satisfies one of the following conditions:

- G is a p-group with $H_p(G) \neq G$ and $|G| > p$;
- G is a Frobenius group;
- G is a group of Hughes-Thompson type;
- G is isomorphic to $PGL(2, p^h)$, with p being an odd prime;
- G is isomorphic to $PSL(2, p^h)$, with p being a prime;
- G is isomorphic to a *Suzuki* group $G(q)$, $q = 2^h$, $h > 1$.

Another theorem, which is a rephrase of Lemma 1.0.4, might come in handy:

Theorem 2.0.9. *Let G be a group. For $g \in G$ an element with order n, and $k \in \mathbb{Z}$, the order of g^k is $\dfrac{n}{\gcd(n,k)}$.*

Another related definition is the *Davenport Constant*, originally a problem proposed by H. Davenport in [8].

Definition 2.0.10. Davenport's Constant, *$D(G)$, is defined to be the smallest D such that every sequence of D elements contains a non-empty zero-sum subsequence.*

Chapter 3: Zero Sum and Isomorphism

Another notion that might be useful is the relationship between zero sum property and isomorphic groups. We shall start by a reminder of isomorphism.

Definition 3.0.11. *A* group isomorphism *is a bijective function, ϕ, between two groups, G and H that preserves the group operation. That is,*

$$\phi(a \cdot b) = \phi(a) \star \phi(b)$$

where $a, b \in G$, $\phi(a), \phi(b) \in H$, and the operation in G is $\cdot$ and the operation in H is $\star$.

Also remembering the following property from Algebra I, given two groups, G and H being isomorphic, with the group isomorphism $\phi : G \to H$, ϕ has the following properties:

i. $\phi^{-1} : H \to G$ is also an isomorphism.

ii. $\phi(e_G) = e_H$ and $\phi^{-1}(e_H) = e_G$.

iii. For $a \in G$, $\phi(a^n) = [\phi(a)]^n$. This also means that a and $\phi(a)$ have the same order.

Given the above, we shall posit that two isomorphic groups either both have zero sum or none does.

Theorem 3.0.12. *Given two isomorphic groups, G and H, and the isomorphism $\phi : G \to H$, if G has the zero sum property, then H does, too, and vice-versa.*

Proof. Let $\cdot$ be the operation for the group G and $\star$ be the operation for the group H. Let $\phi : G \to H$ be the isomorphism. Further, let G have the zero sum sequence, $g_1, g_2, \ldots, g_n$ where $g_1 \cdot g_2 \cdot \ldots \cdot g_n = e_G$.

From the properties of isomorphism, we have that $\phi(g_1), \phi(g_2), \ldots, \phi(g_n) \in H$, and since isomorphism preserves group operation, we have that $\phi(g_1 \cdot g_2 \cdot \ldots \cdot g_n) = \phi(g_1) \star \phi(g_2) \star \ldots \star \phi(g_n)$. Since $g_1 \cdot g_2 \cdot \ldots \cdot g_n = e_G$, and $\phi(e_G) = e_H$, we have that $\phi(g_1) \star \phi(g_2) \star \ldots \star \phi(g_n) = e_H$.

Also note that in isomorphic groups, $|g_k| = |\phi(g_k)|$. This gives us the same sum of reciprocal orders between both sides. That is, $\frac{1}{|g_1|} + \frac{1}{|g_2|} + \ldots + \frac{1}{|g_n|} = \frac{1}{|\phi(g_1)|} + \frac{1}{|\phi(g_2)|} + \ldots + \frac{1}{|\phi(g_n)|}$. Which gives us the other part of zero-sum, that is, the sum of the reciprocal orders.

Therefore, we have that $\phi(g_1), \phi(g_2), \ldots, \phi(g_n)$ is the zero sum sequence for H.

Since if $\phi : G \to H$ is an isomorphism, then $\phi^{-1} : H \to G$ is also an isomorphism, and we can use the same argument substituting G with H, and ϕ with ϕ^{-1} to show that the zero sum sequence in H would result in a zero sum sequence in G if one were to put in the elements in the zero sum sequence in H into ϕ^{-1}.

□

In fact, we can argue that if two groups are isomorphic, they have the same Davenport constant as follows:

Theorem 3.0.13. *Given two isomorphic groups, G and H, and the isomorphism $\phi : G \to H$, the Davenport constant for G, $D(G)$, and the Davenport constant for H, $D(H)$ are the same.*

Proof. Let us begin by assuming that $D(H) > D(G)$. This means that we can construct a sequence of $D(G)$ elements in H where there is no zero sum subsequence. However, in G, if we have a sequence of $D(G)$ elements, we would always have a zero sum subsequence.

Since G and H are linked by the isomorphism ϕ, we have that the sequences in G with $D(G)$ elements in them have images where the elements of this image of sequence consist of images of the elements in the sequence as mapped by ϕ.

This will mean that if any subsequence of the sequences of $D(G)$ elements in G sums up to identity (which some must, by definition of Davenport number, see Definition 2.0.10), we will have the image of that sequence sums up to identity in H (as an aside, with the same sum of reciprocal orders). Which means that we have a contradiction since we can find a subsequence that comes up to identity.

One can show contradiction for $D(H) < D(G)$ using the same argument backward, and using ϕ^{-1} instead of ϕ.

□

Chapter 4: Zero Sum in Partitionable Groups

The general idea here is to construct cyclic groups within some symmetric and alternating groups using each element as generators. In symmetric groups of more than 3 elements, there will be elements that comes in 2 different cyclic groups. In this case, we shall absorb the ”smaller” group into the ”larger” one (since the whole cyclic group with the smaller order would be a subset of the group with the larger order). Repeating this process, we will end up with a partition of cyclic groups covering the whole symmetric groups (except for the identity element, which is present in all of the cyclic groups).

Noticing that if we were to attempt to create a sequence as long as possible where the identity is never included, or composited up from the elements already in the sequence, we will start by adding the generators (or any elements in the cyclic groups whose power is relatively prime to the order of the cyclic group itself) one less times that the order of the cyclic group. If we do this for all cyclic groups, we will end up with the order of the whole group minus one. Using the pigeon-hole principle, we will need to either add one element from one of the cyclic groups, which means that it will composited up to identity, or add identity. Either cases will get us a subsequence will composite up to identity. This we will show explicitly in examples to come.

The second part of the zero sum requirement is that the sum of the reciprocal of orders be less than or equal to one. Piggy-backing on the zero sum property of the partitions, plus the Theorem 2.0.9 and the result from Hurlbert, showing that we have the correct number of elements will provide the correct sum of reciprocal of orders.

This section will show that a partitionable group in which the partitions all have zero-sum property will itself have zero-sum property. In constructing this sequence, the basic idea is to figure out how many copies can one put into the sequence without compositing up to identity and getting the sum of the reciprocal of the orders less than or equal to 1. To do this, we will explore the idea of creating cyclic group partitions in the Symmetric Group, and using the conjugate class and power of permutation cycles to generalise the orders of powers of elements.

First, we shall show that:

Theorem 4.0.14. *If G is a finite group partitionable into subgroups H_1, H_2, $\ldots$, H_k, where each of the H_i's has zero-sum property, then G has zero-sum property.*

To this end, we shall attempt to construct a set where it is not possible to have a sequence of $|G|$ elements where none of the subsequence in this sequence will "add up" to identity with the sum of reciprocal of order being less than or equal than one. To this end, we shall use the Pigeon-Hole principle to prove this theorem as follows:

Proof. Assume that each subgroup of G, H_1, H_2, $\ldots$, H_k, has the zero sum property, if we have a set of $|H_i|$ elements of H_i, then there is a subsequence h_{i_1}, h_{i_2}, $\ldots$, h_{i_d} such that $h_{i_1}h_{i_2}\ldots h_{i_d} = e$ where e is identity and $\sum_{j=1}^{d} \frac{1}{|h_{i_j}|} \leq 1$.

By the definition of partitionable group, we have that

i. $|G^*| = |G| - 1$

ii. $|H_1^*| + |H_2^*| + \ldots + |H_k^*| = |G^*|$, and that

iii. $|H_i^*| = |H_i| - 1$.

If we have a sequence S of $|G|$ elements from G, none of them identity (hence, from G^*), then by point (ii) above, and pigeon-hole principle, at least for one i, S must have more than $|H_i^*|$ elements from H_i^*, and from point (iii), it will be at least $|H_i|$ elements.

Since H_i has the zero sum property, this particular subsequence will give us the zero sum subsequence with the correct sum of reciprocal orders.

□

We will use the above to show that each of the groups A_3, S_3, S_4, S_5, and A_6 have the zero sum property. Further, we shall show that the Dihedral Group, D_n have the zero-sum property for all $n \in \mathbb{N}$. As noted below, the challenge to show that a partitionable group has the zero-sum property is to find a partition in such a way that each partition has the zero-sum property.

4.1 Zero Sum and A_3

A_3 is a cyclic group, generated by (123), and since cyclic groups are abelian, therefore we have the following:

Observation 4.1.1. *The group A_3 has the zero sum property.*

4.2 Zero Sum and S_3

S_3 has only 6 elements, and therefore, easy to list. We will start with this group to illustrate the idea.

Table 4.1: Count of elements of S_3

Order	Conjugacy Class		Count
1	3 1-cycle	( )	1
2	1 2-cycle	(1 2)	3
3	1 3-cycle	(1 2 3)	2

We will now group the elements based on cyclic groups partitioning. In this case, the cyclic group for (1 2 3) and (1 3 2) would be equal, since (1 2 3) · (1 2 3) = (1 3 2).

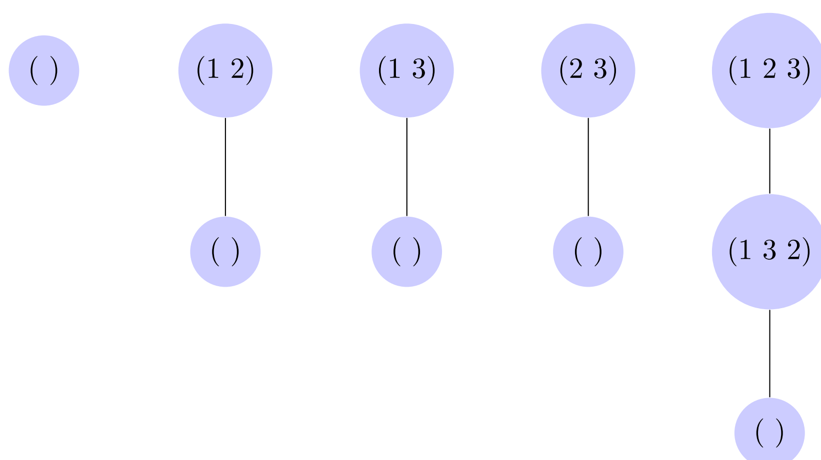

Figure 4.1: Cyclic groupings in S_3

Note that

$$S_3^* = \langle(123)\rangle^* \cup \langle(12)\rangle^* \cup \langle(13)\rangle^* \cup \langle(23)\rangle^*.$$

Since cyclic groups are abelian, every part of S_3 in the above union has the zero sum property. This means that we have the following:

Observation 4.2.1. *The zero sum property holds for S_3.*

4.3 Zero Sum and A_4

A_4 has 12 elements. We will start by listing the conjugacy classes as follows:

Table 4.2: Count of elements of A_4

Order	Conjugacy Class		Count
1	4 1-cycle	( )	1
2	2 2-cycle	(1 2) (3 4)	3
3	1 3-cycle	(1 2 3)	8

We can draw the above in the form of a graph as follows:

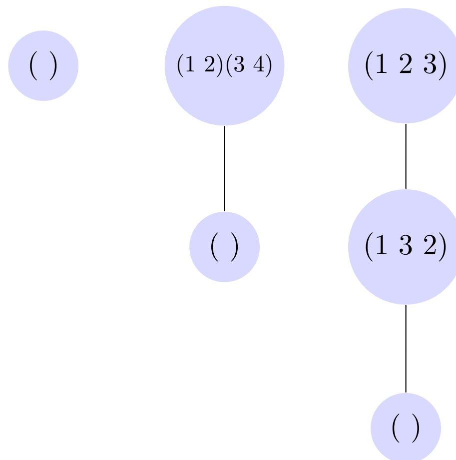

Figure 4.2: Cyclic groupings in A_4

Based on the above, we have the following union:

$$\begin{aligned} A_4^* &= \langle(12)(34)\rangle^* \cup \langle(13)(24)\rangle^* \cup \langle(14)(23)\rangle^* \\ &\cup\langle(123)\rangle^* \cup \langle(124)\rangle^* \cup \langle(134)\rangle^* \cup \langle(234)\rangle^*. \end{aligned}$$

Since all of the cyclic groups have the zero sum property, we have the following:

Observation 4.3.1. *The zero sum property holds for A_4.*

4.4 Zero Sum and S_4

S_4 has 24 elements, and therefore is a bit more complicated. At this point, we also start to see some cyclic group that's subgroup of another cyclic subgroup. First, let's start by listing the element count of each conjugacy class:

Table 4.3: Count of elements of S_4

Order	Conjugacy Class		Count
1	4 1-cycle	( )	1
2	1 2-cycle	(1 2)	6
	2 2-cycle	(1 2) (3 4)	3
3	1 3-cycle	(1 2 3)	8
4	1 4-cycle	(1 2 3 4)	6

Since we have 4 times as many elements as S_3, we will just show the cyclic groups in representative of conjugacy classes. Remembering that conjugacy class is a partition, which means that no element can belong to 2 conjugacy classes at the same time, and that if a and b are in the same conjugacy class, then a^k and b^k would be in the same conjugacy class, we start by listing all the cyclic group we have for each conjugacy class. Note that this list has elements that are listed in more than one cyclic group.

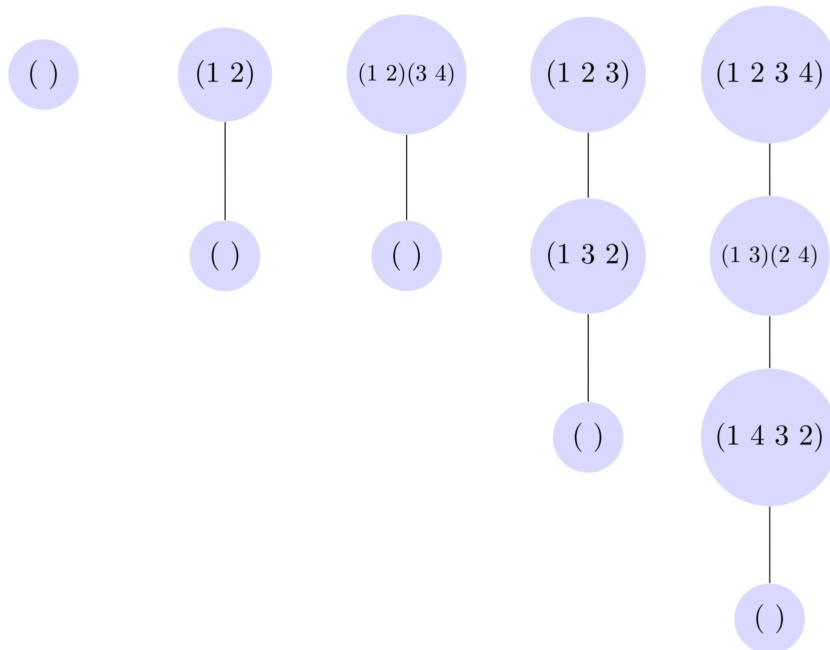

Figure 4.3: Cyclic groupings in S_4

Notice that the cyclic group generated by an element of the conjugacy class (1 2)(3 4) is a cyclic subgroup for the cyclic group generated by an element in the conjugacy class (1 2 3 4). We can "absorb" the cyclic group for the conjugacy class (1 2)(3 4) into the cyclic group for (1 2 3 4). Since there are no more duplicates, we can then rewrite the above diagram into simplified version without any duplicates. This also help us in figuring out how many cyclic groups there are in the group S_4 and the order of each.

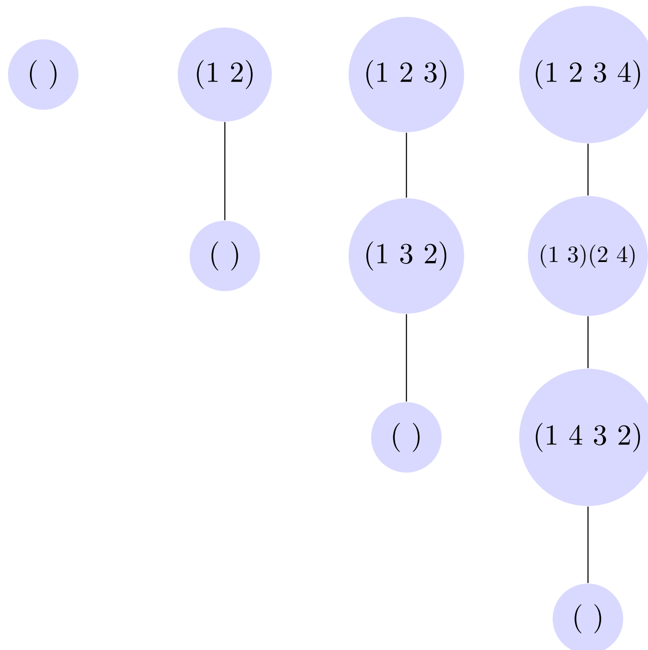

Figure 4.4: Cyclic groupings in S_4 without duplicates

Based on the above list and graph, we can construct the following partition:

$$
\begin{aligned}
S_4^* &= \langle(12)\rangle^* \cup \langle(13)\rangle^* \cup \langle(14)\rangle^* \cup \langle(23)\rangle^* \cup \langle(24)\rangle^* \cup \langle(34)\rangle^* \\
&\quad \cup\langle(123)\rangle^* \cup \langle(124)\rangle^* \cup \langle(134)\rangle^* \cup \langle(234)\rangle^* \\
&\quad \cup\langle(1234)\rangle^* \cup \langle(1243)\rangle^* \cup \langle(1324)\rangle^*.
\end{aligned}
$$

As before, every cyclic group has zero sum property. Note that the group S_4 has 24 elements, which is more than the 21 elements mentioned by Hurlbert in [2]. Based on the above, we have:

Proposition 4.4.1. *The zero sum property holds for S_4.*

4.5 Zero Sum and A_5

Using the same drill, we start with the table as follows:

Table 4.4: Count of elements of A_5

Order	Conjugacy Class		Count
1	5 1-cycle	( )	1
2	2 2-cycle	(1 2)(3 4)	15
3	1 3-cycle	(1 2 3)	20
5	1 5-cycle	(1 2 3 4 5)	24

We then create a graph representing the above:

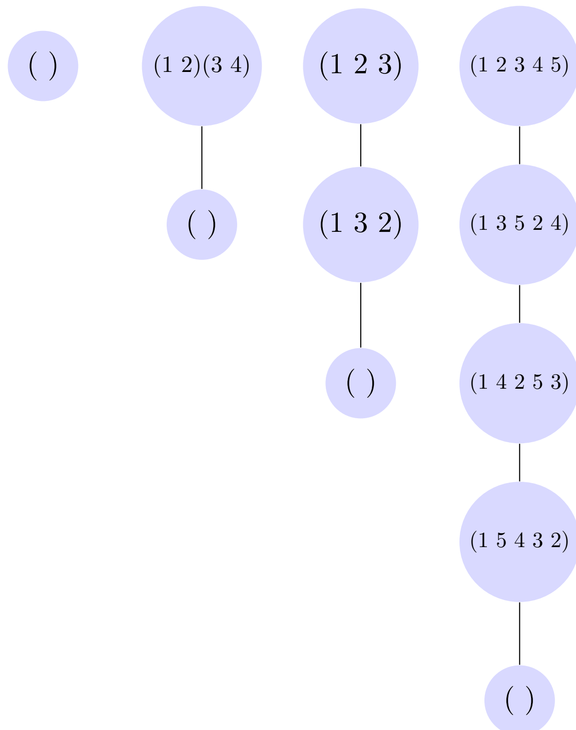

Figure 4.5: Cyclic groupings in A_5

Using the above, we can construct the following partition into cyclic groups:

$$\begin{aligned} A_5^* &= \langle(12)(34)\rangle^* \cup \langle(13)(24)\rangle^* \cup \langle(14)(23)\rangle^* \cup \langle(23)(45)\rangle^* \cup \langle(24)(35)\rangle^* \\ &\cup\langle(25)(34)\rangle^* \cup \langle(13)(45)\rangle^* \cup \langle(14)(35)\rangle^* \cup \langle(15)(34)\rangle^* \cup \langle(12)(45)\rangle^* \\ &\cup\langle(14)(25)\rangle^* \cup \langle(15)(24)\rangle^* \cup \langle(12)(35)\rangle^* \cup \langle(13)(25)\rangle^* \cup \langle(15)(23)\rangle^* \\ &\cup\langle(123)\rangle^* \cup \langle(124)\rangle^* \cup \langle(125)\rangle^* \cup \langle(134)\rangle^* \cup \langle(135)\rangle^* \cup \langle(145)\rangle^* \\ &\cup\langle(234)\rangle^* \cup \langle(235)\rangle^* \cup \langle(245)\rangle^* \cup \langle(345)\rangle^* \\ &\cup\langle(12345)\rangle^* \cup \langle(12354)\rangle^* \cup \langle(12435)\rangle^* \cup \langle(12453)\rangle^* \cup \langle(12534)\rangle^* \cup \langle(12543)\rangle^* \end{aligned}$$

As before, each cyclic groups have zero sum property, therefore, we have that:

Proposition 4.5.1. *A_5 has the zero sum property as well.*

4.6 Zero Sum and S_5

The principle for S_5 is the same as the previous groups. First, we start with the conjugacy class list. The group S_5 has 120 elements in total. We can divide them by cycle type and length as follows:

Table 4.5: Count of elements of S_5

Order	Conjugacy Class		Count
1	5 1-cycle	( )	1
2	1 2-cycle	(1 2)	10
	2 2-cycle	(1 2)(3 4)	15
3	1 3-cycle	(1 2 3)	20
4	1 4-cycle	(1 2 3 4)	30
5	1 5-cycle	(1 2 3 4 5)	24
6	3-cycle 2-cycle	(1 2 3)(4 5)	20

And following the same path, we will the construct the cyclic group with each element as generator. Also, in this case, we shall only list representative of the conjugacy class due to space constraint.

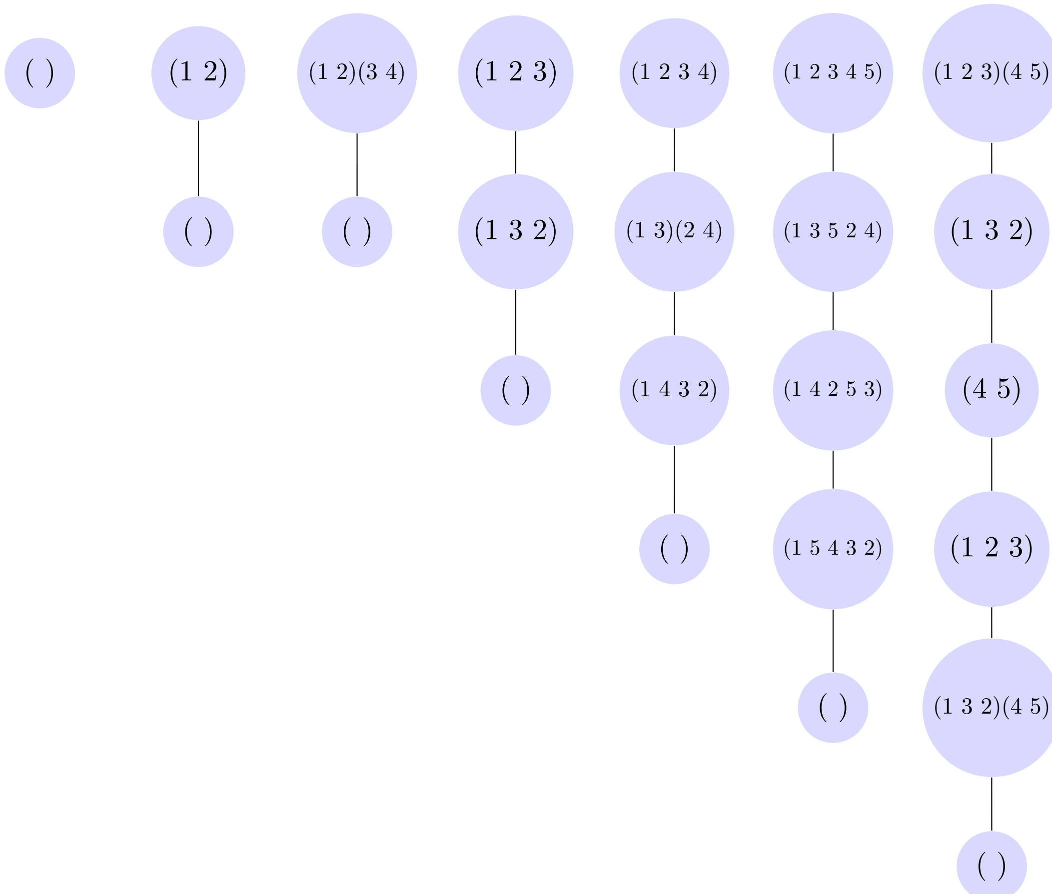

Figure 4.6: Cyclic groupings in S_5

The cyclic groups for the conjugacy classes (1 2) and (1 2 3) are absorbed into the cyclic group for the conjugacy class of (1 2 3)(4 5), while the cyclic group for the conjugacy class of (1 2)(3 4) is absorbed into (1 2 3 4). Removing the 3 cyclic groups, we now get:

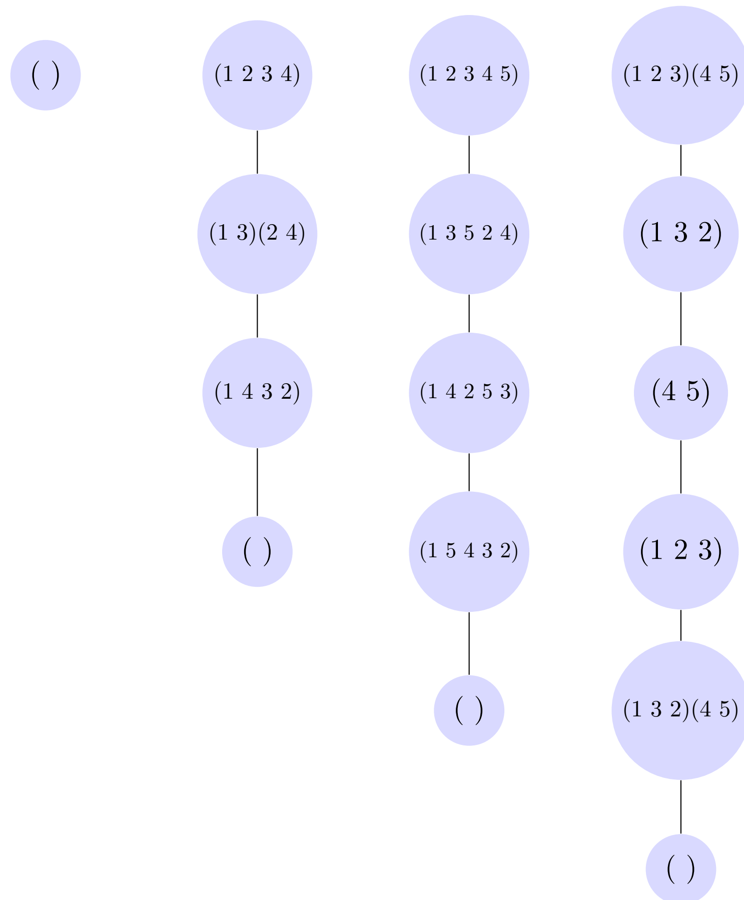

Figure 4.7: Cyclic groupings in S_5 without duplicates

We can therefore construct the following partition:

$$\begin{aligned} S_5^* \quad = \quad & \langle(1234)\rangle^* \cup \langle(1324)\rangle^* \cup \langle(1423)\rangle^* \cup \langle(2345)\rangle^* \cup \langle(2435)\rangle^* \\ & \cup\langle(2354)\rangle^* \cup \langle(1345)\rangle^* \cup \langle(1435)\rangle^* \cup \langle(1354)\rangle^* \cup \langle(1245)\rangle^* \\ & \cup\langle(1425)\rangle^* \cup \langle(1254)\rangle^* \cup \langle(1235)\rangle^* \cup \langle(1325)\rangle^* \cup \langle(1253)\rangle^* \\ & \cup\langle(12345)\rangle^* \cup \langle(12354)\rangle^* \cup \langle(12435)\rangle^* \cup \langle(12453)\rangle^* \cup \langle(12534)\rangle^* \cup \langle(12543)\rangle^* \\ & \cup\langle(123)(45)\rangle^* \cup \langle(124)(35)\rangle^* \cup \langle(125)(34)\rangle^* \cup \langle(134)(25)\rangle^* \cup \langle(135)(24)\rangle^* \\ & \cup\langle(145)(23)\rangle^* \cup \langle(234)(15)\rangle^* \cup \langle(235)(14)\rangle^* \cup \langle(245)(13)\rangle^* \cup \langle(345)(12)\rangle^*. \end{aligned}$$

As before, all the cyclic groups have zero sum property, and therefore,

Theorem 4.6.1. *S_5 has the zero sum property.*

4.7 Zero Sum and A_6

We will go right through to the table of conjugacy class:

Table 4.6: Count of elements of A_6

Order	Conjugacy Class		Count
1	6 1-cycle	( )	1
2	2 2-cycle	(1 2)(3 4)	45
3	1 3-cycle	(1 2 3)	40
	2 3-cycle	(1 2 3) (4 5 6)	40
4	4-cycle 2-cycle	(1 2 3 4) (5 6)	90
5	1 5-cycle	(1 2 3 4 5)	144

In the graphical form, with no repeats, we have:

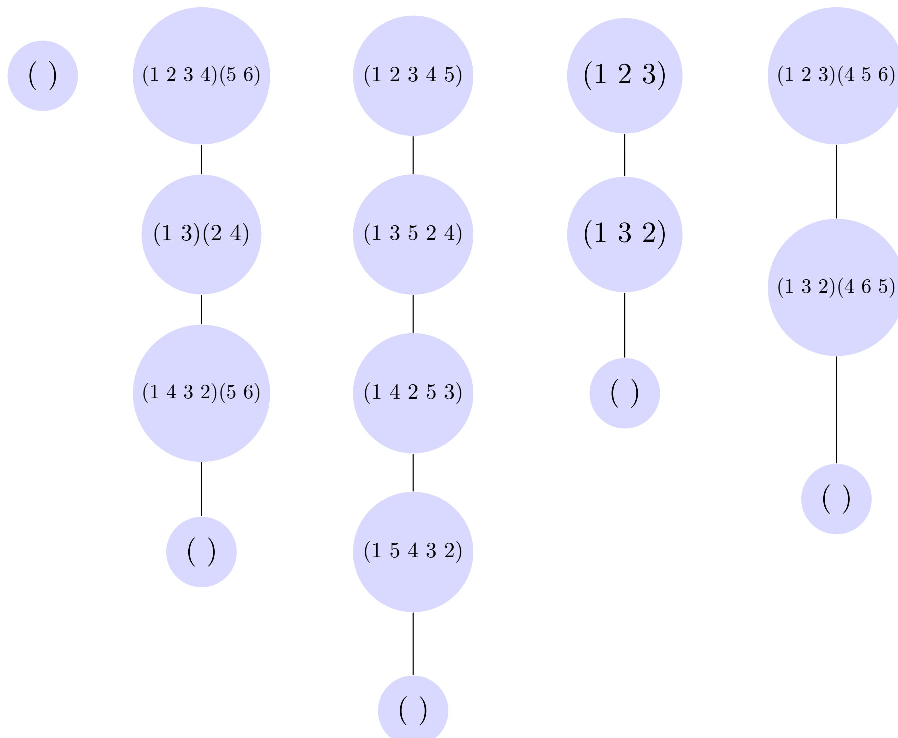

Figure 4.8: Cyclic groupings in A_6 without duplicates

From the above figure, we can construct the following partition:

$$
\begin{aligned}
A_6^* \quad = \quad & \langle(123)\rangle^* \cup \langle(124)\rangle^* \cup \langle(125)\rangle^* \cup \langle(126)\rangle^* \cup \langle(134)\rangle^* \cup \langle(135)\rangle^* \\
& \cup\langle(136)\rangle^* \cup \langle(145)\rangle^* \cup \langle(146)\rangle^* \cup \langle(156)\rangle^* \cup \langle(234)\rangle^* \cup \langle(235)\rangle^* \\
& \cup\langle(236)\rangle^* \cup \langle(245)\rangle^* \cup \langle(246)\rangle^* \cup \langle(256)\rangle^* \cup \langle(345)\rangle^* \cup \langle(346)\rangle^* \\
& \cup\langle(356)\rangle^* \cup \langle(456)\rangle^* \\
& \cup\langle(123)(456)\rangle^* \cup \langle(123)(465)\rangle^* \cup \langle(124)(356)\rangle^* \cup \langle(124)(365)\rangle^* \\
& \cup\langle(125)(346)\rangle^* \cup \langle(125)(364)\rangle^* \cup \langle(126)(345)\rangle^* \cup \langle(126)(354)\rangle^* \\
& \cup\langle(134)(256)\rangle^* \cup \langle(134)(265)\rangle^* \cup \langle(135)(246)\rangle^* \cup \langle(135)(264)\rangle^* \\
& \cup\langle(136)(245)\rangle^* \cup \langle(136)(254)\rangle^* \cup \langle(145)(236)\rangle^* \cup \langle(145)(263)\rangle^* \\
& \cup\langle(146)(235)\rangle^* \cup \langle(146)(253)\rangle^* \cup \langle(156)(234)\rangle^* \cup \langle(156)(243)\rangle^* \\
& \cup\langle(1234)(56)\rangle^* \cup \langle(1243)(56)\rangle^* \cup \langle(1324)(56)\rangle^* \cup \langle(1236)(45)\rangle^* \\
& \cup\langle(1263)(45)\rangle^* \cup \langle(1326)(45)\rangle^* \cup \langle(1235)(46)\rangle^* \cup \langle(1253)(46)\rangle^* \\
& \cup\langle(1325)(46)\rangle^* \cup \langle(1256)(34)\rangle^* \cup \langle(1265)(34)\rangle^* \cup \langle(1625)(34)\rangle^* \\
& \cup\langle(1246)(35)\rangle^* \cup \langle(1264)(35)\rangle^* \cup \langle(1624)(35)\rangle^* \cup \langle(1245)(36)\rangle^* \\
& \cup\langle(1254)(36)\rangle^* \cup \langle(1524)(36)\rangle^* \cup \langle(1456)(23)\rangle^* \cup \langle(1465)(23)\rangle^* \\
& \cup\langle(1524)(23)\rangle^* \cup \langle(1356)(24)\rangle^* \cup \langle(1365)(24)\rangle^* \cup \langle(1635)(24)\rangle^* \\
& \cup\langle(1346)(25)\rangle^* \cup \langle(1364)(25)\rangle^* \cup \langle(1634)(25)\rangle^* \cup \langle(1345)(26)\rangle^* \\
& \cup\langle(1435)(26)\rangle^* \cup \langle(1354)(26)\rangle^* \cup \langle(3456)(12)\rangle^* \cup \langle(3465)(12)\rangle^* \\
& \cup\langle(3645)(12)\rangle^* \cup \langle(2456)(13)\rangle^* \cup \langle(2465)(13)\rangle^* \cup \langle(2645)(13)\rangle^* \\
& \cup\langle(2356)(14)\rangle^* \cup \langle(2365)(14)\rangle^* \cup \langle(2635)(14)\rangle^* \cup \langle(2346)(15)\rangle^* \\
& \cup\langle(2364)(15)\rangle^* \cup \langle(2634)(15)\rangle^* \cup \langle(2345)(16)\rangle^* \cup \langle(2354)(16)\rangle^* \\
& \cup\langle(2534)(16)\rangle^* \\
& \cup\langle(12345)\rangle^* \cup \langle(12346)\rangle^* \cup \langle(12356)\rangle^* \cup \langle(12456)\rangle^* \cup \langle(13456)\rangle^*
\end{aligned}
$$

$$\cup\langle(23456)\rangle^* \cup \langle(23465)\rangle^* \cup \langle(24356)\rangle^* \cup \langle(25436)\rangle^* \cup \langle(12354)\rangle^*$$
$$\cup\langle(12435)\rangle^* \cup \langle(12453)\rangle^* \cup \langle(12534)\rangle^* \cup \langle(12543)\rangle^* \cup \langle(12364)\rangle^*$$
$$\cup\langle(12643)\rangle^* \cup \langle(12365)\rangle^* \cup \langle(12653)\rangle^* \cup \langle(12465)\rangle^* \cup \langle(12654)\rangle^*$$
$$\cup\langle(13465)\rangle^* \cup \langle(13654)\rangle^* \cup \langle(23654)\rangle^* \cup \langle(14256)\rangle^* \cup \langle(14265)\rangle^*$$
$$\cup\langle(14326)\rangle^* \cup \langle(14362)\rangle^* \cup \langle(14356)\rangle^* \cup \langle(14365)\rangle^* \cup \langle(12536)\rangle^*$$
$$\cup\langle(14623)\rangle^* \cup \langle(14652)\rangle^* \cup \langle(14653)\rangle^* \cup \langle(16523)\rangle^* \cup \langle(15326)\rangle^*$$
$$\cup\langle(23546)\rangle^*.$$

Since zero sum property holds for all cyclic groups, this means that:

Theorem 4.7.1. *The zero sum property holds for A_6.*

4.8 Zero Sum and Dihedral Group

The *dihedral group* D_n is the group of symmetries of a regular n-gon, both reflection and rotation. Represent the rotation by c, and the reflection by r, and the group D_n can be represented as

$$D_n = \{c^i r^j : c^n = r^2 = e, rc = c^{-1}r\}$$

where $i \in \{0, 1, 2, \ldots, n-1\}$ and $j \in \{0, 1\}$ for $n \geq 2$ (since if $n = 1$, we have a single point, which basically amount to a Dihedral group containing just the identity).

What is of interest to us here is that the dihedral group D_n is partitionable into cyclic subgroups in the following manner:

$$D_n^* = \left(\bigcup_{0 \leq i \leq n-1} \langle c^i r \rangle^* \right) \cup \langle c \rangle^*$$

where as previously: $X^* = X \setminus \{e\}$. It is readily visible that the first n subgroups would have 2 elements each (with the identity element taken out, they will only have 1 element each), that is $(e, c^i r)$ for $0 \leq i \leq n$, since according to the representation of the dihedral group, $(c^i r)^2 = c^i r c^i r = c^i((c^i)^{-1} r) r = c^i c^{-i} r r = e$. The last group, $\langle c \rangle$ will have n elements, which means that $\langle c \rangle^*$ will have $n-1$ elements, giving us $n+n-1$ elements on the right hand side of the equation.

By Theorem 4.0.14, this means that dihedral group has the zero sum property for all $n \in \mathbb{N}$, which gives us:

Theorem 4.8.1. *For $n \in \mathbb{N}$, the dihedral group D_n has the zero-sum property.*

Chapter 5: Zero Sum in Coverable Groups

We have seen that for partitionable groups, if all the parts of the partitions have zero-sum property, then the group being partitioned would have the same property. This raises the question whether or not a similar line of reasoning could be applied for coverable group (see Definition 2.0.6).

We shall use 2 examples, the Dicyclic Group and the group S_6, to illustrate how Theorem 2.0.9 might possibly be of further use in some cases.

5.1 Zero Sum and Dicyclic Group

Before we come to the dicyclic Group, we shall start with *quaternion group*.

Quaternion Group is a non-abelian group of order eight, formed by the elements $\pm 1, \pm i, \pm j$, and $\pm k$, where 1 is the identity, -1 commutes with other elements in the group, and the following multiplication rules:

- $(-1) \cdot (-1) = 1$,
- $i^2 = j^2 = k^2 = ijk = -1$,
- $ij = k$, $ji = -k$,
- $jk = i$, $kj = -i$, and
- $ki = j$, $ik = -j$.

The *dicyclic Group*, Dic_n, is a subgroup of the unit quaternions generated by:

$$a = e^{\frac{\pi}{n}i} = \cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)$$

$$x = j$$

This group can also be written as:

$$\mathrm{Dic}_n = \{a^k x^l : a^{2n} = 1, x^2 = a^n, xa = a^{-1}x\}$$

where $k \in \{0, 1, \ldots, 2n-1\}$, $l \in \{0, 1\}$. This means that $|\,\mathrm{Dic}_n\,| = 4n$.

Along with this, we have the following multiplication rules:

- $(a^i)(a^j) = a^{i+j}$
- $(a^i)(a^j x) = a^{i+j}x$
- $(a^i x)(a^j) = a^{i-j}x$
- $(a^i x)(a^j x) = a^{i-j}x^2 = a^{i-j+n}$

For $n = 1$, we have $a = \cos\pi + i\sin\pi = -1$, and $x = \sqrt{-1} = i$, and $\mathrm{Dic}_1 = \{i, -1, -i, 1\}$ with 1 being the identity element. This group is basically $\langle i \rangle$, which has the zero-sum property since it is cyclic (isomorphic to $\mathbb{Z}/4\mathbb{Z}$). Therefore, we will assume that $n > 1$.

Based on the representation and the previous definition of $X^* = X \setminus \{e\}$, we can write a covering for Dic_n^* consisting of cyclic groups as follows:

$$\mathrm{Dic}_n^* = \left(\bigcup_{0 \le k \le n-1} \langle a^k x \rangle^*\right) \cup \langle a \rangle^*.$$

Based on the multiplication rules of dicyclic groups, we have $\langle a^k x\rangle = \{a^k x, a^n, a^{n+k}x, 1\}$. Minus the identity, for each k, there are 3 elements in each $\langle a^k x\rangle^*$, with one element being common to all the cyclic groups, a^n. Note that $|a^n| = 2$, and that $|a^k x| = |a^{n+k}x| = 4$. It also happens that $(a^k x)\cdot(a^k x) = (a^{n+k}x)\cdot(a^{n+k}x) = a^n$, and that the sum of reciprocals of order matched $(\frac{1}{4} + \frac{1}{4} = \frac{1}{2})$.

Observing the groups, we see that the intersection between all cyclic groups cover is the group $\langle a^n\rangle = \{e, a^n\}$, which is a cyclic group. We can therefore break the Dicyclic Group into $n + 2$ disjoint "sets": the sets $\{a^k x, a^{n+k}x\}$ for $0 \le k \le n-1$ (let these sets be called C_k for $0 \le k \le n-1$), the set $\{a, a^2, \ldots, a^{n-1}, a^{n+1}, \ldots, a^{2n-1}\}$ (call this set B), and the intersection, $\langle a^n\rangle = \{a^n, e\}$.

Let ℓ be the number of elements of the intersection group, $\langle a^n\rangle$, in the sequence. We can therefore divide the sequence into 3 cases:

$\ell = 2$ Since the intersection group, $\langle a^n\rangle$, is cyclic, and therefore abelian, with order 2, that means that we have our zero sum.

$\ell = 1$ If the one element from the intersection group is e, then we have our zero sum subsequence, namely e. If the one element from the intersection group is a^n, then if we were to avoid zero sum, we can only have 1 element each from the C_k's (since 2 elements from any, combined with a^n will result in zero sum), and only n elements from B, otherwise, with a^n, this will give us the zero sum. This will give us $2n$ elements in the sequence. Since we have $4n$ elements, we will have a zero sum subsequence.

$\ell = 0$ This will give us $2n - 2$ elements from B, and 2 elements from each C_k (otherwise, we have a zero sum subsequence). We need $4n$ elements, which means that we need 2 extra from one of the groups. If the extra 2 elements come from one of the C_k, then we will have our zero sum from the 4 elements from that C_k. Otherwise, if the extra 2 elements come from B, we will then have zero sum from the elements of that group. Otherwise, we might have 1 element from one of the C_k and another from B. This means that we have $2n-1$ elements from B. If we have 2 different elements from C_k, then we will have zero sum, since the 2 different elements from C_k are inverses of each other. Otherwise, if we have 2 elements from the same C_k next to each other, we will have zero sum, since this will substitute for a^n, and this will cause B to have $2n$ elements. Otherwise, for any 2 elements of C_k's, we can find the inverse of the product in B, since for $0 \leq k_1 \leq n-1$ and $0 \leq k_2 \leq n-1$, we have $a^{k_1}x \cdot a^{k_2}x = a^{k_1-k_2+n}$ with the sum of reciprocal order of $\frac{1}{2}$. This means that using Theorem 1.0.1, we can find an inverse in B (or a subsequence that will result in this inverse), which is $a^{2n-k_1+k_2-n} = a^{n-k_1+k_2}$. Since $\gcd(2n, n-k_1+k_2) \leq n$, then we have that the sum of reciprocal order will be less than 1.

Therefore, it seems we have the following:

Theorem 5.1.1. *For $n \in \mathbb{N}$, the dicyclic group Dic_n has zero-sum property.*

5.2 Zero Sum and S_6

S_6 has 720 elements. Using similar idea, we can list the conjugacy class as before:

Table 5.1: Count of elements of S_6

Order	Conjugacy Class		Count
1	6 1-cycle	( )	1
2	1 2-cycle	(1 2)	15
	2 2-cycle	(1 2)(3 4)	45
	3 2-cycle	(1 2)(3 4)(5 6)	15
3	1 3-cycle	(1 2 3)	40
	2 3-cycle	(1 2 3)(4 5 6)	40
4	1 4-cycle	(1 2 3 4)	90
	4-cycle 2-cycle	(1 2 3 4)(5 6)	90
5	1 5-cycle	(1 2 3 4 5)	144
6	3-cycle 2-cycle	(1 2 3)(4 5)	120
	1 6-cycle	(1 2 3 4 5 6)	120

Using similar groupings for S_5, we get the following cyclic groupings:

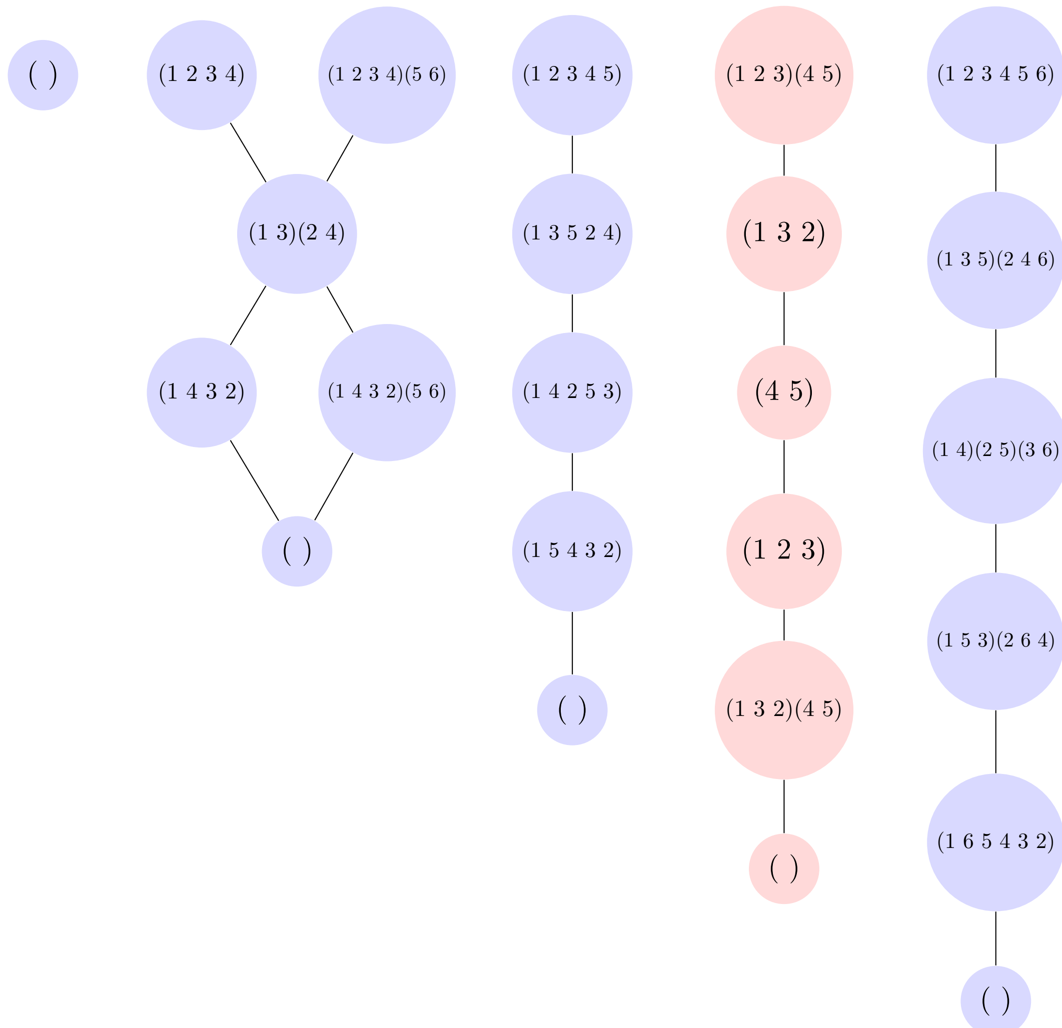

Figure 5.1: Cyclic groupings in S_6 without duplicates.
Note that the $\langle(123)(45)\rangle$ part in red contains intersection within conjugacy class

Using the above table and the above graph, we will have 45 generators of the (1 2 3 4) class, 45 from the (1 2 3 4)(5 6) class, 24 from the (1 2 3 4 5) class, 35 from the (1 2 3)(4 5) class, 35 from the (1 2 3 4 5 6) class. However, since (1 2 3 4) and (1 2 3 4)(5 6) share common powers of 2, with the same possible substitution (the sum of reciprocal orders are equivalent to each other), we have $(45 + 45) \cdot 3 - 45 = 225$ elements from that grouping. Add other groupings, we have $225 + 24 \cdot 4 + 35 \cdot 5 + 35 \cdot 5 = 225 + 144 + 175 + 175 = 719$

elements, at most, if one were to not get any subsequence that will composite up to identity. Therefore, one of the cyclic groupings will have a number of elements of the order of the generator with the appropriate sum of reciprocal orders.

Please also note that (1 2 3)(4 5), (1 2 3)(4 6), and (1 2 3)(5 6) also intersect at (1 3 2), (1 2 3), and the identity, ( ). However, since in the case of intersection, we will get less elements to use, and at this point we already have enough elements required in the sequence to obtain zero-sum, this particular fact of intersection does not matter much.

5.3 Speculations on Zero Sum and Coverable Groups

We know that every group is coverable with cyclic groups based as mentioned in Theorem 2.0.7, by generating cyclic groups from each element and then eliminating the groups that are included in other groups, as we've done for S_4, S_5, A_6, and S_6. Therefore, at this point, we shall assume that if a group is covered, the covers are all cyclic groups, and therefore, have the zero-sum property.

In the S_6 and Dic_n, we observed that the property that we use to be able to use Pigeon-Hole Principle on the whole group is to show that we need to be able to construct something of a *custom sum* on the rest of the group, that is, we can substitute the overlapping element possibly with a "sum" of elements from the rest of the group.

Let the overlapping groups be $\langle g \rangle$ and $\langle h \rangle$, and the overlapping elements be $g^k = h^l$. We will start by stipulating that the rest of the group has the following properties:

- for any $m \geq k$, we can find a sequence of elements g^{k_1}, g^{k_2}, $\ldots$, g^{k_n}, such that $g^k = g^{k_1} \cdot g^{k_2} \cdot \ldots \cdot g^{k_n}$, and that

- $\frac{1}{|g^k|} \leq \frac{1}{|g^{k_1}|} + \frac{1}{|g^{k_2}|} + \ldots + \frac{1}{|g^{k_n}|}$

The first and the second item ensures that the sum of the reciprocal orders will end up with the same reciprocal order as the reciprocal order of the element g^k, which will be substitutable in the other subgroup. The last, ensure the substitutability of the element g^k itself. This means that we can pick one group, say, A with $|A| = a$, and substitute every element to the k-th power with k elements from the second group, say B. We will then need $a - z + k \cdot z$, with z being some items that we are replacing with items from the second group, to guarantee zero-sum.

We are going to illustrate this possible problem in creating our zero-sum sequence by using a subset of S_7. Take for example the cyclic subgroups generated by (1 2 3)(4 5)(6 7) and (1 2 3)(4 5). Both of the cyclic subgroup will be of order $\text{lcm}(3, 2, 2) = \text{lcm}(3, 2) = 6$, and they both have the same elements for powers 2 and 4. The following is the power of elements in the cyclic group and the order for the element to the power listed:

Table 5.2: Power and order of parts of S_7

Power	Order
1	6
2	3
3	2
4	3
5	6
6	1

We will now use the previous method. We will list the elements and "absorb" the appropriate subgroups. This will give us the following diagram:

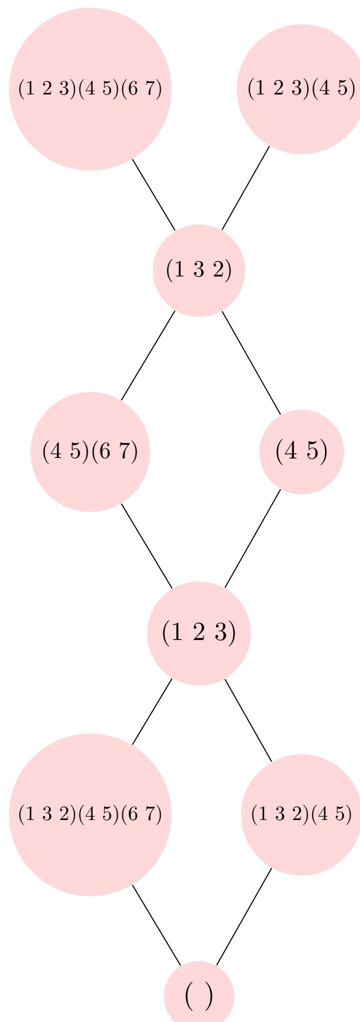

Figure 5.2: (1 2 3)(4 5)(6 7) and (1 2 3)(4 5) overlaps

As we can see, the common elements we have are of order 3, and we can not substitute this with other elements other than 2 of ether the generator, or 2 of the fifth power elements. This means that depending on the number of sets having overlapping elements, and the number of items required in the sequence, it is possible to construct a sequence where one would have either no zero-sum, or elements that sum up to the element, but with the wrong orders.

In the above example, were we to just consider the 9 elements in the diagram, we would find a zero-sum sequence. However, were we to only be able to pick just 6 (which is not an impossible case, since we are now in the realm of overlapping sets), we would be able to construct such a sequence (say, (1 2 3), (1 2 3), (4 5)(6 7), (4 5), (1 2 3)(4 5)(6 7), and (1 2 3)(4 5), in which case we can find a sequence with a sum of identity, with the sum of reciprocal order of more than 1).

We see that on top of this condition, it seems like we have to put more restriction on how many items we need in the sequence compared to how many elements overlapped between the sequences. As we can see in the above example, and in the other examples we've had, we need to put more than the number of elements in the overlapping group minus the overlapping elements to ascertain zero sum.

For our purposes, since we argued previously that all groups can be covered by cyclic groups, we will assume that the covers are all cyclic, and as a consequence, the overlapping "section" of the subgroups are also cyclic. Let the first cyclic group has a elements (this includes the overlapping elements and the identity element), and the cyclic group has b elements. We have c elements in the overlapping group. This will give us a total of $a+b-c$ distinct elements in the union.

To obtain a sequence that does not have any zero-sum property, assuming that no other elements will add up to the overlapping elements, we will need to make sure that we do not have a elements in the first group, and b elements in the second. Therefore, we can only pick $a-1$ elements in the first group. In the second group, we can not have more than $b-1$ elements. In the intersection, we can have at most $c-1$ elements without getting zero-sum.

According to [7], for cosets, the maximum number of items in the sequence we can have without zero-sum is $1+\sum_{i=1}^{r}(n_i-1)$. Which means that we have that in the case of the first group, say A, and the overlapping group, say C, we have that $A \setminus C$ is a group that is generated by C composited by an element of $A \setminus C$. The same case with the second group, say B. A, B, and C are then for our intent and purposes, cosets. We can therefore, use the theorem, getting the sequence size to be at least one more than $1+(a-c-1)+(b-c-1)+(c-1) = 1+a+b-c-3 = a+b-c-2$, giving us a sequence of $a+b-c-1$ (Here, we will also need to consider the sum of the reciprocal orders for elements in the zero-sum subsequence).

However, it has been pointed out that in the case of $\mathbb{Z}_3 \times \mathbb{Z}_2$, if one were to see this as $\{(1,1),(1,0)\} \cup \{(2,1),(2,0)\} \cup \{(0,1),(0,0)\}$, according to Gao and Geroldinger in [7], we can have $3 \cdot (2-1)+1 = 4$ elements in the sequence, and zero sum should be guaranteed. However, if our sequence consists of four $(1,1)$, we will not end up with a zero sum. Therefore, in this case, we do need 6 elements, which means that we might need to calculate all possible Davenport Constant and pick the maximum.

Based on the above, we posit that for a coverable group with overlapping covers where all the covers have zero-sum property, we have to have the overlapping covers to have the following properties:

Either:

i. for any $m \geq k$, we can find a sequence of elements g^{k_1}, g^{k_2}, $\ldots$, g^{k_n}, such that $g^k = g^{k_1} \cdot g^{k_2} \cdot \ldots \cdot g^{k_n}$,

ii. $\dfrac{1}{|g^k|} \leq \dfrac{1}{|g^{k_1}|} + \dfrac{1}{|g^{k_2}|} + \ldots + \dfrac{1}{|g^{k_n}|}$,

However, this is not giving us much of a conclusive condition to determine whether a coverable group has the zero sum property or not. As an example, we will try to look at S_7. As illustrated in the following subsection, aside from immense complication, we can't really conclusively decide whether S_7 has the zero sum or not.

5.3.1 Zero Sum and S_7

In the case of S_7, we are going to show that it might be that the zero-sum property might not hold. Since the graph is pretty large, we will start by enumerating the number of cyclic groups for each generator, accounting for absorbance of smaller cyclic groups as follows:

Table 5.3: Power and order of parts of S_7

Generator	Number of Groups
$\langle(123)(45)\rangle$	266
$\langle(123456)\rangle$	665
$\langle(1234)(56)\rangle$	425
$\langle(12345)(67)\rangle$	273
$\langle(1234)(567)\rangle$	455
$\langle(1234567)\rangle$	120

As we can see in the graph to follow, the cyclic group generated by $\langle(12345)(67)\rangle$ intersects with $\langle(123)(45)\rangle$ at $\langle(12)\rangle$. The cyclic group generated by $\langle(123)(45)\rangle$ intersects with $\langle(1234)(567)\rangle$ at $\langle(123)\rangle$. And finally, $\langle(1234)(56)\rangle$ intersects with $\langle(1234)(567)\rangle$ at $\langle(13)(24)\rangle$.

This is further complicated by multiple cyclic groups generated by the same conjugate group to have intersection. For example, $\{(123)(45), (132), (45), (123), (132)(45), e\}$ and $\{(123)(56), (132), (56), (123), (132)(56), e\}$ intersects at (132) and (123). There are 6 possibilities for each group of intersections. Also notice that $\{(126)(45), (162), (45), (126), (162)(45)\}$ intersects with the original group on (45), and there are 10 for each groups.

If we were to just take the maximum, we would just take one less from the number of distinct elements from $\langle(12345)(67)\rangle$, $\langle(123)(45)\rangle$, $\langle(1234)(567)\rangle$, and $\langle(1234)(56)\rangle$, which is 23. We should then multiply that with the maximum number of groups generated by an element, which is 455. We then have $2323 \cdot 455 = 10465$. Add the other 2 cyclic groups for the elements, we have $10465 + 720 + 3325 = 14510$.

Since we only need 5040 elements in the sequence, we therefore are unable to say conclusively whether the group S_7 has the zero sum property or not.

The following is the graph constructed for cyclic group covering of S_7, taking into account that some cyclic groups are "absorbed" into larger ones.

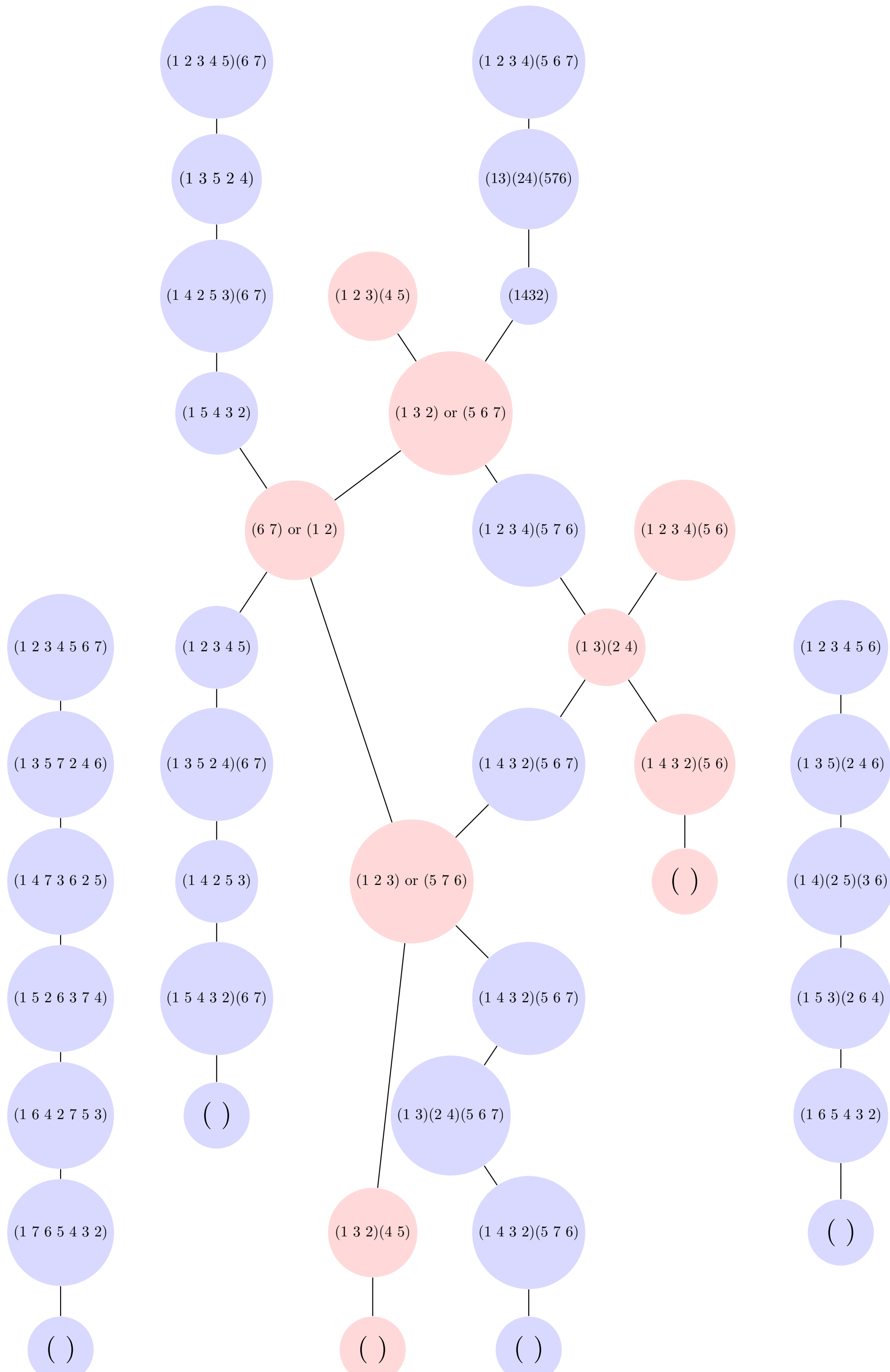

Figure 5.3: S_7 cyclic covering.
Note that parts in red contains intersection within conjugacy class

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Curriculum Vitæ

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