THE GEOMETRY OF THE QUOTIENT STACK ARISING FROM A STACKY FAN

by

David A. Johannsen
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Committee:

______________________	Dr. Rebecca Goldin, Dissertation Director
______________________	Dr. James Lawrence, Committee Member
______________________	Dr. Geir Agnarsson, Committee Member
______________________	Dr. David Marchette, Committee Member
______________________	Dr. David Walnut, Department Chair
______________________	Dr. Richard Diecchio, Interim, Associate Dean for Student and Academic Affairs, College of Science
______________________	Dr. Peggy Agouris, Interim Dean, College of Science
Date: ______________________	Fall 2013 George Mason University Fairfax, VA

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By

David A. Johannsen
Master of Science
The University of North Carolina at Chapel Hill, 1997
Bachelor of Science
The University of Maryland at College Park, 1994
Bachelor of Arts
University of Chicago, 1987

Director: Dr. Rebecca Goldin, Professor
Department of Mathematical Sciences

Fall 2013
George Mason University
Fairfax, VA

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Dedication

I dedicate this dissertation to my wonderful wife and daughter, Di and "Bunn Bunn." Their support and patience and understanding have meant more to me than I will ever be able to convey.

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I wish to thank, first and foremost, Dr. Rebecca Goldin. She has taught me much more mathematics than is conveyed in this dissertation. Additionally, this dissertation would never been completed without her patience and flexibility and her tenacious insistence that I not quit. More than an academic advisor, she has also been a valuable friend.

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Abstract

THE GEOMETRY OF THE QUOTIENT STACK ARISING FROM A STACKY FAN

David A. Johannsen, PhD

George Mason University, 2013

Dissertation Director: Dr. Rebecca Goldin

A quotient stack, $[Z/G]$, is a geometric object that models the quotient of a space, Z, by the action of a Lie group, G, while carrying additional structure at the singularities. Quotient stacks generalize toric varieties, and thus constitute a broad and important class of geometric spaces. In this dissertation, we will exploit a construction given by Borisov, Chen, and Smith that allows one to construct a quotient stack from a particular combinatorial object, called a stacky fan. Our program is to deduce geometric features of the quotient stack from the stacky fan.

Our main results are to determine the component group of the Lie group G from the combinatorics of the stacky fan. In particular, we will give a necessary and sufficient condition on the stacky fan for the corresponding group G to be connected. We will also give a characterization of all the inertia groups of the quotient stack, in terms of the combinatorics of the stacky fan.

Finally, we will turn our attention to the stacky fans that give rise to weighted projective spaces (and fake weighted projective spaces), a very important class of toric varieties. In particular, we will give a characterization of stacky fans that correspond to weighted projective spaces. In the case of 2-dimensional sheared simplices (a special case of the

labeled polytopes of Lerman-Tolman), we give an explicit and complete description of the resulting quotient stack, in terms of the greatest common divisor of positive integers associated to the polytope.

Chapter 1: Introduction

The purpose of this thesis is to describe the geometry of certain stacks that arise as the quotient of a manifold by the action of a compact (but not necessarily connected) torus. In particular, our main result is to give a full description of the local isotropy groups of the quotient stack that arises from a stacky fan. In the case when the fan is polytopal, our results can be considered an extension of those of Lerman and Tolman [19] for labeled polytopes.

The geometric objects with which we are concerned are quotient stacks. Intuitively, quotient stacks model the quotient of a space by a group action while carrying extra structure at the singularities (see Definition 3.6). As the abstract language of stacks can make them difficult for the non-expert, the basic motivation for our investigation is to determine geometric properties of a quotient stack by using an associated combinatorial object. The method for doing this is to exploit the construction of Borisov, Chen, and Smith that allows us to associate a quotient stack to a stacky fan (a construction that we will frequently refer to as the "BCS construction"). Briefly, a stacky fan is a triple (N, Σ, β), where N is a finitely generated $\mathbb{Z}$-module, Σ is a simplicial fan in $\mathbb{R} \otimes_{\mathbb{Z}} N$, and $\beta : \mathbb{Z}^n \to N$ is a module homomorphism that is closely related to the fan Σ (see Definition 3.2). The approach of the BCS construction is to generalize Gale duality to groups with torsion, producing a dual group, $\mathrm{DG}(\beta)$, from the mapping cone construction (see Chapter 4). This allows us to replace the abstract language of stacks with that of elementary algebra and the combinatorial geometry of fans (or, often, polytopes).

Our first result concerns the group G that arises in the BCS construction. In particular, we have been able to give a characterization of the number of connected components of the group G in terms of the stacky fan. More precisely, we have proved the following:

Theorem 5.2. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the associated quotient stack. Let G_0 be the identity component of G, then $G/G_0 \cong N/\mathrm{im}(\beta)$.

Of course an immediate corollary of this Theorem is a characterization of when the group G is connected.

Corollary 2. Let (N, Σ, β) be a stacky fan and $[Z/G]$ the corresponding quotient stack. Then G is connected if and only if $\beta : \mathbb{Z}^n \to N$ is surjective.

In addition to giving explicit description of the group, we have been able to characterize the isotropy groups of the action of G on Z. A special case of our theorem can be found in the paper of Borisov, Chen and Smith [3]. Though the primary focus of [3] was to describe the orbifold Chow ring of a toric Deligne-Mumford stack, their paper does prove a special case of our main result. Namely, Proposition 4.3 in [3] determines the isotropy groups corresponding to the maximal cones in the stacky fan. In contrast, we have obtained a complete decription of the isotropy groups:

Theorem 5.3. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the associated quotient stack. For $z \in Z$ let σ be the minimal cone such that $I_z \subset I_\sigma$. The stabilizer of z is then isomorphic to the torsion subgroup $\mathrm{Tor}(N/N_\sigma)$, where $N_\sigma \subset N$ is the sub-module spanned by $\{\beta(e_i) \mid i \in I_\sigma\}$.

Due to its importance, we give consideration to the special case of a complete simplicial fan that corresponds to the action of a 1-dimensional group. This is the case that includes weighted projective spaces and fake weighted projective spaces. In this case, Sakai [22] gave a description of a stacky polytope corresponding to a given weighted projective space. We here are able to give the converse; that is, we characterize those polytopal stacky fans (i.e., fans that are dual to a simple polytope) that give rise to weighted projective spaces and fake weighted projective spaces under the BCS construction.

Proposition 4. Let (N, Δ, β) be a stacky polytope, and let $\Sigma(\Delta)$ be the dual fan to Δ. The associated toric DM stack $\mathcal{X}(N, \Sigma(\Delta), \beta)$ is a weighted projective space $\mathbb{P}(b_0, \ldots, b_d)$

if and only if $\mathrm{DG}(\beta) \cong \mathbb{Z}$. In this case, the polytope Δ is a simplex, and the weights are determined by the condition that $(b_0, \ldots, b_d)$ generates $\ker \beta \subset \mathbb{Z}^{d+1}$.

The organization of this thesis is as follows. In Chapter 2 we give the historical context for our work from the symplectic perspective. In Chapter 3 we will define the basic objects that we are considering, principally stacky fans and quotient stacks. The next chapter, Chapter 4, outlines the construction that will be fundamental in all that follows, the construction of a quotient stack from a stacky fan. In this chapter we also characterize the failure of the BCS construction to be injective. The chapter that follows, Chapter 5, will contain our results. Of particular interest is the characterization of the local isotropy groups. Also of interest is the characterization of those stacky polytopes that give rise to weighted projective spaces and fake weighted projective spaces.

Traditionally, there are both algebraic and symplectic perspectives on the the correspondence between combinatorics and geometry that is the subject of our investigations. In recent years, the language of stacks has begun replacing the traditional notion of orbifolds (Stake's V-manifolds giving way to étale stacks), so that the algebraic approach has largely subsumed the differential geometric approach. Additionally, the algebraic setting is strictly more general than the traditional differential geometric approach (as there are fans that are not dual to polytopes). We hope to help bridge the gap between the perspectives by providing proofs using both perspectives when it is not excessively cumbersome to do so.

Chapter 2: Historical Background

The object of this chapter is to provide a brief survey of the historical setting for our research. We recall some basic definitions and then state the well-known results of Atiyah/Guillemin-Sternberg that establish the convexity of the image of the moment map. We follow this discussion by recalling the beautiful result of Delzant, and then the generalization given by Lerman-Tolman.

The starting point for our work are the well-known theorems of Atiyah [1]/Guillemin-Sternberg [12], and Delzant [8]. Taken together, these theorems establish a one-to-one correspondence between symplectic toric manifolds and smooth simple rational polytopes. To more firmly establish context for the work that we have completed, we will give precise statements of these theorems (for more detail, consult [2], [5]).

Let M be a smooth manifold. Then a *symplectic form*, ω, on M is a closed non-degenerate 2-form. A *symplectic manifold* is a pair (M, ω); i.e., a manifold and a symplectic form defined on the manifold. Now, we recall the definition of a Hamiltonian action.

Definition 2.1. Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms. We say that the action is **Hamiltonian** if

- there is a G-equivariant map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual Lie algebra to G (and on which G acts by the coadjoint representation);
- the fundamental vector field $v^\sharp$ induced by $v \in \mathfrak{g}$ satisfy $\omega(v^\sharp, \cdot) = -d(\langle \mu, v \rangle)$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing of $\mathfrak{g}$ and its dual, and $d(\cdot)$ is the usual exterior derivative.

Example 1. We give the ubiquitous, but important example of the sphere (adapted from [2]). Consider the 2-sphere, S^2, in $\mathbb{R}^3$. Recall that at $v \in S^2$ we can identify the tangent space, T_vS^2, with the plane orthogonal to v. Then, under this identification, S^2 is a symplectic manifold when we endow it with the symplectic form, $\omega_v(X, Y) = \langle v, X \times Y \rangle$, the usual "area form" on the tangent space. Now consider the action of the circle, S^1 (a compact Lie group), on (S^2, ω) by rotation about the z axis. If we write S^2 in cylindrical coordinates, $(x, y, z) \mapsto (\theta, z)$, where $\theta = \cos^{-1}(x)$ then the action is given by

$$S^1 \times S^2 \to S^2$$

$$(\tau, (\theta, z)) \mapsto (\tau + \theta, z).$$

It is easy to verify that the map $\mu : S^2 \to \mathbb{R}^*$ defined by $(\theta, z) \mapsto z$ is a moment map for this action, and hence the action is Hamiltonian.

We then have a characterization of the image of the moment map for a torus action on a compact connected symplectic manifold. We note that, following convention, we reserve the word "torus" to mean a compact connected Abelian Lie group. We will use the words "disconnected torus" to indicate the product of a torus and a finite group.

Theorem 2.1 (Atiyah[1]/Guillemin-Sternberg[12])**.** For the Hamiltonian action of a compact torus with moment map μ on a compact connected symplectic manifold, the set of fixed points of the action is a finite union of submanifolds, $C_1, \ldots, C_n$. On each of these submanifolds, $\mu(C_j) = x_j$ is constant and the image of μ is the convex hull of the points $\{x_j\}$.

Note that the convex hull of a finite collection of points in a vector space is a convex polytope. Thus, one refers to the image of the moment map as the *moment polytope.*

Example 2. Recall Example 1 above. It is immediately seen that with the Hamiltonian S^1 action described above, the image of the moment map is the line segment $[-1, 1] \subset \mathbb{R}^*$, the convex hull of the image of the fixed points of the action.

We now restrict our attention to the case that the torus that acts has the largest possible dimension. Suppose that a compact connected symplectic manifold (M, μ) has dimension

$2d$ and that there is an effective Hamiltonian action by a compact torus, G, of dimension d. Then we denote this object (M, ω, G, μ) and call it a *symplectic toric manifold.*

Definition 2.2. For a compact torus of dimension d we denote by $N \cong \mathbb{Z}^d$ the natural lattice inside $\mathfrak{t} \cong \mathbb{R}^d$ and let $\Delta \subset \mathfrak{t}^*$ be a convex polytope with n facets. We say that Δ is **Delzant** if it is

- *simple*, that is, if there are d edges incident at each vertex;
- *rational*, that is, if it can be written as an intersection of half-spaces

$$\Delta = \bigcap_{j=1}^{n} \{x \in \mathfrak{t}^* | \langle x, u_j \rangle \geq -\eta_j \in \mathbb{R}\},$$

  where $u_j \in N, j = 1 \ldots, n$. We take u_j to be primitive inward normal to facet j;
- *smooth*, that is, for any vertex $v \in \Delta$ the vectors $u_{i_1}, \ldots, u_{i_d}$ corresponding to facets incident at v span the lattice N (as a $\mathbb{Z}$-module).

We can now state the well-known theorem of Delzant:

Theorem 2.2 (Delzant [8]). Any Delzant polytope, Δ, arises as the image of the moment map of a symplectic toric manifold. Moreover, suppose two symplectic toric manifolds, (M, ω, G, μ) and (M', ω', G, μ'), are such that $\mu(M) = \mu'(M')$, then there exists a G-equivariant symplectomorphism $T : (M, \omega) \to (M', \mu')$.

This then is the complete picture for symplectic toric manifolds, they are "equivalent" to smooth rational simple polytopes.

Remark 2.1. Often it is convenient to speak of Delzant polytopes in $\mathbb{R}^d$ (a practice that we, too, will adopt). Though the Lie aglebra of a d-torus, $(S^1)^d$, is isomorphic to $\mathbb{R}^d$, any such isomorphism is non-canonical. If we begin with a polytope in Euclidean space, then we must adopt a different notion of equivalence. More precisely, suppose that (M, ω, G, μ)

and (M', ω', G', μ') are symplectic toric manifolds. Then there is an automorphism of G (i.e., $G = \phi(G'), \phi \in \mathrm{Aut}((S^1)^d)$) with respect to which (M, ω, G, μ) and (M', ω', G', μ') are equivariantly symplectomorphic if and only if there exists $(A, v) \in \mathrm{AGL}(d, \mathbb{Z})$ such that $\mu(M) = A(\mu'(M')) + v$, where $\mathrm{AGL}(d, \mathbb{Z})$ denotes the affine general linear group.

The above also serves to illustrate the general situation that we seek to investigate: even though the theorem achieves a complete classification of symplectic toric manifolds, it doesn't tell us how to obtain a picture of the geometry of the toric manifold from knowledge of the polytope.

A more significant limitation of Delzant's result is that it applies only in the smooth category. There are, however, essentially toric objects that are not smooth. To construct these objects, we need a brief introduction to symplectic reduction (see [5]). Suppose that a torus T is acting effectively on (M, ω) with moment map $\mu : M \to \mathfrak{t}^*$ (i.e., the action is Hamiltonian). Let ξ be a regular value of μ and denote by $M_\xi := \mu^{-1}(\xi)$. Then the action of T on M restricts to a (Hamiltonian) action of T on M_ξ. We say that the orbit space M_ξ/T is the *symplectic reduction* of (M, ω) (by T), denoted $M//T$. Note that if T acts freely on M_ξ, then the symplectic reduction is a manifold. If the action is only locally free (i.e., all isotropy groups are finite), then the quotient has the structure of a *symplectic orbifold*. We can summarize the above by saying that the category of differentiable manifolds, *Diff*, is not closed under symplectic reduction.

Now consider the case that (M, ω, T, μ) is a symplectic toric manifold and that $S \subset T$ is a subgroup (which need not be connected). Let X_ξ denote the symplectic reduction by S; i.e., $X_\xi := M_\xi/S$. Then there is a "residual" T/S action on X_ξ such that $(X_\xi, \omega, T/S, \mu)$ is a *symplectic toric orbifold*.

A classification of symplectic toric orbifolds was given by Lerman and Tolman [19]. In this paper, the authors enriched the class of geometric objects being considered from manifolds to orbifolds, requiring the simultaneous enlargement of the class of combinatorial objects. The exact class of combinatorial objects that Lerman and Tolman used to capture the geometry of symplectic toric orbifolds are the so-called "labeled polytopes." These polytopes relax the geometry of the polytope from those considered by Delzant by dispensing with the requirement that the normals at each vertex generate $\mathbb{Z}^d$, and include more combinatorial information in the form of a positive integer label for each facet.

Definition 2.3. A convex rational simple polytope in $\mathfrak{t}^*$ (not necessarily smooth) with a positive integer associated to each facet is called a **labeled polytope**.

We make the obvious remark that every Delzant polytope can be given the structure of a labeled polytope in a manner that is consistent with Delzant's theorem, simply by assigning the label 1 to each facet. Thus, labeled polytopes are strictly more general than the Delzant polytopes considered earlier, and the result that we state below subsumes the theorem of Delzant.

Then, in analogy with Delzant's Theorem, we have a classification of symplectic toric orbifolds.

Theorem 2.3 (Lerman-Tolman [19])**.** 1. To every compact symplectic orbifold (M, ω, T, μ) there naturally corresponds a labeled polytope: The image of the moment map, $\mu(M)$, is a rational simple polytope. For every open facet $\dot{F}$ of $\mu(M)$ there exists a positive integer, $n_{\dot{F}}$ such that the structure group of every $x \in \mu^{-1}(\dot{F})$ is $\mathbb{Z}/n_{\dot{F}}\mathbb{Z}$.
2. Two compact symplectic toric orbifolds are isomorphic if and only if their associated labeled polytopes differ only by a translation and the corresponding open facets have the same integer labels.
3. Every labeled polytope arises from some compact symplectic toric orbifold.

The above classification theorem uses the "classical" definition of orbifold, namely a space having an atlas of charts whose images are open sets in $\mathbb{R}^d$ quotient by a finite group. This definition of orbifold is not without difficulties. In particular, it is hard to define the notation of a map of orbifolds in a way that naturally encodes the information about the singularities; i.e., the *structure groups* (see [17]).

2.0.1 The Algebraic Approach

We now shift perspective and briefly outline an algebraic approach to the constructions we have detailed above, due to Cox [6]. In this setting, the geometric objects are toric varieties and the combinatorial objects are fans. Recall that a *toric variety*, X, is an algebraic variety containing a torus as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself (see [10], [7]). Recall, also, that a *fan* in a lattice N is a collection of strongly convex rational polyhedral cones satisfying the property that every face of a cone is also a cone and the intersection of two cones is a face of each (again, see [10]). One then can think of a fan as an object dual to a polytope described above, the rays being given by the (inward) primitive normals to the facets of the polytope. We caution, however, that there are fans that are not dual to polytopes (in fact, examples are known as early as in $\mathbb{R}^3$). The analogous algebraic theory to what we have detailed in the symplectic setting falls under the name of Geometric Invariant Theory (GIT). For a parallel introduction to the algebraic and symplectic perspectives, the reader is encouraged to consult the survey of Thomas, [23], or the lecture notes of Proudfoot, [20].

One way that the combinatorial objects of Lerman-Tolman are more general than those of Delzant is that the polytopes of the correspondence established by Lerman-Tolman are strictly more general (no longer requiring smoothness and by allowing for non-negative facet labels). A different route of generalizing this correspondence between combinatorial and geometric objects was introduced by Borisov, Chen, and Smith [3]. In the correspondences established by Delzant and Lerman-Tolman, from a polytope (resp., labeled polytope) of n facets in $\mathfrak{t}^* \cong \mathbb{R}^d$ one can specify a unique module morphism, $\beta : \mathbb{Z}^n \to \mathbb{Z}^d$. Borisov, Chen, and Smith generalized the combinatorial side of the correspondence is by replacing $\mathbb{Z}^d$ with an arbitrary finitely generated $\mathbb{Z}$-module, N (i.e., allowing N to have a non-trivial torsion component). The map $\beta : \mathbb{Z}^n \to N$ is no longer uniquely determined by the polytope, but is still required to be compatible (see Section 3.1). With this broader class of $\mathbb{Z}$-modules the combinatorial objects that we consider are the *stacky polytopes*, (N, Δ, β), of Sakai [22] and stacky fans of Borisov, Chen, and Smith (see Section 3.1 for the precise definition). The corresponding geometric object is now a quotient stack.

We mention that in the case that N is free, a stacky polytope is (in an obvious way) a labeled polytope that we defined earlier. This fact makes clear that stacky polytopes are a more general combinatorial object than those considered by Lerman and Tolman in their classification, and any classification must subsume Lerman-Tolman's result.

We can continue the (almost dual) algebraic approach described above, and define a fan analogue of the stacky polytope. Borisov, Chen, and Smith [3] defined a *stacky fan*, (N, Σ, β), where a rational simplicial fan takes the place of the polytope in the above definition (see Section 3.1 for a precise definition). We note that despite the order in which we have introduced these "stacky" combinatorial objects here, the work of Borisov, et al. antedates that of Sakai by approximately seven years. Additionally, because the literature of stacky fans is highly developed and the stacky fan is strictly more general than stacky polytopes (recall that for every polytope there is a corresponding fan, but the converse does not hold), statements of theorems and their proofs will use stacky fans as the combinatorial object of consideration.

Given a stacky fan, $\mathbf{\Sigma} := (N, \Sigma, \beta)$, there is a corresponding quotient stack that one can construct, $\mathcal{X}(\mathbf{\Sigma})$, denoted hereafter $[Z/G]$. Recall that a *quotient stack* is category fibered by groupoids that models a Lie group action on a manifold (see, for example, [15], [22]).

When we generalize the combinatorial objects to stacky fans, these quotient stacks are the geometric objects that replace the classical orbifolds of Lerman-Tolman. This association of a quotient stack to a stacky fan is accomplished via a homological construction known as the mapping cone of $\beta : \mathbb{Z}^n \to N$, $\mathrm{Cone}(\beta)$. Though the mapping cone is a general construction from homological algebra, by specific choices of projective resolutions it can be made fairly explicit, thus allowing one to deduce geometric properties of the quotient stack, $[Z/G]$, from the stacky fan, $\mathbf{\Sigma}$.

In the case that one begins with a quotient stack $[Z/G]$ that models the action of a connected Lie group, G, there is also a construction that associates to $[Z/G]$ a stacky fan. This construction again uses techniques of homological algebra. From this algebraic apparatus one can deduce properties of the combinatorial object that are determined by the quotient stack.

Chapter 3: The basic definitions

3.1 Stacky Polytopes and Fans

We now give some details of the combinatorial objects that we will consider, the stacky polytopes and their algebraic counter-part, stacky fans. Though the stacky fans were introduced long before stacky polytopes and though the algebraic formulation is more widely used and strictly more general, we will first introduce the stacky polytope, as defined by Sakai [22]. Our reason for so doing is that the polytopes are intuitively appealing and were the combinatorial objects originally considered in the symplectic formulation.

Definition 3.1. A **stacky polytope** is a triple, (N, Δ, β), consisting of

- N, a finitely generated $\mathbb{Z}$-module of rank d;
- Δ, a simple polytope in $\mathbf{t}^* = (N \otimes_{\mathbb{Z}} \mathbb{R})^*$ with facets $F_1, \ldots, F_n$;
- $\beta : \mathbb{Z}^n \to N$, a homomorphism of $\mathbb{Z}$-modules

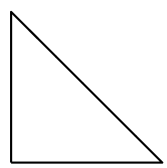
satisfying:

(1) the cokernel of β is finite, and

(2) for $1 \leq j \leq n, \beta(e_j) \otimes 1$ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ is an inward pointing normal to the facet F_j.

Example 3. As an example of a stacky polytope consider a generalization of our illustration of the image of the moment map from the last chapter, Example 2. That is, we consider the triple (N, Δ, β) where N is the free $\mathbb{Z}$-module of rank one (i.e, $\mathbb{Z}$), Δ is the unit interval, $[0, 1]$, and $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ is given in the standard bases by the matrix $(-2 \quad 2)$.

Example 4. Recall that a weighted projective space is the quotient S^{2n+1}/S^1, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ and S^1 acts by $g \cdot (z_0, \ldots, z_n) = (g^{b_0} z_0, \ldots, g^{b_n} z_n)$ and $b_j > 0$ (see Definition 5.1). As we shall demonstrate later (Theorem 5.3), a stacky polytope giving the weighted projective space $\mathbb{P}(105, 70, 42)$ is (N, Δ, β) where

$N = \mathbb{Z}^2 \oplus \mathbb{Z}/7\mathbb{Z}$, $\Delta =$  ,

$\beta : \mathbb{Z}^3 \to \mathbb{Z}^2 \oplus \mathbb{Z}/7\mathbb{Z}$ defined by

$$\beta(e_1) = \begin{pmatrix} -2 \\ -2 \\ |1| \end{pmatrix}$$

$$\beta(e_2) = \begin{pmatrix} 3 \\ 0 \\ |0| \end{pmatrix}$$

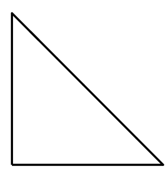
$$\beta(e_1) = \begin{pmatrix} 0 \\ 5 \\ |0| \end{pmatrix}.$$

Note that any Delzant polytope and labeled polytope of Lerman-Tolman described in Chapter 2 can be given the structure of a stacky polytope. In fact, the Lerman-Tolman polytopes include all cases of stacky polytopes where N is free.

Proposition 1. Let N be a free module of rank d, then there is a one-to-one correspondence between stacky polytopes, (N, Δ, β) and labeled polytopes in $(\mathbb{R}^d)^*$.

Proof. Before beginning we remark that condition (2) in the definition of a stacky polytope immediately implies that the polytope is rational. We now define the obvious map from labeled polytopes to stacky polytopes. Let $(\Delta, \{m_i\})$ be a labeled polytope in $(\mathbb{R}^d)^*$ (i.e., Δ is a rational simple polytope). Denote the facets of Δ, $F_1, \ldots, F_n$. Let m_i be the positive integer label associated to facet F_i and let $y_i \in \mathbb{Z}^d$ be the primitive inward-pointing normal to facet F_i. Now, define the map $\phi : (\Delta, \{m_i\}) \mapsto (N, \Delta, \beta)$, where N is the free module of rank d (i.e., $N \cong \mathbb{Z}^d$) and $\beta : \mathbb{Z}^n \to N$ is defined by $\beta(e_i) = m_i y_i$. We also define the map $\psi : (N, \Delta, \beta) \to (\Delta, \{m_i\})$ by taking $m_i = gcd(\{y_i^j\}_{j=1}^d)$, where $y_i = \beta(e_i)$ and y_i^j denotes the jth component of this vector (under an isomorphism of N and $\mathbb{Z}^d$). It's now immediate that ϕ and ψ are inverses and the correspondence is one-to-one. $\square$

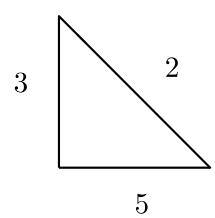
Example 5. Again consider the stacky polytope from Example 4, except suppose that N is free. That is, consider the stacky polytope (N, Δ, β) where

$N = \mathbb{Z}^2,$ $\Delta =$  ,

$\beta : \mathbb{Z}^3 \to \mathbb{Z}^2$ defined by

$$\begin{aligned} \beta(e_1) &= \begin{pmatrix} -2 \\ -2 \end{pmatrix} \\ \beta(e_2) &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ \beta(e_1) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This stacky polytope can be given the stucture of a labeled polytope



Moreover, this stacky polytope and labeled polytope yield the same quotient stack.

We will now introduce the stacky fan, first defined by Borisov, Chen, and Smith [3]. As the stacky fan is strictly more general than the stacky polytope (there are fans that do not arise as duals to polytopes, see [7] for an example) we will proceed to define these objects and state our results in terms of stacky fans. However, when computing examples, we will often revert to the stacky polytope, as the intuitive appeal of drawing a polytope and labeling facets is too strong to resist.

To make this exposition self-contained, we begin with a few standard definitions that we will need. The reader who is not familiar with these constructions is encouraged to consult the recent text by Cox, Little, and Schenk [7] or the well-known text of Fulton [10]. Let N be a lattice of rank d; i.e., an Abelian group isomorphic to $\mathbb{Z}^d$. A *cone* in N is a subset of $N \otimes_{\mathbb{Z}} \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a finite set of vectors $\{\sigma_1, \ldots, \sigma_n\} \subset N$ (called *rays*). We assume that the cones are *strongly convex*; i.e., that they contain no line through the origin. If a cone is generated by rays $\{\sigma_1, \ldots, \sigma_n\}$ then a *face* of the cone is the span of any proper subset, $\{\sigma_{j_1}, \ldots, \sigma_{j_k}\}$. A *fan* is a finite collection of cones such that: each face of a cone is the fan is also in the fan; and any pair of cones in the fan intersects in a common face. A fan is *complete* if the union of the cones is $N \otimes_{\mathbb{Z}} \mathbb{R}$. Finally, a fan is *simplicial* if the generators of each cone are linearly independent over $\mathbb{R}$.

Definition 3.2. A **stacky fan** is a triple, (N, Σ, β) consisting of

- N, a finitely generated $\mathbb{Z}$-module of rank d;
- Σ, a rational simplicial fan with rays $\sigma_1, \ldots, \sigma_n$ in $\mathfrak{t} = N \otimes_{\mathbb{Z}} \mathbb{R}$; and
- $\beta : \mathbb{Z}^n \to N$, a homomorphism of $\mathbb{Z}$-modules.

satisfying the following conditions.

(1) Let $\bar{n}_j$, $j = 1, \ldots, n$ denote the image of $\beta(e_j)$ through the natural map $N \to \mathfrak{t}$. Then $\bar{n}_j$ generates the ray σ_j.

(2) The cokernel of the homomorphism $\beta : \mathbb{Z}^n \to N$ is finite.

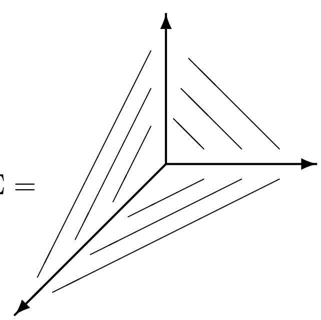
Example 6. Recall the line segment given in Example 3. By taking the normals to each facet, we construct the dual fan and endow it with the structure of a stacky fan. Namely, we take N to be $\mathbb{Z}$, Σ the fan

 ,

and $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ to be the map determined by $\beta(e_1) = 2$ and $\beta(e_2) = -2$. It's immediate that this example satisfies the definition of a stacky fan.

As a second example, we continue Examples 4 and 5.

Example 7. The stacky fan that yields $\mathbb{P}(105, 70, 42)$ is given by taking N and β as in Example 4 and with fan

$\Sigma =$  ,

where the hashing in the figure indicates the 2-dimensional cones in the fan.

These examples illustrate the fact that every stacky polytope yields a stacky fan. Suppose that (N, Δ, β) is a stacky polytope. Let $\{\sigma_1, \ldots, \sigma_n\}$ be the collection of primitive (inward) normals to the facets of Δ (because condition (2) in the definition of a stacky polytope implies Δ is a rational polytope, these primitive normals are well-defined). For each face of a cone in Σ, take the nonnegative span of the collection of normals corresponding to each face (i.e., we can write each face uniquely as the intersection of a collection of facets). Then (N, Σ, β) is a stacky fan.

3.2 Quotient Stacks

In this section we will present a brief introduction to quotient stacks, the geometric objects in the correspondence of interest. Far from being a comprehensive treatment of the important and active research area of stacks, we will be fairly imprecise until restricting our attention to quotient stacks. Even in the setting of quotient stacks, we will develop only those facts that we require for the precise statement of our results. For those who desire a more complete treatment of stacks, we recommend the papers of Fantechi [9], Hochenegger and Witt [15], and Heinloth [14].

3.2.1 Stacks and Quotient Stacks

In what follows, we will take $\mathcal{D}iff$ to be the category whose objects are (second-countable Hausdorff) differentiable manifolds and whose morphisms are $\mathcal{C}^\infty$ maps of manifolds. We begin by recalling some well-known facts and standard definitions. A *principal G-bundle* is a fiber bundle, $\pi : P \to X$ together with a smooth action of G on P such that G preserves the fibers of $\pi : P \to X$ and acts freely and transitively on these fibers. Here, as our interest is the base category $\mathcal{D}iff$, we are assuming the X and P are smooth manifolds and that G a Lie group acting by diffeomorphisms.

In what follows, we will denote by BG the *classifying space* for the a group G. Recall that this space is characterized by the property that for any topological space X, the isomorphism equivalence classes of principal G-bundles over X are in one-to-one correspondence with homotopy equivalence classes of maps $X \to BG$ (for a brief introduction, see [13]). Then we will denote the isomorphism equivalence classes of principal G-bundles over X by $BG(X)$.

We assume that the reader is familiar with the basic definitions of category theory. However, we recall the definitions of small category and groupoid, as they may be less widely known.

Definition 3.3. A **small category** is a category whose objects and morphisms are sets (as opposed to being proper classes).

Example 8. As an example, let $\mathcal{S}et$ be the category whose objects are sets and whose morphisms are functions between sets. Obviously, $\mathcal{S}et$ is not a small category, as the "collection of all sets" is not a set. Less obvious is that $\mathcal{D}iff$ is a small category (most easily seen as a consequence of the Whitney embedding theorem).

Definition 3.4. A **groupoid** is a small category in which every morphism is an isomorphism.

We will now speak generally (and imprecisely) about stacks, before restricting our attention to the special case of interest (namely, quotient stacks). Our presentation of this material has been adopted from the wonderful expository paper of Fantechi, [9].

Let $\mathfrak{S}$ be a category. Then a *category over* $\mathfrak{S}$ is a category $\mathfrak{X}$ with a fixed covariant functor $\pi : \mathfrak{X} \to \mathfrak{S}$. We will say that $\mathfrak{S}$ is the *base category.* We also say that $E \in \mathrm{Ob}(\mathfrak{X})$ *lies over* $X \in \mathrm{Ob}(\mathfrak{S})$ (or *lifts* X or is a *lifting of* X) if $\pi(E) = X$. Similarly, let $\phi \in \mathrm{Mor}(\mathfrak{X})$, say $\phi : E \to F$. Then ϕ *lies over* $f \in \mathrm{Mor}(\mathfrak{S})$ if $\pi(\phi) = f$; i.e., the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \pi(E) & \xrightarrow{f} & \pi(F). \end{array}$$

Example 9. As an example of the above, we take $\mathcal{D}iff$ to be the base category and let $\mathfrak{V}_r$ be the category whose objects are rank r vector bundles over smooth manifolds and whose morphisms are "given by pullback diagrams." By this we mean that if $f : X \to Y$ (i.e, $f \in \mathrm{Mor}(\mathcal{D}\mathrm{iff})$) and $E \in \mathrm{Ob}(\mathfrak{V}_r)$ lies over Y then there is a unique (up to unique isomorphism) $\bar{f} : f^*E \to E$, the morphism lying over f (where f^*E denotes the pullback bundle). Then $\mathfrak{V}_r$ is a category over $\mathcal{D}iff$.

Again eschewing complete generality, we give a less than fully general definition of the fundamental concept necessary to define stacks.

Definition 3.5. A category $\mathfrak{X}$ over $\mathfrak{S}$ is a **category fibered in groupoids** over $\mathfrak{S}$ if for all

$f \in \text{Mor}(\mathfrak{S})$ (i.e., $f : X \to Y$) and for all $F \in \text{Ob}(\mathfrak{X})$ such that $\pi(F) = Y$ (i.e., for any lifting of Y), there exists $\phi : E \to F$ such that $\pi(\phi) = f$ (i.e., ϕ lies over f). Moreover, we require that the lifting be unique up to unique isomorphism; i.e., for any other lifting $\phi' : E' \to F$ there is a unique isomorphism $\psi : E' \to E$ lying over id_X such that $\phi' = \phi \circ f$. Moreover, a **morphism of categories fibered in groupoids** is a functor that commutes with the projections to the base categories. Finally, an **isomorphism of categories fibered in groupoids** is a morphism that is an equivalence of categories.

Example 10. We note that the universal property of pullbacks of bundles implies that $\mathfrak{V}_r$ given in Example 9, is a category fibered in groupoids.

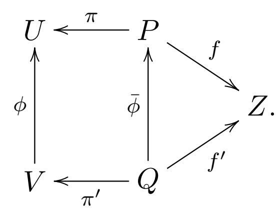
We can finally now say that a *stack* (over a base category) is a category fibered in groupoids whose fibers "glue like bundles." The work required to make this definition precise would take us too far afield. However, we will now specialize to the case of quotient stacks, where we can be quite explicit. Roughly, a quotient stack (over $\mathcal{D}\mathit{iff}$) is a category fibered by groupoids that models the action of a Lie group on a manifold, $G \curvearrowright Z$, denoted hereafter $[Z/G]$. More precisely,

Definition 3.6. Let G be a Lie group acting on a (smooth) manifold Z. The **quotient stack** $[Z/G]$ is the category fibered by groupoids (over $\mathcal{D}\mathit{iff}$) whose objects are all pairs consisting of a principal G-bundle $\pi : P \to X$ and a G-equivariant map $f : P \to Z$ (where the G-equivariance is with respect to the action of G on P given in the principal bundle structure of $\pi : P \to X$). That is for $X \in \text{Ob}(\mathcal{D}\mathit{iff})$, the fiber over X is

$$[Z/G](X) := \{(\pi : P \to X, f : P \to Z) \mid P \in BG(X), f \text{ is } G\text{-equivariant}\}.$$

An object in the category is written as the diagram $U \stackrel{\pi}{\leftarrow} P \stackrel{f}{\rightarrow} Z$, where U is an arbitrary element of $\text{Ob}(\mathcal{D}\mathit{iff})$. We will frequently refer to $U \stackrel{\pi}{\leftarrow} P \stackrel{f}{\rightarrow} Z$ simply as P when no confusion is likely. The morphisms in $[Z/G]$ are given by G-equivariant morphisms of G-bundles; that is a morphism from $V \stackrel{\pi'}{\leftarrow} Q \stackrel{f'}{\rightarrow} Z$ to $U \stackrel{\pi}{\leftarrow} P \stackrel{f}{\rightarrow} Z$ is a pair of smooth maps $(\phi, \bar{\phi})$ such that

$\bar{\phi}$ is G-equivariant and the following diagram commutes



With (quotient) stacks defined as a category, a *morphism of stacks* is a functor $\mathfrak{X} \to \mathfrak{X}'$ that commutes with projections to $\mathcal{D}iff$. An *isomorphism* is a morphism $\mathfrak{X} \to \mathfrak{X}'$ that is an equivalence of categories. Finally, we mention that one "should" define a quotient stack to be a stack that is equivalent to $[Z/G]$ for some Z and G, but that may not be presented as such.

3.2.2 Examples

Explicit computation of (non-trivial) examples can be a bit difficult. However we present two straight-forward examples to help the reader gain some intuition about the information encoded in this structure. We start with the most basic example, the (necessarily trivial) action of a group on a point.

Example 11. Let Z be a point, say $Z = \{pt\}$, and let G be a (compact) Lie group. We will now show that the objects in $[Z/G] = [\{pt\}/G]$ are the classifying space for G, BG (i.e., "all principle G-bundles"). Recall that the objects in $[\{pt\}/G]$ are all pairs of maps, $\{(E \to U, E \to \{pt\}) \mid E \to U$ a principal G-bundle, $E \to \{pt\}$ G-equivariant$\}$. Pictorially, an object in $[\{pt\}/G](U)$ is a diagram

$$\begin{array}{ccc} E & \to & \{pt\}. \\ \downarrow & & \\ U & & \end{array}$$

Of course every map $E \to \{pt\}$ is G-equivariant, so the objects of $[\{pt\}/G]$ are all principal G-bundles (i.e., $[\{pt\}/G] \cong BG$) and, in particular, the fibers over $X \in \mathcal{D}iff$ are all principal G-bundles over X (i.e., $BG(X)$).

We remark that this classifying space for G certainly "encodes" the Lie group G (as the fibers of the bundle). Thus, the quotient stack allows us to recover the group. We make the (now obvious) remark that the quotient stack contains substantially more information than is contained in the quotient (topological) space $\{pt\}/G \cong \{pt\}$ (in particular, the quotient stack "remembers G and how it acts," as opposed to the quotient space which only "knows" the orbits of G).

As a second example we consider the case that Z is a compact manifold and G acts freely on Z. In this case, the quotient space, Z/G, is a manifold. We show that the quotient stack, $[Z/G]$, "is" this manifold. Before starting this example we recall another result from category theory, namely the contravariant version of the Yoneda lemma. Suppose that A is an object of category $\mathcal{C}$. For any object $C \in \mathrm{Ob}(\mathcal{C})$ we have a set of morphisms $\mathrm{Mor}(C, A)$. Then a morphism $f : B \to C$ induces a map of sets, $\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A)$ by precomposition with f; that is, induces a contravariant functor $h_A : \mathcal{C} \to \mathcal{S}et$, where $\mathcal{S}et$ is the category whose objects are sets and whose morphisms are maps of sets. Yoneda's lemma states that the functor h_A is fully faithful; i.e., determines A up to unique isomorphism (informally, if one knows all maps into A, then one knows A). In fact, Yoneda's lemma asserts that the functor h_A is a contravariant equivalence of categories between $\mathcal{C}$ and the subcategory of representable functors.

Example 12. Let G be a compact Lie group acting freely on the compact manifold Z. Recall that $\mathrm{Ob}([Z/G]) = \{X \overset{\pi}{\leftarrow} P \overset{f}{\to} Z \mid \pi : P \to X$ a principal G-bundle, $f : P \to Z$ a G-equivariant map $\}$. Let $X \overset{\pi}{\leftarrow} P \overset{f}{\to} Z \in \mathrm{Ob}([Z/G])$. Recall that $\pi : P \to X$ is a principal G-bundle, so there is a G-action that preserves and acts transitively on each fiber. Thus, $P/G \cong X$. Also, because Z is compact and G acts transitively on Z, Z/G is a smooth manifold. Finally, because $f : P \to Z$ is G-equivariant, it descends to a (well-defined) map on the quotients, $\bar{f} : X \to Z/G$. Thus, the objects in the fiber over X, $[Z/G](X)$, are equivalent to $\mathcal{C}^\infty(X, Z/G)$, the set of all smooth maps $X \to Z/G$ (i.e., the morphisms in the category $\mathcal{D}\mathit{iff}$). As X is an arbitrary smooth manifold, the objects in the quotient stack are the category $\mathcal{C}^\infty(\cdot, Z/G)$. Finally, recall that the Yoneda Lemma implies that the contravariant functor $h_Z : \mathcal{D}\mathit{iff} \to \mathcal{S}et$ determines Z up to unique isomorphism. Thus, $\mathcal{C}^\infty(\cdot, Z/G)$ is equivalent to the manifold Z/G.

Chapter 4: The BCS construction — Stacky Fan to Quotient Stack

We now review the generalized Gale duality construction given in Borisov, Chen and Smith [3] which allows us to associate a quotient stack, $[Z/G]$, to a stacky fan, (N, Σ, β). There are (at least) two slightly different formulations. The first is the symplectic, where Z is a level set of a moment map at a regular value, while G is a (possibly not connected) compact Abelian Lie group. In the second approach, Z is an affine variety and G is an algebraic torus (again, possibly not connected). We will begin our summary of the construction with homological preliminaries that are common to both perspectives, then we will outline the construction of a quotient stack from the symplectic perspective. This subsection will be followed by a description of the algebraic construction of a quotient stack from a stacky fan.

4.1 Homological preliminaries

Let (N, Δ, β) be a stacky polytope or (N, Σ, β) a stacky fan. We will first construct a $\mathbb{Z}$-module homorophism, $\beta^\vee : (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$. Here, $\mathrm{DG}(\beta) := H^1(\mathrm{Cone}(\beta)^*)$ is the first cohomology group of the dual mapping cone of β, a standard homological construction on a map of chain complexes. The details of this construction are given below.

We begin by choosing projective resolutions $\mathbf{E}$ and $\mathbf{F}$ of $\mathbb{Z}^n$ and N, respectively. Then $\beta : \mathbb{Z}^n \to N$ lifts to a map of chain complexes $\beta : \mathbf{E} \to \mathbf{F}$. Recall that for any map of chain complexes, $\beta : \mathbf{E} \to \mathbf{F}$, we can construct the mapping cone, $\mathrm{Cone}(\beta)$. We recall the definition now. Suppose that $\mathbf{E} = (E_*, d_E)$ and $\mathbf{F} = (F_*, d_F)$ are chain complexes; i.e.,

$$\mathbf{E} = \cdots \to E_n \xrightarrow{d_E^n} E_{n-1} \xrightarrow{d_E^{n-1}} \cdots E_0 \xrightarrow{0} 0$$

and similarly for $\mathbf{F}$. Then the *mapping cone* of β, $\mathrm{Cone}(\beta)$, is the chain complex $\mathbf{E}[1] \oplus \mathbf{F}$; i.e., the degree n part of $\mathrm{Cone}(\beta)$ is $E_{n-1} \oplus F_n$ and the differential is given by $d^n_\beta(e, f) = (-d^{n-1}_E(e), d^n_F(f) - \beta(e))$ where $e \in E_{n-1}$ and $f \in F_n$.

The mapping cone, $\mathrm{Cone}(\beta)$, fits into a short exact sequence of chain complexes

$$0 \to \mathbf{F} \to \mathrm{Cone}(\beta) \to \mathbf{E}[\mathbf{1}] \to 0.$$

Now, because $\mathbf{F}$ is a projective $\mathbb{Z}$-module, we get a short exact sequence of cochain complexes

$$0 \to \mathbf{E}[\mathbf{1}]^* \to \mathrm{Cone}(\beta)^* \to \mathbf{F}^* \to 0.$$

Associated to this short exact sequence of cochain complexes, there is the long exact sequence in cohomology. This long exact sequence contains the portion

$$\cdots \to N^* \to (\mathbb{Z}^n)^* \to H^1(\mathrm{Cone}(\beta)^*) \to \mathrm{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \to 0. \tag{4.1}$$

Let $\mathrm{DG}(\beta) := H^1(\mathrm{Cone}(\beta)^*)$ and take $\beta^\vee : (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$ to be the map from the l.e.s. in cohomology given in (4.1).

The above homological algebra is (up to natural isomorphism) independent of the choices of resolutions and lifts, so we now make some choices so that we can do explicit computations. In particular, we take especially nice projective resolutions $\mathbf{E}$ and $\mathbf{F}$. First, because N is a finitely generated $\mathbb{Z}$-module of rank d, the Fundamental Theorem of Finitely Generated Abelain Groups implies

$$N \cong \mathbb{Z}^d \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_r\mathbb{Z},$$

where $a_1 \mid \cdots \mid a_r$. We fix an isomorphism, once and for all. Thus, there is an integer matrix Q such that

$$0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to 0$$

is a projective resolution of N. In fact, in the standard bases for $\mathbb{Z}^r$ and $\mathbb{Z}^{n+r}$

$$[Q] = \begin{pmatrix} & \mathbf{0}_{d\times r} & \\ a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}.$$

In what follows, the presence of bracket notation, $[\cdot]$, will indicate that we have chosen bases and are indicating a matrix. Additionally, the map $\beta : \mathbb{Z}^n \to N$ lifts to a map $B : \mathbb{Z}^n \to \mathbb{Z}^{d+r}$. Of course, $\mathbb{Z}^n$ is free, hence projective. Thus, we can take $E_0 = \mathbb{Z}^n$ and $E_i = 0$ for $i \neq 0$ to be a projective resolution of $\mathbb{Z}^n$. Once we make these choices, we have that $\mathrm{Cone}(\beta)$ is the following chain complex:

$$0 \to \mathbb{Z}^{n+r} \xrightarrow{[BQ]} \mathbb{Z}^{d+r} \to 0.$$

We then construct the following diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & (\mathbb{Z}^{d+r})^* & \xrightarrow{i} & (\mathbb{Z}^{d+r})^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle [BQ]^*} & & \downarrow{\scriptstyle Q^*} & & \\ 0 & \longrightarrow & (\mathbb{Z}^n)^* & \xrightarrow{\iota} & (\mathbb{Z}^{n+r})^* & \xrightarrow{\pi} & (\mathbb{Z}^r)^* & & \end{array},$$

where i is the identity map, ι is inclusion in the first n coordinates, and π is projection to the last r coordinates. Applying the Snake Lemma to this diagram yields the exact sequence ,

$$\ker\left(0 \to (\mathbb{Z}^n)^*\right) \longrightarrow \ker\left([BQ]^*\right) \longrightarrow \ker\left(Q^*\right) \longrightarrow \mathrm{coker}\left(0 \to (\mathbb{Z}^n)^*\right) \longrightarrow$$
$$\mathrm{coker}\left([BQ]^*\right) \longrightarrow \mathrm{coker}\left(Q^*\right).$$

Substituting, this exact sequence becomes,

$$0 \longrightarrow \ker([BQ]^*) \longrightarrow (\mathbb{Z}^d)^* \longrightarrow (\mathbb{Z}^n)^* \longrightarrow (\mathbb{Z}^{n+r})^*/\mathrm{im}([BQ]^*)$$
$$\longrightarrow (\mathbb{Z}^n)^*/Q^* \cong \mathrm{Tor}(N). \tag{4.2}$$

Thus,

$$\mathrm{DG}(\beta) \cong (\mathbb{Z}^{n+r})^*/\mathrm{im}([BQ]^*),$$

and the map $\beta^\vee$ is the composition of the inclusion$(\mathbb{Z}^n)^* \to (\mathbb{Z}^{n+r})^*$ (in the first n components) and the quotient map $(\mathbb{Z}^{n+r})^* \to (\mathbb{Z}^{n+r})^*/\mathrm{im}([BQ]^*)$.

Example 13. We now revisit Example 6, the stacky fan corresponding to the line segment with 2 at each endpoint. More specifically, consider the stacky fan (N, Σ, β), where $N = \mathbb{Z}$, Δ is the 1-dimensional fan generated by the rays 1 and -1, and $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $\beta(e_1) = 2$ and $\beta(e_2) = -2$.

We now construct the dual group, $DG(\beta)$. We take projective resolutions of $\mathbb{Z}^2$,

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \| & & \\ & & E_0 & & \end{array}$$

and of $N = \mathbb{Z}$

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \\ & & F_0. & & \end{array}$$

Then $\beta : \mathbb{Z}^2 \to N$ lifts to a chain map $\beta : \mathbf{E} \to \mathbf{F}$

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle B} & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

where B is simply the matrix $(2\ -2)$.

We then construct the mapping cone of β

$$\mathrm{Cone}(\beta): \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^2 \oplus 0 & \longrightarrow & 0 \oplus \mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & E_0 \oplus F_1 & & E_{-1} \oplus F_0 & & \end{array}$$

where the only non-trivial boundary map is

$$d^1_\beta : E_0 \oplus F_1 \to E_{-1} \oplus F_0$$

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, 0 \right) \mapsto (0, 2y - 2x).$$

Now $\mathrm{Cone}(\beta)$ fits into a short exact sequence of chain complexes, $0 \to F \to \mathrm{Cone}(\beta) \to E[1] \to 0$. More explicitly,

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}^2 \oplus 0 & \longrightarrow & \mathbb{Z}^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0 \oplus \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

One then takes the dual complex, $0 \to E[1]^* \to \mathrm{Cone}(\beta)^* \to F^* \to 0$, which gives a long exact sequence in cohomology, $\ldots \to 0 \to \mathbb{Z}^* \to (\mathbb{Z}^2)^* \to H^1(\mathrm{Cone}(\beta))\,(= DG(\beta)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \ldots$

From the explicit form given in BCS, we have

$$DG(\beta) = (\mathbb{Z}^2)^*/\mathrm{im}(B^*) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

and $\beta^\vee$ is the projection $(\mathbb{Z}^2)^* \to (\mathbb{Z}^2)^*/\mathrm{im}(B^*) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We can write the map explicitly as $(x, y) \mapsto (x + y, x \mod 2)$.

Example 14. We compute one more example, this time, we pick a module with torsion. As above, take the stacky fan (N, Σ, β), where Δ is the 1-dimensional fan generated by the rays 1 and -1, however take $N = \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\beta : \mathbb{Z}^2 \to \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ defined by $\beta(e_1) = (2, 0)$ and $\beta(e_2) = (-2, 1)$.

We specify the obvious projective resolutions, first of $\mathbb{Z}^2$

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \| & & \\ & & E_0 & & \end{array}$$

and then of N

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{Q:=\binom{0}{3}} & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & F_1 & & F_0 & & \end{array}$$

Then β lifts to a map of these chain complexes, $\beta : \mathbf{E} \to \mathbf{F}$

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle B} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \end{array}$$

where

$$B := \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$$

Now, we have

$$[BQ] = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

and

$$\begin{array}{ccccccc} & 0 & \longrightarrow & \mathbb{Z}^3 & \xrightarrow{[BQ]} & \mathbb{Z}^2 & \longrightarrow 0 \\ & & & \| & & \| & \\ \text{Cone}(\beta): & & & \mathbb{Z}^2 \oplus \mathbb{Z} & & 0 \oplus \mathbb{Z}^2 & \\ & & & \| & & \| & \\ & & & E_0 \oplus F_1 & & E_{-1} \oplus F_0 & \end{array}$$

Omitting the intermediate steps illustrated in the previous example, we have $\mathrm{DG}(\beta) = (\mathbb{Z}^3)^*/\mathrm{im}([BQ]^*)$. Now,

$$[BQ]^* = \begin{pmatrix} 2 & 0 \\ -2 & 1 \\ 0 & 3 \end{pmatrix} \overset{SNF}{\longleftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix},$$

where "$\overset{SNF}{\longleftrightarrow}$" indicates the corresponding Smith normal form of a matrix. The given Smith normal form then implies $\mathrm{DG}(\beta) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

In the next two subsections, we will take as our starting point a portion of the exact sequence in (4.1)

$$0 \to N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} \mathrm{DG}(\beta). \tag{4.3}$$

4.2 The symplectic perspective

Given the homological apparatus constructed above, we now outline the symplectic construction of a quotient stack corresponding to the stacky fan (N, Σ, β) (or, the stacky polytope (N, Δ, β)). In this development, the compact Lie group S^1 will play a central role. Because S^1 is an injective $\mathbb{Z}$-module, the functor $\mathrm{Hom}(\cdot, S^1)$ is exact. Applying $\mathrm{Hom}(\cdot, S^1)$ to the exact sequence in (4.3), we have an exact sequence of Lie groups

$$G \xrightarrow{\rho} \mathbb{T}^n \xrightarrow{\sigma} \mathbb{T} \to \{1\},$$

where $G := \mathrm{Hom}(\mathrm{DG}(\beta), S^1)$, $\mathbb{T}^n := \mathrm{Hom}((\mathbb{Z}^n)^*, S^1)$, $\mathbb{T} := \mathrm{Hom}(N^*, S^1)$, and the homomorphisms ρ and σ are induced by $\beta^\vee$ and β^*, respectively.

This (compact Abelian) Lie group, G, is the group that will constitute half of the quotient stack, $[Z/G]$. It remains to define the compact symplectic manifold, Z on which G acts (in a Hamiltonian fashion). In analogy with the Delzant and Lerman-Tolman theorems above, we will construct Z as the level set of a moment map derived from the standard moment map, $\mu_0 : \mathbb{C}^n \to (\mathfrak{t}^n)^* (\cong \mathbb{R}^n)$, where $\mathfrak{t}^n$ is the Lie algebra to $\mathbb{T}^n \cong (S^1)^n$, and the Lie group homomorphism, $\rho : G \to \mathbb{T}^n$.

Recall that $\mathbb{T}^n$ acts on $\mathbb{C}^n$ via $(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) \mapsto (t_1 z_1, \ldots, t_n z_n)$, where $(t_1, \ldots, t_n) \in \mathbb{C}^n$ and $|t_j| = 1$ for $1 \leq j \leq n$. Let $\omega_0 : \mathbb{C}^n \to \mathbb{R}$ be the map $\omega_0(z) = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, then $(\mathbb{C}^n, \omega_0)$ is a symplectic manifold (ω_0 is just the standard symplectic form on $\mathbb{C}^n$). Moreover, the action above is by symplectomorphisms. If we now define

$$\mu_0 : \mathbb{C}^n \to \mathfrak{t}^* \cong (\mathbb{R}^n)^*$$

$$z \mapsto \pi \sum_{j=1}^n |z_j|^2 e^j,$$

where $\{e_j\} \subset \mathbb{R}^n$ is the standard basis and $\{e^j\}$ is the dual basis for $(\mathbb{R}^n)^*$, then μ_0 is a moment map for this $\mathbb{T}^n$ action (i.e., the action is Hamiltonian).

Now, suppose that G is a compact Lie group and that $\rho : G \to \mathbb{T}^n$ is a homomorphism. Then composition with the $\mathbb{T}^n$ action defined above gives an action of G on $\mathbb{C}^n$. Moreover, this action is via symplectomorphisms (when $\mathbb{C}^n$ is equipped with the standard symplectic form, ω_0). Denote the induced linear map of $\rho : G \to \mathbb{T}^n$ by $D\rho : \mathfrak{g} \to \mathbb{R}^n$, where $\mathfrak{g}$ is the Lie algebra to G and we have identified the tangent space of $\mathbb{T}^n$ with $\mathbb{R}^n$. Finally, denote the adjoint map $(D\rho)^* : (\mathbb{R}^n)^* \to \mathfrak{g}^*$ and let $w^j := (D\rho)^*(e^j)$, then

$$\mu : \mathbb{C}^n \to \mathfrak{g}^*$$

$$z \mapsto \pi \sum_{j=1}^{n} |z_j|^2 w^j \tag{4.4}$$

is a moment map for the G-action on $\mathbb{C}^n$ (i.e., the action is Hamiltonian). In this case, the $\{w^j\}$ are called the *weights* of the G-action.

Let $\tau \in \mathfrak{g}^*$ be a regular value of μ. Then $\mu^{-1}(\tau)$ is a smooth submanifold of $\mathbb{C}^n$ (in fact, an ellipsoid whose semi-axis lengths are related to the weights of ρ) and the G-equivariance of μ implies that the action of G on $\mathbb{C}^n$ restricts to an action of G on $\mu^{-1}(\tau)$. We take $Z := \mu^{-1}(\tau)$, and $[Z/G]$ is the quotient stack the properties of which we will be interested in what follows.

Example 15. We now revisit our earlier Example 3, a line segment with a 2 at each endpoiont. That is, the stacky polytope (N, Δ, β) with $N \cong \mathbb{Z}$, $\Delta = [0, 1]$, and $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ given in the standard bases by the matrix $[\beta] = (2 \quad -2)$.

With these choices, $[BQ] = [B] = (2 \quad -2)$, and $\mathrm{DG}(\beta) \cong (\mathbb{Z}^2)^*/\mathrm{im}(2 \quad -2)^* \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^*$. Then, $G = \mathrm{Hom}(\mathrm{DG}(\beta), S^1) \cong S^1 \times \mathbb{Z}/2\mathbb{Z}$, a "disconnected torus." One can verify that the isomorphism $\mathrm{DG}(\beta) \to (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^*$ is given by $(a, b) + \mathrm{im}(2 \quad -2)^* \mapsto (a+b, b \bmod 2)$. It then follows that the homomorphism, $\rho : (S^1 \times \mathbb{Z}/2\mathbb{Z}) \cong G \to \mathbb{T}^2$ is given by $(t, (-1)^k) \mapsto (t, (-1)^k t)$.

Now that we have the group and know its action, we need finally to determine on what it acts; i.e., Z. Recall that the homomorphism $\rho : G \to \mathbb{T}^2$ induces an action of G on $\mathbb{C}^2$ by composition with the usual action $\mathbb{T}^2 \curvearrowright \mathbb{C}^2$. Recalling (4.4), this action is Hamiltonian with moment map

$$\mu : \mathbb{C}^2 \to \mathfrak{g} \cong \mathbb{R}^*$$

$$(z_1, z_2) \mapsto \pi(|z_1|^2 + |z_2|^2).$$

Then any choice of strictly positive $\tau \in \mathbb{R}$, gives $Z = \mu^{-1}(\tau)$ as an $S^3 \subset \mathbb{C}^2$.

We note that the correspondence guaranteed by the theorems of Delzant and Lerman-Tolman are for polytopes (resp., labeled polytopes) in a particular Lie (sub)-algebra. In practice (as in the example above), we often begin with a polytope in $\mathbb{R}^d$. Of course $\mathbb{R}^d$ is diffeomorphic to the Lie algebra ultimately produced, but there is considerable latitude in choosing a diffeomorphism. This choice is reflected in the fact that two polytopes that differ only by the action of an element of $\mathrm{AGL}(d; \mathbb{Z})$ yield the same quotient stack. See Proposition 2 below for the details.

4.3 The algebraic perspective

In the algebraic approach, we again begin with the exact sequence given in (4.3). In contrast to the preceding section, the algebraic torus, a product of $\mathbb{C}^\times$ (where we here mean $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$) will take the place of the compact torus.

To the exact sequence in (4.3), we apply the functor $\mathrm{Hom}(\cdot, \mathbb{C}^\times)$. This gives the exact sequence

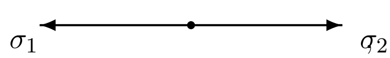
$$G \xrightarrow{\rho} \mathbb{T}^n \xrightarrow{\sigma} \mathbb{T} \to \{1\},$$

where $G := \mathrm{Hom}(\mathrm{DG}(\beta), \mathbb{C}^\times)$, $\mathbb{T}^n := \mathrm{Hom}((\mathbb{Z}^n)^*, \mathbb{C}^\times)$, and $\mathbb{T} := \mathrm{Hom}(N^*, \mathbb{C}^\times)$.

Now that we have group G, we produce a quasi-affine variety (i.e., a Zariski-open subset of an affine variety), Z, on which it acts. Let (N, Σ, β) be a stacky fan (or (N, Δ, β) a stacky polytope) with the rays of Σ (or inward facet normals of Δ) denoted by $\rho_1, \ldots, \rho_n$. Let $\mathbb{C}[z_1, \ldots, z_n]$ be the coordinate ring of $\mathbb{A}^n$ and let $J_\Sigma := \langle \prod_{\rho_j \notin \sigma} z_j \mid \sigma \in \Sigma \rangle$. Finally, set $Z := \mathbb{A}^n \setminus \mathbb{V}(J_\Sigma)$, where $\mathbb{V}(J_\Sigma)$ is the set of points on which the polynomials in J_Σ vanish. The action is derived analogously to what was done in the previous subsection. Recall, that the we have defined the dual map $\beta^\vee : (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$. When we apply $\mathrm{Hom}(\cdot, \mathbb{C}^\times)$, we get a map $\alpha : G \to \mathbb{T}^n (:= (\mathbb{C}^\times)^n)$. Composing with the natural action of $\mathbb{T}^n$ on $\mathbb{A}^n$ (i.e., component-wise multiplication), we have an action of G on $\mathbb{A}^n$. Because $\mathbb{V}(T_\Sigma)$ is a union of coordinate subspaces, Z is G-invariant. This quotient stack, $[Z/G]$, is the geometric object associated to (N, Σ, β) (or, (N, Δ, β)).

Example 16. We now revisit Example 15 in the algebraic setting. Thus, we have stacky fan (N, Σ, β) where $N \cong \mathbb{Z}$,

$\Sigma :=$



and $\beta(e_1) = 2, \beta(e_2) = -2$.

Just as in the previous example, $G \cong S^1 \times \mathbb{Z}/2\mathbb{Z}$. Additionally, it's immediate that $J_\Sigma = \langle z_1, z_2 \rangle \subset \mathbb{C}[z_1, z_2]$ and, thus, $\mathbb{V}(J_\Sigma) = \{0\}$ and $Z = \mathbb{A}^2/\{0\}$.

4.4 Injectivity of the BCS construction

We begin by making a simplifying assumption: **In the remainder of this section we suppose that N is a free module.** We next note the fact that the BCS construction does not yield a quotient stack, rather, the construction produces a representative of an equivalence class of stacks. To clarify this remark, note that the first step in the construction is to select (completely arbitrarily) projective resolutions $\mathbf{E}$ and $\mathbf{F}$ of $\mathbb{Z}^n$ and N, respectively. Additionally, one also must choose a lift of the module map $\beta : \mathbb{Z}^n \to N$ to a map of these resolutions $\mathbf{B} : \mathbf{E} \to \mathbf{F}$. None of these choices is unique, nor in any way canonical. Moreover, different choices of resolutions and lifts do produce different groups, $\mathrm{DG}(\beta)$. Note, however, that different groups that result are isomorphic (see [3]).

We illustrate with a particular example to make the above more clear.

Example 17. Consider the stacky polytope (N, Δ, β) where N is the free module of rank 1, Δ is the closed interval $[-1, 1]$, and $\beta(e_i) = -1^{i+1}$, for $i = 1, 2$ (i.e., in the standard bases, $[\beta] = (1 \quad -1)$). We will consider two different projective resolutions of $\mathbb{Z}^2$. Namely

1. $\mathbf{E}: \qquad 0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0$, and

2. $\mathbf{E}': \qquad 0 \to \mathbb{Z} \to \mathbb{Z}^2 \oplus \mathbb{Z} \to \mathbb{Z}^2 \to 0.$

Then take the obvious resolution of $\mathbb{Z}$, namely $\mathbf{F} : 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ and the obvious lifts of β. We omit the tedious exercise in drawing commutative diagrams and simply note that we have

1. Using the resolution $\mathbf{E}$, $\mathrm{DG}(\beta) = H^1(\mathrm{Cone}(\beta)^*) = (\mathbb{Z}^2 \oplus 0)^* / \left(\begin{pmatrix} -z \\ z \end{pmatrix}, 0 \right) \cong \mathbb{Z}$,

   and

2. using the projective resolution $\mathbf{E}'$, $\mathrm{DG}(\beta) = H^1(\mathrm{Cone}(\beta)^*) = \Big(\ker \big(((\mathbb{Z}^2 \oplus \mathbb{Z}) \oplus 0) \to \mathbb{Z} \oplus 0 \big) \Big) / \Big(\mathrm{im} \big(0 \oplus \mathbb{Z} \to (\mathbb{Z}^2 \oplus \mathbb{Z}) \oplus 0 \big) \Big) \cong \mathbb{Z}$

These cohomology groups are quite obviously isomorphic, but are (equally obviously) not "the same."

Thus, we have exhibited a case where, given a stacky polytope, different choices of resolutions and lifts result in isomorphic but not identical stacks. We now turn our attention to a related question: Can one begin with different stacky polytopes and still make choices of resolutions and lift to get isomorphic $\mathrm{DG}(\beta)$? The answer to this question is: "yes."

Proposition 2. Suppose that (N, Δ, β) is a stacky polytope with N a free $\mathbb{Z}$-module of rank d. Let $(A, v) \in \mathrm{AGL}(d; \mathbb{Z})$. Denote by (N, Δ', β') the stacky polytope obtained by applying (A, v) to Δ and β; i.e., $\Delta' = A(\Delta) + v$ and $\beta' = \mathcal{A} \circ \beta$, where $\mathcal{A} = (A^{-1})^*$ and the $(\cdot)^*$ indicates the adjoint map of dual spaces). Then $\mathcal{X}(N, \Delta, \beta) \cong \mathcal{X}(N, \Delta', \beta')$.

Remark 4.1. We remind the reader that the map $\beta : \mathbb{Z}^n \to N$ is derived from the facet normals of Δ and thus β is independant of any translation of Δ. Thus, we will not again mention the vector v in the remainder of the proof of this Proposition.

The algebraic proof: We first note that $A, \mathcal{A} \in \mathrm{Aut}(\mathbb{Z}^d)$. Via a choice of isomorphism $N \cong \mathbb{Z}^d$, we shall abuse notation and consider $A, \mathcal{A} \in \mathrm{Aut}(N)$. We may then consider $\beta, \beta' : \mathbb{Z}^n \to N$.

We now will show that $\mathrm{DG}(\beta) \cong \mathrm{DG}(\beta')$. Recall that we are allowed to chose the resolutions, so we take projective resolutions of $\mathbb{Z}^n$ given by

$$\mathbf{E}, \mathbf{E}' : 0 \to \mathbb{Z}^n \xrightarrow{id} \mathbb{Z}^n \to 0$$

and projective resolutions of $N \cong \mathbb{Z}^d$

$$\mathbf{F} : 0 \to \mathbb{Z}^d \xrightarrow{id} \mathbb{Z}^d \to 0$$

and

$$\mathbf{F}' : 0 \to \mathbb{Z}^d \xrightarrow{\mathcal{A}} \mathbb{Z}^d \to 0.$$

Then, $\beta, \beta' : \mathbb{Z}^n \to N$ have lifts $\mathbf{B}, \mathbf{B}' : \mathbf{E} \to \mathbf{F}$. Denoting by $B, B' : \mathbb{Z}^n \to \mathbb{Z}^d$ the maps corresponding to $\beta, \beta' : \mathbb{Z}^n \to N$, respectively, we can take the following lifts,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{id} & \mathbb{Z}^n & \longrightarrow & 0 \\ & & {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle B} & & \\ 0 & \longrightarrow & \mathbb{Z}^d & \xrightarrow{id} & N & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{id} & \mathbb{Z}^n & \longrightarrow & 0 \\ & & {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle B'} & & \\ 0 & \longrightarrow & \mathbb{Z}^d & \xrightarrow{\mathcal{A}} & N & \longrightarrow & 0. \end{array}$$

Clearly, $\mathcal{A} \in \mathrm{Aut}(\mathbb{Z}^d)$ implies that the corresponding mapping cones of both resolutions have the same cohomology. Thus, $\mathrm{DG}(\beta) \cong \mathrm{DG}(\beta')$.

It's obvious that there is a one-to-one correspondence of cones in the normal fans to Δ and Δ' (as $A \in \mathrm{GL}(d;\mathbb{Z})$, it is non-singular). Thus, in the algebraic construction given in BCS, we have $Z = Z'$.

Finally, we show that the groups $G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$ and $G' := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta'), \mathbb{C}^*)$ have the same action on Z. Recall that the action is determined by a homomorphism $G \to (\mathbb{C}^*)^n$, composed with the natural action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$. Moreover, this homomorphism is the homomorphism induced by applying $\mathrm{Hom}(\cdot, \mathbb{C}^*)$ to the map $\beta^\vee : (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$. Consider the portions of the long exact sequences in cohomology that arise from the short exact sequence of cochain complexes,

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\mathrm{Cone}(\beta)^*) \to \mathrm{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \to 0$$

$$N^* \xrightarrow{(\beta')^*} (\mathbb{Z}^n)^* \xrightarrow{(\beta')^\vee} H^1(\mathrm{Cone}(\beta')^*) \to \mathrm{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \to 0.$$

Now, note that $\mathrm{im}((\beta')^*) = \mathrm{im}((\mathcal{A} \circ \beta)^*) = \mathrm{im}(\beta^* \circ A^{-1}) = \mathrm{im}(\beta^*)$, as $A \in \mathrm{GL}(d;\mathbb{Z})$. Thus, $\beta^\vee = (\beta')^\vee$ and G and G' have the same action on Z. □

We remark that the construction here works more generally. Namely, if we have chosen resolutions $\mathbf{E}$ and $\mathbf{F}$ and a lift $\mathbf{B}$ of β, we can always take the same resolutions $\mathbf{E}' = \mathbf{E}$ and $\mathbf{F}' = \mathbf{F}$, except that the boundary map $F_1' \to F_0'$ differs from the boundary map $F_1 \to F_0$ by composition with $\mathcal{A}$. Finally, choose the lift $\mathbf{B}' : \mathbf{E} \to \mathbf{F}$ by $B_i' = B_i$ for $i > 0$ and $B_0' = \beta'$. With these choices, it's obvious that $\mathrm{DG}(\beta) \cong \mathrm{DG}(\beta')$. Moreover, the invertibility of A (and consequently $\mathcal{A}$) is precisely what we needed for these resolutions to have the same cohomology (as $\ker(\beta) = \ker(\beta')$, "dead on").

The symplectic proof: Suppose that Δ is a polytope in $\mathbb{R}^d$ with primitive inward normals $y_1, \ldots, y_n$. Let $A \in \mathrm{GL}(d; \mathbb{Z})$ and let $\Delta' = A(\Delta)$. Note that Δ' has primitve inward normals $y_1', \ldots, y_n'$, where $y_i' = (A^{-1})^* y_i$, for $1 \leq i \leq n$.

Now, as in the BCS construction, let $B, B' : \mathbb{Z}^n \to \mathbb{Z}^{d+r}$ be lifts of β and β', respectively. Furthermore, let $[B]$ (resp, $[B']$) be the matrix of β (resp, β') in the standard bases. Finally, let $[\mathcal{A}]$ be the matrix of $(A^{-1})^*$. Then $[B'] = [\mathcal{A}][B]$. As, $[\mathcal{A}]$ is an invertible matrix of determinant 1, note that $[B]$ and $[B']$ have the same Smith normal form. Thus, $[BQ]$ and $[B'Q]$ also have the same Smith normal form and $\mathrm{DG}(\beta) = \mathrm{DG}(\beta')$. Consequently, the same group, G, is associated to both (N, Δ, β) and (N, Δ', β') .

Next, suppose that $v \in \ker(\beta)$. Then, as $\beta'(v) = (A^{-1})^*(\beta(v)) = 0$, we have that $v \in \ker(\beta')$; i.e., $\ker(\beta') \subset \ker(\beta)$. Now, suppose that $v \in \ker(\beta')$, that is $0 = \beta'(v) = (A^{-1})^* \beta(v)$. Then $(A^{-1})^*$ invertible implies that $\beta(v) = 0$ and $\ker(\beta') \subset \ker(\beta)$. Thus, $\ker(\beta) = \ker(\beta)$. As the weights of the action of G are given by generators for $\ker(\beta)$, we have demonstrated that the group G acts with the same weights in both associated quotient stacks.

Finally, in the symplectic formulation of the the association, the weights associated with the action of G give the moment map. As Z is the preimage of a regular value of the induced moment map (see Sakai [22]), we have that both stacky objects have the same associated compact manifold, Z. Thus, as both stacky objects have the same level set, group and group action, we have that they both have the same associated quotient stack, $[Z/G]$. $\square$

A few brief comments are probably in order. The first is to note that this result is not counter to Delzant's theorem. The well-known theorem of Delzant (and later generalization by Lerman-Tolman) assert a one-to-one correspondence between polytopes (up to translation) and toric symplectic objects (manifolds or orbifolds). We remind the reader that this correspondence pertains to polytopes in a particular Lie (sub-)algebra. The $\mathbb{R}^d \cong N \otimes \mathbb{R}$ in the proposition above is only (non-canonically) diffeomorphic to such a Lie algebra. The ambiguity of $\mathrm{GL}(d;\mathbb{Z})$ that we have shown above results from different choices of identification. The second comment pertains to the choice of $\mathrm{GL}(d;\mathbb{Z})$ vs. $\mathrm{SL}(d;\mathbb{Z})$. The proof that we have given above was for $A \in \mathrm{GL}(d;\mathbb{Z})$. However, if one is concerned with the symplectic formulation of the correspondence, then acting on Δ by a matrix of determinant minus one results in a change of sign for the resulting symplectic form. Third, note that translation of the polytope does not change the facet normals and, thus, doesn't change the map β. Therefore, the invariance that we have shown is actually invariance under an action of $AGL(d;\mathbb{Z})$.

4.5 Explicit form of Z

In this section we will give an explicit description of the manifold Z on which the group G acts. In the symplectic setting, we will show that Z is compact. In fact, if $\dim(G) = 1$ and the moment map μ is proper, then Z is a compact ellipsoid. We will make these assertions precise below (Section 4.5.1). Following the symplectic development will give a description of Z in the algebraic formulation (Section 4.5.2), in which Z is a quasi-affine variety.

We first introduce some notation for what follows and establish some general facts. Let (N, Σ, β) be a polytopal stacky fan and suppose that Δ is the corresponding polytope (i.e., suppose that the primitive rays of Σ, say $\rho_1, \ldots, \rho_n$, are the (inward) primitive normals to the facets of the simple rational convex polytope Δ). Then Δ is a rational simple n-simplex in $(\mathbb{R}^d)^* \cong (N \otimes_{\mathbb{Z}} \mathbb{R})^*$ (the polytope is, in fact, an n-gon as the fan is assumed to be simplicial). Furthermore, $\beta : \mathbb{Z}^n \to N$ has finite cokernel (i.e., has rank d). If we denote

$\hat{\rho}_j := \rho_j \otimes_{\mathbb{Z}} 1$ to be the image of x under the natural map $N \to N \otimes \mathbb{R}$, then $\beta(e_j) = c_j \hat{\rho}_j$, where c_j is a positive integer.

We also let G be the compact Abelian Lie group that is associated to (N, Σ, β) using the BCS construction and let $\rho : G \to \mathbb{T}^n$. Fix isomorphisms $\mathfrak{g} \cong \mathbb{R}^d$ and $\mathrm{T}_e\mathbb{T}^n \cong \mathbb{R}^n$, and take the standard bases for $\mathbb{R}^d$ and $\mathbb{R}^n$.

4.5.1 Description of Z in the symplectic setting

Let $Z := \mu^{-1}(c)$, where c is a regular value for the moment map for the induced action of G on $\mathbb{C}^n$; i.e., $\mu : \mathbb{C}^n \to \mathfrak{g}^*$.

Theorem 4.1. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the corresponding quotient stack (i.e., $[Z/G] = \mathcal{X}(N, \Sigma, \beta)$). Suppose that the moment map for this action, $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, is proper and that τ is a regular value of μ. Then $Z = \mu^{-1}(\tau)$ is a compact manifold.

Proof. Recall that $\mu : \mathbb{C}^n \to \mathfrak{g}^*$ is given by

$$\mu \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sum_{j=1}^{n} |z_j|^2 w^j,$$

where $w^j := (D\rho)^*(e^j)$. Thus, if $\rho : G \to \mathbb{T}^n$ is smooth, then μ is smooth and any point in the positive orthant of $\mathbb{R}^d$ is a regular value of μ. Furthermore, recall the exact sequence

$$0 \to N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} \mathrm{DG}(\beta) \longrightarrow \mathrm{Tor}(N) \to 0,$$

where $\mathrm{rank}(\mathrm{im}(\beta)) = d$ and $\mathrm{rank}(\mathrm{Tor}(N)) = 0$. Exactness then implies $\mathrm{rank}(\mathrm{DG}(\beta)) = (n - d)$; i.e., that $\dim(G) = n - d$. It follows from (what is usually called) the *regular value theorem* of differential topology that Z is smooth submanifold of $\mathbb{C}^n$ whose codimension is $n - d$ (i.e., $\dim(Z) = d$). Because μ is proper, Z is compact.

□

Corollary 1. When G has dimension 1, then Z is a codimension one ellipsoid in $\mathbb{R}^n$.

4.5.2 Description of Z in the algebraic setting

We now turn our attention to developing the description of Z as a psuedo-affine variety in $\mathbb{C}^n$. Let Σ be a rational simplicial fan with rays $\rho_1, \ldots, \rho_n$. Recall that in Section 4.3 we already defined Z to be a quasi-affine variety in $\mathbb{C}^n$. We will now provide further details and introduce some notation that will be useful for what follows.

Just as we did in Chapter 2, we define a monomial ideal,

$$J(\Sigma) := \langle \prod_{\rho_i \not\subset \sigma} z_i \mid \sigma \in \Sigma \rangle \subset \mathbb{C}[z_1, \ldots, z_n]$$

and $\mathbb{V}(J(\Sigma))$ is the algebraic set associated to this ideal (i.e., the locus of points on which the polynomials in $J(\Sigma))$ vanish).

For $\sigma \in \Sigma$, let $I_\sigma = \{i \mid \rho_i \subset \sigma\}$; that is, I_σ are the indices of rays in the cone σ. It is convenient to let $J_\sigma = \{1, \ldots, n\} \setminus I_\sigma$; i.e., the complement of I_σ. Finally, for $z \in \mathbb{C}^n$, let $I_z = \{i \mid z_i = 0\}$. Then

$$\begin{aligned} V(J(\Sigma)) &= \bigcap_{\sigma \subset \Sigma} \{(z_1, \ldots, z_n) \mid \prod_{\rho_i \not\subset \sigma} z_i = 0\} \\ &= \bigcap_{\sigma \subset \Sigma} \{(z_1, \ldots, z_n) \mid \prod_{i \in J_\sigma} z_i = 0\}. \end{aligned}$$

This allows us to write Z explicitly, in terms of the fan, as

$$\begin{aligned} Z = \mathbb{C}^n \setminus V(J(\Sigma)) &= \bigcup_{\sigma \subset \Sigma} \{(z_1, \ldots, z_n) \mid z_i \neq 0 \text{ whenever } i \in J_\sigma\} \\ &= \{(z_1, \ldots, z_n) \mid I_z \subset I_\sigma \text{ for some } \sigma \in \Sigma\}. \end{aligned} \tag{4.5}$$

Chapter 5: Results

In this chapter we will describe the quotient stack $[Z/G]$ that arises under the BCS construction from a stacky fan. We begin by discussing the properties of the group G that we can deduce from the stacky object. In the next section of this chapter we give a description of the action of G on Z that is encoded in a stacky object. We devote the final section to the special case of $\dim(G) = 1$, as this case includes the much studied weighted projective spaces and fake weighted projective spaces.

5.1 The group G

Recall that in the previous chapter we discussed the space on which the Lie group G acts; we now turn our attention to the group itself. In this section we will characterize the group in terms of the combinatorics of the stacky object. As the theory of stacky fans subsumes that of stacky polytopes (except for scale information) we will generally use this formulation in stating results.

We begin with an obvious remark. Recall that $G = \mathrm{Hom}(\mathrm{DG}(\beta), S^1)$, thus G is connected if and only if $\mathrm{Tor}(\mathrm{DG}(\beta)) = \{0\}$. Though this remark has the benefit of being easy to state and prove and characterizes a fundamental property of the Lie group G, it is not of much use, in practice. What we would like is a characterization of G in terms of the combinatorics of the stacky fan, (N, Σ, β).

A first step is to recall that we can write the invariant factor form of a quotient module, $\mathrm{DG}(\beta)$ by determining the Smith normal form of $[BQ]^*$ (i.e., the non-trivial diagonal terms in the Smith form of $[BQ]^*$ give the invariant factors of the quotient module and the number of zeros on the diagonal gives the rank). In light of this fact, our preceding remark becomes:

G is connected if and only if the Smith normal form of $[BQ]^*$ contains only zeros and ones. There is an obvious corollary to this remark:

Observation 5.1. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the corresponding quotient stack. If

$$\mathrm{SNF}([BQ]^*) = \begin{pmatrix} I_{d+r} & 0 \\ 0 & 0 \end{pmatrix}$$

then $G \cong \mathbb{T}^k$ (here, $\mathrm{SNF}([BQ]^*)$ denotes the Smith normal form of $[BQ]^*$).

We now state our main theorem for this section. This theorem gives a complete characterization of G, in terms of the combinatorics of the stacky fan (N, Σ, β).

Theorem 5.2. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the associated quotient stack. Let G_0 be the identity component of G, then $G/G_0 \cong N/\mathrm{im}(\beta)$.

Proof. As $G \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^\times)$ it is sufficient to prove that $\mathrm{Tor}(\mathrm{DG}(\beta)) \cong N/\mathrm{im}(\beta)$. We show this in two steps:

1. $\mathrm{Tor}(\mathrm{DG}(\beta)) \cong (\mathbb{Z}^d \oplus \mathbb{Z}^r)/\mathrm{im}(\beta)$
2. $(\mathbb{Z}^d \oplus \mathbb{Z}^r)/\mathrm{im}(\beta) \cong N/\mathrm{im}([BQ])$

We begin by showing 1, above. We compute $\mathrm{Tor}(\mathrm{DG}(\beta))$ by computing $\mathrm{Ext}^1(\mathrm{DG}(\beta), \mathbb{Z})$, as these are (naturally) isomorphic. Thus, we will choose a projective resolution of $\mathrm{DG}(\beta)$, apply $\mathrm{Hom}(\cdot, \mathbb{Z})$ to the resolution, and then calculate the resulting long exact sequence in homology of the resulting sequence to determine $\mathrm{H}_1(\cdot)$. Recalling that $[BQ]^*$ is injective, consider the following projective resolution of $\mathrm{DG}(\beta)$:

$$0 \to (\mathbb{Z}^{d+r})^* \overset{[BQ]^*}{\to} (\mathbb{Z}^{n+r})^* \overset{\pi_{\mathrm{DG}(\beta)}}{\to} \mathrm{DG}(\beta) \to 0.$$

Applying $\mathrm{Hom}(\cdot, \mathbb{Z})$, we have

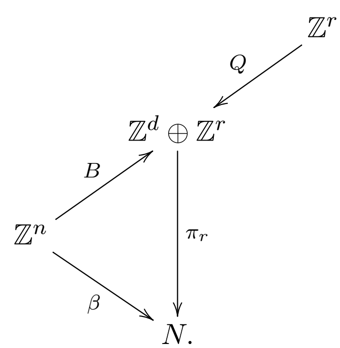
$$0 \leftarrow \mathbb{Z}^{d+r} \leftarrow \mathbb{Z}^{n+r} \rightarrow \mathrm{Hom}(\mathrm{DG}(\beta), \mathbb{Z}) \leftarrow 0.$$

It then follows from the long exact sequence in homology that $\mathrm{Ext}^1(\mathrm{DG}(\beta), \mathbb{Z}) \cong \mathbb{Z}^{d+r}/\mathrm{im}([BQ])$, as desired.

We now show 2, from above. We will proceed by showing that

$$\ker \begin{pmatrix} \mathbb{Z}^d \oplus \mathbb{Z}^r \\ \downarrow \pi_r \\ N \cong Z^d \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_r\mathbb{Z} \\ \downarrow \\ N/\mathrm{im}(\beta) \end{pmatrix} \cong \mathrm{im}([BQ]).$$

It then follows that $(\mathbb{Z}^d \oplus \mathbb{Z}^r)/\mathrm{im}([BQ]) \cong N/\mathrm{im}(\beta)$. Consider the following diagram



(5.1)

Referring to 5.1 above, take $\alpha \in \mathbb{Z}^d \oplus \mathbb{Z}^r$ such that $\pi_r(\alpha) \in \mathrm{im}(\beta)$. Let $x \in \mathbb{Z}^n$ be such that $\pi_r(\alpha) = \beta(x)$. Then $\pi_r(B(x)) = \pi_r(\alpha) \implies B(x) - \alpha \in \ker(\pi_r) = \mathrm{im}(Q)$. Thus, it must be that $\alpha = B(x) + Q(y)$ for some $y \in \mathbb{Z}^r$. So, this implies $\ker(\mathbb{Z}^d \oplus \mathbb{Z}^r \to N \to N/im(\beta)) \subset \mathrm{im}([BQ])$. The converse (i.e., $\mathrm{im}([BQ]) \subset \ker(\mathbb{Z}^d \oplus \mathbb{Z}^r \to N \to N/\mathrm{im}(\beta)))$ is obvious. Thus, we have equality and have established 2. □

An obvious, but useful, corollary is the following:

Corollary 2. Let (N, Σ, β) be a stacky fan and $[Z/G]$ the corresponding quotient stack. Then G is connected if and only if $\beta : \mathbb{Z}^n \to N$ is surjective.

Finally, we note:

Corollary 3. If $\beta : \mathbb{Z}^n \to N$ is surjective, then $G \cong \mathbb{T}^k$.

Remark 5.1. We showed in Theorem 5.2 that $G/G_0 \cong \text{coker}(\beta)$. Now, applying the Snake Lemma to the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^{n+r} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle [BQ]} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^{d+r} & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

shows that $\text{coker}(\beta) \cong \text{coker}([BQ])$. Hence, we can determine the invariant factor decomposition of the component group of G simply by computing the Smith normal form of $[BQ]$.

Given a stacky fan (N, Σ, β) we can construct the quotient stack $[Z/G]$. If, instead, we are interested in quotients by connected groups, we show now that we can construct a related stacky fan (N_0, Σ_0, β_0) whose associated quotient stack is $[Z/G_0]$, where G_0 is the connected component of the identity of G.

We begin by letting $N_0 := \text{im}(\beta)$. Thus, N_0 is a submodule of N and is obviously finitely generated (for example, by the images of any basis for $\mathbb{Z}^n$). Note now that the fact that β has finite cokernel implies that $N_0 \otimes \mathbb{R}$ is isomorphic to $N \otimes \mathbb{R}$. Thus, we take Σ_0 to be the fan in $N_0 \otimes \mathbb{R}$ corresponding to Σ under this isomorphism. Finally, let $\beta_0 : \mathbb{Z}^n \to N_0$ be the map given by β, with restricted codomain.

Proposition 3. Let (N, Σ, β) be a stacky fan with $\mathcal{X}(N, \Sigma, \beta) = [Z/G]$ its associated quotient stack. If (N_0, Σ_0, β_0) is the stacky fan constructed above, then $\mathcal{X}(N_0, \Sigma_0, \beta_0) = [Z/G_0]$.

Proof. We have already remarked that N_0 is finitely generated, so it's immediate that (N_0, Σ_0, β_0) defines a stacky fan. Moreover, as Σ and Σ_0 contain identical combinatorial information, $Z_\Sigma = Z_{\Sigma_0}$.

What remains is to show that, under the quotient stack construction in [3], the group action determined by $\mathcal{X}(N_0, \Sigma_0, \beta_0)$ acts in the same manner as the identity component of G (where we are tacitly identifying Z_Σ and Z_{Σ_0} in light of the preceding remark). We will proceed by applying Lemma 2.3 from [3], which relates a commutative diagram of $\mathbb{Z}$-modules to an induced diagram of dual cones. Consider the following diagram of short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{=} & \mathbb{Z}^n & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & \operatorname{coker}(\beta) & \longrightarrow & 0.
\end{array}$$

Using the obvious and natural identification of $\mathrm{DG}(\{0\} \to \operatorname{coker}(\beta))$ with $\operatorname{coker}(\beta)$, the lemma gives us the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & (\mathbb{Z}^n)^* & \xrightarrow{=} & (\mathbb{Z}^n)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta^\vee} & & \downarrow{\scriptstyle (\beta_0)^\vee} & & \\
0 & \longrightarrow & \operatorname{coker}(\beta) & \longrightarrow & \mathrm{DG}(\beta) & \longrightarrow & \mathrm{DG}(\beta_0) & \longrightarrow & 0.
\end{array}$$

Thus, we see that $\mathrm{DG}(\beta_0)$ and $\mathrm{DG}(\beta)$ have the same rank. This implies, then, that $\mathrm{Hom}(\mathrm{DG}(\beta_0), \mathbb{C}^\times)$ has the same dimension as G. Also, by definition of β_0 and Corollary 2, it follows that the group $\mathrm{Hom}(\mathrm{DG}(\beta_0), \mathbb{C}^\times)$ is connected. Thus, $\mathrm{Hom}(\mathrm{DG}(\beta_0), \mathbb{C}^\times) = G_0$. Moreover, the action on Z_Σ is induced by applying $\mathrm{Hom}(\cdot)$ to the composition above,

$(\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} \mathrm{DG}(\beta) \to \mathrm{DG}(\beta_0)$. Thus, the action of G_0 is by its inclusion into G. □

5.2 The action $G \curvearrowright Z$

The results of this section were obtained jointly with Rebecca Goldin, Megumi Harada, and Derek Krepski while collaborating at McMaster Univeristy in March 2012. Again, we will use the language of stacky fans for our results in this section, as it is both more convenient and general than that of stacky polytopes.

Recall that in (4.5) we were able to describe the quasi-affine variety on which G acts in terms of the cones of the fan. Namely, we wrote

$$Z = \{(z_1, \ldots, z_n) \mid I_z \subset I_\sigma \text{ for some } \sigma \in \Sigma\}.$$

Because Σ is simplicial, we can outline a construction that allows us to associate with any point $z \in Z$ an associated minimal cone $\sigma_z \subset \Sigma$. We remind the reader of some definitions that we introduced in Section 4.5.2. For $\sigma \in \Sigma$, let $I_\sigma = \{i \mid \rho_i \subset \sigma\}$; that is, I_σ are the indices of rays in the cone σ and let $J_\sigma = \{1, \ldots, n\} \setminus I_\sigma$; i.e., the complement of I_σ. Recall also that for $z \in \mathbb{C}^n$, we let $I_z = \{i \mid z_i = 0\}$. Now introduce the notation $Z_\sigma = \{(z_1, \ldots, z_n) \mid I_z \subset I_\sigma\}$.

There is a decomposition of Z, as inclusion of cones, $\sigma' \subset \sigma$ induces an inclusion $Z_{\sigma'} \subset Z_\sigma$. Also, because Σ is assumed to be a simplicial fan, for any $(z_1, \ldots, z_n) \in Z_\sigma$ we can specify a cone $\sigma_z \subset \sigma$ given by the (non-negative) span of the rays $\{\rho_i\}$, with $i \in I_z$. Again, because the fan Σ is simplicial, the number of rays ρ_i with $i \in I_\sigma$ is equal to the dimension of σ. Thus, any subset of these rays spans a face of σ and is therefore in the fan; i.e., $\sigma_z \subset \Sigma$. We conclude, for any point $z \in Z$, we can write $z \in Z_{\sigma_z}$, where the cone σ_z satisfies $I_{\sigma_z} = I_z$. Moreover, $\sigma_z \subset \sigma'$ for any σ' such that $z \in Z_{\sigma'}$.

Theorem 5.3. Let (N, Σ, β) be a stacky fan and let $[Z/G]$ be the associated quotient stack. For $z \in Z$ let σ be the minimal cone such that $I_z \subset I_\sigma$. The stabilizer of z is then isomorphic to the torsion subgroup $\mathrm{Tor}(N/N_\sigma)$, where $N_\sigma \subset N$ is the sub-module spanned by $\{\beta(e_i) \mid i \in I_\sigma\}$.

Proof. Note that under the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$ the stabilizer of a point $z \in \mathbb{C}^n$ in $\mathbb{T}^n$, $\mathrm{stab}(z)$, is given by $\mathrm{stab}(z) = \{(t_1, \ldots, t_n) \mid t_i = 1 \text{ if } z_i \neq 0\}$. We thus make the following definition: for any subset $I \subset \{1, \ldots, n\}$ and its complement J,

$$\mathbb{T}^I = \{t = (t_1, \ldots, t_n) \mid i \in J \text{ implies } t_i = 1\} \subseteq \mathbb{T}^n.$$

Then $\mathrm{stab}(z) = \mathbb{T}^{I_z}$. Note that $\mathbb{T}^I$ is the kernel of the map $\mathbb{T}^n \to \mathbb{T}^{|J|}$ given by projection onto the coordinates indicated by J with cardinality $|J|$.

Recall that in the construction we are using, G acts on Z via a homomorphism to $\mathbb{T}^n$ (composed with the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$, restricted to Z). Thus, the isotropy associated to a point z is given by the kernel of the map

$$G \longrightarrow \mathbb{T}^n \xrightarrow{\pi} \mathbb{T}^{|J_z|}$$

where J_z is the complement of I_z.

Again recalling the notation introduced in the discussion above, note that the stabilizer of each point $z \in Z_{\sigma_z}$ depends only on the cone σ_z, since $I_{\sigma_z} = I_z$. With this in mind, we let Γ_σ denote the kernel of the composition

$$G \longrightarrow \mathbb{T}^n \to \mathbb{T}^{|J_\sigma|}, \tag{5.2}$$

which is the stabilizer of each point z in Z_σ with $\sigma = \sigma(z)$.

The kernel of the composition (5.2) arises by applying $\mathrm{Hom}(\cdot, \mathbb{C}^\times)$ to the composition

$$(\mathbb{Z}^{|J_\sigma|})^* \xrightarrow{\pi^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} \mathrm{DG}(\beta),$$

where π^* is inclusion of the relevant factors. We denote this composition by $f := \beta^\vee \circ \pi^*$.

Now, recalling that $\mathbb{C}^\times$ is injective as a $\mathbb{Z}$-module, we note that the kernel of (5.2) is $\mathrm{Hom}(\mathrm{coker}(f), \mathbb{C}^\times)$. We will show below that $\mathrm{coker}(f)$ is finite, and thus $\mathrm{Hom}(\mathrm{coker}(f), \mathbb{C}^\vee)$ and $\mathrm{coker}(f)$ are isomorphic.

We now let $N_\sigma \subset N$ denote the subgroup spanned by the elements $\beta(\epsilon_i)$ where $i \in I_\sigma$, and let $\beta_\sigma : \mathbb{Z}^{I_\sigma} \to N_\sigma$ denote the restriction of β to $\mathbb{Z}^{I_\sigma}$ together with its codomain.

We claim that $\beta_\sigma : \mathbb{Z}^{I_\sigma} \to N_\sigma$ is an isomorphism, and thus that N_σ is free. To establish this fact, recall that (N, Σ, β) a stacky fan implies that the fan Σ is simplicial and that $\beta(e_i) \otimes 1$ lies on the rays $\rho_i \subset \sigma$. In particular, Σ simplicial implies that the $\{\beta(e_i) \otimes 1\}_{i \in I_\sigma}$ are linearly independent in $N \otimes \mathbb{R}$. Thus, $\mathrm{rank}(N_\sigma) = |I_\sigma|$ and we see that β_σ is a surjective homomorphism of modules of the same rank. Noting that the domain of β_σ, $\mathbb{Z}^{I_\sigma}$, is free, this map must be injective as well. Thus, $N_\sigma \cong \mathbb{Z}^{I_\sigma}$.

Next, we consider $\mathrm{DG}(\beta_\sigma)$. As $\beta_\sigma : \mathbb{Z}^{I_\sigma} \to N_\sigma$ is an isomorphism, so is any lift, B_σ. Also, N_σ has no torsion, so $\mathrm{DG}(\beta_\sigma) \cong (\mathbb{Z}^{I_\sigma})^*/\mathrm{im}[B_\sigma]^* \cong 0$; i.e., $\mathrm{DG}(\beta_\sigma)$ is trivial.

Consider the following diagram, whose rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{|I_\sigma|} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{|J_\sigma|} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_\sigma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_J} & & \\
0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N/N_\sigma & \longrightarrow & 0
\end{array}$$

By Lemma 2.3 in [3], we get the following commutative diagram with exact rows, using the fact that $\mathrm{DG}(\beta_\sigma)$ is trivial.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}^{|J_\sigma|})^* & \xrightarrow{\pi^*} & (\mathbb{Z}^n)^* & \longrightarrow & (\mathbb{Z}^{|I_\sigma|})^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta^\vee} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{DG}(\beta_J) & \xrightarrow{\cong} & \mathrm{DG}(\beta) & \longrightarrow & 0 & &
\end{array}$$

We identify $f = \beta^\vee \circ \pi^*$ with the left vertical arrow.

Deriving the long exact sequence in cohomology from the exact sequence of mapping cones as in (2.3) from [3] to $\beta_J : \mathbb{Z}^{|J_\sigma|} \to N/N_\sigma$, gives

$$(N/N_\sigma)^* \to (\mathbb{Z}^{|J_\sigma|})^* \to \mathrm{DG}(\beta_J) \to \mathrm{Ext}^1(N/N_\sigma, \mathbb{Z}) \to 0$$

and thus $\mathrm{coker}(f) \cong \mathrm{Ext}^1(N/N_\sigma, \mathbb{Z}) \cong \mathrm{Tor}(N/N_\sigma)$. □

It is convenient at this point to introduce notation for these isotropy groups. Let $z \in Z$ and let σ be the minimal cone such that $I_z \subset I_\sigma$. We will denote the stabilizer of z in G by Γ_σ.

Neither the statement nor proof of Theorem 5.3 indicate how an isotropy group can be realized as a subgroup of G. However, in [11], Krepski was able to give a non-canonical (depending on choices of projective resolutions) realization of $\mathrm{Tor}(N/N_\sigma) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$. The interested reader is directed to Remark 3.4 and Proposition 3.16 in [11].

It is useful to note that Theorem 5.3 allows us to compute the isotropy groups in a quite straightforward manner, using the Smith normal form of a matrix. Recall that we showed in the proof of Theorem 5.3, that the isotropy group Γ_σ is isomorphic to the cokernel of the composition $\mathbb{Z}^{|I_\sigma|} \to \mathbb{Z}^n \xrightarrow{\beta} N$. If we choose the projective resolution of N given by $0 \to \mathbb{Z}^l \xrightarrow{Q} \mathbb{Z}^{d+l} \to N$ and a lift of β denoted by $B : \mathbb{Z}^n \to \mathbb{Z}^{d+l}$ and we let B_σ denote the restriction of B to $\mathbb{Z}^{|I_\sigma|}$, then $\mathrm{coker}([B_\sigma Q])$ and $\mathrm{coker}(\beta)$ are isomorphic. It follows then, that the non-trivial entries on the diagonal of the Smith normal form of $[B_\sigma Q]$ (i.e. those entries that are neither 0 nor 1) give the invariant factors of the isotropy group.

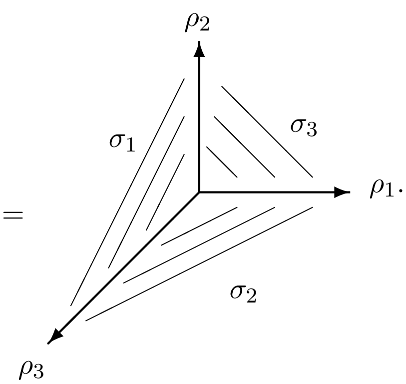
Example 18. We revisit our earlier example (see Example 4) and use Theorem 5.3 (and the remarks immediately preceding this example) to compute the isotropy groups of the corresponding quotient stack. Recall that we are considering the stacky fan (N, Σ, β) where

$N = \mathbb{Z}^2 \oplus \mathbb{Z}/7\mathbb{Z}$ and

$$[BQ] = \begin{pmatrix} -2 & 3 & 0 & 0 \\ -2 & 0 & 5 & 0 \\ 1 & 0 & 0 & 7 \end{pmatrix}$$

and with fan (the rays and cones of which we've labeled in the figure below)

$\Sigma =$



Then we compute the isotropy groups, using Theorem 5.3. We begin with Γ_{σ_1}:

$$[B_{\sigma_1}Q] = \begin{pmatrix} -2 & 0 & 0 \\ -2 & 5 & 0 \\ 1 & 0 & 7 \end{pmatrix}$$

the adjoint of which has Smith normal form

$$\mathrm{SNF}([B_{\sigma_1}Q]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 70 \end{pmatrix}$$

Thus, applying our remark from above, we find

$$\begin{aligned}\Gamma_{\sigma_1} &\cong \mathbb{Z}^3/\mathrm{SNF}([B_{\sigma_1}Q]) \\ &\cong \mathbb{Z}/70\mathbb{Z}.\end{aligned}$$

Similarly, we find

$$\mathrm{SNF}([B_{\sigma_2}Q]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 42 \end{pmatrix}$$

and

$$\mathrm{SNF}([B_{\sigma_3}Q]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 105 \end{pmatrix}.$$

Thus, we see that

$$\Gamma_{\sigma_2} \cong \mathbb{Z}/42\mathbb{Z} \text{ and } \Gamma_{\sigma_3} \cong \mathbb{Z}/105\mathbb{Z}.$$

5.3 When $\dim(G) = 1$

We now turn our attention to the special case that $\dim(G) = 1$ (i.e., $\mathrm{rank}(\mathrm{DG}(\beta)) = 1$). This is the case that gives quotient stacks corresponding to weighted projective spaces and the so-called fake weighted projective spaces. In fact, we will show in Section 5.3.1 that in the case that the group G arising from a staky fan is connected and 1-dimensional (i.e.,

$G \cong S^1$) then $[Z/G]$ is a weighted projective space. We will show in Section 5.3.2 that if G is 1-dimensional, but not connected, the resulting quotient stack is a fake weighted projective space. Finally, in Sections 5.3.3 and 5.3.4, we will again be drawn to the intuitive appeal of polytopes. Specializing to a class of polytopes that we have called *labeled sheared simplices* in 2-dimensions, we will give an explicit description of G in terms of the primitive normal to the non-coordinate facet and the facet labels (involving their greatest common divisor).

5.3.1 When G connected — Weighted Projective Spaces

We recall the definition of weighted projective space.

Definition 5.1. Let $b_0, \dots, b_n$ be positive integers and let S^1 act on $S^{2n+1} \subset \mathbb{C}^{n+1}$ via $g \cdot (z_0, \dots, z_n) \mapsto (g^{b_0} z_0, \dots, g^{b_n} z_n)$. The quotient stack $[S^{2n+1}/S^1]$ is a **weighted projective space**, denoted $\mathbb{P}(b_0, \dots, b_n)$. The vector $(b_0, \dots, b_n)$ is called the **weight vector** and we refer to the entries as **weights**. Alternatively, one can define $\mathbb{P}(b_0, \dots, b_n)$ to be the quotient stack $[(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times]$ acting as above.

Note that we are defying convention in that we do not require that the weights to be relatively prime; i.e., the action $S^1 \curvearrowright S^{2n+1}$ may not be effective. We choose this definition because the restriction to effective actions is unnatural in the context of the work that we have undertaken.

Proposition 4. Let (N, Δ, β) be a stacky polytope, and let $\Sigma(\Delta)$ be the dual fan to Δ. The associated toric DM stack $\mathcal{X}(N, \Sigma(\Delta), \beta)$ is a weighted projective space $\mathbb{P}(b_0, \dots, b_d)$ if and only if $\mathrm{DG}(\beta) \cong \mathbb{Z}$. In this case, the polytope Δ is a simplex, and the weights are determined by the condition that $(b_0, \dots, b_d)$ generates $\ker(\beta) \subset \mathbb{Z}^{d+1}$.

Before we prove Proposition 4 we need to establish a lemma, the proof of which was first shown to me by Dr. Jim Lawrence.

Lemma 1. Let Δ be a regular lattice d-dimensional polytope in $\mathbb{R}^d$ with $d+1$ facets. We denote the primitive inward normal to the ith facet by y_i. Let B be the matrix whose ith column is given by the ith primitive normal, $[B] = [y_1 \quad \dots \quad y_n]$, which we view as a map of lattices, $B : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$. Let $\ker(B) = \langle v \rangle_{\mathbb{Z}}$, where $v \in \mathbb{Z}^{d+1}$ is a primitive lattice vector. Then we can assume that $v_i \geq 0$ for $1 \leq i \leq d+1$.

Proof of Lemma 1. Recall that Δ has a unique representation as an intersection of half-spaces, $\Delta = \cap_{i=1}^{d+1} H_{i,b_i}$ where $H_{i,b_i} = \{x \in \mathbb{R}^d | \langle x, y_i \rangle \geq b_i \in \mathbb{R}\}$.

Consider the convex hull of the normals, $Q = \text{conv}(\{y_i\}_{i=1}^n) \subset \mathbb{R}^d$. Then either $0 \in Q$ or there is a hyperplane, H, containing the origin such that $H \cap Q = \emptyset$.

First, suppose that $0 \in Q$. Then there are $a_i \in \mathbb{R}$ where $a_i \geq 0$ and $\sum_{i=1}^{d+1} a_i = 1$ such that $0 = \sum_{i=1}^{d+1} a_i y_i$ (since Q is the convex hull of the $\{y_i\}$). Moreover, because 0 and the y_i are integers, we can take the $\{a_i\}$ to be rational. Now let $a \in \mathbb{R}^{d+1}$ be the vector whose ith component is a_i. Then $Ba = 0$. Let α be the least common multiple of the denominators of the $\{a_i\}$, and set $v := \alpha a$. Then $v \in \ker(B) \subset \mathbb{Z}^{d+1}$ is the primitive vector that we seek.

Now, suppose that $0 \notin Q$ and suppose that the separating hyperplane H is specified by $u \in \mathbb{R}^d$; i.e, H is given by $H = \{w \in \mathbb{R}^d | \langle w, u \rangle = 0\}$, where u is non-zero. Clearly, all the y_i lie on one side of H, as their convex hull misses H. Thus, for $1 \leq i, j \leq d+1$, $\langle u, y_i \rangle$ and $\langle u, y_j \rangle$ have the same sign. Changing u to $-u$ if necessary, we assume that all these inner products are positive. But, for any $x \in \Delta$, all points on the ray $x + cu$ (for $c \geq 0$) are contained in Δ (because they satisfy the inequalities specifying Δ). This is a contradiction, as Δ is a bounded polytope. □

Proof of Proposition 4. First, suppose that $\mathcal{X}(N, \Sigma(\Delta), \beta)$ is a weighted projective space. Then, by definition, G must be connected and have dimension 1. Thus, there is nothing to prove in this direction.

Suppose now that in the quotient stack $[Z/G] := \mathcal{X}(N, \Sigma(\Delta), \beta)$ we have that G is connected and has dimension 1. Recall that, as we are assuming that G is connected, $\beta : \mathbb{Z}^{d+1} \to N$ is surjective (see Corollary 2). Furthermore, $\text{rank}(N) = d$, which in turn implies that $\text{Tor}(N)$ is at most cyclic and Δ is a (combinatorial) simplex. Without loss of generality, we write $N \cong \mathbb{Z}^d \oplus \mathbb{Z}/g\mathbb{Z}$. For clarity of exposition, we will divide the proof into two cases: N free and $\mathrm{Tor}(N) \cong \mathbb{Z}/g\mathbb{Z}$.

Case 1: Suppose that N is free. Then $\beta : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ and, thus, must have non-trivial kernel. Let $b' = ({b'}_0, \ldots, {b'}_d) \in \ker(\beta)$. Now suppose that $\gcd(\{{b'}_i\}_{i=0}^d) = h$. Define $b_i := {b'}_i/h$, for $0 \leq i \leq d$, then $0 = \beta({b'}_0, \ldots, {b'}_d) = h\beta(b_0, \ldots, b_d)$. Because $h \neq 0$, we must have $\beta(b_0, \ldots, b_d) = 0$. Thus, $b = (b_0, \ldots, b_d)$ can be assumed to be a primitive vector (or, rather, a generator of the kernel must be primitive). Note also that because the columns of β correspond to (inward) normals of the polytope, Δ, we may assume that $b_i \geq 0$ for $0 \leq i \leq d$ (by Lemma 1).

We now construct an isomorphism $DG(\beta) \cong \mathbb{Z}$. We define $f_b : (\mathbb{Z}^{d+1})^* \to \mathbb{Z}$ to be the map $v \mapsto \langle v, b\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual pairing of $\mathbb{Z}^{d+1}$ and its dual.

We claim that $\ker(f_b) = \mathrm{im}(\beta^*)$. We start by showing that $\mathrm{im}(\beta^*) \subset \ker(f_b)$. Take $v \in (\mathbb{Z}^{d+1})^*$ such that $v = \beta^* x$, for some $x \in (\mathbb{Z}^d)^*$. Then $f_b(v) = \langle v, b\rangle = \langle \beta^* x, b\rangle = \langle x, \beta b\rangle = 0$, as $b \in \ker(\beta)$.

We now show $\ker(f_b) \subset \mathrm{im}(\beta^*)$. Let $v \in \ker(f_b) \subset (\mathbb{Z}^{d+1})^*$. Claim that we must have $v = \beta^* x$, for some $x \in (\mathbb{Z}^d)^*$. Recalling that $\beta : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ is surjective, consider the exact sequence

$$0 \to \ker(\beta) \xrightarrow{\iota} \mathbb{Z}^{d+1} \xrightarrow{\beta} \mathbb{Z}^d \to 0$$

and its dual (exact) sequence

$$(\ker(\beta))^* \xleftarrow{\iota^*} (\mathbb{Z}^{d+1})^* \xleftarrow{\beta^*} (\mathbb{Z}^d)^* \leftarrow 0 \tag{5.3}$$

Now let $u \in \ker(\beta)$ and consider $\langle \iota^* v, u\rangle$. Because b generates $\ker(\beta)$ there must exist $m \in \mathbb{Z}$ such that $u = mb$. Then we have $\langle \iota^* v, u\rangle = \langle v, \iota u\rangle = \langle v, mb\rangle = 0$, as $v \in \ker(f_b)$. However, u was arbitrary and, thus, $\iota^* v = 0$. Exactness in (5.3) implies $v \in \mathrm{im}(\beta^*)$.

Thus, we have shown that $\ker(f_b) = \mathrm{im}(\beta)$. It then follows that f_b induces a well-defined isomorphism (for which we will use the same notation), $f_b : DG(\beta) \xrightarrow{\cong} \mathbb{Z}$. Moreover, this explicit isomorphism allows us to write $\beta^\vee : (\mathbb{Z}^{d+1})^* \to DG(\beta)$ as $\beta^\vee : (\mathbb{Z}^{d+1})^* \to \mathbb{Z}$ whose explicit form (under this isomorphism) is $v \mapsto \langle v, b \rangle$.

Thus, we have the following sequence of maps

$$(\mathbb{Z}^{d+1})^* \xrightarrow{\beta^\vee} DG(\beta) \xrightarrow{f_b} \mathbb{Z}$$

and applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, S^1)$ yields

$$S^1 \xrightarrow{f_b^*} G \xrightarrow{(\beta^\vee)^*} \mathbb{T}^{d+1}.$$

If we denote the compositions $\sigma : (\mathbb{Z}^{d+1})^* \to \mathbb{Z}$ and $\tau : S^1 \to \mathbb{T}^{d+1}$ and take $v \in (\mathbb{Z}^{d+1})^*$, then $\tau(t)(v) = t(\sigma(v)) = t(\langle v, b \rangle)$. Applying this composition to the standard basis for $(\mathbb{Z}^{d+1})^*$ shows that $\tau : t \mapsto (t^{b_0}, \ldots, t^{b_d})$.

Case 2: Suppose that $N \cong \mathbb{Z}^d \oplus \mathbb{Z}/g\mathbb{Z}$. Again, recall that $\mathrm{DG}(\beta) \cong \mathbb{Z}$ implies that G is connected and thus $\beta : \mathbb{Z}^{d+1} \to N \cong \mathbb{Z}^n \oplus \mathbb{Z}/g\mathbb{Z}$ is surjective. Thus, there exists $b' \in \mathbb{Z}^{d+1}$ such that $\beta(b') = (0, 1)$ (here using that $N \cong \mathbb{Z}^d \oplus \mathbb{Z}/g\mathbb{Z}$). Denote $\gcd(\{b'_i\}_{i=0}^d) = h$, then let $b_i = b'_i/h$ for $0 \le i \le d$. Then $(0, 1) = \beta(b') = \beta(hb) = h\beta(b)$, which implies that $\beta(b) = (0, y)$ with $hy = 1$ in $\mathbb{Z}/g\mathbb{Z}$. In particular, h, y are both relatively prime to g and $\beta(b)$ generates $0 \oplus \mathbb{Z}/g\mathbb{Z}$. Because b is primitive, it follows that gb generates $\ker(\beta)$.

Now construct $f_{gb} : DG(\beta) \xrightarrow{\cong} \mathbb{Z}$ as before (using gb as the generator of $\ker(\beta)$, instead of b). Again, we will have that the map $\rho : G \to \mathbb{T}^{d+1}$ will be given by $t \mapsto (t^{gb_0}, \ldots, t^{gb_d})$, as it is the pull-back of $\beta^\vee : (\mathbb{Z}^{d+1})^* \to DG(\beta)$. □

We can now state a couple of straightforward corollaries to Proposition 4.

Corollary 4. The polytopal stacky fan (N, Σ, β) corresponds to a weighted projective space if and only if β is surjective. The weighted projective space is effective if $\mathrm{Tor}(N) = \{0\}$.

Corollary 5. Let (N, Δ, β) be a stacky polytope such that Δ is simplex. Then G_0 acts freely if and only if $(1, 1, \ldots, 1) \in \ker(\beta)$.

5.3.2 When G is not connected — fake weighted projective spaces

As in the previous subsection, suppose that (N, Δ, β) is a stacky polytope and that $[Z/G] = \mathcal{X}(N, \Delta, \beta)$. In the case that $\dim(G) = 1$ but G is not connected, we no longer have that $[Z/G]$ is equivalent to a weighted projective space. However, when $\mathrm{rank}(\mathrm{DG}(\beta)) = 1$ and $\mathrm{Tor}(\mathrm{DG}(\beta)) \neq \{1\}$ the resulting quotient stack is equivalent to the quotient of a weighted projective space by a finite group, a *fake weighted projective space* (see [16] and [4]).

We must give a slight generalization to the standard definition of fake weighted projective spaces in the setting of stacks.

Definition 5.2. A **fake weighted projective space** is a stack quotient $\mathcal{W}/\Lambda$, where Λ is a finite Abelian group acting (in the sense of group actions on stacks, see [18] and [21]) on a weighted projective space $\mathcal{W}$.

In the preceding subsection we showed that stacky polytopes giving rise to a connected 1–dimensional group G correspond to weighted projective spaces (Proposition 4). In [11] the analogous result is proved for 1–dimensional disconnected groups G and fake weighted projective spaces:

Proposition 5. Let (N, Δ, β) be a stacky polytope, and let $\Sigma(\Delta)$ be the fan dual to Δ. The associated toric Deligne-Mumford stack $\mathcal{X}(N, \Sigma(\Delta), \beta)$ is a fake weighted projective space $\mathbb{P}(b_0, \ldots, b_d)/\Lambda$ if and only if $\mathrm{rank}(\mathrm{DG}(\beta)) = 1$. In this case, the polytope Δ is a simplex, and the weights are determined by the condition that $(b_0, \ldots, b_d)$ generates $\ker(\beta)$.

5.3.3 Sheared Simplices

Recall that we previously defined labeled polytopes as stacky polytopes with N a free module. It turns out that it is still difficult to give a succinct description of the quotient stack corresponding to an arbitrary labeled polytope (in terms of greatest common divisors and least common multiples of the components of the facet normals and the nonnegative facet labels) – the arithmetic quickly becomes too cumbersome. If we further restrict the class of polytopes that we are considering to something that we have named *sheared simplices* (see Definition 5.3), then we can give (at least in the case of a 2-simplex) a complete description of the quotient stack.

The basic notion of a sheared simplex in $\mathbb{R}^d$ is a labeled polytope contained in the first orthant, where d of the facets are contained in the d coordinate hyperplanes and the remaining facet is an affine hyperplane lying in the first orthant. The nicest example is the standard d-simplex in $\mathbb{R}^d$ (i.e., the convex hull of the origin and the standard basis vectors, $\{e_i\}_{i=1}^d$). If we now consider the convex hull of 0 and $\{a_i e_i\}_{i=1}^d$, where $a_i \in \mathbb{Z}^+$ for $1 \leq i \leq d$, we have (up to scale and assignment of facet labels) all sheared simplices. Note that the requirement that we consider the convex hull of positive integer multiples of the usual basis vectors ensures that the polytope is rational.

As we prefer to state our results more generally using stacky fans, we will adopt a different formal definition.

Definition 5.3. Suppose that $\mathbf{a} = (a_1, \ldots, a_d)$ is a vector of positive integers such that the greatest common divisor of $\{a_j\}$ is one, and let N be a free module of rank d. Let (N, Σ, β) be the stacky fan whose rays are $\rho_j = e_j$ for $1 \leq j \leq d$, where $\{e_j\}_{j=1}^d$ is the standard basis for $N \otimes \mathbb{R}$, and $\rho_0 = -\mathbf{a}$, and whose cones are the cones over all proper subsets of the $\{\rho_j\}_{j=0}^d$. Suppose further that $\beta : \mathbb{Z}^{d+1} \to N$ is defined by $\beta(e_j) = m_j e_j$ for $1 \leq j \leq d$ and $\beta(e_0) = -m_0 \mathbf{a}$, where $m_j \in \mathbb{Z}^+$ for $0 \leq j \leq d$. We call the stacky polytope dual to this fan a **sheared simplex** and denote it $\Delta(\mathbf{a})$. Finally, a **labeled sheared simplex** is the stacky polytope $(N, \Delta(\mathbf{a}), \beta)$, and $\{m_i\}_{i=0}^d$ are the **labels**.

We remark that if a simplex, Δ, has a smooth vertex (in the sense that the facet normals at that vertex generate the lattice, $\mathbb{Z}^d$, as a $\mathbb{Z}$-module), then their exists $A \in \mathrm{GL}(d;\mathbb{Z})$ such that $A(\Delta)$ is a sheared simplex (see Proposition 2). Thus, this class of stacky polytopes is larger than it might first seem.

Note that our assumptions (most notably that N is free) mean that the objects that we are considering are examples of the labeled polytopes considered in Lerman-Tolman (hence the suggestive use of "labels" for the $\{m_i\}$). Note also that though weighted projective spaces are examples of labeled sheared simplices, not every labeled sheared simplex is a weighted projective space (for example, G need not be connected in this class of examples).

Theorem 5.4. Let $(N, \Delta(\mathbf{a}), \beta)$ be a labeled standard simplex. Then G_0 acts freely if and only if all labels are the same (i.e., $m = m_0 = \cdots = m_d$).

Proof. ($\Leftarrow$) Suppose that $m = m_0 = \cdots = m_d$. Then

$$[B]^* = \begin{pmatrix} -m & \cdots & -m \\ m & & 0 \\ & \ddots & \\ 0 & & m \end{pmatrix}$$

Then, it's obvious that $[B]^*$ has the following Smith normal form (SNF),

$$\mathrm{SNF}([B]^*) = \begin{pmatrix} m & & 0 \\ & \ddots & \\ 0 & & m \\ 0 & \cdots & 0 \end{pmatrix}.$$

This, in turn, implies that $G = S^1 \times \mathbb{Z}/m\mathbb{Z} \times \cdots \times \mathbb{Z}/m\mathbb{Z}$. Moreover, $\ker(\beta) = \langle (1, 1, \ldots, 1) \rangle_{\mathbb{Z}}$, so that

$$\begin{aligned} \rho|_{G_0} : G_0 &\to \mathbb{T}^{d+1} \\ (t,1) &\mapsto \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix} \end{aligned}$$

and G_0 acts freely.

($\Rightarrow$) Suppose that G_0 acts freely. Because $\Delta(\mathbf{a})$ is a simplex all facets of which but one are contained in coordinate hyperplanes,we know that $[B]$ has the following form

$$[B] = \begin{pmatrix} -m_0 & m_1 & 0 & \\ \vdots & & \ddots & \\ -m_0 & 0 & & m_d \end{pmatrix}$$

Then G_0 acting freely implies that $\ker([B]) = \langle(1,1,\ldots,1)\rangle_{\mathbb{Z}}$, so that we must have $m_i = m_j$ for all $0 \leq i,j \leq d$. □

Theorem 5.5. Let $(N, \Delta(\mathbf{a}), \beta)$ be a labeled sheared simplex which is not the standard simplex (i.e., $\mathbf{a} \neq (1,\ldots,1)$). If G_0 acts freely then the labeling is not the constant labeling (i.e., there exist $0 \leq i,j \leq d$ such that $m_i \neq m_j$).

Proof. Suppose that G_0 acts freely and $m_i = m_j (= m)$ for $0 \leq i,j \leq d$. Then

$$[B] = \begin{pmatrix} -ma_1 & m & 0 & \\ \vdots & & \ddots & \\ -ma_d & 0 & & m \end{pmatrix}.$$

Then $\ker([B]) = \langle (1, a_1, \ldots, a_d) \rangle_{\mathbb{Z}}$, and

$$\begin{aligned} \rho|_{G_0} : G_0 &\to \mathbb{T}^n \\ (t, 1) &\mapsto \begin{pmatrix} t \\ t^{a_1} \\ \vdots \\ t^{a_d} \end{pmatrix}. \end{aligned}$$

But if there is some $1 \leq i \leq d$ such that $a_i \neq 1$, then, under the action of G_0 on Z, z_{a_i} has stabilizer isomorphic to $\mathbb{Z}/a_i\mathbb{Z}$ and this contradicts the assumption that G_0 acts freely. $\square$

Corollary 6. Let $(N, \Delta(\mathbf{a}), \beta)$ be a labeled sheared simplex with weights $\{m_j\}_{j=0}^d$. If G_0 acts effectively on Z, then one of the following holds:

1. the labeling is non-trivial and the sheared simplex $\Delta(\mathbf{a})$ is not the standard simplex, or
2. $\Delta(\mathbf{a})$ is the standard simplex, and the labeling $\{m_j\}_{j=0}^d$ is constant.

Proposition 6. Let $(N, \Delta(\mathbf{a}), \beta)$ be a labeled sheared simplex with labels $\{m_0 = 1, m_1 \ldots, m_d\}$. Then $\mathcal{X}(N, \Delta(\mathbf{a}), \beta)$ is an effective weighted projective space if and only if the following two conditions are satisfied:

1. $\gcd(m_i, m_j) = 1$, for all $i \neq j$
2. $\gcd(a_i, m_i) = 1$, for all i.

Proof. Recall that the primitive normal to the facet not contained in a coordinate hyperplane is given by $(-a_1, \ldots, -a_d)^T$.

($\Leftarrow$) Assume i) and ii) above. Let $B : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ be a lift of $\beta : \mathbb{Z}^{d+1} \to N$, then in the standard bases

$$[B] = \begin{pmatrix} -a_1 & m_1 & 0 & \dots & 0 \\ -a_2 & 0 & m_2 & \dots & 0 \\ & & \vdots & & \\ -a_{d-1} & 0 & \dots & m_{d-1} & 0 \\ -a_d & 0 & \dots & 0 & m_d \end{pmatrix}$$

and the columns of $[B]$ span $\mathbb{Z}^d$, as a $\mathbb{Z}$-module. Thus, G is connected. Moreover, G is parametrized in $\mathbb{T}^{d+1}$ as

$$G = \begin{pmatrix} t^m \\ t^{\frac{a_1 m}{m_1}} \\ t^{\frac{a_2 m}{m_2}} \\ \vdots \\ t^{\frac{a_d m}{m_d}} \end{pmatrix},$$

where $m = \operatorname{lcm}(\{m_i\})$ and $t \in \mathbb{R}/\mathbb{Z}$. Then the relative primality assumptions imply that this action is effective.

($\Rightarrow$) Now suppose that $\mathbb{P}(b_0, \dots, b_d)$ is an effective weighted projective space. Then Proposition 4 and Corollary 4 together imply that there is a labeled sheared simplex, $(N, \Delta(\mathbf{a}), \beta)$ such that $\mathbb{P}(b_0, \dots, b_d) = \mathcal{X}(N, \Delta(\mathbf{a}), \beta)$ with labels $\{m_0, m_1, \dots, m_d\}$. Additionally, we assume that $m_0 = 1$.

Now suppose that there exists $i, j \in \{1, \dots, d\}$ such that $\gcd(m_i, m_j) = k > 1$. Without loss of generality, suppose that $m_1 = kb$ and $m_2 = kc$ for some $b, c \in \mathbb{Z}^+$. Let, $B : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$

be a lift of $\beta : \mathbb{Z}^{d+1} \to N$, then in the standard bases

$$[B] = \begin{pmatrix} -a_1 & kb & 0 & \dots & 0 \\ -a_2 & 0 & kc & \dots & 0 \\ & & \vdots & & \\ -a_{d-1} & 0 & \dots & m_{d-1} & 0 \\ -a_d & 0 & \dots & 0 & m_d \end{pmatrix}$$

and the columns of β span a sublattice of index k in $\mathbb{Z}^{d+1}$. This implies that G is not connected. Thus, $\mathcal{X}(N, \Delta(\mathbf{a}), \beta)$ is not a weighted projective space.

Suppose now that there exists $i \in \{1, \dots, d\}$ such that $\gcd(a_i, m_i) = k > 1$, some $k \in \mathbb{Z}^+$. Then, again, the columns of $[B]$ span a sublattice of index k in $\mathbb{Z}^{d+1}$ and G is not connected. Hence, $\mathcal{X}(N, \Delta(\mathbf{a}), \beta)$ is not a weighted projective space. $\square$

Proposition 7. Suppose that $(N, \Delta(\mathbf{a}), \beta)$ is a labeled sheared simplex with labels $\{m_0 = 1, m_1, \dots, m_d\}$ and that $\mathcal{X}(N, \Delta(\mathbf{a}), \beta)$ is a weighted projective space. Then denote by z_j the point in $Z(:= \mu^{-1}(c))$ that corresponds to vertex j in $\Delta(\mathbf{a})$ (note that $z_j = \text{const} \cdot e_j$). Then for $j = 1, \dots, d$ the isotropy at z_j is isomorphic to

$$\mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_{j-1}\mathbb{Z} \times \mathbb{Z}/m_{j+1}\mathbb{Z} \times \mathbb{Z}/m_d\mathbb{Z} \times \mathbb{Z}/a_j\mathbb{Z} \cong \mathbb{Z}/\left(\left(\prod_{i \neq j} m_i\right) a_j\right)\mathbb{Z}.$$

The isotropy at z_0 is

$$\mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_d\mathbb{Z}.$$

Proof. Let $\hat{N} = \text{im}_{\mathbb{Z}}(\{\beta(e_i)\}_{i=1}^{d+1})$ and $\hat{N}_j = \text{im}_{\mathbb{Z}}(\{\beta(e_i)\}_{i \neq j})$. Then the isotropy at z_j is isomorphic to $\hat{N}/\hat{N}_j$. Since $(N, \Delta(\mathbf{a}), \beta)$ is a weighted projective space, we have $\hat{N} = \mathbb{Z}^d$, and the result now follows from the relative primality assumptions. $\square$

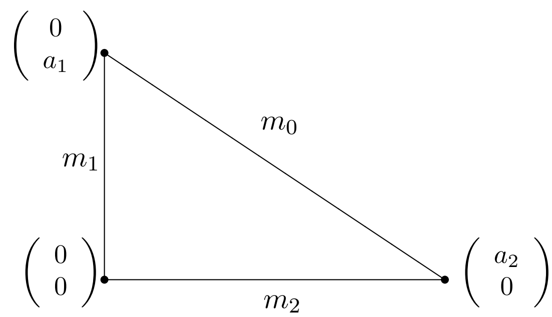


Figure 5.1: An arbitrary sheared simplex in the plane.

5.3.4 2-dimensional labeled sheared simplices

As a consequence of Theorem 5.3 (and the remarks following the proof), we can now determine the isotropy groups corresponding to a labeled sheared simplex in the plane.

Let $\mathbf{a} = (a_1, a_2)$ be a primitive vector in the positive quadrant, and suppose $(\mathbb{Z}^2, \Delta(\mathbf{a}), \beta)$ is a labeled sheared simplex with labels $\{m_0, m_1, m_2\}$. Explicitly, $\Delta(\mathbf{a})$ is the convex hull of the origin together with $(a_2, 0)$ and $(0, a_1)$, with assigned labels m_1 to the edge along the y-axis, m_2 to the edge along the x-axis, and m_0 to the remaining edge (see Figure 5.1).

The cone dual to the origin is simply the cone spanned by the positive coordinate axes; therefore, the corresponding isotropy group is easily seen to be $\mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z}$ (by Theorem 5.3). The cone σ_1 (resp. σ_2) dual to the vertex $(0, a_1)$ (resp. $(a_2, 0)$) has ray generators $\mathbf{a}$ and e_1 (resp. e_2). Below, we shall describe the isotropy Γ_{σ_1}, noting that Γ_{σ_2} is obtained by interchanging m_1 and m_2.

We begin with a Lemma describing the Smith normal form of an integer matrix with exactly one zero entry.

Lemma 2. Let

$$A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in \mathrm{M}(2; \mathbb{Z}),$$

with $abc \neq 0$. Let $g = \gcd(a, b, c)$, then the Smith normal form of A is

$$\begin{pmatrix} g & 0 \\ 0 & bc/g \end{pmatrix}.$$

Proof. Suppose that $\gcd(a, b) = d$. Then, we claim that there exist $x, y \in \mathbb{Z}$ such that $xa + yb = d$, and $\gcd(x, d) = 1$.

To prove this claim, we first note that it is equivalent to the following: Suppose $u, v \in \mathbb{Z}$ are relatively prime. Consider the set of solutions $X = \{x \mid xu + yv = 1 \text{ for some } y \in \mathbb{Z}\}$. For any given integer d, there is some $x \in X$ so that $\gcd(x, d) = 1$.

In this latter formulation, let $d \in \mathbb{Z}$ be given and suppose that x_0 is any solution to $x_0u + yv = 1$. Recall that all solutions are then of the form $x = x_0 + tv$ with $t \in \mathbb{Z}$. Moreover, $x_0u + yv = 1$ implies that $\gcd(x_0, v) = 1$. Then, showing that there is $x \in X$ such that $\gcd(x, d) = 1$ is equivalent to showing that there is $t \in \mathbb{Z}$ such that $\gcd(x_0 + tv, d) = 1$. We will construct such an integer, t.

Suppose that $d = p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s}$ is the prime factorization of d. Let $t = \prod p_i$ such that p_i does not appear in the prime factorization of either x_0 or v. Because x_0 and v are relatively prime, it follows that in the sum $x_0 + tv$ each prime in the factorization of d appears exactly once. That is, each p_i divides exactly one of x_0 or tv. Thus, d cannot divide the sum and $\gcd(x, d) = 1$.

Choosing x, y such that $ax + by = d$ immediately implies that

$$\begin{pmatrix} x & -b/d \\ y & a/d \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}).$$

Furthermore,

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \begin{pmatrix} x & -b/d \\ y & a/d \end{pmatrix} = \begin{pmatrix} d & 0 \\ cx & -bc/d \end{pmatrix}.$$

We now use the fact that we proved above and choose x so that $\gcd(x, d) = 1$. Then, for such an x, we have that $\gcd(cx, d) = \gcd(c, d) = \gcd(a, b, c) = g$. Furthermore, there exist $p, q \in \mathbb{Z}$ such that $p(cx) + qd = g$, and thus

$$\begin{pmatrix} q & p \\ -cx/g & d/g \end{pmatrix} \in \text{ SL } (2; \mathbb{Z}).$$

Moreover,

$$\begin{pmatrix} q & p \\ -cx/g & d/g \end{pmatrix} \begin{pmatrix} d & 0 \\ cx & -bc/d \end{pmatrix} = \begin{pmatrix} g & -cbp/d \\ 0 & -bc/g \end{pmatrix}.$$

Recall that $d \mid b$ and that $g \mid c$, so $g \mid (cbp/d)$. Thus, by elementary column operations

$$\begin{pmatrix} g & -cbp/d \\ 0 & -bc/g \end{pmatrix} \longleftrightarrow \begin{pmatrix} g & 0 \\ 0 & -bc/g \end{pmatrix}.$$

Finally, note that $g \mid (bc/g)$, so the above is the Smith normal form of the matrix A. $\square$

We now can give the explicit form of the isotropy groups of a labeled sheared simplex in the plane.

Proposition 8. Consider the labeled sheared simplex, $(\mathbb{Z}^2, \Delta(\mathbf{a}), \beta)$, with labels $\{m_0, m_1, m_2\}$. Then the isotropy corresponding to the vertex $(0, a_1)$, is $\Gamma_{\sigma_1} \cong \mathbb{Z}/g\mathbb{Z} \oplus \mathbb{Z}/((m_0 m_1 a_1)/g)\mathbb{Z}$, where $g = \gcd(m_0, m_1)$.

Proof. We consider the map $\beta : \mathbb{Z}^3 \to \mathbb{Z}^2$ given, in the standard bases for $\mathbb{Z}^3$ and $\mathbb{Z}^2$, by the matrix

$$[B] = \begin{pmatrix} -m_0 a_2 & m_1 & 0 \\ -m_0 a_1 & 0 & m_2 \end{pmatrix}.$$

As we want to compute the isotropy at the point corresponding to $v = (0, a_1)$, we consider the fan, σ, that is generated by the normals $\mathbf{a}$ and $(1, 0)$; i.e., the primitive inward normals of the facets incident at v. Then,

$$B_\sigma = \begin{pmatrix} -m_0 a_2 & m_1 \\ -m_0 a_1 & 0 \end{pmatrix}.$$

Now, applying Lemma 2, the Smith normal form of B_σ is

$$\begin{pmatrix} g & 0 \\ 0 & m_0 m_1 a_1 / g \end{pmatrix}$$

and the result follows. □

Though Proposition 8 gives the general form of the isotropy group at the vertex $(0, a_1)$ of a sheared simplex, it can be instructive to consider several special cases to illustrate the interplay of the facet labels and the geometry of the sheared simplex.

We summarize the isotropy at $(0, a_1)$ in Table 5.1:

It is perhaps interesting to note that in no case does the greatest common divisor of a facet label and a component of the vector $\mathbf{a} = (a_1, a_2)$ appear as an invariant factor in the isotropy groups. That is, there is no relationship between the facet labels and the geometry of the sheared simplex.

Labels	**Lengths**	Γ_{σ_1}
$m_0 = m_1 = m_2 = 1$	$a_1 = a_2 = 1$	$\{1\}$; i.e., smooth
	a_1, a_2 arbitrary	$\mathbb{Z}/a_1\mathbb{Z}$
m_0, m_1, m_2 arbitrary	$a_1 = a_2 = 1$	$\mathbb{Z}/m_0\mathbb{Z} \oplus \mathbb{Z}/m_1\mathbb{Z}$
	a_1, a_2 arbitrary	$\mathbb{Z}/g\mathbb{Z} \oplus \mathbb{Z}/(m_0m_1a_1/g)\mathbb{Z}$

Table 5.1: The isotropy group Γ_{σ_1} corresponding to the vertex $(0, a_1)$ of a labeled sheared simplex $(\mathbb{Z}^2, \Delta(\mathbf{a}), \beta)$, highlighting various special cases. Here, $g = \gcd(m_0, m_1)$.

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Curriculum Vitae

David A. Johannsen earned a Bachelor of Arts in Economics from the University of Chicago in 1987. After service in the United States Marine Corps, he began studies in mathematics, earning a Bachelor of Science in mathematics from the University of Maryland at College Park in 1994. He then attended the University of North Carolina at Chapel Hill, earning a Master of Science degree in mathematics in 1997. Following a short period of employment as a software quality engineer with MicroStrategy in Vienna, Virginia, he began work as a Scientist at the Naval Surface Warfare Center, Dahlgren Division. Mr. Johannsen remains employed by the Navy, now as a Lead Scientist.