TWO DIMENSIONAL MICROMECHANICS BASED COMPUTATIONAL MODEL
FOR SPHERICALLY VOIED BIAXIAL SLABS (SVBS)

by

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of
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Fall Semester 2014
George Mason University
Fairfax, VA
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This dissertation is dedicated to my caring wife Hiwot, my daughter Liana, and my mother Belaynesh.
ACKNOWLEDGEMENTS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Tables</td>
<td>vii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>viii</td>
</tr>
<tr>
<td>List of Abbreviations or Symbols</td>
<td>xi</td>
</tr>
<tr>
<td>Abstract</td>
<td>xii</td>
</tr>
<tr>
<td>CHAPTER 1: INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Statement of the Problem</td>
<td>3</td>
</tr>
<tr>
<td>1.3 Purpose and Contribution of the Study</td>
<td>4</td>
</tr>
<tr>
<td>1.4 Significance of the Study</td>
<td>5</td>
</tr>
<tr>
<td>1.5 Dissertation Organization</td>
<td>6</td>
</tr>
<tr>
<td>CHAPTER 2: LITERATURE REVIEW</td>
<td></td>
</tr>
<tr>
<td>2.1 History and Benefits of SVBS</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Review of Current Practices on Design of SVBS</td>
<td>12</td>
</tr>
<tr>
<td>2.2.1 Flexural Design</td>
<td>12</td>
</tr>
<tr>
<td>2.2.2 Shear Design</td>
<td>13</td>
</tr>
<tr>
<td>2.2.3 Deflections</td>
<td>14</td>
</tr>
<tr>
<td>2.3 Review of Experimental Research on SVBS</td>
<td>14</td>
</tr>
<tr>
<td>2.4 Review of Micromechanics Approach</td>
<td>16</td>
</tr>
<tr>
<td>2.5 Modeling Approaches</td>
<td>24</td>
</tr>
<tr>
<td>2.6 Extended Discussion on Homogenization</td>
<td>28</td>
</tr>
<tr>
<td>CHAPTER 3: MICROMECHANICAL HOMOGENIZATION OF SVBS</td>
<td>35</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>35</td>
</tr>
<tr>
<td>3.2 Geometry of the RVE &amp; Assumptions</td>
<td>37</td>
</tr>
<tr>
<td>3.3 ANSYS FE Modeling</td>
<td>41</td>
</tr>
<tr>
<td>3.3.1 Extensional Stiffness</td>
<td>48</td>
</tr>
<tr>
<td>3.3.2 Flexural and Torsional Stiffness</td>
<td>57</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table | Page  
--- | ---  
Table 1-1 BubbleDeck slab configurations (BubbleDeck 2006) | 4  
Table 3-1: Material input data | 42  
Table 3-2: Geometrical Data | 42  
Table 3-3: Applied boundary conditions and prescribed displacements to determine $A_{11}$ and $A_{21}$ | 48  
Table 3-4: Applied boundary conditions and prescribed loads for determining $A_{33}$ | 55  
Table 3-5: Applied boundary conditions and prescribed displacement for determining $D_{11}$ and $D_{21}$ | 59  
Table 3-6: Applied boundary conditions and prescribed displacement for determining $D_{33}$ | 63  
Table 3-7: Applied boundary conditions and prescribed loads for determining $E_{55}$ | 67  
Table 4-1: Comparison of the non-zero [A] coefficients determined by micromechanics based FE and analytical methods for plain concrete solid slab | 85  
Table 4-2: Comparison of the non-zero [D] coefficients determined by micromechanics based FE and analytical methods for plain concrete solid slab | 93  
Table 4-3: Comparison of the non-zero Transverse shear, [E], stiffness values determined by micromechanics based FE and analytical methods for the plain concrete solid slab | 99  
Table 6-1: Geometric parameters and computed stiffness values of SVBS230 and SOLID230 | 121  
Table 6-2: Geometric parameters and computed stiffness values of SVBS280 and SOLID280 | 122  
Table 6-3: Geometric parameters and computed stiffness values of SVBS340 and SOLID340 | 123  
Table 6-4: Geometric parameters and computed stiffness values of SVBS390 and SOLID390 | 124  
Table 6-5: Geometric parameters and computed stiffness values of SVBS450 and SOLID450 | 125  
Table 6-6: Specific stiffness values of SVBS and solid slabs | 129  
Table 6-8: Material input data | 137  
Table 6-9: Geometrical configurations of models | 138  
Table 6-10: Number of nodes and elements used in the analyses | 140
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 2-1: Typical spherically voided biaxial slab under construction (Wrap 2012)</td>
<td>10</td>
</tr>
<tr>
<td>Figure 2-2: Compression and tension zones for flexural capacity of SVBS</td>
<td>13</td>
</tr>
<tr>
<td>Figure 2-3: Shear capacity test summary (BubbleDeck 2006)</td>
<td>14</td>
</tr>
<tr>
<td>Figure 2-4: Concept of micromechanics based model development procedure</td>
<td>30</td>
</tr>
<tr>
<td>Figure 3-1: Global and local Cartesian coordinate systems</td>
<td>38</td>
</tr>
<tr>
<td>Figure 3-2: Representative volume element extraction from SVBS 340</td>
<td>39</td>
</tr>
<tr>
<td>Figure 3-3: FE model of the RVE</td>
<td>43</td>
</tr>
<tr>
<td>Figure 3-4: Contact regions in the RVE</td>
<td>44</td>
</tr>
<tr>
<td>Figure 3-5: Meshing result for the analysis model</td>
<td>45</td>
</tr>
<tr>
<td>Figure 3-6: SOLID186 geometry</td>
<td>46</td>
</tr>
<tr>
<td>Figure 3-7: SOLID187 geometry</td>
<td>47</td>
</tr>
<tr>
<td>Figure 3-8: Boundary conditions and prescribed displacement to determine A_{11} and A_{21} of SVBS340; (a) the face with no displacement in all three directions; (b) the face with displacement of 300 mm in the x-directions and zero in the other two directions; (c) and (d) the faces with zero displacement in the z-direction but free on the remaining two directions</td>
<td>49</td>
</tr>
<tr>
<td>Figure 3-9: Displacement contour plot when ( \varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 0 )</td>
<td>51</td>
</tr>
<tr>
<td>Figure 3-10: Resultant force, N_{1}, at face x=0 when ( \varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 0 )</td>
<td>51</td>
</tr>
<tr>
<td>Figure 3-11: Resultant force, N_{2}, at face z=L_{z} when ( \varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 0 )</td>
<td>52</td>
</tr>
<tr>
<td>Figure 3-12: Resultant force, N_{2}, at face z=0 when ( \varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 0 )</td>
<td>52</td>
</tr>
<tr>
<td>Figure 3-13: (a) Face on which fixed support applied, and (b) face on which moment is applied to determine A_{33}</td>
<td>55</td>
</tr>
<tr>
<td>Figure 3-14: Directional deformation contours under applied moment to determine A_{33}</td>
<td>56</td>
</tr>
<tr>
<td>Figure 3-15: Face on which shear force is applied to determine A_{33}</td>
<td>56</td>
</tr>
<tr>
<td>Figure 3-16: Directional deformation contours under applied shear force to determine A_{33}</td>
<td>57</td>
</tr>
<tr>
<td>Figure 3-17: Boundary conditions and prescribed rotations to determine D_{11} and D_{21}; (a) the face with fixed BC, (b) and (c) the faces with zero displacement only in z-directions, and (d) the face with prescribed rotation of 1 rad about the z-axis</td>
<td>60</td>
</tr>
<tr>
<td>Figure 3-18: Deformation contour when determining D_{11} and D_{21}</td>
<td>61</td>
</tr>
<tr>
<td>Figure 3-19: Resultant moments, M_{1} and M_{2}, when determining D_{11} and D_{21}</td>
<td>62</td>
</tr>
<tr>
<td>Figure 3-20: Boundary conditions and remote displacement applied on the FE model to determine D_{33}; (a) the face with fixed support, (b) the face with prescribed rotation</td>
<td>63</td>
</tr>
<tr>
<td>Figure 3-21: Deformation contour when determining D_{33}</td>
<td>64</td>
</tr>
<tr>
<td>Figure 3-22: Resultant moment, M_{R1}, at the fixed face when determining D_{33}</td>
<td>65</td>
</tr>
</tbody>
</table>
Figure 3-23: Boundary condition and prescribed moment to determine $E_{55}$; (a) face with fixed support (b) face with prescribed moment in the z-direction

Figure 3-24: Directional deformation contour under applied moment to determine $E_{55}$

Figure 3-25: Prescribed remote force in the negative y direction

Figure 3-26: Directional deformation contour under the applied force to determine $E_{55}$

Figure 4-1: Representative volume element extraction from plain concrete slab

Figure 4-2: FE model of the RVE for plain concrete (a) dimensions of the RVE, and (b) generated mesh

Figure 4-3: Boundary conditions and prescribed displacement to determine $A_{11}$ and $A_{21}$; (a) the face with no displacement in all three directions, (b) the face with displacement of 300 mm in the x-directions and zero in the other two directions, (c) and (d) the face with zero displacement in the z-direction but free on the other two directions

Figure 4-4: Deformation contour when determining extensional stiffness of the plain concrete RVE

Figure 4-5: Reaction forces on restrained faces for extensional stiffness of the plain concrete RVE

Figure 4-6: Plan view of the original and deformed shapes of the RVE under unit in-plane shear strain

Figure 4-7: Boundary conditions and prescribed moment to determine $A_{33}$; (a) the face with fixed support, and (b) the face with applied moment

Figure 4-8: Directional deformation contour in z direction under applied moment, (a) overall and (b) face at $x=L_x$

Figure 4-9: Boundary conditions and remote force to determine $A_{33}$; (a) the face with fixed support, and (b) the face with applied force

Figure 4-10: Directional deformation contour in z direction under applied force, (a) overall and (b) face at $x=L_x$

Figure 4-11: Prescribed displacement and boundary conditions applied to determine $D_{11}$ and $D_{21}$

Figure 4-12: Deformation contour for determining $D_{11}$ and $D_{12}$

Figure 4-13: Resultant moments at the restrained faces; (a) $M_1$ at face $x=0$, and (b) $M_2$ at face $z=0$ and $z=L_z$

Figure 4-14: Boundary conditions and prescribed unit strain for $D_{33}$, (a) fixed support, (b) unit torsional rotation

Figure 4-15: Deformed shape under unit torsional rotation (a) isometric view and (b) plan view

Figure 4-16: Deformed shape under unit torsional rotation (a) isometric view and (b) plan view

Figure 4-17: Boundary condition and prescribed moment to determine $E_{55}$; (a) face with fixed support (b) face with prescribed moment in the z-direction

Figure 4-18: Directional deformation contours due to the prescribed moment; (a) body in the y-axis (b) on the face at $x=L_x$ in the y-axis

Figure 4-19: Boundary condition and prescribed force to determine $E_{55}$; (a) face with fixed support (b) face with prescribed force in the negative y-direction
Figure 4-20: Directional deformation contours due to the prescribed force; (a) body in the y-axis (b) on the face at x=Lx in the y-axis ................................................................. 98

Figure 5-1: Notations and sign conventions (a) Rotations of mid-surface normal and (b) Slopes of plate surfaces ........................................................................... 101

Figure 5-2: Geometry and orientation of SVBS for macromechanical analysis ........... 103

Figure 5-3: Mid-plane deflection of the SVBS ......................................................... 107

Figure 5-4: Bending moment in x-direction, Mx ..................................................... 108

Figure 5-5: Bending moment in z-direction, Mz ..................................................... 108

Figure 5-6: Torsional moment, Mxz ................................................................. 109

Figure 5-7: Maximum stress on the long direction, σxz, on the top face ................. 109

Figure 5-8: Maximum bending stress on the short direction, σx on the top face ....... 110

Figure 5-9: In-plane shear stress, σxz, of the SVBS ............................................. 110

Figure 5-10: Mid-plane deflection of the SVBS .................................................. 114

Figure 5-11: Bending moment in x-direction, Mx ................................................ 114

Figure 5-12: Bending moment in z-direction, Mz ............................................... 115

Figure 5-13: Torsional moment, Mxz ................................................................. 115

Figure 5-14: Maximum stress on the long direction, σxz on the top face ............. 116

Figure 5-15: Maximum bending stress on the short direction, σx, on the top face ...... 116

Figure 5-16: In-plane shear stress, σxz, of the SVBS ......................................... 117

Figure 6-1: Typical SVBS and equal thickness solid slab models (a) SVBS, and (b) solid slab ................................................................. 120

Figure 6-2: Extensional stiffness properties, A11, for SVBS and solid slabs .......... 126

Figure 6-3: In-plane shear stiffness properties, A33, for SVBS and solid slabs .... 127

Figure 6-4: Bending stiffness properties, D_{11}, for SVBS and solid slabs .......... 127

Figure 6-5: Torsional stiffness properties, D_{33}, for SVBS and solid slabs .......... 128

Figure 6-6: Transverse shear stiffness properties, D_{11}, for SVBS and solid slabs .......... 128

Figure 6-7: Geometry of second RVE type (RVE-2) ........................................... 131

Figure 6-8: Meshing result for the analysis model of RVE-2 ............................. 132

Figure 6-9: Boundary conditions for $A_{11}$ and $A_{21}$ of RVE-2 ....................... 133

Figure 6-10: Directional deformation contour plot when $\varepsilon_1 = 1$, and $\varepsilon_2 = \varepsilon_3 = 0$ of RVE-2 ................................................................. 134

Figure 6-11: Boundary conditions and prescribed rotation to determine bending stiffness of RVE-2 ................................................................. 135

Figure 6-12: Deformation contour of RVE-2 .................................................... 136

Figure 6-13: Geometrical configuration of M1-SVBS, (a) isometric view, and (b) profile view in the longitudinal direction ................................................................. 139

Figure 6-14: Geometrical configuration of M2-SVBS, (a) isometric view, and (b) profile view in the longitudinal direction ................................................................. 140

Figure 6-15: Mesh for M1-SVBS ........................................................................ 141

Figure 6-16: Maximum deflection versus applied load, (a) for short span models and (b) for long span models ................................................................. 143
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
<th>Also known as</th>
<th>AKA</th>
<th>aka</th>
</tr>
</thead>
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<tr>
<td>Axial Stiffness</td>
<td></td>
<td></td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Bending Stiffness</td>
<td></td>
<td></td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>Boundary Condition</td>
<td></td>
<td></td>
<td>BC</td>
<td></td>
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<tr>
<td>Coupling Stiffness</td>
<td></td>
<td></td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Finite Element</td>
<td></td>
<td></td>
<td>FE</td>
<td></td>
</tr>
<tr>
<td>One Dimension</td>
<td></td>
<td></td>
<td>1D</td>
<td></td>
</tr>
<tr>
<td>Reinforced Concrete</td>
<td></td>
<td></td>
<td>RC</td>
<td></td>
</tr>
<tr>
<td>Reinforcing Steel</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Representative Volume Element</td>
<td></td>
<td></td>
<td>RVE</td>
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<td>Spherically Voided Biaxial Slab</td>
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<td>SVBS</td>
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<tr>
<td>Three Dimension</td>
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<td>3D</td>
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<td>Transverse Shear Stiffness</td>
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</tr>
</tbody>
</table>
ABSTRACT

TWO DIMENSIONAL MICROMECHANICS BASED COMPUTATIONAL MODEL FOR SPHERICALLY VOIED BIAXIAL SLABS (SVBS)

Wondwosen Belay Ali, Ph.D.

George Mason University, 2014

Dissertation Director: Dr. Girum S. Urgessa

The demand for reducing the weight-to-stiffness and strength ratios of solid reinforced concrete slabs, while improving their span limit, has led to the emergence of spherically voided biaxial slab (SVBS) systems. However, the presence of the spherical voids makes the structural analysis of SVBS inherently challenging because of the heterogeneous and periodically varying complex cross-sectional geometry. Current analysis procedures heavily rely on proprietary manufacturer information with a recommendation of treating SVBS as solid reinforced concrete slabs and applying global reduction factors without giving regard to the microstructures of the SVBS itself. This dissertation presents new load-response analysis models that use a finite element based micromechanical homogenization procedure for predicting the elastic structural behavior of SVBS. Firstly, a representative volume element (RVE) or unit cell of a selected SVBS has been established and analyzed to determine the equivalent RVE stiffness properties, i.e. the
extensional stiffness matrix [A], the coupling stiffness matrix [B], and the bending stiffness matrix [D]. Independent unit strains and unit curvatures are prescribed to the RVE in ANSYS, and the resulting resultant moments and forces are used to calculate the [A], [B] and [D] matrices. A procedure combining finite element analysis and Timoshenko’s beam theory is proposed for determining the transverse shear stiffness matrix [E] with an acceptable margin of error. The micromechanical homogenization procedure is then verified by applying it to a homogenous isotropic RVE for which results can be determined analytically. Secondly, the ABDE matrices of the RVE are used to analyze a full-scale 2D orthotropic SVBS floor system by Mindlin-Reissner and Kirchhoff-Love plate theories which are programmed in MATLAB. Thirdly, parametric studies are presented investigating stiffness variations in five different configurations of SVBS, selection of alternative RVE type, and the effect of span length on SVBS efficacy. The results from this dissertation can play a role in the development of SVBS design guidelines, which currently do not exist in the US.
CHAPTER 1: INTRODUCTION

1.1 Motivation

One of the most commonly used techniques in slab systems is the use of reinforced concrete. This is because reinforced concrete provides high resistance to compressive and bending stresses; it yields rigid members with minimum deflection; it has long service life with low maintenance cost; and it has great resistance to fire. However, the main disadvantages of a concrete slab are its high weight-to-strength and high weight-to-stiffness ratios. These disadvantages are especially pronounced in the case of multi-story buildings or buildings with difficult foundation layouts, and for long slab spans in particular. Recently, the use of spherically voided biaxial concrete slab (SVBS) system, which uses hollow plastic balls as infill material, has become an emerging solution to reduce the weight-to-strength and weight-to-stiffness ratios. This slab system relies on removing some of the least effective concrete from the middle of a floor slab, where the slab is principally less stressed in flexure, thereby dramatically reducing the structural dead weight by up to about 35% (Marias et al. 2010). This reduction in dead weight of the slab also reduces the magnitude of loads that are transferred to columns, and footings of buildings. After all, a slab is by far the heaviest structural component in reinforced concrete buildings.
A great deal of analytical research on modeling of solid, ribbed and hollow core reinforced concrete slabs has been conducted over the past few decades. However, research on the analytical modeling of spherically voided biaxial concrete slabs is still in its infancy. So far only little research has been conducted on SVBS and the research has focused mainly on conducting either physical experiments or simulations using 3D non-linear finite element software, namely ANSYS, ABAQUS, and DIANA.

The results obtained from the research conducted on SVBS have been compared to corresponding results of solid flat slabs of equal dimensions (Chung et al. 2011, Lai 2010). With the incorporation of reduction factors obtained from the comparison results, the analytical methods developed for flat solid slabs are modified to calculate structural responses and capacities of SVBS systems (Schnellenbach-Held and Pfeffer 2002, BubbleDeck 2006). This simplification is partly attributed to the difficulty of modeling the stiffness parameters due to the geometric complexity of SVBS. The voids formed by the plastic balls are non-prismatic; they are discrete volumes in a two-dimensional array. Hence, the conventional plate theory and the grillage method do not simulate the actual mechanisms well. While the former requires a homogenous medium, the latter is developed for one directional hollow core slabs. The simplification may also partly be attributed to the fact that the system is relatively new to the structural engineering community.
1.2 Statement of the Problem

The current analytical method for computing structural responses and capacities of an SVBS system is only approximate and insufficient. This is partly due to the fact that the method did not consider the slab’s non-prismatic configuration; rather it uses the analytical formulation developed for prismatic (solid) slabs with the inclusion of reduction factors obtained from physical test comparisons. A high factor of safety is usually involved to compensate for the uncertainties inherited with the existing method of analysis. Generally, achieving substantiation of structural capacities and behavior of SVBS systems by physical experiments alone can be prohibitively expensive because of the number of specimens and components required to characterize all material systems, loading scenarios and boundary conditions. Finite element (FE) simulation plays a crucial role in the analysis of the mechanical behavior of structural elements built with complex geometry like SVBS. However, FE analysis on large scale SVBS leads to the need for unstructured meshes, large numbers of finite elements including contact elements, and it requires large amounts of memory and CPU time. Hence, analyzing SVBS as a 3D structure using finite-element analysis is very time consuming and uneconomical in most cases.

The heterogeneity of the SVBS system causes heterogeneity of the stress-strain state, which in turn leads to challenges in the mathematical description of the structural responses of the SVBS systems. It is obvious that the distribution of stresses and deformations in the slab system is dependent upon its stiffness in extension, flexure, torsion and shear. Assessing the stiffness of a SVBS system under a given configuration
is of vital importance for structural engineers. Moreover, mechanics based analytical formulations that consider the actual cross-sectional geometric configuration to analyze stiffness of SVBS systems do not exist. Hence, optimizing a given SVBS design is often challenging. For example, BubbleDeck, one of the companies providing SVBS, recommends same types of slab configurations for two floor systems having spans of 18 m and 11 m, as shown in Table 1-1.

<table>
<thead>
<tr>
<th>Type</th>
<th>Thickness mm</th>
<th>Ball Diameter mm</th>
<th>Span m</th>
<th>Mass kg/m²</th>
<th>Concrete On Site m³/m²</th>
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<tr>
<td>BD230</td>
<td>230</td>
<td>180</td>
<td>7-10</td>
<td>370</td>
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<td>460</td>
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<td>BD340</td>
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<td>9-14</td>
<td>550</td>
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<td>315</td>
<td>10-16</td>
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<tr>
<td>BD450</td>
<td>450</td>
<td>360</td>
<td>11-18</td>
<td>730</td>
<td>0.25</td>
</tr>
</tbody>
</table>

### 1.3 Purpose and Contribution of the Study

The main purpose of this study is developing a fast running and reasonably accurate finite element based micromechanics model for analyzing SVBS. The model accounts for the physical description of the configuration of the voids, the properties of the SVBS constituents, knowledge of continuum mechanics, and first order shear deformation theories for the derivation of the effective macroscopic constitutive behavior of the SVBS system.
The main contribution of this study is the development of an analytical constitutive model for SVBS that gives an effective extensional stiffness matrix \([A]\), a coupling stiffness matrix \([B]\), and a bending stiffness matrix \([D]\) and transverse shear stiffness terms, \(E_{55}\) and \(E_{66}\), by using a finite element based micromechanical approach. Additional objectives of this study are listed as follows:

- Compare the stiffness values of SVBS and solid slabs of equal thickness.
- Compare the specific stiffness values of SVBS and solid slabs of equal thickness.
- Validate the finite element based micromechanical approach developed in this dissertation by applying it on isotropic homogenous (plain concrete) slab, for which results can be obtained analytically.
- Investigate the effect of shear deformation on SVBS by comparing results from first order shear deformable theory (FSDT) and Kirchhoff-Love theory (KLT).
- Investigate the effect of span length on the structural behavior of SVBS and solid slabs.

1.4 Significance of the Study

This study presents efficient new two dimensional mechanics based computational model for analyzing SVBS systems. This model is expected to provide useful insight and guidance for structural analysis and design of non-prismatic slabs with periodical heterogeneities in general and spherically voided biaxial slabs in particular without relying solely on expensive procedures such as physical experimentations and full scale detailed three dimensional finite element analyses.
The model presented in this dissertation can be included as part of future design and analysis manuals. For example, a recent inquiry of the BubbleDeck North American office revealed that design and analysis manuals for the US market do not exist yet, and they are being translated from European codes. However, the European code approaches solely rely on analyzing SVBS as a solid slab with capacity reduction factors. This study will address the gap in the lack of mechanics based models and may impact future design practices as the use of SVBS system is expected to increase widely.

The research approach can also be extended to develop finite element based micromechanics analytical models for other types of slabs that have non-prismatic cross section with periodic nature such as two way waffle slabs, ribbed slabs, U-Boot Beton slabs, etc.

1.5 Dissertation Organization

Chapter one presents an overview of the dissertation including the motivation of the research, the statement of the problem, the purpose and contribution of the study, and the significance of the study.

Chapter two is devoted to the discussion of the background for the dissertation research mainly history and benefits of SVBS, review of current practices on design of SVBS, review of experimental research on SVBS, and review of research conducted on micromechanics. Detailed discussion on the theory of micromechanics, especially on the concept of homogenization, mathematics of homogenization and periodic boundary conditions are also presented.
Chapter three discusses micromechanical homogenization of SVBS, particularly the determination of extensional, bending and shear stiffness values (ABDE stiffness matrix) of an SVBS with distinct configurations based on finite element based micromechanics homogenization technique. The procedures involved in the finite element model building using ANSYS are also presented in this chapter, including but not limited to extraction of representative volume element (called RVE-1), type and number of elements used in the mesh generation, pertinent boundary conditions under different simulations and material mechanical properties. Finite element outputs in terms of deformation contours, resultant moments and forces are also provided. These resultant moments and forces are the basis to determine the ABDE stiffness matrix of the SVBS model.

Chapter four presents the verification of the finite element based micromechanics analysis procedures used in chapter three by implementing similar simulation procedures on plain concrete (isotropic) solid slab. The plain concrete (isotropic) solid slab is selected for verification because of the availability of generally accepted analytical formulations derived based on plate theory to determine the ABDE stiffness matrix. The ABDE stiffness values determined from the FE based micromechanical simulations are compared to the stiffness values computed using analytical formulations.

Chapter five presents a 2D macromechanical analysis to determine the behavior and response of a full-scale SVBS floor system. In this chapter, the ABDE stiffness values determined in chapter three from the RVE are used to determine deflection, moments and stresses at different parts of a full-scale slab under a particular loading,
geometry and boundary conditions. A simply supported slab under a uniform pressure load is analyzed using MATLAB codes written as part of this dissertation. The first order shear deformation theory (FSDT), also known as Mindlin-Reissner, and Kirchhoff-Love theory (KLT) are used in the analysis. Outputs of the MATLAB code including three dimensional plots of deflection, moments and stresses are presented.

Chapter six includes parametric studies comparing the ABDE stiffness values of selected SVBS configuration to solid slabs of the same thickness. The specific stiffness values of the SVBS and the solid slabs are also compared to investigate the efficiency of the SVBS. The parametric study also includes determination of stiffness values using a second type of representative volume element (called RVE-2) and comparing the results with stiffness values of RVE-1 determined in chapter three. The effect of span length on SVBS efficacy is also discussed.

Chapter seven presents conclusions containing key findings of the research, recommendations, limitation of the study and future work. The appendix contains the MATLAB codes written as part of this dissertation for computing deflection, moments and stresses of an SVBS macromechanically given the RVE stiffness values developed micromechanically in chapter three.
CHAPTER 2: LITERATURE REVIEW

2.1 History and Benefits of SVBS

The demand for lighter and safer concrete floor system in order to improve the span limit has stimulated the need to study new structural configurations in floor systems for long time. In the 1950s, hollow-core slab were invented (Metric Concrete Industries 2012). These are one-way spanning concrete slabs with hollow cylinders. The hollow core system was created to reduce the weight of the concrete from the system. This concept removes some of the concrete from center of the slab, where it is less useful to decreases the dead weight of the slab system. However, its application is limited to one-way spanning floor system only as its hollow cavities significantly decrease the slab resistance to shear and fire, thus reduce its structural integrity.

Recently, new developments on spherically voided slab systems have been made in Europe. These developments have enhanced the efficiency of concrete slabs to unprecedented spans reaching nearly 20 m. While the concept of removing the least working concrete to minimize the dead load remains the same, the improvement comes from the fact that these are biaxially loaded slabs with the load distribution of a two-way slab (Mota 2010). Currently, the two most known companies that are providing the spherically voided biaxial slab systems throughout the world are BubbleDeck and Cobiax. In the late 1990’s, a Danish Engineer named Jorgen Breuning invented
BubbleDeck (Lia 2010). In early 2000s, Cobiax was invented in Switzerland. Both slab systems use fundamentally the same techniques, which use spheres made of recycled industrial plastic to create air voids around the middle region of the slab. However, they have some differences in the arrangement of reinforcing steels. The manufacturers’ literatures often provide the available sizes of the spheres, the spacing requirements and the general configuration of the slabs.

Generally, spherically voided biaxial slabs have been used in floor systems when conventional sections fail to satisfy certain aspects of design such as high strength-to-weight, high stiffness-to-weight ratios or deflection. The inherently high flexural rigidity of such slabs mostly comes from the two solid facing plates separated by a hollow core. Figure 2-1 shows a typical biaxially reinforced concrete flat slab system that uses grids of hollow plastic balls as void formers.

![Figure 2-1: Typical spherically voided biaxial slab under construction (Wrap 2012)](image)
Few selected notable projects that used SVBS for the construction of their slab systems are listed as follows: The Millennium Tower in Rotterdam, Holland; University of Wisconsin-Madison’s La Bahn Hockey Arena, USA; National Stadium of Poland in Warsaw, Poland; Le Coie Housing in London, United Kingdom; Elbphilharmonie Opera House in Hamburg, Germany; The Union of European Football Association (UEFA) in Switzerland; Altra Sede, Tower Block in Milan, Italy; York University Life Science Building in Toronto, Canada; The Miami Art Museum in Miami, Florida, USA; Kempinski Hotel in Jeddah, Saudi Arabia; and Sogn Arena in Oslo, Norway.

SVBS systems are receiving significant attention because of the following benefits (Teja et al. 2009, Marais et al. 2010):

- They give an exceptional degree of freedom in architectural design-choice of shape, long cantilevers, long spans/ deck areas with fewer supporting points – no beams, fewer columns and carrying walls results in flexible and easy changeable buildings.
- They allow interior design of building to be easily altered throughout the lifetime of the building.
- They reduce concrete where it is less needed and the subsequent weight reduction makes longer spans possible - up to 20 m between columns without beams.
- The reduced weight of the SVBS system allows a reduction in concrete and steel in other structural elements such as floors, columns, and footings, which in turn reduces both the total building weight and cost.
- They also eliminate beams and drops, resulting in reduced floor-to-floor heights.
- The reduced dead weight also results in lower seismic forces induced to structures.
- They increase sustainability ratings because of the reduction in the volume of concrete needed for a typical structure.

2.2 Review of Current Practices on Design of SVBS

In spite of the advantages offered by SVBS system, so far only a few hundred residential, high rises and industrial structures have been constructed using the system (Lia 2010). This may be because advances in analytical models are still in infancy and there is a limited understanding about the structural behavior of SVBS. The current SVBS design methodologies, as discussed below, are limited to comparisons with the behavior of a similar solid slab.

2.2.1 Flexural Design

A flat solid slab model is assumed for flexural design. The flexural capacity is calculated based on the outer 'shell' of concrete and steel on the compression side, and the steel on the tension side as shown in Figure 2-2. A design check is carried out to ensure that the concrete compression zone remains outside of the depth of the spherical void formers. When checking the compression zone, if the compression zone is found to be encroached into the ball, the manufacturers will offer appropriate guidance on determining the permissible compression zone that can be used in the calculation of the flexural strength (Whittle and Taylor 2009). The voids are positioned in the middle of the cross-section, where concrete has limited effect, while maintaining solid sections in top
and bottom sections where high stresses exist. Hence, the SVBS is fully functional with regards to both positive and negative bending.

![Figure 2-2: Compression and tension zones for flexural capacity of SVBS](image)

### 2.2.2 Shear Design

Shear design of SVBS is carried out by treating the slab again as a flat solid slab with the inclusion of reduction factors. Manufacturers recommend that the shear strength of a solid slab of the same depth should be reduced by a factor of 0.55-0.6 to obtain the design shear resistance for the SVBS. Shear is usually critical near columns due to the punching effect of the columns on the slab (Abramski et al. 2010). According to punching shear tests conducted on SVBS of thickness 230 mm and 450 mm at the technical university of Denmark the comparison with corresponding flat solid slabs showed that the smallest shear capacity of a voided slab is about 60% of the capacity of a solid slab with equal thickness. This occurred where the ratio of the distance from imposed force to support divided by deck thickness is about 3.0 as shown in Figure 2-3.
2.2.3 Deflections

Tests carried out to determine the reduced stiffness of SVBS showed that the stiffness of a voided slab is conservatively about 0.87 times the stiffness of a solid slab. However, deflection calculations based on this stiffness is deemed very approximate. Hence, finite element modeling with adjusted stiffness and strength parameters is the recommended practice (BubbleDeck Slab Properties 2006).

2.3 Review of Experimental Research on SVBS

Since the invention and use of BubbleDeck and Cobiax in early 2000’s, very limited experimental results have been published in open literature besides technical manuals and design guidelines. This section summarizes the experimental research carried out to investigate the structural responses and capacities of SVBS.

Schnellenbach-Held and Pfeffer (2002) investigated the effect of cavities on the punching shear capacity, and the behavior and mode of failure of biaxial hollow slabs.
Punching tests and nonlinear 3-D finite element analyses were carried out on biaxial hollow slabs of different configurations. The test results were consistently in agreement with the finite element analysis results. They showed that the void formers did not affect the crack pattern when compared to that of a solid slab. However, the punching shear capacity of the biaxial hollow slabs was found out to be smaller than that of the solid slab depending on the number of the void formers crossing the punching crack. Hence, a simple modification factor that depends on the number of void formers crossing the control perimeter was suggested.

Abramski et al. (2010) presented the shear behavior of the Cobiax - spherically voided biaxial slab using a combination of experiments and 3-D nonlinear finite element analysis software called DIANA. Thirteen large scale tests and corresponding nonlinear finite element analyses were conducted. The shear strength and crack patterns obtained from the experiments were in good agreement with the corresponding shear strength and crack patterns predicted from the nonlinear finite element analyses. The shear strength of all the tested SVBS was found to be at least 50 % of the shear strength of solid slabs of same dimensions.

Marais et al. (2010) experimentally investigated the shear resistance and their short-term elastic deflections of spherically voided biaxial slabs. Due to the loss of aggregate interlock associated with the voids, the shear resistance of the SVBS without steel cages is suggested to be taken as 55 % of a solid slab with the same thickness whereas with steel cage the tests showed that the shear resistance is increased to 85 %. The stiffness of the SVBS slab is determined to be approximately 90 % of a solid slab with
the same thickness. The authors suggested that the voided biaxial slab be designed in a similar fashion as a solid RC flat slab having subtracted the concrete weight displaced by the spheres.

Chung et al. (2011) presented experimental studies of the flexural capacities of biaxial hollow slab with donut type void formers. Three specimens, solid RC slab, a biaxial hollow slab with cage and a biaxial hollow slab without cage, were tested using a multi-loading frame. The peak strengths of the biaxial slabs with and without cage were found to be about 94 % and 99 % of that of the solid slab, respectively. The test results also showed that the crack pattern and behavior were different by the presence or absence of cage.

2.4 Review of Micromechanics Approach

An SVBS system is a periodic composite structure because of the non-homogeneous constituent materials, their geometry and topology that makes up the slab system. Characterization of the mechanical properties of composites is very important for determining the structural capacities and behaviors of such structures, which can preferably be done by micromechanics approach as it integrates the relevant materials properties of the constituents, their geometry and topology. With advancements in the use of composite materials as structural elements, micromechanics based modeling has become an important means of understanding their behaviors. The micromechanics methods used to develop constitutive models on some periodic composite structures, such as woven textile composites beams (Sankar and Marrey 1993), corrugated cardboard (Aboura et al. 2004), and piezoelectric smart composite shells with hexagonal
honeycomb configuration (Saha et al. 2007) are the inspirations for this dissertation. A review of some relevant research that utilizes micromechanics approach of determining behavior of composite structures is included below.

Libove and Hubka (1951) developed a theory for corrugated-core sandwich plates that focused on the prediction of displacements and membrane stresses based on Reissner–Mindlin plate theory, which considers shear deformation. However, the secondary bending stresses caused by the shear deformation in the direction perpendicular to the web plate were not captured in this research.

Caillerie (1984) showed that an in-plane periodic thin plate could be modeled as a homogeneous Love-Kirchhoff plate if the typical size of the unit cell that generates the plate is very small in comparison with the typical in-plane size of the structure.

Sankar and Marrey (1993) used a unit cell model analysis to determine the flexural stiffness properties of a textile composite beam. The unit cell was modeled using eight-node plane strain finite elements. Three linearly independent deformations, which were pure extension, pure shear and pure bending, were applied to the unit cell model. From the actions required to cause such deformations, the extensional, flexural and shear stiffness of the textile composite beam were calculated. Special constraints elements were also used to consider periodic boundary conditions on the end faces of the unit cell. This approach was validated by applying to isotropic and biomaterial beams, and making comparisons of results with the beam theory and the lamination theory. The comparisons showed excellent agreement. After validation of the model, the method was used to obtain the stiffness coefficients of a plain weave composite modeled as a beam. However,
the transverse shear stiffness coefficient was obtained by a modified method in which the shear strain energy in the beam model was equated to the shear strain energy of the unit cell in the finite element model.

Fung et al. (1994) determined the elastic constants for a Z-core sandwich panel by transforming the panels into an equivalent two dimensional homogenous orthotropic thick plate. The values of the elastic constants were validated based on the comparisons of results obtained from the transformed two dimensional thick plate bending theory with a detailed three dimensional finite element model. These comparisons showed that excellent agreement was obtained between the analytical and finite element models.

Bendaryck (2000) used NASA’s Micromechanics Analysis Code with Generalized Method of Cells (MAC/GMC) to predict the elastic properties of plain weave polymer matrix composites (PMCs). A two-step homogenization procedure was utilized that could enable the accurate prediction of woven PMC elastic properties with MAC/GMC instead of the traditional one step three-dimensional homogenization procedure that had been used in conjunction with MAC/GMC for modeling woven composites, which was believed to be inaccurate due to the lack of shear coupling inherent to the procedure. The results from the two step MAC/GMC homogenization procedure compared favorably with results from several previous models developed for woven composites and experiments.

Buannic et al. (2003) computed the effective properties of corrugated core sandwich panels based on micromechanics approach. Due to their periodic structure, the homogenization theory was used, based on the asymptotic expansion method. First the
unit cell, which was made of the core and the facings, was analyzed. The purpose of homogenization was to substitute the thin initial heterogeneous structure with an equivalent homogeneous plate and its effective properties. By comparing the response of the equivalent homogeneous plates with that given by detailed finite element models of sandwich panels, the results were shown to be in good agreement.

Talbi et al. (2004) developed an analytical model to predict the elastic behavior of corrugated cardboard. The model was based on micromechanical homogenization approach that took into account the geometrical and mechanical properties of the corrugated cardboard constituents. When the elastic behavior of the corrugated cardboard was analyzed, the model used point-wise lamination approach using the classical laminate theory. A unit cell representative of the corrugated cardboard was defined. This unit cell considered the skins and the undulated fluting as an assembling of many infinitesimal elements of unidirectional lamina oriented at different angles. An experiment was also carried out to obtain both the in-plane elastic properties of each constituents and the corrugated cardboard. The analytical model was validated by comparing it with the experimental results. Furthermore, three dimensional finite element analyses were conducted. Results from these analyses were compared with the simplified homogenized model. The simplified homogenized model is shown to be adequately accurate and ten times faster than the three dimensional finite element approach. Hence, the analytical model could be used for effectively analyzing corrugated cardboard panel in the preliminary and optimum design stages. A parametric study was also presented investigating the effect of geometrical parameters on in-plane elastic properties.
Milani et al. (2005) presented a micromechanical model for homogenized limit analysis of masonry walls subjected to out-of-plane loading condition. In the homogenization process, a polynomial expansion for the stress fields along with subdividing the masonry thickness into several layers was used. Then, a linear optimization approach was applied to obtain a failure surface for a homogenized masonry work. The comparisons with both experimental data and previously developed incremental numerical procedures showed that the proposed micromechanical model provided reliable results.

Martinez et al. (2006) used a representative volume element based micromechanics approach to investigate a composite truss-core sandwich panel as an alternative for an Integral Thermal Protection System (ITPS) under different web angles. This system is required to protect a space vehicle from both extreme reentry temperatures and mechanical loads. The sandwich structure is modeled as an equivalent orthotropic thick plate continuum. The representative volume element was analyzed to compute the extensional stiffness matrix, coupling stiffness matrix, bending stiffness, and the transverse shear stiffness terms by taking strain energy and equilibrium equations. After these stiffness matrices were verified using three dimensional finite elements, a closed form solution was derived based on first order shear deformable plate theory. The results showed that the composite truss-core sandwich panel with rectangular webs resulted in a weak extensional, bending, and transverse shear stiffness and that maximum plate deflection was greatest for 48° web angle configuration.
Saha et al. (2007) developed a micromechanical model for piezoelectric smart composite shells with hexagonal honeycomb configuration. The effective elastic and piezoelectric coefficients of the homogenized shell were determined from solutions of a derived set of three-dimensional (3D) unit cell problems. Hence, closed-form expressions for the effective coefficients pertaining to hexagonal honeycomb shells could be obtained.

Romanoff and Varsta (2007) improved the theory developed by Libove and Hubka (1951) for the bending response of unsymmetrical web-core sandwich structure. The theory included the periodic nature of local deformation and resulting stress components. The theory also considered the effect of thick-face-plates. This improved theory can be used to analyze structures even with very low shear stiffness. The plates’ response was evaluated by transforming the actual discrete core into an equivalent homogenous continuum thick plate. This approach utilized analytical formulations for the determination of the equivalent stiffness properties of the plate. For a given load and boundary conditions, the internal forces and displacements were obtained from a finite element method. Thereafter, the internal forces and displacement were applied on the actual periodic structures to predict the stresses based on analytical approaches. A three dimensional finite element was used to validate the theory under various combinations of load and boundary conditions. A good agreement was observed between the three dimensional finite element analysis and the developed theory.

Cecchi and Sab (2007) presented a new procedure for the homogenization of orthotropic three dimensional periodic plates to find equivalent two dimensional
Reissner-Mindlin plates that accounts for the shear effects of the plate by extending homogenous Love-Kirchhoff model. A three dimensional boundary value problem was used on the unit cell that generates the periodic plate to determine the shear constants. The results from this procedure on a unit cell was compared with results from three dimensional finite elements and found that they were in good agreement. This homogenization procedure is then applied to periodic brickwork panels under cylindrical bending conditions. The results were compared with full three dimensional finite elements on heterogeneous and an equivalent Love-Kirchhoff plate models to investigate the discrepancy among these three models as a function of the slenderness of the panel. The results indicated that the proposed thick two dimensional Reissner-Mindlin model is reliable as it captures the effect of shear in a brick panel.

Cecchi and Sab (2009) proposed a homogenized procedure for finding the bending stiffness of a two dimensional regular lattice with random local interactions. The kinematic and static approaches were utilized to obtain the limits of the homogenized moduli. This procedure was applied to a masonry structure with periodic bonds and the results were validated by comparing it with results from a discrete model. The model showed that when the heterogeneities in a structure has periodicity, the average response of the periodic discrete model becomes very close to the response of the deterministic homogenized model as far as the size of the periodic discrete model is very small compared to the structure.

Talbi et al. (2009) presented an analytical homogenization model for corrugated cardboard and its numerical implementation in a shell element. In this model, the
geometric and mechanical properties of the corrugated board components were considered in determining an elastic stiffness matrix to the generalized strains and stresses for an equivalent orthotropic plate. The notion of the procedure is that instead of using a local constitutive law (law of strains–stresses) in every point, the homogenization process leads to a generalized constitutive law (relating the generalized strains with the resultant stresses) for an equivalent homogenous plate. Reissner-Mindlin thick plate approach with shear correction factor of 1.2 was used. The comparison of the results obtained by the present model, a complete 3D shell modeling and the experiments showed that homogenized model was efficient and accurate.

Challagulla et al. (2010) presented an asymptotic homogenization model for smart composite orthotropic grid reinforced shells. In this model, asymptotic expansions, method of two scales and the homogenization technique were involved. Very often composite and smart structures are used in the form of thin plates and shells. Besides, these structures have a periodic or nearly periodic configuration with a period much smaller than their overall dimensions; hence asymptotic homogenization techniques become particularly attractive.

Attanayake et al. (2011) developed a micromechanical based macromechanical analysis model for orthotropic bridge decks. The model simplified the complex non-homogeneous deck structures for three dimensional load response analyses. This model also refined the analyses when compared with the current bridge design codes. It used a selection of representative volume element from which extensional, flexural, torsional and shear stiffness were determined by the use of finite element analyses. The model was
applied on a box-beam bridge superstructure as an example. In addition, the model was verified by conducting a static patch load analysis and carrying comparison of results with conventional three dimensional finite elements. The conclusion was that the macromechanical model developed offered a simplified and reasonably accurate three dimensional representation of orthotropic bridge structure when compared to finite element modeling. Unlike the three dimensional finite element model, it requires less expertise and insignificant computational time.

Xia et al. (2012) developed a homogenization based analytical model that could be valid for any corrugation shape used in roof structures in buildings and morphing aircraft skin. The model was developed based on a simplified geometry for a unit-cell and the stiffness properties of the original sheet. Energy and equivalent force methods were used to estimate the equivalent stiffness of the corrugated panels by analyzing the representative volume element during homogenization. The stiffness coupling was also taken into account for the sake of accuracy. Finite element analyses were used to verify the results. The difference between the stiffness properties from the developed model and the finite element model was less than 1%.

2.5 Modeling Approaches

Continuum mechanics deals with the analysis of the kinematics and mechanical behavior of materials modeled as solids consisting of material points and material neighborhoods, by assuming that the material distribution, the stresses and the strains within an infinitesimal material neighborhood of a typical point are essentially uniform
(Nemat-Naser and Hori 1993). However, at a relatively smaller scale, the infinitesimal material neighborhood may be characterized by local heterogeneity, due to its differing constituent material components, shape variations, presence of voids, etc. Hence, the actual stress and strain fields in the vicinity of the heterogeneity cannot be uniform at this level. The voids from the plastic balls and the reinforcing bars embedded in the SVBS system make it a distinctly heterogeneous medium, for which the principle of continuum mechanics cannot be applied directly.

This heterogeneous nature has a significant impact on the observed macroscopic behavior of the SVBS system. Various phenomena occurring on the macroscopic level originate from the mechanics of the underlying microstructure. The overall behavior of micro-heterogeneous materials depends strongly on the size, shape, spatial distribution and properties of the microstructural constituents and their respective interfaces.

Determination of the macroscopic characteristics of heterogeneous media is an essential problem in many engineering applications. Studying the relation between microstructural phenomena and the macroscopic behavior not only allows to predict the behavior of existing structures, but also provides a tool to design a configuration of microstructure in such a way that the resulting macroscopic behavior exhibits the required characteristics. For instance, the deflection, \( w \), of a rectangular SVBS with assumed flexural rigidity, \( D \), under a distributed uniform load, \( q \), can be determined by the biharmonic formula given in Equation 2-1:

\[
DV^4w = q(x, y) 
\]  

(2-1)
However, the flexural rigidity of the slab, $D$, is unknown due to the complicated internal geometry of the SVBS system. This is partly why this dissertation is very important in advancing the computational methods for complicated geometries used in civil engineering.

Generally, there are three fundamentally different approaches to obtain continuum mechanics based constitutive equations for predicting the structural behavior (responses) of an SVBS. These are:

- Phenomenological approach,
- Numerical approach,
- Micromechanics approach.

The phenomenological approach is based on conducting experiments on test specimens whose dimensions are large compared to the representative volume element (RVE) to determine all stiffness matrices. The stiffness matrices obtained from the experiments are used to determine parameter estimates for an assumed constitutive model to predict the macroscopic/structural behavior of the slab. This approach certainly offers the best advantage for isotropic materials, because the response characteristics of the materials can be well described based on few tests. However, the anisotropic and non-homogeneous nature of the slab system greatly magnifies the labor and expense involved in the experimental determination of the overall mechanical properties. To describe a full three-dimensional constitutive model of the SVBS, its mechanical properties in all three orthogonal planes have to be obtained, which is almost impractical using testing results.
only. Therefore, this approach is not selected in my research for developing an analytical model for SVBS.

The numerical approach is based on a detailed finite element analysis, which is a powerful technique for carrying out structural analysis. This approach can be applied to predict the mechanical properties of SVBS. However, conducting 3D finite element analysis on a large (full scale) SVBS is very challenging since the geometry is very complex and the three dimensional mesh generations is very laborious and time-consuming, particularly if contact algorithms are involved. The computational effort to analyze an SVBS using a detailed three dimensional finite-element can be greatly reduced by modeling it as an equivalent thick-orthotropic plate. In this dissertation, finite element analysis will be used to analyze only the RVE as a part of the micromechanics approach discussed below.

The micromechanics approach assumes a constitutive behavior for each component material contained within a test specimen that can be used by the phenomenological approach described above. Typically, the behavior of the component materials is described by less complex constitutive models than the composite material because component models are usually assumed to be isotropic. An RVE based micromechanics method decouples analysis of a composite material into analyses at the local (micro) and global (macro) levels. The local level analysis models the microstructural details to determine effective elastic properties. The local level analysis can also be used to calculate the relationship of the effective or average RVE strain to the local strain within the RVE. The non-homogenous structure is then replaced by an
equivalent homogeneous material with the calculated effective properties using either analytically or finite element analysis. The global level analysis calculates the effective or average stress and strain within the equivalent homogeneous structure. This process of averaging the micromechanics solution to obtain the locally averaged constitutive equations is often termed as homogenization. When a homogenization procedure is applied to the micromechanics solution, the resulting macroscopic equations should, in principle, be equivalent to that which would be obtained by the phenomenological approach. The added benefit of micromechanics is that if local stress and strain estimates are needed, they can be computed using the relationship between the average and local strain obtained from the local analysis. This procedure is termed as localization.

2.6 Extended Discussion on Homogenization

Homogenization is a method to determine the apparent or overall properties of a heterogeneous material replacing it with an equivalent homogeneous material (Temizer and Zohdi 2006). An SVBS has a periodic discrete unit cell also known as representative volume element (RVE), which is the building block of the slab system and very small compared to the typical in-plane size of the slab. As a periodic composite structure, the slab is characterized by two different spatial scales: (1) a micro-scale and (2) a macro-scale. A micro-scale is in the order of a representative volume element as shown in Figure 2-4. A macroscopic scale is of the same order of magnitude as the overall in-plane dimensions of the structure. Because the physical system of equations describing the behavior of such structures will have both of these scales coupled together, its solution becomes very difficult. The structural length, L, of the slab greatly exceeds the RVE
length, $\ell$ ($\lambda = \ell/L << 1$), implying that the necessary condition is met for the homogenization method to be valid (Manevitch et al. 2002). Homogenization provides the ability to determine structure level properties such as deflection and vibration from micro-(constituent) level properties while localization provides the ability to determine constituent level responses from macro-(structure) level results such as strains and stresses (Arnold et al. 2010). The main benefit of this method is that the periodicity is transferred into an equivalent homogenous stiffness of the SVBS, reducing the number of unknowns of the general shell/plate-bending problem significantly. In general, the homogenization is conducted using detailed 3D finite element analyses; however, where heterogeneity is relatively less complex, mathematics of homogenization can be potentially applied.
In order to understand the key point of the homogenization procedure, let us consider components that make up an RVE of the SVBS. The micro-constitutive law that governs each component is given by the standard elastic constitutive law. On the other hand, the strains and stresses at the macro-level are directly related to the structural level analysis. The homogenized effective macro-level response of SVBS can be obtained based on average stress and strain theorems (Kassem 2009). The average strain theorem states that for any perfectly bonded material within the RVE and for an exterior homogeneous displacement given on the entire boundary of the RVE, the volume average of the strain is the applied displacement on the boundary. On the other hand, average
stress theorem states that for a given uniform external load on a given RVE the volume average of the forces within the RVE is identical to the given force on the boundary (Loehnert 2004).

On the macro-level, an RVE is considered as a single point with a homogenized constitutive law. The macro-stress, \( \overline{\sigma}_{ij} \), is usually defined as the volume average stress in an RVE, \( \langle \sigma_{ij} \rangle \) and is given by Equation 2-2.

\[
\overline{\sigma}_{ij} = \langle \sigma_{ij} \rangle_{\Omega} = \frac{1}{V} \int \sigma_{ij} \, d\Omega
\]  

(2-2)

where \( \Omega \) is the domain of the RVE and \( V \) is its volume. Similarly, the volume average strain, \( \overline{\varepsilon}_{ij} \), and the volume average strain energy density, \( \overline{w} \), in an RVE are given by Equations 2-3 and 2-4, respectively.

\[
\overline{\varepsilon}_{ij} = \langle \varepsilon_{ij} \rangle_{\Omega} = \frac{1}{V} \int \varepsilon_{ij} \, d\Omega
\]  

(2-3)

\[
\overline{w} = \frac{1}{V} \int w \, d\Omega = \frac{1}{V} \int \left( \frac{\sigma_{ij} \varepsilon_{ij}}{2} \right) \, d\Omega = \frac{1}{V} \int \left( \frac{D_{ijkl} \varepsilon_{ij} \varepsilon_{kl}}{2} \right) \, d\Omega = \frac{1}{V} \int \left( \frac{C_{ijkl} \sigma_{ij} \sigma_{kl}}{2} \right) \, d\Omega
\]  

(2-4)

where \( w \) is strain energy density, \( D_{ijkl} \) are local (component) stiffness coefficients and \( C_{ijkl}(C = D^{-1}) \) are local (component) compliance coefficients. Moreover, the macroscopic strain should satisfy Equation 2-5.

\[
\overline{w} = \frac{\overline{\sigma}_{ij} \overline{\varepsilon}_{ij}}{2}
\]  

(2-5)
The effective properties represented by effective stiffness $D_{ijkl}$ or effective compliance $C_{ijkl}$ of the SVBS can be given in terms of the average stress and strain as shown in Equation 2-6.

$$\bar{\sigma}_j = \bar{D}_{ijkl} \bar{e}_{kl}, \quad \bar{e}_j = \bar{C}_{ijkl} \bar{\sigma}_j$$  \hspace{1cm} (2-6)

or by equivalence of the strain energy as shown in Equation 2-7 and 2-8.

$$\frac{\bar{\sigma}_j \bar{e}_j}{2} = \frac{1}{V} \int \left( \frac{\sigma_{ij} \varepsilon_{ij}}{2} \right) d\Omega$$  \hspace{1cm} (2-7)

$$\frac{\bar{D}_{ijkl} \bar{e}_{ij} \bar{e}_{kl}}{2} = \frac{1}{V} \int \left( \frac{D_{ijkl} \varepsilon_{ij} \varepsilon_{kl}}{2} \right) d\Omega$$  \hspace{1cm} (2-8)

The linearity of the stress-strain relation for elastic body leads to Equation 2-9.

$$\bar{D}_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$$  \hspace{1cm} (2-9)

The effective quantities of the stress, strain and strain energy can be calculated by corresponding boundary values with surface average procedures. In the case of small strain, the strains are given by Equation 2-10.

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$  \hspace{1cm} (2-10)

where $u_i$ are displacement components. Applying the divergence theorem in Equation 2-3 yields to Equation 2-11.

$$\bar{\varepsilon}_{ij} = \frac{1}{2V} \int (u_i n_j + u_j n_i) d\Gamma$$  \hspace{1cm} (2-11)

where $\Gamma$ is the boundary of the RVE, and $n_i$ is the outward normal vector on boundary $\Gamma$. 

32
By using the divergence theorem again, the volume average stresses in Equation 2-2 can also be expressed as Equation 2-12.

\[
\bar{\sigma}_j = \frac{1}{2V} \int (T_i x_j + T_j x_i) d\Gamma
\]  

(2-12)

where \(x_i\) are the Cartesian coordinates, \(T_i\) are the traction components acting on the surface of the RVE. It is obtained from Equation 2-12 that the volume average stresses are related only to the tractions on the boundary of the RVE.

In addition, the average strain energy can be expressed by the boundary values according to the work-energy principle given in Equation 2-13.

\[
\bar{w} = \frac{1}{V} \int \left( \frac{\sigma_{ij} \varepsilon_{ij}}{2} \right) d\Omega = \frac{1}{V} \int T_i u_i d\Gamma
\]  

(2-13)

SVBS can be regarded as a periodic array of its RVE. Therefore, the periodic boundary conditions must be applied to the RVE. This implies that each RVE in the SVBS has the same deformation mode and there is no separation or overlap between the neighboring RVEs. The periodicity conditions on the boundary \(\partial V\) of the RVE is given by Equation 2-14 (Manevitch et al. 2002, Xia et al. 2003).

\[
u_i = \bar{\varepsilon}_{ij} x_j + u_i^* \]  

(2-14)

where \(\bar{\varepsilon}_{ij}\) is the average strain \(u_i^*\) is the periodic (fluctuating) part of the displacement components on the boundary surfaces and it is generally unknown and dependent on the applied global loads. For instance, considering an RVE under multi-axial loading condition, the displacements on a pair of opposite boundary surfaces are given by Equation 2-15 and 2-16.
\begin{align}
\mathbf{u}_i^{j^+} &= \bar{\varepsilon}_{ij} \mathbf{x}_k^{j^+} + \mathbf{u}_i^* \quad (2-15) \\
\mathbf{u}_i^{j^-} &= \bar{\varepsilon}_{ij} \mathbf{x}_k^{j^-} + \mathbf{u}_i^* \quad (2-16)
\end{align}

where index $j^+$ means along the positive $x_j$ direction and index $j^-$ means along the negative $x_j$ direction. Subtracting Equation 2-16 from Equation 2-15 results in Equation 2-17.

\begin{align}
\mathbf{u}_i^{j^+} - \mathbf{u}_i^{j^-} &= \bar{\varepsilon}_{ij} (\mathbf{x}_k^{j^+} - \mathbf{x}_k^{j^-}) = \bar{\varepsilon}_{ij} (\Delta x_k^j) = c_i^j \ (i, j = 1, 2, 3) \quad (2-17)
\end{align}

For the RVE, $\Delta x_k^j$ is constant, therefore the periodic boundary conditions is obtained. The constants $c_i^j$ represent the average stretch, contraction, or shear deformation of the RVE. This form of boundary conditions meets the requirement of displacement periodicity and continuity. Equation 2-17 shows that although the difference of the displacements for the corresponding points on the two opposite boundary surfaces are specified, the individual displacement component is still a function of the coordinates.
CHAPTER 3: MICROMECHANICAL HOMOGENIZATION OF SVBS

3.1 Introduction

The objective of micromechanical homogenization theory is to establish the macroscopic behavior of a system which is ‘microscopically’ heterogeneous by describing the effective (overall) characteristics of its heterogeneity (for instance, its bending stiffness, extensional stiffness or conductivity). The heterogeneous material is replaced by a ‘homogeneous’ material whose global characteristics are a good representation of the actual system.

SVBS is a heterogeneous composite structure in which building a representative continuum model is a challenge. Consequently, there is a need to determine the mechanical properties of the SVBS in order to predict the structural capacity and behavior of the slab system using the micromechanical homogenization method. This method provides overall behavior of the SVBS system from known properties of its constituents (concrete, rebars) through analysis of a periodic representative volume element (RVE) (Aboudi, 1991, Xia et al 2002, Kassem 2009). In the macromechanical approach, on the contrary, the heterogeneous structure of the slab is replaced by a homogenous medium with anisotropic properties.

This chapter presents the global mechanical properties, i.e. the ABDE stiffness matrix, of the SVBS as determined by micromechanics based detailed 3D-finite element
(FE) simulations in the absence of experimental data. FE simulations play a crucial role in the analysis of the mechanical behavior of structural elements built with complex microstructure composite materials such as SVBS. In order to define microstructural details, FE analysis of full scale SVBS (macroscopic level) without homogenization often leads to the need for unstructured meshes and large numbers of finite elements. This fact frequently makes it impossible to perform numerical analyses on the mechanical behavior of such structural components, due to the large amounts of required computer memory and CPU time. In this particular context, the homogenization method leads to significant computational savings.

In micromechanical homogenization, a representative volume element (RVE) containing all the geometric and constitutive information of the SVBS is employed. The method is certainly a valuable research tool and also a viable alternative to the costly and often time-consuming laboratory experiments. Achieving substantiation of structural performance by testing alone can be prohibitively expensive because of the number of specimens and components required to characterize all material systems, loading scenarios, and physical and boundary conditions.

The use of FE-models is also a valuable part of the method because the micromechanics require several parameters at the micro-level which are difficult to determine at a macro-level analysis using only analytical methods. Thus, the homogenization method implemented herein to obtain the equivalent material properties of SVBS can solve the challenge of building a continuum model for SVBS.
In the homogenization process used here, the RVE is subjected to eight linearly independent deformations (including displacements and rotations). In general, while one of the deformations is prescribed to be non-zero, all other deformations are prescribed to be zero by applying the appropriate boundary conditions. The equivalent reaction forces and moments at the constrained boundaries of the RVE are computed from the FE-model analysis using ANSYS. Then, based on the FE analysis results, the equivalent material properties such as the stiffness coefficients of the RVE are determined. In general, four important homogenization procedures are utilized for predicting the stiffness parameters or elastic constants of SVBS using a micromechanical based three-dimensional FE analysis. These are

1. Identification and modeling of RVE
2. Application of suitable boundary conditions to generate characteristic deformation modes
3. FE solution to the deformation modes
4. Determination of effective properties from evaluation of reaction forces and moments

The detailed case by case procedures are presented in sections 3.2 through 3.4.

### 3.2 Geometry of the RVE & Assumptions

A typical unit of SVBS is selected to serve as a representative volume element (RVE). The RVE is the primary block of the structure that repeats itself. The RVE is modeled numerically to calculate the equivalent stiffness coefficients (elastic constants)
of SVBS material. The following assumptions are made: the SVBS is macroscopically homogeneous, linearly elastic, macroscopically transversely isotropic and initially stress free (no thermal stress); the rebars are homogeneous, linearly elastic, isotropic, regularly spaced, and perfectly aligned and bonded with concrete; and the concrete is homogeneous, linearly elastic and isotropic.

In this dissertation, a Cartesian coordinate system shown in Fig. 3.1 is used. The global coordinate system (x, z, y) and local coordinate system (1, 2, 3) can be interchangeably used when dealing with RVE. The in-plane location is defined by x-z or 1-2 axes whereas the out of plane location is defined by y or 3 axis.

![Figure 3-1: Global and local Cartesian coordinate systems](image)

38
For the sake of detailed demonstration of the homogenization process used to obtain the ABDE stiffness matrix, one of the typical SVBS configurations is selected. This configuration, namely SVBS 340, is shown in Fig. 3.2. Similar approaches are used later on to determine the ABDE stiffness coefficients of SVBSs and comparative solid slabs of multiple configurations. These stiffness coefficients are presented in chapter 5.

![Figure 3-2: Representative volume element extraction from SVBS 340](image)

There are two RVEs with distinct geometries that can be equally regarded as the smallest building block for the SVBS 340. The first RVE is described by the region
surrounded by vertical cutting planes passing through the middle of the concrete regions surrounding a spherical void former in all four sides as shown in Fig 3.2. The second RVE is the region bounded by vertical cutting planes passing through the center of four closest plastic balls. Either one of them can be interchangeably used, but in this chapter the first RVE type is exclusively used. The second RVE type is used for comparison purpose in chapter 6.

The RVE describes a representative part of the SVBS with all relevant components and statistical homogeneity. Therefore, the properties of the RVE can be assumed to be equivalent to the properties of the SVBS. The statistical homogeneity requires the dimensions of the RVE, $l_x$ and $l_z$, to meet the criterion $l_x << a$ and $l_z << b$, where $a$ and $b$ are the length and width of the SVBS. The RVE for SVBS 340 has external dimensions of 300 mm in length, 300 mm in width, and 340 mm in depth. The diameter of the void is 270 mm and is located at the center of RVE. It has a total of 8 #4 rebars with a diameter of 13 mm as shown in Figure 3-2.

Equations 3-1, 3-2 and 3-3 show the normal force, moment, and transverse shear relationships for the RVE respectively, where $\varepsilon_1$, $\varepsilon_2$, and $\gamma_{12}$, are the normal and shear strains at the middle surface; $\kappa_1$, $\kappa_2$, $\kappa_3$ are the curvatures; and $\gamma_{13}$ and $\gamma_{23}$ are the transverse shear strains.

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix}$$  (3-1)
\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 
\end{bmatrix} =
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} +
\begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{bmatrix}
\begin{bmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]
\[ (3-2) \]

\[
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} =
\begin{bmatrix}
E_{55} & 0 \\
0 & E_{66}
\end{bmatrix}
\begin{bmatrix}
\gamma_{13} \\
\gamma_{23}
\end{bmatrix}
\]
\[ (3-3) \]

where \( N_1, N_2 \) and \( N_3 \) are the normal and shear forces per unit width, \( M_1 \) and \( M_2 \) are the bending moments per unit width in the \( x \) and \( z \) directions respectively, \( M_3 \) is the twisting moment per unit width and \( Q_1 \) and \( Q_2 \) are the transverse shear forces per unit width.

The SVBS being considered here is symmetric around the middle surface as far as geometry and material properties are concerned. For that reason, the coupling matrix, \( B_{ij} \), vanishes. Further simplifications can be made if the SVBS is extensionally balanced (no shear coupling), i.e. \( A_{13} = A_{31} = A_{23} = A_{32} = 0 \), or flexurally balanced (no coupling between bending and twisting), i.e. \( D_{13} = D_{31} = D_{23} = D_{32} = 0 \).

When performing FE analyses, the strains and curvatures are prescribed and the resulting reaction moments and/or forces are obtained. Then, by applying the load vs. deformation relationships, the stiffness coefficients are determined. The reaction resultant forces and moments are written in terms of the middle surface extensional strains, curvatures and in-plane and transverse shear strains with the ABDE stiffness matrix.

### 3.3 ANSYS FE Modeling

This section explains some of the important procedure used to create the FE analysis model. There are series of tasks that have to be completed for the FE model to be
constructed and run properly. Models can be created using the command prompt line input or the Graphical User Interface (GUI). For FE model presented in this chapter, the GUI was utilized to create the model. The appropriate group in the ‘Toolbox’ is selected with the Analysis Systems group. The appropriate template, Static Structural, is selected.

The ‘Engineering Data’ cell is used to input material properties and engineering data for the analysis. Then, the material properties of concrete and steel are selected and defined. The isotropic elasticity material properties used in this chapter are provided in Table 3-1. The geometric parameters are provided in Table 3-2. The geometry of the RVE is built by launching the ‘Design Modeler’ as shown in Figure 3-3.

### Table 3-1: Material input data

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Concrete</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density (kg/m$^3$)</td>
<td>2300</td>
<td>7850</td>
</tr>
<tr>
<td>Young’s Modulus (GPa)</td>
<td>30.0</td>
<td>200</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.18</td>
<td>0.3</td>
</tr>
<tr>
<td>Bulk Modulus (GPa)</td>
<td>15.625</td>
<td>166.7</td>
</tr>
<tr>
<td>Shear Modulus (GPa)</td>
<td>12.712</td>
<td>76.923</td>
</tr>
</tbody>
</table>

### Table 3-2: Geometrical Data

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length in x-direction (mm)</td>
<td>300</td>
</tr>
<tr>
<td>Length in y-direction (mm)</td>
<td>340</td>
</tr>
<tr>
<td>Length in z-direction (mm)</td>
<td>300</td>
</tr>
<tr>
<td>Diameter of the void (mm)</td>
<td>270</td>
</tr>
<tr>
<td>Volume (mm$^3$)</td>
<td>$2.0294 \times 10^7$</td>
</tr>
<tr>
<td>Mass (kg)</td>
<td>48.364</td>
</tr>
<tr>
<td>Saving of Concrete (%)</td>
<td>35%</td>
</tr>
</tbody>
</table>
All boundary conditions, prescribed displacements and/or loads are then defined by the ‘Setup’ cell followed by ‘Edit’ or double-clicking on the ‘Model’ cell in the same analysis system schematic. The ‘Mechanical’ application opens and displays the geometry. In this part, materials are assigned to the constituent parts. The model has a total of 9 parts, 8 of them are rebars and the remaining is concrete. The stiffness behavior is also defined here.

Contact regions between rebars and the surrounding concrete are selected as bonded connection. The contact regions are made up of faces whereas the bodies are solids. These faces are detected by the program automatically. Bonded connection implies that no sliding or separations between faces are allowed. Figure 3-4 shows the eight defined bonded regions used in this model.
When the ‘Meshing’ application is launched from the ANSYS Workbench Project Schematic, the physics preference is set based on the type of the system being edited. For a mechanical model system, such as the one used in this analysis, the mechanical physics preference is selected. The preferred meshers for mechanical analysis are the patch conforming meshers (Patch Conforming Tetrahedrons and Sweeping) for solid bodies. Relevance of 40 with medium sizing is selected, and the number of nodes and elements generated are found to be 751,592 and 490,516, respectively. Figure 3-5 shows the
generated mesh for the current FE analysis. The simulations are conducted using default elements generated.

![Figure 3-5: Meshing result for the analysis model](image)

SOLID187 element type is used for the concrete. SOLID186 element type is used for the rebars. CONTRA174 and TARGE170 types of elements are used for modelling the non-linear interactions between the concrete part and the rebars. SOLID187 element is a higher order 3-D, 10-node element. SOLID187 has a quadratic displacement behavior and is well suited to modeling irregular meshes. The element is defined by 10 nodes having three degrees of freedom per node: translations in the nodal x, y, and z directions. The element has plasticity, hyperelasticity, creep, stress stiffening, large deflection, and large strain capabilities. It also has mixed formulation capability for simulating
SOLID186 is also a higher order 3-D 20-node solid element that exhibits quadratic displacement behavior. The element is defined by 20 nodes having three degrees of freedom per node: translations in the nodal x, y, and z directions. The element supports plasticity, hyperelasticity, creep, stress stiffening, large deflection, and large strain capabilities. It also has mixed formulation capability for simulating deformations of nearly incompressible elastoplastic materials, and fully incompressible hyperelastic materials (ANSYS, Inc., 2012).
Figure 3-7: SOLID186 geometry

The ‘Mechanical’ application is used to step up the static structural analysis. The force convergence is activated with the default settings (0.5% tolerance about a program calculated value). One load step is defined. An overall analysis time of 1 second was used. This is not a physical time, but rather an arbitrary parameter within ANSYS which defines load steps. A value of 1 second is used to conveniently relate output results as a percentage of the applied load. Program controlled auto time stepping is used, meaning that the program will divide the load into 50 increments and apply them individually. In reality, the deformations of the RVE are small, but unit strains (very large deformations) are prescribed in the FE analysis. Hence, the large deformation effect is deactivated to avoid calculating a new stiffness matrix at each sub-step based on the deformed geometry. Boundary conditions, prescribed deformations or rotations and/or forces and
rotations are defined based on the particular stiffness coefficient being determined, and these are described in the following subsections.

### 3.3.1 Extensional Stiffness

The extensional stiffness coefficients can readily be determined by prescribing unit values to each strain component, independently, while retaining the remaining strain components fixed by providing necessary constraints to prevent rigid body translation and rotation of the RVE. The constitutive relation between the in-plane forces and the mid-plane strains yields the in-plane stiffness properties of the structure.

Let us first consider the case where $\varepsilon_1 = 1$, and $\varepsilon_2 = \gamma_{12} = 0$. The boundary conditions and prescribed displacements provided to simulate this case are given in Table 3-3 and Figure 3-8.

<table>
<thead>
<tr>
<th>Face of the FE-Model</th>
<th>Type of Boundary Conditions/ Prescribed Displacements</th>
<th>Displacement Component in x, y, z directions, resp. (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>@ x=0, on y-z plane</td>
<td>Displacement</td>
<td>0, free, free</td>
</tr>
<tr>
<td>@ x=Lx, on y-z plane</td>
<td>Displacement</td>
<td>300, free, free</td>
</tr>
<tr>
<td>@ y=0, on x-z plane</td>
<td>Free</td>
<td>free, free, free</td>
</tr>
<tr>
<td>@ y=Ly, on x-z plane</td>
<td>Free</td>
<td>free, free, free</td>
</tr>
<tr>
<td>@ z=0, on x-z plane</td>
<td>Displacement</td>
<td>free, free, 0</td>
</tr>
<tr>
<td>@ z=Lz, on x-z plane</td>
<td>Displacement</td>
<td>free, free, 0</td>
</tr>
</tbody>
</table>
The displacement contour under the given prescribed displacement and boundary conditions is shown in Figure 3-9. The resultant force at the restrained face at \( x = 0 \) (on \( y-z \) plane) under the prescribed unit strain is found to be \( R_1 = 1.7804 \times 10^9 \) N from ANSYS output as shown in Figure 3-10 and the reaction normal forces acting on both transverse
faces (at \( z = 0 \) and \( z = L_z \)) is found to be each \( R_2 = 2.653 \times 10^8 \text{ N} \) as shown in Figure 3-10. Hence, \( N_1 \) and \( N_2 \), which are forces per unit width at faces \( x = 0 \) and \( z = 0 \), are determined to be \( 5.9347 \times 10^6 \text{ N/mm} \) and \( 8.843 \times 10^5 \text{ N/mm} \), respectively using the following equations.

\[
N_1 = \{ (R_1 = \text{Resultant reaction force acting on the y-z face at } x=0)/L_z \} \quad (3-4)
\]
\[
N_2 = \{ (R_2 = \text{Resultant reaction force acting on the x-y face at } z=0)/L_x \} \} \quad (3-5)
\]

The extensional stiffness properties, \( A_{11} \) and \( A_{21} \) are then determined from the following relationships, which are derived using Equation 3-1. \( N_1 = A_{11} \varepsilon_1 \) and \( N_2 = A_{21} \varepsilon_1 \). Because the FE model is symmetric, there is no coupling between extension and bending, i.e. \( B_{ij} = 0 \), and the strains remain the same across the surface, \( \varepsilon^0 = \varepsilon_1 \).

Therefore \( A_{11} = N_1 \), and \( A_{21} = N_2 \). Only a known uniaxial strain is applied to the FE model in order to obtain the resultant reaction forces, which in turn are used to determine the extensional stiffness properties.

Similarly, a unit uniaxial strain, \( \varepsilon_2 = 1 \) is applied where \( \varepsilon_1 = \varepsilon_3 = 0 \) and the resultant forces can be calculated and then used to determine \( A_{12} \), and \( A_{22} \). However, since the geometry in the x and z directions are the same, \( A_{22} = A_{11} \) and \( A_{12} = A_{21} \).
Figure 3-9: Displacement contour plot when $\varepsilon_1 = 1$, and $\varepsilon_2 = \varepsilon_3 = 0$

Figure 3-10: Resultant force, $N_1$, at face $x=0$ when $\varepsilon_1 = 1$, and $\varepsilon_2 = \varepsilon_3 = 0$
According to the stiffness matrix in Equation 3-1, there is no coupling between shear forces and tensile deformations. Theoretically, the stiffness coefficient $A_{33}$ is obtained by prescribing pure shear strain in the RVE. However, this would be only valid if the thickness of the model being considered is too small causing the moment accompanying the shear force to become negligible. The RVE model being considered
here is proportional in dimensions in all three coordinate directions. Therefore, it is
difficult to prescribe pure shear strain to the model without having a coupling effect with
flexure because a shear force associated with the apparent shear strain applied in one of
the faces will produce a moment as well.

Homogenization theory assumes that every RVE is subjected to identical
deformations which are valid under extensional forces or bending moments. Whenever a
shear force is applied at the face of the RVE, the shear force is constant along the entire
length, and also results in a bending moment. However, the bending moment will vary
linearly along the RVE because by definition the shear force is \( \frac{\partial M}{\partial x} \), which is the
gradient of bending moment. This variation of the bending moment along the length of
the structure violates the homogenization assumption that every RVE is subjected to
identical force and moment resultant and deformations. Therefore, only analyzing the FE
model to obtain the average transverse shear stiffness of the RVE is not possible. Hence,
a different approach that combines the FE and the analytical method based on
Timoshenko’s beam theory is utilized to determine the shear stiffness. Consider the RVE
as a cantilever beam of length, \( L_x \). The beam (RVE) is clamped at the face \( x = 0 \) (on
plane \( y-x \)). The beam is subjected to a couple and an in-plane shear force alternatively at
the free face. The boundary conditions and applied loads are described in Table 3-4 and
Figure 3-13. The tip deflections of the beam are obtained from FE analysis for the applied
moment and in-plane shear force. Remote force is used in ANSYS to apply the in-plane
shear force. The remote force option rather than the force option is used when the force is
supposed to act equally in all five faces (four rebar faces and one concrete face) at the
face \( x = L_x \). The tip deflections are also determined analytically based on Equation 3-6 and 3-7, and correlated with the corresponding results obtained from FE analysis to determine the in-plane shear stiffness.

\[
v_{tip} = \frac{ML_x^2}{2EI}
\]

(3-6)

\[
v_{tip} = \frac{FL_x^3}{3EI} + \frac{FL_x}{bA_{33}}
\]

(3-7)

where \( v_{tip} \) is the tip deformation in \( z \) direction, \( M \) is the applied moment on \( x = L_x \) in negative \( y \)-directions, \( EI \) is the flexural rigidity of the RVE, \( b \) is the length of the RVE in the \( z \)-direction, and \( F \) is an applied force in the \( z \)-direction at face \( x = L_x \).

In the model being considered, first an arbitrary end couple, \( M_y = 2 \times 10^{10} \) N-mm is applied at the tip of the beam in the negative \( y \) direction and the corresponding tip deflection was determined from the finite element analysis output. Under this moment, the tip deflection in the \( z \) direction is obtained to be 50.236 mm as shown in Figure 3-14. The tip deflection is equated with Equation 3-6 above and the flexural rigidity (\( EI \)) of the RVE is determined to be \( 1.7915 \times 10^{13} \) N-mm\(^2\). The couple is then removed and an arbitrary transverse force, \( F_z = 2 \times 10^7 \) N, is applied at the tip of the cantilever beam (RVE). The tip deflection obtained from the finite element output after analysis is 23.226 mm as shown in Figure 3-16. This tip deflection is then equated with Equation 3-7 above, and then \( bA_{33} \) is determined to be \( 4.5528 \times 10^8 \) N. In the case of the in-plane shear, \( b \) is 0.34 m and \( A_{33} \) is determined to be \( 1.3390 \times 10^6 \) N/mm. The deformation contours of the RVE under the applied moment and shear force are shown in Figure 3-14 and 3-16.
Table 3-4: Applied boundary conditions and prescribed loads for determining $A_{33}$

<table>
<thead>
<tr>
<th>Face of the FE-Model</th>
<th>Type of Boundary Conditions/ Loads</th>
<th>Displacement Component in x, y, z direction, resp. (mm)</th>
<th>Rotation/Loads Component in x, y, z direction, resp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>@ $x=0$, on $y-z$ plane</td>
<td>Fixed</td>
<td>0, 0, 0</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>@ $x=L$, on $y-z$ plane</td>
<td>Moment</td>
<td>-</td>
<td>0, $-2 \times 10^{10}$ N-mm, 0</td>
</tr>
<tr>
<td>@ $y=0$, on $x-z$ plane</td>
<td>Remote Force</td>
<td>-</td>
<td>0, 0, $2 \times 10^{7}$ N</td>
</tr>
<tr>
<td>@ $y=L$, on $x-z$ plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ $z=0$, on $x-z$ plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ $z=L$, on $x-z$ plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
</tbody>
</table>

Figure 3-13: (a) Face on which fixed support applied, and (b) face on which moment is applied to determine $A_{33}$
Figure 3-14: Directional deformation contours under applied moment to determine $A_{33}$

Figure 3-15: Face on which shear force is applied to determine $A_{33}$
Figure 3-16: Directional deformation contours under applied shear force to determine $A_{33}$

Considering the in-plane stiffness coefficients obtained from finite element analysis in this section, the sub-matrix $[A]$ can be written as follows:

$$[A] = \begin{bmatrix} 59.347 & 8.843 & 0 \\ 8.843 & 59.347 & 0 \\ 0 & 0 & 13.390 \end{bmatrix} \times 10^5 \text{ N/mm}$$  \hspace{1cm} (3-8)

### 3.3.2 Flexural and Torsional Stiffness

The purpose of this section is to determine flexural and torsional stiffness so that Equation 3-2 is generated. The flexural stiffness is determined by prescribing unit values to each curvature component while retaining other strain components zero by providing the necessary constraints to the RVE. After imposing a unit curvature in one-direction,
the associated restraint moments are then calculated about the neutral axis of the cross section. This is repeated for the orthogonal directions.

In order to determine $D_{11}$ and $D_{21}$ of the RVE model, a fixed boundary condition is provided at the face $x = 0$ and unit rotation, $\kappa_z = 1$ rad, is prescribed at the face $x = L_x$ by providing ‘Remote Displacement’ in ANSYS. A Remote Displacement is a type of boundary condition that allows both displacements and rotations to be applied at an arbitrary remote location in space. The origin of the remote location can be specified under ‘Scope’ in the Details view by picking, or by entering the xyz coordinates directly in the Analysis setting in ANSYS. The default location is at the centroid of the geometry. The displacement and rotation are specified under ‘Definition’. The location and the direction of a Remote Displacement can be defined in the global coordinate system or in a local Cartesian coordinate system. During simulation, ‘Rigid’ rather than ‘Deformable’ is specified because ‘Rigid’ behavior does not allow the scoped geometry (i.e. the face at $x=L_x$) to deform. Note that this option has no effect if the boundary condition is scoped to a rigid body in which case a ‘Rigid’ behavior is always used.

Displacement boundary conditions are provided on y-z plane at faces $z = 0$ and $z = L_z$ as shown in Figure 3-17. The displacement in the z-direction is restrained at the lateral faces at $z = 0$ and $z = L_z$. Table 3-5 summarizes the boundary conditions and prescribed rotation applied to determine the flexural rigidities, $D_{11}$ and $D_{21}$. Figure 3-8 shows the deformation contours under the given boundary conditions and prescribed rotations. Under this prescribed unit rotation, the moments at the fixed face ($x = 0$) and the lateral faces ($z = 0$ and $z = L_z$) are obtained to be $8.8932 \times 10^{10}$ N·mm and $1.6834 \times$
10^{10} \text{ N-mm}, respectively. The resultant moments at the restrained faces from finite element analysis are the same as the flexural stiffness values, $D_{11}$ and $D_{21}$.

$$D_{11} = \{ M_1 = \text{Resultant reaction moment acting on the face at } x = 0 \} \quad (3-9)$$

$$D_{21} = \{ M_2 = \text{Resultant reaction moment acting on the faces at } z = 0 \text{ and } z = L_z \} \quad (3-10)$$

Hence, $D_{11}$ and $D_{21}$ are equal to $8.8932 \times 10^{10} \text{ N-mm}$ and $1.6834 \times 10^{10} \text{ N-mm}$, respectively.

<table>
<thead>
<tr>
<th>Face of the FE-Model</th>
<th>Type of Boundary Conditions/ Prescribed Displacements</th>
<th>Displacement Component in x, y, z direction, resp. (mm)</th>
<th>Rotation Component in x, y, z direction, resp. (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>@ $x=0$, on y-z plane</td>
<td>Fixed</td>
<td>0, 0, 0</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>@ $x=L_x$, on y-z plane</td>
<td>Remote Displacement</td>
<td>Free, Free, Free</td>
<td>Free, Free, 1.0</td>
</tr>
<tr>
<td>@ $y=0$, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ $y=L_y$, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ $z=0$, on x-z plane</td>
<td>Displacement</td>
<td>Free, Free, 0</td>
<td>-</td>
</tr>
<tr>
<td>@ $z=L_z$, on x-z plane</td>
<td>Displacement</td>
<td>Free, Free, 0</td>
<td>-</td>
</tr>
</tbody>
</table>
Since the configuration of the FE model in the x and y directions are similar, $D_{11} = D_{22}$ and $D_{21} = D_{12}$. There is no coupling between the pure flexural (bending) moments and twisting (torsional) moments in this model.
Figure 3-18: Deformation contour when determining $D_{11}$ and $D_{21}$
In order to determine the torsional stiffness, $D_{33}$, the RVE is subjected to a prescribed unit torsional rotation ($\kappa_x = 1$ rad) at the centroid of the face at $x = L_x$ (on plane $y$-$z$) by the use of ‘Remote displacement’. A fixed boundary condition is provided at the face $x = 0$ while keeping the rest of the faces free as shown in Figure 3-20. The boundary conditions and prescribed rotations used are summarized in Table 4 below.
Table 3-6: Applied boundary conditions and prescribed displacement for determining $D_{33}$

<table>
<thead>
<tr>
<th>Face of the FE-Model</th>
<th>Type of Boundary Conditions/ Prescribed Displacements</th>
<th>Displacement Component in x, y, z direction, resp. (mm)</th>
<th>Rotation Component in x, y, z direction, resp. (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>@ x=0, on y-z plane</td>
<td>Fixed</td>
<td>0, 0, 0</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>@ x=Lx, on y-z plane</td>
<td>Remote Displacement</td>
<td>Free, Free, Free</td>
<td>1.0, Free, Free</td>
</tr>
<tr>
<td>@ y=0, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ y=Ly, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ z=0, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
<tr>
<td>@ z=Lz, on x-z plane</td>
<td>Free</td>
<td>Free, Free, Free</td>
<td>Free, Free, Free</td>
</tr>
</tbody>
</table>

Figure 3-20: Boundary conditions and remote displacement applied on the FE model to determine $D_{33}$; (a) the face with fixed support, (b) the face with prescribed rotation

Figure 3-21 shows the deformation contour obtained from ANSYS when the above specified boundary conditions and unit rotations are applied. Figure 3-22 shows
the resultant reaction moment obtained at the fixed support. Under the prescribed unit
torsional rotation and boundary conditions, \( M_{R1} \) is obtained to be \( 5.1135 \times 10^{10} \) N-mm.

From the reaction moment, \( M_{R1} \), the torsional stiffness, \( D_{33} \), can be determined from
Equation 3-11.

\[
D_{33} = \{ M_{R1} = \text{Resultant reaction moment acting on the face at } x = 0 \} \quad (3-11)
\]

Considering all the flexural and torsional coefficients determined from ANSYS,
the sub-matrix \([D]\) can be written as follows:

\[
[D] = \begin{bmatrix}
8.8932 & 1.6834 & 0 \\
1.6834 & 8.8932 & 0 \\
0 & 0 & 5.1135
\end{bmatrix} \times 10^{10} \text{N-mm} \quad (3-12)
\]

Figure 3-21: Deformation contour when determining \( D_{33} \)
3.3.3 Coupling Stiffness

FE analysis is also used to determine the terms of the coupling stiffness values, both the bending-extension coupling stiffness values and the shear-extension coupling values. In general, the values are obtained to be several orders of magnitude smaller than the terms in the bending and extensional stiffness values. Hence, the coupling stiffness values can be rationally approximated to be zero. Also, since the models considered here are symmetric about its middle surface in terms of both material and geometry, the bending-extensional coupling stiffness sub-matrix, $[B]$, is zero (Hyes 2008).
3.3.4 Transverse Shear Stiffness

In general, concrete slabs (plain concrete, RC or SVBS) are considered as thick plates. Thin plates and thick plates follow different formulations in plate-bending behavior. Thick plates follow Mendlin-Reissner formulation, which accounts for out-of-plane shear behaviors, whereas, thin plates follow Kirchhoff’s formulation, which neglects transverse shear deformation (Szilard 2004). Hence, to effectively homogenize the 3D SVBS into an equivalent 2D plate, transverse shear stiffnesses ($E_{55}$ and $E_{66}$) should be determined. In this section, a method that combines finite element analysis and analytical methods is used to obtain the transverse shear stiffness as described in the determination of in-plane shear stiffness coefficient, $A_{33}$. To obtain the transverse shear stiffness, $E_{55}$, the RVE itself is considered as a one dimensional cantilever beam.

Prescribed moment and remote force are applied at the tip ($x = L_x$) of the cantilever beam and their corresponding tip deflections are determined from ANSYS. Then, Timoshenko’s beam theory is used to determine the transverse shear stiffness of the model using Equation 3-7. The beam is clamped at the face $x = 0$ (y-z plane), as shown in Table 3-7 and Figure 3-23. ‘Rigid Behavior’ is selected for the face at $x = L_x$ for the remote displacement in the ‘Details’ view. First, an arbitrary moment, $M_z = -2 \times 10^{10}$ N-mm is applied at the centroid of the face at $x = L_x$ as shown in the 3-23, and a maximum tip deflection of 34.739 mm is obtained as shown in Figure 3-24. Then, an arbitrary shear force, $F_y = -2 \times 10^7$ N, is applied at the face $x = L_x$ as shown in Figure 3-25, and a tip deflection of 21.664 mm is obtained. Based on these outputs and applied
remote force and moment, the flexural rigidity (EI) and transverse shear stiffness, $E_{55}$, are analytically calculated as $2.5907 \times 10^{13} \text{N-mm}^2$ and $1.3590 \times 10^6 \text{N/mm}$, respectively.

Because of the geometry and material similarity in the x and z directions, the transverse shear stiffness on the x-y plane and y-z plane are the same. These results in $E_{55} = E_{66}$.

Considering the transverse shear stiffness coefficients obtained from FE and analytical analyses in this section, the sub-matrix $[E]$ can be written as follows:

$$[E] = \begin{bmatrix} 1.3590 & 0 \\ 0 & 1.3590 \end{bmatrix} \times 10^6 \text{N/mm} \quad (3-13)$$

Coefficients $E_{56}$ and $E_{65}$ are zero since there is no coupling between the transverse shears in the orthogonal directions.

| Table 3-7: Applied boundary conditions and prescribed loads for determining $E_{55}$ |
|----------------------------------|-----------------|-----------------|-----------------|
| Face of the FE-Model            | Type of Boundary Conditions/ Loads | Displacement Component in x, y, z direction, resp. (mm) | Rotation/Loads Component in x, y, z direction, resp. |
| @ x=0, on y-z plane              | Fixed           | 0, 0, 0         | 0, 0, 0         |
| @ x=L_x, on y-z plane            | Moment          | -               | 0, 0, -2 x 10^{10} N-mm |
|                                  | Remote Force    | -               | 0, -2 x 10^7 N, 0 |
| @ y=0, on x-z plane              | Free            | Free, Free, Free| Free, Free, Free |
| @ y=L_y, on x-z plane            | Free            | Free, Free, Free| Free, Free, Free |
| @ z=0, on x-y plane              | Free            | Free, Free, Free| Free, Free, Free |
| @ z=L_z, on x-y plane            | Free            | Free, Free, Free| Free, Free, Free |
Figure 3-23: Boundary condition and prescribed moment to determine $E_{55}$; (a) face with fixed support (b) face with prescribed moment in the $z$-direction

Figure 3-24: Directional deformation contour under applied moment to determine $E_{55}$
Figure 3-25: Prescribed remote force in the negative y direction

Figure 3-26: Directional deformation contour under the applied force to determine $E_{55}$
In conclusion, this chapter presented the determination of the ABDE matrix that relates internal forces/moments of a given SVBS RVE to its corresponding internal effects (strains/curvatures). SVBS 340 was used as an example and the same procedures can be repeated to determine the ABDE matrices of all SVBS configurations.
CHAPTER 4: INDEPENDENT VERIFICATION OF THE FE BASED MICROMECHANICAL MODELING APPROACH

4.1 Overview

The effectiveness and prediction capability of the FE based micromechanical models developed in chapter 3 need to be verified independently. One approach of verifying the models is through comparison with results of micromechanical plain concrete slab models. The procedure of determining the stiffness values of [A], [B], [D] and [E] matrices used for the SVBS is repeated for an RVE of a plain concrete solid slab and the formation of ABDE matrix is discussed below. A plain concrete solid slab can be reasonably assumed to be an isotropic thick plate. A plain concrete solid slab is selected for validation because of the availability of analytical relationships for deriving stiffness properties as well as describing the plate behavior is helpful for verification of the micromechanical model development procedure. Development of ABDE matrix is demonstrated based on a similar example used in the previous chapter.

In order to determine the stiffness properties of the plain concrete solid slab, an RVE (unit cell) is extracted as shown in Figure 4.1. Since the slab is solid and isotropic, the length and width dimensions of the RVE can be arbitrary as long as the RVE has the same thickness as the entire slab under consideration. However, for the sake of comparisons, the RVE is extracted in such a way that it has equal length, width and thickness dimensions of BD340. The FE models are generated using SOLID187 element
for the analysis. Resultant forces and moments produced at restrained faces of the RVE model for prescribed displacements and rotations are directly obtained from ANSYS outputs as demonstrated in the previous sections.

Figure 4-1: Representative volume element extraction from plain concrete slab

Figure 4-2 below shows the dimensions of the RVE extracted and the generated mesh used in this chapter.
4.2 Extensional Stiffness

As described in chapter 3, extensional stiffness values are determined by prescribing unit values to each strain component independently, while retaining all other strain components as zero. The zero strain components are achieved by providing necessary boundary conditions to prevent rigid body translation and rotation of the RVE. After the resultant forces acting on the restrained faces of the RVE are obtained from the ANSYS output, the forces are divided by the width of the respective restrained faces of the RVE to determine the extensional stiffness \( (A_{ij}) \) using Equation 4-1.

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

(4-1)

where \( N \) is the force vector, \( A \) is the extensional stiffness matrix and \( \varepsilon \) the strain vector.
The FE models used to determine $A_{11}$ and $A_{21}$ coefficients are shown in Figure 4-1. The displacements of the face on the y-z plane at $x = 0$ are constrained from translational displacement. The face at $x = L_x$ is given a prescribed displacement of $L_x$ in the positive x-direction. Hence, a unit strain is given to the FE model in x-direction ($\varepsilon_1 = 1$). Prescribed displacements with zero displacements in z directions are also provided on faces at $z = 0$ and $z = L_z$ to determine $A_{21}$. The prescribed displacements are equivalent to $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ and $\varepsilon_3 = 0$. $A_{11}$ and $A_{21}$ are determined from Equation 4-2 and 4-3, respectively.

$$A_{11} = \left\{ \left( N_1 = \text{Resultant reaction force acting on the Y-Z face at } X=0 \right) / L_Z \right\}$$ \hspace{1cm} (4-2)

$$A_{21} = \left\{ \left( N_2 = \text{Resultant reaction force acting on the X-Y? face at } Z=0 \right) / L_x \right\}$$ \hspace{1cm} (4-3)

As described above, the RVE model selected herein has dimensions of $L_x = 300$ mm, $L_z = 300$ mm, and $h = 340$ mm.
Figure 4-3: Boundary conditions and prescribed displacement to determine $A_{11}$ and $A_{21}$; (a) the face with no displacement in all three directions, (b) the face with displacement of 300 mm in the x-directions and zero in the other two directions, (c) and (d) the face with zero displacement in the z-direction but free on the other two directions

The resultant force at the constrained face at $x = 0$ (on y-z plane) under the prescribed unit strain is found to be $R_1 = 3.1625 \times 10^9$ N from ANSYS output as shown in Figure 4-4 and the reaction normal forces acting on both transverse faces (at $z = 0$ and
is found to be each \( R_2 = 5.6924 \times 10^8 \) N shown in Figure 3-10. Hence, \( N_1 \) and \( N_2 \), which are forces per unit width at faces \( x = 0 \) and \( z = 0 \), are determined to be \( 1.0542 \times 10^7 \) N/mm and \( 1.8975 \times 10^6 \) N/mm, respectively using the following equations.

\[
N_1 = \left\{ \frac{(R_1 = \text{Resultant reaction force acting on the y-z face at } x=0)}{L_z} \right\} \tag{4-4}
\]

\[
N_2 = \left\{ \frac{(R_2 = \text{Resultant reaction force acting on the x-y face at } z=0)}{L_x} \right\} \tag{4-5}
\]

Using Equation 4-1, \( A_{11} \) and \( A_{21} \) are determined to be \( 1.0542 \times 10^7 \) N/mm and \( 1.8975 \times 10^6 \) N/mm, respectively. Similar approaches can be used to determine \( A_{22} \) and \( A_{12} \). However, in this case, \( L_x \) is equal to \( L_z \), which implies \( A_{22} = A_{11} \) and \( A_{21} = A_{12} \).

Analytically, the in-plane stiffness coefficients \( A_{11}, A_{22}, A_{21}, \) and \( A_{12} \) are calculated from Equations 4-6 and 4-7.

\[
A_{11} = A_{22} = \frac{Eh}{1 - \nu^2} = 1.0541 \times 10^7 \text{ N/mm} \tag{4-6}
\]

\[
A_{21} = A_{12} = \frac{\nu Eh}{1 - \nu^2} = 1.8974 \times 10^6 \text{ N/mm} \tag{4-7}
\]

where \( E \) is the modulus of elasticity of the concrete \( (3 \times 10^4 \text{ N/mm}^2) \); \( \nu \) is the Poisson’s ratio of concrete \( (0.18) \); \( h \) is the thickness of the model \( (340 \text{ mm}) \). The stiffness values obtained analytically are then compared with the FE results. Based on Equations 3-6 and 3-7, stiffness values are determined as force per unit length. Hence, the FE model for RVE can be generated with arbitrary dimensions of length and width while keeping the thickness of the RVE the same as that of the entire slab.
The deformation contour is shown in figure 4-4 below.

Figure 4-4: Deformation contour when determining extensional stiffness of the plain concrete RVE

Figure 4-5: Reaction forces on restrained faces for extensional stiffness of the plain concrete RVE
As discussed in chapter 3, there is no coupling between shear forces and tensile deformations. Therefore, a shear loading in the orthotropic principle coordinate system leads to a shear distortion without tensile deformation. Theoretically, anti-symmetric boundary conditions are specified in order to determine the equivalent in-plane shear stiffness coefficient $A_{33}$ because an anti-symmetric strain state is caused by in-plane shear forces.

Coefficient $A_{33}$ of the plain concrete RVE is determined using the FE model shown in Figure 4-2. In order to provide a unit in-plane shear strain, the face at $x = 0$ (on y-z plane) is restrained from rotations and displacements in all directions while the opposite face at $x = L_x$ (on y-z plane) is provided with a translational displacement of $L_x$ in the z- direction. These boundary conditions provide $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\gamma_{12} = 1$. Figure 4-6 shows the plan view of the original and deformed shapes of the RVE under unit in-plane shear strain. However, there is moment and shear interaction when pure shear is simulated using FE models. Therefore, a deformation resulting from an applied shear force is expected to have both flexural and shear deformation components. Hence, a combination of FE and an analytical method based on Timoshenko’s beam theory is used to obtain the shear stiffness coefficient, $A_{33}$. 
In principle, shear strain does not depend on the RVE length in the transverse directions. The method described in chapter 3, a combination of FE and analytical method based on Timoshenko’s beam theory, is also utilized here to determine the shear stiffness of plain concrete.

First an arbitrary end couple, $M_y = 2 \times 10^{10}$ N-mm is applied to the tip of the beam in the negative y direction as shown in Figure 4-7 and the corresponding average tip deflection (deflection at the face $x = L_x$) was determined from the finite element analysis output. Under this moment, the tip deflection in the z direction is obtained to be 38.532 mm as shown in Figure 4-8. At all points on the free face of the model, equal deformations are observed. The tip deflection is equated with Equation 3-6 described in chapter 3 and the flexural rigidity (EI) of the RVE is determined to be $2.3357 \times 10^{13}$ N-
mm² (theoretically, EI = 2.295 x 10^{13} \text{ N} \cdot \text{mm}^2, the difference is only 1.7%).

Figure 4-7: Boundary conditions and prescribed moment to determine \( A_{33} \); (a) the face with fixed support, and (b) the face with applied moment
Figure 4-8: Directional deformation contour in z direction under applied moment, (a) overall and (b) face at x = L_a.
The end couple is then removed and an arbitrary transverse remote force, $F_z = 2 \times 10^7$ N, is applied at the tip of the cantilever beam (RVE) as shown in Figure 4-9. Unlike the free face deflection due to end couple, the free face deflection due to the remote force is not constant. The average tip deflection obtained from the finite element output after analysis is 13.333 mm by averaging the maximum and minimum deformations in the $z$ direction at the free end (at face $x = L_x$) as shown in Figure 4-10. This average tip deflection is then equated with Equation 3-7 given in chapter 3, and then $bA_{33}$ is determined to be $10.6638 \times 10^8$ N. In the case of in-plane shear, $b$ is the height of the RVE, which is 0.34 m, and then $A_{33}$ is determined to be $3.1364 \times 10^6$ N/mm.

**Figure 4-9:** Boundary conditions and remote force to determine $A_{33}$; (a) the face with fixed support, and (b) the face with applied force
Figure 4-10: Directional deformation contour in z direction under applied force, (a) overall and (b) face at x=L.
Shear area is an important factor to determine shear stiffness analytically. The shear area is typically presented in the form of shear deformation coefficient, shear correction factor or form factor as presented in literature (Rosinger and Ritchie 1977). The shear coefficients and the form factor are defined as the ratio of the gross area of the section to the shear area of the section as shown in Equation 4-8.

\[ \kappa = \frac{A}{A_v} \]  

(4-8)

where \( \kappa \) is the form factor, shear correction factor or shear deformation coefficient, \( A \) is the gross sectional area and \( A_v \) the shear area of the section. For rectangular beams, the shear factor is obtained using Equation 4-9.

\[ \kappa = \frac{6 + 5v}{5(1 + v)} \]  

(4-9)

The coefficient \( A_{33} \) can be calculated using Equation 4-10 in terms of the shear:

\[ A_{33} = \frac{GA_v}{b} \]  

(4-10)

\( A_{33} \) is calculated analytically as \( 3.2623 \times 10^6 \) N/mm based on the shear rigidity value of the selected slab, \( G = 1.2712 \times 10^4 \) N/mm², width \( b = 340 \) mm, thickness \( h = 300 \) mm, and shear factor \( \kappa = 1.169 \).

Considering the in-plane stiffness coefficients obtained from finite element analysis in this section, sub-matrix \([A]\) can be written as follows:
As discussed previously, stiffness values $A_{13}$, $A_{33}$, $A_{23}$, and $A_{32}$ are zero because of axial and shear deformations do not have coupling effect to each other. The in-plane stiffness results between the analytical model and the FE analysis are compared and summarized in Table 4-1.

**Table 4-1: Comparison of the non-zero [A] coefficients determined by micromechanics based FE and analytical methods for plain concrete solid slab**

<table>
<thead>
<tr>
<th>Coefficient of Sub-matrix [A]</th>
<th>Stiffness of a 340 mm thick slab, $E = 3 \times 10^{10}$ N/m², and $\nu = 0.18$</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11} = A_{22}$</td>
<td>FE Results: $10.5420 \times 10^6$ N/mm, Analytical Results: $10.5410 \times 10^6$ N/mm</td>
<td>$0.009 %$</td>
</tr>
<tr>
<td>$A_{21} = A_{12}$</td>
<td>FE Results: $1.8975 \times 10^6$ N/mm, Analytical Results: $1.8974 \times 10^6$ N/mm</td>
<td>$0.005 %$</td>
</tr>
<tr>
<td>$A_{33}$</td>
<td>FE Results: $3.1364 \times 10^6$ N/mm, Analytical Results: $3.2622 \times 10^6$ N/mm</td>
<td>$4.011 %$</td>
</tr>
</tbody>
</table>

The extensional stiffness results between the analytical model and the FE analysis are within 5%, particularly the axial stiffness values are in excellent agreement.

### 4.3 Flexural and Torsional Stiffness

As stated in the previous chapter, the flexural and torsional stiffness values are determined by prescribing unit values to each curvature component while retaining other curvature components fixed by providing necessary constraints to the RVE model described in the beginning of this chapter. After imposing a unit curvature in one-
direction while making other curvatures zero by providing necessary constraints to the model, the associated restraint moment is then determined from ANSYS and compared with the analytical stiffness value.

\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_3
\end{bmatrix} = 
\begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{bmatrix}
\begin{bmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]

(4-12)

In order to determine \(D_{11}\) and \(D_{21}\) of the RVE model, a fixed boundary condition is provided at the face \(x = 0\) and unit rotation (\(\kappa_1 = 1\) rad) is prescribed at the face \(x = L_x\) by providing remote displacement in ANSYS. Displacement boundary conditions are provided on \(y-z\) plane at faces \(z = 0\) and \(z = L_z\) as shown in Figure 4-11. Displacement in the \(z\)-direction is restrained at the lateral faces at \(z = 0\) and \(z = L_z\). Rigid analysis option is selected in ANSYS. The deformation contour is shown in Figure 4-12.
Figure 4-11: Prescribed displacement and boundary conditions applied to determine $D_{11}$ and $D_{21}$
Based on the boundary conditions and the prescribed unit rotation, the moments at the fixed face \((x = 0)\) and the lateral faces \((z = 0\) and \(z = L_z\)) are obtained to be \(1.0225 \times 10^{11}\) N-mm and \(1.8986 \times 10^{10}\) N-mm, respectively. These resultant moments at the restrained faces obtained from the finite element analysis are shown in Figure 4-13, and correlated with flexural stiffness values \(D_{11}\) and \(D_{21}\) as follows:

\[
D_{11} = \{ M_1 = \text{Resultant reaction moment acting on the face at } x = 0 \} \quad (4-13)
\]

\[
D_{21} = \{ M_2 = \text{Resultant reaction moment acting on the faces at } z = 0 \text{ and } z = L_z \} \quad (4-14)
\]
Hence, based on the FE results and Equations 4-13 and 4-14, $D_{11}$ and $D_{21}$ are $1.0225 \times 10^{11}$ N-mm and $1.8986 \times 10^{10}$ N-mm, respectively.

Figure 4-13: Resultant moments at the restrained faces; (a) $M_1$ at face $x=0$, and (b) $M_2$ at face $z=0$ and $z=L_z$

Analytically, the flexural stiffness can be determined based on Equations 4-15 and 4-16:

$$D_{11} = \frac{Eh^3}{12(1-v^2)} = 1.0155 \times 10^{11} \text{ N-mm} \quad (4-15)$$

$$D_{21} = \frac{vEh^3}{12(1-v^2)} = 1.8279 \times 10^{10} \text{ N-mm} \quad (4-16)$$

The above procedure can be repeated for the orthogonal direction to determine $D_{22}$ and $D_{21}$. Since the geometric configuration of the RVE is the same in the $x$ and $z$ directions, $D_{11} = D_{22}$ and $D_{21} = D_{12}$.

In order to determine the torsional stiffness, $D_{33}$, the RVE model is subjected to a prescribed unit torsional rotation ($\kappa_3 = 1$ rad) at the centroid of the face at $x = L_x$ (on
plane \( y-z \)). A fixed boundary condition is provided at the face \( x = 0 \) while keeping the rest of the faces free as shown in Figure 4-14.

![Figure 4-14: Boundary conditions and prescribed unit strain for D33, (a) fixed support, (b) unit torsional rotation](image)

Under the above boundary conditions and prescribed rotation, the deformed shape is shown in Figure 4-15.
The resultant moment at the restrained face obtained from the finite element analysis is shown in Figure 4-16, and correlated with flexural stiffness $D_{33}$ using Equation 4-17:

$$D_{33} = \{ M_1 = \text{Resultant reaction moment acting on the face at } x = 0 \} \quad (4-17)$$

Under the above described prescribed unit torsional rotation and boundary conditions, $D_{33}$, which is the same as $M_1$, is $6.4136 \times 10^{10} \text{ N-mm}$. 

Figure 4-15: Deformed shape under unit torsional rotation (a) isometric view and (b) plan view
Analytically, the torsional stiffness of the RVE model is the torsional moment (torque) required to twist the face of the RVE model and can be calculated using Equation 4-18:

$$D_{33} = \beta G b^3 h \kappa$$

(4-18)

where $\kappa$ is the twist rate, $\kappa = \frac{\Delta \phi}{\Delta L}$, $\Delta \phi = 1$ rad; $\Delta L = 300$ mm ($L_x$), $h$ is the longer cross-sectional dimension (340 mm), and $b$ is shorter cross-sectional dimension (300 mm); $\beta = 0.163$, which is obtained from standard engineering table (Ugural and Fenster 2003). Hence, the torsional stiffness, $D_{33}$, is calculated analytically as $6.3405 \times 10^{10}$ N-mm.
Considering all the flexural and torsional coefficients determined from micromechanics modeling using ANSYS, sub-matrix \([D]\) can be written as follows I Equation 4-19:

\[
[D] = \begin{bmatrix}
10.225 & 1.8986 & 0 \\
1.8986 & 10.225 & 0 \\
0 & 0 & 6.4136
\end{bmatrix} \times 10^{10} \text{ N-mm}
\] (4-19)

Stiffness values \(D_{13}, D_{31}, D_{23}\) and \(D_{32}\) are zero because of the fact that there is no coupling between flexure and torsion.

The flexural and torsional stiffness results between the analytical model and the FE analysis are compared and summarized in Table 4-2.

<table>
<thead>
<tr>
<th>Coefficient of Sub-matrix ([D])</th>
<th>Stiffness of the slab of 340 mm thick, (E = 3 \times 10^4) N/mm(^2), and (\nu = 0.18)</th>
<th>% Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_{11} = D_{22})</td>
<td>(1.0225 \times 10^{11}) N-mm</td>
<td>0.7 %</td>
</tr>
<tr>
<td>(D_{21} = D_{12})</td>
<td>(1.8986 \times 10^{10}) N-mm</td>
<td>3.7 %</td>
</tr>
<tr>
<td>(D_{33})</td>
<td>(6.4136 \times 10^{10}) N-mm</td>
<td>1.1 %</td>
</tr>
</tbody>
</table>

The flexural and torsional stiffness results between the analytical model and the FE analysis are obtained to be within 4\%, the micromechanics based FE models are in good agreement with the corresponding analytical methods.
4.4 Coupling Stiffness

Similar to the findings and justifications given in section 3.3.3, both the bending-extension coupling stiffness values and the shear-extension coupling values are approximated to be zero.

4.5 Transvers Shear Stiffness

Coefficient $E_{ss}$ of the RVE module considered here is determined similar to the method utilized for determining the in-plane shear stiffness coefficient, $A_{33}$. As previously discussed in chapter 3, it is difficult to prescribe a pure transverse shear displacement without having a bending effect. Hence, prescribed moment and force are applied at the tip ($x = L_x$) of a cantilever beam (RVE) and their corresponding tip deflections are determined from ANSYS. Then, Timoshenko’s beam theory is used to determine the transverse shear stiffness of the model. The beam is clamped at the face $x = 0$ (y-z plane). First, a moment, $M_z = -3 \times 10^{10}$ N-mm is applied at the centroid of the face at $x = L_x$ as shown in the Figure 4-17, and the deflection of the face at $x$ equals to $L_x$ is obtained to be 45.053 mm as shown in Figure 4-18. Then, a shear force, $F_y = -5 \times 10^5$ N, is applied at the face at $x = L_x$ as shown in Figure 4-19, and the average tip deflection of 29.196 mm is obtained as shown in Figure 4-20. Based on the FE results and Equations 3-6 and 3-7, the flexural rigidity (EI) and transverse shear stiffness ($E_{ss}$) are determined to be $2.996 \times 10^{13}$ N-mm$^2$ and $3.5266 \times 10^6$ N/mm, respectively.
Figure 4-17: Boundary condition and prescribed moment to determine E55; (a) face with fixed support (b) face with prescribed moment in the z-direction
Figure 4-18: Directional deformation contours due to the prescribed moment; (a) body in the y-axis (b) on the face at x=Lx in the y-axis
Because of the geometry and material similarity in the x and z directions, the transverse shear stiffness on the x-y plane and y-z plane are the same. This implies $E_{55} = E_{66}$.

Considering the transverse shear stiffness coefficients obtained from finite element analysis in this section, sub-matrix $[E]$ can be written as follows:

$$[E] = \begin{bmatrix} 3.5266 & 0 \\ 0 & 3.5266 \end{bmatrix} \times 10^6 \text{ N/mm}$$

Stiffness values, $E_{56}$ and $E_{65}$, are zero since there is no coupling between the transverse shears in the orthogonal directions.

Figure 4-19: Boundary condition and prescribed force to determine $E_{55}$; (a) face with fixed support (b) face with prescribed force in the negative y-direction.
Figure 4.20: Directional deformation contours due to the prescribed force; (a) body in the y-axis (b) on the face at $x=L_x$ in the y-axis
Analytically, the transverse shear stiffness $E_{55}$ for shear rigidity value $G = 1.2712 \times 10^4$ N/mm$^2$, width $b = 300$ mm, height $h = 340$ mm and shape factor $\kappa = 1.1695$ is calculated to be $3.6957 \times 10^6$ N/mm.

The transverse shear stiffness results between the analytical model and the FE analysis are compared and summarized in Table 4-3.

<table>
<thead>
<tr>
<th>Coefficient of Sub-matrix $[E]$</th>
<th>Stiffness of the slab of 340 mm thick, $E = 3 \times 10^4$ N/mm$^2$, and $\nu = 0.18$</th>
<th>% Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{55} = E_{66}$</td>
<td>$3.5266 \times 10^6$ N/mm</td>
<td>$3.6957 \times 10^6$ N/mm</td>
</tr>
</tbody>
</table>
CHAPTER 5: MACROMECHANICAL ANALYSIS OF SVBS

5.1 Overview

The main objective of this chapter is to present a macromechanical analysis of SVBS using the ABDE stiffness matrix determined from micromechanical modeling of the RVE discussed in chapter 3. The analysis is conducted using plate theory. A plate by definition is a planar structure which has a small thickness in comparison to the planar dimensions. Forces applied on a plate are usually assumed to be perpendicular to the plane of the plate. A plate subjected to a load perpendicular to its plane is in a state of bending and transverse shear. A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two dimensional problem. If the plate is made out of concrete, it is typically referred to as a slab.

Based on the typical dimensions of SVBS shown in Table 1.1, the typical thickness to span ratio can be as high as 4%. Therefore, the displacement based first order shear deformation theory (FSDT), also known as Mindlin-Reissner or thick plate theory, is selected for the micromechanical analysis of SVBS. In this theory, the effect of the transverse shear force on the deformation of the SVBS is included. The other widely cited plate theory, Kirchhoff-Love theory (KLT) or thin plate theory, under-predicts deflections and stresses (Reddy, 2006). However, it is used for comparison purposes in this chapter.
In FSDT, the transverse shear strain distribution is assumed to be constant through the plate thickness. Therefore, a shear correction factor is required to account for the strain energy due to shear deformation. In general, these shear correction factors are shape dependent and are discussed in section 4.2.

Defining notations and establishing sign conventions is an important step in analyzing the SVBS. Let us consider that a given SVBS is placed in the $xz$ plane. Representation of plate surface slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial z}$ by the right hand rule produces arrows that point in negative $x$ and positive $z$ directions respectively. Both surface slopes and rotations are required for plate elements. Signs and subscripts of rotations and slopes are reconciled by replacing $\theta_x$ by $-\varphi_z$ and $\theta_z$ by $\varphi_x$ as shown in Figure 5-1.

![Figure 5-1: Notations and sign conventions (a) Rotations of mid-surface normal and (b) Slopes of plate surfaces](image)

Displacement, $w$, is positive in the $y$-direction; the rotation, $\varphi_x$, is positive if it leads to positive displacements in the $x$-directions at positive $y$-side of the mid plane; and
the rotation, $\varphi_z$, is positive if it leads to positive displacements in the z-direction on the positive y-side.

The macromechanical analysis of SVBS as an orthotropic plate is based on the following assumptions.

1. The slab has a constant thickness with known orthotropic stiffness values.
2. No membrane forces will occur due to support constraints or large deflections. The mid-plane of the slab will experience no strain after the load is applied.
3. A straight line normal to the mid-plane of the slab in an unloaded state remains a straight line after application of the load; however, it does not need to be normal to the mid-plane of the plate.
4. The displacements are small. Therefore, strains involved are infinitesimal.
5. The out of plane stress, $\sigma_{yy}$, in the direction normal to the mid-plane is negligible.

5.2 Example Analysis of a Simply Supported SVBS

Consider a simply supported rectangular SVBS of length $a$, and width $b$, and thickness $h$, is subjected to transverse loading $P_0 = P_0(x, z)$ acting on the top surface (i.e. at $y = h/2$) as shown in Figure 5-2. The SVBS occupies a region defined by the bounds given in Equation 5-1 the x - y - z right-handed Cartesian coordinate system:

$$0 \leq x \leq a; \quad 0 \leq z \leq b; \quad -\frac{h}{2} \leq y \leq \frac{h}{2} \quad (5-1)$$
Figure 5-2: Geometry and orientation of SVBS for macromechanical analysis

The mid-plane coincides with the x-z plane of a right handed orthogonal coordinate system x, y, z. The y-axis is perpendicular to the unloaded plate. If a vertical line over the thickness of the plate that is normal to the mid-plane is considered, it is subjected to a displacement w in y-direction, a rotation $\varphi_x$ and rotation $\varphi_y$.

Mathematically, the simply supported boundary conditions can be described by Equations 5-2 and 5-3.

\[
\begin{align*}
x = [0, a]: w &= 0, \quad \frac{\partial w}{\partial z} = 0 \\
z = [0, b]: w &= 0, \quad \frac{\partial w}{\partial x} = 0
\end{align*}
\]
Applying Navier’s approach, which is performed by approximating the deflection and load-functions through double Fourier sine series, the slab is subjected to a vertical load given by Equation 5-4 (Jawaad 2004).

\[
P_z = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right)
\]

(5-4)

where \( P_{mn} = \frac{16P_0}{\pi^2 mn} \) for uniform loads, and \( m \) and \( n \) are odd positive integers. The deflection and rotations of the SVBS are also expressed by Fourier approximation as shown in Equations 5-5 through 5-7.

\[
w(x, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right)
\]

(5-5)

\[
\varphi_x(x, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right)
\]

(5-6)

\[
\varphi_z(x, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi z}{b} \right)
\]

(5-7)

where \( w(x, z) \) is the vertical deflection in the \( y \)-direction, \( \varphi_x(x, z) \) and \( \varphi_y(x, z) \) are the rotations along the \( x \) and \( z \) directions, \( A_{mn} \), \( B_{mn} \) and \( C_{mn} \) are amplitudes for the deflection and the rotations of the SVBS.

The kinematics (strain-displacement) relationships based on FSDT are described by Equations 5-8 and 5-9 (Blaauwendraad, 2010)

\[
\kappa_1 = \frac{\partial \varphi_x}{\partial x}, \quad \kappa_2 = \frac{\partial \varphi_y}{\partial z}, \quad \kappa_3 = \frac{1}{2} \left( \frac{\partial \varphi_x}{\partial z} + \frac{\partial \varphi_y}{\partial x} \right),
\]

(5-8)
\[ \gamma_{13} = \varphi_x + \frac{\partial w}{\partial x}, \quad \gamma_{23} = \varphi_z + \frac{\partial w}{\partial z} \]  \hspace{1cm} (5-9)

where \( \kappa_1 \) and \( \kappa_2 \) are bending curvatures, \( \kappa_3 \) is torsional strain, and \( \gamma_{13} \) and \( \gamma_{23} \) are transverse shear strains.

The equilibrium relations based on FSDT are given by Equation 5-10.

\[ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xz}}{\partial z} - Q_x = 0, \quad \frac{\partial M_{xz}}{\partial x} + \frac{\partial M_z}{\partial z} - Q_z = 0, \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_z}{\partial z} + P_a = 0 \]  \hspace{1cm} (5-10)

The unknown constants \( A_{mn}, B_{mn} \) and \( C_{mn} \) are obtained by substituting the constitutive relations in the form of the assumed deformations (Equations 5-5 to 5-7) into the differential equation of equilibrium (Equation 5-10). This results in a system of three linear equations and three unknowns (Equation 5-11). Solving for the unknown constants, the deflection and rotations can be obtained at a given \( x \) and \( z \) coordinates for \( m \) and \( n \).

The convergence of the solution is evaluated and it is obtained that the results in the series converges very well for \( m = n = 15 \).

\[
\begin{bmatrix}
E_{66}\left(\frac{m\pi}{a}\right) & E_{55}\left(\frac{n\pi}{b}\right) & E_{66}\left(\frac{m\pi}{a}\right)^2 + E_{55}\left(\frac{n\pi}{b}\right)^2 \\
(D_{12} + 0.5D_{33})\left(\frac{mn\pi^2}{ab}\right) & D_{22}\left(\frac{n\pi}{b}\right)^2 + 0.5D_{33}\left(\frac{m\pi}{a}\right)^2 + E_{45} & E_{55}\left(\frac{n\pi}{b}\right) \\
D_{11}\left(\frac{m\pi}{a}\right)^2 + 0.5D_{33}\left(\frac{n\pi}{b}\right)^2 + E_{66} & (D_{12} + 0.5D_{33})\left(\frac{mn\pi^2}{ab}\right) & E_{66}\left(\frac{m\pi}{a}\right)
\end{bmatrix}
\begin{bmatrix}
B_{mn} \\
C_{mn} \\
A_{mn}
\end{bmatrix} = \begin{bmatrix}
-P_{mn} \\
0 \\
0
\end{bmatrix}
\]

(5-11)

where \( E_{55} \) and \( E_{66} \) are transverse shear stiffness values; \( D_{11}, D_{22} \) and \( D_{21} \) are flexural stiffness values and \( D_{33} \) is the torsional stiffness values of the SVBS determined in chapter 3.
The bending moments $M_x$, $M_z$ and $M_{xz}$ produce bending stresses $\sigma_{xx}$, $\sigma_{zz}$ and $\sigma_{xz}$, respectively. Similarly, the transverse shear forces $Q_x$ and $Q_z$ produce $\sigma_{xy}$ and $\sigma_{zy}$, respectively. In order to determine the bending and transverse shear stresses, the relations shown in Equations 5-12 through 5-16 are used in which integrations through the thickness of the SVBS are involved.

$$M_x = -\int_{h}^y y\sigma_{xx} \, dy$$  \hspace{1cm} (5-12)

$$M_z = -\int_{h}^y y\sigma_{zz} \, dy$$  \hspace{1cm} (5-13)

$$M_{xz} = -\int_{h}^y y\sigma_{xz} \, dy$$  \hspace{1cm} (5-14)

$$Q_x = -\int_{h}^y \sigma_{xy} \, dy$$  \hspace{1cm} (5-15)

$$Q_z = -\int_{h}^y \sigma_{zy} \, dy$$  \hspace{1cm} (5-16)

Appendix 1 includes a computer program written in MATLAB to determine the bending moments, stresses, deflections and shear forces using FSDT based equations.

Consider SVBS340 with a length of 12 m and a width of 10 m under a vertical uniform distributed load of 0.02 N/mm\(^2\) (MPa). SVBS340 is chosen for illustrative purpose because its stiffness values have already been determined from the FE-based micromechanical modeling in chapter 3. The bending and shear stiffness values of SVBS 340, as determined in chapter 3, are shown below:
\[
\begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{33}
\end{bmatrix} = \begin{bmatrix}
8.8932 & 1.6834 & 0 \\
1.6834 & 8.8932 & 0 \\
0 & 0 & 5.1135
\end{bmatrix} \times 10^{10} \text{ N-mm}
\]

\[
\begin{bmatrix}
E_{55} & 0 \\
0 & E_{66}
\end{bmatrix} = \begin{bmatrix}
1.3590 & 0 \\
0 & 1.3590
\end{bmatrix} \times 10^6 \text{ N/mm}
\]

Using these stiffness values and selected dimensions of the slab under a uniformly distributed load of 0.02 N/mm², the MATLAB program was run to determine mid-plane deflection of the slab, bending moment in x-direction, bending moment in z-direction, twisting moment, maximum bending stress in the x-direction, maximum bending stress in the z direction and in-plane shear stress. The analysis results are presented in Figure 5-3 to 5-9.

Figure 5-3: Mid-plane deflection of the SVBS
Figure 5-4: Bending moment in x-direction, $M_x$

Figure 5-5: Bending moment in z-direction, $M_z$
Figure 5-6: Torsional moment, $M_{\alpha\alpha}$

Figure 5-7: Maximum stress on the long direction, $\sigma_x$, on the top face
Figure 5-8: Maximum bending stress on the short direction, $\sigma_z$ on the top face

Figure 5-9: In-plane shear stress, $\sigma_{xz}$, of the SVBS

For comparing the deflection, moments and stresses found using FSDT, an equivalent isotropic thin plate formulation based on Kirchhoff-Love theory (KLT), aka classical plate theory, is utilized. The governing kinematics, constitutive, and equilibrium
equations described in Equations 5-4 to 5-16 are still valid in KLT with the exception of the terms associated with transverse shear. In KLT, the transverse shear stiffness values are assumed to be infinitely large.

After proper rearrangement and substitutions of equations based on Navier’s approach, the final expression for the deflection \( w(x, y) \) of an isotropic thin plate with flexural rigidity, \( D \), subjected to a uniformly distributed load is given by Equation 5-17 (Ugural and Fenster 2003):

\[
w(x, y) = \frac{16P_0}{\pi^3 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \left(\frac{\pi m x}{a}\right) \sin \left(\frac{\pi n y}{b}\right), \text{ for } m, n = 1, 3, 5, 7, \ldots
\]

(5-17)

Using the Hookean relationship for isotropic plate based on the isotropic material parameters, \( E \) and \( \nu \), the moments are given by Equations 5-18 through 5-20:

\[
M_x = -\frac{Eh^3}{12(1-\nu^2)} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial z^2} \right)
\]

(5-18)

\[
M_z = -\frac{Eh^3}{12(1-\nu^2)} \left( \nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]

(5-19)

\[
M_{xz} = -\frac{Eh^3}{12(1+\nu)} \left( \frac{\partial^2 w}{\partial x^2} \right)
\]

(5-20)

where \( E \) is the ‘equivalent’ modulus of elasticity of the SVBS and \( \nu \) is the ‘equivalent’ Poisson’s ratio of the SVBS as determined from the Equation 5-21 and 5-22, respectively:
\[ \nu = \frac{A_{21}}{A_{11}} \]  \hspace{2cm} (5-21)

\[ D_{11} = \frac{Eh^3}{12(1 - \nu^2)} \]  \hspace{2cm} (5-22)

From Equations 3-8 and 3-13, \( A_{11}, A_{21} \) and \( D_{11} \) are 5.9347 \times 10^6 \, \text{N/mm}, 8.8430 \times 10^5 \, \text{N/mm} \) and 8.8932 \times 10^{10} \, \text{N-mm} \) and 1.6834 \times 10^{10} \, \text{N-mm}, respectively for the SVBS under consideration. Substituting these values back to Equations 5-21 and 5-22 results in \( \nu = 0.1490 \) and \( E = 1.6834 \times 10^4 \, \text{N/mm}^2 \).

By inserting the expression for the deflection \( w(x, y) \) into Equations 5-18 to 5-20, the bending moments and torsions per unit length are given by Equations 5-23 to 5-25.

\[ M_x = \frac{16P_0}{\pi^4} \sum_{m} \sum_{n} \left( \frac{m}{a} \right)^2 + \nu \left( \frac{n}{b} \right)^2 \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi z}{b} \right) \frac{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}{mn} \]  \hspace{2cm} (5-23)

\[ M_z = \frac{16P_0}{\pi^4} \sum_{m} \sum_{n} \left( \frac{n}{b} \right)^2 + \nu \left( \frac{m}{a} \right)^2 \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi z}{b} \right) \frac{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}{mn} \]  \hspace{2cm} (5-24)

\[ M_{xz} = -\frac{16P_0}{\pi^4} (1 - \nu) \sum_{m} \sum_{n} \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi z}{b} \right) \frac{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi z}{b} \right)}{ab} \]  \hspace{2cm} (5-25)

Using the moments determined above, the corresponding stresses can be obtained using the Equations 5-26 to 5-28
\[
\sigma_x = -\frac{192P_0}{\pi^4h^3} y \sum_{m} \sum_{n} \left[ \left( \frac{m}{a} \right)^2 + v \left( \frac{n}{b} \right)^2 \right] \frac{\sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right)}{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{5-26}
\]

\[
\sigma_z = -\frac{192P_0}{\pi^4h^3} y \sum_{m} \sum_{n} \left[ \left( \frac{n}{b} \right)^2 + v \left( \frac{m}{a} \right)^2 \right] \frac{\sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi z}{b} \right)}{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{5-27}
\]

\[
\sigma_{xz} = -\frac{192P_0}{\pi^4h^3} y(1-v) \sum_{m} \sum_{n} \left[ \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi z}{b} \right) \right] \frac{ab}{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{5-28}
\]

Appendix 2 includes a computer program written in MATLAB to determine deflections, bending moments and stresses using KLT based equations. The analysis results are presented in Figures 5-10 to 5-16.
Figure 5-10: Mid-plane deflection of the SVBS

Figure 5-11: Bending moment in x-direction, $M_x$
Figure 5-12: Bending moment in z-direction, $M_z$

Figure 5-13: Torsional moment, $M_{xz}$
Figure 5-14: Maximum stress on the long direction, $\sigma_x$, on the top face

Figure 5-15: Maximum bending stress on the short direction, $\sigma_z$, on the top face
Figure 5-16: In-plane shear stress, $\sigma_{xz}$ of the SVBS

The deflection, moment and stress outputs based on KLT and FSDT agree very well. The maximum values were compared and found to be within 10% difference with the exception of the torsional moment and its corresponding stress. However, KLT results are less than the corresponding FSDT results because KLT does not take into account the effect of transverse shear deformations.

KLT is less accurate than FSDT in the sense that elements in KLT are assumed to remain perpendicular to the mid-plane, yet equilibrium requires that stress components $\sigma_{xy}, \sigma_{zy}$ are non-zero (which would cause these elements to deform). FSDT is more accurate, but it still makes the assumption that the out of plane stress, $\sigma_{yy}$, equal to zero. It is also worth noting that both FSDT and KLT theories are approximations of the exact three-dimensional equations of elasticity.
By using similar development, the approach presented could be extended to other loading cases and boundary conditions.
CHAPTER 6: PARAMETRIC STUDIES

6.1 Overview

The overall objective of this chapter is to conduct parametric studies to address the following points:

I. Extend the FE based micromechanical analyses described in chapter 3 so that the ABDE stiffness matrices of SVBS systems with five different geometric configurations (depth and diameter of void former) are determined.

II. Compare the ABDE stiffness matrix of the SVBS with the five different configurations with the ABDE stiffness matrix values of solid slabs with equal thickness. In addition, compare specific stiffness values of these SVBS systems and their corresponding solid slabs.

III. Determine the stiffness values of SVBS340 using a geometrically different second RVE type (RVE-2), and compare it with the stiffness values of SVBS340 that were determined based on the first RVE type (RVE-1) in chapter 3.

IV. Investigate the effect of span length on structural response (maximum deflection) of a SVBS and solid slab with equal thickness.

6.2 Comparison of SVBS and Solid Slabs with Equal Depth

The procedures described in chapters 3 and 4 are fully repeated in this chapter to determine and compare the stiffness values of SVBS and solid slabs with five different
configurations, namely SVBS230, SOLID230, SVBS280, SOLID280, SVBS340, SOLID340, SVBS390, SOLID390, SVBS450 and SOLID450. The material properties provided in Table 3.1 are also used for all slabs considered in this chapter. The ABDE stiffness values of these slabs are determined using the FE based micromechanics homogenization. Tables 6.1 through 6.5 show the geometrical parameters and the stiffness values of SVBS230 vs. SOLID230, SVBS280 vs. SOLID280, SVBS340 vs. SOLID340, SVBS390 vs. SOLID390, and SVBS450 vs. SOLID450, respectively. Figure 6-1 shows typical SVBS and the corresponding solid slab with equal depth. The only geometrical difference between the SVBS and their corresponding solid slabs is the presence of void formers that cause the spherical voids at the middle region of SVBS.

Figure 6-1: Typical SVBS and equal thickness solid slab models (a) SVBS, and (b) solid slab.
Table 6-1: Geometric parameters and computed stiffness values of SVBS230 and SOLID230

<table>
<thead>
<tr>
<th>Parameters</th>
<th>SVBS230</th>
<th>SOLID230</th>
<th>Ratio, %</th>
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</thead>
<tbody>
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</tr>
<tr>
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<td>230</td>
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</tr>
<tr>
<td>Void Former Diameter (mm)</td>
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<td>-</td>
<td></td>
</tr>
<tr>
<td>Length (mm)</td>
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<td>210</td>
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<td>Width (mm)</td>
<td>210</td>
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<td></td>
</tr>
<tr>
<td>Void Diameter/Thickness</td>
<td></td>
<td>78.26</td>
<td></td>
</tr>
<tr>
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<td>23.33</td>
<td>69.89</td>
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<td>4#4</td>
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<td>Concrete Cover (mm)</td>
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<td>30</td>
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</tr>
<tr>
<td><strong>Stiffness</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$A_{11}$ (N/mm)</td>
<td>4.60E+06</td>
<td>7.83E+06</td>
<td>58.76</td>
</tr>
<tr>
<td>$A_{12}$ (N/mm)</td>
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<td>1.57E+06</td>
<td>43.20</td>
</tr>
<tr>
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<td>2.76E+06</td>
<td>42.91</td>
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<tr>
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<td>$D_{21}$ (N-mm)</td>
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</tr>
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<td>$E_{55}$ (N/mm)</td>
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<td>2.61E+06</td>
<td>45.21</td>
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Table 6-2: Geometric parameters and computed stiffness values of SVBS280 and SOLID280

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<td>255</td>
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<td>Void Diameter/Thickness</td>
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<td>30</td>
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<tr>
<td><strong>Stiffness</strong></td>
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<tr>
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<td>$E_{55}$ (N/mm)</td>
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Table 6-3: Geometric parameters and computed stiffness values of SVBS340 and SOLID340

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<th>Ratio, %</th>
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<td>340</td>
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<td></td>
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<td>Length (mm)</td>
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</tr>
<tr>
<td>Width (mm)</td>
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<td>300</td>
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</tr>
<tr>
<td>Void Diameter/Thickness</td>
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<td>30</td>
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</tr>
<tr>
<td><strong>Stiffness</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$A_{11}$ (N/mm)</td>
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<td>3.36E+06</td>
<td>39.83</td>
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<tr>
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<td>1.07E+11</td>
<td>83.50</td>
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<tr>
<td>$D_{21}$ (N-mm)</td>
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<tr>
<td>$D_{33}$ (N-mm)</td>
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<td>$E_{55}$ (N/mm)</td>
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Table 6-4: Geometric parameters and computed stiffness values of SVBS390 and SOLID390

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<td>390</td>
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<td>Length (mm)</td>
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<td>345</td>
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<tr>
<td>Concrete Cover (mm)</td>
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<td>30</td>
<td></td>
</tr>
<tr>
<td><strong>Stiffness</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A_{11} (N/mm)</td>
<td>6.53E+06</td>
<td>1.26E+07</td>
<td>51.86</td>
</tr>
<tr>
<td>A_{12} (N/mm)</td>
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<td>2.40E+06</td>
<td>41.74</td>
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<td>A_{33} (N/mm)</td>
<td>1.49E+06</td>
<td>3.81E+06</td>
<td>39.11</td>
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<td>D_{11} (N-mm)</td>
<td>1.28E+11</td>
<td>1.60E+11</td>
<td>79.71</td>
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<tr>
<td>D_{21} (N-mm)</td>
<td>2.54E+10</td>
<td>3.07E+10</td>
<td>82.94</td>
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<tr>
<td>D_{33} (N-mm)</td>
<td>7.46E+10</td>
<td>9.80E+10</td>
<td>76.14</td>
</tr>
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<td>E_{55} (N/mm)</td>
<td>1.73E+06</td>
<td>4.20E+06</td>
<td>41.05</td>
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</table>
As shown in Tables 6.1 through 6.5, the stiffness values of the SVBS are generally less than the stiffness values of the corresponding solid slabs with equal depths. The extensional and shear stiffness values of the SVBS are determined to be significantly less than that of the solid slabs. Hence, in situations where shear and axial loads are considerable, SVBS may not be economically feasible. However, slabs typically are subjected to high bending (flexural) loading but minimal axial loading (except near column locations or in heavy concentrated force occupancy use). Therefore, SVBS is
economically feasible because there is a significant reduction in weight afforded by using SVBS with only a small reduction in flexural stiffness values.

Figures 6-2 through 6-6 show bar graphs for the stiffness values obtained for the five SVBS configurations and solid slabs equal thickness considered in this section.

![Graph showing extensional stiffness properties, $A_{11}$, for SVBS and solid slabs](image)

**Figure 6-2:** Extensional stiffness properties, $A_{11}$, for SVBS and solid slabs
Figure 6-3: In-plane shear stiffness properties, $A_{33}$, for SVBS and solid slabs

Figure 6-4: Bending stiffness properties, $D_{11}$, for SVBS and solid slabs
Figure 6-5: Torsional stiffness properties, $D_{33}$, for SVBS and solid slabs

![Graph showing $D_{33}$ properties for SVBS and solid slabs](image)

Figure 6-6: Transverse shear stiffness properties, $D_{11}$, for SVBS and solid slabs

![Graph showing $E_{55}$ properties for SVBS and solid slabs](image)
In addition to the ABDE stiffness values of the aforementioned SVBS and solid slabs, their specific stiffness values are also compared. Specific stiffness values in this dissertation are determined by dividing the stiffness values to the weight of the slabs. In other words, specific stiffness is one measure of the efficiency of a given material. Higher values of specific stiffness indicate good practice, with great potential in saving material and self-weight. Table 6-6 summarizes the specific stiffness of the SVBS and solid slabs considered in this chapter.

<table>
<thead>
<tr>
<th>Slab Configurations</th>
<th>Specific Stiffness</th>
<th>Specific Stiffness</th>
<th>Specific Stiffness</th>
<th>Specific Stiffness</th>
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<tbody>
<tr>
<td></td>
<td>A₁₁ (N/mm/kg)</td>
<td>A₃₃ (N/mm/kg)</td>
<td>D₁₁ (N-mm/mm/kg)</td>
<td>D₃₃ (N-mm/mm/kg)</td>
</tr>
<tr>
<td>SVBS230</td>
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<td>2.15E+09</td>
<td>1.05E+09</td>
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<tr>
<td>SOLID230</td>
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<td>1.18E+05</td>
<td>1.79E+09</td>
<td>9.18E+08</td>
</tr>
<tr>
<td>% Difference</td>
<td>-15.97</td>
<td>-38.56</td>
<td>20.11</td>
<td>14.38</td>
</tr>
<tr>
<td>SVBS280</td>
<td>1.85E+05</td>
<td>4.59E+04</td>
<td>2.10E+09</td>
<td>1.06E+09</td>
</tr>
<tr>
<td>SOLID280</td>
<td>2.21E+05</td>
<td>6.88E+04</td>
<td>1.75E+09</td>
<td>8.98E+08</td>
</tr>
<tr>
<td>% Difference</td>
<td>-16.29</td>
<td>-33.28</td>
<td>20.00</td>
<td>18.04</td>
</tr>
<tr>
<td>SVBS340</td>
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<td>2.87E+04</td>
<td>1.91E+09</td>
<td>1.10E+09</td>
</tr>
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<td>4.78E+04</td>
<td>1.51E+09</td>
<td>9.24E+08</td>
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<td>-39.96</td>
<td>26.49</td>
<td>19.05</td>
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<td>2.16E+04</td>
<td>1.85E+09</td>
<td>1.08E+09</td>
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<td>3.57E+04</td>
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<td>23.33</td>
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<td>% Difference</td>
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<td>-38.29</td>
<td>23.18</td>
<td>17.90</td>
</tr>
</tbody>
</table>

Table 6-6: Specific stiffness values of SVBS and solid slabs
The most important stiffness value for slabs under bending is $D_{11}$. As discussed in chapter 5, conventional slabs are typically subjected mainly to transverse loading, and they resist the loadings mainly through bending. Comparing the $D_{11}$ values at Table 6-6, the conclusion is that SVBS are consistently more efficient than solid slabs with equal depth. The difference observed is in the range of 20.00 % - 26.49 %. Hence, for the same weight of concrete, SVBS is shown to provide a substantially higher bending stiffness value. It is worth noting that the stiffness values computed in this chapter do not include the self-weight of the RVEs as well, which implies that the overall load acting on the SVBS are considerably less than the corresponding solid slabs with equal depth. This weight reduction is the main advantage of the SVBS systems as described in section 2-1.

### 6.3 Comparison of RVE Types (RVE-1 and RVE-2)

Nemat-Nasser and Hori (1993) defined an RVE for a material point of a continuum mass as a material volume which is statistically representative of the infinitesimal material neighborhood of that point. Accordingly, an RVE must be large enough to include a substantial number of the microscopic heterogeneities in order to be a statistical representative of the whole microstructure. At the same time, it should remain small enough to be considered as a volume element of continuum mechanics. Hence, the RVE serves as a dividing boundary between continuum theories and microscopic theories. An RVE thus represents the limit of heterogeneity. For scales larger than the RVE, the heterogeneities are smeared out to obtain an equivalent homogeneous medium. Hence, alternatives to RVE-1 shown in Figure 3-3 must be examined. RVE-2, as shown in Figure
6-7, is an alternative geometry for a microscopically heterogeneous slab, mainly SVBS 340. The geometry of SVBS340 is provided in Table 6-3.

Based on the definition of an RVE, the two RVEs described above can be interchangeably used for homogenizing the SVBS they represent. In this section, RVE-2 is analyzed to determine its stiffness values that are more prevalent and the results are compared with the results obtained from the analysis of RVE-1 presented in chapter 3.

The material properties provided in Table-1 are used for analyzing RVE-2. Due to the difference in geometry to RVE-1, the generated mesh in this section is different from
the mesh used in chapter 3. Same relevance of 40 with medium sizing is selected, and the number of nodes and elements generated are found to be 625,082 and 405,866 respectively. Figure 6-8 shows the generated mesh. The simulations are conducted using default elements that are generated in a similar fashion as described in chapter 3.

![Meshing result for the analysis model of RVE-2](image)

**Figure 6-8: Meshing result for the analysis model of RVE-2**

In determining extensional stiffness values, $A_{11}$ and $A_{21}$, similar procedures (boundary conditions and prescribed displacements) as described in Table 3-3 are implemented. Figure 6-9 shows the boundary conditions applied on the faces to
determine $A_{11}$ and $A_{21}$. Figure 6-10 shows the deformation contour in the x-directions resulted in due to the prescribed displacement and boundary conditions.

**Figure 6-9: Boundary conditions for $A_{11}$ and $A_{21}$ of RVE-2**
Figure 6-10: Directional deformation contour plot when $\varepsilon_1 = 1$, and $\varepsilon_2 = \varepsilon_3 = 0$ of RVE-2

The resultant force at the restrained face at $x = 0$ under the prescribed unit strain is obtained to be $R_1 = 1.7805 \times 10^9$ N and the reaction normal forces acting on both transverse faces (at $z = 0$ and $z = L_z$) is obtained to be $R_2 = 2.6536 \times 10^8$ N from the FE analysis. Hence, $A_{11}$ and $A_{21}$, which are forces per unit width at faces $x = 0$ and $z = 0$, are determined to be $5.9350 \times 10^6$ N/mm and $8.8453 \times 10^5$ N/mm. These extensional stiffness values are almost equal to the results of RVE-1 analysis. The percentage difference is less than 0.1%.

The bending stiffness values, $D_{11}$ and $D_{21}$, are also determined for RVE-2. The boundary conditions and prescribed rotation provided in section 3.3.2 are repeated here as shown in Figure 6-11. Figure 6-12 shows the deformation contour of RVE-2 under the boundary conditions and the prescribed rotation given in Figure 6-11.
Figure 6-11: Boundary conditions and prescribed rotation to determine bending stiffness of RVE-2
Under this prescribed unit rotation and boundary condition, the resultant moments at the restrained faces are the same as the flexural stiffness values, $D_{11}$ and $D_{21}$, which are obtained to be $8.9753 \times 10^{10}$ N-mm and $1.6796 \times 10^{10}$ N-mm, respectively. These values are again almost equal to the results of RVE-1 analysis. The percentage difference is less than 0.1 %. The very small discrepancy may be attributed to the difference in meshing.

The conclusion from these comparisons is that the two RVE types (RVE-1 and RVE-2) can be used interchangeably without affecting the prevalent stiffness values.
6.4 Effect of Span Length on Deflection of SVBS and Solid Slabs

This section presents deflection comparisons for two short span (M1) and two long span (M2) SVBS and solid slab models having an equal depth of 34 cm and subjected to uniform loads. The models are simulated using ANSYS after smearing the reinforcing steel with concrete on the outer solid portion. This is an approximate FE analysis since the reinforcing steel is not discretely modeled. Modeling reinforcing steel discretely for slab sizes considered in this section requires extremely large amounts of memory and CPU time. However, the approximate finite element analysis provides a good insight on the effect of void formers on the structural responses of short and long span SVBS and solid slabs. The ratio of steel to concrete in the smeared steel concrete layers is 6.47% by volume and the material properties of the outer portions of the slabs are averaged accordingly. The general material properties used for pure concrete, smeared steel concrete, and steel are provided in Table 6-7 below.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Pure Concrete</th>
<th>Steel</th>
<th>Smeared Steel Concrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density (kg/m³)</td>
<td>2400</td>
<td>7850</td>
<td>3272.8</td>
</tr>
<tr>
<td>Compressive Ultimate strength (MPa)</td>
<td>40.0</td>
<td>250</td>
<td>62.8</td>
</tr>
<tr>
<td>Compressive Yield Strength (MPa)</td>
<td>-</td>
<td>250</td>
<td>16</td>
</tr>
<tr>
<td>Tensile Yield Strength (MPa)</td>
<td>-</td>
<td>250</td>
<td>16</td>
</tr>
<tr>
<td>Tensile Ultimate Strength (MPa)</td>
<td>4.0</td>
<td>460</td>
<td>21.63</td>
</tr>
<tr>
<td>Young’s Modulus (GPa)</td>
<td>40.0</td>
<td>200</td>
<td>59.8</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.2</td>
<td>0.3</td>
<td>0.245</td>
</tr>
<tr>
<td>Bulk Modulus (GPa)</td>
<td>22.2</td>
<td>166.7</td>
<td>39.08</td>
</tr>
<tr>
<td>Shear Modulus (GPa)</td>
<td>16.7</td>
<td>76.9</td>
<td>24.0</td>
</tr>
<tr>
<td>Bilinear Isotropic Hardening Yield</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yield Strength (MPa)</td>
<td>250</td>
<td></td>
<td>15.0</td>
</tr>
</tbody>
</table>
Table 6-8 shows the general geometrical configurations of the M1 and M2 models.

Table 6-8: Geometrical configurations of models

<table>
<thead>
<tr>
<th>Model name</th>
<th>Dimensions in X, Y and Z Axis</th>
<th>Void former diameter</th>
<th>Volume of material</th>
<th>Mass of material</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1-SVBS340</td>
<td>7.1m, 0.34m, 4.8m</td>
<td>0.27m</td>
<td>2.1 m³</td>
<td>5597.2 kg</td>
</tr>
<tr>
<td>M1-SOLID340</td>
<td>7.1m, 0.34m, 4.8m</td>
<td>-</td>
<td>2.9 m³</td>
<td>7472.9 kg</td>
</tr>
<tr>
<td>M2-SVBS340</td>
<td>13.4m, 0.34m, 13.4m</td>
<td>0.27m</td>
<td>11.1 m³</td>
<td>29479 kg</td>
</tr>
<tr>
<td>M2-SOLID340</td>
<td>13.4m, 0.34m, 13.4m</td>
<td>-</td>
<td>15.3 m³</td>
<td>39373.2 kg</td>
</tr>
</tbody>
</table>

All the four models consist of a 27 cm thick pure concrete, sandwiched between 3.5 cm thick top and bottom smeared concrete steel-layers. The smallest distance amongst the closest spherical voids is 30 mm in both transverse and longitudinal directions. While the short span slab models (M1) are analyzed as column supported at the four corners, the long span models (M2) are analyzed as fixed supported in all four edges. To minimize the effect of support conditions on the behavior SVBS, voids are avoided around the support. Steel plates, as a cushion, are provided to avoid punching shear failure around the columns when analyzing the short span slab models under uniform distributed loads.

Since the slabs under consideration have symmetries both in the transverse and longitudinal directions, only quarters models are studied. Frictionless supports are provided at the faces of the quarter symmetry. Because M1 and M2-Solid Slabs have the
same geometry and boundary configurations as their corresponding SVBS excluding the voids, only models of SVBS are displayed in Figure 6-13 and Figure 6-14 for illustrative purposes.

Figure 6-13: Geometrical configuration of M1-SVBS, (a) isometric view, and (b) profile view in the longitudinal direction.
For all solid and SVBS models, medium relevance solid elements were used. However, to capture the effect of the voids in SVBS, the advanced size function for curvature is selected. The mesh metrics known as skewness, which is a measure of mesh quality, is checked for all models and found to be in the acceptable range. Table 6-9 summarizes the number of nodes and elements used in the four models. Figure 6-15 shows the mesh created for M1-SVBS model as a representative example.

Table 6-9: Number of nodes and elements used in the analyses.

<table>
<thead>
<tr>
<th></th>
<th>M1-SVBS</th>
<th>M1-Solid</th>
<th>M2-SVBS</th>
<th>M2-Solid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodes</td>
<td>80630</td>
<td>37688</td>
<td>264083</td>
<td>37474</td>
</tr>
<tr>
<td>Elements</td>
<td>38969</td>
<td>6235</td>
<td>134398</td>
<td>5776</td>
</tr>
</tbody>
</table>
Nonlinear analysis option is used in this parametric study. Note that this section (6.4) is not related to the elastic analyses presented in chapters 3-5, but rather uses smeared FE modeling for the sole purpose of determining load-deflection curves and commenting on the effect of span length on SVBS efficacy compared to solid slabs.

By default, ANSYS uses the Newton-Raphson method to solve nonlinear problems. In nonlinear analysis, the total load applied to a finite element model is divided into a series of load increments called load steps (Lee 2012). At the completion of each incremental solution, the stiffness matrix of the model is adjusted to reflect nonlinear changes in structural stiffness before proceeding to the next load increment. The large deformation effect is also activated so that at each sub-step, a new stiffness matrix is calculated based on the deformed geometry. The automatic time stepping in ANSYS predicts and controls load step sizes for which the maximum and minimum load step
sizes are required. Fifteen and ten load steps have been used for M1 and M2 models, respectively. While a total of 100 KPa is incrementally applied to M1-SOLID340 and M1-SVBS340 models, only 50 KPa is applied to M2-SOLID340 and M2-SVBS340 models to evaluate the resistance of the slabs.

The primary output of the non-linear finite element analyses is maximum deflection. Figure 6-16 shows the maximum deflections corresponding to the incremental loads obtained from the analyses. The results showed that the deflections in both short and long span SVBS models are relatively higher than the corresponding deflections in the solid slabs. This is attributed to the fact that the SVBS is less stiff than solid slabs of equal dimensions. However, the difference in deflection between the long span slabs (M2-SVBS and M2-SOLID) is determined to be much less than the short span slabs (M1-SVBS and M1-SOLID). Generally, as the span gets longer, the slab becomes less stiff. At the same time, the self-weight increase associated with the increase in span length causes larger deflections and it is more pronounced on solid slabs than SVBS. This explains why the difference in deflection between the long span models is small.
Figure 6-16: Maximum deflection versus applied load, (a) for short span models and (b) for long span models.
CHAPTER 7: CONCLUSIONS

This chapter presents the summary of the research, highlights key findings and provides recommendations for future study.

7.1 Summary

The demand for reducing the weight-to-stiffness and strength ratios of solid reinforced concrete slabs, while improving their span limit, has led to the emergence of spherically voided biaxial slab (SVBS) systems. However, the presence of the spherical voids makes the structural analysis of SVBS inherently challenging because of the heterogeneous and periodically varying complex cross-sectional geometry. Current analysis procedures heavily rely on proprietary manufacturer information with a recommendation of treating SVBS as solid reinforced concrete slabs and applying global reduction factors without giving regard to the microstructures of the SVBS itself. This dissertation presents new load-response analysis models that use a finite element based micromechanical homogenization procedure for predicting the elastic structural behavior of SVBS. Firstly, a representative volume element (RVE) or unit cell of a selected SVBS has been established and analyzed to determine the equivalent RVE stiffness properties, i.e. the extensional stiffness matrix [A], the coupling stiffness matrix [B], and the bending stiffness matrix [D]. Independent unit strains and unit curvatures are prescribed to the RVE in ANSYS, and the resulting resultant moments and forces are used to calculate the
[A], [B] and [D] matrices. A procedure combining finite element analysis and Timoshenko’s beam theory is proposed for determining the transverse shear stiffness matrix [E] with an acceptable margin of error. The micromechanical homogenization procedure is then verified by applying it to a homogenous isotropic RVE for which results can be determined analytically. Secondly, the ABDE matrices of the RVE are used to analyze a full-scale 2D orthotropic SVBS floor system by Mindlin-Reissner and Kirchhoff-Love plate theories which are programmed in MATLAB. Thirdly, parametric studies are presented investigating stiffness variations in five different configurations of SVBS, selection of alternative RVE type, and the effect of span length on SVBS efficacy.

### 7.2 Conclusions

The main outcomes from this research are summarized below.

- The FE based micromechanical procedures are used to determine the extensional stiffness matrix [A], the coupling stiffness matrix [B], and the bending stiffness matrix [D], and the transverse shear stiffness matrix [E] for a representative volume element (RVE) of a selected SVBS (i.e. SVBS340).

- The FE based micromechanical procedures are independently verified by applying the procedures on a plain concrete slab. The stiffness values obtained are shown to be within 5% of the results from analytical equations available for homogenous and isotropic plates.

- An approximate procedure combining FE analysis and an analytical equation based on Timoshenko’s beam is presented for determining the shear stiffness values
because of the difficulty in modeling a state of deformation resulting from pure shear forces.

- Slabs are generally subjected to transverse loads that result in high flexural stresses but minimal axial and shear stresses (except near column locations or in heavy concentrated force occupancy use). Therefore, the in-plane bending stiffness properties, $D_{11}$ and $D_{22}$, are the governing properties in the structural behavior and capacities of slabs. The in-plane bending stiffness properties of SVBS are determined to be 79.25% - 83.83% of the corresponding stiffness properties of solid slabs with equal depth. This implies that the flexural capacities of SVBS are not significantly compromised due to the void formers. However, the concrete weight saving of the SVBS is significantly high, in the range of 30.11% - 35.69%.

- With reference to the specific in-plane bending stiffness properties, SVBS are shown to be consistently more efficient than solid slabs with equal depth. Hence, for the same weight of concrete, a substantially higher specific bending stiffness value in the range of 20.00% - 26.49% is obtained for SVBS.

- The FE based micromechanical analysis results show that there is a major loss of shear capacity of SVBS compared to solid slabs; hence either avoiding void formers in the vicinity of considerable shear forces such as columns or the use of shear reinforcement is recommended.

- The axial stiffness properties of SVBS are significantly lower than solid slabs of equal thickness; hence, SVBS should not be used where there is a considerable axial load.
Since SVBS has Vierendeel-like cross-sections, it is sensitive to shear distortion. Therefore, the first order shear deformable plate theory (FSDT) accounting for shear deformation is necessary for analyzing SVBS. Comparison of results from FSDT and the counterpart plate theory that ignores shear deformation effects (KLT) show that the deflections, moments and stresses are significantly underestimated by KLT.

Two different types of RVEs are studied. Both RVEs (called RVE-1 and RVE-2) are shown to produce similar results and they can be used interchangeably.

Simplified and approximate full size FE analysis also showed that the maximum deflection of short span SVBS is considerably greater than solid slab of equal depth; however, the maximum deflection becomes comparable when the span length increases. This is because self-weight of the slabs has more pronounced impact on long span slabs than short span slabs. Therefore, the use of SVBS is recommended for long span slabs.

To make FE based micromechanical approach a practical design tool, it is essential to develop a database (library) of the ABDE stiffness values for SVBS of different configuration. Hence, this research has documented the ABDE stiffness values for the five known configurations of SVBS presented in Table 1-1.

### 7.3 Suggested Future Studies

The following future studies are suggested to improve the analysis procedures developed in this dissertation.

- A holistic analysis model that couples the micromechanical and the macromechanical analyses automatically can be developed; a parallel strategy may be adopted to
simultaneously determine the stiffness values of the RVE using the micromechanical analysis and the structural responses (moment, forces, stresses, deflection, etc.) of the SVBS using the macromechanical approach. Each of the RVE during the micromechanical analysis can be possibly assigned to each numerical integration point on the macromechanical finite element analysis.

- In this dissertation, the FE based micromechanics approach is used to predict the overall macroscopic stiffness properties of SVBS; however, the approach can be expanded to integrate the prediction of the overall macroscopic strength properties of SVBS by incorporating failure theories. Failure theories that are applicable to composite structures such as Tsai-Hill criterion can be studied. In this dissertation, surface cracking due to environmental loads (thermal and drying shrinkage) are not incorporated. Hence, adjustments for the stiffness values can be made for the effect of cracking. Accurate idealization of cracking in terms of crack width, crack height and crack spacing may be challenging because of uncertainties in estimating them effectively.

- RVE stiffness values determined in this dissertation are from elastic analysis, which is sufficient from a practical standpoint. However, non-linear analysis of the mechanical behavior of an RVE for SVBS can be included in future studies.

- In this dissertation, perfect bond is assumed between concrete and steel. However, failure in concrete slabs can be attributed to slippage between concrete and steel. The micromechanical approach presented here can be expanded to include slippage criterion.
• Very few published papers exist regarding experimental tests on SVBS. These papers tend to lack one or more pertinent information to compare to the FE based micromechanical model developed here. As part of future study, an experimental program can be developed to validate and synergize the FE based micromechanical analysis model or its refinement.

• In addition to the experimental program, a full 3D detailed finite element analyses can be considered to validate the FE based micromechanical analysis model.

• A database or library in the form of charts or tables that present the ABDE stiffness values of SVBS systems with a wide range of geometrical configurations and material properties can be prepared so that they become readily available for practicing structural engineers
% Structural Response of Slab based on FSDT Orthotropic plate theory
% this program calculates the deflection, moments and stresses
% based on thick plate theory using Navier's method that uses Fourier
% expansions for orthotropic slab
% Results are presented in the form of plots.

clear all
close all
clc

%%% INPUT PARAMETERS

a=12000;% Length of the slab in mm
b=10000;% Width of the slab in mm
step=200;% No. of grids
loop=25;% This is number of iteration
h=340; % This is thickness of the slab
z=h/2;% Distance of extreme fibers from mid-plane
P0=0.02;% Load applied including self-weight of the slab in MPa

%% Stiffnesses

D11=8.8932e10;% D11 in N-mm as determined from micromechanics
D22=8.8932e10;% D22 in N-mm as determined from micromechanics
D12=1.6834e10;% D12 in N-mm as determined from micromechanics
D33=5.1135e10;% D33 in N-mm as determined from micromechanics
E55=1.3590e6;% E55 in N-mm as determined from micromechanics
E66=1.3590e6;% E55 in N/mm as determined from micromechanics

%% CALCULATION BASED ON LOOPS

x=0:step:a;
y=0:step:b;
Amn=zeros(3,3);
qmn=zeros(3,1);
w_0=zeros(length(x),length(y));
phix=w_0;
phiy=w_0;
for m=1:2:loop
    for n=1:2:loop
        for i=1:length(x)
            for j=1:length(y)
                qmn=16*P0/(m*n*pi^2);
                Qmn=[0 0 qmn]';
                Amn=-[D11*(m*pi/a)^2+0.5*D33*(n*pi/b)^2+E66
                        (D12+0.5*D33)*(m*n*pi^2/(a*b)) E66*(m*pi/a);
                        (D12+0.5*D33)*(m*n*pi^2/(a*b))
                        D22*(n*pi/b)^2+0.5*D33*(m*pi/b)^2+D15
                        E55*(m*pi/a) E55*(n*pi/b)
                        E55*((m*pi/a)^2)+E55*((n*pi/b)^2)];
                for jj=1:3
                    e=Amn\Qmn;
                end
                bmn=e(1);
                cmn=e(2);
                amn=e(3);
            end
        end
    end
end

for m=1:2:loop
    for n=1:2:loop
        for i=1:length(x)
            for j=1:length(y)
                w_0(i,j)=amn*sin(m*pi*x(i)/a)*sin(n*pi*y(j)/b);
                phi_x(i,j)=bmn*cos(m*pi*x(i)/a)*sin(n*pi*y(j)/b);
                phi_y(i,j)=cmn*sin(m*pi*x(i)/a)*cos(n*pi*y(j)/b);
                phi_xx(i,j)=bmn*(m*pi/a)*sin(m*pi*x(i)/a)*sin(n*pi*y(j)/b);
                phi_yy(i,j)=cmn*(n*pi/b)*sin(m*pi*x(i)/a)*sin(n*pi*y(j)/b);
                phi_xy(i,j)=((bmn*n*pi/b)+(cmn*m*pi/a))*cos(m*pi*x(i)/a)*cos(n*pi*y(j)/b);
                eps_55(i,j)=(cmn+amn*n*pi/b)*sin(m*pi*x(i)/a)*cos(n*pi*y(j)/b);
                eps_66(i,j)=(bmn-amn*m*pi/a)*cos(m*pi*x(i)/a)*sin(n*pi*y(j)/b);
                mm=[D11 D12 0; D12 D22 0; 0 0 D33]*[phi_xx(i,j),phi_yy(i,j),
                     phi_xy(i,j)]';
                mx_0(i,j)=mm(1);
                mz_0(i,j)=mm(2);
                mxz_0(i,j)=mm(3);
                qqx=[E55 0;0 E66]*[eps_55(i,j) eps_66(i,j)]';
                qqqx=qqx(l);
qqy=qq(2);
end
end
w_T=w_T+w_0;
mx_T=mx_T+mx_0;
mz_T=mz_T+mz_0;
mxz_T=mxz_T+mxz_0;
qx=qqx+w_0;
qy=qqy+w_0;
end
end
w=w_T;
mx=mx_T;
my=mz_T;
mxxy=mxz_T;
Qx=qx;
Qy=qy;
sigmax=-12*z.*mx_T/h^3;
sigmaxz=-12*z.*mz_T/h^3;
sigmaxz=-12*z.*mxz_T/h^3;
sigzx=Qx/h; % Average Shear Stresses

% RESULTS IN PLOTS
%---------------------------------------------------------------------
figure(1)
surf(w')
xlabel('x')
ylabel('z')
title('w')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(w)) max(max(w))])
figure(2)
surf(mx')
xlabel('x')
ylabel('z')
title('mx')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(mx)) max(max(mx))])
figure(3)
surf(my')
xlabel('x')
ylabel('z')
title('my')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(my)) max(max(my))])
figure(4)
surf(mxxy')
xlabel('x')
ylabel('z')
title('mxxy')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(mxxy)) max(max(mxxy))])
figure(5)
surf(sigmax)
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigmaz)) max(max(sigmaz))])
figure(6)
surf(sigmaz')
xlabel('x')
zlabel('z')
title('sigmaz')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigmaz)) max(max(sigmaz))])
figure(7)
surf(sigmaxz')
xlabel('x')
zlabel('z')
title('sigmaxz')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigmaxz)) max(max(sigmaxz))])
APPENDIX 2: MATLAB CODE-KLT SVBS ANALYSIS

% Structural Response of Slab based on KLT isotropic approach
% This program calculates the deflection, moments and stresses
% based on Thin Plate Navier's method that uses Fourier expansions for
% an equivalent isotropic plate. Results are presented in the form of
% plots.
%-------------------------------------------------------------------
clear all
close all
clc

%% INPUT PARAMETERS
%-------------------------------------------------------------------
%-------------------------------------------------------------------
a=12000;% Length of the slab in mm
b=10000;% Width of the slab in mm
step=200;% No. of grids
loop=25;% This is number of iteration
E=2.6179*10^4;% Equivalent modulus of elasticity of the slab in MPa
nu=0.1893;% Equivalent Poisson’s ratio of the slab
h=340;% Height of the slab
y=h/2;% Distance of extreme fibers from centroid
P0=0.02;% Load applied including self-weight of the slab in MPa

%% CALCULATION BASED ON LOOPS
%-------------------------------------------------------------------
%-------------------------------------------------------------------
x=0:step:a;
z=0:step:b;
w_0=zeros(length(x),length(z));
mx_0=w_0;
mz_0=w_0;
mxz_0=w_0;
w_T=w_0;
mx_T=w_0;
mz_T=w_0;
mxz_T=w_0;
D=E*h^3/(12*(1-nu^2));
for m=1:2:loop
  for n=1:2:loop
    for i=1:length(x)
      for j=1:length(z)
        w_0(i,j)=sin(m*pi*x(i)/a)*sin(n*pi*z(j)/b)/((m^2/a^2+n^2/b^2)^2*m*n);
\[
\begin{align*}
\text{mx}_0(i,j) &= (\frac{(m/a)^2 + \nu(n/b)^2}{(m^2/a^2+n^2/b^2)^2}) \cdot \sin(m\pi x(i)/a) \cdot \sin(n\pi z(j)/b) \\
\text{mz}_0(i,j) &= (\frac{(n/b)^2 + \nu(m/a)^2}{(m^2/a^2+n^2/b^2)^2}) \cdot \sin(m\pi x(i)/a) \cdot \sin(n\pi z(j)/b) \\
\text{mxz}_0(i,j) &= \cos(m\pi x(i)/a) \cdot \cos(n\pi z(j)/b) / (a \cdot b) \\
\end{align*}
\]

\[
\begin{align*}
\text{w}_T &= \text{w}_T + \text{w}_0; \\
\text{mx}_T &= \text{mx}_T + \text{mx}_0; \\
\text{mz}_T &= \text{mz}_T + \text{mz}_0; \\
\text{mxz}_T &= \text{mxz}_T + \text{mxz}_0; \\
\end{align*}
\]

\[
\begin{align*}
\text{w} &= -16P0/(\pi^6 D) \cdot \text{w}_T; \\
\text{mx} &= 16P0/(\pi^4) \cdot \text{mx}_T; \\
\text{mz} &= 16P0/(\pi^4) \cdot \text{mz}_T; \\
\text{mxz} &= -16P0/\pi^4 (1-\nu) \cdot \text{mxz}_T; \\
\text{sigma}_x &= -192P0y/h^3\pi^4 \cdot \text{mx}_T; \\
\text{sigma}_z &= -192P0y/h^3\pi^4 \cdot \text{mz}_T; \\
\text{sigma}_xz &= -192P0y/h^3\pi^4 (1-\nu) \cdot \text{mxz}_T; \\
\end{align*}
\]

```matlab
w_T = w_T + w_0;
mx_T = mx_T + mx_0;
mz_T = mz_T + mz_0;
mxz_T = mxz_T + mxz_0;

w = -16*P0/(pi^6*D).*w_T;
mx = 16*P0/(pi^4).*mx_T;
mz = 16*P0/(pi^4).*mz_T;
mxz = -16*P0/pi^4*(1-nu).*mxz_T;
sigma_x = -192*P0*y/(h^3*pi^4).*mx_T;
sigma_z = -192*P0*y/(h^3*pi^4).*mz_T;
sigma_xz = -192*P0*y/(h^3*pi^4)*(1-nu).*mxz_T;

figure(1)
surf(w')
xlabel('x')
ylabel('z')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(w)) max(max(w))])
title('Deflection')
figure(2)
surf(mx')
xlabel('x')
ylabel('z')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(mx)) max(max(mx))])
title('mx')
figure(3)
surf(mz')
xlabel('x')
ylabel('z')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(mz)) max(max(mz))])
title('mz')
figure(4)
surf(mxz')
xlabel('x')
ylabel('z')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(mxz)) max(max(mxz))])
title('mxz')
figure(5)
surf(sigma_x')
xlabel('x')
ylabel('z')
```
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigma_x)) max(max(sigma_x))])
title('Sigma X')
figure(6)
surf(sigma_z')
xlabel('x')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigma_z)) max(max(sigma_z))])
title('Sigma z')
figure(7)
surf(sigma_xz')
xlabel('x')
axis([0 max(a,b)/step+1 0 max(a,b)/step+1 min(min(sigma_xz)) max(max(sigma_xz))])
title('Sigma xz')
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BIOGRAPHY

Wondwosen Belay Ali grew up in Ethiopia. He attended Mekelle University, where he received his Bachelor of Science in Civil Engineering in 2004. He also received his Master of Science in Geotechnics and Geohazards from Norwegian University of Science and Technology in 2008, and his Master of Science in Civil Engineering, emphasis on Structures, from Texas Tech University in 2010. He has been studying towards his doctoral degree in Civil and Infrastructure Engineering at George Mason University since August 2011.