SURVEY OF FUZZY SET THEORY IN ACTUARIAL LIFE MODELING

by

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of
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Bachelor of Science
George Mason University, 2012

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DEDICATION

This thesis is dedicated to the entire Boni family in Abidjan, Ivory Coast, my affectionate companion Ornella Pitah, my lovely friends and professors whose advices guided me to completion of this masterwork; and last but not least the all-mighty Lord Jesus Christ.
I would like to thank the many friends, relatives, and supporters who walked with me on this perilous path: Dr. Douglas Eckley, Dr. Tim Sauer and Dr. David Singman who allowed my scientific curiosity to wander to its supremum and for their invaluable help. The late Dr. Richard O’beirne, who taught me everything about Actuarial Science and inspired me in the notions of Financial Mathematics. Finally, this is a tribute to Magloire & Reine BONI for perfectly fulfilling their parental duties, along with Mackenzie Boni, Karine BONI & Ornella Pitah for their constant motivation and lasting hours of debate on my future.
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ABSTRACT

SURVEY OF FUZZY SET THEORY IN ACTUARIAL LIFE MODELING

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George Mason University, 2016
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This thesis describes Actuarial science and Fuzzy Logic as relatively recent fields of mathematics introducing methods for containing uncertainty and vagueness in the line of business in which it is being used. Whereas actuaries work on the financial risk in (re)insurance of future events, ‘Fuzzicists’ aim at modeling the degree to which such events may occur. In the process of researching and writing, the author conducted a literature search and review of Fuzzy Set Theory with a structural approach to actuarial modeling. Following the recent development and discoveries of fuzzy logic, life insurance actuaries gained ultramodern modeling techniques, replacing the sole use of probabilities that had started to become insufficient. This thesis is slated to span the applications of Fuzzy Mathematics in the actuarial modeling of Life Contingencies.
PART 1: STRUCTURAL APPROACH TO FUZZY LOGIC AND ACTUARIAL MODELING

1. Introduction

The year 1965 marked the birth of fuzzy logic as forefather Lotfi A. Zadeh published his paper entitled “Fuzzy Sets” in the journal of Information and Control. He introduced an alternative logic to the well-known Boolean logic that an event is either True (=1) or False (=0) or, as formerly stated in Aristotle’s law of excluded middle, an element is either contained in a set or not contained in a set. From this point of view, only a few elements of the real world can be properly represented, for everything has to be black and white and there are no shades of grey. On a note from Albert Einstein “So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.” Fuzzy logic proposes that sets of objects had boundaries not sharply defined, awarding elements to be contained in a set to a grade of membership. Today on its 50th year anniversary, it has evolved into a whole field of mathematics with its very own analysis, operations and rules. Pioneer work in the theory of Fuzzy Sets extends to actuarial science and, specifically for our purposes, life contingencies models. We must introduce and define the basic engineering of these fields.
1.1. Life Contingent Events

Actuarial Science is the field of study that uses mathematical and statistical methods to assess risk in insurance, finance, and other industries/professions. The study is frequently associated with insurance and stock markets where its principles are commonly applied. In Life Insurance, actuaries aim to assess the risk of losing a life, to an insurer by modeling the policyholder's life expectancy. A person’s life becomes a probabilistic set and its distribution is represented with various assumptions. One future lifetime becomes a random variable, and probabilities of death or survival are calculated. “Life contingencies” is a term used to describe survival models for human lives and resulting cash flows that start/stop contingent upon the state of a human life.

1.2. Fuzzy Logic

The traditional way of thinking in mathematics complies with Boolean logic (Boole 1847) and dates to 300 B.C. with Aristotle’s law of the excluded middle. It states that for any two contradictory propositions (i.e. where one proposition is the negation of the other) one must be true (1), and the other false (0). Later adapted to algebra as X must be either in a set A or not in A.

Fuzzy logic is rather an extended logic dealing with linguistic ambiguity and handling the concept of partial truths. Truth values of a variable may be any real number between 0 and 1. By linguistic ambiguity arises matters of daily life having two or more aspects whose boundaries have not been unanimously agreed upon. One such example is the temperatures for Cold, Hot and Warm. The three functions are plotted in Figure 1,
where the sense of transition between these aspects is more visible. Other examples may have more or less variables, such as Young-Adult-Old; Short-Tall, etc.

Fuzzy Set Theory (Zadeh 1965) is the mathematical field based on Fuzzy logic, dealing with the sets whose elements have degrees of membership. Define a Fuzzy set as a pair \((U, m)\) for a set \(U\) and a membership function \(m: U \rightarrow [0, 1]\). For each \(x \in U\), the value \(m(x)\) is the grade of membership of \(x\) in the fuzzy set. In particular, \(x\) is not included if \(m(x) = 0\), and \(x\) is fully included if \(m(x) = 1\). That means for all \(x \in U\): \(m(x) \in (0,1)\), \(x\) is at the same time partly included and partly not included, hence the concept of sets with no sharp boundaries. Classical sets are special cases of fuzzy sets called crisp sets, with membership function \(m: U \rightarrow \{0,1\}\).
1.3. Literature Review

Most insurance executives deal better with the crisp/traditional logic, and often transform imprecise statements into rigid rules. This is the case of Belgian insurers using fuzzy statistical evidence, such as "Young drivers provoke more automobile accidents" to set up the rating rule "Drivers under 23 years old will pay $150 deductible if they provoke an accident" (Lemaire 1990). Thus, the initial statement was distorted and “Young” was equated to "under 23," when 23 is only perhaps 80% young.

Since 1965, the count of publications on Fuzzy set theory has grown to exceed 50,000 today (Chen et al). We have experienced what is called a fuzzy boom since the 1990s thanks to pioneers in actuarial science such as Shapiro, Lemaire and Liu. Today, there are more researchers in Fuzzy Logic than in Actuarial Science, with important contributions from Japan, China & Russia. The evolution of the study in the literature...
started with linguistic variables and fuzzy sets, followed by fuzzy numbers arithmetic, fuzzy inference systems and fuzzy linear programming, and more recently fuzzy clustering with soft computing. It may be found in a variety of applications such as helicopter autopilot, home electronics, vehicle control, camera stabilization. The figure below from Zimmerman’s “Fuzzy Sets Theory and its applications” 2001 provides a better grasp of the evolution.

![Figure 3. Survey of Evolution (Zimmerman 2001)](image)

1.4. The Current Research

This investigation aims at presenting the applications of Fuzzy Set Methodology in an actuarial science framework with focus on modeling life contingencies. The
approach is meant to define where actuarial science and fuzzy logic intersect. First, the
traditional mechanism of life insurance will be explained, from the underwriting process
to the classification of policies in preference classes. This will require a review of
probability theory in human life modeling, with the customary use of survival models and
life tables for premium calculations. Second, the applications of fuzzy mathematics will
be fully described through a series of theorems and definitions, including fuzzy rules,
analysis, and clustering algorithms.

The second part of the research will survey the use of the aforementioned
applications in life insurance. This will involve the translation of the medical records of
applicants for life insurance using a fuzzy decision-making process, followed by the
classification of the policyholders by risk levels using a fuzzy system of preference
classes. Next, the author will remodel actuarial survival probabilities, insurance benefits
and premium computations using fuzzy parameters, to eliminate the inaccuracy caused by
the fluctuation of interest rates. Finally, each policyholder’s risk derived from actuarial
life tables using only the age factor will be rearranged in fuzzy clusters.

2. Traditional actuarial Life

2.1. Market Setup: Underwriting and Preferred Lives

Life insurance is a contract between a person and an insurance company. The
insurance company promises a compensation to a designated person with a certain
amount of money upon death of the insured, in return for periodic payments. Life
insurance has its own jargon and the designated person is called the beneficiary; the
money received upon death is the sum insured or death benefit; the insured is also called
the policyholder and his/her payments are called premiums. Underwriting is the process of determining which risk class an insured belongs to, based on several factors such as age, gender, physical condition, medical history, financial background, personal habits, profession, hobbies, etc. Underwriting is an important aspect for an insurance company: it classifies the applicants according to their level of risk, protects the company from fraud or identity thefts, benefits consumers by keeping the insurance more affordable with low premiums, and helps the solvency of the company.

The consumer chooses the type of insurance coverage he/she wishes. The traditional insurance contracts are:

- **Whole life insurance** pays a lump sum benefit on the death of the policyholder whenever it occurs. Premiums are often payable up to a maximum age (80).
- **Term insurance** pays a lump sum benefit on the death of the policyholder, provided death occurs before the end of a specified term. Subtypes of these contracts include term insurance renewable every year, and term insurance convertible to whole life insurance.
- **Pure Endowment** pays the insured himself if he survives a specified period but pays nothing in case of a death prior the specified date.
- **Endowment insurance** offers a lump sum benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs first.

(Term insurance + pure endowment).

Life insurance policies may involve a single premium at the beginning of the contract, or a series of premiums payable until death/end of term. Another type of
insurance contract to consider is *Life Annuity*. An annuity in financial mathematics is a contract that offers a regular series of payments to the buyer. If the annuity depends on survival of an individual, it is called a ‘life annuity’ and the recipient is the *annuitant*. Note that the beneficiary can be the policyholder himself in this case. These contracts are often purchased by older lives to provide income in retirement. They include:

- *Whole life annuity* pays until the death of the annuitant.

- *Term life annuity* pays up to an agreed upon date, and may stop upon death of the annuitant, if sooner.

Other types of insurance contracts that are of more recent vintage and are more attuned to the current economy are:

- *With-profit insurance* shares profits earned on the invested premiums are shared with the policyholders in the form of cash dividend, reduced premiums, or increased sum insured

- *Universal life insurance* puts premiums into an investment fund and deducts insurance charges from the fund periodically.

Once the type of contract is selected, underwriters do a classification of the risk level of the applicant following the guidelines of the insurance company executives. Insurance Risk Classes are groups of people with similar characteristics and risk level.

The classes may be defined based on age, sex, income and physical condition. In life insurance, an underwriting decision is made whereby the applicant may be either denied coverage or put into one of the following generic classes:
- **Preferred Class**: healthy, middle-aged individuals with stable income and a better than average risk of insolvency or mortality. They are charged lower (preferred) rates.

- **Standard Class**: This is for a more or less healthy person, complying with the definition of normal or typical risk the carrier desired to insure. This class is charged the standard rate.

- **Substandard/Rated Class**: In this class are applicants for higher coverage with only tolerable physical or medical condition. They represent an above average risk and are charged higher premium rates than the Preferred and the Standard classes.

- **Postponed**: Occasionally cases are postponed until additional information is gathered, some time passes, or until negative factors change in favor of assigning a risk class. For instance, a person with a medical condition could be subject to additional tests, such as a stress test to check cardiovascular functions.

- **Declined**: if the client represents an uninsurable risk, coverage is declined.

This is the type of underwriting scheme used by most life insurance carriers, in particular by the American International Group (AIG) in its ‘Field Underwriting Guide’. Other derivatives schemes can be found as in Figure 4, which shows the risk classes Deloitte Consulting uses for ‘Predictive modeling for Life Insurance.’ Note to the reader, declination of insurance coverage does not reflect the health or likelihood of dying, but solely a risky venture for the solvency of the company.
Life insurers agree that mortality is unevenly distributed in the population. For instance, with all factors being equal, females outlive males; the use of tobacco is detrimental to health and critically affects mortality. United States mortality tables are divided by sex, race and ethnicity and all give different statistics (See Tables 5, 6, 7 & 8).

Preferred programs expanded in the 1980s with the HIV/AIDS scare (Hughes 2012). Prior to this date, a single premium rate was charged for each age/sex cohort with occasionally a nonsmoker discount. All 35 year-old male customers would pay the same premium. This method had some negative outcomes by creating a pooling of risks – customers of higher mortalities offset costs of favorable mortalities, and vice-versa. To address this problem, carriers began demanding blood samples to determine if an applicant was HIV-positive. Initially, this additional information would only suggest denial of coverage. But eventually, blood panels revealed a wealth of data on a person’s well-being, later used to assess mortality risk in more refined ways.

Figure 4. Predictive Modeling for Life Insurance by Deloitte Consulting, LLP
The main objectives for a preference program are to have rates in line with each risk profile, to reduce premium cross-subsidization, and to shape the industry into a competitive market. By design these programs operate more closely to individual risk setting. But the practice of grouping lives has been more prominent in life insurance sales historically, and requires less underwriting.

Preferred programs are complicated to develop and require a chief pricing actuary. A preferred premium must correlate to the expected risk profile of the best class. Selection decisions outside the agreed-upon bounds (i.e. exceptions) can affect the distribution of risks and profitability. If underwriting guidelines are applied liberally, more applicants will qualify for the best class. On the other hand, if they are too rigorous, premium rates will be too high, driving away applicants the preferred class was designed to attract (Hughes 2012). Most companies require that clients apply for a stated minimum death benefit of perhaps $250,000 in order to qualify for preferred status. Industry surveys say that 10-45% of all applicants qualify for preferred classification.

2.2.  Survival Probability Models

2.2.1.  Survival Models In Statistics

In Probability theory, the sample space \( \Omega \) for a random phenomenon is the set of all possible outcomes (Ross 2013). The event \( E \) is any subset of the sample space and the probability of event \( E \) occurring is \( P(E) \in [0,1] \). There exists a duality property between an event \( E \) and its complement \( P(E^C) + P(E) = 1 \). Two important equalities in probability are:

\[
\forall A, B \in \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
and the conditional probability of A given B is

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

A random variable X is a numerical outcome of a random experiment. The distribution of X is the collection of outcomes and their probabilities. X is said to be discrete if it has a countable number of outcomes; and continuous if it has an infinite continuum of possible values (e.g. blood pressure, weight). The cumulative distribution function (or cdf) is given by: \( F_X(x) = P(X \leq x) \) or \( F_X(x) = P(X = x) \).

The derivative of the cdf is called the probability density function (or pdf) in the continuous case, and the probability mass function (or pmf) in the discrete case. It is given by:

\[ f_X(x) = \frac{dF_X(x)}{dx} \]

Each distribution has an expected value denoted

\[ E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx \quad \text{and} \quad E[X] = \sum_{x=-\infty}^{\infty} x \cdot P(X = x) \]

and a variance \( V[X] = E(X - E[X])^2 \).

With the above considerations in mind, the author will attempt to define an actuarial survival model.

The distribution of a future lifetime may be represented using probability theory. Actuaries and Statisticians agree that any survival function for a lifetime distribution must satisfy the following conditions:

i. All lives must die before a stated terminal age \( \omega \).
ii. Survival functions are non-increasing over time.

iii. The probability that a life aged x survives the next t minutes goes to 1 as \( t \to 0 \).

Let \((x)\) denote a “life aged x”, for \( x \geq 0 \). Death may occur at any age greater than \( x \), and the future lifetime of \((x)\) is a continuous random variable denoted \( T_x \). That means \( T_0 \) represents the future lifetime at birth or for a life aged 0; and \( x + T_x \) represents the age-at-death for \((x)\). Denote the cdf of \( T_x \) as \( F_x \), so \( F_x(t) \) is the probability that \((x)\) does not survive beyond age \( x + t \):

\[
F_x(t) = P[T_x \leq t]
\]

The complement \( S_x \) is the survival function and \( S_x(t) \) is the probability that \((x)\) survives at least \( t \) more years defined as:

\[
S_x(t) = 1 - F_x(t) = P[T_x > t]
\]

The previous conditional probability formula gives the following relations:

\[
F_x(t) = P[T_x \leq t] = P[T_0 \leq x + t | T_0 > x] = \frac{P[x < T_0 \leq x + t]}{P[T_0 > x]} = \frac{F_0(x + t) - F_0(x)}{S_0(x)}
\]

\[
S_x(t) = \frac{S_0(x + t)}{S_0(x)} \quad \text{or} \quad S_0(x + t) = S_0(x) \cdot S_x(t)
\]

Condition (i) can be translated as \( S_x(0) = 1 \ \forall x \leq \omega \), the terminal age assumed, and condition (ii) as \( \lim_{t \to \infty} S_x(t) = 0 \). In order for the mean and the variance of \( T_x \) to exist, other assumptions need to be made:

1. \( S_x(t) \) is differentiable for all \( t > 0 \). Note this means that \( \frac{d}{dt} S_x(t) \leq 0 \forall t > 0 \).
2. \( \lim_{t \to \infty} t \cdot S_x(t) = 0 \) and \( \lim_{t \to \infty} t^2 \cdot S_x(t) = 0 \).
2.2.2. Actuarial Survival Models

Actuarial Mathematics uses another notation for life models: the International Actuarial Notation. Following the development of the survival models from Dickson et al (2009), the previously defined quantities are expressed using this notation as follows:

\[ tP_x = S_x(t) \quad \text{and} \quad tq_x = 1 - tP_x = F_x(t). \]

Also, a deferred mortality probability \( u|\tau q_x \) is defined as the probability that \( (x) \) dies between ages \( x + u \) and \( x + u + t \):

\[ u|\tau q_x = P[u < T_x \leq u + t] = S_x(u) - S_x(u + t) = uP_x - u+tP_x \]

The last term is derived following the above conditional survival formula in simple steps:

\[ u+tP_x = S_x(u + t) = S_x(t) \cdot S_{x+t}(u) = tP_x \cdot uP_{x+t} \]

Deaths do not exactly occur at integer ages taken as points of time. To properly represent this, actuaries compound the possibilities of death on infinitesimal periods of life \( \Delta t \) for a life \( (x) \). We define the force of mortality at fixed age \( x \) by \( \mu_x \) (Daykin, Macdonald, 2004):

\[ \mu_x = \lim_{dt \to 0^+} \frac{1}{dt} \frac{P[T_0 \leq x + dt|T_0 > x]}{P[T_x \leq dt]} \]

\[ = F_x'(0) = -S_x'(0) = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x) \]

This quantity will help connect others. Note the equation for the pdf of the lifetime distribution can be calculated as:

\[ f_x(t) = tP_x \mu_{x+t} = \frac{d}{dt} tP_x = -\frac{d}{dt} tP_x \]
This is because by definition the pdf \( f_x(t) \) is the derivative of the \( F_x(t) \), the complement of \( S_x(t) \). Finally, the force of mortality and the survival functions can be newly expressed as the equations below, whose measurement can be clarified in Figure 5:

\[
\mu_{x+t} = \frac{f_x(t)}{S_x(t)}
\]

\[
tp_x = S_x(t) = e^{-\int_0^t \mu_{x+s} ds} \quad \text{and} \quad tq_x = \int_0^t s p_x \mu_{x+s} ds
\]

The mean of \( T_x \) is the expected future lifetime of \( x \). It is denoted \( \dot{e}_x \), for the complete expectation of life and is the following:

\[
\dot{e}_x = E[T_x] = \int_0^\infty t \, tp_x \mu_{x+t} \, dt.
\]

The second moment of \( T_x \) is \( E[T_x^2] \) and is expressed as:

\[
E[T_x^2] = \int_0^\infty t^2 \, tp_x \mu_{x+t} \, dt = \int_0^\infty t^2 \left( -\frac{d}{dt} \, tp_x \right) dt
\]

\[
= -t^2 \, \bigg|_{0}^{\infty} dp_x + \int_0^\infty 2t \, tp_x \, dt
\]
\[ = 2 \int_0^\infty t \, t p_x \, dt. \]

The variance of \( T_x \) is

\[ V[T_x] = E[T_x^2] - (\dot{e}_x)^2 \]

The **curtate future lifetime** is defined as the integer part of future lifetime \( K_x \) for a life \((x)\)

\[ K_x = \lfloor T_x \rfloor, \]

obtained by the floor function.

The actuarial science literature contains three pioneer efforts to find the exact force of mortality (Huang et al 2011). First is the law of *Abraham de Moivre* (1725) in “Annuities upon Lives,” with the quantity \( \omega \) denoting the **ultimate age**, i.e. the terminal age at which no lives remain. This is usually taken to be \( \omega = 100 \) or \( \omega = 120 \). *De Moivre’s law of mortality* is expressed as:

\[
\mu_{x+t} = \frac{1}{\omega - x - t} \quad \text{and} \quad t p_x \mu_{x+t} = \frac{1}{\omega - x}
\]

Second is the **Gompertz law of mortality**. In this case, Gompertz argues that life decays exponentially over time. For flexibility, constants \( B \) and \( c \) are given values such that \( 0 < B < 1 \) and \( c > 1 \). For a life aged \( x > 0 \),

\[
\mu_{x+t} = B c^{x+t} \quad \text{and} \quad t p_x \mu_{x+t} = B c^{x+t} e^{B t} e^{B t} c^x (1-c^t)
\]

Another law is that of *Makeham* (1860), which extends the Gompertz force of mortality by adding a term \( A \geq -B \) that is not age-related for accidental deaths and improves the fit of the model to mortality data at younger ages. For a life aged \( x > 0 \),

\[
\mu_{x+t} = A + B c^{x+t} \quad \text{and} \quad t p_x \mu_{x+t} = (A + B c^{x+t}) e^{-A t} e^{B t} e^{B t} c^x (1-c^t).
\]
Unless stated otherwise, the assumptions in this research will be the Makeham’s law of mortality with parameters \( A = 0.00022, B = 2.7 \times 10^{-6} \) and \( C = 1.124 \) (Dickson et al. 2009).

Finally is the Weibull assumption (1951) that mortality is modelled as follows:

\[
\mu_{x+t} = k(x + t)^n \quad \text{and} \quad tP_x \mu_{x+t} = k(x + t)^n e^{\frac{k}{n+1}[x^{n+1}-(x+t)^{n+1}]}
\]

2.2.3. Life Tables

Life tables are tables containing the mortality statistics that allow the calculation of life expectancy for a group of individuals. Given the survival models as above, we choose the parameters \( A, B \) and \( c \) to fit the data of the life table in use. Consider a life table starting from age \( x_0 \) to a maximum age \( \omega \). \( l_x \) for \( x_0 \leq x \leq \omega \) is a function for the number of persons alive of age \( x \). Define \( l_{x_0} \) to be any positive number and call it the radix of the table, and for \( 0 \leq t \leq \omega - x_0 \),

\[
l_{x_0+t} = l_{x_0} \quad tP_{x_0} \quad \text{and} \quad tP_x = \frac{l_{x+t}}{l_x}
\]

The number of deaths occurring during the interval between \( l_x \) and \( l_{x+1} \) is denoted \( d_x \):

\[
d_x = l_x - l_{x+1} = l_x q_x
\]

where \( q_x \) is the probability that a person dies between \( x \) and \( x + 1 \).
Individuals of the same age in life tables are assumed to have the same life expectancy. See Tables 5, 6, 7 and 8 and Figure 19 in Appendix, excerpted from ‘United States Life Tables of 2011’ by the Division of National Vital Statistics.

2.3. Benefits and Premium Calculations

2.3.1. Financial Mathematics: Interest Theory

The notation in actuarial science is consistent with that of interest theory. Define \( i \) to be the constant interest rate, and the force of interest or continuously compounded interest rate \( \delta \) :

\[
\delta = \log(1 + i), \quad 1 + i = e^\delta
\]

The discount rate, or the present value of $1 in one year, is expressed as:

\[
v = \frac{1}{1 + i} = e^{-\delta}.
\]

The nominal rate of interest compounded \( p \) times per year is expressed as:

\[
(i(p)) = p \left( (1 + i)^{\frac{1}{p}} - 1 \right) \iff 1 + i = \left( 1 + \frac{i(p)}{p} \right)^p
\]

The effective rate of discount per year is \( d = 1 - v = iv = 1 - e^{-\delta} \).
The nominal rate of discount compounded $p$ times per year is

\[ d(p) = p\left(1 - \frac{1}{v^p}\right) \iff v = \left(1 - \frac{d(p)}{p}\right)^p. \]

### 2.3.2. Insurance benefits & Annuities

The continuous future lifetime benefit random variable $Z$ combines both the survival distribution and interest theory. It is the present value of a benefit of $1$ payable immediately on death:

\[ Z = v^T_x = e^{-\delta T_x}. \]

Given $(x)$, the **International Actuarial Notation** defines the method for notation of the *expected present value* (EPV) of a life insurance contract in actuarial science. $A_{x:n}$ is the present value of insurance for $(x)$ paying $1$ on the insured event for $n$ years; $a_{x:n}$ is the present value of an annuity for $(x)$ paying $1$ per annum for $n$ years at the end of each year. To simplify the word editing, we will remove the bars above and to the right of $n$. Thus $A_{x:n} = A_{x:n}$ and $a_{x:n} = a_{x:n}$

The notation symbols and letter meanings are explained below using an example containing all of them. Denote an insurance contract of $n$ years starting $u$ years from now with interest compounded $m$ times a year and benefit payable at the beginning of the year with

\[ u_{x:n} \bar{A}^{1(m)}_{x:n} \text{ or } u_{x:n} \bar{A}^{(m)}_x \]

- $u$ is the *deferred period,*
- \( (m) \) represents the interest rate i compounding frequency, and
- the superscript “1” is the endowment indicator –

when placed above \( x \), it indicates that benefit is payable only if \( (x) \) dies within \( n \) years, and when placed above \( n \), it indicates that benefit is payable if \( (x) \) survives \( n \) years. The absence of a superscript means that the insurance pays on the earliest of death or \( n \)-years.

- The mark above “A” or annuity “a” represents the time of payment –

For \( \ddot{A}_x \) or \( \ddot{a}_x \), the line is for continuity i.e. payment is made continuously or immediately at the moment of death. For \( \dot{A}_x \) or \( \dot{a}_x \), the double dot indicates that payments are made at the beginning of the year. For \( A_x \) or \( a_x \), the absence of mark indicates that payments are made at the end of the year.

For instance, the EPV of a life insurance of $1 benefit payable immediately on death is:

\[
\bar{A}_{x:n}^1 = E[Z] = E[e^{-\delta T_x}] = \int_0^n e^{-\delta t} t p_x \mu_x + t dt
\]

and the EPV of a life insurance of $1 benefit payable at the end of year of death is:

\[
A_{x:n}^1 = \sum_{t=0}^{n} v^t t p_x \mu_x + t
\]

As \( n \to \infty \) or for a maximum age \( \omega \), \( n \to \omega \), the quantities become \( A_x \) or \( a_x \).

The second moment of the death benefit EPV is of the form \( m[\bar{A}_{x:n}] \), \( m[\ddot{A}_{x:n}] \), or \( m[A_{x:n}] \) where the upper-left superscript “2” is for double force of interest. For example:

\[
2\bar{A}_{x:n} = \int_0^n e^{-2\delta t} t p_x \mu_x + t dt + e^{-2\delta n} n p_x
\]

The variance of the EPV and for a sum insured \( S \) is given by:

20
\[ V[Z] = V[e^{-\delta T_x}] = \frac{2 \bar{A}_{x:n|}}{\bar{A}_{x:n|}^2} - (\bar{A}_{x:n|})^2 \]

\[ V[SZ] = V[Se^{-\delta T_x}] = S^2 \left( \frac{2 \bar{A}_{x:n|}}{\bar{A}_{x:n|}^2} - \bar{A}_{x:n|}^2 \right) \]

The present value RV for the annuity payment series, Y are those of interest theory with the time now a (curtate) lifetime distribution.

\[ a_{K+1} = \frac{1 - v^{K+1}}{d} \]

All other results can be derived, such as the following relationships between whole insurance and annuity contracts, and between annuities:

\[
\begin{align*}
\bar{a}_x &= \frac{1 - A_x}{d} ; \quad \bar{A}_x = \frac{1 - \bar{A}_x}{d} \\
\end{align*}
\]

\[
\begin{align*}
a_{x:n|} &< a_{x:n|}^{(m)} < \bar{a}_{x:n|} < \bar{a}_{x:n|}^{(m)} < \bar{a}_{x:n|} \\
\end{align*}
\]

2.3.3. **Premium Calculations**

Consider the *net premium*; that is, the remaining part of the premium once the company expenses are removed. Other concepts are the *premium income* and the *insurance benefit outgo*, both life contingent. The *net future loss function* is \( L_0^n \) for a sum insured \( S \) and premium rate \( P \) for life insured (x). It is calculated at curtate/integer ages \( K_x = \lfloor T_x \rfloor \).

\[
L_0^n = PV \text{ of benefit} - PV \text{ of premiums.}
\]

\[
L_0^n = S v^{\min(K_x+1,n)} - P \bar{a}_{\min(K_x+1,n)}
\]

The expected value \( E[L_0^n] \) involve the actuarial accumulation functions.

\[
E[L_0^n] = S A_{x:n|} - P \bar{a}_{x:n|}
\]
The Equivalence principle states that the net premium has to be set such that \( E[L^n_0] = 0 \).

This is meant to give a fair price for coverage. High prices drive away the customers, and low prices put the company solvency in danger. The equation for the Net premium is:

\[
P = S \frac{\bar{A}_{x:n}}{\bar{a}_{x:n}} = S\left(\frac{1}{\bar{a}_{x:n}} - d\right)
\]

**Example 2-1.** Consider a 20-year endowment insurance with sum insured $100,000 issued to a life aged 45 under which the death benefit is payable immediately at death.

Using Makeham’s law with an interest rate of 5% per year, find the net premium payable in a year if premiums are payable annually.

**Solution.** Use the equation above for \( P \). From interest theory,

\[
e^\delta = 1.05 \text{ and } d = 1 - \frac{1}{1.05} = .0479619;
\]

The SSSM gives the mortality rate for \( x = 45 \) and for \( t \in [0,20] \),

\[
\mu_x = .00022 + 27 \times 10^{-6} \cdot 1.124^x \text{ and } \ t_{45} = e^{-\int_0^t \mu_{45+s}ds}
\]

If premium payments are made annually and are life contingent, then the present value is

\[
\bar{a}_{45:20} = \int_0^{20} e^{-\delta t} t_{45} dt = \int_0^{20} 1.05^{-t} e^{-\int_0^t .00022 + 27 \times 10^{-6} \cdot 1.124^{45+s} ds} dt = 12.9295
\]

Finally, the net premium follows.

\[
P = S \left(\frac{1}{\bar{a}_{45:20}} - d\right) = 100,000 \left(\frac{1}{12.9295} - .047619\right) = $ 2972.35.
\]

**2.3.4. Modeling Issues**
Classical probability theory has limited effectiveness when dealing with problems in which some of the principal sources of uncertainty are non-statistical in nature. Life tables group the risk of death by age and give the statistical mean as life expectancy. The mortality forces have parameters to help fit the data found in life table. Arguing the efficacy of the model leads to the pooling of risks by life insurance companies, whereby all individuals of the same risk class are insured at the same rate. Higher mortalities may offset costs of favorable mortalities (and vice-versa). Preferred programs are a refinement of the concept.

Age is inevitably correlated to mortality, as over time one’s health decays. In reality, each individual has his own life expectancy depending not only on age but on his physical condition, gender, family medical history, financial background, and many other factors. Fuzzy mathematics, and precisely fuzzy statistical clustering, is the science able to combine these attributes.

Furthermore, premium calculations often use a constant rate \( i \) and a force of mortality \( \delta_t \) over time. The actuary may use time-series regression to forecast future interest rates but this does not allow for extreme events ranked as a “Black Swan” events. The Black Swan theory of Nicholas Taleb (2001) describes extreme outliers in a theatrical way that shows a major impact, and yet causes can only be found after the fact through retrospective analysis. Fuzzicists use the theory of possibility to extend the grasp of probability. Hence, the next section presents Zadeh’s engineering and its applications.
3. Fuzzy Mathematics

Let us adopt the formal framework for mathematicians to present new concepts: Definitions, Theorems and Examples.

3.1. Fuzzy Set Methodology

The theory for classical/crisp sets remains the same as it is normally defined by a collection of elements or objects. \( x \in X \) that can be finite, countable, or infinite. Each element can either belong to or not belong to a set \( A \subseteq X \). A good general reference for the theory of fuzzy sets is H.J. Zimmerman "Fuzzy Set Theory and its Applications" (2001). The notation given in that text is used for this thesis.

3.1.1. Fuzzy Set Theory

**Definition 3-1.** If \( X \) is a collection of objects \( x \), then a *fuzzy set* \( A \) in \( X \) is a set of ordered pairs: \( A = \{ (x, m_A(x)) | x \in X \} \) and \( m_A : X \to M \) is the membership function of \( A \) for all \( x \) in \( X \), where \( M \) is a bounded subset of \( \mathbb{R}^+ \) or \([0, \infty)\) called the membership space.

The membership function is not limited to values between 0 and 1. If \( \sup_x m_A(x) = 1 \), then the fuzzy set \( A \) is *normal*. A fuzzy set can always be *normalized* by dividing \( m_A(x) \) by its supremum as shown below.

\[
\frac{m_A(x)}{\sup_x m_A(x)}
\]

One may omit elements with membership grade of 0 in writing the fuzzy sets. This is illustrated below.
Example 3-1. (Finite set.) Let $X = \{1, 2, 3, ..., 10\}$ the set of houses with $x$ the number of bedrooms. The fuzzy set "comfortable house for 4 individuals" is

$$A = \{(1,.2), (2,.5), (3,.8), (4,1), (5,.9), (6,.4)\}$$

(Infinite Set) Let $X = \mathbb{R}$, and $A$= “real numbers considerably larger than 25”

$$m_A(x) = \begin{cases} 
0, & \text{if } x \leq 25 \\
\frac{1}{1 + (x - 25)^{-2}}, & x > 25 
\end{cases}$$

$B$= “real numbers almost equal to 10” = \{(x, m_B(x)) | m_B(x) = [1 + (x - 10)^2]^{-1}\}

Definition 3-2. The Support of a fuzzy set $A$, $S(A)$ is the crisp set of all $x \in X$ such that $m_A(x) > 0$. The crisp set of elements in $A$ with grade of membership greater or equal to $\alpha \in M$ is called the $\alpha$-level set or $\alpha$-cut: $A_\alpha = \{x \in X | m_A(x) \geq \alpha\}$.

Note $A'_\alpha = \{x \in X | m_A(x) > \alpha\}$ is called "strong $\alpha$-level set" or "strong $\alpha$-cut”.

Example 3-2. Recall the finite set of Example 3-1. The following are $\alpha$-level sets:

$$A_0 = S(A) = \{1, 2, 3, 4, 5, 6\} A_{.5} = \{2, 3, 4, 5\} A_{.8} = \{3, 4, 5\} A_1 = \{4\}$$

The strong $\alpha$-level set for $\alpha = .8$ is $A'_{.8} = \{4, 5\}$

Definition 3-3. For finite $A$, the cardinality is defined as

$$|A| = \sum_{x \in X} m_A(x) \text{ or } |A| = \int_X m_A(x) \, dx,$$

and the relative cardinality of $A$ is $\|A\|$ when divided by $\text{card}(X)$, the number of elements.

$$\|A\| = \frac{|A|}{\text{card}(X)}$$
Example 3-3. Same A as in finite set of example 3-1, \( |A| = .2 + .5 + .8 + 1 + .9 + .4 = 3.8 \) and the relative cardinality \( ||A|| = \frac{3.8}{10} = .38 \)

Definition 3-4. (Zadeh 1968) The intersection of fuzzy sets A and B is \( C = A \cap B \) and the membership function is defined pointwise by
\[
m_C(x) = \min\{m_A(x), m_B(x)\}, \quad x \in X.
\]
The union is \( D = A \cup B \) and the membership function is defined pointwise by
\[
m_D(x) = \max\{m_A(x), m_B(x)\}, \quad x \in X
\]

Definition 3-5. The complement of a normal fuzzy set is the set \( A^c \) and the membership function is defined by \( m_{A^c}(x) = 1 - m_A(x), \quad x \in X \)

Example 3-4. Recall the finite set of example 3-1, and let B be the fuzzy set "large house" with \( B = \{(3, .4), (4, .7), (5, .9), (6, 1), (7, 1), (8, 5)\} \). Then the intersection is
\[
C = A \cap B = \{(3, .4), (4, .7), (5, 9), (6, 4)\}
\]
and the union \( D = A \cup B = \{(1, .2), (2, .5), (3, .8), (4, 1), (5, .9), (6, 1), (7, 1), (8, 5)\} \)
The complement "not large house" may have small or extra-large houses:
\[
B^c = \{(1, 1), (2, 1), (3, .6), (4, .3), (5, 1), (8, .5), (9, 1), (10, 1)\}
\]
Now, consider the infinite set A in example 3-1, A="real numbers considerably larger than 25" and \( C="x \cong 26" \) with \( m_C(x) = \frac{1}{1+(x-26)^4} \)

Their intersection and union are:
Definition 3-6. A type 2 fuzzy set is a fuzzy set whose membership function is also a fuzzy set on the space M. More generally, for some integer $m > 1$, a type $m$ fuzzy set in $X$ is a fuzzy set whose membership values are type $m$-1 fuzzy sets on $M$.

From a practical point of view, type $m$ fuzzy sets for $m \geq 3$, are extremely difficult to measure or visualize. Examples are fuzzy sets with membership function as a probabilistic set (Hirota 1981), or an intuitionistic fuzzy set of ordered triples (Atanassov and Stoeva 1983).

Definition 3-7. A fuzzy number $K$ is a fuzzy subset of the real line whose membership function is a continuous mapping defined by $m_K: \mathbb{R} \rightarrow M \cap [0,1]$ and is represented solely by $(a_1, a_2, a_3, a_4)$ such that:

- $m_K(x) = 0$ for $x \in (-\infty, a_1] \cup [a_4, \infty)$,
- $m_K(x)$ increases linearly on $[a_1, a_2]$,
- $m_K(x) = 1$ for $x \in [a_2, a_3]$,
- $m_K(x)$ decreases linearly on $[a_3, a_4]$.

In the case $a_2 = a_3$, it is a triangular fuzzy number; otherwise it is a trapezoidal.
3.1.2. Fuzzy Set-Theoretic Operations

Operations in this section apply to both normal and other fuzzy sets over the same underlying set.

**Operation 3-1.** (Zadeh 1965) The *algebraic sum* of two fuzzy sets (probabilistic sum)

\[ A + B \text{ is defined by } m_{A+B}(x) = m_A(x) + m_B(x) - m_A(x) \cdot m_B(x) \]

The *algebraic product* of two fuzzy sets \( A \cdot B \) is defined by \( m_{A\cdot B}(x) = m_A(x) \cdot m_B(x) \)

**Operation 3-2.** The bounded sum of fuzzy sets \( A \oplus B \) is defined by

\[ m_{A\oplus B}(x) = \min\{1, m_A(x) + m_B(x)\}. \]

The bounded difference \( A \ominus B \) is defined by \( m_{A\ominus B}(x) = \max\{0, m_A(x) + m_B(x) - 1\} \).

---

Figure 6. Trapezoidal and Triangular Fuzzy Number
**Operation 3-3.** The *Cartesian product* of fuzzy sets $A_1, \ldots, A_n$ in $X_1, \ldots, X_n$ respectively is a subset of $X_1 \times \ldots \times X_n$ with membership function

$$m_{(A_1 \times \ldots \times A_n)}(x) = \min\{m_{A_i}(x_i) | x = (x_1, \ldots, x_n), x_i \in X_i\}$$

**Operation 3-4.** The $n^{th}$ *power* of a fuzzy set $A$ is $A^n$ with membership function

$$m_{A^n}(x) = [m_A(x)]^n, \ x \in X$$

This mapping is a *concentration* if $n > 1$, and a *dilation* if $n < 1$. When combined in order to increase the membership grade of certain elements and/or reduce the grade of others, it is an *intensification*.

**Example 3-5.** Define the fuzzy sets $J = \{(a, .5), (b, 1), (c, .2)\}$ and $H = \{(a, ,3), (b, .7)\}$

The above definitions are then illustrated by the following:

$$J + H = \{(a, .65), (b, 1), (c, .2)\}$$

$$J \cdot H = \{(a, .15), (b, .7)\}$$

$$J^2 = \{(a, .25), (b, 1), (c, .04)\}$$

$$J \oplus H = \{(a, .8), (b, 1), (c, .2)\}$$

$$J \ominus H = \{(b, .7)\}$$

$$J \times H = \{[(a; a), .3], [(b; a), .3], [(c; a), 0], [(a; b), .5], [(b; b), .7], [(c; b), .2]\}$$

*Min* and *max* for intersections and unions are not generally smooth functions. Their explicit formula is a sequence of interval brackets or piecewise-continuous functions. Fuzzy mathematics, though named “fuzzy” as if to emphasize imprecision, focuses on improving precisions. Fuzzicists eventually decided to consider softer
definitions of “intersection”. Many were suggested, with all satisfying the following properties (Dubois & Prade 1980):

i. Cumulative effects: $m_{A \cap B}(x) \leq \min\{m_A(x), m_B(x)\}$, if $m_A(x) < 1$, $m_B(x) < 1$

ii. Interactions between criteria: Assume $m_A(x) < m_B(x) < 1$. Then the effect of a decrease of $m_A(x)$, on $m_{A \cap B}(x)$ may depend on $m_B(x)$.

iii. Compensations between criteria: if $m_A(x) < 1$, $m_B(x) < 1$ the effect of a decrease of $m_A(x)$ on $m_{A \cap B}(x)$ can be erased by an increase of $m_B(x)$ (unless $m_B(x) = 1$)

Earlier mathematics applications of triangular norm satisfy these criteria (Menger 1942). Hence, intersections can be defined by t-norms and unions by a t-conorms.

Definition 3-8. A t-norm is any bivariate function $t$ from $M \times M \rightarrow [0,1]$ such that for all fuzzy sets $A$, $B$, $C$ and $D$ in $X$ each with its own membership function, $t(a, b)$ is commutative, associative and monotonic, and $\forall x \in X$:

$$t(0,0) = 0,$$

$$t(m_A(x), 1) = t(1, m_A(x)) = m_A(x)$$

$$t(m_A, m_B) = t(m_B, m_A) \text{ commutativity.}$$

$$t(m_A, t(m_B, m_C)) = t(t(m_A, m_B), m_C) \text{ associativity.}$$

if $m_A \leq m_C$ and $m_B \leq m_D$, then $t(m_A, m_B) \leq t(m_C, m_D) \text{ monotonicity.}$

Definition 3-9. A t-conorm or s-norm is a commutative, associative and monotonic bivariate function $s$ from membership space $M \times M \rightarrow [0,1]$ such that

$$\forall x \in X, \quad s(1,1) = 1; \quad s(m_A(x), 0) = s(0, m_A(x)) = m_A(x).$$
For mathematical derivations, proofs and other t-norms the reader is referred to Klement et al. (1994)

**Theorem 3-5.** (Alsina 1985). *t-norms* and *t-conorms* are dual such that the function *t* is defined as 
\[ t(m_A, m_B) = 1 - s((1 - m_A), (1 - m_B)). \]

**Lemma 3-1.** (Hamacher 1978). The intersection of two fuzzy sets A and B may be defined as a *t-norm* with
\[ t(m_A, m_B) = \frac{m_A \cdot m_B}{p + (1 - p)[m_A + m_B - m_A \cdot m_B]}, \quad \text{for } p \geq 0 \]
and the union of two fuzzy sets A and B is defined as a *t-conorm* with
\[ s(m_A, m_B) = \frac{(p' - 1)m_A \cdot m_B + m_A + m_B}{1 + p' \cdot m_A \cdot m_B}, \quad \text{for } p' \leq -1 \]

**Lemma 3-2.** (Yager 1980). The intersection of two fuzzy sets A and B may be defined as a *t-norm* with
\[ t(m_A, m_B) = 1 - \min\{1, [(1 - m_A)^p + (1 - m_B)^p]^{1/p}\}, \quad \text{for } p \geq 1 \]
and the union of two fuzzy sets A and B is defined as a *t-conorm* with
\[ s(m_A, m_B) = \min\left\{1, (m_A^p + m_B^p)^{1/p}\right\}, \quad \text{for } p \geq 1. \]

For intersections or unions between more than two fuzzy sets, the method recommended is to merge them two-by-two or progressively. This means, first combine \( A \cap B \), then \( (A \cap B) \cap C \), then \( (A \cap B \cap C) \cap D \), etc., as t-norms and t-conorms are associative and commutative by Definition 3-8.
Note that the Hamacher norm for \( p = 1 \) corresponds to the t-norm “Algebraic product”. Taking the infinity norm for the Yager norm (as \( p \to \infty \)), both the intersection and the union give the minimum operator and the maximum operator, respectively. Many other norms are used in Fuzzy Set Theory, see Table 9 for a list of common t-norms and s-norms (Bonissone and Decker 1986).

An important arithmetic result in Dubois & Prade (1978, 1980), for the sum and product of fuzzy numbers, is associativity and commutativity of inverses.

**Theorem 3-2.** (Dubois, Prade 1980) Let \( A, B \) be trapezoidal fuzzy numbers with membership notations \( m_A(x) = U_A(x) \) and \( m_B(x) = U_B(x) \). \( U_{A1} \) is the increasing part of \( U_A(x) \) on \([a_1, a_2]\), and \( U_{A2} \) is the decreasing part on \([a_3, a_4]\). Their inverses are

\[
V_{A1} = U_{A1}^{-1} \quad \text{and} \quad V_{A2} = U_{A2}^{-1}.
\]

Then the sum \( C = A \oplus B \) has membership functions on \([a_1, a_2]\) and \([a_3, a_4]\)

\[
U_{C1} = \left(U_{A1}^{-1} + U_{B1}^{-1}\right)^{-1} \quad \text{or} \quad V_{C1} = V_{A1} + V_{B1}
\]

\[
U_{C2} = \left(U_{A2}^{-1} + U_{B2}^{-1}\right)^{-1} \quad \text{or} \quad V_{C2} = V_{A2} + V_{B2}
\]

and the product \( D = A \odot B \) has membership functions on \([a_1, a_2]\) and \([a_3, a_4]\)

\[
U_{D1} = \left(U_{A1}^{-1} \cdot U_{B1}^{-1}\right)^{-1} \quad \text{or} \quad V_{D1} = V_{A1} \cdot V_{B1}
\]

\[
U_{D2} = \left(U_{A2}^{-1} \cdot U_{B2}^{-1}\right)^{-1} \quad \text{or} \quad V_{D2} = V_{A2} \cdot V_{B2}
\]

3.2. **Fuzzy clustering algorithms**
In Data mining, clustering techniques are used to put together objects showing similar characteristics within the same group, and to separate objects with different characteristics. To do so, one must write algorithms that permit iterations before stabilizing. These clustering techniques are made for detection and handling of noisy data or outliers. There are two approaches: **Hard clustering** and **Soft clustering**. Hard data clustering divides data elements into clusters in such a way that one data item can belong to one cluster only. This is the crisp version for data mining. Soft clustering, also known as *fuzzy clustering*, allocates data elements to one or more clusters based on their membership levels in the different clusters.

**Fuzzy C-Means** (Dunn 1973) is the most popular and efficient technique of soft computing. Note this is the fuzzy logic version of the most popular hard clustering method: *K-means algorithm*. Other names for it are Soft computing and Fuzzy K-means. The fuzzy c-means (FCM) algorithm requires steps such as the calculation of cluster centers, assignment of points to centers by taking their Euclidian distances, and continuous iteration until the cluster centers stabilizes (Thomas 2012).

**Definition 3-10.** Consider the set of data $X = \{x_1, \ldots, x_n\}$, and $V_{c,n}$ the set of real $c \times n$ matrices ($2 \leq c \leq n$). The matrix $\bar{U} = [m_{jk}] \in V_{c\times n}$, with $m_{jk} \in [0,1]$, $1 \leq j \leq c$, $1 \leq k \leq n$ is called a **fuzzy-c partition** if it satisfies the following conditions [Bezdek 1981]:

$$\sum_{j=1}^{c} m_{jk} = 1 \quad \text{and} \quad 0 < \sum_{j=1}^{n} m_{jk} < n$$
Example 3-6. Let \( X = \{x_1, x_2, x_3\} \). A fuzzy 2-partitions can be

\[
\bar{U}_1 = \begin{bmatrix}
1 & 0.5 & 0 \\
0.5 & 0 & 1
\end{bmatrix}
\text{ or }
\bar{U}_2 = \begin{bmatrix}
0.8 & 0 & 0.1 \\
0.2 & 1 & 0.9
\end{bmatrix}.
\]

In \( \bar{U}_1 \), \( x_1 \) and \( x_3 \) are fully included in clusters \( c_1 \) and \( c_2 \) respectively, and \( x_2 \) is equally contained in both. In \( \bar{U}_2 \), \( x_2 \) is fully included in clusters \( c_1 \), when \( x_1 \) and \( x_3 \) are still 20% and 10% in cluster 2, respectively. Note the fuzzy c-partition conditions are met: the sum of each column is 1, and the row sums are always less than 3, the number of data.

Fuzzy c-means became more popular for symmetric data such as \( \bar{U}_1 \), where the K-means algorithm would fail. This is due to the presence of midpoints (\( m_{ik} = 0.5 \)). Eventually the algorithm would insert it in a random cluster when it may as well belong to another.

Example 3-7. This is the case for the popular butterfly example (Zimmerman 1994).

![Figure 7. The midpoint bias of the Butterfly problem in data mining](image)

Let us define an algorithm to find these fuzzy c-partitions. For an FCM algorithm, it is necessary to choose a few parameters. These are the desired number of clusters \( c \).
(2 ≤ c ≤ n); an exponential weight \( r \) (1 < \( r \) < \( \infty \)) often called the fuzzy parameter; the type of norm \( ||\cdot|| \) (here the Euclidean distance will serve as norm); and a termination criterion \( \varepsilon > 0 \). A method to initialize the membership matrix \( \tilde{U}^{(l)} \in V_{c,n} \), for \( l \geq 0 \) for each iteration is also necessary. Here membership values and cluster centers are given by:

\[
\forall \ 1 \leq i \leq c, \ 1 \leq j \leq c \text{ and } 1 \leq k \leq n,
\]

\[
m_{jk} = \left[ \sum_{i=1}^{c} \frac{\|x_k - c_j\|}{\|x_k - c_i\|} \right]^{-2/r-1} \quad \text{with} \quad c_j = \frac{\sum_{k=1}^{N} m_{jk}^r \cdot x_k}{\sum_{k=1}^{N} m_{jk}^r}
\]

To summarize the steps:

**Step 1.** Choose \( c, r, \varepsilon \).

**Step 2.** Initialize \( \tilde{U}^{(l)} = \left[ m_{jk} \right]^{(l)} \in V_{c,n} \), for \( l \geq 0 \), set \( l = 0 \).

**Step 3.** Calculate the \( c \) fuzzy cluster centers \( \{ c_j^{(l)} \} \) by using \( \tilde{U}^{(l)} \).

**Step 4.** Calculate the new membership matrix \( \tilde{U}^{(l+1)} \) by using \( \{ c_j^{(l)} \} \). If \( x_k \neq c_j^{(l)} \), Else set \( m_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \).

**Step 5.** Calculate \( \Delta = \| \tilde{U}^{(l+1)} - \tilde{U}^{(l)} \| \).

**Step 6.** If \( \Delta > \varepsilon \), set \( l = l + 1 \) and go to **Step 2.** If \( \Delta \leq \varepsilon \to \text{stop} \)

**Example 3-7.(continued)** The data of the butterfly were processed with a fuzzy 2-means algorithm. Choose \( c = 2, \varepsilon = .01, m = 1.25 \) with the Euclidean norm. In 6 iterations the clustering results in the memberships and cluster centers as shown in Figure 8. The
butterfly fuzzy c-partition gives the membership level of each point in its cluster. The partition is:

\[ \bar{U} = \begin{bmatrix}
.99 & 1 & .99 & 1 & 1 & .99 & .47 & .01 & 0 & 0 & .01 & 0 & .01 \\
.01 & 0 & .01 & 0 & 0 & 0 & .01 & .53 & .99 & 1 & 1 & 1 & .99 & 1 & .99
\end{bmatrix} \]
To improve the fuzzy c-partitions, one only needs to increase the fuzzy parameter $m$ and choose $c$ suitably. The same process is done in Figure 9 for $m = 2$ and all other default parameters. The new butterfly fuzzy c-partition is

$$\hat{U} = \begin{bmatrix}
.86 & .97 & .86 & .94 & .99 & .94 & .86 & .5 & .14 & .06 & .01 & .06 & .14 & .03 & .14 \\
.14 & .03 & .14 & .06 & .01 & .06 & .14 & .5 & .86 & .94 & .99 & .94 & .86 & .97 & .86
\end{bmatrix}.$$
PART 2: APPLICATIONS TO LIFE CONTINGENCIES

This part illustrates the use of the above applications in a life insurance setting. It involves the translation of medical records for applicants through fuzzy decision-making processes. Through a series of case studies, we will observe the classification of policyholder risks using a fuzzy system for preference; the definition of survival probability formula using fuzzy parameters to counter interest rate fluctuations; and lastly, the clustering of policyholder sociodemographic data by the fuzzy c-means algorithm.

1. Classification of Preferred Policyholders in Life Insurance

Let X be the set of applicants for a life insurance. The carrier may have a set of pricing policies or a preference program. For instance, one carrier may offer a bonus of 15% more coverage, if the applicant is not a smoker or has not smoked for a minimum of 12 months prior to application. Another may be even more generous and give a bonus of 50% more coverage with no increases in premium if the applicant achieves the highest degree of health defined by the company. This corresponds to an applicant who has not smoked for a year, a resting pulse of 72 or below, a blood pressure below 134/80, a cholesterol reading below 200, and does not participate in hazardous sports. To reach the perfection standard laid out by the CDC/WHO (Centers for Disease Control and
Prevention/World Health Organization), the applicant must follow weekly exercise programs, be within Body Mass Index (BMI) specified height/weight restrictions, and have no family history of deaths prior to 50 years old due to kidney/heart disease, stroke or diabetes. However, this collection of individuals is extremely uncommon. Thus, for marketing purposes, the company may want to accept the preferred status for a person lacking only a few of these criteria. Fuzzy Set Theory will be very resourceful in modeling these preference classes for underwriting purposes.

Case Study 1. (Lemaire 1990) For simplicity, limit the study to 4 variables $t_i$, for $i = 1, 2, 3, 4$ and the fuzzy sets: Cholesterol in Blood (A), Systolic Blood Pressure (B), BMI (C) and Cigarette Consumption (D). Each applicant $x \in X$ is represented with its information by $x(t_1, t_2, t_3, t_4)$. Lemaire uses the following membership functions for the fuzzy variables.

$t_1$ Blood Cholesterol (mg/dl)

$$m_A(x, t_1) = \begin{cases} 1, & \text{if } t_1 \leq 200 \\ 1 - 2 \left( \frac{t_1 - 200}{40} \right)^2, & \text{if } 200 \leq t_1 \leq 220 \\ 2 \left( \frac{240 - t_1}{40} \right)^2, & \text{if } 220 \leq t_1 \leq 240 \\ 0, & \text{if } 240 < t_1 \end{cases}$$

$t_2$ Blood Pressure (mmHg)
Choose at random an applicant \( x = x(210mg/dl, 145mmHg, 112\%, 0) \)

The fuzzy set \( E = A \cap B \cap C \cap D \) determines how fit a customer is for the preferred program. Recall the t-norms for intersection of fuzzy sets in Theorem 3-5 and Table 9.

The pricing actuary of the life insurance carrier will choose which operator works best among the following.

\[
m_B(x, t_2) = \begin{cases} 
1, & \text{if } t_2 \leq 130 \\
1 - 2 \left( \frac{t_2 - 130}{40} \right)^2, & \text{if } 130 \leq t_2 \leq 150 \\
2 \left( \frac{170 - t_1}{40} \right)^2, & \text{if } 150 \leq t_2 \leq 170 \\
0, & \text{if } 170 < t_2
\end{cases}
\]

\( t_3 \) Body Mass Index (\( \% \))

\[
m_C(x, t_3) = \begin{cases} 
0, & \text{if } t_3 \leq 60 \\
2 \left( \frac{t_3 - 60}{25} \right)^2, & \text{if } 60 \leq t_3 \leq 72.5 \\
1 - 2 \left( \frac{85 - t_3}{25} \right)^2, & \text{if } 72.5 \leq t_3 \leq 85 \\
1, & \text{if } 85 < t_3 \leq 110 \\
1 - 2 \left( \frac{t_3 - 110}{20} \right)^2, & \text{if } 110 \leq t_3 \leq 120 \\
2 \left( \frac{130 - t_3}{20} \right)^2, & \text{if } 120 \leq t_2 \leq 130 \\
0, & \text{if } 130 < t_3
\end{cases}
\]

\( t_4 \) Cigarette Consumption

\[
m_D(x, t_4) = \begin{cases} 
1, & \text{if } t_4 = 0 \\
0, & \text{if } t_4 > 0
\end{cases}
\]
- **Minimum operator**,  
  \[ m_E(x; 210, 145, 112, 0) = \min(.875, .71875, .98, 1) = .71875 \]

- **Algebraic product**,  
  \[ m_E(x) = (.875)(.71875)(.98)(1) = .6163 \]

- **Bounded difference**,  
  \[ m_E(x) = \max[0, .875 + .71875 + .98 + 1 - 3] = .57375 \]

- **Hamacher operator** for \( p=1/2 \),  
  \begin{align*} 
  m_E(x; 210, 145) &= \frac{(.875)(.71875)}{.5 + (1-.5)[.875 + .71875 - (.875)(.71875)]} = .6402 \\
  m_E(x; 210, 145, 112, 0) &= \frac{(.6402)(.98)}{.5 + (1-.5)[.6402 + .98 - (.6402)(.98)]} = .629 
  \end{align*}

- **Yager operator** for \( p=2 \),  
  \[ m_E(x) = 1 - \min\{1, [(1 -.875)^2 + (1 -.71875)^2 + (1 -.98)^2]^{1/2}\} = .69157 \]

Thus, no computation of \( x(210, 145, 112, 0) \) health will give a preference status if the requirement is 100% grade of membership. Note that a smoker is never preferred; every operator gives 0% membership. The pricing actuary may allow a few infringements to perfection with a statement such as: “*An applicant is considered preferred if he meets at least 75% of the requirements of the CDC/WHO health index.*” This step may require, in crisp set theory, the creation of new membership functions. If the actuary uses only the minimum operator which is the strictest operator, underwriters will obtain rules defined by  
\[ t_1 \leq 214.2; \quad t_2 \leq 144.2; \quad 76.2 \leq t_3 \leq 117.1; \quad t_4 = 0 \]
This means, any applicant with information not in one of these intervals will be below 75% and cannot be a preferred policyholder.

Fuzzicists may, as in Definition 3-2, take the alpha-level set to refine the results and create classes such as the one from Section 2.1. Take, for any $\alpha$, $E_\alpha$ to be the crisp set of policyholders with grade of membership greater than $\alpha$. Choose $\alpha = 75\%$, so that after evaluating the membership of an applicant in the fuzzy set $E$ above, he becomes preferred if $m_E(x) \geq 0.75$. Clearly, the policyholder $x(210, 145, 112, 0)$ is still not part of the preferred program under any operator/t-norm.

Then, we build another preference class, as for Deloitte Consulting (Figure 4).

Say “An applicant is considered Superpreferred if he meets at least 75% of the requirements of the CDC/WHO health index, and he is considered preferred if he qualifies from 65% to 75%.” The same process works and the actuary does not need to build membership functions. By taking the alpha cut $E_{65}$, the applicant $x(210, 145, 112, 0)$ falls in the range for the preferred program benefits only if the actuary decides to use minimum operator or Yager t-norm with $p=2$. Otherwise $x(210, 145, 112, 0)$ may fall in the standard class or some residual classes.

In reality, each criterion has its own importance. To show this difference, Fuzzicists use the operations of concentration, dilation, and intensification. Suppose blood pressure better predicts future health complications, while cholesterol level does less well. The actuary may then concentrate the fuzzy number $t_1$ for the cholesterol by taking the square; while dilating $t_2$ blood pressure by taking the square root. Then we have the following:
- Min operator,

\[ m_E(x; 210, 145, 112, 0) = \min(.875^2, \sqrt[5]{.71875}, .98, 1) = .7656 \]

- Algebraic product, \( m_E(x) = (.875^2)\sqrt[5]{.71875}(.98)(1) = .6361 \)

- Bounded difference,

\[ m_E(x) = \max[0, .875^2 + \sqrt[5]{.71875} + .98 + 1 - 3] = .59267 \]

- Hamacher operator for \( p=1/2 \),

\[
m_E(x; 210, 145) = \frac{(.875^2)(.71875)^{5}}{.5 + (1 - .5)[.875^2 + .71875^{5} - (.875^2)(.71875)^{5}]} = .6608
\]

\[
m_E(x; 210, 145, 112, 0) = \frac{(.6608)(.98)}{.5 + (1 - .5)[.6608 + .98 - (.6608)(.98)]} = .6497
\]

- Yager operator for \( p=2 \),

\[
m_E(x) = 1 - \min[1, [(1 -.875^2)^2 + (1 -.71875^5)^2 + (1 -.98)^2]^{1} = .7198
\]

Then \( x(210, 145, 112, 0) \in E_{.75} \) when using the minimum operator, i.e. it is a Superpreferred policy, and \( x(210, 145, 112, 0) \in E_{.65} \) when using the Hamacher operator for \( p = .5 \) and the Yager operator for \( p = 2 \), making him a Preferred policy.

This shows how fuzzy decision-making processes can be used to translate medical records and facilitate the classification of policyholder risks. It is in fact a faster and simpler process for underwriters. Preferred classes offer bonuses on coverage but the author has yet to show how to calculate these benefits and premiums, using fuzzy set theory. For that it is necessary to have fuzzy survival functions.
2. Fuzzy Survival Probability

The future lifetime may be represented as a fuzzy random variable (FRV) when one adds to it some linguistic variables; a basis for fuzzy logic. Assume for a moment the awful event where a medical doctor tells someone they only have a short time left to live. *Short future lifetime (S), Medium future lifetime (M), and Long future lifetime (L)* can be considered FRVs over the lifetime probability space $\Omega$ mentioned above. In this case, fuzzy sets and survival probabilities are combined. This scenario is best illustrated in Puri and Ralescu (1986) and Shapiro (2013). The following case study puts the problem in context.

**Case Study 2.** (Shapiro 2013) Consider the task of giving post-retirement planning advice to new retirees. At this juncture, it may be necessary to know how far their future lifetime will extend. The linguistic lifetime scale S, M, L cited above can be retained. Puri and Ralescu describe a function $T$ that assigns a membership value to each retiree death probability time event $\omega_i \in \Omega$. This is done so that $T(\omega_i)$ is equated to the highest of the membership functions $m_S(\omega_i), m_M(\omega_i), m_L(\omega_i)$. Retirement is assumed to be 65 years of age, so future lifetime starts from $x = 65$. Using the Gompertz law of mortality, we build simplistic fuzzy survival probabilities for S, M, L.

Sivanandam et al. (2007) offers a catalog of methods for the development of membership functions (MF) but a simplistic model for a fuzzy set $A$ is as follows:

$$m_A(x) = \begin{cases} \frac{x - x_L}{x_M - x_L}, & x_L \leq x \leq x_M \\ \frac{x - x}{x_U - x}, & x_M \leq x \leq x_U \\ \frac{x_U - x}{x_U - x_M}, & x_U \leq x \leq x_U \\ 0, & othwewise \end{cases}$$
where $x_L$ is the lower bound, $x_M$ is the midpoint and $x_U$ is the upper bound of the fuzzy number. In the same fashion, the lifetime scale MFs and their graphs are defined below (Figure 10).

$$m_S(t) = \begin{cases} 
1, & 0 \leq t \leq 5 \\
\frac{15 - t}{10}, & 5 \leq t \leq 15 \\
0, & \text{otherwise}
\end{cases}$$

$$m_M(t) = \begin{cases} 
\frac{t - 10}{5}, & 10 \leq t \leq 15 \\
\frac{20 - t}{5}, & 15 \leq t \leq 20 \\
0, & \text{otherwise}
\end{cases}$$

$$m_L(t) = \begin{cases} 
\frac{x - 15}{10}, & t \leq 25 \\
0, & \text{otherwise}
\end{cases}$$

(Figure 10). Short, Medium, Long future lifetime membership for 65 years old
However, this only gives the lifetime as a fuzzy variable. The purpose is to make it a *fuzzy random variable*; that is a RV for which each value has a membership grade in the linguistic scale considered. Figure 11 offers a simple representation of this:

![Figure 11. A Fuzzy random variable representation](image)

Each event of death $\omega_i \in \Omega$ has probability density $P(\omega_i) \in \mathbb{R}$ (contained in [0,1] if normalized), and this event has a degree of membership in the three groups Short, Medium or Long future lifetime. Finally, $T(\omega_i)$ outputs the highest degree of membership of $\omega_i$, and is also a fuzzy random variable:

$$T(\omega_i) = \max\{m_S(\omega_i), m_M(\omega_i), m_L(\omega_i)\}$$

Now, let us consider the *fuzzy death risk* in a future lifetime. It weighs the risk of death in the whole linguistic groups instead of each individual event $\omega_i \in \Omega$. This means, for instance, if $m_S(\omega_i) > 0$ for all $i \in [1, n]$, such that every $\omega_1 ... \omega_n \in S$, then the risk
of the short lifetime is \( P(T(S)) \), the probability of the FRV short future lifetime \( S \). It is equal to the expectation of the membership function in general (Zadeh 1968), such that

\[
P(T(S)) = \int_0^n m_5(t) f_x(t) dt = \int_0^n m_5(t) t p_x \mu_{x+t} dt = E[m_5].
\]

In our case:

\[
P(T(S)) = \begin{cases} 
  \int_0^n 1 \cdot t p_x \mu_{x+t} dt, & 0 \leq t \leq 5 \\
  \int_0^n 15 - t \cdot t p_x \mu_{x+t} dt, & 5 \leq t \leq 15 \\
  0, & \text{otherwise}
\end{cases}
\]

Or equivalently,

\[
P(T(S)) = \begin{cases} 
  n q_x, & 0 \leq t \leq 5 \\
  \frac{1}{10} \left( 15 n q_x - 1 \cdot \dot{e}_x \right), & 5 \leq t \leq 15 \\
  0, & \text{otherwise}
\end{cases}
\]

Note that \( \dot{e}_x \) is the complete expectation of life and \( f_x(t) \) uses the Makeham’s law of mortality for \( x \geq 65 \) as the assumed age for retirement. This gives the mortality probability. Using Definition 3-5 for the complement of a fuzzy variable, the finding of the survival probabilities becomes a simple process.

\[
P(T(S)^C) = \begin{cases} 
  n p_x, & 0 \leq t \leq 5 \\
  \frac{10 + \dot{e}_x - 15 n q_x}{10}, & 5 \leq t \leq 15 \\
  0, & \text{otherwise}
\end{cases}
\]

Hence, this combines the membership function and the mortality probabilities as in Figure 12. The same argument derives the Medium and Long fuzzy survival probabilities.
3. Computation of Fuzzy Premiums

This section gives more numerical results for better understanding of the theory.

**Case Study 3.** (Buckley 1987, Lemaire 1990) *Fuzzy interest rates*

Compute the net single premium of an insurance benefit $S = 1000$ on a 10-year pure endowment policy, issued to a life $x$ aged (55), with $\text{10}_5 = .87$ using a fuzzy interest rate $i$. The interest rate $i$ (approximately 6%) is defined as a fuzzy probabilistic set (Hirota 1981). This is the trapezoidal fuzzy number below.
\[
m_i(z) = \begin{cases} 
0, & \text{if } z \leq 0.03 \\
50z - 1.5, & \text{if } 0.03 < z \leq 0.05 \\
1, & \text{if } 0.05 < z \leq 0.07 \\
4.5 - 50z, & \text{if } 0.07 < z \leq 0.09 \\
0, & \text{if } 0.09 < z 
\end{cases}
\]

The net single premium is expressed as the actuarial present value of a pure endowment:

\[
S \cdot nE_x = S \cdot np_x v^n = S \cdot np_x (1 + \tilde{i})^{-n}
\]

The tilde (\(\sim\)) above the “\(i\)” is meant to differentiate the fuzzy variables from the non-fuzzy (or crisp) ones. By plugging in the quantities from our assumption, we obtain the fuzzy present value below.

\[
S \cdot 10E_{55} = S \cdot 10p_{55} (1 + \tilde{i})^{-10} = 1000 \times 0.87(1 + \tilde{i})^{-10}
\]

Following Theorem 3.6, take the inverse of the membership function of the interest rate \(m_i(z)\). Precisely, \(m_{i1}^{-1}(z)\) and \(m_{i2}^{-1}(z)\), i.e.

\[
m_{i1}^{-1}(z) = 0.03 + 0.02z \quad \text{and} \quad m_{i2}^{-1}(z) = 0.09 - 0.02z
\]

It is the inverses that go through all the computations, and as for normal piecewise functions, we have:

\[
1000 \times 0.87(1 + m_{i1}^{-1}(z))^{-10} = 870(1.09 - 0.02z)^{-10}, \quad \text{and}
\]

\[
1000 \times 0.87(1 + m_{i2}^{-1}(z))^{-10} = 870(1.03 + 0.02z)^{-10}
\]

Again, take the inverses of the two new results in order to have the membership functions for the fuzzy set of \(S \cdot 10E_{55}\). Notice the change in the intervals for \(z\) since the exponent “-10” is negative. We obtain the following membership function and the corresponding graph:
Figure 13. Membership function for a 10-year continuous life insurance for a 55 year-old. In this case however, not only the interest rate is fuzzy but so is the survival probability. For notation purposes, write $\tilde{p} = \tilde{p}_{10\bar{p}_{55}}$, for the fuzzy short-term survival probability for a 55 year-old. One may relate to the Short future lifetime in Case Study 4. (Lemaire 1990) Fuzzy interest Rates and fuzzy survival probabilities

Using the same assumptions as Case study 3 above, we compute the net single premium of a pure endowment with sum insured $S = $1000 for a life aged $x = 55$ years.
Study 2. Below is the membership function for the triangular fuzzy number $\tilde{p}$ along with its graph:

$$m_p(z) = \begin{cases} 
0, & \text{if } (z \leq 0.77) \cup (z \leq 0.97) \\
m_{p1}(z) = 10z - 7.7, & \text{if } 0.77 < z \leq 0.87 \\
m_{p2}(z) = 9.7 - 10z, & \text{if } 0.87 < z \leq 0.97 
\end{cases}$$

For convenience, we use a simpler membership function for the interest rate $i$, also approximately 6%. It is defined as the triangular fuzzy number below.

$$m_i(x) = \begin{cases} 
m_{i1}(z) = 50z - 2, & \text{if } 0.04 < z \leq 0.06 \\
m_{i2}(z) = 4 - 50z, & \text{if } 0.06 < z \leq 0.08 \\
0, & \text{otherwise} 
\end{cases}$$
The new actuarial present value of the pure endowment can be expressed taking into account the new fuzzy variable by

\[
S \cdot nE_x = S \cdot \bar{p}(1 + \bar{i})^{-10}
\]

To find the membership function, we must use Theorem 3-6. First, we determine the inverses of \(m_{p1}^{-1}(z)\) and \(m_{p2}^{-1}(z)\), precisely

\[
m_{p1}^{-1}(z) = .77 + .1z \quad \text{and} \quad m_{p2}^{-1}(z) = .97 - .1z
\]

Next, the inverse of the membership function of the interest rate \(m_i(x)\):

\[
m_{i1}^{-1}(z) = .04 + .02z \quad \text{and} \quad m_{i2}^{-1}(z) = .08 - .02z
\]

To facilitate the calculation, we use the algorithm for multiplication of two trapezoidal fuzzy numbers from Dutta et al (2011) along with Theorem 3.6:

\[
m_{x,y}(z) = \begin{cases}
-(b-a)p + (q-p)a + \sqrt{(b-a)p + (q-p)a)^2 - 4(b-a)(q-p)(ap-z) & , \quad ap \leq z \leq bq \\
2(b-a)(q-p) \\
-(r-q)c + (c-b)r + \sqrt{(r-q)c + (c-b)r)^2 - 4(c-b)(r-q)(cr-z) & , \quad bq \leq z \leq cr \\
2(b-a)(q-p)
\end{cases}
\]
Hence, by the multiplication of two fuzzy numbers and multiplication of a fuzzy number by a scalar, we choose \( X = S \cdot (1 + i)^{-10} \) and \( Y = \bar{p} \) to do the multiplication and obtain these new membership functions:

\[
m_X(z) = m_{S \cdot (1+i)^{-10}}(z) = \begin{cases} 
m_{X1}(z) = 54 - 50 \left( \frac{z}{1000} \right)^{-1/10}, & \text{if } 463.2 < z \leq 558.4 \\
m_{X2}(z) = 50 \left( \frac{z}{1000} \right)^{-1/10} - 52, & \text{if } 558.4 < z \leq 675.6 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
m_Y(z) = m_{\bar{p}}(z) = \begin{cases} 
0, & \text{if } (z \leq .77) \cup (z \leq .97) \\
m_{p1}(z) = 10z - 7.7, & \text{if } .77 < z \leq .87 \\
m_{p2}(z) = 9.7 - 10z, & \text{if } .87 < z \leq .97
\end{cases}
\]

The corresponding values are therefore:

\[
a = 463.2, b = 558.4, c = 675.6, \quad \text{and} \quad p = .77, q = .87, r = .97
\]

So that the combination of both membership functions by the multiplication algorithm gives:

\[
m_{S \cdot nEx}(z) = \begin{cases} 
-6.2828 + .0525\sqrt{38.08z} + 728.136, & \text{if } 356.66 < z \leq 485.81 \\
7.7323 - .0427\sqrt{46.88z} + 2122.9, & \text{if } 485.81 < z \leq 655.33 \\
0, & \text{otherwise}
\end{cases}
\]

Figure 16. Membership function of the net single premium of a pure endowment
The premiums computed with the fuzzy logic approach reveal more information than our usual crisp premiums. First, it gives a bargaining advantage to the underwriters/company to discuss premium rates. Second, since the premium is calculated for each interest rate $i \in [0.3, 0.9]$, the fuzzy approach provides a range for the premium that entails with the probability that a change in interest rates will happen. That means, not only are you aware of the rate change according to the fluctuation in interest rates, but also the chance of that event occurring. Third, with the fuzzy survival probability, the insurance company is equipped with a short term lifetime probability distribution, allowing for accidental deaths or black-swan events. Another important application of fuzzy set theory is the segmentation of the policyholders into clusters by sociodemographic traits.

4. Fuzzy Insurance Benefits

Case Study 5. Consider the crisp models of distribution of future lifetime given by the laws of DeMoivre, Gompertz, Makeham and Weibull. One wants to find expressions for the insurance benefit of an $n$-year continuous life insurance (Huang et al 2011). In this case, the only fluctuating variable is the interest rate $i$ (approximately 6%) from Case Study 3. This is the trapezoidal fuzzy number below.
\[ m_i(z) = \begin{cases} 
0, & \text{if } z \leq .03 \\
50z - 1.5, & \text{if } .03 < z \leq .05 \\
1, & \text{if } .05 < z \leq .07 \\
4.5 - 50z, & \text{if } .07 < z \leq .09 \\
0, & \text{if } .09 < z 
\end{cases} \]

Note that the general equation for the insurance benefit is:

\[ \bar{A}_{x:n}^1 = \int_0^n S e^{-\tilde{\delta} t} t P_x \mu_{x+t} dt \]

Again, the tilde (\(\sim\)) is meant to differentiate the fuzzy variables from the non-fuzzy (or traditional) ones. The force of interest \(\tilde{\delta} = \log(1 + i)\) is inherently fuzzy. Its membership function is:

\[ m_{\tilde{\delta}}(z) = \begin{cases} 
0, & \text{if } z \leq \log(1.03) \text{ and } \log(1.09) < z \\
50e^z - 51.5, & \text{if } \log(1.03) < z \leq \log(1.05) \\
1, & \text{if } \log(1.05) < z \leq \log(1.07) \\
54.5 - 50e^z, & \text{if } \log(1.07) < z \leq \log(1.09) 
\end{cases} \]

This is found by taking the inverse of the pieces of the membership function of the interest rate \(m_i(z)\). Precisely, \(m_{i1}^{-1}(z) = .03 + .02z\) and \(m_{i2}^{-1}(z) = .09 - .02z\).

To go through the computations:

\[ \log(1 + m_{i1}^{-1}(z)) = \log(1.03 + .02z), \quad \text{and} \]
\[ \log(1 + m_{i2}^{-1}(z)) = \log(1.09 - .02z) \]

And taking the inverse one more time:

\[ m_{\tilde{\delta}1} = 50e^z - 51.5 \quad \text{and} \quad m_{\tilde{\delta}2} = 54.5 - 50e^z \]

Using the DeMoivre assumption, the insurance benefit for \(n\)-year continuous life insurance is newly expressed as:
\[
A_{x:n} = \frac{S}{\omega - x} \int_0^n e^{-\delta t} dt = \frac{S}{\omega - x} \cdot \frac{1}{\delta} \cdot 1 - e^{-\delta n}
\]

So the membership value \( \forall n < \omega \in \Omega, S > 0 \) is:

\[
m_{x:n}^A(z) = \frac{S}{\omega - x} \cdot m_{1-e^{-\delta n}}(z) = \frac{S}{\omega - x} \cdot m_{1} \cdot m_{1-e^{-\delta n}}(z)
\]

\[
= \frac{S}{\omega - x} \cdot \frac{1 - (1 + m_t(z))^{-n}}{\log(1 + m_t(z))}
\]

Since \( \frac{S}{\omega - x} \) is a real number or a constant, we may focus on the membership function. To simplify the calculation, we split the membership function into two pieces and we combine them at the end. As in Case Study 4, we can rename the quantities to facilitate multiplication.

Since \( m_x = m_{\frac{1}{\delta}} = \frac{1}{\log(1+m_t)} \), we only need to take the inverse of the above results for the force of interest:

\[
\frac{1}{\log(1 + m_{i1}^{-1}(z))} = \frac{1}{\log(1.03 + .02z)}, \quad \text{and}
\]

\[
\frac{1}{\log(1 + m_{i2}^{-1}(z))} = \frac{1}{\log(1.09 - .02z)}
\]

Again, taking the inverses of the two new results in order to have the membership functions for the fuzzy set of \( m_{\frac{1}{\delta}} \):

\[
m_x(z) = m_{\frac{1}{\delta}}(z) = \begin{cases} 
0, & \text{if } z \leq \log(1.09)^{-1} \text{ and } \log(1.03)^{-1} < z \\
54.5 - 50e^{1/z}, & \text{if } \log(1.09)^{-1} < z \leq \log(1.07)^{-1} \\
1, & \text{if } \log(1.07)^{-1} < z \leq \log(1.05)^{-1} \\
50e^{1/z} - 51.5, & \text{if } \log(1.05)^{-1} < z \leq \log(1.03)^{-1}
\end{cases}
\]
Next, the membership functions for the fuzzy set of $m_Y = m_{1-e^{-\delta n}} = 1 - (1 + m_t)^{-n}$.

Using the same argument as above, the inverses are:

$$1 - \left(1 + m_{i1}^{-1}(z)\right)^{-n} = 1 - (0.02z + 1.03)^{-n}, \quad \text{and}$$

$$1 - \left(1 + m_{i2}^{-1}(z)\right)^{-n} = 1 - (1.09 - 0.02z)^{-n}$$

Taking the inverses of the two new results in order to have the membership functions for the fuzzy set of $m_{1-e^{-\delta n}}$:

$$m_Y(z) = m_{1-e^{-\delta n}}(z) = \begin{cases} 
50(1 - z)^{-1/n} - 51.5, & \text{if } 1 - 1.03^{-n} < z \leq 1 - 1.05^{-n} \\
1, & \text{if } 1 - 1.05^{-n} < z \leq 1 - 1.07^{-n} \\
54.5 - 50(1 - z)^{-1/n}, & \text{if } 1 - 1.07^{-n} < z \leq 1 - 1.09^{-n} \\
0, & \text{otherwise}
\end{cases}$$

The next step is to find $m_{X \cdot Y} = m_{1-e^{-\delta n}}$ by following the technique of Dutta et al. (2011) and Taleshian and Rezvani (2011) to multiply the two trapezoidal fuzzy numbers:

$$m_{X \cdot Y}(z)$$

$$= \begin{cases} 
m_{X \cdot Y_1}(z) = \frac{-(aq + bp - 2ap) + \sqrt{(aq - bp)^2 - 4(b - a)(q - p)z}}{2(b - a)(q - p)}, & \text{if } ap \leq z \leq bq \\
1, & \text{if } bq \leq z \leq cr \\
\frac{(s - r)d - (d - c)s}{2(d - c)(s - r)} - \sqrt{\left((s - r)d - (d - c)s\right)^2 - 4(d - c)(s - r)(ds - z)} & \text{if } cr \leq z \leq ds
\end{cases}$$

Here, the corresponding letters are constants and we have:

$$a = \log(1.09)^{-1}, b = \log(1.07)^{-1}, c = \log(1.05)^{-1}, d = \log(1.03)^{-1} \quad \text{and}$$

$$p = p_n = 1 - 1.03^{-n}, q = q_n = 1 - 1.05^{-n}, r = r_n = 1 - 1.07^{-n}, s = s_n = 1 - 1.09^{-n}$$

So that the combination of both their membership functions by the multiplication algorithm gives:
As \( n \in \mathbb{Z} \) for years in integer form, we can approximate the membership function of the insurance benefit for term life insurance of \( n \) years or for whole life insurance.

Whole life insurance can be treated by taking the limit of \( n \) to infinity. A simple but lengthy result for \( n = 1 \) gives the membership functions below, along with the graph of the functions:

\[
m_{X_Y_1}(z) = \frac{-3.2p_n - 11.6(q_n - p_n) + \sqrt{(3.2p_n + 11.6(q_n - p_n))^2 - 12.8(q_n - p_n)(11.6p_n - z)}}{6.4(q_n - p_n)}
\]

\[
m_{X_Y_2}(z) = \frac{13.3s_n - 33.8(s_n - r_n) + \sqrt{(13.3s_n + 33.8(s_n - r_n))^2 - 53.2(s_n - r_n)(33.8s_n - z)}}{26.6(s_n - r_n)}
\]

\[
m_{\delta}(1-e^{-\delta}) = \begin{cases} 
0, & \text{if } z \leq \frac{1 - 1.03^{-1}}{\log(1.09)} \text{ and } \frac{1 - 1.09^{-1}}{\log(1.03)} < z \\
-0.308 + \sqrt{0.015 + 0.237z} & \text{if } \frac{1 - 1.03^{-1}}{\log(1.09)} < z \leq \frac{1 - 1.05^{-1}}{\log(1.07)} \\
1.18 & \text{if } \frac{1 - 1.05^{-1}}{\log(1.07)} < z \leq \frac{1 - 1.07^{-1}}{\log(1.05)} \\
1.68 - \sqrt{2.78 - 0.912z} & \text{if } \frac{1 - 1.07^{-1}}{\log(1.05)} < z \leq \frac{1 - 1.09^{-1}}{\log(1.03)}
\end{cases}
\]

Figure 17. Membership function of insurance benefit for a continuous life insurance
In the following page, we have the full expression of the membership function with the sequences for all $n$ expressed.
\[ m_{\lambda_{1,n}}(z) = \frac{S}{\omega - \chi} \cdot m_{1} \frac{1 - e^{-\lambda}}{\lambda} 
\]

\[
\begin{cases} 
0, & \text{if } z \leq \frac{1 - 1.03^{-n}}{\log(1.09)} \text{ and } \frac{1 - 1.09^{-n}}{\log(1.03)} < z \\
-3.2(1 - 1.03^{-n}) - 11.6(1.03^{-n} - 1.05^{-n}) + \sqrt{(3.2(1 - 1.03^{-n}) + 11.6(1.03^{-n} - 1.05^{-n}))^2 - 12.8(1.03^{-n} - 1.05^{-n})(11.6(1 - 1.03^{-n}) - z)} & \text{if } \frac{1 - 1.03^{-n}}{\log(1.09)} < z \leq \frac{1 - 1.05^{-n}}{\log(1.07)} \\
6.4(1.03^{-n} - 1.05^{-n}) & \text{if } \frac{1 - 1.03^{-n}}{\log(1.09)} < z \leq \frac{1 - 1.07^{-n}}{\log(1.05)} \\
1, & \text{if } \frac{1 - 1.05^{-n}}{\log(1.07)} < z \leq \frac{1 - 1.07^{-n}}{\log(1.05)} \\
13.3(1 - 1.09^{-n}) + 33.8(1.07^{-n} - 1.09^{-n}) - \sqrt{(13.3(1 - 1.09^{-n}) + 33.8(1.07^{-n} - 1.09^{-n}))^2 - 53.2(1.07^{-n} - 1.09^{-n})(33.8(1 - 1.09^{-n}) - z)} & \text{if } \frac{1 - 1.07^{-n}}{\log(1.05)} < z \leq \frac{1 - 1.09^{-n}}{\log(1.03)} \\
26.6(1.07^{-n} - 1.09^{-n}) & \text{if } \frac{1 - 1.07^{-n}}{\log(1.05)} < z \leq \frac{1 - 1.09^{-n}}{\log(1.03)} 
\end{cases}
\]
The approach is similar for the other life distribution laws (Gompertz, Makeham, Weibull). The same function $\frac{1-e^{-\delta_n}}{\delta}$ is used in every assumption, only to be multiplied by their very specific survival probability of $i\Pr_x \mu_x t$. Recall section 2.2.2 for these actuarial survival probabilities. Thus:

- The insurance benefit of the n-year continuous life insurance using Gompertz assumption

$$m_{A_{x,\tilde{n}}}^1(z) = S \cdot \int_0^t BC^x + t e^{B \ln C^x (1 - C^t)} dt \cdot m_{1-e^{-\delta_n}}(z)$$

- The insurance benefit of the n-year continuous life insurance using Makeham assumption

$$m_{A_{1-x,\tilde{n}}}^1(z) = S \cdot \int_0^t (A + BC^x + t) e^{-At + B \ln C^x (1 - C^t)} dt \cdot m_{1-e^{-\delta_n}}(z)$$

- The insurance benefit of the n-year continuous life insurance using Weibull assumption:

$$m_{A_{x,\tilde{n}}}^1(z) = S \cdot \int_0^t k(x + t)^n e^{k(n+1)[x^{n+1}-(x+t)^{n+1}]} dt \cdot m_{1-e^{-\delta_n}}(z)$$
5. Fuzzy Clustering of Policyholders

A Non-Life Approach

Age is always treated as a factor in mortality, because inevitably the older you get the more susceptible to die you are. The goal for preference classes is to have premium rates according to each policyholder’s risk profile. It is common practice to group policyholders for reasons of efficiency; even if the rating operates by design more effectively on an individual basis. The premium rate must be estimated for the group. Each underwriting structure regroups policyholders by blocks of ages, and most premium estimations are based on these classes. The crisp concept uses curtail age (section 2.2.2); that is the integer age at the last birthday. An example of grouping is Brockham and Wright (1991):


Limitations to the model occur when we wish to assign a 30-year-old to a group. Given the random nature of insurance, the evidence favoring the 4th group over the 5th group is not likely to be conclusive. The theory of Fuzzy Set aims to resolve this ambiguity. In the next Case Study, Verrall & Yakoubov provide a fuzzy approach to grouping policyholder ages using past claims for non-life insurance.


This is a case study of general insurance that can easily be applied to life insurance. The data for the study consist of approximately 50,000 motor policies of all ages. The youngest ages are grouped under the label “<25” and the oldest under “83+” to
remove the absent data or discrepancies. We only focus on two types of claims: Body Injuries (BI) and the Material Damage (MD). For each age, we retrieve the number of claims, or *Frequency*, along with the total cost of the claim, or *Severity*. For each claim many factors enter in the cause of an accident rather than age only. The Department of Motor Vehicles “Unit for Accidents: Causes and Prevention” gives an exhaustive list among which the most often cited are age, gender, driver’s years of exposure, car group, mechanical failure, location and road conditions and weather conditions. Most insurance companies use only age as the key to grouping policyholders. Hence, to remove distortion due to the uneven mix of policyholder age, we adjust the data as follows:

\[
\text{Adjusted Frequency} = \text{frequency} \times \text{severity}
\]

Table 1 displays the Frequency and Severity of the MD and BI claims in the data set used in Verrall et al (1999). The exposure tab gives a numerical value for the number of earned driver years. The abbreviation AdjFreq means Adjusted Frequency and CruPrem mean Crude Premium. It is the a priori estimation of what a policyholder of age \( x \) will need to pay in premium, before any extra fees such as taxes, state surcharges, transaction fees, etc. It equals the expenses the insurance carrier is at risk of incurring for the policyholder \( x \); hence:

\[
\text{Crude Premium} = MD \text{AdjFreq} + BI \text{AdjFreq}
\]
Table 1. Frequency and Severity of Claims

<table>
<thead>
<tr>
<th>Age</th>
<th>Frequency</th>
<th>Severity</th>
<th>AdjFreq</th>
<th>CrudePrem</th>
<th>Exposure</th>
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</thead>
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<td></td>
<td>MD</td>
<td>BI</td>
<td>MD</td>
<td>BI</td>
<td>MD</td>
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<tr>
<td>&lt; 25</td>
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Notes: AdjFreq = Adjusted Frequency; and CrudePrem = Crude Premium.
A fuzzy c-means algorithm (recall Section 3.2) is applied to the adjusted data for the optimal number of clusters $c = 6$ (Bezdek 1981). The exponential weight or fuzzy parameter $r = 2$; $||\cdot||$ is the Euclidean norm; the termination criterion $\varepsilon = 0.05$. For each cluster, we are to determine the center of the adjusted MD, the adjusted BI, and the Crude Premiums. The cluster centers (centroids) derived from Table 1 are allotted in Table 2.

### Table 1: Frequency, Severity, and Adjusted Data

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<th>Age</th>
<th>Frequency MD</th>
<th>Frequency BI</th>
<th>Severity MD</th>
<th>Severity BI</th>
<th>AdjustFreq MD</th>
<th>AdjustFreq BI</th>
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<th>Exposure</th>
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*Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.*
Table 2. Clusters centers or centroids.

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</table>

*Notes: MDADJ = MD Adjusted Frequency; and BIADJ = BI Adjusted Frequency.*

The resulting table of centroids is then used to calculate the membership of each age to a cluster. We use the adjusted frequencies and determine the membership values for the corresponding age. As in Brockham and Wright (1991), we wish to create age groups for underwriters. With less computing skills or tools, underwriting procedure would require crisp age groups. Hence, to separate the element that belongs to more than one cluster, we proceed with an alpha-cut of 20%. The level set of the fuzzy membership is shown in Table 3.
Table 3. 20%-level set of the membership

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</tbody>
</table>
Underwriting usually anticipates that risk progress smoothly with age, and so age groupings must be adjacent. Verrall et al. uses a risk measure to determine which adjacent ages are in the same group. This is the following equality:

\[ R_i = \frac{1}{\|i\|} \sum_{\text{ages in } i} \sum_{k} \mu_{jk} \|v_k\| \]

From the beginning of the case study, it is assumed that <25 and 83+ were whole groups. Applying the risk measure to the 20 percent-cut data gives the results as in the Table 4 below. Hence, for the six clusters, we can count 7 age groups:

\[ \leq 25, 26 - 27, 28 - 31, 32 - 47, 48 - 51, 52 - 68, 69 + \]

Table 4. Risk Measure per Age group

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<th>3</th>
</tr>
</thead>
<tbody>
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<td>( R_i )</td>
<td>406.29</td>
<td>135.65</td>
<td>114.79</td>
</tr>
<tr>
<td>Group, i</td>
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<td>5</td>
<td>6</td>
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<tr>
<td>( R_i )</td>
<td>90.15</td>
<td>100.15</td>
<td>71.78</td>
</tr>
</tbody>
</table>

In figure 18, we have a comparative graph for the crude risk premium and premium based on risk groups. The group premium gives a very good fit of the model for crude premium. The fuzzy clustering allows the creation of risk groups with very smooth transition between ages. The accuracy may slightly be off at groups 1 and 2, for the little information available for these drivers; hence such high premiums.

Age is only one factor among many others, yet the insurance industry made it as the primary indicator of risk in any type of coverage.
This experiment can be applied to a life contingency study of multiple state models. Simply replace the Material Damage (MD) claims by Disability Claims and the Body Injuries (BI) by Death Claims.
CONCLUSIONS AND FURTHER RESEARCH

This paper presented successful concepts and techniques of fuzzy set theory as used in the actuarial science environment. The applications focused on life contingencies and life insurance from the underwriting to claims. Fuzzy actuarial mathematics offers a new and promising way of treating uncertainty, with a useful addition to the modeling tools. The Actuarial Research Clearinghouse is correct to forecast a drastic increase of interest for fuzzy methods in the future. New research may shift toward multiple state models including joint-survivorship, disability, sickness, retirement or withdrawal. Also involved are the hybrid models and the company-sponsored insurance with multiple options. Perhaps these are building blocks that will suggest other fields to develop fuzzy set applications.
Table 5. Life Table for the total population of United States, 2011

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Probability of dying between ages x and x + 1</th>
<th>Number surviving to age x</th>
<th>Number dying between ages x and x + 1</th>
<th>Person-years lived between ages x and x + 1</th>
<th>Total number of person-years lived above age x</th>
<th>Expectation of life at age x</th>
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<td>95,522</td>
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<td>0.000173</td>
<td>95,247</td>
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<td>94,792</td>
<td>94,617</td>
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<td>93,882</td>
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<td>93,428</td>
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</table>

See footnote at end of table.
Table 6. Life Table for the total population of United States, 2011 (continued)

<table>
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<th>Age (years)</th>
<th>Probability of dying between ages x and x + 1</th>
<th>Number surviving to age x</th>
<th>Number dying between ages x and x + 1</th>
<th>Person-years lived between ages x and x + 1</th>
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Table 7. Expectation of Life by age, race, ethnicity and sex in United States, 2011

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<th>Non-Hispanic white¹</th>
<th>Non-Hispanic black¹</th>
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</table>

¹Life tables by Hispanic origin are based on death rates that have been adjusted for race and ethnicity misclassification on death certificates.

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<th>Age</th>
<th>All races and origins</th>
<th>White</th>
<th>Black</th>
<th>Hispanic</th>
<th>Non-Hispanic white</th>
<th>Non-Hispanic black</th>
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1 Life tables by Hispanic origin are based on death rates that have been adjusted for race and ethnicity misclassification on death certifies.

Figure 19. Life expectancy at birth by origin, race and sex from 2006 to 2011
Table 9. Standard results for t-norms and t-conorms for Fuzzy set operations

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<td>Einstein product-sum</td>
<td>$\frac{xy}{1+(1-x)(1-y)}$</td>
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<tr>
<td>Bounded difference-sum</td>
<td>$\max(0, x + y - 1)$</td>
<td>$\min(1, x + y)$</td>
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<tr>
<td>Dubois and Prade (1980b) operators ($0 \leq p \leq 1$)</td>
<td>$\frac{xy}{\max(x,y,p)}$</td>
<td>$1 - \frac{(1-x)(1-y)}{\max[(1-x),(1-y),p]}$</td>
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REFERENCES


Jean Guy Daniel Boni graduated from his hometown Saint Viateur High School, Abidjan, Cote d’Ivoire, in 2008. He moved to the United States for secondary education where he followed two-year undergraduate coursework at Marymount University in Arlington, VA; before graduating from George Mason University in 2012 with a Bachelor of Science in Mathematics and a concentration in Actuarial Science. As of 2015, he completed his graduate work in Mathematical Science along with a Graduate Certificate in Actuarial Science. Conducting several researches in the field of actuarial science, he worked at the National Social Security Fund in Cote d’Ivoire, where efforts are being made to reform the pension and retirement programs.