The Identity of Zeros of Higher and Lower Dimensional Filter Banks and The Construction of Multidimensional Nonseparable Wavelets

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at George Mason University

By

Sirak Belayneh
Master of Science
George Mason University, 1990
Bachelor of Science
Addis Ababa University, 1976

Director: Professor Edward J. Wegman
School of Information Technology and Engineering

Fall Semester 2008
George Mason University
Fairfax, VA

DEDICATION

Dedicated to my daughters Meron and Mekedas who bring love and joy to my world and in memory of my parents Ayelech Tessema, Belayneh Woldeyes, my sister Aster Belayneh and my friend Nolawi Abebe.

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my advisor, Professor Edward Wegman, for his friendship, his excellent advise through all my graduate study. I am also very thankful to members of my defense committee, Professor Stephan Nash, Professor David Walnut, Professor Kristine Bell and Professor Yasmin Said for their time and assistance in reviewing my dissertation.

I also would like to thank Professor Daniel Menasce, Lisa Nodler and Laura Harrison of the Dean's office for their cooperation and encouragement.

Finally I would like to thank my wife, Fikirte Worku, my relatives and friends who stood by me and encouraged me through the years. Special thanks also go to my brother Elias for his encouragement and support.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS AND SYMBOLS

AC	Analysis Component
BSNR	Blurred Signal-To-Noise Ratio
CWT	Continuous-time Wavelet Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
FCO	Face Centered orthorhombic
FFT	Fast Fourier Transform
FIR	Finite Impulse Response
FPT	Forward Polyphase Transform
FS	Fourier Series
IPT	Inverse Polyphase Transform
ISNR	Improvement to Signal-To-Noise Ratio
MRA	Multiresolution Analysis
MRFB	Multirate Filter Bank
MRI	Magnetic Resonance Imaging
MSE	Mean Square Error
ON	Orthonormal
PR	Perfect Reconstruction
PRFB	Perfect Reconstruction Filter Bank
QAR	Quantitative Autoradiology
QM	Quadratic Mirror
QMF	Quadratic Mirror Filter
R-function	Reisz function
SC	Synthesis Component
SNR	Signal to Noise Ratio
S.O.	Semi Orthogonal
STFT	Short-Time Fourier Transform
WS	Wavelet Series
WT	Wavelet Transform

ABSTRACT

THE IDENTITY OF ZEROS OF HIGHER AND LOWER DIMENSIONAL FILTERBANKS AND THE CONSTRUCTION OF MULTIDIMENSIONAL NONSEPARABLE WAVELETS

Sirak Belayneh, Ph.D.

George Mason University, 2008

Dissertation Director: Prof. Edward J. Wegman

This dissertation investigates the construction of nonseparable multidimensional wavelets using multidimensional filterbanks. The main contribution of the dissertation is the derivation of the relations zeros of higher and lower dimensional filtersbanks for cascade structures. This relation is exploited to construct higher dimensional regular filters from known lower dimensional regular filters. Latter these filters are used to construct multidimensional nonseparble wavelets that are applied in the reconstruction and denoising of multidimensional images.

The relation of discrete wavelets and multirate filterbanks was first demonstrated by Meyer and Mallat. Latter, Daubechies used this relation to construct continuous wavelets using the iteration of filterbanks. Daubechies also set the necessary conditions on these filer banks for the construction of continuous wavelets. These conditions also known as the regularity condition are critical for the construction of continuous wavelet basis form iterated filterbanks.

In the single dimensional case these regularity conditions are defined in terms of the order of zeros of the filterbanks . The iteration of filterbanks with higher order zeros results in fast convergence to continuous wavelet basis. This regularity condition for the single dimensional case has been extended by Kovachevic to include the multidimensional case. However, the solutions to the regularity condition are often complicated as the orders and dimensions increase.

In this dissertation the relations of zeros of lower and higher dimensional filters based on the definition of regularity conditions for cascade structures has been investigated. The identity of some of the zeros of the higher and lower dimensional filterbanks has been established using concepts in linear spaces and polynomial matrix description. This relation is critical in reducing the computational complexity of constructing higher order regular multidimensional filterbanks. Based on this relation a procedure has been adopted where one can start with known single dimensional regular filterbanks and construct the same order multidimensional nonseparable regular filterbanks . These filterbanks are then iterated as in the one dimensional case to give continuous multidimensional nonseparabke wavelets. The conditions for dilation matrices with better isotropic transformation has also been revisited. Several examples are used to illustrate the construction of these multidimensional nonseparable wavelets. Finally, these nonseparable multidimensional wavelet basis are used in the reconstruction and denoising of multidimensional images and the results are compared to those obtained by separable wavelets.

CHAPTER 1

INTRODUCTION

1.1 Overview

In the last two decades three quite independently developed concepts in the areas of applied mathematics, signal processing and computer vision have emerged into one unified theory. In applied mathematics, dilation and translation have been used for long time to generate families of functions from single prototype function. Fourier analysis and more recently wavelet analysis are such examples. The works of A. Grossman and J. Morlet grew to what is known today as wavelets [28]. In the area of signal processing, the concept of multirate filter banks evolved from the simple idea of splitting signals into channels so as to be able to reconstruct them without spectral overlap (aliasing) [7], [59]. A. Croisier, D. Esteban and C. Galand first observed the phenomena in 1976 [18]. Subsequently, F. Mintzer, M. J. T. Smith and T. P. Barnwell [2], [57], [71], [72], [82] developed the idea of perfect reconstruction filter banks (PRFB). In computer vision, the concept of multiresolution analysis developed as a method of successive refinement of signals or pyramidal coding schemes [1], [8], [33]. S. Mallat and Y. Meyer [51], [52], [56] used the concept of MRA to arrive at octave band splitting.

The connection between wavelet analysis, multirate filtering and multiresolution analysis was gradually established. Mallat and Meyer [51], [56] were the first to show such relation when they used the concept of MRA to arrive at octave band splitting in the late eighties. Later Daubechies established the relation between the compactly supported wavelet basis and iterated low pass branch of a particular subband scheme [21], [22]. Since then much research has been done to lay a firm theoretical bases for the unified approach, which in turn has contributed to the refinement of techniques in the individual areas [80]. For example concepts used to develop non-separable multidimensional filter banks are applied to develop nonseparable wavelet basis.

1.2 Problem Statement and Motivation

My work is primarily motivated by the desire to understand the fine work of Daubechies, Vetterli and Kovachevic and applying the concepts to multidimensional image reconstruction [20], [85], [38], [36]. Later the work focused on solving the problem posed by J. Kovachevic and M. Vetterli [39] of establishing the relation between the zeros of higher and lower dimensional regular filters. This question has not been addressed previously. Establishing the above relation contributed to the idea of constructing higher dimensional filter from lower dimensional filters.

The goal of this dissertation was to extend some of the concepts pertaining to the construction of wavelets from iterated filter banks to multiple dimensions. These include:

* establishing the relation of regularity condition of lower and higher dimensional filters.
* developing methods for the construction of regular multidimensional filter banks starting from known lower dimensional regular filter banks using the above relation.
* constructing multidimensional nonseparable wavelet basis using the method of iteration and the above constructed filter banks.
* setting the conditions on dilation matrices for better isotropic transformation
* applying the above wavelets for the reconstruction and denoising of

multidimensional images.

Substantial effort has been made to achieve all of the above stated goals. The nonseparable wavelets have been used for the reconstruction and denoising of 2-D images. The results show a more isotropic treatment of images than the separable wavelets. However, the extension of the result to 3-D images has been computationally cumbersome and required additional research in 3-D convolution and programming skills for the manipulation of 3-D data.

1.3 Overview of the Dissertation

The first part of this thesis will concentrate on reviewing the basic concepts of wavelet analysis, multirate filter banks and multiresolution analysis. A brief description of the basic tenet and historical evolution of these three concepts are made. The concept of Fourier analysis and its limitations that lead to the evolutionary transformation to wavelet analysis are discussed. The mathematical concepts underlying wavelet analysis are described. The fundamental theorems in multiresolution analysis and the concept of successive approximation or refinement in signal processing is summarized. Finally, the section ends with the discussion of basic concepts in multirate filter bank such as sampling rate change and perfect reconstruction. The different techniques of analysis and synthesis of perfect reconstruction filter banks are described.

The second part discusses the interrelationship of the above concepts and the evolution of these relations into a unified theory in the one-dimensional case. An outline derivation of one concept based on the assumptions of another is made. Special attention is given to the construction of wavelet basis from iterated filter banks and the regularity condition.

The third part discusses the extension of the above concepts to more than one dimensions. The final objective is to extend the concept of one-dimensional construction of continuous wavelets form iterated filter banks to multiple dimensions. The method of constructing multidimensional wavelet basis based on the iteration of regular, multidimensional nonseparable filter banks will be of special significance. This requires basic knowledge of the fundamentals and techniques of construction of multidimensional filter banks [92], [93]. All of the single dimensional concepts such as downsampling perfect reconstruction filter banks and so on will be revisited with the aim of extending them to multiple dimensions. Extension of these concepts requires a definition of a whole series of new concepts that are widely different from those in the single dimensional case. First these new concepts such as lattice, separability, coset vectors and so on which are specific to multidimensional filters are elaborated. Even those common in one dimension such as down and upsampling and perfect reconstruction are redefined within the multidimensional frame.

The fourth part which is the core of this dissertation, discusses regularity of filter banks. As in the one-dimensional case the construction of multidimensional wavelets heavily depends on the construction of regular multidimensional filer banks. First the basic definition of regularity is extended to multiple dimensions. Then the existence of the relations of the zeros in the lower and higher dimensional filter banks are proved. This relation between lower dimensional and higher dimensional cascade filters is then used to construct higher dimensional regular filters from lower dimensional ones in the next section.

The fifth part discusses the different alternative methods of construction of multidimensional wavelets. The main emphasis will be on the construction of multidimensional wavelets using the iteration method. The regular multidimensional filter banks constructed above will be used to build nonseparable multidimensional wavelets.

The six part will deal with image reconstruction and denoising algorithms and the special advantages of using the multidimensional nonseparable wavelet transform. The nonseparable multidimensional wavelet basis constructed above will be used to reconstruct and denoise multidimensional images. In this section illustrative examples are used used to demonstrate the concepts.

The final section summarizes the results and contributions as well as proposals for future work.

CHAPTER 2

WAVELETS, MRA and FILTER BANKS

2.1 INTRODUCTION:

The background knowledge essential to the understanding of this research is outlined in this section. The basic concepts of wavelet analysis, multiresolution analysis and multirate filter banks and the interconnection between them are fundamental in the construction of the multidimensional wavelet bases. It is essentially these concepts that will be latter extended to construct the multidimensional wavelet bases. The material in this chapter is based on the literature and fine works of C. K. Chui, I. Daubechies, M. Vetterlli and J. Kovacevic, Y. Meyer , E. Viscito and others [15], [21], [85], [56], [87].

I start by reviewing essential concepts from discrete-time signal processing theory. Then a brief look of each of the above three concepts along with their historical evolution is made.

2.2 FROM FOURIER TO WAVELET TRANSFORM:

2.2.1 FOURIER ANALYSIS:

Fourier analysis has been the foundation of harmonic analysis and constitutes two components, Fourier transform and Fourier series. The Fourier transform *F* of a measurable function *f* is defined as

$$F(\omega) = \int_0^{2\pi} f(x)\, e^{-nix} dx \qquad 2.1$$

The Fourier transform has limitations in the study of the spectral behavior of analog signal. First, full knowledge of the signal in the time-domain is required and second if the signal is nonstationary the entire spectrum is affected.

The Fourier series representation of any f in $L^2(0,2\pi)$ is given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \qquad 2.2$$

where the constants c_n are the Fourier coefficients of f and are defined by

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \qquad 2.3$$

$L^2(0,2\pi)$ denotes the collection of measurable functions f defined on the interval $(0,2\pi)$ with

$$\int_0^{2\pi} |\mathrm{f(x)}|\mathrm{dx} < \infty. \qquad 2.4$$

The Fourier series representation has two distinct features. First the function f is decomposed into the sum of infinitely many mutually orthogonal components $g_n(x) = c_n e^{inx}$ since $\omega_n(x) = e^{inx}$ form orthonormal basis of $L^2(0,2\pi\ )$. The second distinct feature is that the orthogonal basis ω_n is generated by dilation of an angle function $\omega(x) = e^{ix}$.

The Fourier series representation also satisfies the Parseval identity indicating that $l^2(0,2\pi)$, $L^2(0,2\pi)$ to be isometric to each other.

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad 2.5$$

Let us now consider measurable functions in $L^2(R)$. Since every function in $L^2(R)$ must decay to zero at $\pm\infty$ the sinusoidal functions $\omega_n(x) = e^{inx}$ do not belong to $L^2(R)$. The wavelets ψ that generate $L^2(R)$ must on one hand have fast decay and on the other hand must cover the whole real line.

The above limitation of Fourier analysis lead to the development of the following transforms.

2.2.2 GABOR TRANSFORM:

D. Gabor [26] used the Gaussian function, which is an optimal window for time localization of signals.

$$g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4}\alpha} \qquad 2.6$$

where $\alpha > 0$.

The Gabor transform of $f \epsilon L^2(R)$ defined below localizes the frequency of f around $t=\beta$.

$$G_\beta^\alpha(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) g_\alpha(t-\beta)dt. \qquad 2.7$$

Instead of localization of the Fourier transform of f we may interpret it as windowing of the signal function f by using the window $G_{\beta,\omega}^\alpha$ and the Gabor transform becomes

$$\begin{aligned} G_\beta^\alpha(\omega) &= \langle f, G_{\beta,\omega}^\alpha \rangle \\ &= \int_{-\infty}^{\infty} f(t)\overline{G_{\beta,\omega}^\alpha(t)}dt \end{aligned} \qquad 2.8$$

and the Parseval identity can be applied to give

$$<f, G^{\alpha}_{\beta,\omega}> = \frac{1}{2\pi}<\hat{f}, \hat{G}^{\alpha}_{\beta,\omega}> \tag{2.9}$$

$$G^{\alpha}_{\beta}(\omega) = \frac{e^{-i\beta\omega}}{2\sqrt{\pi\alpha}} G^{\frac{1}{4\alpha}}_{\omega}(-\beta). \tag{2.10}$$

The window Fourier transform of f with window g_{α} at $t=\beta$ agrees with the "window inverse Fourier transform" with window $\frac{g_1}{4\alpha}$ at $\eta=0$. The product of these two windows would give the area of the time-frequency window.

$$(2\Delta_{g_{\alpha}})(2\Delta_{\frac{g_1}{4\alpha}}) = 2 \tag{2.11}$$

$$<f, G^{\alpha}_{\beta,\omega}> = <\hat{f}, H^{\alpha}_{\beta,\omega}>$$

$$\text{where } H^{\alpha}_{\beta,\omega}(\eta) = \left(\frac{e^{i\beta\omega}}{2\sqrt{\pi\alpha}}\right) e^{-i\beta\eta} \frac{g_1}{4\alpha}(\eta-\omega).$$

In general any window function ω can be defined, but must satisfy the requirement

$$t\omega(t) \epsilon L^2(R) \tag{2.12}$$

and also

$$|t|^{\frac{1}{2}}\omega(t) \epsilon L^2(R). \tag{2.13}$$

2.2.3 SHORT-TIME FOURIER TRANSFORM (STFT):

If $\omega \epsilon L^2(R)$ is chosen such that ω and its Fourier transform $\hat{\omega}$ satisfy the above condition, then the window Fourier transform called STFT of f is defined as follows with rigid window at $t=b$.

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\overline{\omega(t-b)}dt \tag{2.14}$$

ω and $\hat{\omega}$ satisfy

$$\Delta_{\omega}\Delta_{\hat{\omega}} \geq \frac{1}{2}. \tag{2.15}$$

The optimal window is obtained when we have the equality. The Gabor transform with the Gaussian window is STFT with the smallest time-frequency window. The Gabor transform does not give extra freedom to achieve other desirable properties, which a larger window can give. The larger window gives extra degree of freedom for introducing dilation to make time-frequency window flexible.

The analysis of signals using STFT gives a time-frequency window that is still rigid. It does not give the flexibility to adjust the time window based on the frequency. Normally a narrow time-window is required to locate high frequency phenomena precisely and wide time-window to analyze low frequency signals.

Thus to overcome this resolution limitation of STFT $\Delta t, \Delta f$ the basic wavelet is defined with the flexibility of time-frequency window, which narrows automatically with high frequency phenomena and widens otherwise. This is possible by making Δf proportional to the central window frequency f that is

$$\frac{\Delta f}{f} = c. \qquad 2.16$$

2.2.4 WAVELETS:

The limitation in the rigidity of the time-frequency window of STFT forced the introduction of the integral wavelet transform. Instead of windowing the Fourier and inverse Fourier transform as in the STFT, the integral wavelet transform windows the function and its Fourier transform directly allowing flexibility of the time-frequency window using dilation parameters.

The integral wavelet transform is defined in terms of the basic wavelet $\Psi \epsilon L^2(R)$ as

$$W_\Psi f(b,a) = |a|^{\frac{-1}{2}} \int_{\infty}^{\infty} f(t)\overline{\Psi(\frac{t-b}{a})}\, dt, \qquad 2.17$$

where Ψ is a basic wavelet if $\Psi \in L^2(R)$ and satisfies the admissibility condition

$$C_\Psi = \int_{-\infty}^{\infty} \frac{|\hat{\Psi}(\omega)|^2}{|\omega|} < \infty. \tag{2.18}$$

If Ψ and $\hat{\Psi}$ satisfy the admissibility condition, then the basic wavelet Ψ provides the time-frequency window with area $4\Delta\hat{\Psi}\Delta\Psi$. Under the assumptions mentioned above Ψ is a continuous function and from the finiteness of C_Ψ equation 2.18 $\Psi(0)$ or

$$\int_{-\infty}^{\infty} \Psi(t)dt = 0. \tag{2.19}$$

Thus the basic wavelet Ψ is defined as

$$\Psi_{b,t}(t) = |a|^{\frac{-1}{2}} \Psi(\frac{(t-b)}{a}), \tag{2.20}$$

and the wavelet transform is given by

$$W_\Psi f(b,a) = <f\,,\Psi_{b,a}>. \tag{2.21}$$

Ψ can be used as a basic wavelet only if the inversion exists. The function f has to be reconstructed from the values of $W_\Psi f(b,a)$. Any formula that expresses every $f \in L^2(R)$ in terms of $W_\Psi f(b,a)$ is called Inversion Formula and the kernel function $\hat{\Psi}$ to be used in this formula will be called a dual of the basic wavelet Ψ. The recovery of f from $W_\Psi f$ depends on the domain of (a,b). The conditions of recovery of f for the special dyadic case where $b=k/2^j$, $a=1/2^j$, and $j, k \in Z$ are covered under Reisz basis (see Appendix A).

An R-function (Reisz function) $\Psi \epsilon L^2(R)$ is called an R-wavelet (wavelet) if there exists a function $\tilde{\Psi} \epsilon L^2(R)$ such that $\{\Psi_{j,k}\}$ and $\{\Psi^{j,k}\}$ are dual bases of $L^2(R)$. If Ψ is an R-wavelet, then $\tilde{\Psi}$ is called dual wavelet corresponding to Ψ.

Every wavelet Ψ orthogonal or not generates a wavelet series representation of any $f \epsilon L^2(R)$,

$$f(x) = \sum_{j,k=-\infty} C_{j,k} \Psi_{j,k}(x), \qquad 2.22$$

where $C_{j,k}$ is the wavelet transform of f relative to the dual $\tilde{\Psi}$.

If Ψ is an orthogonal wavelet the decomposition is also orthogonal and unique. However, there are class of wavelets $\{\Psi_{j,k}\}$ called semi-orthogonal that can be used to generate $L^2(R)$ without requiring the orthonormal properties for each j, that is

$$<\Psi_{j,k} , \Psi_{j,l}> = \delta_{k,l}. \qquad 2.23$$

A wavelet Ψ in $L^2(R)$ is called semiorthonormal (s.o.) wavelet if the Reisz basis $\{\Psi_{j,k}\}$ it generates satisfies

$$<\Psi_{j,k} , \Psi_{l,m}> = 0 \qquad j \neq l , \; j,k,l,m \epsilon Z. \qquad 2.24$$

Every semiorthonormal wavelet generates orthogonal decomposition of $L^2(R)$, and if Ψ is nonorthogonal wavelet (not s.o.) being Reisz basis has a dual $\tilde{\Psi}$ and the pair $\{\Psi,\tilde{\Psi}\}$ satisfies the biorthogonal properties

$$<\Psi_{j,k}\ ,\ \Psi_{l,m}> = \delta_{j,l}\delta_{k,m} \qquad j,k,l,m \in Z. \qquad 2.25$$

This can be used to generate biorthogonal decomposition of $L^2(R)$, and is used to construct linear phase filter banks.

The following figure shows the difference between the time-frequency localization of STFT and wavelet transform.

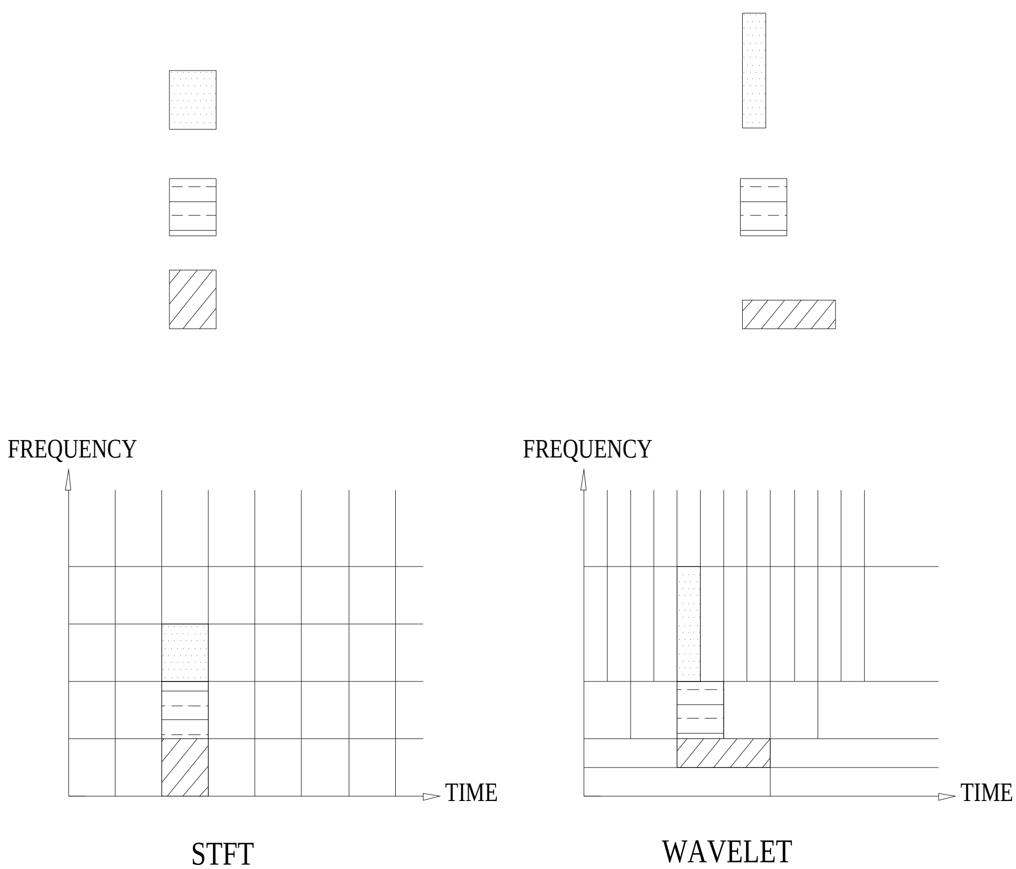

Figure 2.1: STFT and WAVELET

2.3 MULTIRESOLUTION ANALYSIS:

The concept of multiresolution analysis, as introduced by Mallat and Meyer [51], [52], [56] can be viewed as successive approximation or successive refinement of a signal. Signals are analyzed at different resolutions each time adding details when going from coarser to finer resolution. P. J. Burt and E. H. Adelson [11] used the idea for image analysis and compression in the Laplacian pyramid scheme [34], [61].

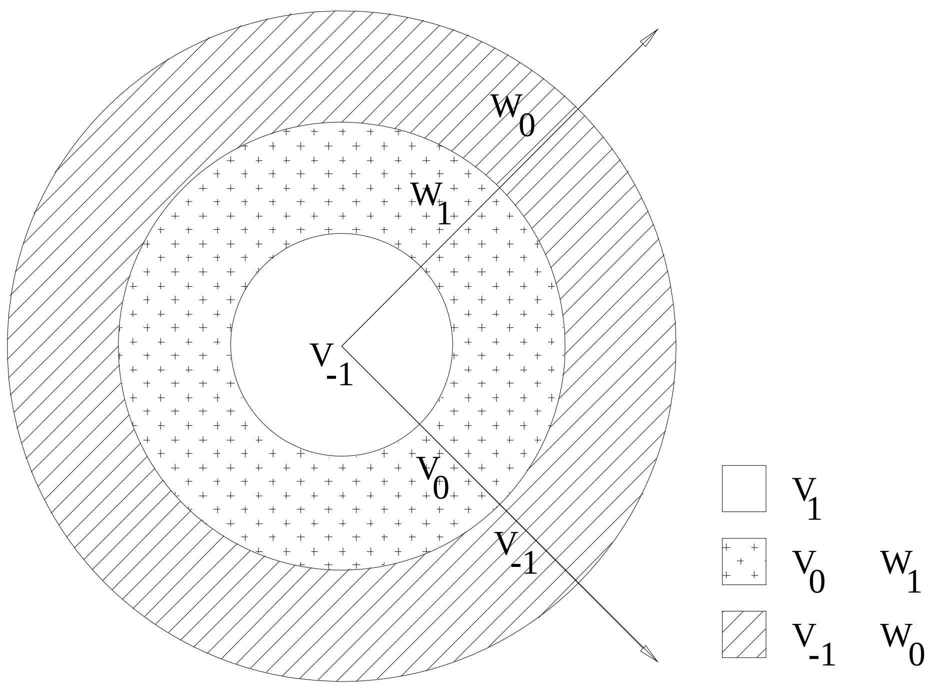

Figure 2.2: Multiresolution Analysis

The square integrable function *f* is written as a limit of successive approximation, each of which is a smoothed version of *f*, with concentric smoothing functions. The successive approximations are also required to have some translational invariance.

In general the multiresolution analysis consists of

1. embedded closed subspaces $V_m \subset L^2(R), m \epsilon Z,$

2. such that

$$\bigcap_{m \epsilon Z} V_m = 0 \qquad 2.26$$

$$\overline{\bigcup_{m \epsilon Z} V_m} = L^2(R),$$

3. and if $f(t)$ is in the space V_m then $f(2t)$ is in $V_{m\text{-}1}$

$$f(t) \epsilon V_m \Leftrightarrow f(2.t) \epsilon V_{m-1}, \qquad 2.27$$

4. Finally, there exists a $\Phi \epsilon V_0$ such that for all $m \epsilon Z$, Φ_{mn} constitutes an unconditional basis for V_m, with

$$V_m = \overline{span\{\Phi_{mn}\}} \qquad n \epsilon Z \qquad 2.28$$

and there exists $0<A \leq B<\infty$ such that for all C_n, $n \epsilon Z$, such that

$$A\sum_n |C_n|^2 \leq \|\sum_n C_n \Phi_{mn}\|^2 \leq B\sum_n |C_n|^2. \qquad 2.29$$

If P_m denote the orthogonal projection onto V_m, then

$$\lim_{m\to\infty} P_m f = f \quad for\ all\ f \epsilon L^2(R). \qquad 2.30$$

The successive projections $P_m f$ as m increases correspond to approximations of f on a finer and finer scale.

Let us now define $W_j, j \in Z$ to be the orthogonal complement of V_j in V_{j-1}

$$V_{j-1} = V_j \oplus W_j \tag{2.31}$$

and

$$W_j \perp W_{j'}, \text{ if } j \neq j'. \tag{2.32}$$

Since $W_j \subset V_{j'} \perp W_{j'}$ and if $j > j'$ and $j < J$

$$V_j = V_J \oplus \bigoplus_{k=0}^{J-j-1} W_{J-k}, \tag{2.33}$$

which implies the decomposition of $L^2(R)$ into mutually orthogonal subspaces, i.e.

$$L^2(R) = \bigoplus_{j \in Z} W_j. \tag{2.34}$$

Furthermore W_j inherits the scaling property of V_j

$$f \in W_j \Leftrightarrow f(2^j) \in W_0. \tag{2.35}$$

As with V_0, there exists a vector Ψ such that its integer translates span W_0, that is

$$W_0 = \overline{\{span\{\Psi_{0n}\}}. \tag{2.36}$$

From the scaling property above it follows

$$W_m = \overline{\{span\{\Psi_{mn}\}}. \tag{2.37}$$

Once the existence of Ψ is shown, it remains to construct this function. In the sequel the general construction of orthonormal wavelet basis is outlined.

A function Φ with Φ_{0n} an orthonormal basis for V_0 is determined. Because $\Phi \in V_0 \subset V_{-1} = \overline{span\{\Phi(2x-n)\}}$, there exists h_n such that

$$\Phi(x) = \sum_n h_n \Phi(2x-n) \qquad 2.38$$

and by definition

$$\Psi(x) = \sum_n (-1)^n C_{n-1} \Phi(2x+n)$$
$$= \sum_n g_n \Phi(2x+n).$$

These equations are called the "two-scale relations" of the scaling and wavelet functions. The corresponding Ψ_{0n} and Ψ_{mn} will constitute an orthonormal basis of W_0 and W_m respectively. $\{\Psi_{mn}, m,n \in Z\}$ constitutes an orthonormal wavelet basis for $L^2(R)$. The two scale sequences h_n and g_n defined above are the only quantities needed to produce the coefficients of a wavelet multi-resolution decomposition.

2.4 MULTIRATE FILTER BANK

2.4.1 Multirate

In many applications of digital signal processing, it is necessary to make changes in the sampling rate of a signal. One common example is the frequency conversion between television standards (60 Hz US and 50 Hz European). Systems that employ multiple sampling rates in the processing of digital signals are called multirate systems. The process of sampling rate conversion in the digital domain can be viewed as a linear filtering operation. Let us consider the two basic sampling rate conversions.

2.4.1.1 Down-Sampling

The first is down-sampling by a factor D called decimation where every Dth sample of the discrete sequence $x(n)$ is kept and the rest is discarded. Its input/output relation is given by

$$y(n) = x(Dn). \qquad 2.39$$

In the Fourier and *Z-domain* this relationship can be written as follows

$$\widehat{Y}(\omega) = \frac{1}{D}\sum_{k=0}^{D}\widehat{X}\left(\frac{\omega-2\pi k}{D}\right) \qquad 2.40$$

$$Y(z) = \frac{1}{D}\sum_{k=0}X(W_D^k z^{\frac{1}{D}}). \qquad 2.41$$

The down sampler is a linear, but periodically shift-varying operation as shown on Figure 2.3. To avoid aliasing, we must first reduce the bandwidth of $x(n)$ to $\omega_{max.} = \frac{\pi}{D}$ before down-sampling.

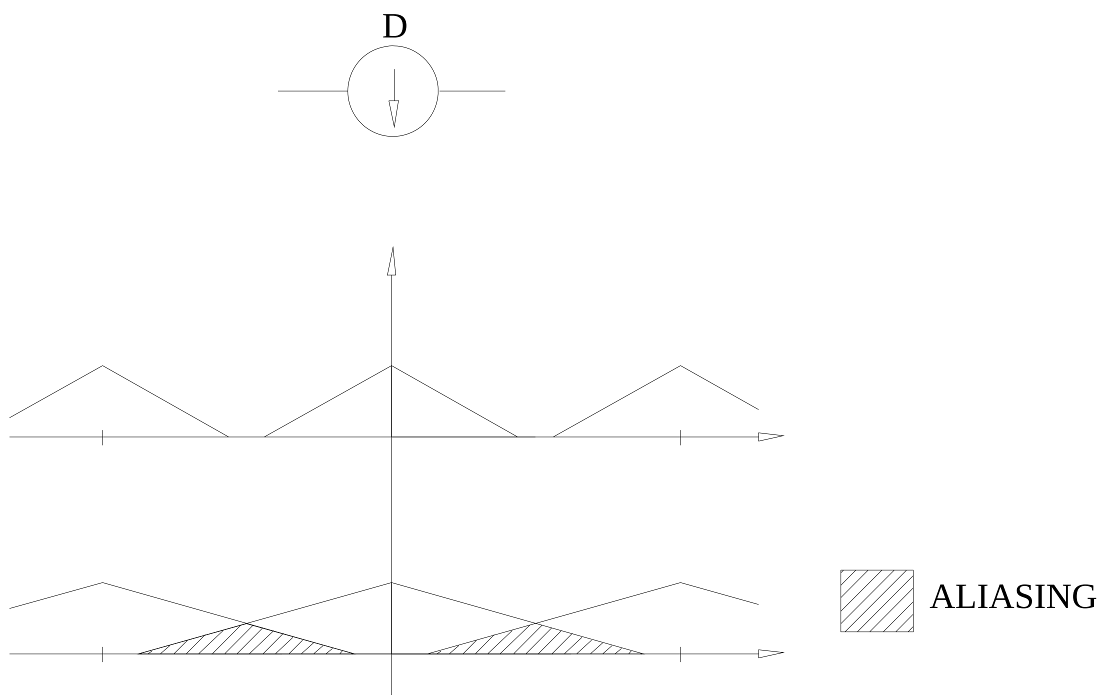

Figure 2.3: Downsampling

2.4.1.2 Up-Sampling

The second is up-sampling by a factor U called interpolation where U-1 zeroes are introduced between each successive values of the input signal $x(n)$. If input/output relationship is given by

$$y(n) = x(\tfrac{n}{U}) \quad \text{if } n = Uk \qquad 2.42$$

$$= 0 \quad \text{.if } n \neq Uk, k=0,1,\ldots$$

The Fourier and z-transform of the relations are given by

$$\widehat{Y}(\omega) = \widehat{X}(U\omega) \qquad 2.43$$

$$Y(z) = X(z^U).$$

From Figure 2.4 we see the production of the signal at points $2k\pi/U$ in the spectrum. To remove the unwanted images one would need a filter with the cut-off frequency $\frac{\pi}{U}$ to follow the up-sampler.

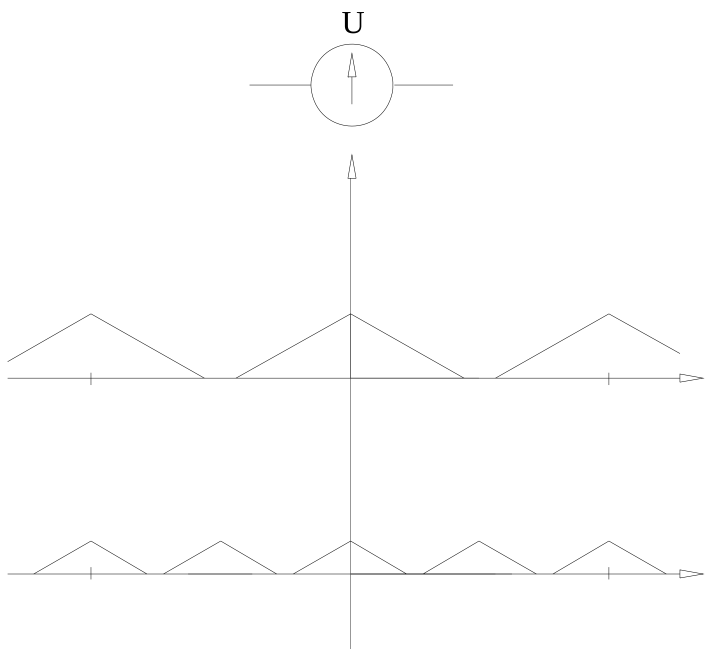

Figure 2.4: Upsampling

For the general case of sampling-rate conversion by a rational factor $\frac{U}{D}$, a cascade of the above two basic blocks can be used. The sampling rate conversion is achieved by first performing interpolation by the factor U and then decimation of the output of the interpolator by the factor D. The filters can be combined to a single filter with cut-off frequency that is $min(\frac{\pi}{U}, \frac{\pi}{D})$ prevent aliasing and multiple imaging. There are several kinds of filter realizations of a decimator and interpolator and it is essential to consider the efficiency of these filters in the design process. [63].

2.4.2 Perfect Reconstruction Filter Bank

Subband systems consist of three components: analysis, transmission and synthesis as shown below.

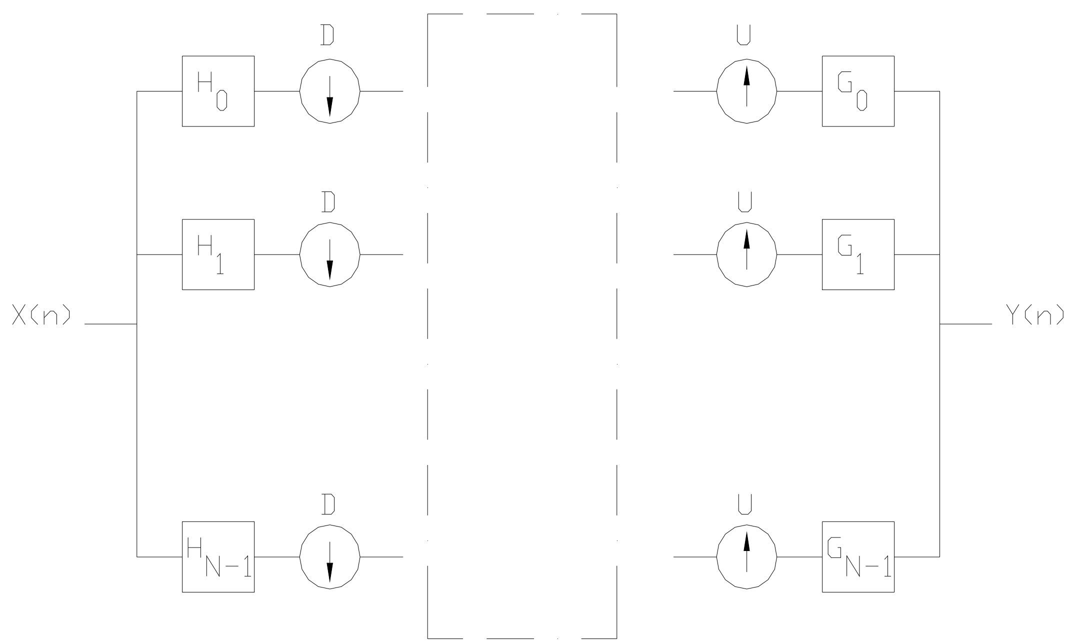

Figure 2.5: Multirate Filter Bank

The analysis section decomposes the signal into *N* subband components and each subband component is formed by filtering and resampling according to their corresponding Nyquist frequency. The signal is then coded and transmitted. At the synthesis section both up-sampling and filtering is done to reconstruct an approximation to the input signal.

The aim in filter bank system design is to achieve perfect reconstruction by eliminating distortions. The first distortion is due to aliasing arising from the down-sampling of signals that are not perfectly band limited. The second is phase and magnitude distortion. These distortions are eliminated by imposing design constraints on both the analysis and synthesis filters. Aliasing in the sub-band components of the analysis bank can be exactly canceled at the output of the synthesis bank by appropriately choosing the synthesis filter. Some restrictions on the analysis filters are necessary if certain properties such as FIR or stability conditions are to be desired. Other criteria are imposed on the design of the analysis and synthesis filters in order to eliminate phase and magnitude distortion allowing the output to be exact replica of the input within time shift and scaling.

2.4.3 Two Channel PRFB

In what follows construction and analysis of two-channel filter bank is used to demonstrate the above principles. Several methods are used to analyze the two-channel PRFB, which is a special case of the *N*-channel case and the methods can be extended to the *N*-channel case.

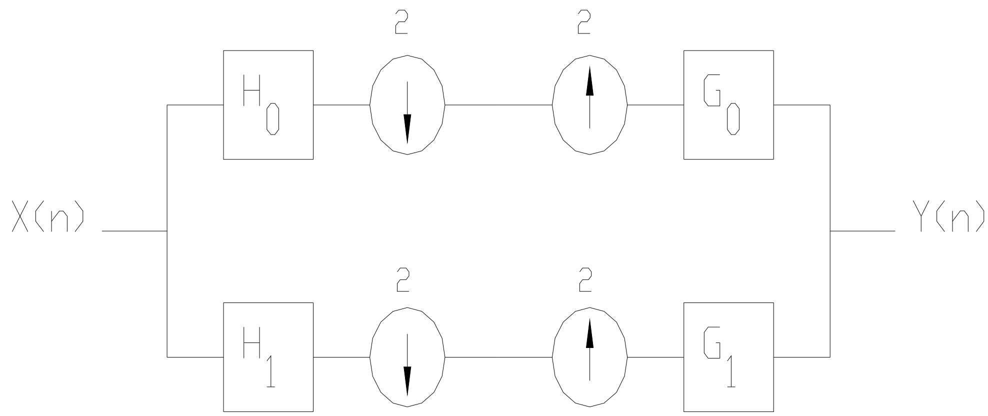

Figure 2.6: Two Channel PRFB

2.4.3.1 Time Domain Analysis

From the figure above the input signal is split into two bands, the lowpass and highpass filtered version of the signal [58]. Assume the input signal is square integrable discrete sequence $x(n)$, $n \in Z$. The convolution with the filter having impulse response $[h_0(0), h_0(1), \ldots h_0(L-1)]$ followed by subsampling by 2, corresponds to matrix multiplication of the infinite signal vector $x=[\ldots, x(-1), x(0), x(1), \ldots]$ by

$$H_0 = \begin{bmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & h_0(L-1) & h_0(L-2) & h_0(L-3) & h_0(L-4) & . & . \\ . & . & 0 & 0 & h_0(L-1) & h_0(L-2) & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \qquad 2.44$$

The same can be done with the highpass filter. Perfect reconstruction is obtained if and only if

$$G_0H_0 + G_1H_1 = I, \qquad 2.45$$

and if the orthogonality constraint, namely $G_0 = H_0^*$ and $G_1 = H_1^*$ is satisfied, then

$$H_0^* H_0 + H_1^* H_1 = I. \qquad 2.46$$

The orthogonality feature makes the synthesis filter the same as the analysis filter within time reversal.

2.4.3.2 Polyphase Domain Analysis

Another way of looking at the system is to decompose both the signals and the filters into polyphase components

$$h(n) = h(Nn + 1). \qquad 2.47$$

The polyphase components for the above case are just the even and odd subsequences of filters and signals. Thus the lowpass filter can be decomposed into the following

$$H_0(z) = H_{01}(z^2) + z^{-1} H_{02}(z^2), \qquad 2.48$$

where H_{0i} is the ith polyphase component of the filter H_0 and is given by

$$H_{0i}(z) = \sum_n h_0(2n+i) z^{-n}. \qquad 2.49$$

Similarly the input signal is decomposed into two sequences, but in reverse fashion accounting for the shift-reversal when convolving the two components,

$$X(z) = X_0(z^2) + zX_1(z^2) \tag{2.50}$$

$$X_i = \sum_n X(2n-i)z^{-n}.$$

The output of the system in terms of the polyphase decomposition is

$$Y(z) = [1 \quad z]G_p(z^2)H_p(z^2)x_p(z^2), \tag{2.51}$$

where G_p and H_p are matrices containing the polyphase components of the analysis and synthesis filters respectively, and x_p contains the polyphase components of the input signal. The vector $(1\ z)$ and $(1\ z^{-1})$ are called the inverse and forward polyphase transforms. Perfect reconstruction is possible if and only if

$$G_p(z)H_p(z) = I. \tag{2.52}$$

The matrix $T_p = G_pH_p$ is called the transfer polyphase matrix. Aliasing is canceled if T_p is pseudo-circulant matrix and perfect reconstruction if and only if T_p is a pseudo-circulant delay. In case of FIR filter perfect reconstruction is possible if and only if the determinant of the analysis polyphase matrix is a delay. And if orthogonality is assumed, then the condition for perfect reconstruction is

$$\widehat{H}_p(z)H_p(z) = I, \tag{2.53}$$

where $\widehat{H}$ denotes transposition of the matrix, conjugation of the coefficients and substitution of z by z^{-1}.

2.4.3.3 Modulation Domain Analysis

The system can also be analyzed by directly finding the *z*-domain expressions for all the signals. The output will then be

$$Y(z) = \tfrac{1}{2}(G_0(z)\ G_1(z)) \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix}. \qquad 2.54$$

The solution to the above equation gives cancellation of aliasing. One such solution is QMF (quadrature mirror filter) [81].

$$H_0(z) = H(z) \quad G_0(z) = H(z)$$

$$H_1(z) = H(-z) \quad G_1(z) = -H(-z) \qquad 2.55$$

$$Y(z) = \tfrac{1}{2}[X(z)\ X(-z)] \begin{bmatrix} H_0(z)H_0(z)-H_0(-z)H_0(-z) \\ 0 \end{bmatrix}$$

The conditions for perfect reconstruction then becomes

$$H_0^2(z) - H_0^2(-z) = 2 \qquad 2.56$$

2.5 UNIFIED APPROACH

In this section, I will demonstrate the interconnection between the wavelets, multiresolution analysis, and multirate filter banks discussed in the previous section. These interconnections between the three concepts have created new venues for the enhancement and elaboration of each of the particular concepts. Applications in one area have simulated theoretical development in another area. Of particular significance to this dissertation is the development of wavelet bases through the iteration of filter banks.

Most of the work in this area was first developed by S.Mallat and Y. Meyer [51], [52], [56] and later extended by I. Daubechies, M. Veterlli and others [12], [13], [16], [21], [65], [66].

2.5.1 WAVELET & MRA

The construction of wavelets from multiresolution analysis consists of a ladder of spaces $\{V_j\}$, $j \in Z$ and a special function $\Phi \in V_0$. In what follows the construction of wavelets from multiresolution analysis is outlined.

If we define W_j , $j \in Z$ such that

1. W_j is the orthogonal complement of V_j in $V_{j\text{-}1}$ i.e.

$$V_{j-1} = V_j \bigoplus W_j \qquad 2.57$$

and

2. $W_j \perp W_k$, $if\ j \neq k$.

Then $L^2(R)$ can be decomposed into mutually orthogonal subspaces and W_j inherits the scaling property of V_j.

$$L^2(R) = \bigoplus_{j \in Z} W_j. \qquad 2.58$$

As with V_0, there exits a vector Ψ such that its integer translates span W_0, i.e.

$$W_0 = \overline{span\{\Psi_{0n}\}}. \qquad 2.59$$

Once the existence of Ψ is shown it remains to construct this function. The " two-scale relation " of the scaling and wavelet functions is used.

1. First a function Φ with Φ_{0n} , an orthonormal basis for V_0 is determined.

Because

$\Phi \epsilon\ V_1 \subset V_{-1}$ there exists h_n such that

$$\Phi(x) = \sum_n h_n \Phi(2x-n). \tag{2.60}$$

2. By definition

$$\Psi(x) = \sum_n (-1)^n C_{n-1} \Phi(2x+n)$$
$$= \sum_n g_n \Phi(2x+n). \tag{2.61}$$

The above shows the construction of wavelets from multiresolution analysis consisting of $\{V_j\}, j \epsilon Z$ and a special function $\Phi \epsilon V_0$.

The converse can also be proved for a space of band-limited functions and an appropriately defined scaling function Φ. Consider the following example where V_{-1} is a space of band limited functions with frequencies defined in the interval $[-\pi,\pi]$ and the scaling function Φ as $sinc(t)$. Then Φ and its integer translates $\Phi(t-k)$ form orthonormal basis for V_{-1}. Similarly if V_0 is the space of band-limited functions with frequencies in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$, then $(1/\sqrt{2})\Phi(t/2)$ and its integer translates constitute the orthonormal basis for V_0. V_0 is a subspace of V_{-1}. And if W_0 is the space of functions band limited to $[-\pi,-\frac{\pi}{2}] \cup [\frac{\pi}{2},\pi]$, then it is the orthogonal complement to V_0 in V_{-1}. In general if V_j is the space of band-limited functions with frequencies in the interval $(-2^{-j+1}\pi, 2^{-j+1}\pi)$ and by scaling the following relation is obtained,

$$V_j \subset V_{j-1}, j \epsilon Z$$
$$V_{j-1} = V_j \bigoplus W_j \quad j \epsilon Z, \tag{2.62}$$

where W_j , $j \epsilon Z$ is the space of band-limited functions with

frequencies in the interval $(-2^{-j+1}\pi,2^{-j}\pi) \cup (2^{-j}\pi,2^{-j+1}\pi)$.

And using the monotone sequential continuity theorem [67] the other properties of the multiresolution analysis are satisfied.

2.5.2 FILTER BANK & MRA

If $h_0(n)$ and $h_1(n)$ are ideal lowpass and highpass filters respectively, the multiresolution analysis will therefore have interpretation in the discrete time case through the following filter banks. The lowpass filtered version of the signal representing the coarse resolution, while the highpass-filtered version represents the fine resolution.

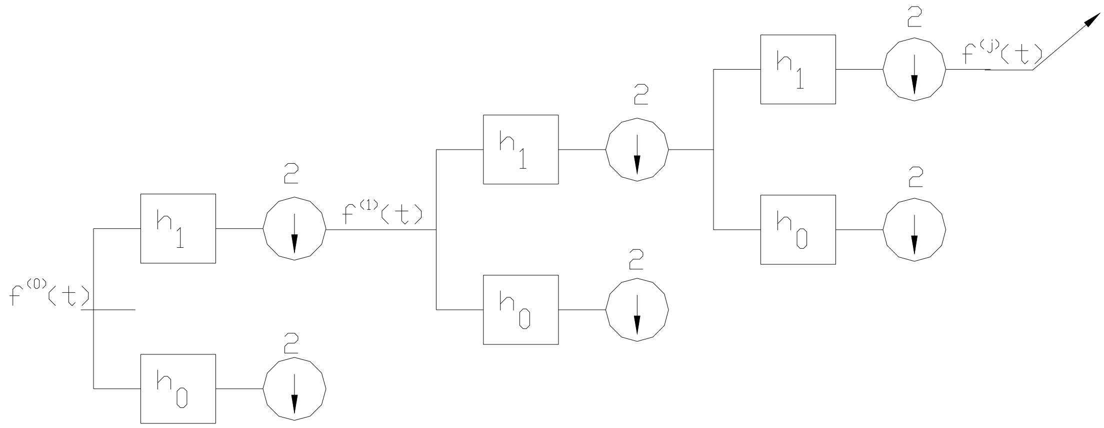

Figure 2.7: Iterated Filter Bank

2.5.3 FILTER BANKS & WAVELETS:

Given wavelets and scaling functions satisfying the "two-scale relation" and the orthonormality of their integer translates, perfect reconstruction filter banks are constructed. From the orthonormality of the integer translates

$$< \phi(t), \phi(t\text{-}k) > = \delta_k. \qquad 2.63$$

$$\text{where } \delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Substituting for the scaling function using the "two-scale relation"

$$< h_0(n), h_0(n\text{-}2k) > = \delta_k. \qquad 2.64$$

Defining H_0 as a matrix with rows $h_0(n)$, $h_0(n\text{-}2)$..., the above relation can be written as

$$H_0 H_0^* = I \qquad 2.65$$

$$H_1 H_1^* = I.$$

As shown above the compactly supported wavelets basis lead to the construction of PRFB.

However, the converse is not always true. In figure 2.7 above the upper branch is an infinite cascade of filters followed by subsampling by 2. The equivalent filter for a cascade of *j* blocks is given as

$$H^{(j)}(Z) = \prod_{j=0}^{j-1} H(Z^{2j}) \qquad 2.66$$

followed by sampling by 2^j. As j increases the length of $H^{(j)}$ increases to infinity. Instead we will consider the piecewise continuous function, $f^{(j)}(x)$, from the discrete iterated filter i.e.

$$f^{(j)} = 2^{\frac{1}{2}} h^{(j)}(n) \qquad \frac{n}{2^j} \leq t < \frac{n+1}{2^j}. \qquad 2.67$$

The fundamental question is to find out whether and to what function $f^{(j)}(t)$ converges as $j \rightarrow \infty$ given $f^{(0)}(t)$ as an indicator function over the unit interval [0,1). The limiting behavior depends on the nature of the filters. In order to construct wavelets of compact support the filters must exhibit certain regularity condition. Under this regularity condition the iterated function converges to a continuous function $\Phi(t)$. The wavelet function is then obtained using the two-scale relation.

2.6 SUMMARY

In this chapter the basic concepts of wavelet analysis, multiresolution analysis and multirate filter banks has been outlined. The historical development of wavelet analysis from Fourier analysis as a result of the limitations of the latter has been discussed. Several methods of analysis of perfect reconstruction filter banks has been discussed using the two channel case as an example. Finally the essential connections between the three concepts and their convergence to a single theory is shown. Basically, the fundamental concepts essential to the understanding of the subsequent chapters are laid down. The next chapter will deal the extension of these concepts to multiple dimensions.

CHAPTER 3

MULTIDIMENSIONAL EXTENSION

3.1 INTRODUCTION

Previous sections have been restricted to one-dimensional concepts. In the following sections, I will extend these concepts to more than one dimension. The ultimate objective of these extensions is to build multidimensional wavelets which are the topic of the subsequent chapters. Most of the materials in this section are based on the results of studies by Allenbach, Kovacevic, Vetterli , Lin, et al. (see [87], [38], [39], [84], [85]). In the first section the background material such as multidimensional sampling and multirate filter banks is outlined and latter the general multidimensional concepts are discussed. At each stage of the development of the concepts, typical two- and three-dimensional examples are constructed for illustration. All notations and definitions applicable to this section such as multidimensional z-transform and sampling are included in Appendix C.

3.2 PRELIMINARIES

Unlike the one-dimensional case, the multidimensional multirate signal processing is completely based on a new set of concepts such as lattice, coset vectors and so on [87]. In this section a brief introduction of these fundamental concepts is made.

3.2.1 Multidimensional Sampling

In single dimensional uniform sampling of signals, the sampling period T or its reciprocal the sampling density (sampling rate $1/T$) completely define the process. Unlike the one-dimensional case, multidimensional sampling is represented by an integer lattice Λ defined as the set of all linear combinations of $\boldsymbol{n}$ basis vectors $\boldsymbol{n} = [n_1, n_2, ...n_n]^t$ with integer coefficients. The sampling sublattice Λ_D is generated by the sampling matrix D and is the set of integer vectors $\boldsymbol{m} = D\boldsymbol{n}$ for some integer vector $\boldsymbol{n}$. The proper definition of a sublattice requires a nonsingular sampling matrix with integer-valued entries. Isotropic transformations in addition require the sampling matrices to have equal singular value decomposition [47],[86]. A given sublattice can be generated by a number of sampling matrices, each of which is related by a linear transformation represented by unimodular integer matrix.

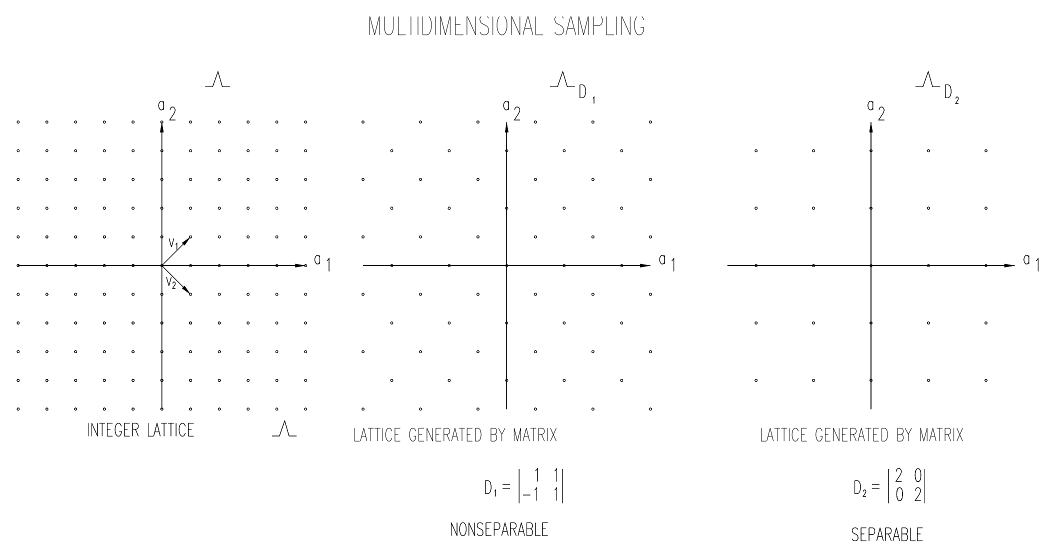

Figure 3.1: Multidimensional Sampling Lattice

The sampling lattice can be separable or nonseparable depending on whether the sampling matrix is diagonal or not. The simplest form of sampling is the orthogonal sampling generated by diagonal matrix. It appears when the sampling is done along each dimension separately.

The unit cell is the set of points such that the disjoint union of its copies shifted to all of the lattice points gives the input lattice. The number of input lattice samples contained in the unit cell represents the reciprocal of the sampling density and is given by $N=det(D)$. The fundamental parallelepiped formed by $\boldsymbol{n}$ basis vectors is an important unit cell. The Voronoi cell is another unit cell whose points are closer to the origin than any other lattice points as shown in Figure 3.2 below.

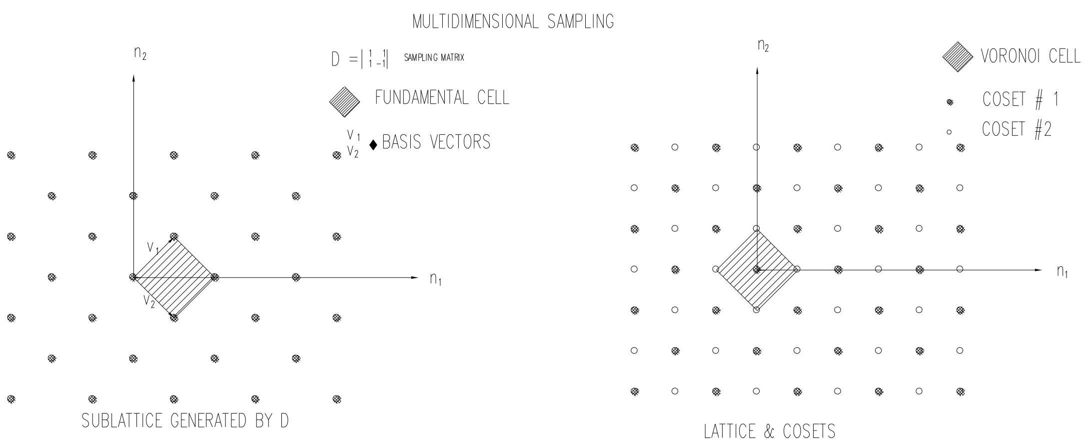

Figure 3.2: Multidimensional Sampling Lattice and Cosets

Each point in a unit cell constitutes a vector and these vectors are called coset vectors associated with D. There are N such vectors $\{\boldsymbol{k}_0,\boldsymbol{k}_1,...\boldsymbol{k}_{N-1}\}$ with $\boldsymbol{k}_0$ defined as a zero vector. A coset of a sublattice is the set of points obtained by shifting the entire sublattice by an integer shift vectors $\boldsymbol{k}_l$. There are exactly N distinct cosets and their union gives the input lattice Λ.

Another important concept is the unit cell in the frequency domain. If a signal to be sampled is band limited to the unit cell, no overlapping of spectra will occur and the signal can be reconstructed from its samples. The reciprocal or modulation lattice is actually the Fourier transform of the original lattice and its points represent the points of replicated spectra in the frequency domain.

In general, the relation between the sampling matrix and the modulation or reciprocal matrix is defined by

$$\widehat{\mathrm{D}} = 2\pi(D^{-t}). \qquad 3.1$$

The columns of $\widehat{D}$ and D represent the reciprocal and sampling vectors respectively.

3.2.1.1 DOWN SAMPLING AND DECIMATION:

A downsampler shown below samples an input signal $x(\boldsymbol{n})$ by mapping points on the sublattice Λ_D to Λ and discarding the rest.(See detail in [15], [87])

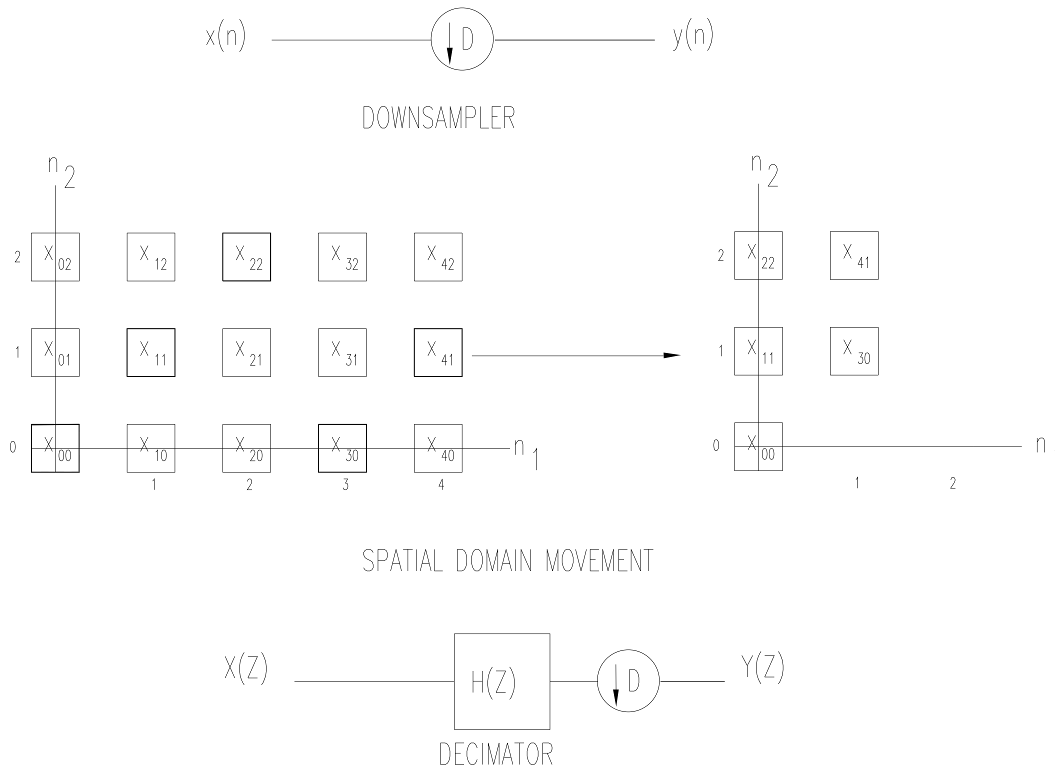

Figure 3.3: Multidimensional Sampling Downsampler

The time, Fourier, and *z*-domain expressions for the output of a downsampler are given by

$$y(\boldsymbol{n}) = x(D\boldsymbol{n}) \qquad 3.2$$

$$\widehat{Y}(\boldsymbol{\omega}) = \frac{1}{N}\sum_{\boldsymbol{k}\,\epsilon\,U_c^t} X(D^{-t}\boldsymbol{\omega}-2\pi D^{-t}\boldsymbol{k}) \qquad 3.3$$

$$Y(\boldsymbol{z}) = \frac{1}{N}\sum_{\boldsymbol{k}\,\epsilon\,U_c^t} X(W_{D^{-1}}(2\pi\boldsymbol{k})\circ \boldsymbol{z}^{D^{-1}}) \qquad 3.4$$

where $\boldsymbol{\omega}$ is n-dimensional real vector, $\boldsymbol{z}$ is n-dimensional complex vector and $\boldsymbol{n}, \boldsymbol{k}$ are n-dimensional integer vectors.

From the Fourier expression, we see that the output at each frequency $\boldsymbol{\omega}$ is formed by summing the input at a set of N aliasing frequencies

$$D^{-t}\boldsymbol{\omega} - 2\pi D^{-t}\boldsymbol{k}, \quad \boldsymbol{k} \in U_c^{t}. \qquad 3.5$$

All but one of these aliasing components have to be zero for alias cancellation. Assume ρ to be the set of frequencies in the admissible passband, then ρ_p, $p=1,\ldots,N-1$ are sets obtained by shifting ρ by the aliasing offsets $2\pi D^{-t}\boldsymbol{k}_p$,

$$\rho_p = \{\boldsymbol{\omega} : \boldsymbol{\omega} + 2\pi D^{-t}\boldsymbol{k}_p \in \rho\}. \qquad 3.6$$

The frequency domain effect of downsampling can be interpreted as a mapping of the baseband frequency region (ρ) and the $N-1$ other identically shaped frequency regions $(\rho_p, p=1,\ldots,N-1)$ to the unit frequency cell. The process of prefiltering with a filter $H(\boldsymbol{\omega})$ is to remove all but one of the above frequency regions. The downsampler with the filter forms the decimator and the shape of the decimator filter has to be selected. Generally there are many admissible passbands for a given downsampling matrix D. The fundamental property of an admissible passband is defined by the relation

$$\rho \cap \rho_p = \Phi \quad p=1,\ldots,N-1. \qquad 3.7$$

The intersection of ρ and any of ρ_p's must be empty for an admissible passband. If two frequencies that are separated by an aliasing offset are both in ρ the passband is inadmissible. Hence, a passband ρ is inadmissible if there exists $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ in ρ and some integer vector $\boldsymbol{m}$ such that

$$\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 2\pi D^{-t}\boldsymbol{k}_p + 2\pi\boldsymbol{m}. \qquad 3.8$$

Regardless of its shape, an admissible passband can have a hypervolume no larger than *1/N* of the size of unit frequency cell. However, with maximum hypervolume passbands, there exist frequencies at the boundary surfaces of the passbands that alias onto their own negatives. These frequencies are called self-aliasing frequencies and are defined as $(\boldsymbol{\omega}_1=\boldsymbol{\omega}_2=2\boldsymbol{\omega}_{sa})$

$$\boldsymbol{\omega}_{sa} = \pi D^{-t}\boldsymbol{k}_l + \pi\boldsymbol{m}. \qquad 3.9$$

There are more than one self-aliasing frequencies for each $\boldsymbol{k}_p$ corresponding to different values of $\boldsymbol{m}$. These self-aliasing frequencies are useful to characterize the admissible passbands. Actually the self-aliasing frequencies are essential in defining the boundaries of the passband with maximum possible hypervolume. If a filter with the given passband has real coefficients then ρ must be symmetric about $\boldsymbol{\omega} = \boldsymbol{0}$ and the existence of these self-aliasing frequencies is guaranteed. These concepts are illustrated using two examples in subsequent sections.

3.2.1.2 UPSAMPLING AND INTERPOLATION:

The process of upsampling maps a signal on the input lattice Λ to another signal that is nonzero only at points on the sampling sublattice Λ_D as shown below.

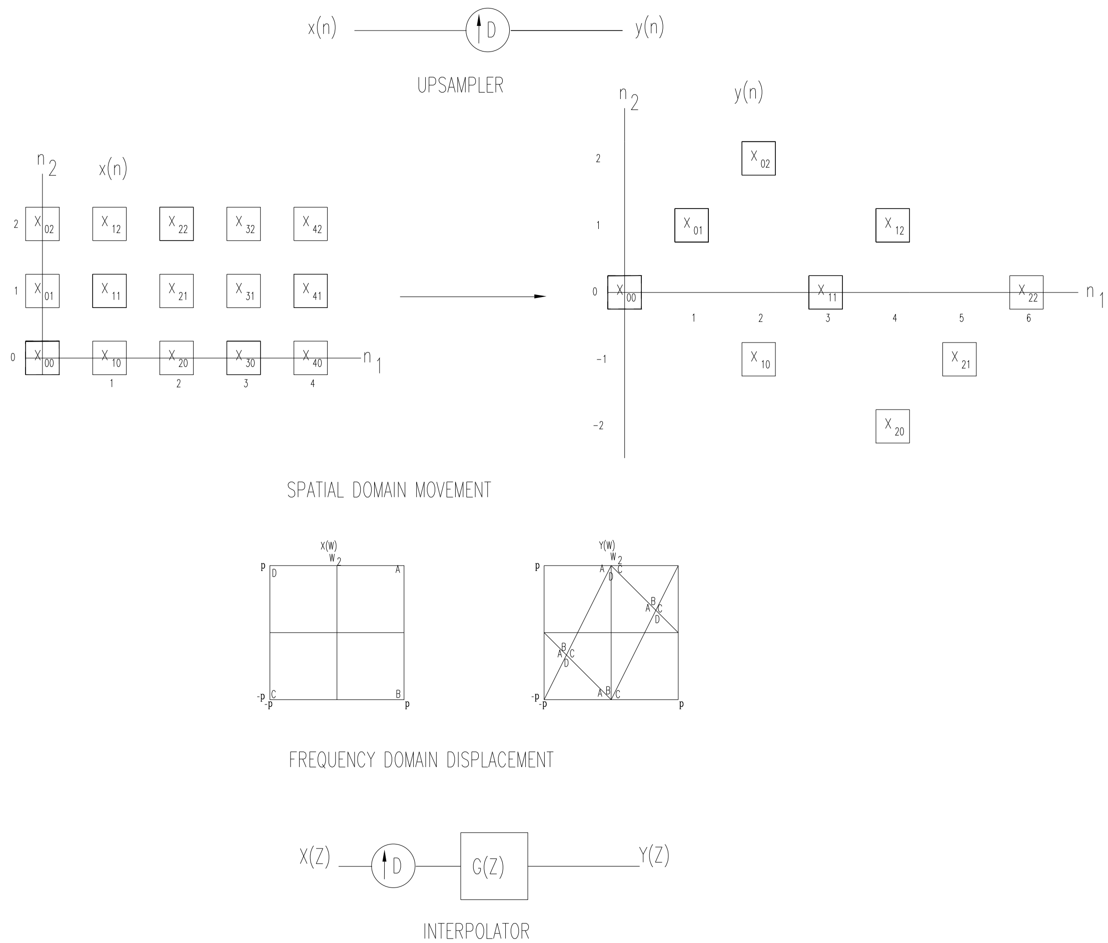

Figure 3.4: Multidimensional Sampling Upsampler

The time, frequency and *z-transform* expressions of the output of the upsampler are given by

$$y(\boldsymbol{n}) = \begin{cases} x(D^{-1}\boldsymbol{n}) & \text{if } D^{-1}\boldsymbol{n} \in \Lambda \\ 0 & \text{if } D^{-1}\boldsymbol{n} \notin \Lambda, \end{cases} \qquad 3.10$$

$$Y(\boldsymbol{\omega}) = X(D^{t}\boldsymbol{\omega}), \qquad 3.11$$

$$Y(\mathbf{z}) = X(\mathbf{z}^{D}). \tag{3.12}$$

The output of the upsampler in the frequency domain results in the reduction of the passband and the skewing of their orientation.

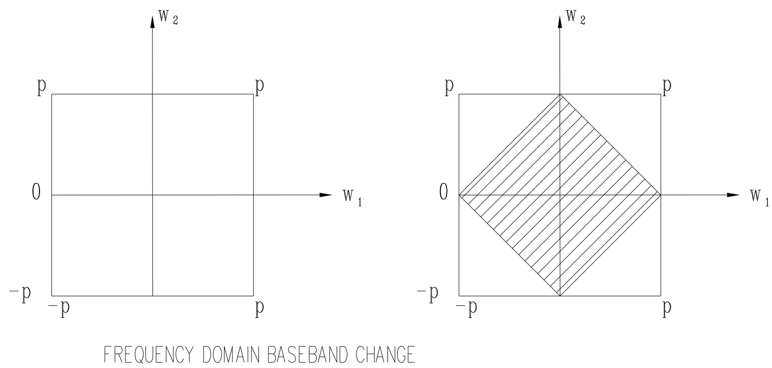

Figure 3.5: Frequency Domain Baseband Change

As a result images of the unit cell are mapped to regions surrounding the baseband resulting in N complete images of one period of $X(\omega)$ in $Y(\omega)$. It is therefore essential for an upsampler to be followed by a filter $G(\omega)$ that stops all but one of the images. An interpolator is the combination of the upsampler followed by such a filter.

3.2.2 POLYPHASE DECOMPOSITION

The polyphase decomposition of a signal is an important tool in multirate signal processing. If $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{N-1}$ are the set of coset vectors associated with the sampling matrix D, then the pth polyphase component of $x_p(\boldsymbol{n})$ of the signal $x(\boldsymbol{n})$ is formed by shifting $x(\boldsymbol{n})$ by $-\boldsymbol{k}_p$ and downsampling the result:

$$x_p(\boldsymbol{n}) = x(D\boldsymbol{n} + \boldsymbol{k}_p). \qquad 3.13$$

A signal can be recovered from its polyphase components by

$$X(\mathbf{z}) = \sum_{\boldsymbol{k} \in \Lambda} x(D\boldsymbol{n}+\boldsymbol{k})\mathbf{z}^{-\boldsymbol{k}}. \qquad 3.14$$

3.2.3 EXAMPLES

The following two typical examples used by M. Veterlli and J. Kovacevic are used to illustrate the above concepts [41]. These 2-*D* and 3-*D* examples are also used in subsequent sections.

Example 1: Two-dimensional sampling

Two-dimensional quincunx sampling is generally represented by

$$\Lambda_Q = \{(n_1 + n_2)^t \mid n_1 + n_2 = 2k, n_i , k \in \mathcal{Z}\}. \qquad 3.15$$

Let us consider the following specific quincunx sampling matrix given by

$$D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad 3.16$$

The determinant of D^t being equal to 2 there are 2 distinct coset vectors $\boldsymbol{k}_p$ defined below

$$\boldsymbol{k}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 3.17$$

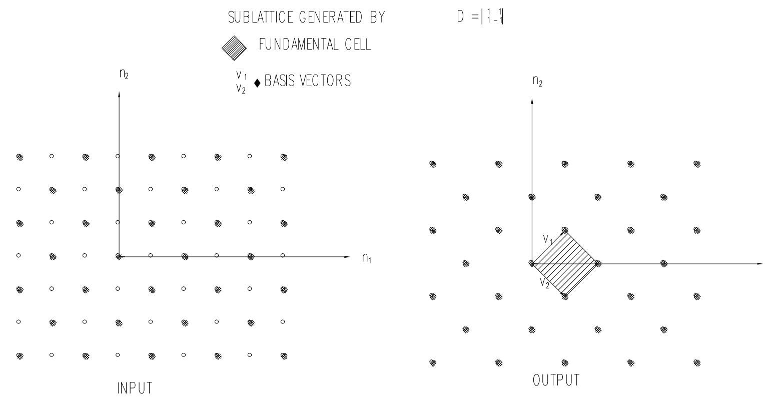

Figure 3.6: Bases Vectors and Fundamental Cell

The aliasing offsets are defined as $2\pi D^{-t}\boldsymbol{k}_p$ and the self-aliasing frequencies can be determined for each k_p.

For $\boldsymbol{k}_p$:

$$\boldsymbol{\omega}_{sa} = \tfrac{\pi}{2}(D^{-t})\boldsymbol{k}_p + \pi\boldsymbol{m} \qquad 3.18$$

$$= \tfrac{\pi}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\boldsymbol{k}_p + \pi\boldsymbol{m},$$

$$\boldsymbol{\omega}_{sa,1} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{m} = \boldsymbol{0} \quad \boldsymbol{\omega}_{sa,2} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \boldsymbol{m} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad 3.19$$

$$\boldsymbol{\omega}_{sa,3} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ -1 \end{bmatrix}, \boldsymbol{m} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_{sa,4} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{m} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad 3.20$$

A complete set of the four aliasing frequencies for D are shown in the figure below.

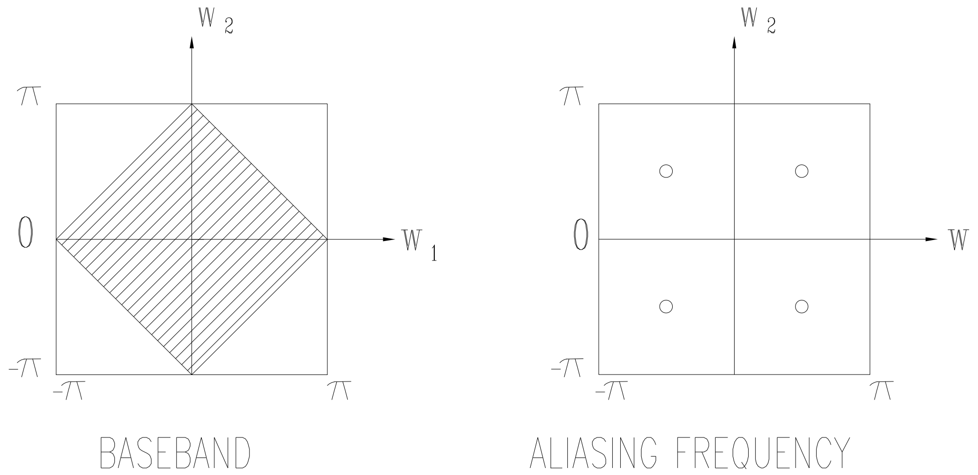

Figure 3.7: Example of Aliasing Frequencies for D

A number of admissible passbands for decimation filters with real impulse responses can be drawn using these frequencies. These self-aliasing frequencies define the boundaries of the passbands with maximum possible hypervolume of $\frac{1}{N}(2\pi)^D$. It is possible to use horizontal, vertical, bandpass filters for the sampling matrix as shown in Figure 3.8 below.

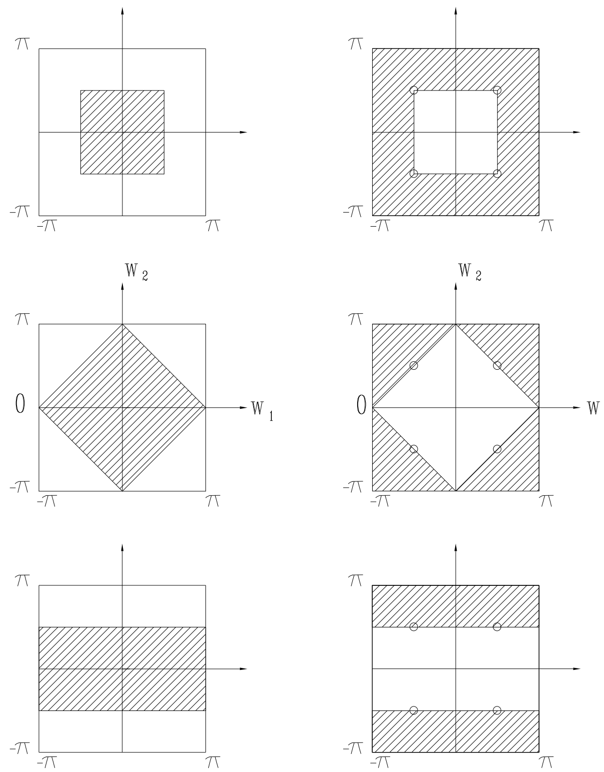

Figure 3.8: Admissible Passbands Generated by D

Example 2: Three dimensional sampling

The face centered orthogonal (FCO) sampling lattice is described by [31]

$$\Lambda_Q = \{(n_1 + n_2 + n_3)^t \mid n_1 + n_2 + n_3 = 2k, n_i , k \in \mathcal{Z}\}. \qquad 3.21$$

Let us consider the following specific sampling matrix

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \qquad 3.22$$

Because the determinant of D^t is equal to 2, there are 2 distinct coset vectors $\boldsymbol{k}_p$ defined below

$$\boldsymbol{k}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{k}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \qquad 3.23$$

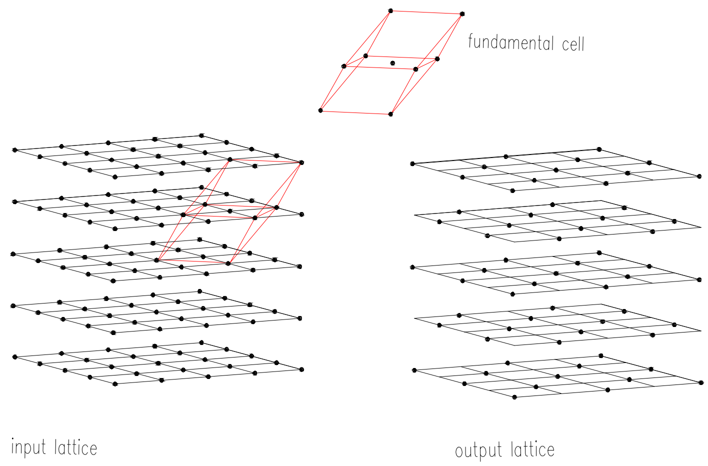

Figure 3.9: Coset Vectors Generated by FCO Sampling

The aliasing offsets are defined as $2\pi D^{-t}\boldsymbol{k}_p$ and the self-aliasing frequencies can be determined for each $\boldsymbol{k}_p$, $p \neq 0$.

For $\boldsymbol{k}_p$:

$$\boldsymbol{\omega}_{sa} = \tfrac{\pi}{2}(D^{-t})\boldsymbol{k}_p + \pi\boldsymbol{m} \qquad 3.24$$

$$= \tfrac{\pi}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\boldsymbol{k}_p + \pi\boldsymbol{m},$$

$$\boldsymbol{\omega}_{sa,1} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{m}=\boldsymbol{0} \quad \boldsymbol{\omega}_{sa,2} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix},$$

$$\boldsymbol{\omega}_{sa,3} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{\omega}_{sa,4} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\omega}_{sa,5} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_{sa,6} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix},$$

$$\boldsymbol{\omega}_{sa,7} = \tfrac{\pi}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{\omega}_{sa,8} = \tfrac{\pi}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{m}=\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

The complete set of the eight aliasing frequencies for D that lie on the boundary of any maximum-volume passband are shown below.

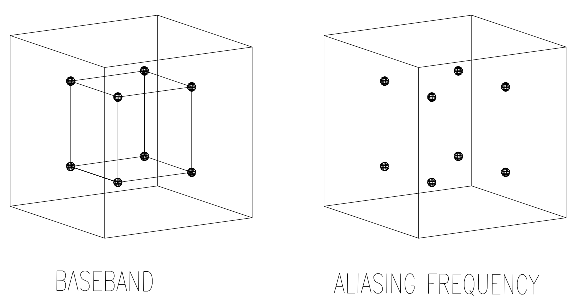

Figure 3.10: Baseband and Aliasing Frequency FCO Sampling

Again a number of admissible passbands for filters with real coefficients can be drawn using these frequencies as in the two-dimensional case. These self-aliasing frequencies define the boundaries of the passbands. It is possible to use cubic, spherical, or cylindrical filters for the sampling matrix as shown in Figure 3.11. The figures are projections since they represent 3-*D* volumes.

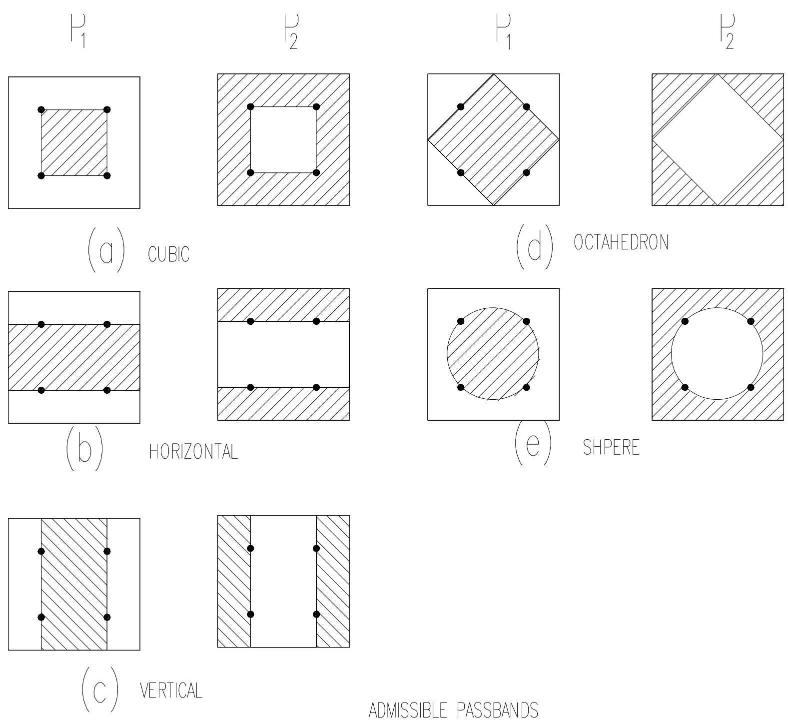

Figure 3.11: Admissible Passband for FCO Sampling

For a given sampling matrix several admissible passbands can be determined as shown above. These passbands represent the admissible subbands in the multidimensional filter banks and constitute an important factor in the design of multidimensional perfect reconstruction filter banks (PRFBs). The size and shape of these subbands that are determined by the sampling matrix D determine the number of channels and design of filters in PRFBs. Hence, a choice of any particular passband has to be carefully selected based on alias cancellation and other filter design considerations, including conditions for perfect reconstruction. In what follows the conditions for perfect reconstruction are discussed.

3.3 MULTIDIMENSIONAL FILTER BANKS

One-dimensional filter banks have been extensively studied and several design methods have been successfully developed. Recently there has been a lot of interest in filter banks in multiple dimensions. However, the extension of filter banks into multiple dimensions is still an area of extensive research. Here I will briefly review some of the important concepts and results of recent research in multidimensional filter banks. (See details in [87], [78], [91], [13], [41]). The multirate filter bank can be considered as a hierarchy with sampling and filtering constituting the basic operations at the lowest level.

A multidimensional N-channel filter bank is shown below. The discussion is restricted to uniform band, maximally sampled filter banks in which there are $N=det(D)$ channels with the same sampling matrix D.

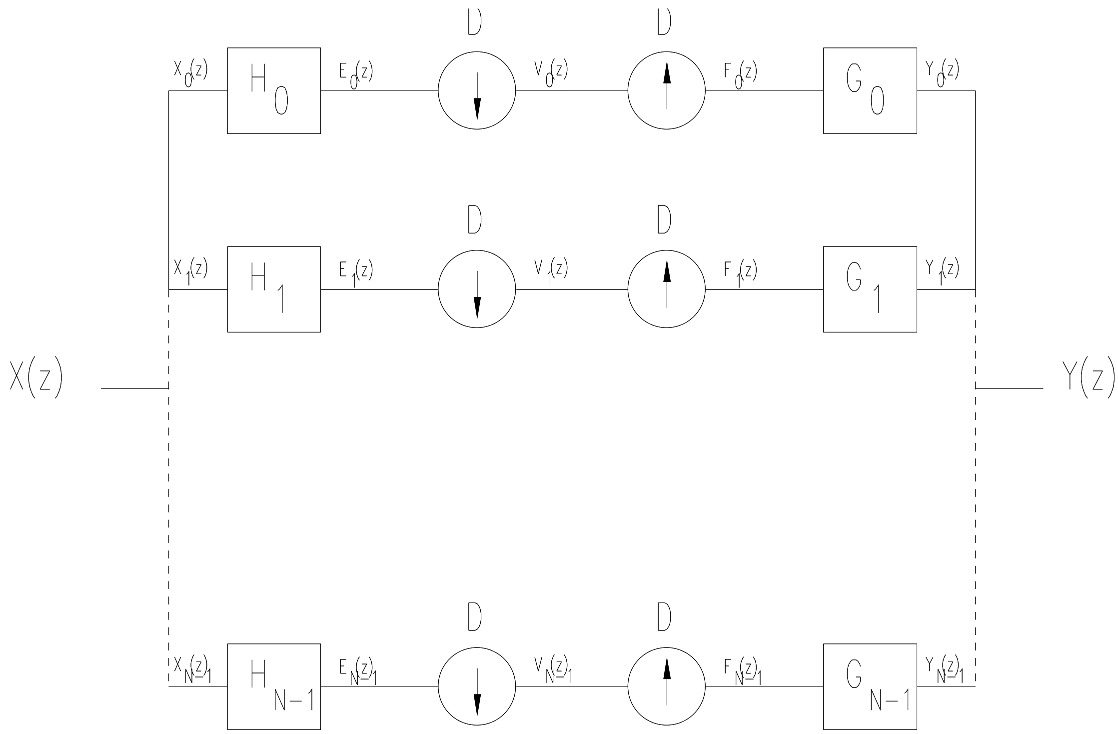

Figure 3.12: Multidimensional PRFB

3.3.1 MODULATED ANALYSIS:

The modulated analysis is used for the analysis and design of PRFB's. The modulated versions of the input signal is given by

$$\hat{\mathbf{x}}_m(\boldsymbol{\omega}) = \{\hat{X}(\boldsymbol{\omega} - 2\pi(D^{-t}).\boldsymbol{k})\}_{\boldsymbol{k} \in U_c^t}, \qquad 3.25$$

$$\boldsymbol{x}_m(\mathbf{z}) = \{X(W_{D^{-1}}(2\pi.\boldsymbol{k}).\ \mathbf{z})\}_{\boldsymbol{k} \in U_c^t}. \qquad 3.26$$

The modulated versions of the filters is given by

$$\widehat{H}_m(\boldsymbol{\omega}) = \{\widehat{H}_i(\boldsymbol{\omega}- 2\pi(D^{-t}).\boldsymbol{k})\}, \boldsymbol{k} \in U_c^t , i \in \{0, 1, ...N-1\}, \qquad 3.27$$

$$H_m(\mathbf{z}) = \{H_i(W_{D^{-1}}(2\pi.\boldsymbol{k}). \mathbf{z})\}, \boldsymbol{k} \in U_c^t , i \in \{0, 1, ...N-1\}. \qquad 3.28$$

The output of the system after upsampling and filtering in the synthesis bank is given by

$$\widehat{Y}(\boldsymbol{\omega}) = \frac{1}{N} (\widehat{G}_0(\boldsymbol{\omega}).\widehat{G}_1(\boldsymbol{\omega})...\widehat{G}_{N-1}(\boldsymbol{\omega})).\widehat{H}_m(\boldsymbol{\omega}).\widehat{x}_m(\boldsymbol{\omega}), \qquad 3.29$$

$$Y(\mathbf{z}) = \frac{1}{N} (G_0(\mathbf{z}).G_1(\mathbf{z})...G_{N-1}(\mathbf{z})).H_m(\mathbf{z}).x_m(\mathbf{z}), \qquad 3.30$$

$$Y(\mathbf{z}) = \frac{1}{N} (G_0(\mathbf{z}).G_1(\mathbf{z})...G_{N-1}(\mathbf{z})).H_{AC}(\mathbf{z})X(\mathbf{z}), \qquad 3.31$$

where the (m,p)th element of $H_{AC}(\mathbf{z})$ is $H_m(W_{D^{-1}}(2\pi\boldsymbol{k}_p)\mathbf{z})$.

The filter bank output is identical to the input if and only if

$$\begin{bmatrix} G_0(\mathbf{z}) \\ \vdots \\ G_{N-1}(\mathbf{z}) \end{bmatrix} = NH_{AC}^{-1}(\mathbf{z}) \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \qquad 3.32$$

To achieve perfect reconstruction it is necessary and sufficient that the analysis component (AC) matrix be invertible and the vector of synthesis filters be chosen as the first column of $H_{AC}^{-1}(\mathbf{z})$. However, the design of such a filter bank is not easy as it is often difficult to invert an arbitrary AC matrix. Besides, it may result in higher order analysis filters and there is no simple way to ensure the stability of the synthesis filters. The analysis and design of PRFB's is easily done using the polyphase form.

3.3.2 POLYPHASE ANALYSIS:

The polyphase approach was introduced for the analysis and design of one-dimensional PRFB's. A convenient way of alias cancellation in multidimensional system is to decompose both signals and filters into polyphase components each corresponding to one of the cosets of the output lattice. The polyphase decomposition of the input signal is given by

$$X(\mathbf{z}) = \sum_{\boldsymbol{k} \in U_c^t} \mathbf{z}^{-\boldsymbol{k}} X_{\boldsymbol{k}}(\mathbf{z}^D) = \boldsymbol{p}_i^t x_p(\mathbf{z}^D) \qquad 3.33$$

and

$$X_{\boldsymbol{k}}(\mathbf{z}) = \sum_{\boldsymbol{n} \in \mathcal{Z}^{\boldsymbol{n}}} x(D\boldsymbol{n} - \boldsymbol{k}).\boldsymbol{z}^{-\boldsymbol{n}} \qquad 3.34$$

where $\boldsymbol{p}_i(\mathbf{z}) = \{\mathbf{z}^{\boldsymbol{k}}\}_{\boldsymbol{k} \in U_c^t}$ is the vector of the inverse polyphase transform

and $\boldsymbol{x}_p(\mathbf{z})$ is the vector containing the polyphase components of the input signal

$$\boldsymbol{x}_p(\mathbf{z}) = \{X_{\boldsymbol{k}}(\mathbf{z})\}_{\boldsymbol{k} \in U_c^t}. \qquad 3.35$$

Similarly, the polyphase components of the filter $H(z)$ could be defined as

$$H(\mathbf{z}) = \sum_{\boldsymbol{k} \in U_c^t} \mathbf{z}^{-\boldsymbol{k}} H_{\boldsymbol{k}}(\mathbf{z}^D) = \boldsymbol{p}_i^t h_p(\mathbf{z}^D), \qquad 3.36$$

and

$$H_{\boldsymbol{k}}(\mathbf{z}) = \sum_{\boldsymbol{n} \in \mathcal{Z}^{\boldsymbol{n}}} h(D\boldsymbol{n} - \boldsymbol{k}).\boldsymbol{z}^{-\boldsymbol{n}}, \qquad 3.37$$

where $\boldsymbol{p}_f(\mathbf{z}) = \{\mathbf{z}^{-\boldsymbol{k}}\}_{\boldsymbol{k} \in U_c^t}$ is the vector of the forward polyphase transform and $h_p(\mathbf{z}) = \{H_{\boldsymbol{k}}(\mathbf{z})\}_{\boldsymbol{k} \in U_c^t}$ is the vector containing the polyphase components of the filter.

Thus a single-input linear shift-variant system can be expressed as a multi-input shift-invariant system. To obtain the polyphase expansion first we have to select $\{\boldsymbol{k}_0, \boldsymbol{k}_1 ... \boldsymbol{k}_{N-1}\}$, the coset vectors associated with the sampling matrix D. Then we have to determine the polyphase components of the input signal and each of the analysis filter in

Figure 3.12 above. As in the one-dimensional case signals at the output of the analysis bank can be represented in terms of the input signal, forward polyphase transform $\boldsymbol{p}_f(\mathbf{z})$ and the analysis polyphase matrix $h_p(\mathbf{z})$ as

$$E(\mathbf{z}) = [h_p(\mathbf{z}^D)p_f(\mathbf{z})]X(\mathbf{z}) \qquad 3.38$$

$$= h(\mathbf{z})X(\mathbf{z}),$$

where $h^t(\mathbf{z}) = [H_0(\mathbf{z}),....H_{N-1}(\mathbf{z})]$,

$$\boldsymbol{p}_f^t = [\mathbf{z}^{-\boldsymbol{k}_0}, \mathbf{z}^{-\boldsymbol{k}_1}, ...\mathbf{z}^{-\boldsymbol{k}_{N-1}}],$$

and

$$h_p(\mathbf{z}) = \begin{bmatrix} H_{0,0}(\mathbf{z}) & ... & ... & H_{0,N-1}(\mathbf{z}) \\ H_{1,0}(\mathbf{z}) & ... & ... & H_{1,N-1}(\mathbf{z}) \\ . & ... & ... & . \\ H_{N-1,0}(\mathbf{z}) & ... & ... & H_{N-1,N-1}(\mathbf{z}) \end{bmatrix}. \qquad 3.39$$

The bank of the analysis filters $h(\mathbf{z})$ in the Figure 3.12 can be replaced by $h_p(\mathbf{z}^D)\boldsymbol{p}_f^t$. The decimator can be moved to the left of the polyphase matrix and the argument of $h_p(.)$ is changed from $\mathbf{z}^D$ to $\mathbf{z}$. This gives the front end of the polyphase structure shown on Figure 3.13.

Similarly, the output signal at the synthesis bank can be represented in terms of the input channel signals, the synthesis polyphase matrix $G_p(\mathbf{z})$ and the inverse polyphase transfrom $p_i(\mathbf{z})$.

$$Y(z) = [\boldsymbol{p}_{ic}^t G_p(\mathbf{z}^D)]F(\mathbf{z}) \qquad 3.40$$

$$= g(\mathbf{z})F(\mathbf{z}),$$

where $g^t(\mathbf{z}) = [G_0(\mathbf{z})...G_{N-1}(\mathbf{z})]$

$$\boldsymbol{p}_{ic}^t = [\mathbf{z}^{\boldsymbol{k}_0}, \mathbf{z}^{\boldsymbol{k}_1}, ...\mathbf{z}^{\boldsymbol{k}_{N-1}}],$$

$$G_p(\mathbf{z})=\begin{bmatrix} G_{0,0}(\mathbf{z}) & \dots & \dots & G_{0,N-1}(\mathbf{z}) \\ G_{1,0}(\mathbf{z}) & \dots & \dots & G_{1,N-1}(\mathbf{z}) \\ . & \dots & \dots & . \\ G_{N-1,0}(\mathbf{z}) & \dots & \dots & G_{N-1,N-1}(\mathbf{z}) \end{bmatrix}. \qquad 3.41$$

The bank of the synthesis filters $g(z)$ in Figure 3.12 are replaced by $G_p(\mathbf{z}).\boldsymbol{p}_{ic}^t$. Moving the up-sampler to the right changes the argument from $\boldsymbol{z}^D$ to $\boldsymbol{z}$. The final equivalent synthesis structure represents the back end of the polyphase structure shown below.

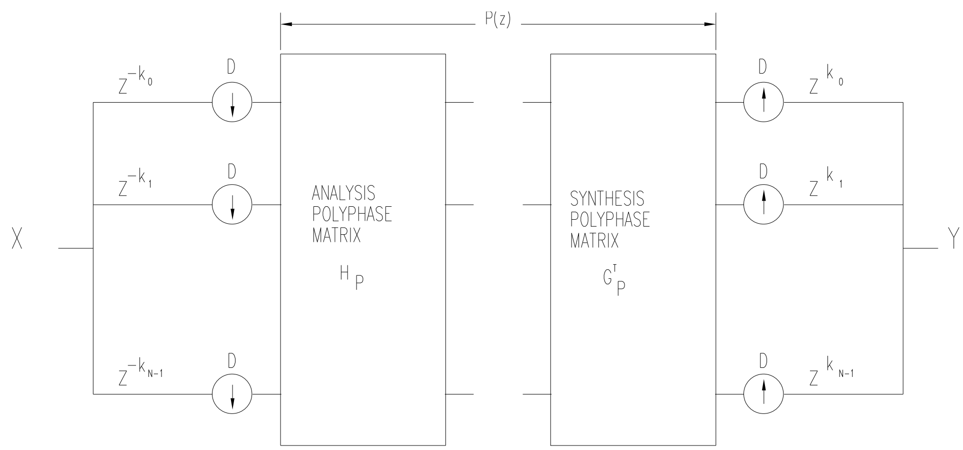

Figure 3.13: Multidimensional Analysis/Synthesis Filter Bank

Combining the two polyphase filters we get the transfer polyphase matrix.

$$T_p(\mathbf{z}) = G_p^t(\mathbf{z})h_p(\mathbf{z}). \qquad 3.42$$

The input output relation of the analysis/synthesis filters is given by

$$\begin{aligned} Y(\mathbf{z}) &= \boldsymbol{p}_{ic}^t(\mathbf{z}).G_p(\mathbf{z}).h_p(\mathbf{z}^D).x_p(\mathbf{z}^D) \qquad 3.43 \\ &= \boldsymbol{p}_{ic}^t(\mathbf{z}).T_p(\mathbf{z}^D).x_p(\mathbf{z}^D). \end{aligned}$$

The following set of conditions are set based on the above input/output relations (See [40]):

1. Aliasing is cancelled if and only if the inverse polyphase transform vector p_i is the left eigenvector of the transfer polyphase matrix in the upsampled domain $T_p(\mathbf{z}^D)$ that is

$$\boldsymbol{p}_i^t . T_p(\mathbf{z}^D) = T(\mathbf{z}).\boldsymbol{p}_i^t, \tag{3.44}$$

$T(z)$ the eigenvalue of $T_p(z^D)$ is a scalar polynomial and defines the overall transfer function of the filter bank. If we let $\boldsymbol{z}^{k_{N-1}} \epsilon\ U_c^t$ denote the vector that makes p_{ic}^tcausal, then

$$Y(\mathbf{z}) = \mathbf{z}^{k_{N-1}} \boldsymbol{p}_i^t . T_p(\mathbf{z}^D).x_p(\mathbf{z}^D), \tag{3.45}$$

and since $X(\mathbf{z}) = \boldsymbol{p}_i^t\ x_p(\mathbf{z}^D)$ we obtain

$$Y(\mathbf{z}) = \mathbf{z}^{k_{N-1}} T(\mathbf{z}).X(\mathbf{z}). \tag{3.46}$$

Thus the output is a scalar multiple of the input.

2. Perfect reconstruction is achieved if and only if the eigenvalue $T(\mathbf{z})$ associated with the eigenvector $\boldsymbol{p}_i^t$ in 1 is monomial i.e.

$$T(\mathbf{z}) = c.\boldsymbol{z}^{-k}. \tag{3.47}$$

And since $\boldsymbol{z}^{k_{N-1}} \boldsymbol{z}^{-k} = \boldsymbol{z}^{-\boldsymbol{n}}$,

$$Y(\mathbf{z}) = c\boldsymbol{z}^{-\boldsymbol{n}} X(\mathbf{z}). \tag{3.48}$$

The output is a shifted version of the input.

3. Perfect reconstruction with FIR filters is achieved if and only if the determinant of the analysis polyphase matrix is monomial i.e.

$$det(h_p(\mathbf{z})) = \boldsymbol{z}^{-\boldsymbol{k}}. \qquad 3.49$$

If we choose $G_p(\mathbf{z}) = adj(H_p(\mathbf{z}))$ then $G_p(\mathbf{z})$ is FIR if $H_p(\mathbf{z})$ is FIR and T_p becomes a diagonal matrix of delays,

$$Y(\mathbf{z}) = \boldsymbol{z}^{\boldsymbol{k}_{N-1}}\boldsymbol{p}_i^t(\mathbf{z}).I.x_p(\mathbf{z}^D) \qquad 3.50$$

$$= \boldsymbol{z}^{-\boldsymbol{n}}X(\mathbf{z}).$$

The two representations polyphase and modulated domain analysis are time-frequency representation and are related by Fourier Transform.

3.3.3 ADDITIONAL CONSTRAINTS

Once perfect reconstruction is obtained some additional requirements can be imposed on the filter banks. The most important ones are orthogonality and linear phase [4], [5].

3.3.3.1 ORTHOGONAL CASE:

As was mentioned in the one-dimensional case, a matrix is paraunitary if it satisfies

$$\widehat{H}(z)H(z) = H(z)\widehat{H}(z) = c \times I. \qquad 3.51$$

This matrix becomes orthogonal on the unit hypercircles $(z_i = e^{j\omega_i}, i=1..n)$. If the analysis polyphase matrix is orthogonal, then

$$H_p(z)\widehat{H}_p(z) = I. \quad 3.52$$

By choosing the following matrix, a perfect reconstruction system is obtained.

$$G_p(z) = z^{-k}\widehat{H}_p(z). \quad 3.53$$

Suppose the modulation matrix $H_m(z)$ defined above is orthogonal, then

$$\widehat{H}_i(\omega- 2\pi(D^{-t}).k)=\sum_{k \in U_c^t} H_i(W_{D^{-1}}(2\pi.k).z)\widehat{H}_i(W_{D^{-1}}(2\pi.k).z) \quad 3.54$$

$$=N\delta_{ij}.$$

If $(G_0(z)....G_{N-1}(z))$ as the first row of $H_m(z)$ perfect reconstruction is achieved. And if $H_i(z)\widehat{H}_j(z)$ is real it is the z-transform of the cross correlation sequence $r_{ij}(n) = <h_i(k), h_j(k+n)>$. This means that each filter is orthogonal to its translates with respect to the lattice. Thus the set $\{h_i(\mathbf{k}+D\mathbf{n}), i=0,...N-1, \mathbf{k},\mathbf{n} \in \mathcal{Z}^n\}$ is an orthonormal set and is the lattice extension of the orthogonality relations with respect to the shifts in the one-dimensional case.

3.3.3.2 LINEAR PHASE:

If a real filter is linear phase, it can be written as

$$H(z) = a.D(z).\widehat{H}(z), \quad 3.55$$

where a denotes symmetry and

$$D(z) = z^{-(P+Q)} = \prod_{i=1}^{n} z_i^{-(p_i+q_i)}. \tag{3.56}$$

And $P = (p_1, p_2, \ldots p_n)$, $Q = (q_1, q_2, \ldots. q_n)$ are the corners on the main hyperdiagonal of the parallelpiped around a polyphase represented in the space of its exponents.

The analysis filters can be written as

$$\begin{aligned} h(z) &= (H_0(z), \ldots\ldots H_{N-1}(z))^t \\ &= H_p(z^D) p_f(z). \end{aligned} \tag{3.57}$$

If all the filters are linear phase, then

$$H_p(z^D).p_f(z) = \begin{bmatrix} a_0.D_0(z).\widehat{H}_0(z) \\ . \\ . \\ a_{N-1}.D_{N-1}(z).\widehat{H}_{N-1}(z) \end{bmatrix} \tag{3.58}$$

$$= a.\Delta(z)\widehat{H}_p^t(z^D).p_f^t,$$

$$\text{where } a = \begin{bmatrix} a_0 & & & & \\ & a_1 & & & \\ & & . & & \\ & & & . & \\ & & & & a_{N-1} \end{bmatrix},$$

and

$$\Delta(z) = \begin{bmatrix} D_0(z) & & & & \\ & D_1(z) & & & \\ & & . & & \\ & & & . & \\ & & & & D_{N-1}(z) \end{bmatrix}.$$

The above can be either used to test linear phase of the filter banks or put a constraint in the design of linear phase filters.

3.3.4 EXAMPLES:

Here in this subsection the previous examples are used to demonstrate the application of the theory just developed. The above two examples, quincunx lattice in 2-*D* and the FCO face-centered orthorhombic in 3-*D*, have sampling matrices with determinant equal to two. The maximally decimated filter bank will have two channels shown below.

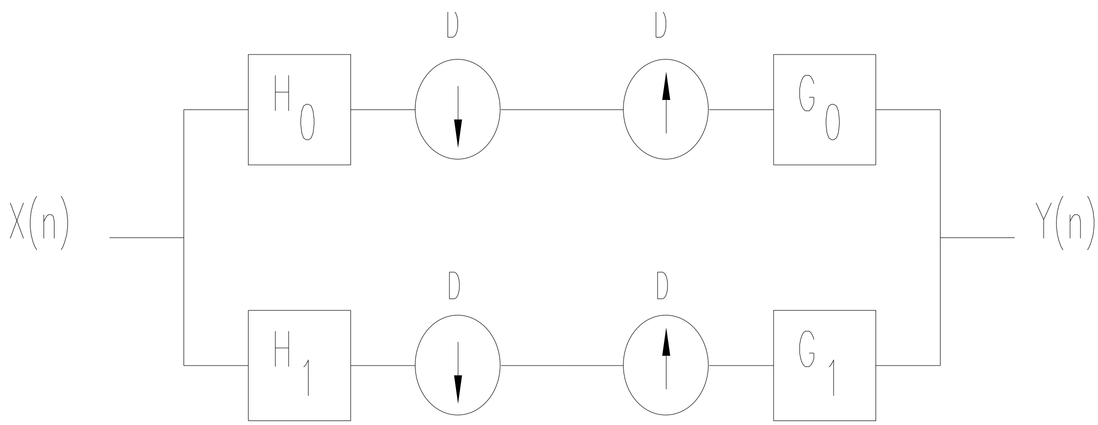

Figure 3.14: Multidimensional Two Channel PRFB

Suppose that the desired band split are those shown in Figure 3.11 b and d then the low and high frequency regions will be p_0 and $p_1$respectively. Since $p_1$is the complement of p_0 any frequency in one of the sub-bands can be trasferred to a frequency in the other sub-band by spatial-domain modulation of $(-1)^{\sum_i n_i}$. Hence if the passband of $H_0(z)$ approximates p_0 then $H_1(z) = H_0(-z)$ approximates the sub-band p_1. The input output in the modulation domain is given by

$$Y(z) = \tfrac{1}{2}\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} x(z) \\ x(-z) \end{bmatrix} \qquad 3.59$$

The general quadratic mirror filter (QMF) solution for alias cancellation for the above two examples is given by

$$H_0(z) = H(z) \qquad H_1(z) = H(-z) \qquad 3.60$$

$$G_0(z) = H(z) \qquad G_1(z) = -H(-z).$$

In the polyphase domain we have

$$Y(z) = p_{ic}^t(z).G_p(z^D)h_p(z^D).x_p(z^D) \qquad 3.61$$

$$= p_{ic}^t(z).T_p(z^D).x_p(z^D),\ p_{ic}^t(z) = (z^{-k_1}\, 1).$$

From shapes of the band split chosen and QM symmetry for the quincunx and FCO sampling, we obtain

$$H_1(z) = H_0(-z^{-I}) \qquad 3.62$$

$$H_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ -H_{01}(z^{-I}) & H_{00}(z^{-I}) \end{bmatrix} \qquad 3.63$$

$$G_p(z) = \begin{bmatrix} H_{00}(z^{-I}) & H_{10}(z^{-I}) \\ -H_{01}(z) & H_{00}(z) \end{bmatrix}, \qquad 3.64$$

and for perfect reconstruction (PR) FIR filter bank, the following has to be satisfied

$$det(H_p(z)) = cz^{k_1}, \tag{3.65}$$

which reduces to

$$H_{00}(z)H_{00}(z^{-I}) + H_{01}(z)H_{01}(z^{-I}) = 1. \tag{3.66}$$

For quincunx:

$$H_{00}(z_1,z_2)H_{00}(z_1^{-1},z_2^{-1}) + H_{01}(z_1,z_2)H_{01}(z_1^{-1},z_2^{-1}) = 1. \tag{3.67}$$

For FCO:

$$H_{00}(z_1,z_2,z_3)H_{00}(z_1^{-1},z_2^{-1},z_3^{-1}) + H_{01}(z_1,z_2,z_3)H_{01}(z_1^{-1},z_2^{-1},z_3^{-1}) = 1 \tag{3.68}$$

In addition to the prefect reconstruction requirement the following two additional constraints may be considered in the general case.

1. Orthogonal:

If the analysis filter is paraunitary, then

$$H_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ cz^{-k_1}H_{01}(z^{-I}) & -cz^{-k_1}H_{00}(z^{-I}) \end{bmatrix} \tag{3.69}$$

$H_1(z)$ is completely specified by $H_0(z)$ $(H_1(z) = -z^{-k_1}\widehat{H}_{00}(-z)$, the two polyphase components are the same size and the polyphase components of $H_0(z)$ have to satisfy the power complementary property above. This requirement automatically excludes the desired diamond shaped passband because both filters have the same symmetry and size.

2. Linear Phase:

For the above examples the general linear phase testing conditions reduces to

$$H_p(z^D)\begin{pmatrix} 1 \\ z_1^{-k_1} \end{pmatrix} = \begin{pmatrix} a_0 & \\ & a_1 \end{pmatrix}\begin{pmatrix} D_0(z) & \\ & D_1(z) \end{pmatrix} H_p(z^{-D})\begin{pmatrix} 1 \\ z_1^{k_1} \end{pmatrix} \qquad 3.70$$

where a_0 and a_1 represent the symmetry of h_0 and h_1

$D_0(z_1,z_2)$, $D_1(z_1, z_2)$ are monomials related to the size of the filters.

For the two-channel real FIR case linear phase and orthogonality are mutually exclusive.

3.4 SYNTHESIS OF MULTIDIMENSIONAL FILTER BANKS:

In the following sections, we look at design structures, which meet the perfect reconstruction conditions derived above. One of the desirable properties for the analysis and synthesis banks is that they should both consist of finite-impulse response filters thus eliminating problems related to stability. Given an analysis filter that is FIR, then a synthesis filter bank of perfect reconstruction system is FIR if and only if the determinant of the polyphase matrix of the analysis filter bank is monomial in z. There are two fundamental approaches to satisfying the above perfect reconstruction conditions. One possibility is to construct the analysis polyphase matrix $h_p(z)$ so that its inverse may readily be found. This approach is used for one dimensional PRFB's and needs to be extended to multiple dimensions. The second possibility sees the polyphase matrix equation not as that relating the analysis and synthesis filters, but as a set of scalar equations relating the impulse response coefficients.

3.4.1 NUMERICAL OPTIMIZATION:

Numerical optimization is used to solve the set of scalar equations relating the impulse response coefficients. Additional conditions such as FIR and equal size filters are incorporated as a set of quadratic equality constraints. The constraints are nonlinear and nonconvex and several standard nonlinearity constrained optimization methods are used to design the filter banks. Symmetry is also used to reduce the number of variables and independent constraints. However, due to the large numbers of variables and constraints the optimization is often complex.

3.4.2 CASCADE:

The condition to achieve PR for FIR analysis and synthesis filters requires that the det$(H_p(z))$ to be a shift. These shift vectors are factorizable and so is the determinant. A polynomial matrix whose determinant is factorizable can itself be factorized. The determinant of a product is a product of the determinants. Thus if $H_p(z)$ is constructed from the cascade(product) of matrices which satisfy condition (3.48) then $H_p(z)$ will also satisfy the same condition. Polynomial matrix factorization methods can be used to design cascade structures for $H_p(z)$ (see [21], [14], [16], [29]). Thus cascade structures can be used to find set of filters so that $H_p(z)$ meets the perfect reconstruction condition stated above.

$$H_p(z) = U_n D_n(z) \cdots U_1 D_1(z) U_0 \qquad 3.71$$

where U_i are arbitrary constant coefficient matrices

$D_i(z)$ are diagonal matrices of shifts.

Cascade structures not only have very low complexity, but also have the advantages of generating higher order filters from lower-order ones. Several approaches are possible depending on which constraints have to be met. One can require the filter bank to be paraunitary, linear phase, or both [73], [74].

3.4.2.1 ORTHOGONAL:

In the orthogonal case the cascade if formed by combining orthogonal building blocks, i.e. orthogonal and diagonal delay matrices.

$$H_p(z) = U_0 \prod_{i=1}^{n} U_i D_i(z) \qquad 3.72$$

where U_i are unitary matrices

$D_i(z)$ are diagonal matrices of delays.

For the two channel n-dimensional case the real FIR orthogonal polyphase matrix H_p can be written as

$$H_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ cz^{-k}\widehat{H}_{01}(z) & -cz^{-k}H_{00}(z) \end{bmatrix}. \qquad 3.73$$

The following cascade will produce PR set containing two filters of the same size:

$$H_p(z) = U_0 \prod_{j=1}^{k} \prod_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & z_i^{-1} \end{pmatrix} U_{ij}, \qquad 3.74$$

where U_{ij} is unitary, i.e.

$$U_{i1,2} = \begin{pmatrix} 1 & -a_{i1,2} \\ a_{i1,2} & 1 \end{pmatrix}.$$

3.4.2.2 LINEAR PHASE:

Similarly cascade structures can be used along with the linear phase testing conditions defined above to design linear phase systems.

$$H_p(z) = U_0 \prod_{i=1}^{n} U_i D_i(z), \qquad 3.75$$

where U_i are symmetric matrices

$D_i(z)$ are diagonal matrices of delays.

3.4.2.3 EXAMPLE:

We will examine the design issues regarding the above two examples. Let us first consider the cascade structure that generates filters that are either orthogonal or linear phase. For the quincunx sampling matrix $D=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we have

$$H_p(z_1,z_2) = U_0 \left[\prod_{i=1}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} U_{1_i} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} U_{2_i} \right]. \qquad 3.76$$

The orthogonal filters U_{j_i} have to be unitary

$$U_i = \frac{1}{\sqrt{1+a_i^2}} \begin{pmatrix} 1 & -a_i \\ a_i & 1 \end{pmatrix}. \qquad 3.77$$

The smallest size filters will be

$$H_p(z_1,z_2)=d.\begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix}\begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix}.. \qquad 3.78$$

$$..\begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix},$$

with

$$d = \frac{1}{\sqrt{\prod_{i=0}^{2}(1+a_i^2)}}.$$

The smallest lowpass filter obtained from the above cascade would be

$$\begin{aligned} H_{00}(z) &= \sum h(Dn)z^{-n} && 3.79 \\ &= \sum h(n_1-n_2, n_1+n_2)z_1^{-n_1}z_2^{-n_2} \\ &= d.(1-a_0a_1z_1^{-1}-a_1a_2z_2^{-1}-a_0a_2z_1^{-1}z_2^{-1}), \\ H_{01}(z) &= \sum h(Dn + k_1)z^{-n} \\ &= \sum h(n_1-n_2+1, n_1+n_2)z_1^{-n_1}z_2^{-n_2} \\ &= d.(-a_2 + a_0a_1a_2z_1^{-1}-a_1z_2^{-1}-a_0z_1^{-1}z_2^{-1}), \end{aligned}$$

$$h_0(n_1,n_2) = d.\begin{pmatrix} 0 & -a_0a_2 & -a_0 & 0 \\ -a_1a_2 & -a_1 & -a_0a_1 & a_0a_1a_2 \\ 0 & 1 & -a_2 & 0 \end{pmatrix}. \qquad 3.80$$

The highpass filter is obtained by modulation and time reversal of the lowpass filter.

For the linear phase filter U_{j_i} has to be symmetric

$$U_i = \begin{pmatrix} 1 & a_i \\ a_i & 1 \end{pmatrix}, \qquad 3.81$$

and the smallest size filter will be

$H_p(z_1,z_2)$

$$= \begin{pmatrix} 1 & a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ a_2 & 1 \end{pmatrix}. \qquad 3.82$$

The lowpass filter obtained from the above cascade will be

$$H_{00}(z) = \sum h(Dn)z^{-n} \qquad 3.83$$

$$=\sum h(n_1-n_2,n_1+n_2)z_1^{-n_1}z_2^{-n_2}$$

$$=1+a_0a_1z_1^{-1}+a_1a_2z_2^{-1}+a_0a_2z_1^{-1}z_2^{-1},$$

$$H_{01}(z) = \sum h(Dn + k_1)z^{-n}$$

$$=\sum h(n_1-n_2+ 1,n_1+n_2)z_1^{-n_1}z_2^{-n_2}$$

$$=a_2 + a_0a_1a_2z_1^{-1}+a_1z_2^{-1}+a_0z_1^{-1}z_2^{-1},$$

$$h_0(n_1,n_2) = \begin{pmatrix} 0 & a_0a_2 & a_0 & 0 \\ a_1a_2 & a_1 & a_0a_1 & a_0a_1a_2 \\ 0 & 1 & a_2 & 0 \end{pmatrix} \qquad 3.84$$

$$= \begin{pmatrix} 0 & a_2 & 1 & 0 \\ a_1a_2 & a_1 & a_1 & a_1a_2 \\ 0 & 1 & a_2 & 0 \end{pmatrix}.$$

For FCO sampling matrix $D=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, the cascade structures generates the following filters

$$H_p(z_1,z_2,z_3)$$
$$= U_0 \left[\prod_{i=1}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} U_{1_i} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} U_{2_i} \begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix} U_{3_i} \right]. \qquad 3.85$$

Again for the orthogonal filters U_{j_i} have to be unitary

$$U_i = \frac{1}{\sqrt{1+a_i^2}} \begin{pmatrix} 1 & -a_i \\ a_i & 1 \end{pmatrix}. \qquad 3.86$$

And for the smallest size filters we obtain

$$H_p(z_1,z_2,z_3) = d.\begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix}$$
$$\begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}, \qquad 3.87$$

with

$$d= \frac{1}{\sqrt{\prod_{i=0}^{3}(1+a_i^2)}}.$$

The lowpass filter will be

$$H_{00}(\mathbf{z}) = \sum h(D\mathbf{n})\mathbf{z}^{-\mathbf{n}} \qquad 3.88$$

$$=\sum h(n_1+n_2,n_1+n_3,\ n_2+n_3)z_1^{-n_1}z_2^{-n_2}z_3^{-n_3}$$

$$=d.\ (1-a_0a_1z_1^{-1}-\ a_1a_2z_2^{-1}-\ a_0a_2z_1^{-1}z_2^{-1}-\ a_2a_3z_3^{-1}\ +\ a_0a_1a_2a_3z_1^{-1}z_3^{-1}-$$
$$a_1a_3z_2^{-1}z_3^{-1}-a_0a_3z_1^{-1}z_2^{-1}z_3^{-1}),$$

$$H_{01}(z) = \sum h(Dn\ +\ k_1)z^{-n} \qquad 3.89$$

$$=\sum h(n_1+n_2+\ 1,n_1+n_3+1,\ \ n_2+n_3+1)z_1^{-n_1}z_2^{-n_2}z_3^{-n_3}$$

$$=d.(-a_3+\ a_0a_1a_3z_1^{-1}\ +\ a_1a_2a_3z_2^{-1}\ +\ a_0a_2a_3z_1^{-1}z_2^{-1}\ -\ a_2z_3^{-1}\ +$$
$$a_0a_1a_2z_1^{-1}z_3^{-1}\ -\ a_1z_2^{-1}z_3^{-1}-a_0z_1^{-1}z_2^{-1}z_3^{-1}).$$

$h_0(n_1,n_2,n_3)$ is given by the following coefficients

$h_0(0,0,0) =1$ $\qquad$ $h_0(1,1,1) =-a_3$ $\qquad$ 3.90

$h_0(1,1,0) = -a_0 a_1$	$h_0(2,2,1) = a_0 a_1 a_3$
$h_0(1,0,1) = -a_1 a_2$	$h_0(2,1,2) = a_1 a_2 a_3$
$h_0(2,1,1) = -a_0 a_2$	$h_0(3,2,1) = a_0 a_2 a_3$
$h_0(0,1,1) = -a_2 a_3$	$h_0(1,2,2) = -a_2$
$h_0(1,2,1) = a_0 a_1 a_2 a_3$	$h_0(2,3,2) = a_0 a_1 a_2$
$h_0(1,1,2) = -a_1 a_3$	$h_0(2,2,3) = -a_1$
$h_0(2,2,2) = -a_0 a_3$	$h_0(3,3,3) = -a_0$.

For the linear phase U_{j_i} have to be presymmetric

$$U_i = \begin{pmatrix} 1 & a_i \\ a_i & 1 \end{pmatrix} \qquad 3.91$$

and for the smallest size filters we obtain

$$H_p(z_1,z_2,z_3) = \begin{pmatrix} 1 & a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ a_3 & 1 \end{pmatrix}. \qquad 3.92$$

The low pass filter is given by

$$\begin{aligned} H_{00}(\mathbf{z}) &= \sum h(D\mathbf{n})\mathbf{z}^{-\mathbf{n}} \qquad 3.93 \\ &= \sum h(n_1+n_2, n_1+n_3, n_2+n_3) z_1^{-n_1} z_2^{-n_2} z_3^{-n_3} \\ &= 1 + a_0 a_1 z_1^{-1} + a_1 a_2 z_2^{-1} + a_0 a_2 z_1^{-1} z_2^{-1} + a_2 a_3 z_3^{-1} \\ &\quad + a_0 a_1 a_2 a_3 z_1^{-1} z_3^{-1} + a_1 a_3 z_2^{-1} z_3^{-1} + a_0 a_3 z_1^{-1} z_2^{-1} z_3^{-1}, \end{aligned}$$

$$H_{01}(z) = \sum h(Dn + k_1)z^{-n}$$

$$= \sum h(n_1+n_2+1, n_1+n_3+1,\ n_2+n_3+1)z_1^{-n_1}z_2^{-n_2}z_3^{-n_3}$$

$$=a_3 + a_0a_1a_3z_1^{-1} + a_1a_2a_3z_2^{-1} + a_0a_2a_3z_1^{-1}z_2^{-1} + a_2z_3^{-1} +$$

$$a_0a_1a_2z_1^{-1}z_3^{-1} + a_1z_2^{-1}z_3^{-1} + a_0z_1^{-1}z_2^{-1}z_3^{-1},$$

$h_0(n_1,n_2,n_3)$ is given by the following coefficients

$h_0(0,0,0) = 1$	$h_0(1,1,1) = a_3$	3.94
$h_0(1,1,0) = a_0a_1$	$h_0(2,2,1) = a_0a_1a_3$	
$h_0(1,0,1) = a_1a_2$	$h_0(2,1,2) = a_1a_2a_3$	
$h_0(2,1,1) = a_0a_2$	$h_0(3,2,2) = a_0a_2a_3$	
$h_0(0,1,1) = a_2a_3$	$h_0(1,2,2) = a_2$	
$h_0(1,2,1) = a_0a_1a_2a_3$	$h_0(2,3,2) = a_0a_1a_2$	
$h_0(1,1,2) = a_1a_3$	$h_0(2,2,3) = a_1$	
$h_0(2,2,2) = a_0a_3$	$h_0(3,3,3) = a_0$.	

3.4.3 *N-1 to N* ORDER FILTER:

Generally higher order filters can be obtained by cascading basic blocks while retaining perfect reconstruction property [40]. A higher order linear phase system can also be derived from a lower order ones by choosing

$$H_p^N(z) = H_p^{N-1}(z).D(z).U \qquad 3.95$$

where $D(z) = z^{-k}.J\widehat{D}(z)J$ and U is persymmetric(i.e. $U=JUJ$).

A higher order linear phase system is obtained where filters have the same symmetry as in H_p^{N-1}. Starting with the smallest size linear phase filter with sizes 3 and 5 in *n-1* dimension *n*-dimensioal filters of the same size can be reconstructed [20]. The general solution formulated by Kovacevic is given as follows:

$$H_{00}(z_u^{(n)}) = H_{00}(z_u^{n-1}) + a.z_1^{-1}(z_n^{-1} + z_n) \qquad 3.96$$

$$H_{01}(z_u^{(n)}) = H_{01}(z_u^{n-1})$$

$$H_{10}(z_u^{(n)}) = H_{10}(z_u^{n-1}) + H_c(z_u^{n-1}).z_1^{-1}(z_n^{-1} + z_n) + cz_1^{-2}(z_n^{-2} + z_n^{2})$$

$$H_{11}(z_u^{(n)}) = H_{11}(z_u^{n-1}) + d.z_1^{-1}(z_n^{-1} + z_n)$$

where

$$H_{01}(z_u^{n-1}) = b,\ b \neq 0;$$

$$H_c(z_u^{n-1}) = \frac{d.H_{00}(z_u^{n-1}) + H_{11}(z_u^{n-1})}{b}$$

$$a.d=b.c$$

$$t_{n-1} + 2a.d \neq 0$$

and det $H_p(z_u^{n-1}) = t_{n-1}.z_1^{-2}$ in *n-1* dim

det $H_p(z_u^{n}) = (t_{n-1} + 2ad).z_1^{-2} = t_n.z_1^{-2}$ in *n* dim.

3.4.3.1 EXAMPLE :

Let us consider the smallest size linear phase filters in one-dimension and build a two and three dimensional linear phase filters using the above formula developed by Kovacevic. The polyphase components for the one-dimensional filter is given by

$$H_p(z^2) = \begin{pmatrix} 1+z_1^{-2} & 1 \\ 1+z_1^{-2}+z_1^{-4} & 1+z_1^{-2} \end{pmatrix} \qquad 3.97$$

1. Two dimensional quincunx case:

To construct the two dimensional solution we need to determine

$$H_c(z_1^2) = d.(1+z_1^{-2}) + a.(1 + z_1^{-2}) \qquad 3.98$$

$$= (a+d)(1+z_1^{-2}).$$

For the quincunx sampling matrix

$$H_{00}(z_1z_2^{-1},z_1z_2) = 1 + z_1^{-2} + az_1^{-1}(z_2^{-1} + z_2) \qquad 3.99$$

$$H_{01}(z_1z_2^{-1},z_1z_2) = 1$$

$$H_{10}(z_1z_2^{-1},z_1z_2)=1+z_1^{-2} +z_1^{-4}+(a+d).(1+z_1^{-2})z_1^{-1}(z_2^{-1}+z_2)+ad.z_1^{-2}(z_2^{-2}+z_2^{2})$$

$$H_{11}(z_1z_2^{-1},z_1z_2) = 1 + z_1^{-2} + dz_1^{-1}(z_2^{-1} + z_2),$$

$$h_0(n_1,n_2) = \begin{pmatrix} 0 & a & 0 \\ 1 & 1 & 1 \\ 0 & a & 0 \end{pmatrix}, \qquad 3.100$$

$$h_0(n_1,n_2) = \begin{pmatrix} 0 & 0 & ad & 0 & 0 \\ 0 & a+d & d & a+d & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & a+d & d & a+d & 0 \\ 0 & 0 & ad & 0 & 0 \end{pmatrix}. \qquad 3.101$$

2. Three-dimensional FCO case:

For the FCO sampling case, we start with the two dimensional linear phase filter constructed above and using the same formula above the polyphase components of the two filters are developed. First let us determine H_c

$$H_c(z_u^{n-1}) = \frac{f.H_{00}(z_u^{n-1}) + e.H_{11}(z_u^{n-1})}{H_{01}(z_u^{n-1})} \qquad 3.102$$

$$= f.(1 + z_1^{-2} + az_1^{-1}(z_2^{-1} + z_2)) + e.(1 + z_1^{-2} + dz_1^{-1}(z_2^{-1} + z_2))$$

$$= (e+f)(1+z_1^{-2}) + (af +de)z_1^{-1}(z_2^{-1} + z_2)$$

$$H_{00}(z^{D_{FCO}}) \quad = H_{00}(z_u^{n-1}) + e.z_1^{-1}(z_3^{-1} + z_3) \qquad 3.103$$

$$= 1 + z_1^{-2} + az_1^{-1}(z_2^{-1} + z_2) + e.z_1^{-1}(z_3^{-1} + z_3)$$

$$H_{00}(z^{D_{FCO}}) \quad = H_{01}(z_u^{n-1})$$

$$= 1$$

$$H_{10}(z^{D_{FCO}}) \quad = H_{10}(z_u^{n-1}) + H_c(z_u^{n-1}).z_1^{-1}(z_3^{-1} + z_3) + ef.z_1^{-2}(z_3^{-2} + z_3^{2})$$

$$= 1 + z_1^{-2} + z_1^{-4} + (a+d)(1+z_1^{-2})z_1^{-1}(z_2^{-1} + z_2) + ad.z_1^{-2}(z_2^{-2} + z_2^{2})$$

$$+ (e+f)(1+z_1^{-2}) + (af + de)z_1^{-1}(z_2^{-1} + z_2)z_1^{-1}(z_3^{-1} + z_3) +$$

$$ef.z_1^{-2}(z_3^{-2} + z_3^{2})$$

$$H_{11}(z^{D_{FCO}}) \quad = H_{11}(z_u^{n-1}) + f.z_1^{-1}(z_3^{-1} + z_3)$$

$$= 1 + z_1^{-2} + dz_1^{-1}(z_2^{-1} + z_2) + f.z_1^{-1}(z_3^{-1} + z_3).$$

The two filters $h_0(n_1,n_2,n_3)$ and $h_1(n_1,n_2,n_3)$ are shown in the Figure 3.15 below.

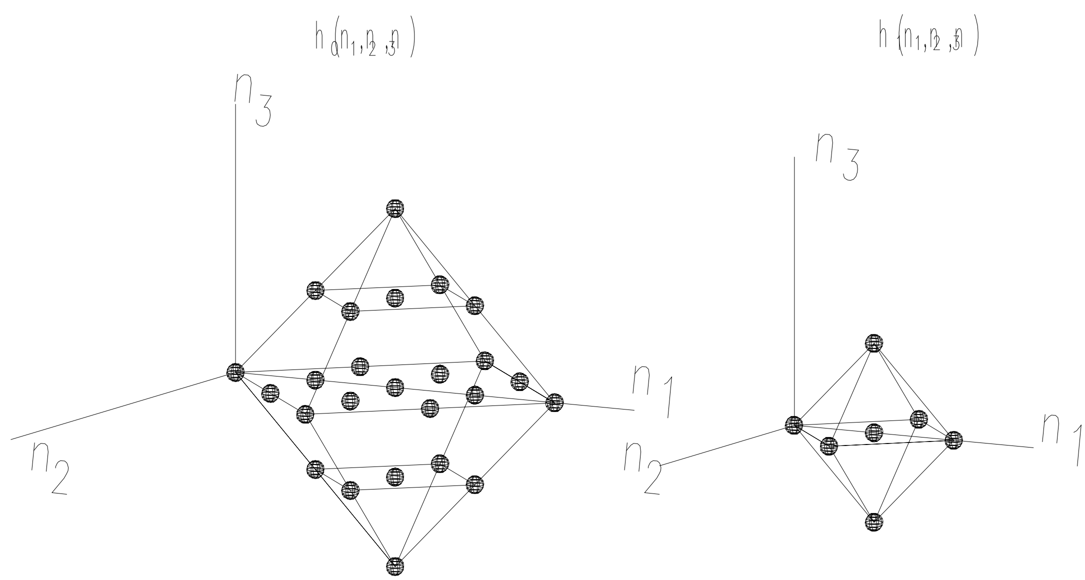

Figure 3.15: 3-D FCO HP and LP Filters

3.4.4 TRANSFORMATION:

Multidimensional filters can be designed using a transformation that maps one-dimensional designs into multidimensional ones. (See [13], [64], [70], [78], [79], [91]) Under these transformations perfect reconstruction and zeros at aliasing frequencies are preserved.

3.4.4.1 SEPARABLE POLYPHASE COMPONENTS:

Multidimensional filter with separable polyphase components can be designed with each polyphase component constituting products of one-dimensional filters. We start with one-dimensional filter having polyphase components $H_i(z)$ and construct n-dimensional components as

$$H_i(z_1,z_2,...,z_n) = H_i(z_1).H_i(z_2)\cdots H_i(z_n),\ i=0,1,2... \qquad 3.104$$

Then the n-dimensional filter with respect to the sampling lattice D (with columns $d_1,\dots,d_n$) is given by

$$H(z_1,z_2,\dots,z_n) = \sum_{k_i \in U_c^t} z^{-k_i} H_i(z^{d_1})\cdots H_i(z^{d_n}). \qquad 3.105$$

The zeros of $H(z)$ map into the zeros of $H(z_1,z_2,\dots,z_n)$.

Let us consider the previous quincunx sampling and the following one-dimensional linear phase filter with polyphase components

$$H_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix} \qquad 3.106$$

$$= \begin{bmatrix} 1+6z^{-2}+z^{-4} & 4+4z^{-2} \\ \frac{1}{16}(1+z^{-2}) & \frac{4}{16} \end{bmatrix},$$

$$\begin{bmatrix} H_0 \\ H_1 \end{bmatrix} = \begin{bmatrix} 1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4} \\ \frac{1}{16}(1+4z^{-1}+z^{-2}) \end{bmatrix}. \qquad 3.107$$

The separable polyphase components are given by

$$H_i(z_1,z_2) = H_i(z_1)H_i(z_2),\ i=0,1 \qquad 3.108$$

$$H_{00}(z_1,z_2) = H_{00}(z_1)H_{00}(z_2)$$

$$= 1+\ 6z_1^{-1}+6z_2^{-1}+z_1^{-2}+z_2^{-2}+36z_1^{-1}z_2^{-1}+6z_1^{-1}z_2^{-2}+6z_1^{-2}z_2^{-1}+ z_1^{-2}z_2^{-2}$$

$$H_{00}(z_1z_2^{-1},z_1z_2) = H_{00}(z_1z_2^{-1})H_{00}(z_1z_2)$$

$$= 1+\ 6z_1^{-1}z_2+\ 6z_1^{-1}z_2^{-1}+z_1^{-2}z_2^{2}+z_1^{-2}z_2^{-2}+\ 36z_1^{-2}+\ 6z_1^{-3}z_2^{-1}+ 6z_1^{-3}z_2\ +\ z_1^{-4}$$

$$H_{01}(z_1,z_2) = H_{01}(z_1)H_{01}(z_2)$$

$$= 16+16z_1^{-1}+16z_2^{-1}+16z_1^{-1}z_2^{-1}$$

$$H_{01}(z_1z_2^{-1},z_1z_2) = H_{01}(z_1z_2^{-1})H_{01}(z_1z_2)$$

$$= 16+16z_1^{-1}z_2+16z_1^{-1}z_2^{-1}+16z_1^{-2}.$$

And the two dimensional lowpass filter is given by

$$h_0(n_1,n_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 16 & 6 & 0 \\ 1 & 16 & 36 & 16 & 1 \\ 0 & 6 & 16 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \qquad 3.109$$

Let us extend the above linear phase filter to a three-dimensional filter with FCO sampling and the separable polyphase components will be

$$H_i(z_1,z_2,z_3) = H_i(z_1)H_i(z_2)H_i(z_3),\ i=0,1 \qquad 3.110$$

$$\begin{aligned}
H_{00}(z_1,z_2,z_3) &= H_{00}(z_1)H_{00}(z_2)H_{00}(z_2)\\
&= 1+6z_1^{-1}+6z_2^{-1}+6z_3^{-1}+z_1^{-2}+z_2^{-2}+z_3^{-2}+36z_1^{-1}z_2^{-1}+36z_1^{-1}z_3^{-1}+\\
&\quad 36z_2^{-1}z_3^{-1}+6z_1^{-1}z_2^{-2}+6z_1^{-2}z_2^{-1}+6z_1^{-1}z_3^{-2}+6z_1^{-2}z_3^{-1}+\\
&\quad 6z_2^{-1}z_3^{-2}+6z_2^{-2}z_3^{-1}+z_1^{-2}z_2^{-2}+z_1^{-2}z_3^{-2}+z_2^{-2}z_3^{-2}+\\
&\quad 36z_1^{-1}z_2^{-1}z_3^{-2}+36z_1^{-1}z_2^{-2}z_3^{-1}+36z_1^{-2}z_2^{-1}z_3^{-1}+6z_1^{-1}z_2^{-2}z_3^{-2}+\\
&\quad 6z_1^{-2}z_2^{-1}z_3^{-2}+6z_1^{-2}z_2^{-2}z_3^{-1}+216z_1^{-1}z_2^{-1}z_3^{-1}+z_1^{-2}z_2^{-2}z_3^{-2}
\end{aligned}$$

$$\begin{aligned}
H_{00}(\mathbf{z}^{D_{FCO}}) &= H_{00}(z_1z_2)H_{00}(z_1z_3)H_{00}(z_2z_3) \qquad 3.111\\
&=1+6z_1^{-1}z_2^{-1}+6z_2^{-1}z_3^{-1}+6z_1^{-1}z_3^{-1}+z_1^{-2}z_2^{-2}+z_2^{-2}z_3^{-2}+z_1^{-2}z_3^{-2}+\\
&\quad 36z_1^{-1}z_2^{-1}z_3^{-2}+36z_1^{-1}z_2^{-2}z_3^{-1}+36z_1^{-2}z_2^{-1}z_3^{-1}+36z_1^{-2}z_2^{-3}z_3^{-3}+\\
&\quad 36z_1^{-3}z_2^{-2}z_3^{-3}+36z_1^{-3}z_2^{-3}z_3^{-2}+z_1^{-4}z_2^{-2}z_3^{-2}+z_1^{-2}z_2^{-4}z_3^{-2}+z_1^{-2}z_2^{-2}z_3^{-4}\\
&\quad +6z_1^{-4}z_2^{-3}z_3^{-3}+6z_1^{-3}z_2^{-4}z_3^{-3}+6z_1^{-3}z_2^{-3}z_3^{-4}+6z_1^{-3}z_2^{-1}z_3^{-2}+\\
&\quad 6z_1^{-3}z_2^{-2}z_3^{-1}+6z_1^{-1}z_2^{-3}z_3^{-2}+6z_1^{-2}z_2^{-3}z_3^{-1}+6z_1^{-1}z_2^{-2}z_3^{-3}+\\
&\quad 6z_1^{-2}z_2^{-1}z_3^{-3}+216z_1^{-2}z_2^{-2}z_3^{-2}+z_1^{-4}z_2^{-4}z_3^{-4}
\end{aligned}$$

$$\begin{aligned}
H_{01}(z_1,z_2,z_3) &= H_{01}(z_1)H_{01}(z_2)H_{01}(z_3)\\
&= 64+64z_1^{-1}+64z_2^{-1}+64z_3^{-1}+64z_1^{-1}z_2^{-1}+64z_1^{-1}z_3^{-1}+\\
&\quad 64z_2^{-1}z_3^{-1}+64z_1^{-1}z_2^{-1}z_3^{-1}
\end{aligned}$$

$$\begin{aligned}
H_{01}(\mathbf{z}^{D_{FCO}}) &= H_{01}(z_1z_2)H_{01}(z_1z_2)H_{01}(z_2z_3)\\
&= 64+64z_1^{-1}z_2^{-1}+64z_2^{-1}z_3^{-1}+64z_1^{-1}z_3^{-1}+64z_1^{-2}z_2^{-1}z_3^{-1}+\\
&\quad 64z_1^{-1}z_2^{-2}z_3^{-1}+64z_1^{-1}z_2^{-1}z_3^{-2}+64z_1^{-2}z_2^{-2}z_3^{-2}.
\end{aligned}$$

And the three-dimensional lowpass filter is given by $h_0(n_1,n_2,n_3)$ with the following coefficients:

$h_0(0,0,0) = 1$	$h_0(2,2,2) = 216$
$h_0(0,1,1) = 6$	$h_0(2,2,3) = 64$
$h_0(0,2,2) = 1$	$h_0(2,2,4) = 1$
$h_0(1,0,1) = 6$	$h_0(2,3,1) = 6$
$h_0(1,1,0) = 6$	$h_0(2,3,2) = 64$
$h_0(1,1,2) = 36$	$h_0(2,3,3) = 36$
$h_0(1,2,2) = 36$	$h_0(2,3,4) = 64$
$h_0(1,2,2) = 64$	$h_0(2,4,2) = 1$
$h_0(1,0,1) = 6$	$h_0(2,4,3) = 64$
$h_0(1,1,0) = 6$	$h_0(3,1,2) = 6$
$h_0(1,1,2) = 36$	$h_0(3,2,1) = 6$
$h_0(1,2,2) = 36$	$h_0(3,2,3) = 36$
$h_0(1,2,2) = 64$	$h_0(3,3,2) = 36$
$h_0(1,0,1) = 6$	$h_0(3,3,3) = 64$
$h_0(1,2,3) = 6$	$h_0(3,3,4) = 6$
$h_0(1,3,1) = 64$	$h_0(3,4,3) = 6$
$h_0(1,3,2) = 6$	$h_0(3,4,4) = 64$
$h_0(2,0,2) = 1$	$h_0(4,2,2) = 1$
$h_0(2,1,1) = 36$	$h_0(4,3,3) = 6$
$h_0(2,1,3) = 6$	$h_0(4,4,4) = 1.$
$h_0(2,2,0) = 1$	

3.4.4.2 McClellan Transformation:

This transformation maps a single dimensional zero-phase filters into zero-phase multidimensional filters (See [77],[90]). A linear phase filter can be written as

$$H(z_1,z_2,...z_n) \quad = d.H_s\left(\frac{z_1+z_1^{-1}}{2}, \frac{z_2+z_2^{-1}}{2},....\frac{z_n+z_n^{-1}}{2}\right), \qquad 3.112$$

$$d \quad = \text{pure delay.}$$

The idea of McClellan transformation is to replace the one-dimensional kernel $k(z)=(\frac{z_1+z_1^{-1}}{2})$ by a multidimensional kernel $k(z_1,z_2,...z_n)$. The filter $H(\ k(z_1,z_2,...z_n))$ remains linear phase as long as $k(z_1,z_2,...z_n)$ is zero-phase. Perfect reconstruction and zeros at aliasing frequencies are preserved. If the determinant of $H_p(k(z))$ is monomial in one-dimension so is the determinant of $H_p(\ k(z_1,z_2,...z_n))$ and perfect reconstruction is preserved. The zeros of order $2N$ at π in one-dimension maps into zeros of order $2N$ at $(\pi,\pi,...\pi)$ in *n-dimensions*.

3.4.4.3 EXAMPLE:

Let us consider the single dimensional linear phase filter given above (3.112). Based on the McClellan transformation, we will construct the two-dimensional linear phase filter for the quincunx case. The above filter can be written in terms of the kernel, $k(z)$, as follows

$$H_0(z) = z^{-k}H_{s_0}(k(z)), \qquad 3.113$$

$$H_1(z) = z^{-l}H_{s_1}(k(z)).$$

Then the polyphase components of the filter can be expressed as

$$H_{00}(z^2) = z^{-k}H_{s_{00}}(k(z)) = z^{-2}\left[4+4\left(\frac{z_1+z_1^{-1}}{2}\right)^2\right] \quad 3.114$$

$$H_{01}(z^2) = z^{-k+1}H_{s_{01}}(k(z)) = z^{-1}\left[8\left(\frac{z_1+z_1^{-1}}{2}\right)\right]$$

$$H_{10}(z^2) = z^{-l}H_{s_{10}}(k(z)) = z^{-1}\left[\frac{1}{8}\left(\frac{z_1+z_1^{-1}}{2}\right)\right]$$

$$H_{11}(z^2) = z^{-l+1}H_{s_{11}}(k(z)) = z^{0}\left[\frac{4}{16}\right]$$

with

$$det\ H_p(z^2) = z^{-(k+l-1)}(P(k(z))-Q(k(z)))$$

where $P(k(z)) = H_{s_{00}}(k(z))H_{s_{11}}(k(z))$

$Q(k(z)) = H_{s_{10}}(k(z))H_{s_{01}}(k(z))$.

Because perfect reconstruction holds

$$P(k(z))= k_1+R(k(z)),\ Q(k(z)) = k_2 + R(k(z)). \quad 3.115$$

If we define $k(z_1,z_2) = \left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}}{2}\right)$, the two-dimensional filters and their polyphase components are given by

$$H_{00}(z_1,z_2) = z_1^{-k}H_{s_{00}}(k(z_1,z_2)) \quad 3.116$$

$$= z_1^{-2}\left[4+4\left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}}{2}\right)^2\right]$$

$$= 1+6z_1^{-2}+z_1^{-4}+2z_1^{-1}z_2+2z_1^{-1}z_2^{-1}+2z_1^{-2}z_2^{2}+2z_1^{-2}z_2^{-2}+2z_1^{-3}z_2+$$
$$2z_1^{-3}z_2^{-1}$$

$$H_{00}(z_1z_2^{-1},z_1z_2)=1+6z_1^{-2}z_2^{2}+z_1^{-4}z_2^{4}+2z_2^{2}+2z_1^{-2}+2z_2^{4}+2z_1^{-4}+2z_1^{-2}z_2^{2}+$$
$$2z_1^{-4}z_2^{2}$$

$$H_{01}(z_1,z_2) = z_1^{-k+1}H_{s_{01}}(k(z_1,z_2))$$

$$=z_1^{-1}\left[8\left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}}{2}\right)\right]$$

$$=4+4z_1^{-2}+4z_1^{-1}z_2+4z_1^{-1}z_2^{-1}$$

$$H_{01}(z_1z_2^{-1},z_1z_2) = 4+4z_1^{-2}z_2^{2}+4z_2^{2}+4z_1^{-2}$$

$$H_{10}(z_1,z_2) \quad = z_1^{-1}H_{s_{10}}(k(z_1,z_2))$$

$$= z_1^{-1}\left[\tfrac{1}{8}\left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}}{2}\right)\right]$$

$$= \tfrac{1}{16}(1+z_1^{-2}+z_1^{-1}z_2+z_1^{-1}z_2^{-1})$$

$$H_{10}(z_1z_2^{-1},z_1z_2) = \tfrac{1}{16}(1+z_1^{-2}z_2^{2}+z_2^{2}+z_1^{-2})$$

$$H_{11}(z_1,z_2) \quad = z_1^{-1+1}H_{s_{11}}(k(z_1,z_2))$$

$$=\tfrac{4}{16}$$

$$H_{11}(z_1z_2^{-1},z_1z_2) = \tfrac{4}{16}$$

$$h_0(n_1,n_2) = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 \\ 1 & 4 & 6 & 4 & 1 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad 3.117$$

$$h_1(n_1,n_2) = \begin{pmatrix} 0 & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{4}{16} & \frac{1}{16} \\ 0 & \frac{1}{16} & 0 \end{pmatrix}. \qquad 3.118$$

If we define $k(z_1,z_2) = \left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}+z_3+z_3^{-1}}{2}\right)$, the three-dimensional filters and their polyphase components are given by

$$H_{00}(z_1,z_2,z_3) = z_1^{-k}H_{s_{00}}(k(z_1,z_2,z_3)) \qquad 3.119$$

$$=z_1^{-2}\left[4+4\left(\frac{z_1+z_1^{-1}+z_2+z_2^{-1}+z_3+z_3^{-1}}{2}\right)^2\right]$$

$$= 28z_1^{-2}+4+8z_1^{-1}z_2+8z_1^{-1}z_3+8z_1^{-1}z_2^{-1}+8z_1^{-1}z_3^{-1}+8z_1^{-3}z_2^{-1}+$$

$$8z_1^{-3}z_3 + 8z_1^{-3}z_3^{-1} + 4z_1^{-2}z_2^{2} + 8z_1^{-2}z_2z_3 + 8z_1^{-2}z_2z_3^{-1} +$$
$$8z_1^{-2}z_2^{-1}z_3 + 8z_1^{-2}z_2^{-1}z_3^{-1} + 4z_2^{-2}z_3^{2} + 4z_1^{-2}z_2^{-2} + 8z_2^{-3}z_3 +$$
$$4z_1^{-2}z_3^{-2} + 4z_1^{-4}$$

$$H_{01}(z_1,z_2,z_3) = z_1^{-k+1}H_{s_{01}}(k(z_1,z_2,z_3))$$
$$= z_1^{-1}\left[8\left(\frac{z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1}}{2} \right) \right]$$
$$= 4 + 4z_1^{-2} + 4z_1^{-1}z_2 + 4z_1^{-1}z_2^{-1} + 4z_1^{-1}z_3 + 4z_1^{-1}z_3^{-1}$$

$$H_{10}(z_1,z_2,z_3) = z_1^{-1}H_{s_{10}}(k(z_1,z_2,z_3))$$
$$= z_1^{-1}\left[\frac{1}{8}\left(\frac{z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1}}{2} \right) \right]$$
$$= \frac{1}{16}(1 + z_1^{-2} + z_1^{-1}z_2 + z_1^{-1}z_2^{-1} + z_1^{-1}z_3 + z_1^{-1}z_3^{-1})$$

$$H_{11}(z_1,z_2,z_3) = z_1^{-1+1}H_{s_{11}}(k(z_1,z_2,z_3))$$
$$= \frac{4}{16}$$

$$H_{00}(\mathbf{z}^{D_{FCO}}) = 28z_1^{-2}z_2^{-2} + 8z_1^{-1}z^{-3} + 8z_1^{-1}z_2^{-2}z_3^{-1} + 8z_1^{-3}z_2^{-2}z_3 + 8z_1^{-2}z_2^{-1}z_3^{-1} +$$
$$8z_1^{-2}z_2^{-3}z_3 + 8z_1^{-4}z_2^{-3}z_3^{-1} + 8z_1^{-3}z_2^{-1} + 4z_1^{-4}z_2^{-4} + 4z_1^{-4}z_2^{-2}z_3^{2} + 4 +$$
$$4z_2^{-2}z_3^{2} + 4z_1^{-2}z_3^{2} + 4z_1^{-2}z_2^{-4}z^{-2} + 8z_2^{-1}z_3 + 8z_1^{-1}z_3 + 8z_1^{-3}z_2^{-4}z^{-1}$$
$$+ 8z_1^{-1}z_2^{-1}z_3^{2}$$

$$H_{01}(\mathbf{z}^{D_{FCO}}) = 4 + 4z_1^{-2}z_2^{-2} + 4z_2^{-1}z_3 + 4z_1^{-2}z_2^{-1}z_3^{-1} + 4z_1^{-1}z_3 + 4z_1^{-1}z_2^{-2}z_3^{-1}$$

$$H_{10}(\mathbf{z}^{D_{FCO}}) = \frac{1}{16}(1 + z_1^{-2}z_2^{-2} + z_2^{-1}z_3 + z_1^{-2}z_2^{-1}z_3^{-1} + z_1^{-1}z_3 + z_1^{-1}z_2^{-2}z_3^{-1})$$

$$H_{11}(\mathbf{z}^{D_{FCO}}) = \frac{4}{16}.$$

3.5 SUMMARY

The basic concepts in multidimensional multirate filter banks have been described. Several methods for the construction of multidimensional filters have been outlined. Even though several methods exist for building these filter banks much more emphasis has been placed on cascade structures for their simplicity of construction. The satisfaction of regularity condition is essential in generating continuous wavelets bases from these multidimensional filter banks. The method of generating regular multidimensional filter banks will be the topic of the subsequent chapter.

CHAPTER 4

ON THE REGULARITY OF MULTIDIMENSIONAL NON-SEPARABLE CASCADE FILTER BANKS

4.1 INTRODUCTION:

The satisfaction of regularity condition is essential in generating continuous wavelet bases from iterated filter banks. In one-dimension regularity of filters is guaranteed by packing zeros at the aliasing frequencies [25],[45]. In multiple dimensions the same could be done, but care must be made to keep the perfect reconstruction requirement simultaneously. In the case of transformation of one dimensional to multidimensional filterbanks, the zeros at aliasing frequencies of the starting filters are preserved. It is therefore essential to design the single dimensional filters with sufficient regularity. With cascade filters the perfect reconstruction requirement is structurally included and the zeros at aliasing frequencies are imposed algebraically using the undetermined coefficients. In what follows the regularity of cascade structures are discussed in detail.

4.2 REGULARITY:

As mentioned previously the satisfaction of regularity condition is necessary to generate continuous wavelet bases from iterated filter banks. In case of one-dimensional filters this condition is guaranteed by packing zeros at the aliasing frequencies. (See [21], [84]). A sufficiently high order of zeros are imposed at aliasing frequencies (or points of repeated spectra). In multiple dimensions, the same could be applied, i.e. impose a zero of order m at the multidimensional aliasing frequencies $2\pi(D^{-t})$ [40], [69]. This is equivalent to setting the following condition.

$$\frac{\partial^{k-1} H_0(z_1,z_2,...z_n)}{\partial^{t_1} z_1 \partial^{t_2} z_2 ......\partial^{t_n} z_n}\Big|_{(-1,-1,...,-1)} = 0 \qquad 4.1$$

$$\text{for } k=1...m, \text{ and } t_i=1,...k-1.$$

However, care must be made to keep the perfect reconstruction requirement simultaneously [83]. With cascade filters the perfect reconstruction requirement is structurally included and the problems of imposing zeros at aliasing frequencies reduces to solving algebraically the undetermined coefficients satisfying these conditions. Generally, the algebraic solutions become cumbersome for large size filters in multiple dimensions. Any method of reducing these algebriac equations becomes critical.

An interesting observation was made by Kovacevic [40] indicating identity of some of the zeros of the higher and lower dimensional filters. Establishing this relation will have significant ramifications to the effort to simplify the solution the above problem. In what follows, an attempt has been made to establish a possible connection between the zeros of the lower and higher dimensional filters.

Here we try to consider the cascade structure and redefine the above regularity condition. We know the scaling filter H_0 is the first element of the polyphase vector H and can be defined in terms of H and a vector I_0 with zeros everywhere except for the first element which is one. It can then be expressed in terms of the polyphase matrix $\mathrm{H_p}$ and its cascade form

$$\begin{aligned} H_0 &= \boldsymbol{I}_0^t H(z_1,z_2,...z_n) \qquad 4.2 \\ &= \boldsymbol{I}_0^t H_p(\boldsymbol{z}^D)\boldsymbol{p}_f \end{aligned}$$

$$= \boldsymbol{I}_0^t U_0.\prod_i U_i D_i(\boldsymbol{z}^{\boldsymbol{d}_i})\boldsymbol{p}_f$$

where $\boldsymbol{d}_i$ = the ith column of the sampling matrix D.

Let us now assume that there exits a lower dimensional cascade such that

$$H_p^{N\text{-}1}(z_1,z_2,....z_{n-1}) = U_0.\prod_i^{n-1} U_i D_i \qquad 4.3$$

Then

$$H_p^{N}(z_1,z_2,....z_n) = H_p^{N\text{-}1}(z_1,z_2,....z_{n-1})U_n D_n(z_n) \qquad 4.4$$

and the scaling filter reduces to the following

$$H_0 = \boldsymbol{I}_0^t H_p^{N\text{-}1}(z^{\boldsymbol{d}_1}, z^{\boldsymbol{d}_2} .... z^{\boldsymbol{d}_{n-1}})U_n D_n(\boldsymbol{z}^{\boldsymbol{d}_n})\boldsymbol{p}_f \qquad 4.5$$

where $\boldsymbol{d}_i$ = the ith column of the sampling matrix D.

From the constraint on the matrix multiplication the size of this lower dimensional filter has to be the same as the higher dimensional one. And for maximally decimated filter bank systems the filter sizes can be defined in terms of the determinant of the downsampling matrices:

$$\text{size of } H_p^N = MxM \qquad 4.6$$

where $M=det\ (D)$

$$\text{size of } H_p^{N\text{-}1} = NxN \qquad 4.7$$

where $N = det(D'_{ij})$ and D'_{ij} is $n{*}n\text{-}1$ dimensional submatrix of D.

From the definition of the determinant of D and the multiplicative constraint on the matrices above

$$det(D)=M=\sum_{j} d_{ij}|D'_{ij}| = det(D'_{ij})= N. \qquad 4.8$$

If there exists a submatrix D' of D having the same determinant as D then there are a series of unimodular transformations that reduce D to the following forms.

$$D \equiv \begin{pmatrix} d_{11} & & 0 & & d_{1n} \\ & & 0 & & \\ 0 & 0 & d_{ij} & 0 & 0 \\ & & 0 & & \\ d_{n1} & & 0 & & d_{nn} \end{pmatrix} \qquad 4.9$$

or

$$D \equiv \begin{pmatrix} D' & 0 \\ 0 & d_{nn} \end{pmatrix}. \qquad 4.10$$

These unimodular transformations of the sampling matrix would not alter the sampling pattern. The coset vectors and the forward polyphase transform remain also the same. Substituting this into the above H_0, we obtain

$$H_0 = \boldsymbol{I}_0^t H_p^{N\text{-}1}(\boldsymbol{z}_{(n\text{-}1)}^{D'})U_n D_n(\boldsymbol{z}_n^{d_{nn}})\boldsymbol{p}_f \qquad 4.11$$

where $\boldsymbol{z}_{(n\text{-}1)}$ is the $n\text{-}1$ dimensional index $\boldsymbol{z}$.

The forward polyphase transform in N dimensions can be expressed in terms of the corresponding forward polyphase transform in $N\text{-}1$ dimensions.

$$\boldsymbol{p}_f^N = T(z_n)\boldsymbol{p}_f^{N-1} \qquad 4.12$$

$$\text{where } T = \begin{pmatrix} z_n^{k_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & z_n^{k_n} \end{pmatrix}$$

is $n*n$ matrix of delays along the nth dimension.

Finally substituting the above term into the scaling filter gives

$$H_0 = \boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})U_n D_n(z_n^{d_{nn}})T(z_n)\boldsymbol{p}_f^{N-1}. \qquad 4.13$$

The regularity condition can be rewritten as

$$\frac{\partial^{k-1}\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})U_n D_n(z_n^{d_{nn}})T(z_n)\boldsymbol{p}_f^{N-1}}{\partial^{t_1}z_1\partial^{t_2}z_2\ldots\ldots\partial^{t_n}z_n}\bigg|_{(-1,-1,..-1)} = 0 \qquad 4.14$$

$$\frac{\partial^{k-1}\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})Q.\boldsymbol{p}_f^{N-1}}{\partial^{t_1}z_1\partial^{t_2}z_2\ldots\ldots\partial^{t_n}z_n}\bigg|_{(-1,-1,..-1)} = 0,$$

$$\text{where } Q(z_n) = U_n D_n(z_n^{d_{nn}})T(z_n)$$

$$\sum_{i=1}^{n} t_i = k\text{-}1,\ t_i \epsilon\ [0..k\text{-}1].$$

For the special case where t_n equals zero this reduces to

$$\frac{\partial^{k-1}\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})Q.\boldsymbol{p}_f^{N-1}}{\partial^{t_1}z_1\partial^{t_2}z_2\ldots\ldots\partial^{t_{n-1}}z_{n-1}}\bigg|_{(-1,-1,...-1)} = 0, \qquad 4.15$$

where $\sum_{i=1}^{n-1} t_i = k\text{-}1,\ t_i = 0..k\text{-}1.$

One way of approaching the problem is to redefine a new N-1 dimensional forward polyphase transform as

$$\boldsymbol{p}_f^{\overline{N-1}} = Q\boldsymbol{p}_f^{N-1}. \qquad 4.16$$

Then the regularity condition reduces to

$$\frac{\partial^{k-1}\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})\boldsymbol{p}_f^{\overline{N-1}}}{\partial^{t_1}z_1\partial^{t_2}z_2......\partial^{t_{n-1}}z_{n-1}}\Big|_{(-1,-1,...-1)} = 0. \qquad 4.17$$

Thus for some forward polyphase transfer function $\boldsymbol{p}_f^{\overline{N-1}}$**,** the n-1 dimensional filter satisfies the regularity condition. Because $Q=U_N D_N T$ is a nonsingular unimodular matrix at $z_n=-1$ the transformation $\boldsymbol{p}_f^{\overline{N-1}} = Q\boldsymbol{p}_f^{N-1}$ is one-to-one. Thus if $\boldsymbol{p}_f^{\overline{N-1}}$, $\boldsymbol{p}_f^{N-1}$ are elements of linear spaces L, L^* respectively then these linear spaces are isomorphic. And if $\boldsymbol{p}_f^{\overline{N-1}} = Q\boldsymbol{p}_f^{N-1}$ is an element of the null space of H_p^{N-1} so is $\boldsymbol{p}_f^{N-1}$ and by the same reasoning so is $\boldsymbol{p}_f^{N}$. Thus some of the zeros of H_p^{N} are shown also to be zeros of H_p^{N-1} [39].

An alternative approach to show the above relation would be to use the concepts of matrix fraction description of a polynomial transfer function $H(z)$ [35]. A rational polynomial matrix can be written as

$$H(z) = \frac{N(z)}{d(z)}. \qquad 4.18$$

The transfer function $H(z)$ can be defined using right or left matrix fraction descriptions.

$$H(z) = N(z)D_R^{-1}(z) \text{ or } D_L^{-1}(z)N(z). \qquad 4.19$$

Let us now consider the following two definitions which are the result of generalizations of 2-*D* polynomial matrices to the *n-D* cases by Z. Lin [48], [49].

Definition 5.1:

Let $A(z) \in C^{mxl}[z]$ be full normal rank, with $m \geq l$. Then $A(z)$ is said to be primitive in $C[z_1][z_2,...z_n]$ if for all fixed $z_{11} \in C$, $A(z_{11}, z_2, ...z_n) \in C^{mxl}[z_1,z_2,...z_n]$ is of full normal rank.

Definition 5.2:

Let $A(z) \in C^{mxl}[z]$ be of full normal rank, with $m \geq l$; $A(z) \in C[z]$ the g.c.d. of all the lxl minors of $A(z)$ and $g(z_1) \in C[z_1]$ the content of $A(z)$. We say that $A(z)$ has a primitive factorization in $C[z_1][z_2,...z_n]$ if $A(z) = L(z)R(z)$ for some $L(z) \in C^{mxl}[z]$, $R(z) \in C^{mxl}[z]$ with det $R(z) = g(z_1)$, and $L(z)$ being primitive in $C[z_1][z_2,..z_n]$.

The existence of the primitive factorization implies the uniqueness of $L(z)$ modulo a right unimodular matrix and $R(z)$ modulo a left unimodular matrix. Hence the invariant polynomials of $L(z)$ and $R(z)$ that determine the poles and zeros of $A(z)$ are not affected by unimodular transformations.

Let us now consider the following factor in the scaling filter

$$A(z) = H_p^{N-1}(z_{(n-1)}^{D'})Q(z_n) \qquad 4.20$$

where $L(z) = H_p^{N-1}(z_{(n-1)}^{D'})$ and $R(z) = Q(z_n)$.

By construction $A(z) = L(z)R(z)$ satisfies the condition for existence of primitive factorization and hence the finite poles and zeros of $A(z)$ are unchanged by the unimodular transformation. We know that $Q(z_n)$ is unimodular when evaluated at $z_n=-1$ and let

$$P^{-1} = Q(z_n)\big|_{z_n=-1} \qquad 4.21$$

$$A^{*}(z) = H_p^{N-1}(z_{(n-1)}^{D'})P.Q(z_n).$$

$A^{*}(z)$ is equivalent to $A(z)$ and substituting in equation 4.15 we get an equivalent relation

$$\frac{\partial^{k-1}\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'})\boldsymbol{p}_f^{N-1}}{\partial^{t_1}z_1\partial^{t_2}z_2......\partial^{t_{n-1}}z_{n-1}}\Big|_{(-1,-1,...-1)} = 0 \qquad 4.22$$

$$\text{where } \sum_{i=1}^{n-1} t_i = k\text{-}1,\ t_i = 0..k\text{-}1.$$

Thus some of the zeros of the higher dimensional filers are shown to be zeros of the lower dimensional filters [6]. This result has significant importance in devising methods for the construction of higher dimensional regular filters from lower dimensional ones.

4.3 TWO-CHANNEL CASE:

Here we restrict our discussion to uniform band, maximally sampled filter banks, in which all the channels share the same downsampling matrix D with determinant equal to two. These sampling matrix are nonseparable and reduce to quincunx and Face-Centered Orthorhombic (FCO) cases in two and three dimensions respectively. The analysis filters can be written as

$$H(\mathbf{z}) = H_p(\mathbf{z}).\boldsymbol{p}_f \qquad 4.23$$

where $H(\mathbf{z}) = [H_0(\mathbf{z})\ ,\ H_1(\mathbf{z})]^t$

$\boldsymbol{p}_f = [\mathbf{z}^{\mathbf{k}_0}\ \mathbf{z}^{\mathbf{k}_1}]^t$, $\boldsymbol{k}_i$ = coset vectors.

The perfect reconstruction filter bank with FIR analysis and synthesis filter has to satisfy the condition

$$det(H_p(\mathbf{z})) = c\mathbf{z}^{\mathbf{d}} \qquad 4.24$$

where $\boldsymbol{d}$ is a delay vector.

For the two channel n-dimensional case the following cascade will produce perfect reconstruction set containing two filters of the same size

$$H_p(\mathbf{z}) = U_0 \prod_{j=1}^{k} \prod_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & z_i^{-1} \end{pmatrix} U_{ij} \qquad 4.25$$

U_{ij} is unitary for the orthogonal case, i.e.

$$U_{i1,2} = \frac{1}{\sqrt{1+a_{i1,2}^2}} \begin{pmatrix} 1 & a_{i1,2} \\ -a_{i1,2} & 1 \end{pmatrix}$$

and

U_{ij} is persymmetric for the linear phase case i.e.

$$U_{i1,2} = \frac{1}{\sqrt{1+a_{i1,2}^2}} \begin{pmatrix} 1 & a_{i1,2} \\ a_{i1,2} & 1 \end{pmatrix}.$$

For the smallest size filters this reduces to

$$H_p(\mathbf{z}) = U_0 \prod_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & z_i^{-1} \end{pmatrix} U_i, \qquad 4.26$$

where U_i equals unitary for orthogonal and persymmetric for the linear phase case and the polyphase components of the analysis filters are given by

$$
\begin{aligned}
H(\mathbf{z}) &= H_p(\mathbf{z}^D).\begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix} \\
&= U_0 \prod_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & z_i^{-d_i} \end{pmatrix} U_i. \begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix} \\
&= \begin{pmatrix} 1 & a_0 \\ -a_0 & 1 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} 1 & a_i \\ -a_i z_i^{-d_i} & z_i^{-d_i} \end{pmatrix} . \begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix}.
\end{aligned}
\qquad 4.27
$$

The scaling filter H_0 is given by the first row of the above vector and the above regularity condition is defined by

$$
\frac{\partial^{k-1} H_0}{\partial^{t_1} z_1 \partial^{t_2} z_2 \ldots\ldots \partial^{t_{n-1}} z_{n-1}} \Big|_{(-1,-1,\ldots-1)} = 0 \qquad 4.28
$$

$$
\frac{\partial^{k-1} I_0 H(\mathbf{z})}{\partial^{t_1} z_1 \partial^{t_2} z_2 \ldots\ldots \partial^{t_{n-1}} z_{n-1}} \Big|_{(-1,-1,\ldots-1)} = 0.
$$

As stated above some of the zeros of the higher dimensional filters constitute also the zeros of the lower dimensional filters. This is illustrated by the following examples.

4.3.1 EXAMPLE:

Let us examine the quincunx and FCO cases for the sampling matrices given in [42].

$$
D_q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad D_{FCO} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}. \qquad 4.29
$$

For the above given sampling matrix D_q the following 2-D polyphase cascade matrix is obtained.

$$H_p(z_1, z_2) = \left[\prod_{i=1}^{k} \begin{pmatrix} 1 & 0 \\ 0 & z_1 \end{pmatrix} U_{1_i} \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix} U_{2_i} \right] U_0 \qquad 4.30$$

and for the orthogonal filters U_{j_i} have to be unitary i.e.

$$U_{j_i} = \frac{1}{\sqrt{1+a_i^2}} \begin{pmatrix} 1 & -a_i \\ a_i & 1 \end{pmatrix}.$$

For the case of the smallest size filter we obtain

$$\begin{aligned} H_p(z_1,z_2) &= U_2 D_2 U_1 D_1 U_0 \qquad 4.31 \\ &= d. \begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix} .. \\ &.. \begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix}, \end{aligned}$$

where $d = \frac{1}{\sqrt{\prod_{i=0}^{2}(1+a_i^2)}}$.

The polyphase components are given by

$$\begin{aligned} H(\mathbf{z}) = d. &\begin{pmatrix} 1 & a_0 \\ -a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-d_1} \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-d_2} \end{pmatrix} .. \\ &.. \begin{pmatrix} 1 & a_2 \\ -a_2 & 1 \end{pmatrix} \begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix} \qquad 4.32 \end{aligned}$$

where $\boldsymbol{p}_f = \begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix} = \begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix}$.

The 2nd order regularity condition is given by

$$\frac{\partial^1 \boldsymbol{I}_0^t H(\mathbf{z})}{\partial^{t_1} z_1 \partial^{t_2} z_2}\Big|_{(-1,-1)} = 0 \qquad 4.33$$

$$\text{where } \sum_i^2 t_i = 1,\ t_i = 0..1,$$

and the zeros of H_0 determined by the direct algebraic method are given by

$$\begin{array}{lll} a_0 = -2 \mp \sqrt{3} & a_1 = \pm\sqrt{3} & a_2 = \pm\sqrt{3} \\ a_0 = -2 \pm \sqrt{3} & a_1 = 0 & a_2 = \pm\sqrt{3}. \end{array} \qquad 4.34$$

For FCO sampling matrix D_{FCO} the cascade structure generates the following filters

$$H_p(z_1,z_2,z_3) = U_0 \left[\prod_{i=1}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} U_{1_i} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} U_{2_i} \begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix} U_{3_i} \right]. \qquad 4.35$$

Again the orthogonal filters U_{j_i} have to be unitary

$$U_i = \frac{1}{\sqrt{1+a_i^2}} \begin{pmatrix} 1 & -a_i \\ a_i & 1 \end{pmatrix}.$$

And for the smallest size filters we get

$$\begin{aligned} H_p(z_1,z_2,z_3) = d. &\begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \ldots \\ &.. \begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}, \end{aligned} \qquad 4.36$$

where $\quad d = \dfrac{1}{\sqrt{\prod_{i=0}^{3}(1+a_i^2)}}.$

The polyphase components are given by

$$H(\mathbf{z}) = d.\begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_1} \end{pmatrix}\begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_2} \end{pmatrix}\cdots$$
$$\cdots\begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_3} \end{pmatrix}\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}\begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix}. \qquad 4.37$$

The 2nd order regularity condition for the 3-*D* FCO is given by

$$\frac{\partial^1 \mathbf{I}_0^t H}{\partial^{t_1} z_1 \partial^{t_2} z_2 \partial^{t_3} z_3}\Big|_{(-1,-1,-1)} = 0 \qquad 4.38$$
$$\text{where } \sum_i^3 t_i = 1,\ t_i = 0..1 \quad .$$

The solutions to the above equation determined by direct algebraic method are given by

$$a_0=0 \quad a_1= -2\pm\sqrt{3} \quad a_2=0 \quad a_3=\pm\sqrt{3} \qquad 4.39$$

$$a_0=0 \quad a_1=-2\mp\sqrt{3} \quad a_2=\mp\sqrt{3} \quad a_3=\pm\sqrt{3}.$$

Some of the solutions are the same as those given above for the 2-*D* case. This relation between the 3-*D* and 2-*D* zeros can be reestablished using the result and method developed above.

First, we know the sampling pattern is not affected by the unimodular transformation of the sampling matrix D_{FCO}.

$$D'_{FCO} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & -4 \\ 1 & 1 & 0 \end{pmatrix} \qquad 4.40$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} D_q & 0 \\ 0 & 1 \end{pmatrix}.$$

Rewriting the polyphase components of the 3-D filters with the new sampling matrix we get

$$H(\mathbf{z}) = d.\begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_1} \end{pmatrix}\begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_2} \end{pmatrix}.. \qquad 4.41$$

$$..\begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-d_3} \end{pmatrix}\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}\begin{bmatrix} \mathbf{z}^{\mathbf{k}_0} \\ \mathbf{z}^{\mathbf{k}_1} \end{bmatrix}$$

$$= H_p^{(N-1)}(z^{D_q})d_3.\begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix}\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix}$$

where $d_3 = \frac{1}{\sqrt{(1+a_3^2)}}$.

The scaling filter H_0 is given by

$$H_0 = I_0 H(\mathbf{z}) \qquad 4.42$$

$$= I_0 H_p^{(2)}(z^{D_q})d_3.\begin{pmatrix} 1 & 0 \\ 0 & z_3^{-1} \end{pmatrix}\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix}.$$

The zeros of H_0 on the constant hyperplane of z_3= -1 is given by the zeros of

$$H_0 = I_0 H_p^{(2)}(z^{D_q})d_3.\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix} \qquad 4.43$$

$$= I_0 H_p^{(2)}(z^{D_q}) d_3 . \begin{pmatrix} 1 & -a_3 \\ -a_3 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix} .$$

The regularity condition can then be written as

$$\frac{\partial^{k-1} H_0}{\partial^{t_1} z_1 \partial^{t_2} z_2}\Big|_{(-1,-1)} = 0 \qquad 4.44$$

$$\frac{\partial^{k-1} \boldsymbol{I}_0 H_p^{(2)}(z_{(2)}^{D_q}) . Q . \boldsymbol{p}_f}{\partial^{t_1} z_1 \partial^{t_2} z_2} . \Big|_{(-1,-1)} = 0$$

$$\text{where } \; Q = d_3 . \begin{pmatrix} 1 & -a_3 \\ -a_3 & -1 \end{pmatrix},$$

$$\sum_i^2 t_i = k\text{-}1,\; t_i = 0, \ldots, k\text{-}1.$$

Because Q is unimodular and $\boldsymbol{p}_f^{N-1}$ equals $\boldsymbol{p}_f$, this will reduce to the following equation according to the above derived relationship.

$$\frac{\partial^{k-1} \boldsymbol{I}_0 H_p^{(2)}(z_{(2)}^{D_q}) \boldsymbol{p}_f}{\partial^{t_1} z_1 \partial^{t_2} z_2}\Big|_{(-1,-1)} = 0 \qquad 4.45$$

$$\text{where } \sum_i^2 t_i = k\text{-}1,\; t_i = 0..k\text{-}1.$$

For the 2nd order regularity condition

$$\frac{\partial^{1} \boldsymbol{I}_0 H_p^{(2)}(z_{(2)}^{D_q}) \boldsymbol{p}_f}{\partial^{t_1} z_1 \partial^{t_2} z_2}\Big|_{(-1,-1)} = 0 \qquad 4.46$$

$$\text{where } \sum_i^2 t_i = 1,\; t_i = 0..1.$$

This is identical to equation (4.43), which is the regularity condition for the quincunx case above. Some of the zeros of the the higher dimensional FCO filters are shown to be the same as the zeros of the lower dimensional quincunx filters.

In fact, the 2nd order zeros determined directly form equation (4.45) above for a_3 arbitrary are given by

$$a_0 = 2 \pm \sqrt{3} \quad a_1 = 0 \quad a_2 = -\frac{4a_3 - 2a_3^2 \pm \sqrt{3} + a_3^2(2 \pm \sqrt{3})}{-1+3a_3^2} \quad a_3 = a_3 \qquad 4.47$$
$$a_0 = 2 \pm \sqrt{3} \quad a_1 = \pm\sqrt{3} \quad a_2 = \frac{-2a_3^2 - 4a_3 \pm \sqrt{3} + a_3^2(2 \pm \sqrt{3})}{-1+3a_3^2} \quad a_3 = a_3.$$

And for values of a_3 away from the singular values of a_2, we obtain identical values as those determined algebraically for the 2-*D* quincunx case above.

4.4 CONSTRUCTION OF REGULAR CASCADE FILTERS:

I proved above that some of the zeros of the higher dimensional filters are also zeros of the lower dimensional filters. I try to exploit this relation to build higher-dimensional regular filters from lower-dimensional regular filters. Starting with lower dimensional filters we will build higher dimensional regular filers recursively. In so doing we will be able to reduce the number of undetermined coefficients and the complexity of the algebraic solution. Let us assume that the zeros of the lower-dimensional filters have been determined using the relation (4.14). Hence the coefficients $a_0 \ldots a_{n-1}$ are determined. Substituting these values into equation (4.13) the only undetermined coefficients will be a_n. But

$$\frac{\partial^{k-1} \boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'}) U_n D_n(z_n^{d_{nn}}) T(z_n) \boldsymbol{p}_f^{N-1}}{\partial^{t_1} z_1 \partial^{t_2} z_2 \ldots\ldots \partial^{t_n} z_n} \Big|_{(-1,-1,\ldots,-1)} = 0 \qquad 4.48$$

and if we assume $t_1 \ldots t_{n-1} = 0$, then this will reduce to

$$\frac{\partial^{k-1} \boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'}) U_n D_n(z_n^{d_{nn}}) T(z_n) \boldsymbol{p}_f^{N-1}}{\partial^{t_n} z_n} \Big|_{(-1,-1,\ldots,-1)} = 0 \qquad 4.49$$
$$\frac{\boldsymbol{I}_0^t H_p^{N-1}(z_{(n-1)}^{D'}) \partial^{k-1} U_n D_n(z_n^{d_{nn}}) T(z_n) \boldsymbol{p}_f^{N-1}}{\partial^{t_n} z_n} \Big|_{(-1,-1,\ldots,-1)} = 0$$

where $t_n = 0 \ldots k-1$.

The above equations will be sufficient to solve a_n and hence complete the determination of all of the coefficients of the scaling filter.

4.4.1 EXAMPLE

Let us construct 3-*D* scaling filter for the FCO sampling matrix using the above technique. Since the determinant of the FCO sampling matrix is 2 the submatrices considered at every stage of the construction must have the also the same value. We start with the single dimensional two channel filter with sampling rate equal to 2.

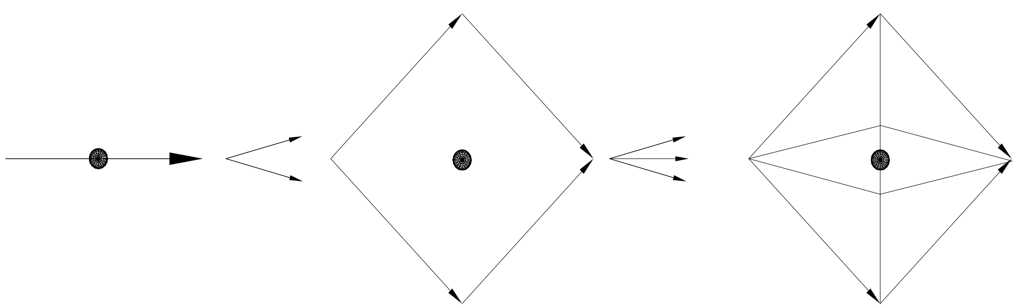

Figure 4.1: Extension From One To Multiple Dimension

$$
\begin{aligned}
H_p &= U_0 D_1 U_1 \\
&= \begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix}
\end{aligned} \qquad 4.50
$$

$$
\begin{aligned}
H_0 &= I_0^t H_p(z_1^2) \boldsymbol{p}_f \\
&= I_0^t \begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-2} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z_1^{-1} \end{pmatrix}
\end{aligned}
$$

where $p_f = [1\ z_1^{-1}]^t$.

The 2nd order regularity condition is given by

$$\frac{\partial^{k-1} H_0}{\partial^{t_1} z_1}\Big|_{(z_1=-1)} = 0 \qquad 4.51$$

where $t_i = 0 \ldots 1 \qquad k = 1 \ldots 2.$

The solution to this problem is given by

$$a_0 = -2 \pm \sqrt{3} \qquad a_1 = \pm\sqrt{3}\,. \qquad 4.52$$

Let us then consider the two-dimensional case with the submatrix

$$D_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \qquad 4.53$$

The polyphase matrix for the two dimensional case will be

$$\begin{aligned} H_p &= U_0 D_1 U_1 D_2 U_2 \qquad 4.54 \\ &= \begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix}, \end{aligned}$$

and the scaling filter will be

$$\begin{aligned} H_0 &= I_0^t H_p(z_1^2) \boldsymbol{p}_f \qquad 4.55 \\ &= I_0^t \begin{pmatrix} 1 & -a_0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-2} \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} .. \\ &.. \begin{pmatrix} 1 & -a_2 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z_1^{-1} \end{pmatrix}, \end{aligned}$$

where $p_f = [1 \; z_1^{-1}]^t$.

The 2nd order regularity condition

$$\frac{\partial^{k-1} H_0}{\partial^{t_1} z_1 \partial^{t_2} z_2}\Big|_{(z_1=-1, z_2=-1)} = 0 \qquad 4.56$$

$$\text{where } \sum_{i}^{2} t_i = 1,\ t_i = 0..1\ \ k = 1..2.$$

The solution to the above given the coefficients $a_0$and a_1 will be

$$a_0 = -2 \pm \sqrt{3} \qquad a_1 = \pm\sqrt{3} \qquad a_2 = 0 \qquad 4.57$$

$$a_0 = -2 \mp \sqrt{3} \qquad a_1 = \pm\sqrt{3} \qquad a_2 = \pm\sqrt{3}\,.$$

And finally the solution to 3-*D* filter coefficients using the above technique and previously determined coefficients is

$$a_0 = -2 \pm \sqrt{3} \quad a_1 = \pm\sqrt{3} \quad a_2 = 0 \qquad a_3 = 2 \pm \sqrt{3} \qquad 4.58$$

$$a_0 = -2 \mp \sqrt{3} \quad a_1 = \pm\sqrt{3} \quad a_2 = \pm\sqrt{3} \quad a_3 = 2 \pm \sqrt{3}.$$

The solution above is identical to that determined by directly solving the equation below.

$$\frac{\partial^{k-1} H_0}{\partial^{t_1} z_1 \partial^{t_2} z_2 \partial^{t_2} z_2}\Big|_{(z_1=-1, z_2=-1, z_3=-1)} = 0 \qquad 4.59$$

$$\text{where } \sum_{i}^{3} t_i = 1,\ t_i = 0..1\ \ k = 1..2.$$

As you may observe the number of solutions increases as the dimensions increase.

4.5 SUMMARY:

The work shows that some of the zeros of higher dimensional cascade filters are related to the zeros of their lower dimensional cascade component filters. This relation is exploited to develop a method for the construction of multidimensional regular filter banks recursively. Starting form lower dimensional regular filters it is possible to build higher dimensional filters with the same degree of regularity. In the next chapter I will utilize these relation to outline practical procedure for the recursive construction of regular multidimensional filters and compactly supported multidimensional wavelets.

CHAPTER 5

DESIGN OF MULTIDIMENSIONAL NON-SEPARABLE WAVELETS

5.1 INTRODUCTION:

The wavelet theory and its connection to multirate filter banks was first described for the one-dimensional case by Mallat and Daubechies et al. in [21], [53], [54]. The perfect reconstruction filter banks computes the discrete-time wavelet transform when the branch with the low-pass filter is iterated. It was also shown that continuous wavelet bases could be obtained by iterating regular low pass filters. The same basic ideas can be applied in multiple dimensions with some major modifications. In the one-dimensional case there are several techniques to design regular filter banks and in the last chapter I have discussed techniques for the design of multidimensional regular filter banks. The final aim is to construct continuous multidimensional wavelet bases by iterating these regular multidimensional filter banks. Here in this chapter I discuss the methods for the construction of multidimensional wavelet in general and the iterated filter bank method in particular.

5.2 TENSOR PRODUCT OF ONE DIMENSIONAL BASIS:

5.2.1 TENSOR PRODUCT OF ONE DIMENSIONAL WAVELETS

The most trivial way of constructing multidimensional wavelet basis for $L^2(R^n)$, is simply to take the tensor product functions generated by n one-dimensional bases [20]. Let the n orthonormal wavelet bases be defined as

$$\psi_{j,k}(x) = 2^{-\frac{j}{2}}\psi(2^{-j}x\text{-}k) \in L^2(R). \qquad 5.1$$

Then the tensor product wavelet is given by

$$\psi_{j_1,k_1;j_2,k_2;\ldots\ldots j_n,k_n}(x_1,x_2,\ldots.x_n) = \psi_{j_1,k_1}\psi_{j_2,k_2}\ldots\ldots\ldots\psi_{j_n,k_n.} \qquad 5.2$$

The resulting functions are wavelets, and an orthonormal basis for $L^2(R^n)$. In this basis, the variables $x_1, x_2,\ldots,x_n$ are dilated separately.

5.2.2 TENSOR PRODUCT OF ONE DIMENSIONAL MRA:

Another construction is one in which the dilation of the resulting orthonormal wavelet basis control all the variables simultaneously [20]. In this construction the tensor product of n one-dimensoinal multiresolution analysis is considered. Define the space $\boldsymbol{V}_j$, $j \in Z$, by

$$\begin{aligned} \boldsymbol{V}_0 &= V_0 \otimes V_0\ldots \oplus V_0 \\ &= \overline{span\ F(x_1,x_2,\ldots,x_n)} \\ &= \overline{f_1(x_1)f_2(x_2)\ldots f_n(x_n)}\ \ f_1, f_2,\ldots,f_n \in V_0 \end{aligned} \qquad 5.3$$

and

$$F \in \boldsymbol{V}_j \Leftrightarrow F(2^j.,2^j.,\ldots.2^j) \in \boldsymbol{V}_0. \qquad 5.4$$

Then the $\boldsymbol{V}_j$ form a multiresolution ladder in $L^2(R^n)$ and satisfy the following conditions:

$$....\boldsymbol{V}_2 \subset \boldsymbol{V}_1 \subset \boldsymbol{V}_0 \subset \boldsymbol{V}_{-1} \subset \boldsymbol{V}_{-2}... \qquad 5.5$$

$$\bigcap_{j \epsilon Z} \boldsymbol{V}_j = \phi$$

$$\overline{\bigcup_{j \epsilon Z} \boldsymbol{V}_j} = L^2(R^n).$$

Because the $\phi(.,-\boldsymbol{n})$, $n \epsilon Z$ constitute an orthonormal basis for V_0, the product functions

$$\boldsymbol{\Phi}_{0;n_1,n_2,...n_n}(x_1,x_2,....x_n) = \phi(x_1-n_1)\ \phi(x_2-n_2)...\phi(x_n-n_n) \qquad 5.6$$

$$n_1,n_2,..n_n \epsilon Z,$$

constitute an orthonormal basis for $\boldsymbol{V}_0$, generated by Z^n-translations of a single function $\boldsymbol{\Phi}$. Similarly, the

$$\boldsymbol{\Phi}_{0;n_1,n_2,...n_n}(x_1,x_2,....x_n) = \phi_{j,n_1}\phi_{j,n_2}....\phi_{j,n_n} \qquad 5.7$$

$$n_1,n_2,..n_n \epsilon Z$$

constitute an orthonormal basis for $\boldsymbol{V}_j$. As in the one-dimensional case, I define for each $j \epsilon Z$, the complement space $\boldsymbol{W}_j$ to be the orthogonal complement in $\boldsymbol{V}_{j-1}$ of $\boldsymbol{V}_j$ i.e.

$$\boldsymbol{V}_{j-1} = \boldsymbol{V}_j \oplus \boldsymbol{W}_j \qquad 5.8$$

and

$$\overline{\bigoplus_{j \epsilon Z} \boldsymbol{W}_j} = L^2(R^n)\Phi\Phi. \qquad 5.9$$

The space $\boldsymbol{W_j}$ will be represented by 2^n-1 orthonormal wavelet bases.

5.3 GENERAL MRA CONSTRUCTION:

One can also consider the case in which one starts from an n-dimensional multiresolution analysis in which V_0 is not a tensor product of n one-dimensional V_0-spaces [20]. The multiresolution structure of the V_j implies that the corresponding scaling function Φ satisfies

$$\Phi(x_1,x_2,....x_n) = \sum_{n_1,n_2,..n_n} h_{n_1,n_2,...n_n} \Phi(2x_1-n_1,2x_2-n_2...2x_n-n_n) \qquad 5.10$$

for some sequence h_n, $n \in Z^n$. The orthonormality of the $\Phi_{\{0;n\}}$ forces the trigonometric polynomial

$$m_0(\xi_1,\xi_2,...\xi_n) = \frac{1}{2}\sum_{n_1,n_2,..n_n} h_{n_1,n_2,...n_n} e^{-(n_1\xi_1+n_2\xi_2+....+n_n\xi_n)} \qquad 5.11$$

to satisfy

$$|m_0(\xi_1,\xi_2,...\xi_n)|^2 + |m_0(\xi_1+\pi,\xi_2,...\xi_n)|^2 .. + |m_0(\xi_1+\pi,\xi_2+\pi,...\xi_n+\pi)|^2 = 1. \qquad 5.12$$

To construct an orthonormal basis of wavelets corresponding to this multiresolution analysis, one has to find 2^n-1 wavelets in V_{-1}, orthogonal to V_0 and to each other. The integer translates of each should also be orthonormal. This implies

$$\widehat{\Psi}^{\lambda}(\xi_1,\xi_2,...\xi_n) = m_\lambda(\xi_1/2,\xi_2/2,...\xi_n/2)\, \widehat{\Phi}(\xi_1/2,\xi_2/2,...\xi_n/2), \qquad 5.13$$

where $\lambda \epsilon (1....2^n-1)$

and the $2^n * 2^n$-dimensional matrix

$$U_{r,s}(\xi_1,\xi_2,...\xi_n) = m_{r-1}(\xi_1+s_1\pi,\xi_2+s_2\pi,...\xi_n+s_n\pi) \quad 5.14$$

is unitary with $r = 1,...,2^n$, and $s = (s_1,s_2,...,s_n) \epsilon \{0,1\}^n$.

Using the polyphase decomposition the m_r's can be defined as

$$2^{n/2}m_r(\xi_1,\xi_2,...\xi_n) = \sum_{s\epsilon(0,1)^n} e^{-i(s_1\xi_1+s_2\xi_2+.....s_n\xi_n)}m_{r,s}(2\xi_1,2\xi_2,...2\xi_n) \quad 5.15$$

and the unitary of is equivalent to the unitary of the polyphase matrix defined by

$$\widetilde{U}_{r,s}(\xi_1,\xi_2,...\xi_n) = m_{r-1,s}(\xi_1,\xi_2,...\xi_n). \quad 5.16$$

The construction therefore requires the determination of $m_1,m_2,...,m_n$ given m_0 such that the above matrix is unitary. These constructions cannot force compact support for the Ψ^j. Even if m_0 is trigonometric polynomial the m_j's may not necessarily be.

5.4 WAVELET BASES FROM ITERATED MULTIDIMENSIONAL FILTER BANKS:

In one dimension, the lowpass filter of a PRFB is iterated to obtain the discrete wavelet transform and given regular filters continuous wavelet transforms are generated. In multiple dimensions, the basic idea remains the same except the dilation matrix is used instead of the dilation factors[44].

Let us consider the equivalent filter that corresponds to the iteration of lowpass filters as in Figure 5.1. Subsampling by D followed by filtering with $H(z)$ is equivalent to filtering with $H(z^D)$ followed by the subsampling by D. Hence, the i stages of lowpass filtering and subsampling by D as shown in the figure below will be equivalent to $H^i(z)$ followed by subsampling by D^i. This equivalent filter is given by

$$H^i(\mathbf{z}) = \prod_{i=0}^{i-1} H(\mathbf{z}^{D^i}) \qquad i=1,2,... \qquad 5.17$$

Or in the frequency domain

$$\widehat{H}^{(i)}(\omega) = \prod_{k=0}^{i-1} \widehat{H}_0((D^t)^i\omega) \qquad i=1,2,..., \qquad 5.18$$

where $\widehat{H}^{(0)}(\omega) = 1$.

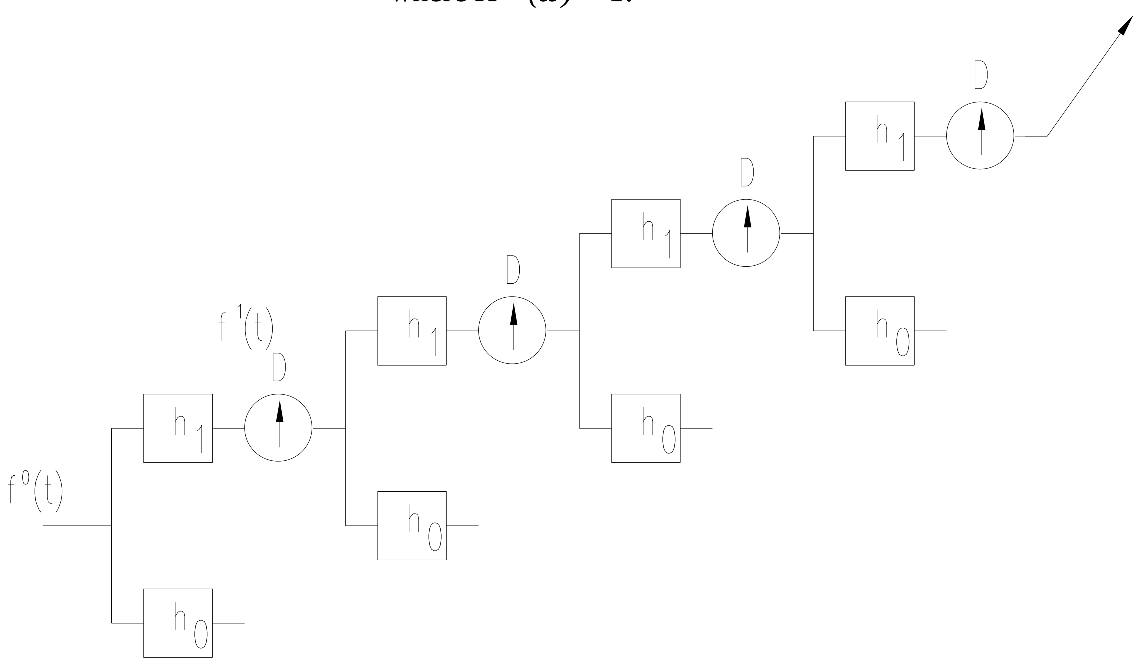

Figure 5.1: Multidimensional Iterated Filter Bank

Let the function $h^{(i)}(n)$ be the impulse response of the iterated filter $\widehat{H}^{(i)}(\omega)$ and, as i increases infinitely, the filter becomes infinitely long. Instead let us consider the function $f^{(i)}$ defined as follows

$$f^{(i)}(\boldsymbol{t}) = N^{\frac{i}{2}}.h^{(i)}(\boldsymbol{n}) \qquad D^i\boldsymbol{t} \in \boldsymbol{n}+[-\tfrac{1}{2}, \tfrac{1}{2})^{\boldsymbol{n}} \tag{5.19}$$

$f^{(0)}(t)$ is the indicator function over the hypercube $[-\frac{1}{2}, \frac{1}{2})^{\boldsymbol{n}}$. The shape of the hyper-regions is determined by the shape of the unit cell belonging to the lattice D^{-1}. As in the one-dimensional case the interest is in the limiting behavior of the iterated function. We have already discussed the regularity of the iterated filter and assume here that the limit $f^{(i)}(\boldsymbol{t})$ exists and is in L^2. This limit is defined as the scaling function associated with the discrete filter $h_0(\boldsymbol{n})$

$$\phi(\boldsymbol{t}) = \lim_{i\to\infty} f^{(i)}(\boldsymbol{t}), \qquad \phi(\boldsymbol{t}) \in L^2. \tag{5.20}$$

The equivalent filter after i steps in terms of the equivalent filter after $(i-1)$ steps will be

$$h^{(i)}(\boldsymbol{n}) = \sum_{\boldsymbol{k}} h_0(\boldsymbol{k}) h^{(i-1)}(\boldsymbol{n}-D^{i-1}\boldsymbol{k}). \tag{5.21}$$

In terms of the iterated functions this becomes

$$h^{(i)}(\boldsymbol{n}) = N^{-\frac{i}{2}}.f^{(i)}(\boldsymbol{t}) \tag{5.22}$$

and

$$h^{(i-1)}(\boldsymbol{n}-D^{i-1}\boldsymbol{k}) = N^{-\frac{i-1}{2}}.f^{(i-1)}(D\boldsymbol{t}-\boldsymbol{k}),$$

for $D^i\boldsymbol{t} \in \boldsymbol{n} + [-\frac{1}{2}, \frac{1}{2})^{\boldsymbol{n}}$,

with some substitution this reduces to

$$f^{(i)}(\boldsymbol{t}) = \sqrt{N}\sum_{\boldsymbol{k}} h_0(\boldsymbol{k}) f^{(i-1)}(D\boldsymbol{t}-\boldsymbol{k}). \qquad 5.23$$

With the assumption that $f^{(i)}(\boldsymbol{t})$ converges to the scaling function and taking the limit of the above equation leads to

$$\phi(\boldsymbol{t}) = \sqrt{N}\sum_{\boldsymbol{k}} h_0(\boldsymbol{k})\phi(D\boldsymbol{t}-\boldsymbol{k}). \qquad 5.24$$

The scaling function satisfies the two-scale equation and $(N-1)$ wavelets are defined

$$\psi(\boldsymbol{t}) = \sqrt{N}\sum_{k} h_i(\boldsymbol{k})\phi(D\boldsymbol{t}-\boldsymbol{k}), \qquad 5.25$$

where $i = 1,2, ...,N-1$.

The orthogonality relations between filters and their translates with respect to the sampling lattice have been outlined previously. The same kind of relationships for the scaling and the N-1 wavelet functions can be drawn.

1. $<\Phi(\boldsymbol{t}), \Phi(\boldsymbol{t-1})> = \delta_l$ the scaling function is orthogonal to its integer translates.

2. $<\Phi(D^i\boldsymbol{t-1}), \Phi(D^i\boldsymbol{t-k})> = N^{-i}\ \delta_{lk}$ scaling function orthogonal to its integer translates at all scales.

3. $<\Psi_m(D^i\mathbf{t}\text{-}\mathbf{1})\ ,\ \Psi_n(D^i\mathbf{t}\text{-}\mathbf{k})> = N^{-i}\ \delta_{mn}\delta_{\mathbf{lk}}$ wavelets are orthogonal to each other and their integer translates

4. $<\Phi(\mathbf{t})\ ,\ \Psi_m(\mathbf{t}\text{-}\mathbf{l})> = 0$ the scaling function is orthogonal to each of the wavelets

5. $<\Psi_m(D^i\mathbf{t}\text{-}\mathbf{1})\ ,\ \Psi_n(D^j\mathbf{t}\text{-}\mathbf{k})> = N^{-i}\ \delta_{ij}\ \delta_{mn}\delta_{\mathbf{lk}}$ wavelets are orthogonal across scales.

5.5 DESIGN OF COMPACTLY SUPPORTED WAVELETS:

The design of multidimensional filters is generally difficult and the requirement of regularity makes it much more involved. This has been discussed in the last two chapters. The cascade structure has been the main focus because of its simplicity and ease in the synthesis of regular multidimensional filter banks. Here in this section, I outline practical procedures of constructing multidimensional wavelets using the theories developed in the last two chapters.

5.5.1 Direct Design

It is stated above that compactly supported multidimensional wavelets can be constructed by iterating regular multidimensional filter banks. In order to use this method, I need first to construct these regular multidimensional filters. The construction of these regular multidimensional filter banks can be accomplished by designing first single dimensional regular filter banks and extending them to multiple dimensions using concepts developed in the previous chapter.

The coefficients of nth order single dimensional orthogonal cascade filters can be solved using the nth order regularity condition.

$$H_p(z) \quad = U_0\prod_{j=1}^{k}\prod_{i=1}^{n}(D_jU_{ij}) \qquad 5.26$$

where $D_j = \begin{pmatrix} 1 & 0 \\ 0 & z_j \end{pmatrix}$ and $U_{ij} = \frac{1}{\sqrt{1+a_{ij}^2}} \begin{pmatrix} 1 & a_{ij} \\ -a_{ij} & 1 \end{pmatrix}$.

For $k=1$, the regularity condition reduces to

$$\frac{\partial^n I_0 H_p(z_1) p_f}{\partial z_1^n}\Big|_{(z_1=-1)} = 0. \qquad 5.27$$

The coefficients a_{ij} for $j=1$ are solved using the above equation. Based on these determined coefficients, the a_{ij} for $j=2$ are solved using the following regularity condition

$$\frac{\partial^n I_0 H_p(z_2) p_f}{\partial z_2^n}\Big|_{(z_2=-1)} = 0. \qquad 5.28$$

Hence using this procedure all the coefficients a_{ij} for $j=1\ldots k$ are solved recursively. The algebraic solution to these nonlinear equations becomes complex as the order n increases.

The solution for the one-dimensional second order system $(n=2)$ gives the same coefficients as Daubechie's four tap filter. The low and high pass filters generated by the orthogonal cascade structure as determined in chapter 4 are given by

$$h_l = [.4829629133\ .8365163043\ .224143869\ -.1294095222]$$

$$h_h = [.1294095222\ .224143869\ -.8365163043\ .4829629133]$$

The iteration of the above filters gives the scaling function for the construction of 2nd order Daubechie's wavelets shown below.

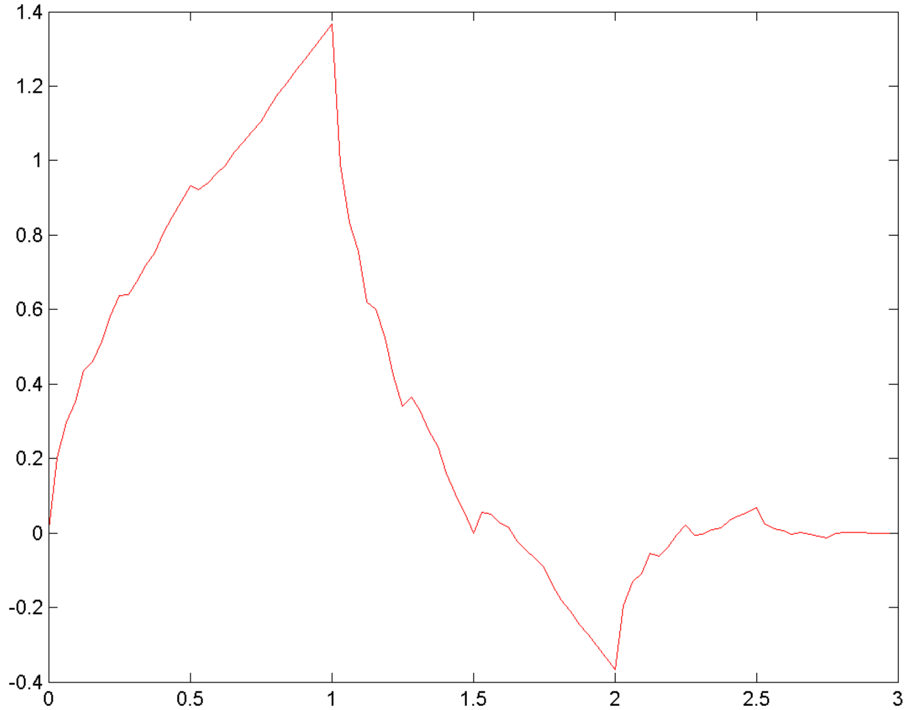

Figure. 5.2: Scaling Function

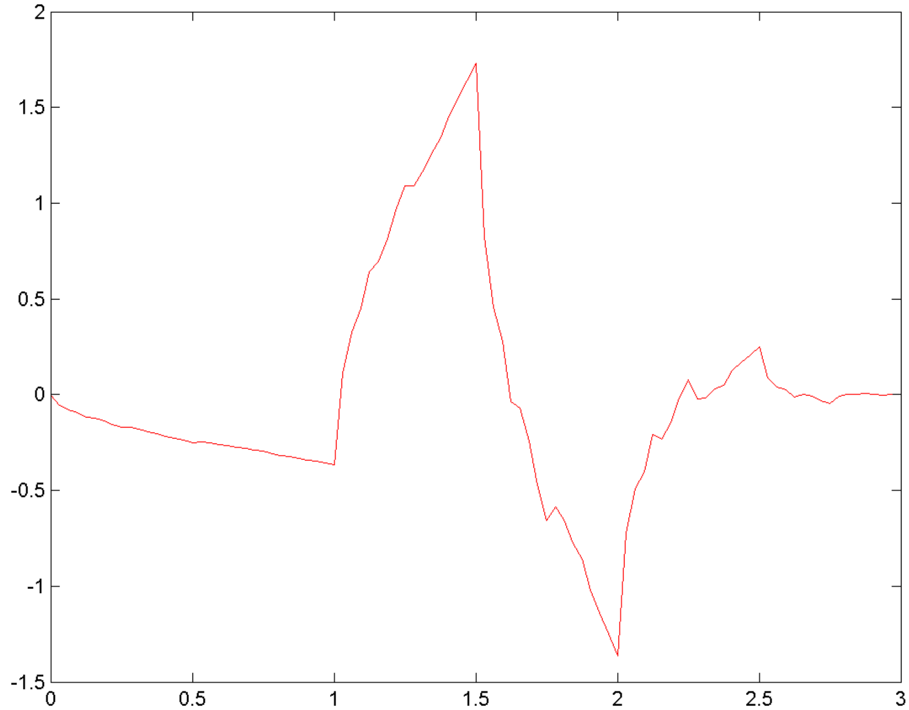

Figure. 5.3: Daubechie's Wavelet

The same methods developed in Chapters 3 and 4 are used to determine the higher-dimensional filter coefficients. The 2nd-order two-dimensional filter coefficients are given by

$$h_l = \begin{bmatrix} 0 & .1120719339 & -.1941142837 & 0 \\ .2414814568 & -.41825815 24 & -.1120719339 & -.06470476120 \\ 0 & .7244443705 & .4182581522 & 0 \end{bmatrix}$$

$$h_h = \begin{bmatrix} 0 & -.4182581522 & .7244443705 & 0 \\ .06470476120 & -.1120719339 & .4182581524 & .2414814568 \\ 0 & .1941142837 & .1120719339 & 0 \end{bmatrix}$$

The 2nd order two dimensional wavelet is constructed using the scaling function obtained by iterating the above filters. The scaling and wavelet functions are shown in the Figure below.

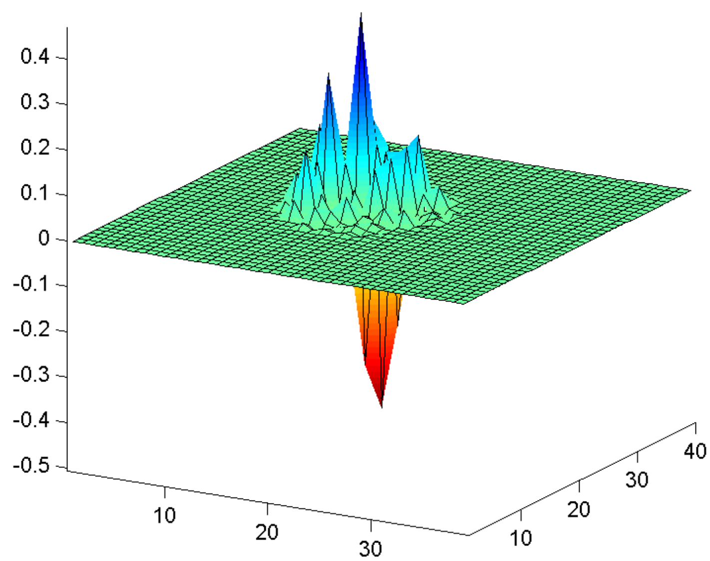

Figure. 5.4: 2-D Scaling function

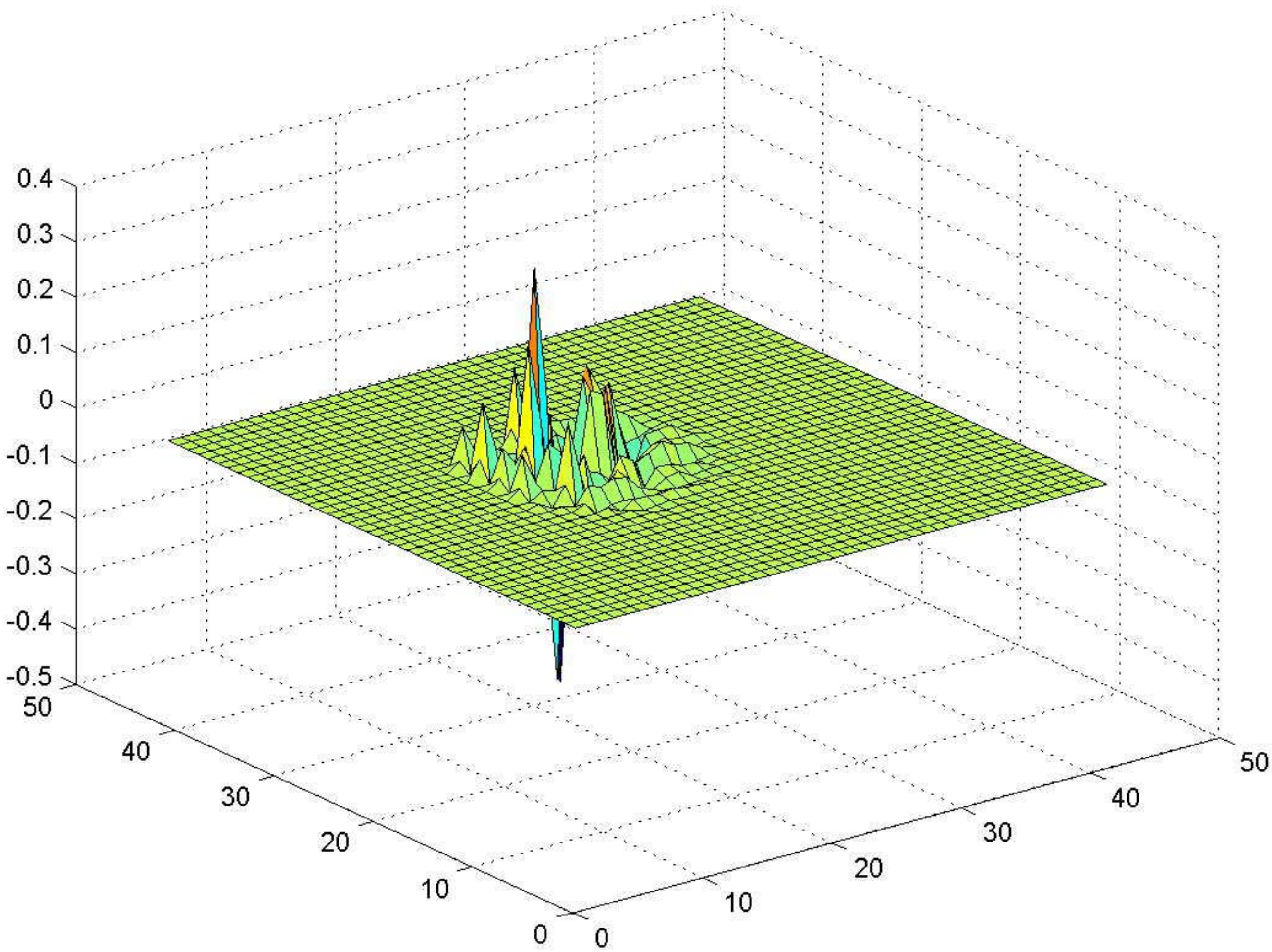

Figure. 5.5: 2-D Wavelet function

Again the same concepts are used to construct the three dimensional filters. These three dimensional filters are iterated to give the 2nd-order, three-dimensional scaling and wavelet functions.

5.5.1 Indirect Design form known one-dimensional wavelets

We are acquainted with several single dimensional compactly supported wavelets such as the Haar, Daubechies, Meyer's, Lemorie, and Mallet [27]. If we choose both ϕ and ψ with compact support the corresponding subband filtering scheme will use only FIR filters. In a two-channel subband filtering scheme, the input signal is convolved with two different filters, one lowpass and one highpass:

$$h_l(n) = \sqrt{2}\int \Phi(t)\overline{\Phi(2t-n)}dt \qquad 5.29$$
$$h_h(n) = \sqrt{2}\int \Psi(t)\overline{\Phi(2t-n)}dt$$

Only a finitely many h_l and h_h are nonzero for a compactly supported ϕ and ψ.

The length of these subband filters depend on the regularity of the wavelet and scaling functions. The coefficients of these filters are given for some of the known compactly supported wavelets and scaling functions. The length of these filters depends on the regularity of the above functions.

A 3rd order 6 coefficient Daubechies scaling function is given by

$$H_0= [0.230377813309 \;\; 0.714846570553 \;\; 0.630880767930 \;\; -.027983769417 \;\; -.187034811719 \;\; 0.030841381836 \;\; 0.932883011667 \;\; -.010597401785] \qquad 5.30$$

A 4th order 8 coeffient Daubechies scaling function is given by

$$H_0=[0.160102397974 \;\; 0.603829269797 \;\; 0.724308528438 \;\; 0.138428145901 \;\; -0.242294887066 \;\; -0.032244869585 \;\; 0.077571493840 \;\; -0.006241490213 \;\; -0.012580751999$$

$0.003335725285]$ 5.31

A *n*th order 23 coefficient symmetric Mallat scaling function

$$H_0 = [-.002\ -.003\ .006\ .006\ -.013\ .012\ -.030\ 0.023\ -.078$$
$$-.035\ .307\ \ 0.542\ 0.307\ -.035\ \ -.078\ .023\ -.030\ .012$$
$$-.013\ .006\ .006\ -.003\ -.002] \qquad 5.32$$

The polyphase matrices for the above scaling functions can be reconstructed using the filter coefficients and the forward polyphase vector for a given sampling density. Based on these polyphase matrices cascade representations can also be formulated. Once these single dimensional cascade representation are drawn the multidimensional filter banks with the same order of regularity are constructed using some of the methods discussed in the previous chapters. For the two-channel case the relation between the low and high-pass filter and the polyphase matrix can be written as

$$H(z) = \begin{bmatrix} h_l(z) \\ h_h(z) \end{bmatrix} = H_p(z^D)p_f(z) \qquad 5.33$$

This polyphase matrix can be written in cascade form as follows

$$\begin{aligned} H_p(z) &= U_0 \prod_{i=1}^{n} D_i U_i \qquad 5.34 \\ &= \begin{pmatrix} 1 & a_0 \\ -a_0 & 1 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & a_i \\ -a_i & 1 \end{pmatrix} \end{aligned}$$

Here n depends on the regularity of the low and high-pass filters. Since the left hand side of 5.34 is known the coefficients of the single dimensional cascade structure on the right hand side can be solved algebraically using the same relation. Let us write the multidimensional cascade structure for the above two channel case.

$$H_p(z) = U_0 \prod_{i=1}^{n} \prod_{j=1}^{m} D_j U_{ij} \qquad 5.35$$

$$= \begin{pmatrix} 1 & -a_0 \\ -a_0 & 1 \end{pmatrix} \prod_{i=1}^{n} \prod_{j=1}^{m} \begin{pmatrix} 1 & 0 \\ 0 & z_j \end{pmatrix} \begin{pmatrix} 1 & a_{ij} \\ -a_{ij} & 1 \end{pmatrix}$$

The a_{ij} coefficients for $j=1$ are the same as in the one dimensional case and what remains to be determined are the rest of the coefficients. Based on the relation of lower and higher dimensional zeros a method has been developed in the previous chapter to construct higher dimensional cascade structures The same method can be applied to determine these coefficients. The a_{ij}'s are solved recursively starting with $j=1$ and solving algebraically the regularity condition at each stage of the process. Once the multidimensional cascade structures are built the filter components are iterated to determine the multidimensional wavelets. Thus starting from known single dimensional wavelets this procedure can be used to build multidimensional wavelets with the same order of regularity. The procedure is illustrated using the following example.

For the 3rd order 6 coefficient Daubechies scaling filter mentioned above

$$H(z)=H_p(z^D)p_f(z)$$

$$= \begin{pmatrix} h_l \\ h_h \end{pmatrix}$$

$$= \begin{pmatrix} 0.2304+0.7149z+0.6309z^2-.0280z^3-.1870z^4+0.0308z^5+0.9330z^6-.0106z^7 \\ -.0106--0.9330z^1+0.0308z^2+.1870z^3+.0280z^4+0.6309z^5-0.7149z^6+0.2304z^7 \end{pmatrix} \qquad 5.36$$

The cascade form representation can be written as

$$H_p(z_1) = \begin{pmatrix} 1 & a0 \\ -a0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{11} \\ -a_{11} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \cdots$$
$$\cdots \begin{pmatrix} 1 & a_{12} \\ -a_{12} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{13} \\ -a_{13} & 1 \end{pmatrix} P_f \qquad 5.37$$

And the coefficients are determined to be

$$\begin{aligned} a0 &= .10588941993701038816276149157133 \\ a_{11} &= -.72872968755654075342697496688881 \\ a_{12} &= 2.5007177900510798710684518794224 \\ a_{13} &= -.35048907241722776708100525117394 \end{aligned} \qquad 5.38$$

Now the above filter can be extended to two dimensional case if we can assume the zeros of the higher dimensional filter to be the same as the lower dimensional one and the polyphase matrix can be written as

$$H_p(z_1, z_2) = \begin{pmatrix} 1 & a0 \\ -a0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{11} \\ -a_{11} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \cdot\cdot$$
$$\begin{pmatrix} 1 & a_{12} \\ -a_{12} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{13} \\ -a_{13} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \cdot\cdot$$
$$\begin{pmatrix} 1 & a_{21} \\ -a_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{22} \\ -a_{22} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_2^{-1} \end{pmatrix} \cdot\cdot$$
$$\begin{pmatrix} 1 & a_{23} \\ -a_{23} & 1 \end{pmatrix}, \qquad 5.39$$

$P_f = [1\, z^{-1}]$.

And solving for the rest of the coefficients we obtain the following results

$$\begin{aligned} a_{21} &= .42679660764767147745871150126469e\text{-}1 \\ a_{22} &= -1.1936170727914372153769761620884 \\ a_{23} &= .32098723435471949832674603244614. \end{aligned} \qquad 5.40$$

The two dimensional scaling filter will be

$$\begin{aligned} H_0 = [&1.0772 + 2.5177 z_1^{-1} - 0.3100 z_1^{-2} + 0.0266 z_1^{-3} + 0.4676 z_2^{-1} + 0.9874 z_1^{-1} z_2^{-1} \\ &+ 3.2420 z_1^{-2} z_2^{-1} - 0.8327 z_1^{-3} z_2^{-1} + 0.0085 z_1^{-4} z_2^{-1} - 0.0148 z_2^{-2} - 1.1812 z_1^{-1} z_2^{-2} \\ &- 3.7952 z_1^{-2} z_2^{-2} + 1.3890 z_1^{-3} z_2^{-2} - 0.2703 z_1^{-4} z_2^{-2} + 0.0460 z_1^{-1} z_2^{-3} - 0.2706 z_1^{-2} z_2^{-3} \\ &+ 2.1385 z_1^{-3} z_2^{-3} - 0.6228 z_1^{-4} z_2^{-3}] \end{aligned} \qquad 5.41$$

$$h_l = \begin{pmatrix} 0 & .0266 & -.3100 & 2.5177 & 1.0772 \\ .0085 & -0.8327 & 3.2420 & 0.9874 & .4676 \\ -0.2703 & 1.3890 & -3.7952 & -1.1812 & -0.0148 \\ -0.6228 & 2.1385 & -.2706 & 0.046 & 0 \end{pmatrix}.$$

SUMMARY:

In this chapter practical procedures for the construction of compactly supported multidimensional wavelets have been outlined. These procedures have been illustrated using practical examples. The relation of zeros between higher and lower dimensional filters have been the key factor for setting these procedures. The multidimensional nonseparable wavelets constructed using iteration of multidimensional filters will be utilized in the reconstruction and denoising of multidimensional images in the next chapter.

CHAPTER 6

APPLICATIONS:IMAGE RESTORATION

6.1 INTRODUCTION

The focus of all previous chapters have been first to construct regular multidimensional nonseparable filter banks and then iterate these filters to construct a compactly supported nonseparable multidimensional wavelets. In the this chapter, I will discuss the application of these multidimensional nonseparable wavelets in multidimensional image processing. Typical goals in image processing include restoration, enhancement, smoothing/extrapolation, synthesis, detection/estimation and spectral factorization [32]. In this chapter, I consider the problem of image restoration in multiple dimensions. The use of wavelet for the task of image restoration and enhancement is relatively new and rapidly emerging concept in image processing. Wavelet methods for de-noising and de-blurring have attracted considerable interest especially in the fields of medical imaging, synthetic aperture radar, extra galactic astronomy, computer vision, molecular spectroscopy and others. The main focus of this chapter is to test and demonstrate the application of the multidimensional nonseparable filterbanks constructed in the previous chapters in the restoration and reconstruction of images.

6.2 PRELIMINARIES

The field of image restoration began several decades ago when scientists were involved in the space programs that produced incredible images of the Earth and our

Solar system [67], [46]. The images obtained from the various planetary missions of the time were subject to many photographic degradation's as a result of substandard imaging environment, vibration in machinery and the spinning and tumbling of the spacecraft. The degradation of images was not a small problem, considering the enormous expense required to obtain these images. Thus the need to develop methods for the retrieval of meaningful information from degraded images became critical. To this end inordinate amount of effort has been put to develop techniques for the restoration of images in the last few decades. As a result some of the most common algorithms in one-dimensional signal processing and estimation theory found their way into the realm of image restoration. A lot of the work done in image processing have their primary roots in classical signal processing approaches to estimation theory, filtering, and numerical analysis.

The wide areas of application of image processing has contributed to the diverse perspectives in the development of the theory. Astronomical imaging is still one of the primary applications of digital image restoration. Extraterrestrial observations of the Earth and the planets were degraded by motion of blur as a result of slow camera shutter speeds relative to rapid spacecraft motion. Images obtained were often subject to noise of one form or another. Often astronomical imaging degradation problems are characterized by Poisson noise which is signal-dependent and has its roots in the photon-counting statistics involved with low light sources. Another type of noise found in these images is Gaussian noise, which often arises from electronic components in the imaging system and broadcast transmission effects.

In the areas of medical imaging, image restoration has played an important role. Restoration has been used for filtering of Poisson distributed film grains noise in chest X-rays, mammogram's, digital angiographic images and the removal of additive noise in

Magnetic Resonance Imaging (MRI) [30]. Another emerging application of image restoration in medicine is in the areas of quantitative autoradiology (QAR).

Image restoration has also wide application in the media and particularly in the movies industry. Image restoration techniques are used for the restoration of blurry, aged Polaroid negative images of the last decade. Image restoration is used not only to restore aged and deteriorated films it is also used to colorize black-and-white films.

In general there are significant works done in the areas of image restoration in several application areas. Image restoration has been applied in law enforcement and forensic science for an number of years. The restoration of blurry photographs of license plates and crime scenes and poor-quality security videotapes is frequent. Another exciting application of image restoration is in the field of image and video coding. Several techniques are being developed to improve coding efficiency, and reduce bit rates of coded images.

Image restoration is being used in many other applications as well. A few of them include restoration of blurry X-ray images of aircraft wings to improve federal aviation inspection procedures, for printing of high quality continuous images, in assembly line manufacturing of electronic parts and in defense applications such as guided missiles which may obtain distorted images due to the effect of pressure differences around the camera mounted on the missile. Image restoration plays a significant role in today's world.

6.3 IMAGE RESTORATION AND RECONSTRUCTION

Image restoration is a field that studies methods used to recover an original scene from degraded observations. Techniques used for image restoration are oriented toward modeling the degradation's, usually blur and noise, and applying an inverse procedure to obtain an approximation of the original scene. Reconstruction are generally separate from image restoration since they operate on a set of image projections and not on a full image. However, restoration and reconstruction techniques do share the same objective of recovering the original image and end up solving the same mathematical problem.

Developing techniques to perform the image restoration task requires the use of models not only for the degradation's, but also for the images themselves. Here we will be concerned with some of the well developed techniques. A more general and comprehensive degradation and restoration model is shown in the Figure 6.1 below.

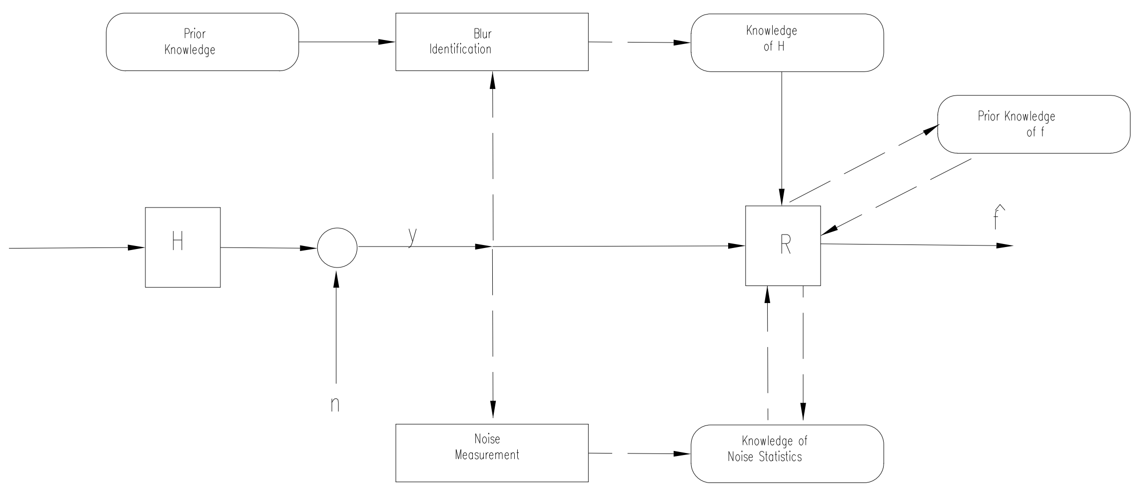

Figure 6.1: Image Degradation Module and Restoration Process

It is evident from this figure that the amount of prior knowledge plays a major part in achieving the best restorations. These are knowledge of degradation, knowledge about the original image and prior knowledge of H, i.e. blur identification.

6.3.1 SOURCE OF IMAGE DEGRADATION

The general discrete model for a linear degradation caused by blurring and additive noise can be given by the following superposition summation

$$y(\boldsymbol{n}) = h(\boldsymbol{n}) * f(\boldsymbol{n}) + n(\boldsymbol{n}) \qquad 6.1$$

$$= \sum_{\boldsymbol{K}} h(\boldsymbol{n}-\boldsymbol{k}) f(\boldsymbol{k}) + n(\boldsymbol{n})$$

where $f(\boldsymbol{n})$ represents the original image

$y(\boldsymbol{n})$ is the degraded image

$n(\boldsymbol{n})$ represents an additive noise introduced by the system usually taken as zero mean Gaussian distributed noise term.

and

* indicates n-dimensional convolution

The blurring process described by equation 6.1 is linear. Often the linear convolution is approximated by circular convolution. This involves treating the image as one period from the n-dimensional periodic signal. The borders of an image are often treated as symmetric extensions of the image. Such approaches seek to minimize the distortion at the borders caused by filtering algorithms which must perform deconvolution over the entire image. When implementing image restoration algorithms, it is very important to consider how the borders of the image are treated, as different approaches can result in very different restored images. The several analytical models are frequently used in equation to represent the shift-invariant image degradation. The models are developed for varied blurs such as motion blur, turbulence blur, etc. and additive noise such as Gaussian and Poisson.

6.3.2 SOME CLASSICAL IMAGE RESTORATION TECHNIQUES

One of the many common approaches to image restoration and solving equation 6.1 is the minimum mean square error (MSE) an optimization based on varied criteria such as BSNR (Blurred Signal-to-Noise Ratio), ISNR (Improvement to Signal-to-Noise Ratio). Even though these criteria do not always reflect the perceptual properties of the human visual system, they serve to provide an objective standard by which to compare different techniques.

Another is the classical direct approach to solving equation 6.1 i.e. finding an estimate of $\widehat{f}$ which minimizes the norm

$$\|\boldsymbol{y} - \boldsymbol{H}\widehat{\boldsymbol{f}}\|^{\boldsymbol{2}} \qquad 6.2$$

Thus providing a least squares fit to the data. This leads to the generalized inverse filter, which is given by the solution

$$(H^T H)\widehat{f} = H^T y \qquad 6.3$$

An alternative approach to solving equation 6.1. in a regularized fashion can lead to direct restoration approaches when considering either a stochastic or a deterministic model for the original image f. In both cases, the model represents prior information about the solution which can be used to make the problem well-posed.

Another approach is the iterative image restoration algorithms. The primary advantages of iterative techniques are that there is no need to explicitly implement the inverse operator and that the process may be monitored as it progresses. Iterative algorithms are very well suited to restoring images suffering from a variety of degradation's, such as linear, nonlinear, spatially varying, or spatially invariant blurs, and signal-dependent noise, because of the flexible framework provided by each approach. One of the most basic deterministic iterative techniques considers solving

$$(H^T H + \alpha C^T C) f \qquad 6.4$$

with the method of successive approximations. This leads to the following iteration for f

$$f_0 = \beta H^T y \qquad 6.5$$

$$f_{k+1} = f_k + \beta [H^T y - (H^T H + \alpha C^T C) f_k] \qquad 6.6$$

This iteration converges if

$0 < \beta < \frac{2}{|\lambda max|}$ where λmax is the largest eigenvalue of the matrix $(H^T H + \alpha C^T C)$.

6.3.3 WAVELETS AND IMAGE RESTORATION

In the last two decades, there has been considerable interest in the use of wavelet transforms for image restoration, reconstruction, feature extraction anedge detection [3], [43]. One method applied by R.R. Coifman [17] and D. L. Donoho [23] has been to use the transform-based thresholding defined below

Transform the noisy data into an orthogonal domain

Apply soft or hard thresholding to the resulting coefficients, thereby suppressing those coefficients smaller than a certain amplitude.

Transform back into the original domain.

The denoising algorithm developed by Mallat et al. which is a wavelet based technique has been developed and refined since it first came out. One of our motivation in this chapter is to test some of these algorithms for multidimensional image processing.

Figure 6.2. shows a basic model of an image restoration system. In the system module, $x(i,j,k)$ is the original $3D$ signal that is degraded by the linear operation of $f(i,j,k)$ and the addition of the noise $n(i,j,k)$ to form the degraded $3D$ signal $d(i,j,k)$. The restoration filter is $g(i,j,k)$ which is convolved with the degraded $3D$ signal $d(i,j,k)$ to form the restored $3D$ signal $y(i,j,k)$. The additive noise is modeled as a Gaussian process (White noise).

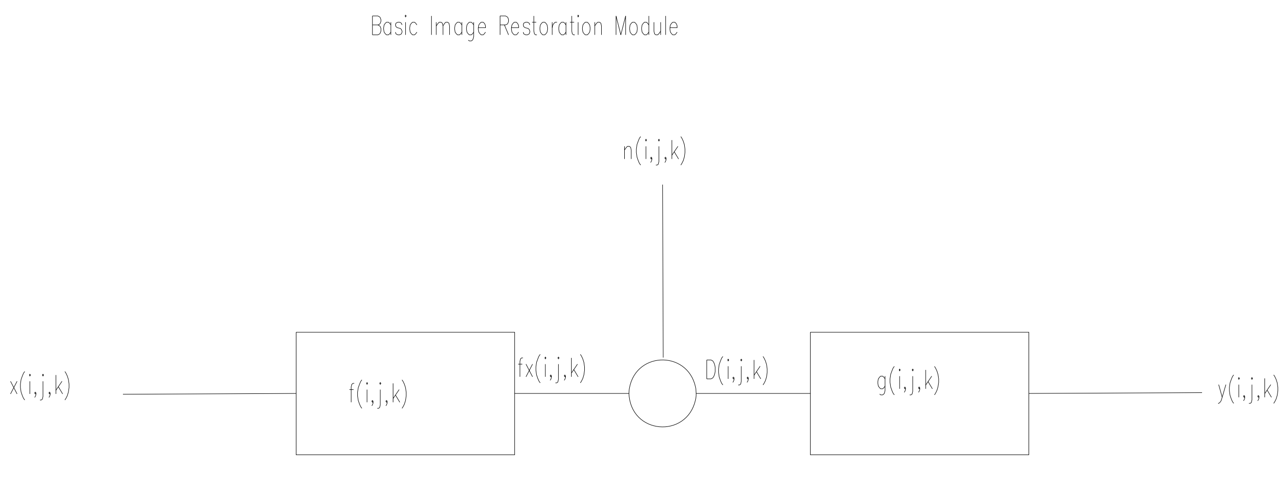

Figure 6.2: Basic Image Resolution Module

In the denoising algorithm, the separation of the signal information from the noise is done by discriminating the signal singularities from the noise singularities. This is done by analyzing the behavior of the wavelet transform maxima across scales [24], [44, [89]. Advantage is taken of the spatial coherence of the signal singular structures to suppress the noise components.

Once the signals have been decomposed into elementary building blocks that are well localized both in space and frequency, the local regularity of the signals can then be characterized by the wavelet transform. The wavelet's modulus maxima across scales are analyzed. Noise fluctuations that dominate the signal singularities are then removed. The remaining local maxima positions and amplitudes are modified, and the "denoised" signal is finally reconstructed from these modulus maxima.

The denoising algorithm consists of three different stages, suppressing the noise components, enhancing the signal components and then reconstructing the "denoised" signal from the remaining local maxima.

6.4 SIMULATION

In this section several simulations have been made to show the implementation of separable and nonseparable multidimensional filterbanks constructed in the previous chapters for image reconstruction. The solution to the filter coefficients has been worked out using Maple 7 [88] and all the simulations have been done using matlab[55]. Several algorithms have been developed to accommodate the multidimensional nonseparable sampling and convolution operations. All of the images used in the simulation and the denoising algorithms used here are obtained and adopted from Mathlab and Wavelab [75] except for the nonseparable cases. The improvement in image restoration as a result of higher order filters is demonstrated. Some of the image libraries in image processing toolbox of matlab are used to demonstrate the results.

6.4.1 IMAGE RECONSTRUCTION

6.4.1.1 2-D IMAGES.

Using the methods developed in the previous chapters a 2-D nonseparable filter is designed from a 1-D filter with 2-order regularity. Figure 6.2 shows a first level image reconstruction using the 2nd order 2-D nonseparable filterbank. Higher levels of image reconstruction can be built based on the same concepts and algorithms developed for the first level.

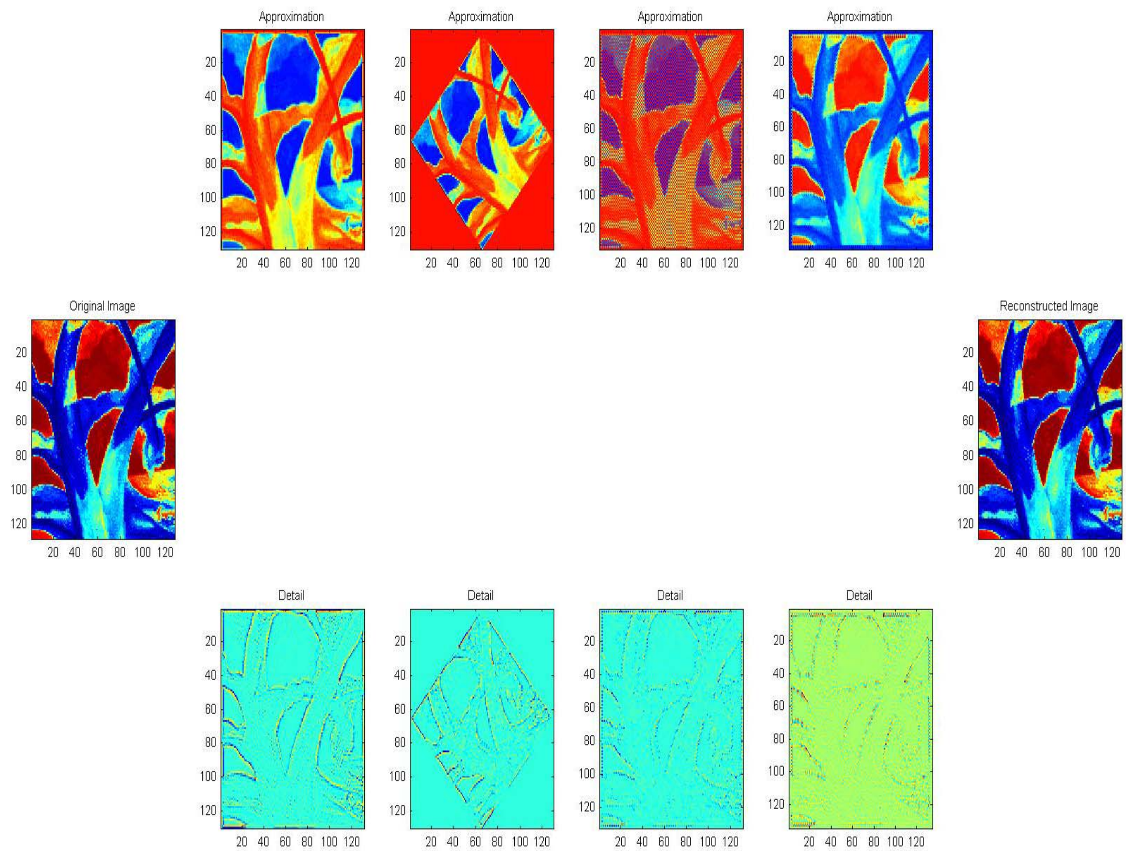

Figure. 6.3: Image Reconstruction 2nd Order Nonseparable Process

The error for the above reconstruction is calculated to be 1.1655e-006.

A second simulation is done for 3-order 2-D nonseparable filter constructed using the same method of the simulation is shown in figure 6.4 below.

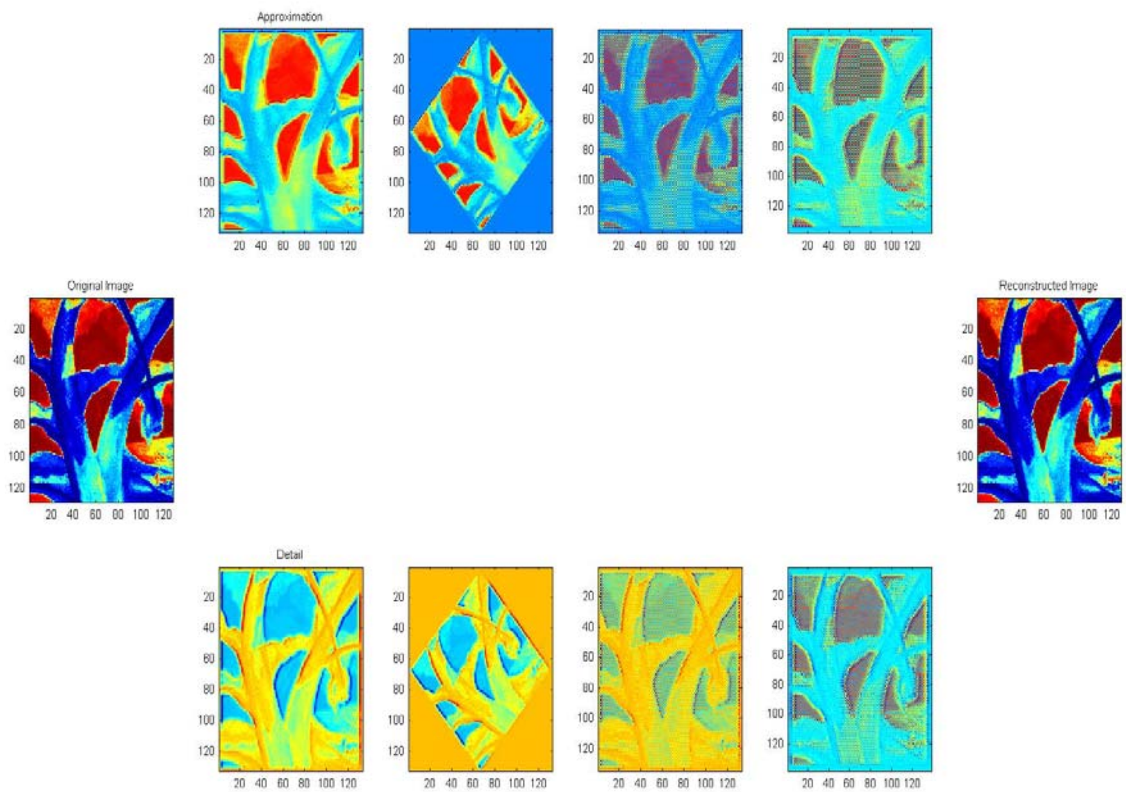

Figure 6.4: Image Reconstruction 3rd Order Nonseparable Filter

The mean square error for the above reconstruction is given as 1.7965e-007. This demonstrates that the higher order filter designed using methods developed above gives better results as is expected.

In order to show the difference of the image reconstruction problem using the separable and nonseparable filters the matlab algorithms have been used to generate images for the separable case built with 1-D dimensional filters with the same degree of regularity. The result of the simulation is shown on Figure 6.5 and Figure 6.6. There is a slight variation in the results.

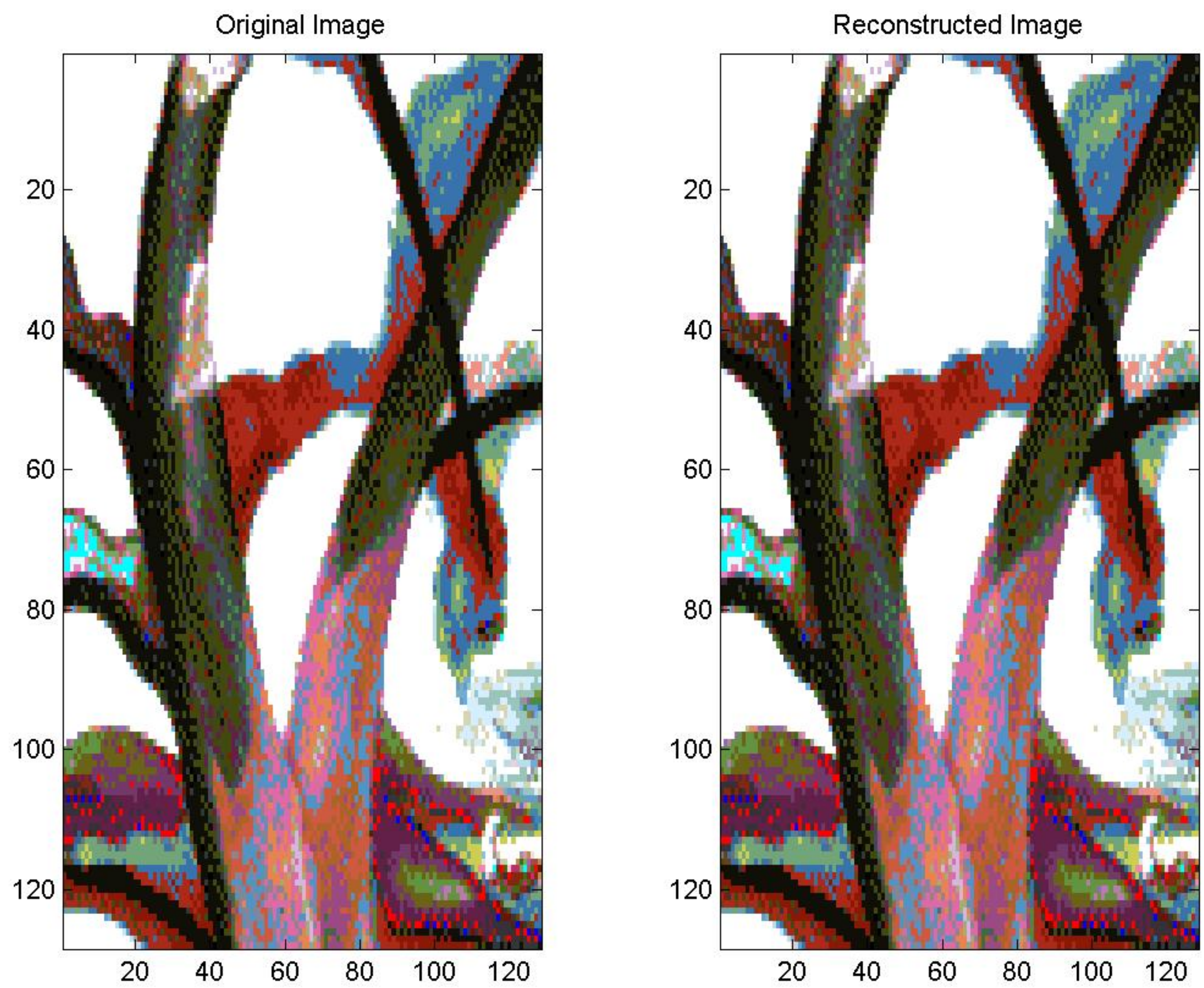

Figure 6.5: Image Reconstruction 2nd Order Separable Filter

The error for the separable case with 2nd order filter is 2.0360e-004.

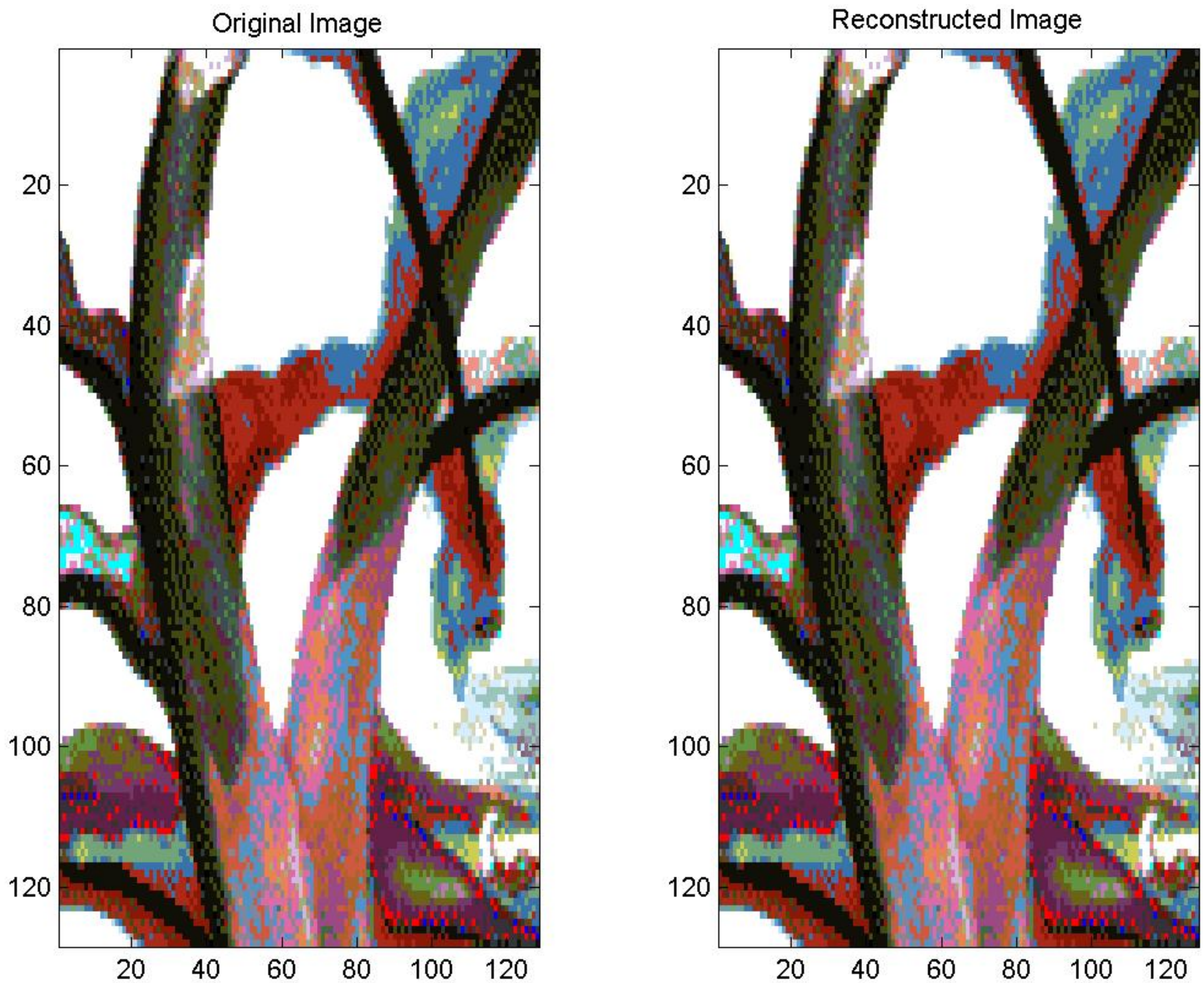

Figure 6.6: Image Reconstruction 3rd Order Separable Filter

The error for the separable case with 3rd order filter is 3.000e-005.

6.4.1.2 3-D Image

The higher order filter designed using the methods developed in the previous chapters have been used to reconstruct 3-D images using the separable method. The results of the simulation is shown in Figures 6.7 and 6.8. However, the nonseparable construction is not shown because of computational and programming complexity of the 3-D nonseparable sampling and convolution.

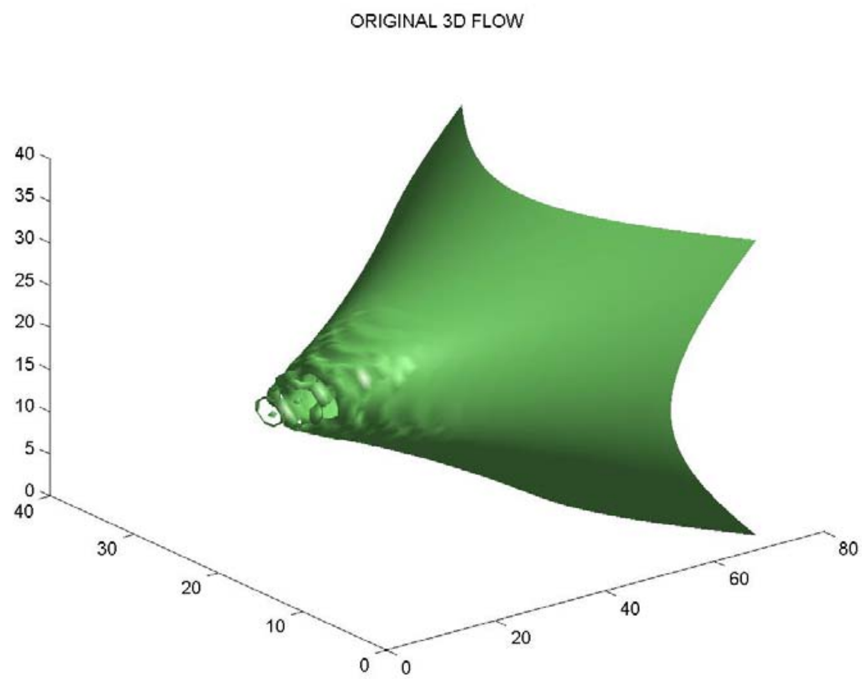

Figure 6.7: Original 3D Flow

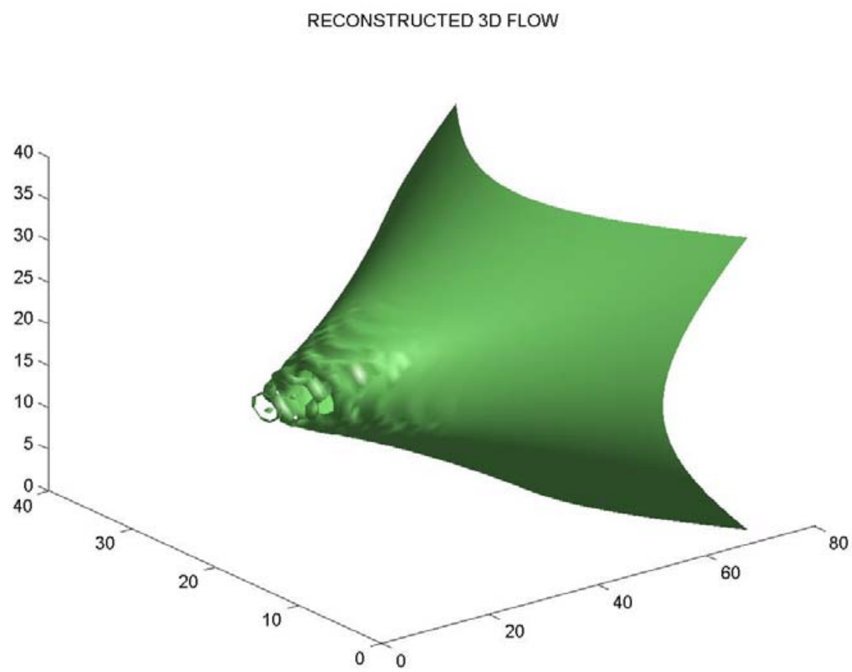

Figure 6.8: Reconstructed 3D Flow

6.4.2 IMAGE DENOISING

The image denoising algorithms hardthresh and softtresh that are part of wavelab version 850 has been used in the nonseparable wavelet decomposition to denoise the noisy image provided [75]. The results for the nonseparable filters developed previously is in shown below.

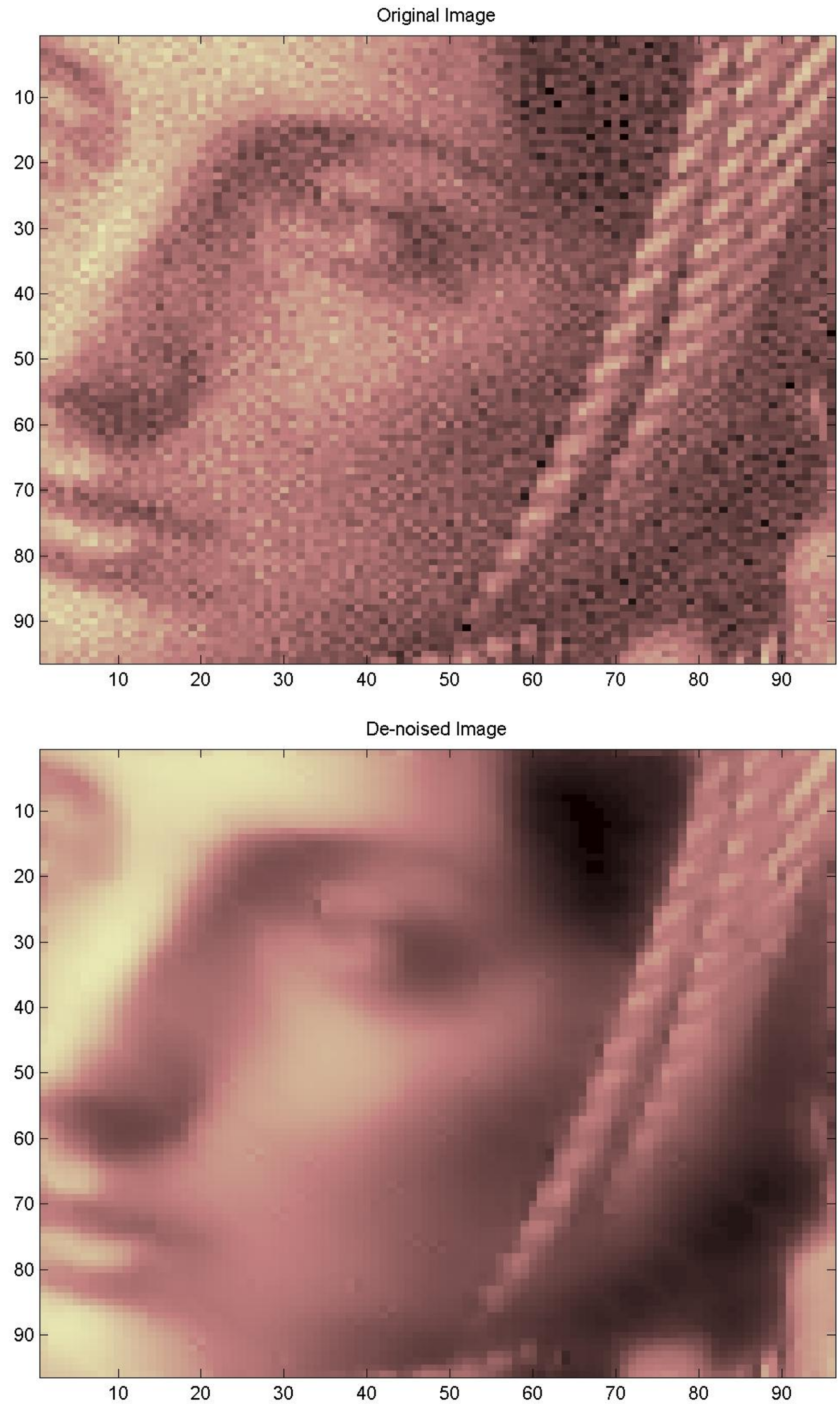

Figure 6.9: Image Denoise 2nd Order Nonseparable Filter

The result of the image denoising using the separable filters is shown below.

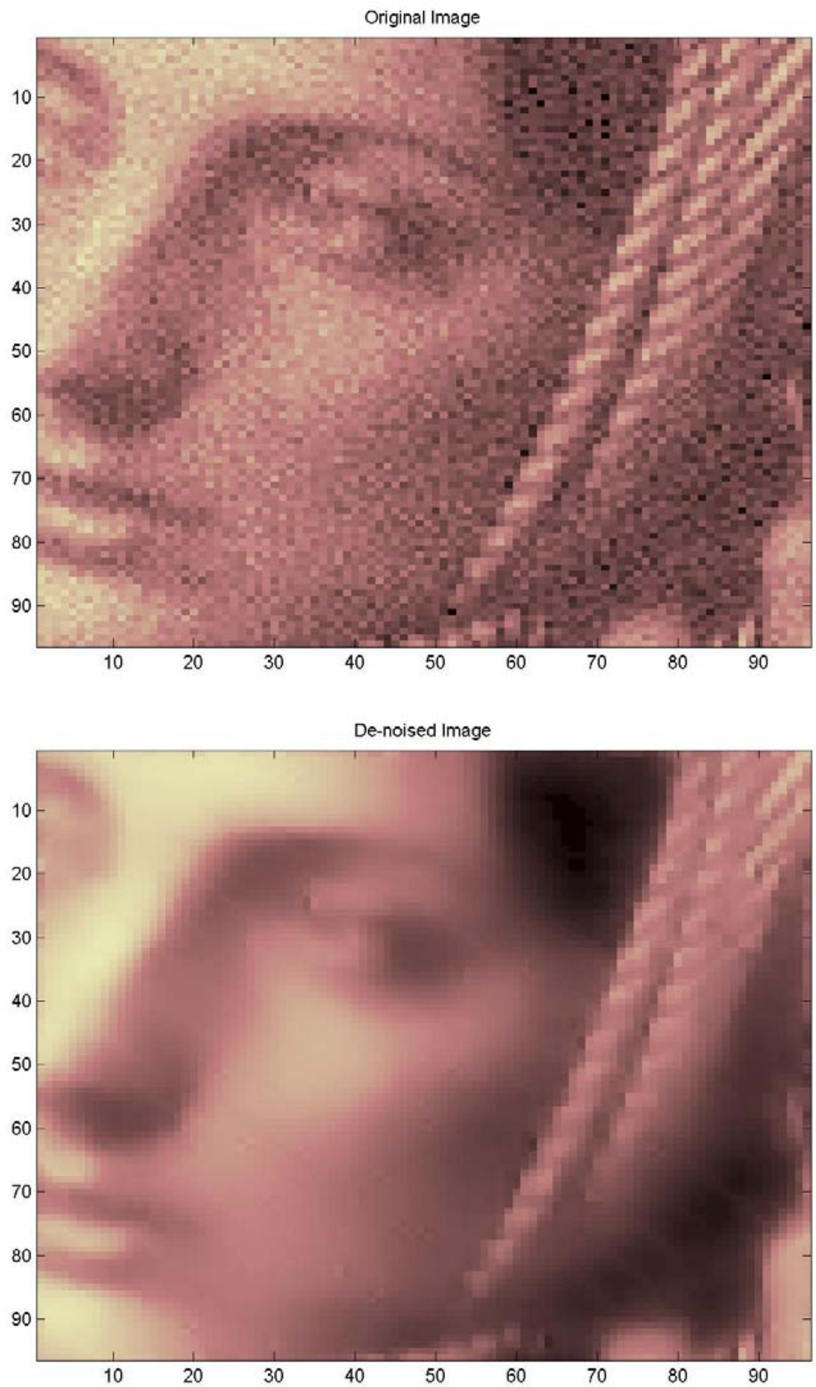

Figure 6.10: Image Denoising Separable

6.6 SUMMARY

In this chapter the multidimensional nonseparable as well as separable filters constructed using methods developed in the previous chapters have been applied for the reconstruction and denoising of images. The results demonstrate that the filters constructed using the methods developed in this dissertation accomplish the desired results as illustrated by the different examples. Higher order filters produce better results than lower order filters. The nonseparable filters with isotropic sampling matrices also give better results compared to the separable filters.

CHAPTER 7

CONCLUSION AND DIRECTIONS

7.1 CONTRIBUTION

In this dissertation multidimensional multirate system theory and applications have been studied. The basic concepts of wavelet analysis, multirate filter banks and multiresolution analysis are summarized. The interrelationship of these concepts and the development of a unified approach in single dimension is outlined. The extension of these concepts to multiple dimensions based on multidimensional sampling and lattice structures has been described. The construction of continuous wavelets from iterated filterbanks depends on the construction of regular filterbanks be it in single or multiple dimensions. This dissertation investigates the conditions of regularity in the multidimensional case and proposes new method for the construction of multidimensional filters for wavelet transforms satisfying constraints of regularity and vanishing moments [76].

In this dissertation the existence of the relations of zeros in the lower and higher dimensional filterbanks are proved for a class of filters with cascade structures. This relation has been first observed but not established by J. Kovacevic and M. Vetterli [39]. Proving this relationship is fundamental for the transformation of a one-dimensional into a two-dimensional into a three-dimensional regular filters. In this dissertation two different approaches has been used to establish this relation. First, basic theorems of linear spaces have been used to show the existence of the relation of zeros of higher and lower dimensional regular filters with cascade structures. Second, theories and concepts of polynomial matrix description are used to prove the same relation.

Based on the above established relation this dissertation develops methods and procedures for the construction of regular nonseparable multidimensional filters. The n-dimensional nonseparable regular filter is constructed using $(n-1)$-dimensional regular filter. And by extension the n-dimensional nonseparable regular filter is constructed using single dimensional regular filters. Based on this relation practical procedures for the recursive construction of regular multidimensional filterbanks and compactly supported multidimensional wavelets has been developed. The above stated relation has been exploited to construct higher dimensional filters banks from known lower dimensional filterbanks with varied degrees of regularity. These filterbanks with non-diagonal sampling lattice are iterated to construct multidimensional nonseparable wavelet basis. The method substantially reduces the computational complexity of constructing the multidimensional filters. The filter coefficients are calculated at each stage of the recursive processes thereby reducing the number of undetermined coefficient and making the algebraic equation much more manageable. The arithmetic operation grows linearly with the order of regularity $O(N)$ instead of $O(N^{|D|})$ where $|D|$ is the determinant of the dilation matrix. Besides, single dimensional filters with proven characteristics and regularity could be easily transformed to multiple dimensions.

In the dissertation several examples have been used to illustrate the concept and construct multidimensional filterbanks starting from known one-dimensional scaling filters with varied degree of regularity. The iteration of these filters gives nonseparable wavelets depending on the lattice structure. The single dimensional second and third order Daubehcies scaling filters are used to construct 2-D and 3-D nonseparable multidimensional wavelet basis. The results obtained agree with published values.

Finally the multidimensional nonseparable wavelets constructed above are used in the reconstruction and denoising of multidimensional images. However, the choice of the dilation matrix that determines the downsampling and upsampling is very important in the image processing [9]. The dilation matrix determines all aspects of the filter design as well as the visual effect. This matrix has to be isotropic to ensure no distortion in image processing. The multidimensional filters with isotropic matrices are employed for the reconstruction and denoising of multidimensional images. The performance of the separable and nonseparable filters with second and third order regularity are compared. Finally a simulation has been done and the results of 2-D images are plotted. The 2-D nonseparable filters constructed with quincunx dilation matrix with equal singular values show no distortions. The calculated errors for the nonseparable filters show improvement over those obtained by separable filters. However, not all dilation matrices and their corresponding nonseparable filterbanks give better results unless the matrices are isotropic. The selection of the dilation matrices requires the satisfaction of additional constraint.

7.2 FUTURE RESEARCH

The design of filter banks based on factorization of polyphase matrices is challenging because of the large number of parameters to be determined. The number is attenuated by imposing linear phase perfect reconstruction and critical sampling constraints. The results are also limited to real valued coefficients. Yet the extension of the results in this dissertation to higher dimensions has been computationally cumbersome and even if the 3-D nonseparable filterbanks are constructed applying them to image reconstruction and denoising requires additional work in 3-D convolution and programming.. The effect of relaxation of the constraints such as oversampling and concatenation of filters with varied features need further research [50], [94]. The choice of dilation (downsampling and upsampling) matrix for optimal filter design and image processing needs to be further investigated. Generating integer valued dilation matrix that are critically sampled and satisfying the isotropic condition is a real challenge [68], [86]. For example, the FCO used to construct 3-D nonseparable filterbank is not isotropic and is liable to image distortions. Despite the last twenty years of research in the wavelet transforms and the continually increasing areas of applications there are several areas that need further research more so in the area of multidimensional nonseparable wavelet construction [19], [60], [62].

APPENDIX A:

REISZ BASIS

A function $\Psi \in L^2(R)$ is said to generate a Reisz basis(or unconditional basis) $\{\Psi_{b,j,k}\}$ with the sampling rate b_0 if both the following properties are satisfied.

1) The linear span $< \Psi_{b,j,k}\ j,k \in Z >$ is dense in $L^2(R)$

2) There exists positive constant A, B with $0<A<B<\infty$ such that

$$A\|C_{j,k}\|_{l^2}^2 \leq \|\sum_{j,k \in Z} C_{j,k}\Psi_{b_0,j,k}\|_{l^2}^2 \leq B\|C_{j,k}\|_{l^2}^2 \qquad \text{A.1}$$

for all $\{C_{j,k} \in\ l^2(Z^2)$

If Ψ generates Riesz basis with $b_0=1$ then Ψ is called R-function. If Ψ is an *R*-function then there is a unique Riesz basis $\{\Psi^{j,k}\}$ *of* $L^2(R)$ which is dual to $\{\Psi_{j,k}\}$ in the sense

$$< \Psi_{j,k}\ , \Psi^{l,m} > = \delta_{j,l}\delta_{k,m} \quad j,k,l,m \in Z \qquad \text{A.2}$$

Hence every function $f \in L^2(R)$ has a unique series expansion

$$f(x) = \sum_{j,k=-\infty} < f, \Psi_{j,k} > \Psi^{j,k}(x) \qquad \text{A.3}$$

Although the coefficients are the values of the wavelet transform of f relative to Ψ the series is not necessarily a wavelet series. To qualify as a wavelet series there must exit some $\tilde{\Psi} \in L^2(R)$ such that the dual basis $\{ \Psi^{j,k}\}$ is obtained form $\tilde{\Psi}$ by

$$\Psi^{j,k}(x) = \tilde{\Psi}_{j,k}(x) \qquad \text{A.4}$$

$$\tilde{\Psi}_{j,k}(x) = 2^{\frac{1}{2}}\Psi(2^{jx}-k)$$

If $\{\Psi_{j,k}\}$ is orthonormal basis of $L^2(R)$ then $\Psi^{j,k} = \Psi_{j,k}$ or $\tilde{\Psi} = \Psi$. In general $\tilde{\Psi}$ does not exit. If Ψ is chosen such that it exists then f can be recovered from the pair $\{\Psi, \tilde{\Psi}\}$ by

$$f(x) = \sum < f, \Psi_{j,k} > \tilde{\Psi}_{j,k}(x) \qquad \text{A.5}$$

$$= \sum < f, \tilde{\Psi}_{j,k} > \Psi_{j,k}(x)$$

APPENDIX: B

NOTATION AND DEFINITIONS

This section contains most of the definitions involving multidimensional z-transform and sampling.

1. Multidimensional z-Transform

1.1 Raising an n-dimensional complex vector $\boldsymbol{z} = (z_1, z_2 \ldots z_n)$ to an n-dimensional integer vector $\boldsymbol{k} = (k_1, k_2 \ldots k_n)$

$$\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} \qquad \text{B.1}$$

1.2 Raising a $\boldsymbol{z}$ to a matrix power D denotes

$$\boldsymbol{z}^D = (z_1^{\boldsymbol{d}_1}, z_2^{\boldsymbol{d}_2}, \ldots z_n^{\boldsymbol{d}_n}) \qquad \text{B.2}$$

where $\boldsymbol{d}_i$ is the column of the matrix D.

1.3 The equivalent Nth root of unity is defined as

$$W_D(\boldsymbol{\omega}) = (e^{-j\langle \omega, \boldsymbol{d}_1 \rangle}, \ldots\ldots e^{-j\langle \omega, \boldsymbol{d}_n \rangle}) \qquad \text{B.3}$$

1.4 The z-transform of a discrete sequence $x(\boldsymbol{k}) = h(k_1, k_2 \ldots k_n)$ is defined as

$$x(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathcal{Z}^n} x(\boldsymbol{k}) \boldsymbol{z}^{-\boldsymbol{k}}$$
$$= \sum_{n_1} \ldots\ldots \sum_{n_N} x(n_1, \ldots n_N) z_1^{-n_1} \ldots\ldots z_N^{-n_N} \qquad \text{B.4}$$

2. Multidimensional Fourier transform

The Fourier transform of the multidimensional function x is given by

$$x(\boldsymbol{\omega}) = \sum_{\boldsymbol{k} \in \mathcal{Z}^n} x(\boldsymbol{k}) e^{-j<\omega,\boldsymbol{k}>} \qquad \text{B.5}$$

3. Lattice:

Lattice : is the set of all linear combinations of *N* basis vectors $\boldsymbol{n} = (n_1, n_2....n_N)$ with integer coefficients.

Sublattice: If every point of the lattice Λ is also a point of the lattice M then Λ is sublattice of *M*. The determinant of Λ is an integer multiple of the determinant of *M*.

Greatest Common Sublattice(gcs (Λ_1, Λ_2): is the set of all points belonging to both $\Lambda_1$and Λ_2 *1   2 (1 ï 2).*

Separable Lattice: is a lattice that can be represented by a diagonal matrix and appears when one-dimensional systems are used in a separable fashion along each dimension.

Unit Cell: is the set of points such that the disjoint union of its copies shifted to all of the lattice points yields the input lattice.

Fundamental parallelepiped *Uc*: is the parallelepiped formed by n basis basis vectors.

Coset: is a lattice formed by shifting the origin of the output lattice to any of the points of the input lattice. The union of all cosets for a given lattice yields the input lattice.

Reciprocal lattice: is the Fourier transform of the original lattice and its points represent the points of replicated spectra in the frequency domain.

Voronoi cell: is the set of points closer to the origin than to any other lattice point.

Unimodular Matrix: is a matrix with determinant equal to ± 1.

The number of input lattice samples contained in the unit cell represents the reciprocal of the sampling density and is given by $N=det(D)$.

APPENDIX: C

DILATION MATRIX

Unlike the one-dimensional case, multidimensional sampling is represented by an integer lattice Λ defined as the set of all linear combinations of $\boldsymbol{n}$ basis vectors $\boldsymbol{n} = [n_1,n_2,.....n_n]^t$ with integer coefficients. The sampling sublattice Λ_D is generated by the sampling matrix D and is the set of integer vectors $\boldsymbol{m} = D\boldsymbol{n}$ for some integer vector $\boldsymbol{n}$. The proper definition of a sublattice requires a nonsingular sampling matrix with integer-valued entries. A given sublattice can be generated by a number of sampling matrices, each of which is related by a linear transformation represented by unimodular integer matrix. The sampling process is separable if it can be represented with a diagonal sampling matrix and nonseparable otherwise.

The role of the sampling matrix becomes crucial when used in dialation equation since iterating on the downsampling process (as in wavelet transform) amounts to a dialtation with integer powers of the sampling matrix. Different dilation matrices lead to different regularity properties on the perfect reconstruction filter bank. The existence and smothness of the wavelet basis functions highly depend on this matrix. The nonseparable matrices although require more resource in terms of memory and computation, have better performance over the separable counterparts. In the past the dilation matirx D defined over the underlying lattice Λ must satisfy the following properties

1) The new lattice Λ_D forms a sublattice of Λ (trivial property when each element of D is interger)
2) The magnitude of each eigenvalue λ_i of D must be strictly larger than 1 to ensure a dilation in each dimension.

However, recent results show that isotropic transformations are only possible if the follwing conditions on singular values of *D* are imposed instead of the conditions 2 above[68].

3) The magnitude of each singular value $\sigma_i$of *D* must be strictly larger than 1. Besides, imposing an additional constraint on the singular values that $\sigma_i = \sigma$ results in better results. However, it is not always possible to find a dialation matrix that meets these constraints.

For example for dimensions greater than 2 no dilation matrix can be a similiarity transformation with determinant 2 [86].

If *D* is an admissible dilation matrix then it should satisfy

$$D^T D = mI \qquad \text{C-1}$$

where m is an integer then

$$|det\, D| = \sqrt{m^N} \qquad \text{C-2}$$

On the other hand for two-channel design

$$|det\, D| = 2. \qquad \text{C-3}$$

Satisfying both constraints C-2, C-3 would require

$$m = \sqrt[N]{4} \qquad \text{C-4}$$

with *m* integer, which is only possible for *N=2.*

It is therefore no *3-D* dilation matrix with determinant 2 exists. The popular FCO and BCC do not satisfy the isotropic transformation constraint. The best practical alternative will be to consider dialation matrices with minimal variance of the singular values.

REFERENCES

REFERENCES

[1] A.N.Akanu and R.A.Haddad, *Multiresolution Signal Decomposition*, San Diego, CA.:Academic Press, Inc., 1992.

[2] T.P.Barnwell III and M.J.T. Smith, "Filter banks for analysis-reconstrution systems: a tutorial", *Circuit and Systems, 1990, IEEE International Symbosium,* vol. 3, 1-3 May 1990, pp.1999-2003.

[3] M. Barrat and O. Lepetit, " Recursive wavelet transform for 2*D* signals" *CVGIP: Graphical Models and Image Processing,* 56(1), pp.106-8, Jan. 1994.

[4] Basu, S.; Han Mook Choi, "On multidimensional linear phase perfect reconstruction filter banks", *Circuits and Systems, ISCAS '94., 1994 IEEE International Symposium on* , vol. 2, pp. 145 -148, 1994.

[5] Basu, S.; Han Mook Choi, "Multidimensional causal, stable, perfect reconstruction filter banks", *Image Processing, 1994. Proceedings. ICIP-94., IEEE International Conference* , vol. 1 , pp. 805 -809, 1994.

[6] S. Belayneh, "On the regularity of multidimensional nonseparable cascade filterbanks", *Africon Conference in Africa, 2002. IEEE AFRICON*. 6th, vol. 1, pp. 305-310, 2002.

[7] A. Benveniste and al. "Multiscale system theory", *IEEE Transactions on Circuits and Systems. Part I, Fundamental Theory and Applications*, vol. 41, no. , pp. 2-15, Jan. 1994.

[8] M. Bhatia , W.C. Karl and A.S.Willsky, "A wavelet-based method for multiscale tomographic reconstruction", *IEEE Transactions on Medical Imaging*, 15(1), pp. 92-101, Feb. 1996.

[9] J. Blanc-Talon et al.(Eds.): Ana Ruedin, "Dilation Matrices for Nonseparable Bidimensional Wavelets" ACIVS 2006, LNCS 4179, pp. 91-102 Springer-Verlag Berlin Heidelberg 2006

[10] Brislawn, C.M. and Wohlberg, B. "The Polyphase-With-Advance Representation and Linear Phase Lifting Factorizations", IEEE Transactions on Signal Processing,Vol.54, No. 6, pp. 2022-2034, June 2006.

[11] P.J.Burt and E.H.Adelson, " The Laplacian pyramid as a compact image code", *IEEE Transactions on Communications.* vol. 31, No. 4, pp. 532-540, April 1983.

[12] Y.T. Chan & K.C. Ho, " Multiresoluton analysis, its link to the discrete parameter wavelet transform, and its initialization", *IEEE Transactions On Signal Processing,* 44(4), pp. 1001-6, Apr. 1996.

[13] T. Chen & P.P.Vaidyanathan, "Multidimensional multirate filters derived from one dimensional filters", *Electronics Letters*, 27(3), pp. 225-228, Jan. 1991.

[14] Ying-Jui Chen; Oraintara, S.; Amaratunga, K.S, "Theory and factorizationfor a class of structurally regular biorthogonal filter banks", IEEE Transactions on Signal Processing, Vol. 54, No. 2, pp. 691-700 February 2006.

[15] C.K.Chui, *An Introduction to Wavelets*, Academic Press, Inc., New York, 1992.

[16] K.W.M.Clulow, G.E. Hayton and A.C.Pugh, "On a structure theorem for n-D MFDs" *Multi-Dimensional Systems: Problems and Solutions, IEE Colloquium on*, pp. 7/1-7/5,10 Jan 1996.

[17] Coifman, R.R and Wickerhauser M. V. "Adapted Waveform ``de-Noising" for Medical Signals and Images", IEEE Engineering in Medicine and Biology, Vol. 14 No. 5 pp. 578--586, September/October 1995.

[18] A.Croisier, D.Esterban, and C. Galand, "Perfect channel splitting by use of interpolation/decimination/tree decomposition techniques", *In Internatinal Conferance on Information Sciences and Systems*, Patras, Greece, pp. 443-446, August 1976,

[19] Da Cunha, A.L.; Do, M.N, "On Two-Channel Filter Banks With Directional Vanishing Moments", IEEE Transactions on Image Processing, Vol. 16, No. 5, pp. 1207-1219, May 2007.

[20] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[21] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Communication in Pure and Applied Mathematics*, 41(7), pp. 961-1005, Sept. 1990.

[22] I. Daubechies, "The wavelet transform, time-frequency localization and signal analysis", *IEEE Transactions on Information Theory*, 36(5), pp 961-1005,

Sept. 1990.

[23] D.L. Donoho, "Nonlinear wavelet methods for recovery of signals, densities, and spectra from indirect and noisy data", *Proceedings of Symposia in Applied Mathematics*, vol. 47, pp. 173-205, 1993.

[24]D. L. Donoho, " De-noising by soft-thresholding", *IEEE Transactions Information Theory,* 41(3), pp. 613-627, May 1995.

[25] Bogdam Dumitrescu and Corneliu Popeea, "Accurate Computation of Compaction Filters With High Regularity" IEEE Signal Processing Letters, Vol. 9 No. 9, pp. 278-281, Sept. 2002

[26] D.Gabor, Theory of Communications, *Journal IEE*, vol. 93, pp. 429-457, 1946

[27] K. Grochenig & W.R.Madyeh, " Multiresolution analaysis, Haar bases, and self-similar tilings of R^n ", *IEEE Transactions on Information Theory,* 38(2), pp. 556-568, March 1992.

[28] A. Grossmann and J. Morlet, "Decomposition of Hardy functions into square integrable wavelets and constant shape", *SIAM Journal of Math. Anal*. 15(4), pp. 723-736, July 1984.

[29] J.P.Guiver & N.K.Bose, " Polynomial matirx primitive factorization over arbitrary coefficient field and related results", *IEEE Transactions on Circuits and Systems,* CAS-29(10), pp. 649-657, Oct. 1982

[30] D. M. Healy & J. B. Weaver, " Two applications of wavelet transform in magnetic resonance imaging", *IEEE Transactions On Information Theory,* 38(2), pp. 840-860, March 1992.

[31] K.Hirano, M.Sakane & M.Z.Mulk, " Design of three-dimensional recursive digital filters", *IEEE Transactions On Circuits and Systems,* CAS-31(6), pp. 550-561, June 1984.

[32] Anil K. Jain, *Fundamentals of Digital Image Processing,* Prentice Hall, New Jersey, 1989.

[33] B.Jawerth, W. Sweldens, "An overview of wavelet based multiresolution analyses", *SIAM Review*, 36(3), pp. 377-412, Sept. 1994.

[34] Z. Jincheng & Y.Baozong, " Volume rendering oriented 3D image compression using wavelet transform", *Proceedings of ICSP*, vol. 2, pp. 1203-1206, 1996

[35] T. Kailath, *Linear Systems*, Englewood Cliffs, NJ: Prentice Hall, 1980.

[36] G.Karlsson & M.Vetterli, " Theory of two-dimensional multirate filter banks", *IEEE Transactions on Acoustics, Speech and Signal Processing*, 38(6), pp. 925-937, June, 1990.

[37] A.N.Kolomogrov & S.V.Fomin, *Introductory Real Analysis*, NY: Dover Publications, Inc. 1970.

[38] Kovacevic, J.; Le Gall, D.J.; Vetterli, M., "Image coding with windowed modulated filter banks" *Acoustics, Speech, and Signal Processing, ICASSP-89., 1989 International Conference on* , pp.1949 -1952 vol.3, 1989.

[39] J.Kovacevic & M. Vetterli, " Design of multidimensional non-separable regular filter banks and wavelets", *Proc. IEEE Int. Conf. Acoust, Speech Signal Processing* pp. IV:389-IV:392 , Mar., 1992.

[40] J.Kovacevic & M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for R^n", *IEEE Transactions on Information Theory*, 38(2), pp. 533-555, March 1992.

[41] J.Kovacevic & M. Vetterli, " Nonseparable two- and three-dimensional wavelets", *IEEE Transactions on Signal Processing*, 43(5), May 1995.

[42] J.Kovacevic & M. Vetterli, " FCO sampling of digital video using perfect reconstruction filter banks", *IEEE Transactions on Image Processing*, 2(1), Jan. 1993.

[43] O.Laligant, F.Truchetet & J.Miteran, "Edge detection by multiscale merging", *Time-Frequency and Time-Scale Analysis, 1994., Proceedings of the IEEE-SP International Symposium on* , pp. 237-240, Aug. 1994.

[44] D.L.Lau, G.R.Arce & N.C.Gallagher, "Robust image wavelet shrinkage for denoising", *Proceedings, International Conference on Image Processing, 16-19 Sep 1996* , vol. 1, pp. 371-374, 1996.

[45] W.M. Lawton, "Necessary and sufficient conditions for constructing orthonormal wavelet bases", *Journal of Mathematical Physics*, 32(1), pp. 57-61, Jan. 1991.

[46] Hua Lee and Glen Wade, *Imaging Technology*, IEEE Press, New York, 1986.

[47] Zhang Lei and Anamitra Makur, "Enumeration of Downsampling Lattice in Two-Dimensional Multirate Systems", IEEE Transactions on Signal Processing, Vol. 56, No. 1, pp. 414-418, January 2008.

[48] Z. Lin, "On matrix fraction descriptions of multivariable linear n-D sfystems" *IEEE Transactions on Circuits and Systems*, 35(10), pp. 1317-1322, Oct. 1988

[49] Z.Lin, "On primitive factorization for 3-D polynomial matrices," *IEEE Transactions On Circuits and Systems-I: Fundamental Theory And Applications*, 39(12), pp. 1024-1027, Dec. 1992.

[50] Lu, Y.M.; Do, M.N, "Multidimensional Directional Filter Banks and Surfacelets" IEEE Transactions on Image Processing,Vol. 16, No.4, pp. 918 - 931, April 2007

[51] S. Mallat, "Multifrequency channel decompositions of images and wavelets models", *IEEE Trans. Acoust., Speech and Signal Proc.*, 37(12), pp. 2091-2110, Dec. 1989.

[52] S. Mallat, " A theory of multiresolution signal decomposition, the wavelet representation", *IEEE Trans. Pattern Recognition and Machine Intelligence*, 11(7), pp.674-693, July, 1989.

[53] S. Mallat & W.L. Hwang, " Singularity detection and processing with wavelets", *IEEE Transactions on Information Theory*, 38(2), pp. 617-643, March 1992.

[54] S. Mallat & S. Zhong, " Characterization of signals from multiscale edges", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(7), pp. 710-32, July 1992.

[55] MathWorks, Inc., *Matlab version 6.1.0. release 12.1,* Boston, Massachussets, 2001.

[56] Y.Meyer, "Orthonormal Wavelets," in *WAV89*, p. 21-37, 1989

[57] F.Mintzer, "Filters for distortion-free two-band multirate filter banks", *IEEE Trans. Acoust., Speech, and Signal Proc*. 33(3), pp. 626-630, June 1985.

[58] Nayebi, K.; Barnwell, T.P., III; Smith, M.J.T., "A general time domain analysis and design framework for exactly reconstructing FIR analysis/synthesis filter banks", IEEE International Symposium on Circuits and Systems, vol. 3, p. 2022-2025, 1990.

[59] M.Padmanabhan & K.Martin, " Filter banks for time-recursive implementation of

transforms", IEEE Transactions on Circuits and Systems. Part II, Analog and Digital Signal Processing, 40(1), pp. 41-50, Jan. 1993.

[60] Patil, B. D.; Patwardhan, P. G.; Gadre, V. M, "On the Design of FIR Wavelet Filter Banks Using Factorization of a Halfband Polynomial" IEEE Signal Processing Letters, Vol. 15, pp. 485 - 488, 2008.

[61] Patwardhan, P.G. and V. M. Gadre, "On Filter Symmetries in a Class of Tree -Structured 2-D Nonseparable Filter-Banks", IEEE Signal Processing Letters, Vol. 13, No. 10, Oct. 2006.

[62] S. Phoong , C.W. Kim & P.P. Vaidyanathan, " A new class of two-channel biorthogonal filter banks and wavelet bases", *IEEE Transactions On Signal Processing*, 43(3), pp. 649-65, Mar. 1995.

[63] L. R. Rabiner and B. Gold, *Theory And Application Of Digital Signal Processing*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1975.

[64] V. Rajaravivarma & P.K.R. Rajan, " Design of multidimensional FIR digital filters using the symmetrical decomposition technique", *IEEE Transactions on Signal Processing*, 42(1), pp. 164-74, Jan. 1994.

[65] O. Rioul & M. Vetterli, "Wavelets and signal processing", *IEEE SP Magazine*, pp.14-37, Oct. 1991.

[66] O. Rioul, "A discrete-time multiresolution theory", *IEEE Transactions on Signal Processing*, 41(8), pp. 2591-2606, Aug. 1993

[67] Azreil Rosenfield, *Picture Processing by Computer*, Academic Press, New York, 1969.

[68] Ana Ruedin, "Polyphase Filter and Polynomial Reproduction conditions for the construction of smooth bidimensional multiwavelets", Lecture Notes in Computer Science, Advanced Concepts for Intelligent Vision Systems, Vol. 4678, pp. 221-232, 2007

[69] Phil Schniter, "Regularity Conditions, Compact Support, And Daubechies's Wavelets", Connexions module: m10495 Vol. 2.8 pp. 1-5, October 5, 2005

[70] E.P.Simoncelli & E.H.Adelson, "Non-separable extenstions of quadrature mirror filters to multiple dimensions", *Proc. of The IEEE*, 78(4), pp. 652-664, April 1990.

[71] M.J.T. Smith and T.P.Barnwell III, "A procedure for designing exact reconstruction filter banks for tree structured sub-band coders", *Proc. IEEE Int. Conf. Acoust., Speech and Signal Proc.* , San Diego, CA., March 1984.

[72] M.J.T. Smith and T.P.Barnwell III, "Exact reconstruction for tree-structured subband coders", *IEEE Trans. Acoust., Speech and Signal Proc.*, 34(3), pp. 431-441, June 1986.

[73] A.K.Soman, P.P.Vaidyanathan & T.Q.Nguyen, " Linear phase paraunitary filter banks: theory, factorizations and designs", *IEEE Transactions on Siganl Processing*, 41(12), pp. 3480-3496, Dec. 1993.

[74] Soman, Anand K.; Vaidyanathan, P.P., " A complete factorization of paraunitary matrices with pairwise mirror-image symmetry in the frequency domain", *IEEE Transactions on Signal Processing*, 43(4), pp. 1002-4, Apr. 1995.

[75] Stanford University, Department of Statistics., *Wavelab version 8.50, ,* Stanford, California, 2006

[76] D. Stanhill & Y.Y. Zeevi, " Two-dimensional orthogonal wavelets with vanishing moments", *IEEE Transactions On Signal Processing*, 44(10), pp. 2579-90, Oct. 1996.

[77] G. Strang & V. Strela, " Short wavelets and matrix dilation equations", *IEEE Transactions on Signal Processing*, 43(1), pp. 108-15, Jan. 1995.

[78] D.B.H.Tay & N.G.Kingsbury, "Flexible design of multidimensional perfect reconstruction FIR 2-band filters using transformations of variables", *IEEE Transactions on Image Processing*, 2(4), pp. 466-480, Oct. 1993.

[79] D.B.H.Tay & N.G. Kingsbury, " Design of nonseparable 3-D filter banks/wavelet Bases Using Transformations of Variables", *IEE Proc. Vis. Image Signal Process*. 143(1), pp. 51-61, Feb. 1996.

[80] David B. H. Tay, "Daubechies Wavelets as Approximate Hilbert-Pairs?", IEEE Signal Processing Letters, Vol. 15, pp. 57-60, 2008.

[81] T.E.Tuncer & T.Q.Nguyen, " General analysis of two-band QMF banks", *IEEE Transactions on Signal Processing*, 43(2), pp. 544-8, Feb. 1995.

[82] P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ:Prentice Hall, 1992.

[83] Venkataraman, Shankar; Levy, Bernard C., "A comparison of design methods for 2-D FIR orthogonal perfect reconstruction filter banks", *IEEE Transactions on Circuits and Systems. Part II, Analog and Digital Signal Processing*, 42(8), pp.525-36, Aug. 1995.

[84] M. Vetterli & Cormac Herley, "Wavelets and filter banks: theory and design", *IEEE Transactions on Information Theory*, 40(9), pp. 2207-2232, Sept.1992.

[85] M.Vetterli & J. Kovacevic, *Wavelets and Subband Coding*, Englewood Cliffs, NJ: Prentice Hall, 1995.

[86] Dimitri Van De Ville, Thierry Blu and Michael Unser, "On the Multidimensional Extension of the Quincunx Subsampling Matrix", IEEE Signal Processing Letterss, Vol. 12, No. 2, pp.112-114, February 2005

[87] E.Viscito & J.P.Allebach, " The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattice", *IEEE Transactions on Circuits and Systems*, 38(1), pp. 29-41, Jan. 1991

.

[88] Waterloo Maple Inc, *Maple version 7*, Ontario, Canada, 2001.

[89] G.W.Wornell & A. V. Oppenheim, " Estimation of fractal signals from noisy measurements using wavelets", *IEEE Transactions On Signal Processing,* 40(3), pp. 612-623, March 1992.

[90] X. Xia, J. Geronimo & al., "Design of prefilters for discrete multiwavelet transforms", *IEEE Transactions On Signal Processing*, 44(1), pp. 25-35, Jan. 1996.

[91] T.Yoshida, A.Nishihara & N.Fujii, "A design method for 3-dimensional band-limiting FIR filters using McClellan transformation", *IEICE Transactions Fundamentals*, E76-A(8), pp. 1283-1292, Aug. 1993.

[92] D. C.Youla & G.Gnavi, "Notes on n-dimensional system theory", *IEEE Transactions on Circuits and Systems*, Cas-26(2), pp. 105-111, Feb. 1979.

[93] D.C. Youla, "The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices", *IEEE Transactions On Circuits and Systems*, CAS-31(6), June 1984.

[94] Zhan, Y. and Heijmans, H.J.A.M, "Non-Separable 2D Biorthogonal Wavelets with two-Row Filters", International Journal of Wavelets, Multiresolution and Information Processing, Vol. 3 No. 1, pp. 1-18, 2005.

CURRICULUM VITAE

Sirak Belayneh graduated from Addis Ababa Univerity with Bachelor of Science Degree in October 1976. He received a Master of Science from George Mason University, Fairfax, Virginia in August 1990. He has over twenty years of experience in the design analysis and review of electrical power system with focus on fault analysis and coordination study. He is a registered professional engineer in the Commonwealth of Virginia, State of Maryland and District of Columbia. He is a senior member of IEEE and past chair of Northern Virginia power engineering society.