ALGEBRA, COMBINATORICS, AND COMPUTATION OF CERTAIN TIGHT CLOSURE INVARIANTS IN STANLEY-REISNER RINGS

by

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Dedication

I dedicate this dissertation to my wife Tabitha. Without a large commitment on her part, this would not have been possible.

Acknowledgments

I would like to thank Dr. Neil Epstein for advising me on this dissertation and for taking a chance on me as his student when I had not yet proved capable. I would like to thank Dr. Geir Agnarsson, Dr. James Lawrence, and Dr. Jay Shapiro for teaching the majority of my coursework; most of which was applicable to this dissertation. I would like to thank the mathematics faculty at University of Lynchburg for getting me started in advanced mathematics.

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Abstract

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Tight closure was first introduced in the 1980's [12] by Hochster and Huneke to answer questions about invariant theory and the Briançon-Skoda theorem. It has since come into its own as a fairly robust theory. The tight closure I^* of an ideal I is named as such because it is, in general, contained in, but not equal to, the integral closure I^- of the same ideal, so it is a "tighter" closure operator than integral closure. Tight closure is notoriously difficult to compute for an arbitary ideal, but with certain rings, this task is less arduous. In this dissertation, we build a a bridge between tight closure theory and combinatorics by way of simplicial complexes and Stanley-Reisner rings. We discuss the specifics of tight closure theory and Stanley-Reisner rings and make special effort to focus on the standard results of both topics that will be most useful to our purposes. We discuss the analogous notions for $*$-reductions and reductions of ideals for tight and integral closure repectively. When we focus our attention on the maximal ideal, $\mathfrak{m}$, of the Stanley-Reisner ring $k[\Delta]$ that is generated by the variables of the ring, we observe that if I is a reduction of $\mathfrak{m}$, then it is also a $*$-reduction of $\mathfrak{m}$. We will determine the the minimal number of generaters of a $*$-reduction of $\mathfrak{m}$, called the $*$-spread of $\mathfrak{m}$, and the intersection of all minimally generated $*$-reductions of $\mathfrak{m}$, called the $*$-core of $\mathfrak{m}$. These notions were introduced by Epstein [6] and

Fouli and Vassilev [7] respectively. We endeavor to describe both in terms of the Stanley-Reisner ring and the simplicial complex of the Stanley-Reisner ring. Finally, we examine $*$-core $\mathfrak{m}$ in specific examples and in slightly more general cases of Stanley-Reisner rings. These include dimension 1 Stanley-Reisner rings, Stanley-Reisner rings with disconnected simplicial complexes, and Stanley-Reisner rings with a graph for a simplicial complex.

Chapter 1: Introduction

1.1 Preliminaries

In general, as per Dummit and Foote [4], a *ring* is a set R with two binary operations $+$ and $\times$ satisfying the following axioms:

i.) $(R, +)$ is an abelian group,

ii.) $\times$ is associative: $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$,

iii.) the distributive laws hold in R: for all $a, b, c \in R$

$$(a + b) \times c = (a \times c) + (b \times c) \text{ and } a \times (b + c) = (a \times b) + (a \times c)$$

The ring R is said to have identity if there is an element $1 \in R$ such that for all nonzero $a \in R$,

$$1 \times a = a \times 1 = a$$

and the ring R is said to be *commutative* if in addition to the above properties, $a \times b = b \times a$ for all $a, b \in R$. All rings in this dissertation are commutative with identity.

An *ideal* I of a commutative ring R is a subring that is closed under multiplication by elements of R. An ideal P is said to be *prime* if when a and b are two elements in R and $ab \in P$, then either $a \in P$ or $b \in P$. A ring is said to be *Noetherian* if and only if every ideal of R is finitely generated and a *field* is a commutative ring in which all nonzero elements have a multiplicative inverse. All rings R in this dissertation are also Noetherian and contain an infinite field.

1.2 Motivation and History

Stanley-Reisner rings have been a topic of special interest in many fields since the 1970's when their properties were first studied by Hochster, Reisner, and Stanley [11][22][24]. They are of special interest because they bridge the gap between algebra and combinatorics by using simplicial complexes to define a ring structure. Simplicial complexes and other combinatorial structures are useful in describing phenomena in other fields. In particular, many problems in algebraic geometry, algebraic statistics, or even bioinformatics can be modeled by posets or simplicial complexes. By using such structures to define commutative rings, we can use algebraic tools to analyze this encoded information.

Tight closure was developed in the 1980's and 1990's by Hochster and Huneke [12][13] as a way to tackle problems in invariant theory and about the Briançon-Skoda theorem. More recently, research in tight closure has shifted some focus to applications to algebraic combinatorics and in particular Stanley- Reisner rings. Recent work along these lines can be found from Enescu and Ilioaea [5] and from Goel, Mukundan, and Verma [9]. This dissertation serves to add to this growing interest between tight closure and combinatorics.

1.3 Summary

In this section, we provide a summary of the remainder of the dissertation.

Chapter 2 presents the basic definitions of simplicial complexes and Stanley-Reisner rings and focus on the interplay between the two. We describe the important characteristics of Stanley-Reisner rings and make mention of some useful theorems that will pertain to the later chapters of the dissertation. We conclude the section by outlining how Stanley-Reisner rings can be viewed as fiber products [19] that use ring information to reflect the structure of the simplicial complex as opposed to typical formulation of quotienting a polynomial ring by a squarefree monomial ideal.

Chapter 3 focuses on closure operators of ideals I of a ring R. Special attention is paid to integral closure, tight closure in characteristic p, and tight closure in equal characteristic

0. Standard first principles are mentioned and the main theorems necessary for the rest of the dissertation are stated as well. The chapter includes definitions of reductions and $*$-reductions which are the primary objects of study in the original research portions of the dissertation.

In chapter 4, we prove that for our primary focus, the graded maximal ideal $\mathfrak{m}$ of a Stanley-Reisner ring $k[\Delta]$, the set of reductions and the set of $*$-reductions of $\mathfrak{m}$ are the same set of ideals. Therefore, we make the choice to prove all the following results in the language of tight closure.

We next discuss how to generate $*$-reductions of $\mathfrak{m}$ in a reliable way, and we make mention of the some of the important linear algebra that can be preformed on the generators of a $*$-reduction I of $\mathfrak{m}$ in order to choose the most useful generating set of I. Specifically we choose a generating set of I in relation to a chosen minimal prime ideal of $k[\Delta]$. If we want I to be minimally generated, we also show that I must have a generating set of size $d = \dim k[\Delta] = \dim \Delta + 1$. Namely, we have the following corollary:

Corollary 4.2.8. *If I is a $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$, then there exists an ideal $J \subseteq I$ of $k[\Delta]$ with d generators such that $J^* = \mathfrak{m}$ i.e $*$-spread of $\mathfrak{m}$ is d.*

Equipped with an adequate description of minimal $*$-reductions I of $\mathfrak{m}$, we finish chapter 4 by exploring what elements all minimal $*$-reductions of $\mathfrak{m}$ have in common. We call the ideal generated by the common elements of the minimal $*$-reductions of $\mathfrak{m}$ the $*$-core of $\mathfrak{m}$. We show $*$-core$(\mathfrak{m})$ is generated by monomials, explore examples for particular choices of Δ, and show that for all Δ, $d = \dim k[\Delta]$, and τ the test ideal of $k[\Delta]$, $*$-core$(\mathfrak{m})$ is bounded the following way:

Theorem 4.3.8. *If Δ is a simplicial complex of dimension $d-1$ on n vertices, then for $\mathfrak{m}$ of $k[\Delta]$,*

$$\mathfrak{m}^{d+1} + \tau\mathfrak{m} \subseteq *\text{-core}\,\mathfrak{m} \subseteq \mathfrak{m}^2.$$

In the last chapter, we explore $*$-core$(\mathfrak{m})$ in special cases of $k[\Delta]$. We reduce the question of exactly which monomials generate $*$-core$(\mathfrak{m})$ to the case where Δ is a connected simplicial complex (i.e. for any choice of two vertices x and y there is an edge path from x to y). We show that when $\dim k[\Delta] = 1$ (i.e. Δ is a discrete collection of points), $*$-core$(\mathfrak{m}) = \mathfrak{m}^2$ and try to say as much as we can about the case where Δ is a simple graph.

Chapter 2: Stanley-Reisner Rings

In this chapter we outline the basics of Stanley-Reisner rings. In Section 2.1 we introduce a combinatorial and set theoretic object called a simplicial complex. We can then use the elements of the complement of the simplicial complex to construct what will be referred to as a Stanley-Reisener ring or a square free monomial algebra. In Section 2.4 we recover a result from Matsumura and Moore [19] that allows us to build a Stanley-Reisner ring constructively in much the same way that we can construct a simplicial complex by its facets.

2.1 Simplicial Complexes

Let $V = \{v_1, \ldots, v_n\}$ be a finite set. A *simplicial complex* Δ on V is a collection of elements from 2^V, the power set of V, such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$, and such that $\{v_i\} \in \Delta$ for $i = 1, \ldots, n$. For example, if $V = \{v_1, v_2, v_3\}$, then

$$2^V = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$$

and if we know $\{v_1\}$ and $\{v_2, v_3\}$ are in Δ, then the set

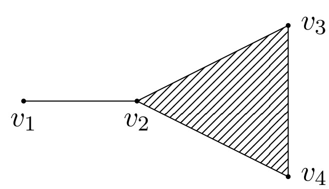
$$\{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_2, v_3\}\}$$

is also a subset of Δ. Further, if $\{v_1, \ldots, v_n\} \in \Delta$, then we define Δ to be a *simplex*. Each element of Δ is called a *face* and a face F is called a *facet* if there exists no other face G in Δ such that $F \subset G$. A simplicial complex Δ can also be defined to be a set of simplices F such that every face of F is also an element of Δ and if F and G are both simplices in Δ, their intersection is a face of both F and G.

The dimension of a face F of Δ is $\dim F = |F| - 1$ where $|F|$ is the number of elements in the face F. The dimension of a simplicial complex is

$$\dim \Delta = \max\{\dim F : F \in \Delta\}.$$

Simplicial complexes can also be represented in a visual way that will be more useful to later parts of this dissertation. If Δ contains the one element set $\{v_i\}$, we can represent this face with a vertex and label it v_i. If the two element set $\{v_i, v_j\}$ is in Δ, we use an edge with the end vertices labeled v_i and v_j. This allows to show that not only is the set $\{v_i, v_j\}$ in Δ, but also the sets $\{v_i\}$ and $\{v_j\}$. For a set of size three, a shaded triangle with labeled vertices is used. After this, presentation becomes a bit more difficult on a two dimensional surface, but in general, if a face has dimension d, it will be represented as a d dimensional triangule with $d + 1$ labeled vertices, i.e. a 0-dimensional face is a point, a 1-dimensional face is an edge, a 2-dimensional face is a triangle, a 3-dimensional face is a tetrahedron, etc. and a d-dimensional face includes every $d - 1$-dimensional face on the same $d + 1$ vertices. As an example, if Δ is the simplicial complex with facets $\{v_1, v_2\}$ and $\{v_2, v_3, v_4\}$, the visual presentation is

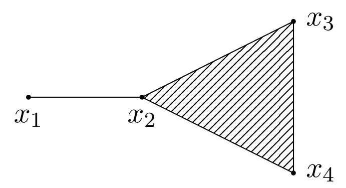


where the triangle corresponds to $\{v_2, v_3, v_4\}$ and the edge that is not part of the triangle corresponds to $\{v_1, v_2\}$.

To conclude this section, we mention some definitions that are specific to this paper. We define a simplicial complex Δ to be *proper* if $\Delta \neq 2^V$. Further for $d < n$, we define $\Delta_{d,n}$ to be the $d - 1$ dimensional (proper) simplicial complex on n vertices such that every facet of Δ is a set of size d and all size d subsets of V are facets. The distinction that d is strictly less than n is important because if $d = n$, then $\Delta_{d,n}$ is a simplex and $d > n$ is impossible.

2.2 Defining Ideal of Δ

For this section, let k be any field and $k[x_1, \ldots, x_n]$ be a polynomial ring over k in n variables. If Δ is a simplicial complex over vertex set $V = \{v_1, \ldots, v_n\}$, define I_Δ to be the ideal of $k[x_1, \ldots, x_n]$ generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \notin \Delta$. If we recall the previous example represented with the diagram



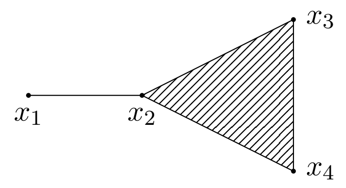
we see that Δ does not contain the faces

$$\{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_4\},$$

which correspond to the monomals x_1x_3, x_1x_4, $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_4$, and $x_1x_2x_3x_4$ respectively. The ideal I_Δ will be generated by these six monomials. Some of these generators are redundant though. In fact,

$$\begin{aligned} I_\Delta &= (x_1x_3, x_1x_4, x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_2x_3x_4) \\ &= (x_1x_3, x_1x_4). \end{aligned}$$

The ideal I_Δ is an example of what is called a *squarefree monomial ideal* because all the generators of I_Δ are squarefree monomials. It should be noted that there exists a one-to-one correspondence between all simplicial complexes on n vertices and all square free monomial ideals of $k[x_1, \ldots, x_n]$ generated by nonlinear elements, so if we care about a particular set of nonlinear squarefree monomials, we can always find a simplicial complex to represent this set of monomials. Since this correspondence exists, we will drop the distinction between vertices v_i and variables x_i, and label both with the same symbols, i.e. our diagram from before can be labeled in the following way:



2.3 Stanley-Reisner Rings

Named for Richard Stanley and Gerald Reisner, who investigated their properties in the 1970's [24][22] (a large debt is also owed to Mel Hochster [11]), we define the *Stanley-Reisner ring* $k[\Delta]$ to be the quotient ring $k[x_1, \ldots, x_n]/I_\Delta$. Since I_Δ is generated by monomials of at least degree two, $k[\Delta]$ is never an integral domain if Δ is proper. When Δ is a simplex, I_Δ is (0) and $k[\Delta] = k[x_1, \ldots, x_n]$. In fact, the ring $k[\Delta]$ is a polynomial ring if and only if Δ is a simplex.

Stanley-Reisner rings have a few useful and interesting properties. The first of these can be found in [3], among other sources, and provides a characterization of all the minimal primes of the ring $k[\Delta]$ as well as the dimension of the ring:

Theorem 2.3.1 ([3], Theorem 5.1.4)**.** *Let Δ be a simplicial complex and k a field then*

$$I_\Delta = \bigcap_F \mathfrak{B}_F$$

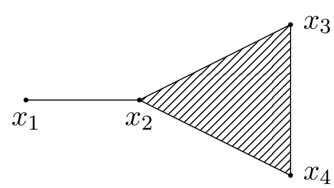
where the intersection is taken over all facets F of Δ, and $\mathfrak{B}_F$ denotes the prime ideal generated by all x_i such that $v_i \notin F$. In particular,

$$\dim k[\Delta] = \dim \Delta + 1.$$

Specifically, Theorem 2.3.1 states that the minimal prime ideals of $k[\Delta]$ are the variable complements of the facets of Δ and that the Krull dimension, which can unambiguously refer to as dimension, the supremum of the lengths of all chains of prime ideals in a ring R, is equal to

$$\dim \Delta + 1 = 1 + \max\{\dim F : F \in \Delta\}.$$

For example, the ring $k[\Delta]$ with $I_\Delta = (x_1x_3, x_1x_4)$, which as we know corresponds to the simplicial complex Δ in the following figure with facets $\{x_1, x_2\}$ and $\{x_2, x_3, x_4\}$



has as its minimal primes $P = (x_3, x_4)$ and $Q = (x_1)$ and $\dim k[\Delta] = 2 + 1 = 3$. Further, Theorem 2.3.1 states that I_Δ is the intersection of the minimal prime ideals of $k[\Delta]$, and we see that

$$P \cap Q = (x_3, x_4) \cap (x_1) = (x_1x_3, x_1x_4).$$

2.4 Stanley-Reisner Rings as Fiber Products

It is natural to view a simplicial complex as a sum of its parts, namely its facets. If we refer to the example we have been using throughout this section, Δ can be viewed as a union of two simplicies Δ_1 and Δ_2 where

$$\Delta_1 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$$

$$\Delta_2 = \{\emptyset, \{x_2\}, \{x_3\}, \{x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\}$$

and the intersection of the two simplicies is $\{\emptyset, \{x_2\}\}$. It is natural to wonder if the Stanley-Reisner ring $k[\Delta]$ of the whole simplicial complex can be viewed as some sort of sum of the parts as well, the parts being $k[\Delta_1]$ and $k[\Delta_2]$.

If we suppose that $\Delta = \Delta_1 \cup \Delta_2$, for simplicial complexes Δ_1 and Δ_2, we see the following:

$$\begin{aligned}\Delta &= \Delta_1 \cup \Delta_2 \\ &= (\Delta_1 - (\Delta_1 \cap \Delta_2)) \cup (\Delta_2 - (\Delta_1 \cap \Delta_2)) \cup (\Delta_1 \cap \Delta_2) \\ &= (\Delta - \Delta_2) \cup (\Delta - \Delta_1) \cup (\Delta_1 \cap \Delta_2).\end{aligned}$$

The last two lines both partition Δ into a disjoint union of subsets. Specifically, each element of Δ is in one and only one of the three pieces.

Since each nonzero squarefree monomial in $k[\Delta]$ can be associated to an element of Δ, the set of nonzero square free monomials in $k[\Delta]$ can be divided up into three disjoint sets similar to the way the elements of Δ are divided into disjoint sets. If a monomial m is not squarefree, we can include it with the set that contains square free monomial of largest degree that divides m. If we again refer to $k[\Delta] = k[x_1, x_2, x_3, x_4]/(x_1x_3, x_1x_4)$, Δ gets divided in the following way:

$$\begin{aligned}\Delta - \Delta_2 &= \{\{x_1\}, \{x_1, x_2\}\} \\ \Delta - \Delta_1 &= \{\{x_3\}, \{x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\} \\ \Delta_1 \cap \Delta_2 &= \{\emptyset, \{x_2\}\}.\end{aligned}$$

If we let D_1,D_2, and $D_\cap$ be the sets of monomomials that correspond to sets of faces $\Delta - \Delta_2$,

$\Delta - \Delta_1$, and $\Delta_1 \cap \Delta_2$ respectively,

$$D_1 = \{ax_1^{t_1}, ax_1^{t_1}x_2^{t_2}\}$$

$$D_2 = \{ax_3^{t_3}, ax_4^{t_4}, ax_2^{t_2}x_3^{t_3}, ax_2^{t_2}x_4^{t_4}, ax_3^{t_3}x_4^{t_4}, ax_2^{t_2}x_3^{t_3}x_4^{t_4}\}$$

$$D_\cap = \{a, ax_2^{t_2}\}$$

where $a \in k\backslash\{0\}$ and $t_i \in \mathbb{N}$.

For any polynomial $g \in k[\Delta]$, we can group the monomial terms of g according to which set D_i, $i \in \{1, 2, \cap\}$, they fall into; i.e. $g = g_1 + g_2 + g_\cap$ with the monomials of g_i associated to elements of D_i for $i \in \{1, 2, \cap\}$.

If Δ and Γ are simplicial complexes with $\Delta \subset \Gamma$, then there exists a natural surjection $\beta : k[\Gamma] \twoheadrightarrow k[\Delta]$. If we let $\Delta = \Delta_1 \cup \Delta_2$, then $\Delta_i \subset \Delta$ for $i = 1, 2$ and there exist natural surjections $\beta_i : k[\Delta] \twoheadrightarrow k[\Delta_i]$ for $i = 1, 2$ such that $\beta_1(g_1 + g_2 + g_\cap) = \overline{g_1 + g_\cap}$ and $\beta_2(g_1 + g_2 + g_\cap) = \overline{g_2 + g_\cap}$. Similarly, there exist surjective homomorphims $f_i : k[\Delta_i] \twoheadrightarrow k[\Delta_1 \cap \Delta_2]$ for $i = 1, 2$ such that $f_1(\overline{g_1 + g_\cap}) = \bar{\bar{g}}_\cap$ and $f_2(\overline{g_2 + g_\cap}) = \bar{\bar{g}}_\cap$ i.e. the following diagram commutes:

$$\begin{tikzcd}
k[\Delta] \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & k[\Delta_2] \arrow[d, "f_2"'] \\
k[\Delta_1] \arrow[r, "f_1"] & k[\Delta_1 \cap \Delta_2]
\end{tikzcd}$$

If we let R_1, R_2 and S be commutative rings and h_1 and h_2 be ring homomorphisms such that $h_i : R_i \to S$, then the *fiber product* of R_1 and R_2 over S is the subring $R_1 \times_S R_2$ of $R_1 \times R_2$ defined by

$$R_1 \times_S R_2 = \{(r_1, r_2) \in R_1 \times R_2 | h_1(r_1) = h_2(r_2)\}$$

with projection maps $\alpha_i : R_1 \times_S R_2 \to R_i$, $i = 1, 2$ such that $\alpha_i(r_1, r_2) = r_i$ and the diagram

$$\begin{tikzcd}
R_1 \times_s R_2 \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & R_2 \arrow[d, "h_2"'] \\
R_1 \arrow[r, "h_1"] & S
\end{tikzcd}$$

commutes. The work of Matsumura and Moore [19] tells us if $R_i = k[\Delta_i]$ and $S = k[\Delta_1 \cap \Delta_2]$ such a fiber product exists and it is isomorphic to $k[\Delta] = k[\Delta_1 \cup \Delta_2]$ i.e there exists an isomorphism $u : k[\Delta] \to k[\Delta_1] \times_{k[\Delta_1 \cap \Delta_2]} k[\Delta_2]$ such that the diagram

$$\begin{tikzcd}
k[\Delta] \arrow[dr, dotted, "u"] \arrow[drr, bend left, "\beta_2"] \arrow[ddr, bend right, "\beta_1"'] & & \\
& k[\Delta_1] \times_{k[\Delta_1 \cap \Delta_2]} k[\Delta_2] \arrow[r, "\alpha_2"'] \arrow[d, "\alpha_1"] & k[\Delta_2] \arrow[d, "f_2"'] \\
& k[\Delta_1] \arrow[r, "f_1"] & k[\Delta_1 \cap \Delta_2]
\end{tikzcd}$$

commutes.

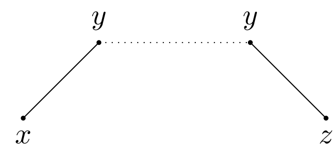
Theorem 2.4.1 ([19], Theorem 3.4)**.** *Let Δ be a simplicial complex, k a ring, and let Δ_1 and Δ_2 be simplicial complexes such that $\Delta = \Delta_1 \cup \Delta_2$. Then*

$$k[\Delta] \cong k[\Delta_1] \times_{k[\Delta_1 \cap \Delta_2]} k[\Delta_2,]$$

the fiber product of $k[\Delta_1]$ and $k[\Delta_2]$ over $k[\Delta_1 \cap \Delta_2]$.

This theorem gives us the power to construct Stanley-Reisner rings facet by facet. As an example, we will construct $k[x, y, z]/(xyz)$ using fiber products. First we will construct

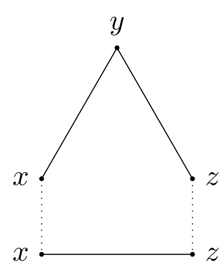
$k[x,y,z]/(xz)$ by gluing at the dotted line



using the diagram

$$\begin{tikzcd}
k[x,y] \times_{k[y]} k[y,z] \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & k[y,z] \arrow[d, "f_2"'] \\
k[x,y] \arrow[r, "f_1"] & k[y].
\end{tikzcd}$$

We must next glue the remaining facet $\{x,z\}$ to the simplicial complex to complete the ring we are constructing. We will glue the edge on in the way depicted in the following picture



by using the diagram

$$\begin{tikzcd}
k[x,y,z]/(xz) \times_{k[x]\times_k k[z]} k[x,z] \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & k[x,z] \arrow[d, "f_2"'] & \\
k[x,y,z]/(xz) \arrow[r, "f_1"] & k[x] \times_k k[z] \arrow[r, "="] & k[x,z]/(xz).
\end{tikzcd}$$

Many of the statements proven in Chapter 5 will rely on this view of Stanley-Reisner rings to make the statement of theorems easier. Much of that later work indicates that the glue part of the diagram, i.e. $k[\Delta_1 \cap \Delta_2]$, may play an important role in determining $*$-core $\mathfrak{m}$, an invariant of a given Stanley-Reisner ring that is covered in depth in Chapters 4 and 5.

Chapter 3: Closure Operators

In this chapter we focus on two closure operators of ideals: integral closure and tight closure. The former, according to Huneke and Swanson [17], has played a role in number theory and algebraic geometry since the nineteenth centry, with the modern formulation arising in the 1930's in the work of Krull and Zariski. The concept was later reimagined in terms of reductions by Northcott and Rees in the 1950's [20]. The latter was first described and developed by Hochster and Huneke [13]. We define both operators and make mention of the theorems pertaining to both that are either first principles or important to this paper.

3.1 Integral Closure

We begin this chapter by introducing the notion of closure of ideals. The following definitions can be found in [17] and it should be noted that $IJ = \{\Sigma a_j b_j | a_j \in I, b_j \in J\}$ and I^i is this type of product of ideals corresponding to i copies of I:

Definition 3.1.1. *Let I be an ideal of a ring R. An element $r \in R$ is said to be integral over I if there exists an integer n and elements $a_i \in I^i$, $i = 1, 2, \ldots, n$, such that*

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements of r that are integral over I is called the integral closure of I and is denoted I^-. If $I = I^-$, then I is said to be integrally closed.

It is important to mention that in the majority of existing literature, the integral closure of I is often denoted $\overline{I}$, but the choice of I^- in this dissertation is considered more modern.

We see easily that $I \subseteq I^-$ because if $b \in I$, let $n = 1$ and $a_1 = -b$, then $b + (-b) = 0$ shows all elements of I are integral over I.

As a more substantial example, let $R = k[x, y]$ be the polynomial ring over the field k in two variables and let $I = (x^2, y^2)$. The element $r = xy$ is in R but not I, however for $n = 2$, $a_1 = 0 \in I^1$ and $a_2 = -x^2y^2 \in I^2$, $r^2 + a_1 r + a_2 = 0$, thus $xy \in I^-$.

In the following theorem, various well known results about integral closure are compiled. Many of the results have analogues in tight closure that will be mentioned in the sequel.

Theorem 3.1.2 (Remark 1.1.3, [17]). *For a ring R be a ring and I, J ideals of R:*

(a) $I \subseteq I^-$.

(b) If $I \subseteq J$, then $I^- \subseteq J^-$.

(c) $I^- \subseteq \sqrt{I}$.

(d) Radical, hence prime, ideals are integrally closed.

(e) The nilradical $\sqrt{0}$ is contained in I^- for every ideal I.

(f) Intersections of integrally closed ideals are integrally closed.

(g) The following property is called persistence: if $\varphi : R \to S$ is a ring homomorphism, then $\varphi(I^-) \subseteq (\varphi(I)S)^-$.

The next theorem relates inclusion of elements in the integral closure of an ideal to minimal prime ideals of the ring R. The theorem has an important analogue in tight closure, which is mentioned in the next section and is also used in later results.

Theorem 3.1.3 (Proposition 1.1.5, [17]). *Let R be a ring, not necessarily Noetherian. Let I be an ideal of R. An element $r \in R$ is in the integral closure of I if and only if for every minimal prime ideal P in R, the image of r in R/P is in the integral closure of $(I+P)P/P$.*

In this dissertation, it is less important to ask what is the integral closure of an ideal $I \subset R$ than it is to ask given an ideal $I \subset R$ which ideals $J \subset R$ are such that $J^- = I$. To that end, we use the following definition from the seminal work of Northcott and Rees [20].

Definition 3.1.4 (Definition 1, [20]). *If I and J are ideals of R then J will be called a reduction of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for at least one integer n.*

This notion of reductions is linked to integral closure by the following theorem:

Theorem 3.1.5 (Corollary 1.2.5, [17]). *Let $J \subset I$ be ideals and let I be finitely generated. Then J is a reduction of I if and only if $I \subseteq J^-$.*

By this theorem, the fact that $(I^-)^- = I^-$,[17, Corollary 1.3.1], and Theorem3.1.2.(a) we see that if J is a reduction of I, then

$$J \subseteq I \subseteq J^- \subseteq I^- \subseteq (J^-)^- = J^-$$

which proves that $J^- = I^-$ if and only if J is a reduction of I.

If we think about integral closure in the context of reductions, the following two theorems are useful, though they are essentially re-imaginings of Theorem 3.1.2 (g) and Theorem 3.1.3.

Theorem 3.1.6 (Lemma 8.1.3 (1), [17]). *Let $R \to S$ be a ring homomorphism and $J \subseteq I$ ideals of R. If J is a reduction of I, then JS is a reduction of IS.*

Theorem 3.1.7 (Lemma 8.1.4, [17]). *Let R be a Noetherian ring, $J \subseteq I$ ideals in R. Then $J \subseteq I$ is a reduction if and only if for every minimal prime P of R, $J(R/P) \subseteq I(R/P)$ is a reduction*

We will need the following two definitions in later sections. Both are from [20].

Definition 3.1.8 (Definition 2, [20]). *A reduction J will be called a minimal reduction of I if no ideal strictly contained in J is a reduction of I.*

Definition 3.1.9 (Definition 3, [20]). *An ideal that has no reduction other than itself will be called a basic ideal.*

We conclude this section with a lemma from Hays [10]. This lemma is the anchor of a later result that gives added significance to this body of work.

Lemma 3.1.10 (Example 2.8, [10]). *Let $k[x_1, \ldots, x_n]$ with $n \geq 2$ be a polynomial ring over a field. Then the maximal ideal $(x_1, \ldots, x_n)$ is basic.*

3.2 Tight Closure

In this section we define tight closure in both prime characteristic p and characteristic 0. We also compile relevant results used in the later portions of this dissertation. The sections for characteristic p and characteristic 0 are separated because tight closure is originally a characteristic p notion [12],[13] and of full understanding of the characteristic p case is required to understand the characteristic 0 case.

3.2.1 Tight Closure in Prime Characteristic p

For this section, let R be a ring of characteristic p. For our purposes, these will be quotients of $k[x_1, \ldots, x_n]$, the polynomial ring in n variables over a field k of characteristic p. Our preference is for k to be infinite. An infinite field of characteristic p can be found by taking the fraction field of a polynomial ring over $\mathbb{F}_p$. We will let R° be the set of elements of R not contained in any minimal prime of R.

If we let I be an ideal of R, we define the q-th Frobenius power of I, $I^{[q]}$, to be the ideal of R generated by the q-th powers of the elements of I where $q = p^e$ with $e \in \mathbb{N} \gg 0$. This is not to be confused with $I^q = \langle \{\prod_{i=1}^{q} a_i : a_i \in I\} \rangle$.

We will now define tight closure of an ideal I of R. A definition can be found in any work about tight closure, for example [12],[13],[16], or [23], but we are using the version found in [3].

Definition 3.2.1. *Let $I \subset R$ be an ideal. The tight closure I^* of I is the set of all elements $x \in R$ for which there exists $c \in R^\circ$ with $cx^{p^e} \in I^{[p^e]}$ for $p^e \gg 0$. One says I is tightly closed if $I = I^*$.*

The following theorem, found in [3], but compiled from other sources, lists many of the basic properties of tight closure. Many of the results are tight closure analogues of integral closure items in Theorem 3.1.2 and Theorem 3.1.3.

Theorem 3.2.2 (Proposition 10.1.2, [3])**.** *Let I and J be ideals in R of characteristic p. Then the following hold:*

(a) I^* is an ideal and $I \subset J \Rightarrow I^* \subset J^*$;

(b) there exists $c \in R^\circ$ with $c(I^*)^{[q]} \subset I^{[q]}$ for $q \gg 0$;

(c) $I \subset I^* = I^{**}$;

(d) $x \in I^*$ if and only if the residue class of x lies in $((I+P)/P)^*$ for all minimal prime ideals P of R.

For our purposes, the most useful of these items is Theorem3.2.2.(d) for which the original source is [1, Lemma 2.10(c)(1)]. Specifically for any Stanley-Reisner ring $k[\Delta]$ and all minimal primes P_i of $k[\Delta]$, $k[\Delta]/P_i$ will be a polynomial ring. We will pair this with the following theorem, which states that all ideals in a polynomial ring over a field of characteristic p are tightly closed.

Theorem 3.2.3 (Theorem 4.4, [13])**.** *If R is a polynomial ring over a field k and I is an ideal of R, then $I^* = I$.*

To determine if $x \in k[\Delta]$ is in I^*, for all minimal primes P, we pass to $k[\Delta]/P$, and $x \in I^*$ if and only if $\overline{x} \in (I+P_i)/P_i$ for all i.

Since simplicial complexes are at their basest levels sets, set containment will play a large role in this dissertation. If $\Delta \subseteq \Delta'$ are simplicial complexes, there exists a natural surjection $f : k[\Delta'] \twoheadrightarrow k[\Delta]$. It will be important to know how tight closure of an ideal is affected by such a map. For this we turn to the following theorem:

Theorem 3.2.4 (Theorem 6.24, [14])**.** *If $f : R \to S$ is a map of finitely generated algebras over a field, I an ideal of R, and $x \in I^*$, then $f(x) \in (f(I)S)^*$.*

3.2.2 Tight Closure in Equal Characteristic 0

When R is a finitely generated algebra over a field k of characteristic 0, one passes to characteristic p models of R and $x \in I^*$ if and only if true for almost all characteristic p models.

All of the results in the previous section have analogues for tight closure in equal characteristic 0. We provide statements and sources for teh most important ones to this research.

The following result for tight closure in equal characteristic 0 is analogous to Theorem3.2.2.(d):

Theorem 3.2.5 (Theorem 2.5.5(n), [15])**.** *Let $I \subset R$ be an ideal of a ring of equal characteristic 0. Then $x \in I^*$ if and only if the residue class of x lies in $((I + P)/P)^*$ for all minimal prime ideals P of R.*

We also retain the fact that ideals of polynomial rings are tightly closed mentioned in 3.2.3 when we pass to characteristic 0. The version of the theorem for characteristic 0 can be found in [15, Theorem 4.1.1].

Theorem 3.2.6. *If R is a polynomial ring over a field k of characteristic 0 and I is an ideal of R, then $I^* = I$.*

The third important theorem is that of persistence of tight closure. The characteristic 0 version is found at [15, Theorem 2.5.5(k)].

Theorem 3.2.7. *If $f : R \to S$ is a map of finitely generated algebras over a field of characteristic 0, I an ideal of R, and $x \in I^*$, then $f(x) \in (f(I)S)^*$.*

Because all of these theorems carry over to the characteristic 0 case from the characteristic p case, we no longer need to make special mention of characteristic when referring to tight closure. Therefore, for the rest of the paper, any statement about tight closure is true for both the characteristic 0 and characteristic p cases.

Now that the basics of tight closure are outlined, the following theorem shows how the tight closure and the integral closure of an ideal I are related. In characteristic p, the statement can be found at [13, Theorem 5.2] and for equal characteristic 0, the result is located at [16, Corollary 6.3, Appendix 1]

Theorem 3.2.8. *Let I be any ideal of the ring R. Then*

$$I^* \subseteq I^-.$$

We conclude this section with another useful result about tight closure in both types of characteristic. Though it is difficult to believe that the following result is first mentioned here, no source could be found. This may be due to the simplistic nature of statement and the fact that it follows easily from early theorems in integral and tight closures. To that end, a proof is provided.

Observation 3.2.9. *Let P be a prime ideal of R. Then*

$$P^* = P.$$

Proof. The tight closure of an ideal I is contained in integral closure of I [13]. All prime ideals are integrally closed by Theorem 3.1.2 (d). If P^- represents the integral closure of a prime ideal P, then

$$P \subseteq P^* \subseteq P^- = P.$$

Thus $P^* = P$. □

3.2.3 $*$-reductions

We concluded our discussion of integral closure with the definition and some results about reductions of an ideal I in R. The following notion is first described in [6] and is the tight closure analog of a reduction.

Definition 3.2.10. *Given ideals $J \subseteq I$, we say J is a $*$-reduction of I if $I \subseteq J^*$ (equivalently $I^* = J^*$). A $*$-reduction J is minimal if for all ideals $K \subsetneq J$, $I \not\subseteq K^*$.*

The remainder of this paper is concerned with $*$-reductions of the maximal ideal $\mathfrak{m}$ in $k[\Delta]$. In the next chapter, we will see that in this case, $*$-reductions and reductions of $\mathfrak{m}$ are identical notions, which renders the need to distinguish between integral and tight closures moot for our purposes.

Chapter 4: $*$-reductions and $*$-core

In this chapter, we will examine the notions of reductions and $*$-reductions more closely. We focus our attention to the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ of the Stanley-Reisner ring $R = k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$. In this situation, it turns out that reductions and $*$-reductions of $\mathfrak{m}$ are the same ideals. We then describe $*$-reductions (and reductions) of $\mathfrak{m}$ in as much generality as possible, paying special attention to the number of generators of a $*$-reduction, the internal linear algebra that occurs between the generators, and monomials that are common between all $*$-reductions of $\mathfrak{m}$ in a given Stanley-Reisner ring.

4.1 Reductions and $*$-reductions

In the previous chapter, we defined both reductions and $*$-reductions of ideals. Epstein observed that any $*$-reduction of an ideal I is also a reduction [6]. In general, however, the reverse is not true. For example, let $R = k[x, y]$ and let $I = (x^2, y^2)$. Since R is a polynomial ring over a field, all ideals are tightly closed by Theorem 3.2.3, so $I = I^* = (x^2, y^2)$. When we attempt to find the integral closure of I, we see that $xy \in (x^2, y^2)^-$ because

$$(xy)^2 + a_1(xy) + a_2 = 0$$

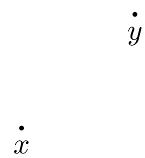
when $a_1 = 0 \in I$ and $a_2 = -x^2y^2 \in I^2$. Hence, I is a reduction of $\mathfrak{m}^2 = (x^2, xy, y^2)$, but not a $*$-reduction of it.

4.1.1 Equality of Reductions and $*$-reductions of $\mathfrak{m}$ in $k[\Delta]$

The example that was just covered is not considered a Stanley-Reisner ring as we are defining them, though most sources allow polynomial rings to be considered Stanley-Reisner. In those cases, a polynomial ring in n variables is the Stanley-Reisner ring associated to the $n-1$ dimensional simplex on n vertices. The result featured in this section applies to these cases, though over all we do not pay much attention to them.

As it turns out, if we keep our focus to the maximal ideals $\mathfrak{m}$ of Stanley-Reisner rings, $*$-reductions and reductions of $\mathfrak{m}$ are the same thing. In the case of $k[\Delta]$ being a polynomial ring, this is easy to see as a direct consequence of Lemma 3.1.10 and Theorem 3.2.3.

If we focus on our definition of a Stanley-Reisner ring, it is beneficial to first look at an example. Let $k[\Delta]$ be the Stanley-Reisner ring corresponding to the simplicial complex consisting of only two vertices:



$$k[\Delta] = k[x,y]/(xy).$$

The set of minimal primes is $\{(x),(y)\}$. Let $I = (x+y)$ be an ideal of $k[\Delta]$. Then,

$$Ik[\Delta]/(x) = yk[y] = (yk[y])^*$$

$$Ik[\Delta]/(y) = xk[x] = (xk[x])^*$$

by Theorem 3.2.3. In $Ik[\Delta]/(x)$, $\overline{x} = \overline{0}$ and in $Ik[\Delta]/(y)$, $\overline{x} = \overline{x+y}$, so by Theorem 3.2.2.(d), $x \in (x+y)^*$. Similarly, $y \in (x+y)^*$. Thus,

$$(x,y) \subseteq (x+y)^* \subseteq (x,y)^* = (x,y)$$

and the tight closure of $(x+y)$ is (x,y).

If we want to determine I^- in the same ring, by Theorem 3.1.7 the equalities,

$$Ik[\Delta]/(x) = yk[y] = \mathfrak{m}k[\Delta]/(x)$$

$$Ik[\Delta]/(y) = xk[x] = \mathfrak{m}k[\Delta]/(y)$$

tell us that $I^- = \mathfrak{m}$ i.e. I is a reduction of $\mathfrak{m}$. Thus in the case of $k[\Delta] = k[x, y]/(xy)$, $I = (x + y)$ is both a reduction and a $*$-reduction of $\mathfrak{m}$.

When concerned with Stanley-Reisner rings, the following theorem provides proof that, in general, if A is the set of reductions of $\mathfrak{m}$ and B is the set of $*$-reductions of $\mathfrak{m}$, then $A \subseteq B$ (or more importantly $A = B$).

Theorem 4.1.1. *Let $k[\Delta]$ be a Stanley-Reisner ring and $\mathfrak{m}$ the maximal ideal of $k[\Delta]$ generated by the images of the variables. Then every reduction of $\mathfrak{m}$ is a $*$-reduction of $\mathfrak{m}$.*

Proof. Let I be a reduction of $\mathfrak{m}$. Then for all minimal primes P in $k[\Delta]$, $(I + P)/P$ is a reduction of $(\mathfrak{m} + P)/P$. Since $k[\Delta]/P$ is a polynomial ring, $(\mathfrak{m} + P)/P$ is basic by Lemma 3.1.10 and therefore $(I+P)/P = (\mathfrak{m}+P)/P$ by Definition 3.1.9. Thus I is also a $*$-reduction of $\mathfrak{m}$. $\square$

From this point forward, all results will apply to both integral closure and tight closure because of Theorem 4.1.1. All theorems will be stated and proven in terms of tight closure because although mentioned first, Theorem 4.1.1 was discovered late in the research. All work prior to this discovery was done in the language of tight closure.

4.1.2 Generating Examples of $*$-reductions of $\mathfrak{m}$

As mentioned above, we will now focus solely on $*$-reductions of $\mathfrak{m}$. If we let $\mathfrak{m} = (x_1, \ldots, x_n)$ in the Stanley-Reisner ring $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$ we can find useful and effective descriptions of the minimal $*$-reductions of $\mathfrak{m}$.

When determining what a $*$-reduction I of $\mathfrak{m}$ must look like, it is best to work backwards and build such an ideal based on what must necessarily be true in order for the given ideal to be a $*$-reduction of $\mathfrak{m}$. By Theorem 3.2.2.(d), it must be true that for a every minimal prime $P = (r_1, \ldots, r_t)$ of $k[\Delta]$ and a $*$-reduction I of $\mathfrak{m}$ with s generators that

$$(I + P)/P = (\mathfrak{m} + P)/P.$$

So if $f_1, \dots, f_s$ are the generators of I, and $f_i = g_i + h_i$ where g_i is the sum of linear summands of f_i and h_i is the sum of the nonlinear summands of f_i, we know

$$(\mathfrak{m} + P)/P = (I + P)/P = ((f_1, \dots, f_s) + P)/P = ((g_1, \dots, g_s) + P)/P,$$

this last equivalence coming from Theorem 4.1.2 i.e. $(g_1, \dots, g_s)$ is a $*$-reduction of $\mathfrak{m}$ and the coefficients of $\overline{g}_1, \dots, \overline{g}_s$, the linear generators of $(I + P)/P$, form an $s \times (n - t)$ matrix that can reduced to row echelon form. Alternatively, $\overline{g}_1, \dots, \overline{g}_n$ form a solvable system of linear equations. When considering all minimal primes of a Stanley-Reisner ring in n variables, the coefficients of the linear terms form an $s \times n$ matrix of values representing a linear system such that for each specific minimal prime P, the columns of the matrix that represent the variables outside of P must on their own form a matrix that can be converted to reduced row echelon form with a leading entry in each column.

For an example, consider the ring $\mathbb{R}[x, y, z]/(xz)$ and the ideal $I = (x+y+z, x+2y+z)$. It is not to hard to check that I is a $*$-reduction of $\mathfrak{m}$. The minimal primes of this ring are $P = (x)$ and $Q = (z)$. If we focus on P, $(I + P)/P$ provides a $2 \times (3 - 1)$ matrix that reduces as follows:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the first column is for the variable y and the second column is for z. Similarly, for Q the resulting matrix of coefficients of the generators of $I + Q/Q$ reduces as such

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for x in the first column and y in the second column.

Because it will be useful in the next section, we provide a second example in the same ring, $\mathbb{R}[x, y, z]/(xz)$, and let the ideal $J = (x + y + 2z, x + 2y + z)$. We will check that J is a $*$-reduction of $\mathfrak{m}$. This time, the ideal P provides the following matrix reduction:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the first column is for the variable y and the second column is for z. Similarly, for Q the matrix reduces as such

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for x in the first column and y in the second column.

The method for finding (linear) examples is surprisingly not complicated. If we continue to focus on $k[\Delta] = \mathbb{R}[x, y, z]/(xy)$, we can observe that the variable complement of $P = (x)$ is the set $\{y, z\}$. The work above tells us that for any $*$-reduction I of $k[\Delta]$, $(I + P)/P \cong (y, z)k[y, z]$. Therefore, it must at least be true that $I = (y + \varphi_1, z + \varphi_2, \ldots, \varphi_s)$ where φ_i is a polynomial in $k[\Delta]$ such that φ_1 and φ_2 have no linear y or z summand respectively. The second minimal prime $Q = (z)$ hints that in I, we must at least be able to find an x term and a y term in distinct generators. If these generators are $y + \varphi_1$ and $z + \varphi_2$ respectively, we can say

$$I = (y + \alpha x + \varphi_1', z + \beta y + \varphi_2', \ldots, \varphi_s)$$

where α and β are nonzero elements of $\mathbb{R}$.

We claim that the ideal $I' = (y + \alpha x, z + \beta y)$ is a $*$-reduction of $\mathfrak{m}$ in most (here all) cases. We will prove a general version of this claim later, but here

$$(I' + (x))/(x) = (y, z + \beta y)k[y, z] = (y, z)k[y, z]$$

because $z = -\beta y + (z + \beta y)$ and

$$(I' + (z))/(z) = (y + \alpha x, \beta y)k[x, y] = (x, y)k[x, y]$$

because

$$x = \alpha^{-1}(y + \alpha x) - (\alpha\beta)^{-1}(\beta y)$$

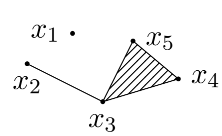
$$y = 0(y + \alpha_x) + \beta^{-1}(\beta y)$$

which shows that I' is a $*$-reduction of $\mathfrak{m}$.

It is beneficial to have an example for which the minimal primes do not all have the same number of generators. Let $k[\Delta] = k[x_1, x_2, x_3, x_4, x_5]/I_\Delta$ where

$$I_\Delta = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5),$$

the Stanley-Reisner ring of the simplicial complex



which has three minimal primes: $P_1 = (x_2, x_3, x_4, x_5)$, $P_2 = (x_1, x_4, x_5)$, and $P_3 = (x_1, x_2)$. We start with the prime P_3 because it has the largest variable complement. It implies that $x_3, x_4,$ and x_5 must be a summand of different generators. In $k[\Delta]$, the simplest ideal that fits this description is $I = (x_3, x_4, x_5)$. The ideal with the next largest complement is P_2. This prime says x_2 and x_3 must be summands in separate generators. We will redefine I with this fact. There are six good ways to do this with nonzero α_1, α_2:

1. $(x_3 + \alpha_1 x_2, x_4 + \alpha_2 x_3, x_5)$

2. $(x_3 + \alpha_1 x_2, x_4, x_5 + \alpha_2 x_3)$

3. $(x_3, x_4 + \alpha_1 x_2, x_5 + \alpha_2 x_3)$

4. $(x_3, x_4 + \alpha_1 x_2, x_5)$

5. $(x_3, x_4 + \alpha_2 x_3, x_5 + \alpha_1 x_2)$

6. $(x_3, x_4, x_5 + \alpha_1 x_2)$

We can now incorporate information from P_1. For nonzero $\beta \in k$, add βx_1 to one of the generators of one of the above ideals. For example, $(x_3 + \alpha_1 x_2 + \beta x_1, x_4 + \alpha_2 x_3, x_5)$. No matter which choice of the above six ideals or which of the three generators in that ideal is chosen to and βx_1 to, the resulting ideal will be a (minimal) $*$-reduction of $\mathfrak{m}$. It is interesting to note given one of the listed six ideals (f_1, f_2, f_3) and any choice of $\beta_1, \beta_2, \beta_3 \in k$, at least one of which is nonzero, $(f_1 + \beta_1 x_1, f_2 + \beta_2 x_1, f_3 + \beta_3 x_1)$ is a (minimal)$*$-reduction of $\mathfrak{m}$.

4.1.3 Echelonization of $*$-reductions

Given a Stanley-Reisner ring $k[\Delta]$ and a $*$-reduction I of $\mathfrak{m}$ there is useful linear algebra we can do on the generating set of I to present the generators of I in the must useful fashion given a particular context. In the previous section, we applied row reductions to the matrices of coefficients of a subset of the variables chosen with respect to the minimal primes of $k[\Delta]$. We will apply the same row reductions on the generators of I without passing to $k[\Delta]/P$ for a minimal prime P. This will give a new generating set of I such that if y, z are distinct variables outside of P, then y and z are summands of different generators of I and if y is a summand of a generator f, then z is not. We will call this process echelonization with respect to P.

As an example, let $I = (x+y+2z, x+2y+z)$ be an ideal of $\mathbb{R}[x,y,z]/(xy)$ with minimal primes $P = (x)$ and $Q = (z)$. We have previously shown that I is a $*$-reduction of $\mathfrak{m}$. If we borrow the row reductions for this example from the previous section, for the ideal P our generating set changes in the following way:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ \frac{1}{3} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}.$$

For the ideal Q, the generators echelonize in the following way:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

The echelonization process has given us two alternate ways to generate I: $(\frac{x}{3}+y, \frac{x}{3}+z)$ and $(x+3z, y-z)$ respectively. We will use echelonization with respect to a chosen P in later results. Because this echelonization process exists, we will often choose to generate a $*$-reduction $I = (f_1, \dots, f_s)$ of $\mathfrak{m}$,with respect to a chosen prime P, by the set of polynomials $\{x_1 + g_1, \dots, x_r + g_r, g_{r+1}, \dots, g_s\}$ where $x_1, \dots, x_r$ are the variables outside of P and the g_i are polynomials that exist in P for $1 \leq i \leq s$.

In the previous examples, the $*$-reductions of $\mathfrak{m}$ that were given or determined were generated by linear polynomials. It is important to note that a $*$-reduction of $\mathfrak{m}$ may be generated by polynomials that are not linear. For example, the ideal $J = (x+y+xz, y+z)$ in $k[x,y,z]/(xyz)$ is a $*$-reduction of $\mathfrak{m}$ and is not generated by linear polynomials. If we let I be an ideal in the same ring generated by the linear parts of the generators of I, i.e. $I = (x+y, y+z)$, we see that I is also a $*$-reduction of $\mathfrak{m}$.

Theorem 4.1.2. *Let $J = (f_1, \dots, f_s)$ be a $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$. Let $f_i = g_i + h_i$ for $1 \leq i \leq s$ where g_i is the polynomial of linear summands of f_i and h_i is the polynomial of nonlinear summands of f_i. If $I = (g_1, \dots, g_s)$, then $I^* = \mathfrak{m}$.*

Proof. Let P be a minimal prime of $k[\Delta]$. Then $(J+P)/P = (\mathfrak{m}+P)/P$. Let $\overline{f}_i$ be the image of f_i in $(J+P)/P$ and let $\overline{x}_j$ be the image of x_j in $(\mathfrak{m}+P)/P$. Then there exist polynomials $a_1, \ldots, a_s$ in $k[\Delta]/P$ such that

$$\overline{x}_j = a_1\overline{f}_1 + \cdots + a_s\overline{f}_s$$

in $(J+P)/P$. Let each $a_i = c_i + b_i$ where c_i is the constant term of a_i and b_i is the sum of every other term. Then

$$\overline{x}_j = \sum c_i\overline{g}_i + \sum b_i\overline{g}_i + \sum a_i\overline{h}_i$$

and since all the terms of $\sum b_i\overline{g}_i + \sum a_i\overline{h}_i$ are of degree greater than one, it must be true that $\sum b_i\overline{g}_i + \sum a_i\overline{h}_i = 0$. Thus for all minimal primes P of $k[\Delta]$,

$$I + P/P = J + P/P = \mathfrak{m}/P$$

and $I^* = \mathfrak{m}$. $\square$

4.2 Analytic Spread and $*$-spread of $\mathfrak{m}$

In the previous section we examined the relationships between the generators of a $*$-reduction of $\mathfrak{m}$. Our aim is try and work exclusively with minimal $*$-reductions of $\mathfrak{m}$ for the rest of the dissertation. To narrow our focus, we need to know the minimal number of generators there are in minimal $*$-reduction of $\mathfrak{m}$ in a given Stanley-Reisner ring $k[\Delta]$. To this end, we make mention of $*$-spread of $\mathfrak{m}$ which was first defined by Epstein [6].

Definition 4.2.1. *Let I be an ideal of R. Suppose that all minimal $*$-reductions of I have the same number of generators. This number is called the $*$-spread of I.*

The $*$-spread of an ideal I in R is tight closure analogous to the integral closure notion of analytic spread, which is the minimal number of generators of a minimal reduction of I. Information about analytic spread can be found in [20]. Because of Theorem 4.1.1, by finding information about $*$-spread of $\mathfrak{m}$ in $k[\Delta]$, we know the corresponding information about analytic spread of $\mathfrak{m}$. This is described in the following corollary to Theorem 4.1.1.

Corollary 4.2.2. *The analytic spread of* $\mathfrak{m}$ *is equal to* $*$-spread $\mathfrak{m}$.

By this Corollary and Corollary 4.2.8, the analytic spread of $\mathfrak{m}$ is

$$d = \dim k[\Delta] = \dim \Delta + 1.$$

The rest of this section is devoted to proving that $*$-spread of $\mathfrak{m}$ is equal to the dimension of the ring. To that end, we start with an easy observation based on Krull's height theorem. For convenience, we will first restate Krull's height theorem. It can be found in most sources on Noetherian commutative algebra, a couple of which are [3, Theorem A.2] [2, Corollary 11.16]

Theorem 4.2.3 (Krull's height theorem)**.** *Let R be a Noetherian ring, and I a proper ideal of height n. Then there exist $x_1, \ldots, x_n \in I$ such that height $(x_1, \ldots, x_i) = i$ for $i = 1, \ldots, n$.*

Observation 4.2.4. *The* $*$-spread *of* $\mathfrak{m}$ *is at least* $d = \dim k[\Delta]$.

Proof. Let I be a $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$ and let $d = \dim k[\Delta]$. Suppose P is a prime of $k[\Delta]$ such that $I \subseteq P$. Then $\mathfrak{m} = I^* \subseteq P^* = P$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} = P$. Thus $\mathfrak{m}$ is the only prime containing I. The height of I is therefore d. By Krull's height theorem, I has at least d generators. $\square$

Though we now know a lower bound for $*$-spread of $\mathfrak{m}$, this does not mean that $d = \dim k[\Delta]$ is for sure the value of $*$-spread of $\mathfrak{m}$. However, the examples of $*$-reductions we have seen so far in this chapter have all been minimal and have all had $d = \dim k[\Delta]$ generators.

Recall that $\Delta_{d,n}$ is the complete $d-1$ dimensional simplicial complex on n variables. Every proper simplicial complex Δ is a subcomplex of $\Delta_{d,n}$ for $d = \dim \Delta + 1$ and n equal to the number of vertices of Δ. The $*$-spread of $\mathfrak{m}$ in $k[\Delta_{d,n}]$ ends up being exactly d and because of the relationship between Δ and $\Delta_{d,n}$, $*$-spread of $\mathfrak{m}$ in $k[\Delta]$ is d as well.

Theorem 4.2.5. *The ideal* $\mathfrak{m}$ *in* $k[\Delta_{d,n}]$ *has minimal* $*$*-reductions with* d *generators.*

Proof. For almost every choice of coefficients $a_{(i,j)} \in k$, we want to show that there exists an ideal $I = (f_1, \ldots, f_d)$ with

$$f_1 = a_{(1,1)}x_1 + \cdots + a_{(1,n)}x_n$$

$$f_2 = a_{(2,1)}x_1 + \cdots + a_{(2,n)}x_n$$

$$\vdots$$

$$f_d = a_{(d,1)}x_1 + \cdots + a_{(d,n)}x_n$$

such that $I^* = \mathfrak{m}$. Let X be the set of variables of $k[\Delta_{d,n}]$. For any minimal prime $P = (X - \{x_{i_1}, \ldots, x_{i_d}\})$ of $k[\Delta_{d,n}]$, I is a $*$-reduction of $\mathfrak{m}$ if $IR/P \cong (x_{i_1}, \ldots, x_{i_d})$ in the ring $k[x_{i_1}, \ldots, x_{i_d}]$. Therefore if we have d generators of I, for any choice of d of the variables, we get a square matrix with the first column corresponding to the coefficients on x_{i_1} across the d linear polynomial generators of I. In all cases, this square matrix must be nonsingular.

Let $\gamma_{j_1,\ldots,j_d}$ be the determinant of the matrix of coefficients for columns $j_1, \ldots, j_d$. These are all nonzero if and only if

$$\gamma = \prod \gamma_{j_1,\ldots,j_d} \neq 0.$$

The product γ is a polynomial in $n \cdot d$ variables and its vanishing cuts out a hypersurface in affine $n \times d$ space. Therefore by [18, Proposition 1.3.], almost any choice of values for $a_{(i,j)}$ will give us the desired ideal I. $\square$

The last part of the above proof states that almost every linearly generated ideal with $d = \dim k[\Delta]$ generators is a $*$-reduction of $\mathfrak{m}$. Therefore, for any Stanley-Reisner ring of the form $k[\Delta_{d,n}]$, examples of minimal $*$-reductions are very easy to find.

The ideals constructed in Theorem 4.2.5 can also be used to find the $*$-spread of $\mathfrak{m}$ in an arbitrary Stanley-Reisner ring $k[\Delta]$:

Theorem 4.2.6. *If $\Delta_{d,n}$ is the smallest complete simplicial complex for which Δ is a proper subcomplex i.e Δ has n vertices and $\dim \Delta = d-1$, then the maximal ideal $\mathfrak{m}$ in $k[\Delta]$ has minimal $*$-reductions with d generators. The $*$-reduction is minimal by Observation 4.2.4.*

Proof. Since $\Delta \subset \Delta_{d,n}$, there exists a natural surjection

$$\varphi : k[\Delta_{d,n}] \twoheadrightarrow k[\Delta].$$

If we define I to be as in Theorem 4.2.5, then $\varphi(I)$ is an ideal of $k[\Delta]$. If $\mathfrak{n}$ is the maximal ideal of $k[\Delta_{d,n}]$ generated by the images of the variables, then

$$\mathfrak{m} = \varphi(\mathfrak{n}) = \varphi(I^*) \subseteq \varphi(I)^* \subseteq \mathfrak{m},$$

where the first inclusion follows from Theorem 3.2.4. Hence $\varphi(I)^* = \mathfrak{m}$ which means $\mathfrak{m}$ has a $*$-reduction with d generators. Thus the $*$-spread $\mathfrak{m} \leq d$. But by Observation 4.2.4 $\dim k[\Delta] = d$, so $*$-spread $\mathfrak{m} \geq d$ as well. Hence $*$-spread $\mathfrak{m} = d$. □

Even though the $*$-spread of $\mathfrak{m}$ is at least d, there do exist $*$-reductions with minimal generating sets of more than d generators. We will show that such a reduction is not minimal. Using iterations of the following theorem we can show that given a $*$-reduction J of $\mathfrak{m}$ with more than d generators, we can find an ideal $I \subseteq J$ that is a minimal $*$-reduction of $\mathfrak{m}$. The following result is analogous to a result of Epstein [6, Theorem 5.1], though his result is for excellent analytically irreducible local domains of characteristic $p > 0$ and the method of proof is different.

Theorem 4.2.7. *Let $d = \dim k[\Delta]$ and let $J = (f_1, \ldots, f_c)$ with $c \geq d+1$. If $J^* = \mathfrak{m}$, then there exists $I = (g_1, \ldots, g_{c-1})$ such that $I \subset J$ and $I^* = \mathfrak{m}$.*

Proof. Let $P_1, \ldots, P_s$ be the minimal primes of $k[\Delta]$ ordered such that for $r \leq s$ and

$1 \leq i \leq r$, if J is as defined above,

$$(f_1, \ldots, f_{c-1})k[\Delta]/P_i \cong Jk[\Delta]/P_i \cong \mathfrak{m}k[\Delta]/P_i$$

and if $i > r$, then

$$(f_1, \ldots, f_{c-1})k[\Delta]/P_i \ncong Jk[\Delta]/P_i \cong \mathfrak{m}k[\Delta]/P_i.$$

Let $Q = (x_{t+1}, \ldots, x_n) \in \{P_{r+1}, \ldots, P_s\}$. Then there exists $f_1', \ldots, f_c' \in k[\Delta]$ such that the linear part of each of these polynomials only include variables from $\{x_{t+1}, \ldots, x_n\}$ such that

$$J = (x_1 + a_1x_t + f_1', \ldots, x_{t-1} + a_{t-1}x_t + f_{t-1}', f_t', \ldots, f_{c-1}', x_t + f_c')$$

and

$$(f_1, \ldots, f_{c-1}) = (x_1 + a_1x_t + f_1', \ldots, x_{t-1} + a_{t-1}x_t + f_{t-1}', f_t', \ldots, f_{c-1}')$$

for some $a_1, \ldots, a_t \in k$. Then letting $g_i = a_ix_t + f_i', 1 \leq i \leq t-1$,

$$(x_1 + g_1, \ldots, x_{t-1} + g_{t-1}, \alpha(x_t + f_c') + f_t', \ldots, f_{c-1}')k[\Delta]/Q \cong \mathfrak{m}k[\Delta]/Q$$

for any nonzero $\alpha \in k$.

We want to show that we can choose the above α in such a way that for $J' = (x_1 + g_1, \ldots, x_{t-1} + g_{t-1}, \alpha(x_t + f_c') + f_t', \ldots, f_{c-1}')$,

$$J'k[\Delta]/P_i \cong \mathfrak{m}k[\Delta]/P_i$$

for all $1 \leq i \leq r$

For each $P_i \in \{P_1, \ldots, P_r\}$ attempt to echelonize the generators of J' as in the beginning of the section except for $\alpha(x_t + f_c') + f_t'$, leave that generator untouched. The echelonization of the other $c-2$ generators will include a leading term in the column for each variable outside of P_i with the exception of at most one of the variables. If every necessary column has a leading term after the echelonization process, then

$$J'k[\Delta]/P_i \cong \mathfrak{m}k[\Delta]/P_i$$

no matter the choice of α. Otherwise, the echelonization misses exactly one column. In this case we need the generator $\alpha(x_t + f'_c) + f'_t$ to accommodate this column. This depends on the choice of α.

Assume that the column this generator is needed to accommodate is the one for the variable y and that β is the coefficient on y and that α_j is the coefficient on x_j in $\alpha(x_t+f'_c)+f'_t$. Part of the echelonization process includes performing a row operation to remove $\alpha_j x_j$ as part of this sum, which potentially alters the coefficients on the terms of $\alpha(x_t + f'_c) + f'_t$, including β. It is therefore necessary for β to not be both equal in magnitude and opposite in sign to the cumulative effect of these row operations i.e. for each γ_j, the coefficient on y in the generator with leading term in the x_j column, $0 \neq \beta - \sum \alpha_j \gamma_j$. Since α only appears once in this equation as part of the construction of β, only at most one value of α will not work for each P_i. Therefore since there are r minimal primes P_i that we need to check, there are at most r values of α that will not work. Since k is infinite, almost any value of α will work.

Repeat this process from the beginning of the proof until $r = s$ and let final resulting $J' = I$. Thus there exists an ideal $I \subset J$ with one less generator than J such that $I^* = \mathfrak{m}$. □

Corollary 4.2.8. *If I is a $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$, then there exists an ideal $J \subseteq I$ of $k[\Delta]$ with d generators such that $J^* = \mathfrak{m}$ i.e $*$-spread of $\mathfrak{m}$ is d.*

Proof. This follows naturally by reverse induction on c. □

4.3 $\operatorname{core} \mathfrak{m}$ and $*\text{-core}\, \mathfrak{m}$

At this point, we have a working knowledge about minimal reductions and minimal $*$-reductions of $\mathfrak{m}$ in $k[\Delta]$. Specifically we know that they all have the same number of generators. It is tempting to examine what other characteristics all minimal reductions and minimal $*$-reductions have in common. For this, we turn to Rees and Sally [21]. They had the idea to intersect all minimal reductions of an ideal I in R to see which elements were common to all reductions. For this they defined the *core* of I, denoted $\operatorname{core}(I)$, to be the intersections of all reductions of I (this definition was sourced from [17, Definition 17.8.8]). In this section, we will examine the notion of core, but for $*$-reductions of $\mathfrak{m}$.

4.3.1 Definition and Examples

We begin with the definition of $*\text{-core}(I)$ for an ideal $I \subset R$. This notion is introduced in [7] and is analogous to the notion of core.

Definition 4.3.1. *Let I be an ideal of a Noetherian ring R. The $*\text{-core}$ of I, denoted $*\text{-core}(I)$ is the intersection of all (minimal) $*$-reductions of I.*

If we return to the ring $k[\Delta] = k[x,y]/(xy)$ for infinite field k, we saw that $(x+y)$ was a (minimal) $*$-reduction of $\mathfrak{m}$. In fact, all minimal $*$-reductions of $\mathfrak{m}$ in this ring are of the form $(x + \lambda y)$, where λ is any nonzero element of k. Then,

$$*\text{-core}\, \mathfrak{m} = \bigcap_{\lambda} (x + \lambda y) = (x^2, y^2).$$

The fact that $(x^2, y^2) \subseteq \bigcap_{\lambda}(x+\lambda y)$ is easy to show: $x^2 = x(x+\lambda y)$ and $y^2 = \frac{1}{\lambda} y(x+\lambda y)$. The containment $\bigcap_{\lambda}(x+\lambda y) \subseteq (x^2, y^2)$ is not as easy to show. It relies on the facts that $*\text{-core}(\mathfrak{m})$ is generated by monomials and if $x, y \in (x + \lambda y)$ for all λ, then the only $*$-reduction of $\mathfrak{m}$ is $\mathfrak{m}$ itself.

The importance of using an infinite field k in the above example cannot be overstated. For example, if $R = k[\Delta_{1,2}] = k[x,y]/(xy)$ where $k = \mathbb{F}_2$, then the only minimal $*$-reduction of $\mathfrak{m}$ is $(x+y)$, which means that the intersection of all minimal $*$-reductions of $\mathfrak{m}$ is $(x+y)$. The upper and lower bound for $*$-core($\mathfrak{m}$) we introduce in the following sections are only guaranteed when k is infinite.

In [7], the authors show that core and $*$-core agree if analytic spread is equal to $*$-spread for normal local domains of characteristic $p > 0$ with infinite perfect residue fields. This fact inspired the following result, though we present it as a corollary to Theorem 4.1.1. Indeed, for any Stanley-Reisner ring $k[\Delta]$, an analogous result is true for the core and $*$-core of $\mathfrak{m}$.

Corollary 4.3.2. *For the maximal ideal* $\mathfrak{m}$ *of* $k[\Delta]$,

$$\operatorname{core} \mathfrak{m} = *\text{-core}\, \mathfrak{m}.$$

In particular, we will show in Theorem 4.3.8 that

$$\mathfrak{m}^{d+1} + \tau\mathfrak{m} \subseteq \operatorname{core} \mathfrak{m} \subseteq \mathfrak{m}^2$$

and in Chapter 5, we will explore $*$-core $\mathfrak{m}$ in different cases. Once again, the results in this section will be valid for core($\mathfrak{m}$) as well as $*$-core($\mathfrak{m}$), though the proofs are done in the language of tight closure.

As mentioned in the above example, $*$-core($\mathfrak{m}$) is generated by monomials. This is due to the following result:

Theorem 4.3.3. *Suppose* $I \subseteq k[\Delta]$ *is an ideal generated by monomials and that* $I = I^*$*. Then* $*$-core I *is generated by monomials.*

Proof. Let $S = k[x_1, \dots, x_n]$, a polynomial ring in n variables over an infinite field k and let I_Δ be the defining ideal of a simplicial complex such that $k[\Delta] = S/I_\Delta$. There exists a group action of $G = (k^\times)^n$ on S, where $k^\times$ is the largest multiplicative subgroup of k, defined by

$$(\lambda_1, \ldots, \lambda_n) \cdot f(x_1, \ldots, x_n) := f(\lambda_1 x_1, \ldots, \lambda_n x_n).$$

The fixed ideals of S under G are the monomial ideals. The action of G on S induces an action on $k[\Delta]$ since I_Δ is a monomial ideal. Let J be an ideal of $k[\Delta]$ such that $J^* = I$. Then for any $g \in G$, $g \cdot J = \{g \cdot a \mid a \in J\}$ is an ideal and $(g \cdot J)^* = I$ by Theorems 3.2.4 and 3.2.7. Thus

$$*\text{-core}\, I = \bigcap_{\{J \mid J^* = I\}} J = \bigcap_{\{J \mid J^* = I\}} g \cdot J = g \cdot \bigcap_{\{J \mid J^* = I\}} J = g \cdot *\text{-core}\, I.$$

Since the group action fixes $*$-core I, $*$-core I must be generated by monomials. $\square$

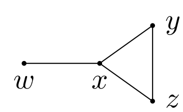
4.3.2 Lower Bound of $*$-core$(\mathfrak{m})$ in $k[\Delta_{d,n}]$

Calculating the $*$-core of any ideal can be difficult, so we want to have a better idea of where we should look for the monomials that generate the $*$-core. The first place we look is to the test ideal τ of the ring R. The test ideal τ is defined by Hochster and Huneke [14] the following way:

$$\tau := \bigcap_{I \text{ ideal of } R} (I : I^*)$$

which by Vassilev [25, Theorem 3.7] can also be defined to be the sum of the annihilating ideals of the minimal primes in a Stanley-Reisner ring $k[\Delta]$. The test ideal is not hard to find in the setting of Stanley-Reisner rings, especially since the minimal primes of a Stanley-Reisner ring $k[\Delta]$ are easy to describe. In our example of $k[\Delta] = k[x, y]/(xy)$, the annihilator of (x) is (y) and the annihilator of (y) is (x), therefore $\tau = (x, y)$.

For a more interesting example, let $k[\Delta] = k[w,x,y,z]/(wy,wz,xyz)$, the Stanley Reisner ring of the simplicial complex



where $\{x,y,z\}$ is not a face of Δ. The minimal primes of Δ are (w,x), (w,y), (w,z), and (y,z). The annihilators of these ideals are

$$ann(w,x) = (y,z) \cap (yz) = (yz)$$

$$ann(w,y) = (y,z) \cap (xz,w) = (xz)$$

$$ann(w,z) = (y,z) \cap (xy,w) = (xy)$$

$$ann(y,z) = (xz,w) \cap (xy,w) = (w)$$

Thus $\tau = (yz,xz,xy,w)$.

The following observation about the test ideal τ from [8, Observation 3.1] provides a computationally based lower bound for the $*$-core of an ideal.

Observation 4.3.4. *Let R be a ring of any equal characteristic with test ideal τ. Let I be an ideal of R. Then $\tau I \subseteq *\text{-core}\, I$.*

In simple cases, τI is exactly the $*\text{-core}\, I$. For example, in $k[x,y]/(xy)$, $\tau\mathfrak{m} = (x,y)^2 = (x^2,y^2) = *\text{-core}\,\mathfrak{m}$. As the dimension of Stanley-Reisner rings increases, this lower bound does not capture all the information about $*\text{-core}\, I$. The ring $k[x,y,z]/(xy)$ has test ideal $\tau = (x,y)$ and

$$\tau\mathfrak{m} = (x,y)\cdot(x,y,z) = (x^2,xz,y^2,yz),$$

but the monomial z^2 is also easily computed to be in $*\text{-core}\,\mathfrak{m}$. The difficulty of computation also tends to increase as simplicial complexes get more complicated. Because of this, we provide a different lower bound for $*$-core $\mathfrak{m}$, namely $\mathfrak{m}^{d+1} \subseteq *$-core $\mathfrak{m}$. Often, this lower bound is exactly $*$-core $\mathfrak{m}$. To show this we need the following lemma.

Lemma 4.3.5. *Let Δ be a $d-1$ dimensional simplicial complex on n vertices. Let $\alpha_1, \alpha_2, \ldots, \alpha_s$, $s < t$, be a partition of the positive integer $t \leq d+1$ with*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s \geq 1; \alpha_1 \geq 2$$

Then all monomials of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_s}^{\alpha_s}$ (with $i_1, \ldots, i_s$ distinct) are in the $*$-core $\mathfrak{m}$ if all monomials of the form $x_{j_1}^{\alpha_1 - 1} x_{j_2}^{\alpha_2} \cdots x_{j_s}^{\alpha_s} x_{j_{s+1}}$ (with $j_1, \ldots, j_{s+1}$ distinct) are in $*$-core $\mathfrak{m}$.

Proof. By making no assumptions about the location of the vertex x_{i_1} in the simplicial complex, it is enough to show that a single monomial of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_s}^{\alpha_s}$ is in $*$-core($\mathfrak{m}$). If so, then we have shown they all are in $*$-core($\mathfrak{m}$). We therefore will let $i_1 = 1$, $i_2 = 2, \ldots, i_s = s$. If $\prod_{i=1}^{s} x_i$ is in I_Δ, we are done. Otherwise, $\{v_1, \ldots, v_s\}$ is a face of Δ. Therefore there exists a prime P of $k[\Delta]$ such that for any minimal $*$-reduction of $I = (f_1, \ldots, f_d)$ of $\mathfrak{m}$, we know by Theorem 4.2.5 and Corollary 4.2.8 that $I = (x_1 + g_1, \ldots, x_s + g_s, g_{s+1}, \ldots, g_d)$ where each g_r, $1 \leq r \leq d$, is a polynomial with no linear terms in the variables $x_1, \ldots, x_s$ and every nonlinear term of g_i is divisible by a variable not from $\{x_1, \ldots, x_s\}$. Multiply $x_1 + g_1$ by $x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$. Then $x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} \cdot g_1$ is a polynomial in which all the terms are divisible by a monomial of the form $x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} y$ for $y \notin \{x_1, \ldots, x_s\}$. Since $x_1 + g_1 \in I$ and each term of $x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} \cdot g_1$ is, by assumption, in $*$-core($\mathfrak{m}$)$\subseteq I$, $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$ must also be in I. Since I and x_1 were arbitrary, $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$ is in every minimal $*$-reduction of $\mathfrak{m}$ and is therefore in $*$-core($\mathfrak{m}$). $\square$

This lemma will be used inductively in the proof of the following theorem:

Theorem 4.3.6. *Let $\Delta_{d,n}$ be the complete $d-1$ dimensional simplicial complex on n*

vertices, and $\mathfrak{m}$ *the maximal ideal of* $k[\Delta_{d,n}]$*. Then*

$$\mathfrak{m}^{d+1} \subseteq *\text{-core}\,\mathfrak{m}.$$

Proof. The ideal $\mathfrak{m}^{d+1}$ is generated by all monomials of degree $d+1$ over the variables $x_1, \dots, x_n$. If we can show all such monomials of are in $*$-core $\mathfrak{m}$, then $*$-core $\mathfrak{m}$ is bounded below by $\mathfrak{m}^{d+1}$. The degree distribution on any monomial corresponds to a partition of $d+1$. The length of a partition, s, is $1 \leq s \leq d+1$. In $k[\Delta_{d,n}]$, any product of $d+1$ distinct variables is 0. Therefore, all products of $d+1$ distinct variables are in $*$-core $\mathfrak{m}$. These correspond to the only partition of length $d+1$.

Suppose that all monomials corresponding to partitions of $d+1$ of length s are in $*$-core $\mathfrak{m}$, $s > 1$. Let $x_{i_1}^{\alpha_1} \cdots x_{i_{s-1}}^{\alpha_{s-1}}$ be a monomial corresponding to a partition of length $s-1$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_s \geq 1$. The exponent $\alpha_1 \geq 2$ by the pigeonhole principle. Then by Lemma 4.3.5, $x_{i_1}^{\alpha_1} \cdots x_{i_{s-1}}^{\alpha_{s-1}} \in *$-core $\mathfrak{m}$ because all monomials of the form $x_{j_1}^{\alpha_1 - 1} \cdots x_{j_{s-1}}^{\alpha_{s-1}} x_{j_s} \in *$-core $\mathfrak{m}$. Thus by induction, $\mathfrak{m}^{d+1} \subseteq *$-core $\mathfrak{m}$. □

4.3.3 Upper Bound of $*$-core($\mathfrak{m}$) in $k[\Delta_{d,n}]$

Unlike the lower bound for $*$-core($\mathfrak{m}$), there does not appear to be an existing upper bound other that $\mathfrak{m}$ itself. However, we did see an example earlier in $k[x,y]/(xy)$ where $*$-core($\mathfrak{m}$) $\subseteq (x^2, y^2) = \mathfrak{m}^2$. This result holds in general for $k[\Delta_{d,n}]$.

Theorem 4.3.7. *Let* $\Delta_{d,n}$ *be the complete* $d-1$ *dimensional simplical complex on* n *vertices where* $d < n$*, and* $\mathfrak{m}$ *the maximal ideal of* $k[\Delta_{d,n}]$ *generated by the variables. Then*

$$*\text{-core}\,\mathfrak{m} \subseteq \mathfrak{m}^2.$$

Proof. Since a simplicial complex of the form $\Delta_{d,n}$ is symmetric in the sense that all vertices are exactly identical except for their name, if one variable x_i is in $*$-core $\mathfrak{m}$, all variables are in $*$-core $\mathfrak{m}$, which forces every minimal $*$-reduction of $\mathfrak{m}$ to be exactly equal to $\mathfrak{m}$. Since $d < n$, this is an impossibility. Thus $*$-core $\mathfrak{m}$ contains no degree one monomials. But $*$-core $\mathfrak{m}$ is generated by monomials and the smallest possible remaining monomial members of $*$-core $\mathfrak{m}$ are the degree two monomials. Thus $*$-core $\mathfrak{m} \subseteq \mathfrak{m}^2$. □

4.3.4 Bounds for $*$-core($\mathfrak{m}$) in $k[\Delta]$

We conclude this chapter by establishing bounds for $*$-core($\mathfrak{m}$) for general $k[\Delta]$. The bounds are established by the previously discussed relationship between Δ of dimension $d-1$ on n vertices and $\Delta_{d,n}$.

Theorem 4.3.8. *If Δ is a simplicial complex of dimension $d-1$ on n vertices, then for $\mathfrak{m}$ of $k[\Delta]$,*

$$\mathfrak{m}^{d+1} + \tau\mathfrak{m} \subseteq *\text{-core}\,\mathfrak{m} \subseteq \mathfrak{m}^2.$$

Proof. We want to show that no variable of the Stanley-Reisner ring $k[\Delta]$ is in the $*$-core of the maximal ideal $\mathfrak{n}$ of $k[\Delta]$ generated by the variables. Showing this will confirm that $*$-core($\mathfrak{n}$) $\subseteq \mathfrak{n}^2$.

Let I represent a linearly generated minimal $*$-reduction of $\mathfrak{m}$ in $k[\Delta_{d,n}]$ of the type crafted in Theorem 4.2.5. Let $f : k[\Delta_{d,n}] \twoheadrightarrow k[\Delta]$ be the natural surjection between the Stanley-Reisner ring of $\Delta_{d,n}$ and the Stanley-Reisner ring associated to Δ, a proper simplicial subcomplex of $\Delta_{d,n}$ of dimension $d-1$ on n vertices, then $f(I)$ is a $*$-reduction of $f(\mathfrak{m}) = \mathfrak{n}$, the maximal ideal generated by the variables in $k[\Delta]$. Without loss of generality, suppose the variable x_1 is in $*$-core($\mathfrak{n}$). Then $x_1 \in f(I)$. All ideals generated by $n-d$ of the variables $k[\Delta_{d,n}]$ are minimal primes of $k[\Delta_{d,n}$. Define $P = (x_{d+1}, \ldots, x_n)$ to be one such prime of $k[\Delta_{d,n}]$, then we can echelonize the matrix of coefficients of the linear terms of the generators of I to get the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & a_{1,d+1} & a_{1,d+2} & \cdots & a_{1,n} \\ 0 & 1 & 0 & \cdots & 0 & a_{2,d+1} & a_{2,d+2} & \cdots & a_{2,n} \\ 0 & 0 & 1 & \cdots & 0 & a_{3,d+1} & a_{3,d+2} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{d,d+1} & a_{d,d+2} & \cdots & a_{d,n} \end{pmatrix}$$

where the coefficient of x_j in the i^{th} row is 1 when $i = j$ and 0 in all other rows for $1 \leq j \leq d$, and for $d+1 \leq j \leq n$, $a_{i,j}$ is the coefficient on x_j in the i^{th} row.

The surjection f preserves the elements of k, so $f(A)$ is the same matrix as A. For simplicity, let $g_1, \ldots, g_d$ be the generators of $f(I)$ as they are presented as rows in $f(A)$. Since $x_1 \in f(I)$, there exist $r_1, \ldots, r_d$ such that $x_1 = r_1 g_q + \cdots + r_d g_d$.

Note that in all the g_i, $x_1, ..., x_d$ only appear as a summand in one generator, so for $2 \leq i \leq d$, r_i is responsible for eliminating x_i in the above equation. This can only happen if $r_i = 0$. Thus $x_1 = r_1 g_q$. Since x_1 is a summand of g_1 with coefficient 1, r_1 is forced to be 1, so $x_1 = g_1$. Thus it must be the case that $a_{1,j} = 0$ for all $j \geq d+1$, and thus $a_{1,j} = 0$ in $k[\Delta_{d,n}]$ as well.

Let $Q = (x_{d+2}, \ldots, x_n, x_1)$ be a minimal prime of $k[\Delta_{d,n}]$. Then $I + Q/Q$ produces the $d \times d$ square matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_{1,d+1} \\ 1 & 0 & \cdots & 0 & a_{2,d+1} \\ 0 & 1 & \cdots & 0 & a_{3,d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{d,d+1} \end{pmatrix}$$

where $a_{1,d+1} = 0$. However,

$$I + Q/Q = \mathfrak{m} + Q/Q = (x_2, \ldots, x_{d+1}),$$

which means $a_{1,d+1}$ must be a value other than 0. Thus x_1, and by symmetry any other variable, is not a element of $*$-core($\mathfrak{n}$)

Preserving the lower bound only requires using Lemma 4.3.5 because the lemma is true for all Δ. All monomials corresponding to the length $d+1$ partition of $d+1$ are zero in $k[\Delta]$, so they are, by default, in every ideal of $k[\Delta]$, including $*$-core $\mathfrak{n}$. If we want to show the monomial $x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ such that $\alpha_1 + \cdots + \alpha_t = d+1$ and $\alpha_i \geq \alpha_{i+1}$ is in $*$-core $\mathfrak{n}$, choose a minimal prime $P = (y_1, \ldots, y_t)$ of $k[\Delta]$ such that $\{x_1, \ldots, x_s\} \cap \{y_1, \ldots, y_t\} = \varnothing$. If no such ideal exists, $x_1^{\alpha_1} \cdots x_s^{\alpha_s} = 0$. Echelonize and reduce the linear variables of the generators of an arbitrary minimal $*$-reduction J and then manipulate the generators so that the nonlinear terms are all divisible by a variable in P. One of the generators of J will be of the form $x_1 + g$ where g is a sum consisting of linear terms in the prime P and nonlinear terms divisible by variables in the prime P. Then if all the terms of $x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} \cdot g$ are in $*$-core $\mathfrak{n}$, then so is $x_1^{\alpha_1} \cdots x_s^{\alpha_s}$. All these terms are in $*$-core $\mathfrak{n}$ by the same induction as in Theorem 4.3.6. $\square$

Chapter 5: Special Cases

Using the techniques and machinery built in previous sections, we can calculate $*$-core $\mathfrak{m}$ for many classes of Stanley-Reisner rings without intersecting every minimal $*$-reduction of $\mathfrak{m}$. What these calculations suggest is that the structure of the simplicial complex plays a significant role in determining the $*$-core of $\mathfrak{m}$. We will first look at what happens when the simplicial complex consists of disjoint components.

5.1 Disconnected Simplicial Complexes

In what may be considered the simplest form, the disconnected components of a simplicial complex are all simplices when considered individually, such as the example $k[x, y]/(xy)$, which is a disjoint union of two 0-faces. It turns out that no matter the dimension of Δ, if Δ is a disjoint union of simplices, then $*$-core($\mathfrak{m}$) is always $\mathfrak{m}^2$.

Proposition 5.1.1. *If Δ is a disjoint union of two or more simplices, then* $*$-core $\mathfrak{m} = \mathfrak{m}^2$.

Proof. For simplices Δ_i with $1 \le i \le r$, let

$$\Delta = \bigcup_{i=1}^{r} \Delta_i$$

such that $\Delta_i \cap \Delta_j = \emptyset$ for any choice of i and j, $i \neq j$. Let the vertices of Δ_i be the set $\{v_{i,1}, \ldots, v_{i,n_i}\}$. Then

$$k[\Delta] = k[x_{1,1}, \ldots, x_{1,n_1}, x_{2,1}, \ldots, x_{2,n_2}, \ldots, x_{r,1}, \ldots, x_{r,n_r}]/I_\Delta$$

where I_Δ is generated by all degree two monomials $x_{i,s}x_{j,t}$, with $i \neq j$, $1 \leq s \leq n_i$, and $1 \leq t \leq n_j$.

Let X be the set of all variables. The ring $k[\Delta]$ has r minimal primes $P_1, \ldots, P_r$ such that $P_i = (X - \{x_{i,1_i}, \ldots, x_{i,n_i}\})$. The annihilator of P_i is the ideal $(x_{i,1_i}, \ldots, x_{i,n_i})$. Therefore the test ideal $\tau = \mathfrak{m}$ and

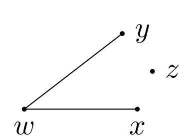
$$\tau \cdot \mathfrak{m} = \mathfrak{m}^2 \subseteq *\text{-core}\,\mathfrak{m}.$$

Since we have shown in Theorem 4.3.8 that an upper bound for $*$-core $\mathfrak{m}$ is always $\mathfrak{m}^2$, $*$-core $\mathfrak{m} = \mathfrak{m}^2$. $\square$

As a corollary to this theorem, the $*$-core $\mathfrak{m}$ in $k[\Delta]$ when $\dim \Delta = 0$ is $\mathfrak{m}^2$. This fact can also be inferred from the previously established bounds for the $*$-core $\mathfrak{m}$ in Theorem 4.3.8 because in any one dimensional ring, $\mathfrak{m}^{d+1} = \mathfrak{m}^2$.

Corollary 5.1.2. *Let $k[\Delta]$ be a Stanley-Reisner ring of dimension 1. Then $*$-core$(\mathfrak{m}) = \mathfrak{m}^2$.*

We will now begin to relax the condition that that the disjoint pieces of the simplicial complex are simplices. We will first look at rings where the complex is a disjoint union between a proper simplicial complex and a simplex such as the ring $k[w, x, y, z]/(xy, wz, xz, yz)$ which is the Stanley-Reisner ring for the simplicial complex



which is the disjoint union of the 0-simplex $\{\emptyset, \{z\}\}$ and the simplicial complex

$$\{\emptyset, \{w\}, \{x\}, \{y\}, \{w, x\}, \{w, y\}\}.$$

Proposition 5.1.3. *Let $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1 is a proper simplicial complex on the variable set $\{x_1, \ldots, x_n\}$ and Δ_2 is a simplex on the variable set $\{y_1, \ldots, y_m\}$ that is disjoint* *from* Δ_1*. Let* $\mathfrak{m}$ *be an ideal of* $k[\Delta]$*. Then*

$$*\text{-core}\,\mathfrak{m} = \varphi^{-1}(*\text{-core}\,\mathfrak{m}_1) + (y_1, \ldots, y_m)^2,$$

where φ *is the natural surjection from* $k[\Delta]$ *to* $k[\Delta_1]$ *and* $\mathfrak{m}_1$ *is the maximal ideal generated by the images of the variables in* $k[\Delta_1]$*.*

Proof. Let $P_1, \ldots, P_s$ be the minimal primes of the ring $k[\Delta_1]$. Then the ideals $Q_i = P_i k[\Delta] + (y_1, \ldots, y_m)$ are minimal primes of $k[\Delta]$ for $1 \leq i \leq s$. In addition to the ideals Q_i, the ideal $(x_1, \ldots, x_n)$ is the only other minimal prime of $k[\Delta]$. If we compute the test ideal τ, we see

$$\text{ann}\,(Q_i) = \text{ann}\,(P_i) \cdot k[\Delta] \cap (x_1, \ldots, x_n) = \text{ann}\,(P_i) \cdot k[\Delta]$$

and

$$\text{ann}\,(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$$

so

$$\tau = \sum_{i=1}^{s} \text{ann}\,(P_i) \cdot k[\Delta] + (y_1, \ldots, y_m).$$

The lower bound we get for the $*$-core $\mathfrak{m}$ by computing $\tau \cdot \mathfrak{m}$ is

$$\tau \cdot \mathfrak{m} = \sum_{i=1}^{s} \text{ann}\,(P_i) \cdot k[\Delta] \cdot (x_1, \ldots, x_n) + (y_1, \ldots, y_m)^2$$

which shows that inclusion of Δ_2 in the simplicial complex Δ results in the inclusion of the degree two monomials in the variables $y_1, \ldots, y_m$ as generators of $*$-core $\mathfrak{m}$. Since any product $x_i y_j$ is in I_Δ, we need only determine which monomials in the variables $x_1, \ldots, x_n$ are generators of the $*$-core $\mathfrak{m}$.

For $d = \dim k[\Delta]$, let $J = (f_1, \ldots, f_d)$ be a linearly generated minimal $*$-reduction of $\mathfrak{m}$

in $k[\Delta]$. Each generator f_i of J can be written $f_i = g_i + h_i$ where g_i is a linear polynomial in the variables $x_1, \ldots, x_n$ and h_i is a linear polynomial in the variables $y_1, \ldots, y_m$. Let φ be the natural surjection from $k[\Delta]$ to $k[\Delta_1]$ and let $\mathfrak{m}_1$ be the maximal ideal of $k[\Delta_1]$ generated by the images of the variables. As we have seen previously,

$$\mathfrak{m}_1 = \varphi(\mathfrak{m}) = \varphi(J^*) \subseteq \varphi(J)^* \subseteq \mathfrak{m}_1,$$

so the ideal $\varphi(J) = (g_1, \ldots, g_d)$ is a $*$-reduction of $\mathfrak{m}_1$. Let α be a monomial in $*$-core $\mathfrak{m}_1$, then for some $\beta_1, \ldots, \beta_d$ in $k[\Delta_1]$, $\alpha = \beta_1 g_1 + \cdots + \beta_d g_d$. If $\overline{\beta}_i$ represents the element in the inverse image of β_i in only the variables $x_1, \ldots, x_n$, then in the ideal J,

$$\overline{\beta}_1 f_1 + \cdots + \overline{\beta}_d f_d = \overline{\beta}_1 g_1 + \cdots + \overline{\beta}_d g_d$$

is equal to the element of the preimage of α that has no terms in the variables $y_1, \ldots, y_m$. The only such element that exists is the monomial α itself in $k[\Delta]$. Thus every monomial in $*$-core $\mathfrak{m}_1$ is in $*$-core $\mathfrak{m}$.

The last thing we must show is that if α is a monomial in the variables $x_1, \ldots, x_n$ and $\alpha \in *$-core $\mathfrak{m}$, then α is also in $*$-core $\mathfrak{m}_1$. Let $d_1 = \dim k[\Delta_1]$ and let $J = (g_1, \ldots, g_{d_1})$ be a minimal $*$-reduction of $\mathfrak{m}_1$. One of the following two things is true about d_1: $d_1 < d = m$ or $d_1 = d$. To extend J to a minimal $*$-reduction of $\mathfrak{m}$, it is important to note that $J \cdot k[\Delta]$ is such that $J \cdot k[\Delta]/Q_j = \mathfrak{m}/Q_j$ for all j. So $J \cdot k[\Delta]$ meets all the criteria to be a $*$-reduction of $\mathfrak{m}$ except for the requirement that $J \cdot k[\Delta]/(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$. In both cases, we can extend J to a minimal $*$-reduction $J' = (f_1, \ldots, f_d)$ of $\mathfrak{m}$ by adding y_i, for $1 \leq i \leq m$, to the ith generator of J, and using 0 as for all possible generators $g_{d_1+1}, \ldots, g_d = g_m$ if $d_1 < d$. Let α be a monomial in $*$-core $\mathfrak{m}$ consisting of only variables $x_1, \ldots, x_n$. Then there exist $\beta_1, \ldots, \beta_d$ in $k[\Delta]$ such that $\alpha = \beta_1 f_1 \cdots + \beta_d f_d$. Each β_i can be written $\beta_i = b_i + b_i'$ such that the b_i are the terms of β_i in the variables $x_1, \ldots, x_n$ and the b_i' are the terms in variables $y_1, \ldots, y_m$. Then

$$\alpha = \beta_1 f_1 + \cdots + \beta_d f_d = b_1 g_1 + \cdots + b_1 g_{d_1} \in J.$$

Thus α is in all minimal $*$-reductions of $\mathfrak{m}_1$ and therefore, is in $*\text{-core}\, \mathfrak{m}_1$. $\square$

We no describe $*\text{-core}\, \mathfrak{m}$ when Δ is the union of at least two disjoint proper simplicial complexes. In short, lift the $*$-core of the maximal ideal in the Stanley-Reisner rings of the indvidual pieces to $k[\Delta]$ and $*\text{-core}\, \mathfrak{m}$ is equal to the sum of these liftings.

Proposition 5.1.4. *Let $\Delta = \Delta_1 \cup \Delta_2$ be the disjoint union of two proper simplicial complexes Δ_1 and Δ_2. Let $\mathfrak{m}_1$ be the maximal ideal generated by the images of the variables in $k[\Delta_1]$ and let $\mathfrak{m}_2$ be defined analogously for $k[\Delta_2]$. Then*

$$*\text{-core}\, \mathfrak{m} = \varphi_1^{-1}(*\text{-core}\, \mathfrak{m}_1) + \varphi_2^{-1}(*\text{-core}\, \mathfrak{m}_2).$$

Proof. Let $k[\Delta_1] = k[x_1, \ldots, x_n]/I_{\Delta_1}$ and $k[\Delta_2] = k[y_1, \ldots, y_m]/I_{\Delta_2}$ and let $d_1 = \dim k[\Delta_1]$ and $d_2 = \dim k[\Delta_2]$. Without loss of generality suppose $d = d_1 \geq d_2$. Let $P_1, \ldots, P_s$ be the minimal primes of $k[\Delta_1]$ and $Q_1, \ldots, Q_t$ be the minimal primes of $k[\Delta_2]$. The minimal primes of $k[\Delta]$ are therefore $P_i k[\Delta] + (y_1, \ldots, y_m)$ for $1 \leq i \leq s$ and $Q_j k[\Delta] + (x_1, \ldots, x_n)$ for $1 \leq j \leq t$. If $I_1 = (g_1, \ldots, g_d)$ is a minimal $*$-reduction of $\mathfrak{m}_1$ in $k[\Delta_1]$ and $I_2 = (h_1, \ldots, h_{d_2})$ is a minimal $*$-reduction of $\mathfrak{m}_2$ in $k[\Delta_2]$, we can make a minimal $*$-reduction in $k[\Delta]$ the following way: let $\overline{g_i}$ be the preimage of g_i in the natural surjection $\varphi_1 : k[\Delta] \twoheadrightarrow k[\Delta_1]$ that has no additional monomials as terms and let $\overline{h_j}$ be defined analogously for h_j across the surjection $\varphi_2 : k[\Delta] \twoheadrightarrow k[\Delta_2]$. Define polyonomials $f_1, \ldots, f_d$ of $k[\Delta]$ the following way:

$$f_i = \begin{cases} \overline{g_i} + \overline{h_i} & \text{for } 1 \leq i \leq d_2 \\ \overline{g_i} & \text{for } d_2 + 1 < i \leq d. \end{cases}$$

Then $I = (f_1, \ldots, f_d)$ is a minimal $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$.

Let α be a nonzero monomial in $*\text{-core}\,\mathfrak{m}$ and let I be a minimal $*$-reduction of $\mathfrak{m}$ of the type defined above. Either α is a product of the x_i variables or it is a product of the y_j variables. Suppose the former. Then for the ideal $I = (f_1, \ldots, f_d)$ of $k[\Delta]$, there exist $a_1, \ldots, a_d$ in $k[\Delta]$ such that

$$\alpha = a_1 f_1 + \cdots + a_d f_d.$$

Since any product of an x and a y is 0, we can assume the individual monomials of all the f and g polynomials are of one of the two types of variables. Since α is all x variables, the sum of the y variables in $a_1 f_1 + \cdots a_d f_d$ is 0. Specifically if $\overline{g_i}$ is the part of f_i with x variables and b_i is the part of a_i with x variables,

$$\alpha = b_1 \overline{g_1} \cdots + b_d \overline{g_d}$$

which shows α to be in the ideal $I' = (\overline{g_1}, \ldots, \overline{g_d})$ of $k[\Delta]$ and the image of α in $k[\Delta_1]$ is in the minimal $*$-reduction $I_1 = (g_1, \ldots, g_d)$ of $k[\Delta_1]$. Since this works for all such I_1, α is a monomial in $*\text{-core}\,\mathfrak{m}_1$. Similarly, if $\beta \in *\text{-core}\,\mathfrak{m}$ is a monomial in only the y variables, the image of $\varphi_2(\beta) \in *\text{-core}\,\mathfrak{m}_2$. Thus

$$*\text{-core}\,\mathfrak{m} \subseteq \varphi_1^{-1}(*\text{-core}\,\mathfrak{m}_1) + \varphi_2^{-1}(*\text{-core}\,\mathfrak{m}_2).$$

Let $I = (f_1, \ldots, f_d)$ be a minimal $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$. Then $\varphi_1(I)$ is a minimal $*$-reduction of $\mathfrak{m}_1$ in $k[\Delta_1]$ and $\varphi_2(I)$ is a $*$-reduction of $\mathfrak{m}_2$ in $k[\Delta_2]$. Let $\varphi_1(f_i) = g_i$ and $\varphi_2(f_i) = h_i$ and let α be a monomial in $*\text{-core}\,\mathfrak{m}_1$. Then there exist polynomial $a_1, \ldots, a_d$ in $k[\Delta_1]$ such that $\alpha = a_1 g_1 + \cdots + a_d g_d$. Let $\overline{a_i}$ be the preimage of a_i in $k[\Delta]$ with no additional monomial terms. Then $\overline{a_1} f_1 + \cdots + \overline{a_d} f_d$ is the monomial preimage of α in $k[\Delta]$. Thus $\varphi_1^{-1}(*\text{-core}\,\mathfrak{m}_1) \subseteq *\text{-core}\,\mathfrak{m}$. Similarly, $\varphi_2^{-1}(*\text{-core}\,\mathfrak{m}_2) \subseteq *\text{-core}\,\mathfrak{m}$. Hence

$$\varphi_1^{-1}(*\text{-core}\,\mathfrak{m}_1) + (\varphi_2^{-1} *\text{-core}\,\mathfrak{m}_2) \subseteq *\text{-core}\,\mathfrak{m},$$

which means

$$*\text{-core}\,\mathfrak{m} = \varphi_1^{-1}(*\text{-core}\,\mathfrak{m}_1) + \varphi_2^{-1}(*\text{-core}\,\mathfrak{m}_2).$$

□

We can infer from propositions 5.1.1, 5.1.3, and 5.1.4 that we now need only explicitly calculate the $*$-core $\mathfrak{m}$ when a simplicial complex is connected. This statement is best summarized through the language of fiber products, i.e. we know how gluing two Stanley-Reisner rings at the field k affects $*$-core($\mathfrak{m}$) in every case.

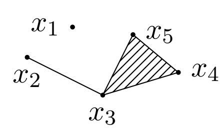
Theorem 5.1.5. *Let $k[\Delta] \cong k[\Delta_1] \times_k k[\Delta_2]$ with $\mathfrak{m}_1$ and $\mathfrak{m}_2$ the maximal ideals of $k[\Delta_1]$ and $k[\Delta_2]$ respectively. Then,*

a. If Δ_1 and Δ_2 are simplices, $*$-core($\mathfrak{m}$) $= \mathfrak{m}^2$

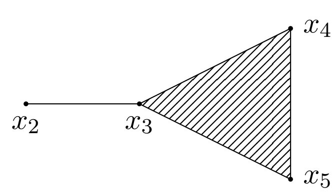
b. If Δ_1 is a simplex and Δ_2 is proper, then $*$-core($\mathfrak{m}$) $= \mathfrak{m}_1^2 k[\Delta] + *$-core($\mathfrak{m}_2$)$k[\Delta]$.

c. If both Δ_1 and Δ_2 are proper, then $*$-core($\mathfrak{m}$) $= *$-core($\mathfrak{m}_1$)$k[\Delta] + *$-core($\mathfrak{m}_2$)$k[\Delta]$.

For example, if we revisit the ring $k[x_1, x_2, x_3, x_4, x_5]/I_\Delta$ where Δ is



let Δ_1 the 0 dimensional simplex of the point x_1 and let Δ_2 be the simplicial complex



then Theorem 5.1.5 b. says $*$-core($\mathfrak{m}$) $= (x_1)^2 + \mathfrak{m}_2^2 k[\Delta]$ where Proposition 5.2.1 gives $*$-core($\mathfrak{m}_2$) in $k[\Delta_2]$.

For the rest of this dissertation, we will assume that no simplicial complexes contain disjoint elements unless otherwise stated.

5.2 Connected Simplicial Complexes

In the above section, we handled both Stanley-Reisner rings of dimension 1 and disjoint simplicial complexes. Therefore, it is tempting to next examine Stanley-Reisner rings of dimension 2 over connected simplicial complexes i.e. simple connected graphs. To this end, we first discover what happens for the fiber product $k[\mathbf{x}, \mathbf{z}] \times_{k[\mathbf{z}]} k[\mathbf{y}, \mathbf{z}]$, where $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are the string of variables on a simplex i.e. when two simplices are glued together at another simplex

Proposition 5.2.1. *Let Δ be a simplicial complex with exactly two distinct facets. Then* $*\text{-core}\,\mathfrak{m} = \mathfrak{m}^2$.

Proof. Let $\Delta = \Delta_1 \cup \Delta_2$ with the vertices of Δ_1 associated to the variable set

$$\{x_1, \ldots, x_n, z_1, \ldots, z_r\}$$

and the vertices of Δ_2 associated to the variable set $\{y_1, \ldots, y_m, z_1, \ldots, z_r\}$ where $\{z_1, \ldots, z_r\}$ is the set of variables associated to the face $\Delta_1 \cap \Delta_2$. The defining ideal of the simplicial complex is $I_\Delta = (\{x_i y_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\})$. The ring $k[\Delta]$ has two minimal primes: $P = (y_1, \ldots, y_m)$ and $Q = (x_1, \ldots, x_n)$. Therefore, the annihilator of P is Q and the annihilator of Q is P.

The test ideal τ is generated by the generators of the two annihilators of the minimal primes i.e. $\tau = (x_1, \ldots, x_n, y_1, \ldots, y_m)$. Thus

$$\tau \cdot \mathfrak{m} = (P + Q) \cdot \mathfrak{m} = P^2 + P \cdot Q + Q^2 + P \cdot (z_1, \ldots, z_r) + Q \cdot (z_1, \ldots, z_r)$$

is contained in $*\text{-core}\,\mathfrak{m}$. To show that $\mathfrak{m}^2 \subseteq *\text{-core}\,\mathfrak{m}$, we need only show that $(z_1, \ldots, z_r)^2 \subseteq *\text{-core}\,\mathfrak{m}$.

Without loss of generality, we may assume that $\dim \Delta_1 \geq \dim \Delta_2$. Then the dimension of Δ is $n + r - 1$. Then for any minimal $*$-reduction I of $\mathfrak{m}$,

$$I = (x_1 + g_1, \dots, x_n + g_n, z_1 + h_1, \dots, z_r + h_r)$$

where each g and each h are polynomials with linear terms in $y_1, \dots, y_m$ and non linear terms divisible by at least one y_i. Since $P \cdot (z_1, \dots, z_r) \subseteq *\text{-core}\,\mathfrak{m}$, For any choice of i, j between 1 and r, $z_i h_j \in I$, so

$$z_i z_j = z_i z_j + z_i h_j - z_i h_j = z_i(z_j + h_j) - z_i h_j \in I.$$

Thus $(z_1, \dots, z_r)^2 \subseteq *\text{-core}\,\mathfrak{m}$ and $*\text{-core}\,\mathfrak{m} = \mathfrak{m}^2$. $\square$

At this point we must introduce a new notion which we will call the linear $*$-core of a linearly generated ideal, which we will be denoted ℓ^*-core:

Definition 5.2.2. *We define the linear* $*$-core *of the ideal I in R to be the intersection of all linearly generated minimal $*$-reductions of I.*

The reason we are introducing this notion is that the nonlinear parts of the generators of a $*$-reduction add a level of complexity that we can avoid using only linear generators. What should be immediately clear is that $*\text{-core}\,I \subseteq \ell^*\text{-core}\,I$. The motivation for introducing this idea is if in any ring R the reverse containment is true, we can discard the nonlinearly generated reductions when calculating $*\text{-core}\,I$. An example of when $*\text{-core}(\mathfrak{m}) = \ell^*\text{-core}(\mathfrak{m})$ is the ring $k[x, y]/(xy)$, but this is not surprising because there are no $*$-reductions of $\mathfrak{m}$ generated by nonlinear generators.

For this discussion, let $R = k[\Delta]$ be a Stanley-Reisner ring of dimension d. Let $J = (f_1, \dots, f_d)$ be linearly generated and $I = (f_1 + g_1, \dots, f_d + g_d)$ where g_i has no linear terms, such that $I^* = J^* = \mathfrak{m}$. Until stated otherwise, this is how I and J will be defined. It is important to remember here that by Theorem 4.1.2, the linear parts of the generators of a minimal $*$-reduction themselves generate a minimal $*$-reduction. For any minimal prime

$P = (x_{m+1}, \dots, x_n)$, we can rewrite the generators of J and I to be

$$J = (x_1 + f'_1, x_2 + f'_2, \dots, x_m + f'_m, \dots, f'_d)$$

$$I = (x_1 + f'_1 + g'_1, x_2 + f'_2 + f'_2, \dots, x_m + f'_m + g'_m, \dots, f'_d + g'_d)$$

and we will rename $x_i + f'_i$ for $1 \le i \le m$ and f'_i for $m+1 \le j \le d$ to be h_i. If a is a monomial of degree q in $\ell^*\text{-core}\,\mathfrak{m}$ then $a = b_1h_1 + \dots + b_dh_d$ for $b_1, \dots, b_d$ which are individually either 0 or homogenous of degree $q-1$. If we carry the same b_i over to I, we get that

$$b_1(h_1 + g'_1) + \dots + b_d(h_d + g'_d) = a + b_1g'_1 + \dots + b_dg'_d$$

where each $b_ig'_i$ is a polynomial with terms of degree $q+1$ or larger. Without knowing much specifically about the $b_ig'_i$ we get the following lemma and two corollaries:

Lemma 5.2.3. *If I contains all monomials of degree $q+1$, then I also contains all monomials of degree q that are in J.*

Corollary 5.2.4. *All monomials of degree d that are in J are also in I.*

Proof. This is a direct consequence of $*\text{-core}\,\mathfrak{m}$ being bound below by $\mathfrak{m}^{d+1}$ and the lemma. □

Corollary 5.2.5. *If $\dim k[\Delta] \le 2$, then $*\text{-core}\,\mathfrak{m} = \ell^*\text{-core}\,\mathfrak{m}$.*

Proof. We know the case for $\dim k[\Delta] = 1$ already, and when $\dim k[\Delta] = 2$, we have all the degree 3 monomials and we get the degree 2 monomials we need from Corollary 5.2.4. □

What Corollary 5.2.5 says is that we can compute the $*\text{-core}\,\mathfrak{m}$ of any simple graph by intersecting only the linearly generated minimal $*$-reductions of $\mathfrak{m}$.

Proposition 5.2.6. *Let the simplicial complex Δ be a cycle graph. Then $*\text{-core}\,\mathfrak{m} = \mathfrak{m}^3$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ for $n \geq 3$ be the vertex set of the one dimensional simplicial complex

$$\Delta = \{\{v_i\} : 1 \leq i \leq n\} \cup \{\{v_i, v_{i+1}\} : 1 \leq i \leq n-1\} \cup \{\{v_1, v_n\}\}$$

and let $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$ be the Stanley-Reisner ring associated to Δ. Then I_Δ is generated by all square free degree two monomials not representing edges in Δ. In all cases, we know that $\mathfrak{m}^3 \subseteq *\text{-core}\,\mathfrak{m}$ by Theorem 4.3.8. We will show that $*\text{-core}\,\mathfrak{m}$ contains no degree two monomials. To do this, we rely on the obvious symmetry of a cycle graph and the contrapositive to Lemma 4.3.5.

Case 1: n is odd. Let $I \subset k[\Delta]$ be the ideal generated by the linear polynomials f_1 and f_2 such that

$$f_1 = x_1 + x_3 + \cdots + x_{n-2} + x_n$$

$$f_2 = x_2 + x_4 + \cdots + x_{n-1} + x_n$$

i.e. f_1 is the sum of the odd number variables and f_2 is the sum of the even number variables plus x_n. By Theorem 3.2.2.(d), $I^* = \mathfrak{m}$. Then by definition, $*\text{-core}\,\mathfrak{m} \subseteq I$. We will show $x_1^2 \notin I$ and is therefore not in $*\text{-core}\,\mathfrak{m}$.

Suppose there exist polynomials g_1 and g_2 in $k[\Delta]$ such that $x_1^2 = g_1 f_1 + g_2 f_2$. We can suppose g_1 and g_2 are linear because all homogeneous degree two polynomials in I will come from products of linear polynomials. Therefore, let

$$g_1 = a_1 x_1 + \cdots + a_n x_n$$

$$g_2 = b_1 x_1 + \cdots + b_n x_n$$

and we will find values for the coefficients in these two polynomials. In the product $f_1 g_1 +$

f_2g_2, we get the following set of linear equations to help find the coefficients of g_1 and g_2:

$$x_1^2 : a_1 = 1$$

$$x_i^2 : \begin{cases} a_i = 0 & i \text{ is odd and } i \neq 1, n \\ b_i = 0 & i \text{ is even} \end{cases}$$

$$x_n^2 : a_n + b_n = 0$$

$$x_i x_{i+1} : \begin{cases} a_i + b_{i-1} = 0 & i \text{ is even} \\ a_{i-1} + b_i = 0 & i \text{ is odd, } 1 < i < n \end{cases}$$

$$x_{n-1}x_n : a_{n-1} + b_{n-1} + b_n = 0 \Rightarrow a_{n-1} + b_n = 0$$

$$x_1 x_n : a_1 + a_n + b_1 = 0 \Rightarrow a_n + b_1 = -1$$

The last two equations we change because we know $a_1 = 1$ and $b_{n-1} = 0$. The last $n+1$ equations listed represent a linear system in $n+1$ variables. The system of equations is inconsistent, which implies that no such coefficients exist. Thus, $x_1^2 \notin I$ and therefore $x_1^2 \notin *\text{-core}\,\mathfrak{m}$. Because of symmetry of the graph, $x_i^2 \notin *\text{-core}\,\mathfrak{m}$ for $1 \leq i \leq n$. By the contrapositive of Lemma 4.3.5, this implies that not all monomials of the form $x_i x_j$, $i \neq j$ are in $*\text{-core}\,\mathfrak{m}$. This can only mean the nonzero monomials, and by rotational symmetry, we can say that the nonzero monomials of the form $x_i x_j$, $i \neq j$ are not in $*\text{-core}\,\mathfrak{m}$. Thus as ideals of $k[\Delta]$, $*\text{-core}\,\mathfrak{m} \subseteq \mathfrak{m}^3$.

Case 2: n is even. The proof of this similar to the case when n is odd. Number the variables of the ring in order around the cycle. Let I be the ideal generated by f_1 and f_2 such that

$$f_1 = x_1 + x_3 + \cdots + x_{n-1} + x_n$$

$$f_2 = x_2 + x_4 + \cdots + x_{n-2} + x_n.$$

Similar to before, f_1 is the sum of the odd numbered variables plus x_n and f_2 is the sum of the even numbered variables. This ideal is a minimal $*$-reduction of $\mathfrak{m}$. It can be shown that x_1^2 is not in this ideal, and consequently, there are no nonzero degree two monomials in $*\text{-core}\,\mathfrak{m}$ and $*\text{-core}\,\mathfrak{m} \subseteq \mathfrak{m}^3$ in both cases. □

The next few results will show that a graph in the absence of cycles will be at the upper bound of $*\text{-core}\,\mathfrak{m}$ instead of the lower. Namely Corollary 5.2.10 will show that if Δ is an acyclic proper simplicial complex of dimension 1 (an acyclic graph on at least three vertices), then $*\text{-core}\,\mathfrak{m} = \mathfrak{m}^2$. First we must introduce two lemmas. The first lemma deals with leaves and their incident edges in a simplicial complex. The second lemma shows that the presence of specific squarefree degree two monomials in $*\text{-core}\,\mathfrak{m}$ allows for the square of certain variables to be in $*\text{-core}\,\mathfrak{m}$. Depending on the point of view, Lemma 5.2.8 can be seen as either a weaker or stronger version Lemma 4.3.5. On one hand, Lemma 5.2.8 focuses on dimension two rings,and on the other, fewer squarefree monomials are required than in Lemma4.3.5.

Lemma 5.2.7. *Let y be a leaf of the simplicial complex Δ and $\{x, y\}$ be the edge incident with y. Then $(y)(x, y) \subset *\text{-core}\,\mathfrak{m}$ in $k[\Delta]$.*

Proof. Let $I = (f_1, \ldots, f_d)$ be a minimal $*$-reduction of $\mathfrak{m}$. Then since $\{x, y\}$ is a facet of Δ, there exists, by echelonization, a presentation of I such that $I = (x + g_1, y + g_2, g_3, \ldots, g_d)$ where g_i has no x or y term. Since y is a leaf and incident with the edge $\{x, y\}$, the only nonzero monomials divisible by y in $k[\Delta]$ are of the form $x^{t_1}y^{t_2}$ for non negative integer t_1 and positive integer t_2. Thus $y(y + g_2) = y^2$ and $y(x + g_1) = xy$. □

Lemma 5.2.8. *Let Δ be a proper simplicial complex on vertex set $\{y, x_1, \ldots, x_{n-1}\}$. If every monomial $x_iy \in *\text{-core}\,\mathfrak{m}$ for $1 \leq i \leq n-1$, then so is y^2.*

Proof. For a minimal prime ideal P of $k[\Delta]$ not containing y, echelonize a minimal $*$-reduction $I = (f_1, \ldots, f_d)$. By echelonizing, we find an element $y + g$ of I such that every summand of g is divisible by at least of the variables x_i. Multiply $y + g$ by y to obtain

$y^2 + yg$. Since each summand of g is divisible by some variable x_i, each summand of yg is divisible by $x_i y$ for some $i \leq n-1$. Since $x_i y \in *\text{-core}\,\mathfrak{m}$ for all $i \in \{1, 2, \dots, n-1\}$, $x_i y \in I$. Hence $yg \in I$ and $y^2 = y(y+g) - yg \in I$. Thus $y^2 \in *\text{-core}\,\mathfrak{m}$. □

Using the two lemmas above, the following theorem shows how $*\text{-core}\,\mathfrak{m}$ is affected when $k[\Delta] = k[\Delta_1] \times_{k[\Delta_1 \cap \Delta_2]} k[\Delta]$ is such that Δ_1 is an acyclic graph, $\Delta_1 \cap \Delta_2$ is a single vertex, and in Δ_1 that single vertex is a leaf of the graph.

Theorem 5.2.9. *Let $\Delta = \Delta_1 \cup \Delta_2$ be a simplicial complex such that Δ_1 is a connected acyclic graph (a tree) on vertex set $\{x, y, z_1, \dots, z_t\}$, $\Delta_1 \cap \Delta_2 = \{\emptyset, \{x\}\}$, and as a vertex of Δ_1, x has degree 1. Then $(x, y, z_1, \dots, z_t)(y, z_1, \dots, z_t) \subset *\text{-core}\,\mathfrak{m}$ for maximal ideal $\mathfrak{m}$ of $k[\Delta]$.*

Proof. Since x as degree 1 in Δ_1, suppose that $\{x, y\}$ is the edge of Δ_1 that lies incident with x. With the exception of the product xy, every product of a variable w of $k[\Delta_2]$ with any of the variables of $k[\Delta_1]$ is an element of I_Δ and thus in $*\text{-core}\,\mathfrak{m}$. We will show by induction on the length s of a path in Δ_1 starting at x, that the product of any two distinct variables of $k[\Delta_1]$ is also $*\text{-core}\,\mathfrak{m}$. Once this established, by Lemma 5.2.8, for each of the variables $z \in \{y, z_1, \dots, z_t\}$, $z^2 \in *\text{-core}\,\mathfrak{m}$ and hence $(x, y, z_1, \dots, z_t)(y, z_1, \dots, z_t) \subset *\text{-core}\,\mathfrak{m}$.

For the base case of our induction, let the longest path starting at x in Δ_1 be equal to 1. Then the path starts at x and ends at y. Thus y is a leaf and by Lemma 5.2.7, $xy \in *\text{-core}\,\mathfrak{m}$.

Suppose that the path of maximal length starting at x in Δ_1 is s and that the theorem is true for all paths of up to length $s-1$. Let I be a minimal $*$-reduction of $\mathfrak{m}$ in $k[\Delta]$. Since $\{x, y\}$ is a facet of Δ, there exists a prime P generated by all other variables. Echelonize I with respect to this prime. Thus $I = (x+g_1, y+g_2, g_3, \dots, g_d)$ such that for $1 \leq i \leq d$, every summand of g_i is divisible by at least one variable that is not x or y. Therefore if we multiply $x+g_1$ by y, every summand of yg_1 is either in I_Δ and is therefore equal to 0 or divisible by yz_i for $1 \leq i \leq t$. If yz_i is nonzero, then the edge incident with both variables exists. This edge lies on a path of length no more than $s-1$ starting at y. By the inductive hypothesis, yz_i

is therefore an element of $*$-core $\mathfrak{m}$. Hence $yg_1 \in *$-core $m \subset I$ and $xy = y(x+g_1) - yg_1 \in I$ for all minimal $*$-reductions I of $\mathfrak{m}$. Thus $(x, y, z_1, \ldots, z_t)(y, z_1, \ldots, z_t) \subset *$-core $\mathfrak{m}$. $\square$

By making Δ_2 in the previous theorem the same as $\Delta_1 \cap \Delta_2$, we find that $*$-core $\mathfrak{m}$ in acyclic graphs is $\mathfrak{m}^2$.

Corollary 5.2.10. *If Δ is an acyclic graph on n vertices $\{x_1, \ldots, x_n\}$ with leaf x_1, then $*$-core $\mathfrak{m} = \mathfrak{m}^2$ in $k[\Delta]$.*

Proof. We need only show that $\mathfrak{m}^2 \subseteq *$-core $\mathfrak{m}$. Let $\Delta_1 = \Delta$ and $\Delta_2 = \{\emptyset, \{x\}\}$. By Theorem 5.2.9 $(x_2, \ldots, x_n) \cdot \mathfrak{m} \subset *$-core $\mathfrak{m}$. By Lemma 5.2.7, $x^2 \in *$-core $\mathfrak{m}$. Thus $\mathfrak{m}^2 \subseteq *$-core $\mathfrak{m}$. $\square$

There is still work to be done on the case where Δ is a graph. For example, the ring $k[w, x, y, z]/(wxy, wz, yz)$ is the Stanley-Reisner ring of a graph, but $*$-core $\mathfrak{m} = \mathfrak{m}^3 + (z)(y, z)$, which at the moment can only be confirmed with Macaulay2 (though the containment $\supseteq$ is confirmed by Theorem 5.2.9 and Theorem 4.3.8.

We close with a conjecture about $*$-core$(\mathfrak{m})$ in $k[\Delta]$ when Δ is a simple graph and a conjecture about the relationship between ℓ^*-core$(\mathfrak{m})$ and $*$-core$(\mathfrak{m})$.

Conjecture 5.2.11. *Let Δ be a simple connected graph.*

a. For any edge $\{x, y\}$ of Δ, $xy \in *$-core$(\mathfrak{m})$ if and only if $\{x, y\}$ and one of x or y are not in a cycle.

b. For any vertex x of Δ, $x^2 \in *$-core$(\mathfrak{m})$ if and only if x is not part of a cycle.

In particular, if Δ is a simple connected graph and $e = \{x, y\}$ is an edge of Δ such that e and x are not in a cycle, then

$$*\text{-core}(\mathfrak{m}) = \mathfrak{m}^3 + \sum_e (x^2, xy).$$

Conjecture 5.2.12. *For all Stanley-Reisner rings $k[\Delta]$, $*$-core$(\mathfrak{m}) = \ell^*$-core$(\mathfrak{m})$.*

This conjecture, if true, would make the computation of $*$-core and core of $\mathfrak{m}$ easier because we could ignore any $*$-reductions of $\mathfrak{m}$ not generated by linear polynomials.

Appendix A: Future Work

There is still much work to be done on this problem. As of the time of writing, it is still a bit of a mystery what $*$-core $\mathfrak{m}$ is when Δ is a graph that is neither tree nor cycle. We can also hope to achieve success in determining $*$-core $\mathfrak{m}$ when $\dim k[\Delta] > 2$. It is my belief that the techniques and machinery developed in this dissertation will help answer these questions.

Tweaking conditions used throughout this dissertation also yields interesting questions. We can relax the condition that the defining ideal of the ring we are working in be generated by squarefree monomials i.e. given an infinite field k and an ideal I of $S = k[x_1, \dots, x_n]$ generated by not necessarily squarefree monomials, determine $*$-reductions, $*$-spread, and $*$-core of $\mathfrak{m}$ in S/I. Such rings behave similarly to Stanley-Reisner rings largely as a result of I decomposing into an intersection of monomial primary ideals. These similarities to Stanley-Reisner rings imply that we can apply a lot of the machinery built in the Stanley-Reisner ring case to describe the aforementioned tight closure invariants in the general monomial algebra case.

We may also ask what happens when we examine reductions of an ideal that is not $\mathfrak{m}$. This line of questioning wass inspired by a question posed to me from Florian Enescu of Georgia State University. He asked what would happen to the aforementioned invariants if we looked at reductions of $\mathfrak{m}^2$. That question resulted in the following theorem:

Theorem A.0.13. *Let $k[\Delta]$ be Stanley-Reisner ring of dimension d. Then the $*$-spread of $\mathfrak{m}^c$ is $\binom{d+(c-1)}{c}$ and*

$$\mathfrak{m}^{d+c} \subseteq * - core\ \mathfrak{m}^c.$$

In general, we can ask what the invariants $*$-core and $*$-spread of an ideal J are when J is generated by monomials.

One question we had early on was happens when the defining ideal is generated by binomials. This is an interesting question because rings defined in this way have many applicaltions to subjects like algebraic geometry and algebraic statistics. Once the monomial cases are resolved, this would be the next natural step to take in research. One of the major hurdles here that we were unable to resolve is when exactly such a ring is normal.

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Curriculum Vitae

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