SOME PROPERTIES OF SIMPLICIAL GEOMETRIES

by

Cynthia S. Merrick
A Dissertation
Submitted to the
Graduate Faculty
of
George Mason University
In Partial fulfillment of
The Requirements for the Degree
of
Doctor of Philosophy
Mathematics

Committee:

_____ Dr.James Lawrence, Dissertation Director

_____ Dr.Geir Agnarsson, Committee Member

_____ Dr.Rebecca Goldin, Committee Member

_____ Dr.Valentina Harizanov, Committee Member

_____ Dr.David Walnut, Department Chair

_____ Dr.Timothy L. Born, Associate Dean for Student and Academic Affairs, College of Science

_____ Dr.Vikas Chandhoke, Dean, College of Science

Date: _____ Summer Semester 2013
George Mason University
Fairfax, VA

Some Properties of Simplicial Geometries

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at George Mason University

By

Cynthia S. Merrick
Master of Science
George Mason University, 2006
Bachelor of Arts
The American University, 1990

Director: Dr.James Lawrence, Professor
Department of Mathematics

Summer Semester 2013
George Mason University
Fairfax, VA

Dedication

I dedicate this dissertation to the many patient people whose encouragement and sacrifices made my research possible. These include my husband, Robert, my children Emily and Stephen. I would also like to thank my advisers and committee: Rebecca Goldin, Valentina Harizanov, James Lawrence, and Geir Agnarsson.

Table of Contents

Page

List of Figures v

Abstract 1

1 Introduction 2

1.1 Definition of a Simplicial Geometry 2

1.2 Recent Related Work 3

1.3 The Geometry G_k^n 4

1.4 Matrix Representation of G_k^n 5

1.5 The Matroid of G_k^n 9

1.6 The Oriented Matroid $M(G_k^n)$ 12

1.7 Circuits 15

2 Matroid Properties 18

2.1 Connectedness of $M(G_3^n)$ 18

2.2 When $M(G_3^n)$ Is Not Regular 21

2.3 Some Large Regular Minors $M(G_3^6)$ 24

2.4 The Special Bases of $M(G_3^6)$ 41

2.5 A Classification of Circuits on Six Vertices 43

2.6 $M(G_3^6)$ Is Ternary 58

3 Zonotope Properties 60

3.1 Simplicial Bases From Special Bases 61

3.2 The Lexicographic Method 63

3.3 Chirotopes 68

Bibliography 76

List of Figures

Figure		Page
1.1	Matrix Representation for G_3^6	7
2.1	Four Odd Edges	27
2.2	Five Odd Edges	29
2.3	Six Odd Edges	31
2.4	Edges in a Circuit An Odd Number of Times	33
2.5	Special Basis as a Triangulation of $\mathbb{R}P^2$	38
2.6	A Perimeter Transformation	39
2.7	A Triangulation of $\mathbb{R}P^2$	42
2.8	Four Triangles Sharing a Common Edge	46
2.9	Attempting to Construct Another Binary Rank 9 Circuit	49
2.10	First Set of Choices In Constructing A Rank 9 Circuit	49
2.11	Second Set of Choices In Constructing A Rank 9 Circuit	50
2.12	Beginning to construct the rank 5 circuit	51
2.13	Looking For A $K_{3,3}$ Subgraph	53
2.14	One-skeleton of the rank 5 circuit	54
2.15	P_1, the one-skeleton of the 3-dimensional crosspolytope	55
2.16	P_2, the one-skeleton of the 3-dimensional cyclic polytope	55
3.1	Two Different Orientations of G_3^3	64
3.2	$M(G_3^4)$ With Weighted Edges	66
3.3	A Lexicographic Matrix for $M(G_3^4)$	66
3.4	A Lexicographic Orientation of $M(G_3^5)$	67
3.5	A Lexicographic Matrix for $M(G_3^5)$	67
3.6	A Non-Lexicographic Orientation of $M(G_3^4)$	68
3.7	Relative Weights in the Lexicographic Matrix	69
3.8	$M(G_3^3)$	70
3.9	Simplicial Orientation for $M(G_3^4)$	71

3.10 Signings Satisfying Grassmann-Plücker Relations on $M(G_3^5)$ 72
3.11 Lexicographic Matrices for Two Chirotopes 72
3.12 Signings Satisfying the Grassmann-Plücker Relation on a Permutation of the Vertices 75

Abstract

SOME PROPERTIES OF SIMPLICIAL GEOMETRIES

Cynthia S. Merrick, PhD

George Mason University, 2013

Dissertation Director: Dr.James Lawrence

Simplicial geometries, whose points are the collection of all k-element subsets of a given (finite) ground set, were described in 1970 by Crapo and Rota [5]. So far, only geometries for which $k \leq 2$ and their duals have been well-studied. In this paper, I address many general questions about simplicial geometries on n vertices, via matroid properties such as the structure of circuits, minors and orientability. I describe the smallest largely unstudied simplicial geometry, on a ground set of six vertices, with $k = 3$, which I call G_3^6. A construction of the only (up to symmetry) non-contractible basis is given, as well as a complete characterization of all circuits that can be built using six or fewer vertices. I prove that the matroid of G_3^6 is ternary, and give two large deleted minors which are regular. I also explore more than one method for finding topes of the associated arrangement of hyperplanes, and describe a specific construction for a simplicial tope.

Chapter 1: Introduction

1.1 Definition of a Simplicial Geometry

In 1970 Henry Crapo and Gian-Carlo Rota [5] described a method for interpreting simplicial complexes as combinatorial geometries. In what they call a simplicial geometry, "points" of the geometry are k-element subsets of a given underlying point-set, for some fixed value of k, satisfying a specifically defined notion of set-closure. We interpret the more abstract k-set as the simplex on k vertices.

In general, a combinatorial geometry is the structure gotten by taking a closure operator defined on subsets of a set V that satisfies the exchange property. The closure operator has the property that, for any $A \subseteq V$, A is a subset of its own closure (denoted $\overline{A}$), and if A is a subset of the closure of any $B \subseteq V$, then $\overline{A} \subseteq \overline{B}$. The exchange property ensures that for any points $a, b \in V$, and $A \subseteq V$, if $a \in \overline{A \cup b}$, and $a \notin \overline{A}$, then $b \in \overline{A \cup a}$.

To obtain the combinatorial geometry known as a simplicial geometry, we define a rank function based on the Betti numbers of a given simplicial complex. (The n-th Betti number for a simplicial complex is the number of $\mathbb{Z}$ summands of the n-th Homology group for the complex.) For a fixed vertex set V of size n, and a fixed number k, we consider all k-subsets of V. We obtain a simplicial complex by associating a k-subset with the simplex with k vertices. Let V_k be the collection of all k-simplexes that are associated with k-subsets of V. Then for some sub-collection $A \subseteq V_k$, define the simplicial complex associated to A as $S(A) = V_0 \cup \cdots \cup V_{k-1} \cup A$. Building simplicial complexes in this way means most Betti numbers are zero. The rank of A is the difference $r(A) = |A| - \beta_{k-1}(S(A))$, where $\beta_{k-1}(S(A))$ is the $(k-1)$-th Betti number of $S(A)$. Since all simplexes with fewer than k

vertices are included in a complex, every cycle of simplexes with less than or equal to $(k-1)$ vertices bounds a face in the complex of size one larger. This places all such cycles in the kernel of the respective boundary map. A basic introduction to obtaining Betti numbers is found in the text by Hatcher [7].

This type of simplicial geometry has been extensively studied when $k = 2$ (they are trivial when $k = 1$). When $k > 2$, much remains to be done. In this paper, we use oriented matroids to study cases where $k \geq 3$.

1.2 Recent Related Work

Many of the results in this paper stem from the existence of an independent set which is not contractible as a simplicial complex. When $n = 6$, this non-contractible set is a basis of the geometry (and the matroid). I show that this basis, which I call a *special basis* is a triangulation of the real projective plane. Such objects have been of interest in recent research.

In 1983, Kalai [8] generalized Cayley's formula, which counts the number of trees in the complete graph, to a formula that counts spanning trees of k-simplicial complexes. Klivans, et al, [9] add to Kalai's generalization by giving a simplicial version of the classical Matrix Tree Theorem. Kirchhoff developed this theorem to apply to graphs in general, from which Cayley derived his formula for complete graphs. The results on general simplicial complexes account for the possibility of torsion in spanning trees of simplexes for $k \geq 3$.

Cordovil [3] and Lindström [11] both settled the question of simplicial matroids being, *a priori*, non-regular and non-binary, with different proofs. Cordovil [2] also wrote a proof of Reid's 3-Simplicial Matroid Theorem. This theorem states that if a matroid is representable over a field F, then it is a minor of a simplicial matroid which is also representable over F, for which $k = 3$.

All the foregoing results give tools for counting or using some of the structure of simplicial matroids (which are the matroids of simplicial geometries), but thus far the literature does not contain much description of the combinatorial structure of these simplicial matroids themselves. This paper serves that purpose. For $k = 3$, I classify circuits of ranks up to and including rank 10, the rank of the smallest non-binary circuit. I clarify the non-regular nature of the matroid of G_3^6, by showing the special basis to be the source of all non-binary circuits, the key to finding the excluded minor $U_{2,4}$ (contained in every non-binary matroid), and a means to finding some of the simplicial topes of the associated hyperplane arrangement. Also, using the unique characteristics of the special basis, I give two ways to delete small sets of elements to create regular minors of G_3^6.

1.3 The Geometry G_k^n

Working from Crapo and Rota's original ideas, we give this definition of G_k^n:

Definition 1.3.1 (The Simplicial Geometry G_k^n)**.** Let $V = \{v_1, ..., v_n\}$ be a ground set of n vertices, and V_k be the set $\{t_i\}$ of k-simplexes associated with all k-tuples of V, for a fixed $k \leq n$. For any subset $S \subseteq V_k$, the closure $cl(S) = S \cup \{t_i | t_i \in V_k,\ r(S \cup t_i) = r(S)\}$. Then G_k^n is the *simplicial geometry* on vertices V with point set V_k.

Comment on Notation: In many examples and results, we have a small number of vertices in mind, so that for readability we name vertices more specifically by lowercase letters of the alphabet. For example, G_3^6 has vertices $V = \{a, b, c, d, e, f\}$ and a point is abc, with boundary edges ab, ac and bc. When we wish to refer to an arbitrary k-tuple, for brevity we write t_i (suggestive of the case when $k = 3$ and our elements are thought of as triangles) instead of using tuple notation.

G_3^6 is the smallest largely unstudied simplicial geometry. We visualize this as the simplicial complex whose 1-skeleton is the complete graph, K_6. Many combinatorial facts and

known properties of simplexes come into play as we describe the structure of simplicial geometries. For example, any two distinct simplexes share at most one facet. For G_3^n, *points* are associated with 3-subsets, where two different 3-sets have no more than two vertices from the ground set in common. Viewing the 3-set as the simplex on three vertices, a triangle, this is equivalent to the property for simplexes that any two distinct triangles share no more than one edge. For any n there are $\binom{n}{3}$ points in the geometry, so that G_3^6 has twenty points. Also, each point of G_3^n is in $n-3$ tetrahedra, and, more generally, any k-simplex of the simplicial geometry G_k^n is in $n-k$ $(k+1)$-simplexes.

1.4 Matrix Representation of G_k^n

Through an arbitrary choice of orientation for each element of G_k^n, we can derive a matrix representation of the points. Let VS^e be the vector space comprised of all linear combinations of $(k-1)$-simplexes. Then each point of G_k^n, regarded as a k-simplex bounded by $(k-1)$-simplexes, is represented as a column vector in VS^e, with non-zero entries in the coordinates which form the boundary of the k-simplex. We motivate our formal definitions with an example, where $n = 6$ and $k = 3$, to build a matrix representation.

Example 1. The edges of the graph K_6 are a basis for VS^e, and index the coordinates of a vector. K_6 has fifteen edges, so the vectors in our vector space have fifteen coordinates. The points of our geometry are triangles. Each point as a vector has three non-zero coordinates corresponding to the edges from K_6 which bound the triangle. Since there are twenty triangles in the triangle complex, the matrix of all triangles will have twenty columns.

Let $\{a, b, c, d, e, f\}$ label the vertices of K_6. By arbitrarily choosing the lexicographic ordering on vertices, we get an implicit ordering on the edges, so that, for example, ab is the edge directed from a to b, and we write $+ab$. For the edge directed the other way,

we write $-ab$. From edge orientations, we can define orientation on the triangles. Let $\sigma : t_i \mapsto \{+t_i, -t_i\}$ be an assignment on each triangle in the graph. The orientation of the triangle is directly related to the orientations of the edges on its boundary via the boundary map. For example, when $\sigma(abc) = +abc$, $\delta : abc \mapsto +ab - ac + bc$, and when $\sigma(abc) = -abc$, $\delta : abc \mapsto -ab + ac - bc$. Then, we express abc as a vector for which each coordinate represents an edge of the graph.

For the remainder of this paper, except when otherwise explicitly stated, we choose the lexicographic ordering on the edges of K_n. When $n = 6$, we have the ordered set: $\{ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef\}$.

Below is a list of the twenty triangles, which we designate as positively oriented:

$$\begin{array}{cccc} abc & ace & bcd & bef \\ abd & acf & bce & cde \\ abe & ade & bcf & cdf \\ abf & adf & bde & cef \\ acd & aef & bdf & def \end{array}$$

Listing each column by a lexicographic ordering of the triangles, Figure 1.1 gives the matrix representation of G_3^6.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

Figure 1.1: Matrix Representation for G_3^6

This 15×20 matrix has rank 10, and so has a 10×20 row-reduced form which we shall sometimes find more convenient to use.

Let $VS^e = \{\Sigma\alpha_i e_i | \alpha_i \in \mathbb{R}\}$ be the vector space of all linear combinations of $(k-1)$-simplexes e_i of G_k^n. By making an arbitrary choice of orientation for each $(k-1)$-simplex, we have a notion of $+e_i, -e_i$. The non-empty support of $\{\alpha_i e_i\}$ consists of precisely the $(k-1)$-tuples that form the boundary of the k-simplex t.

Definition 1.4.1 (Matrix Representation of G_k^n)**.** Let G_k^n be a simplicial geometry with points all k-tuples. Then G_k^n is the simplicial complex on n vertices of all k-simplexes having boundary elements $(k-1)$-simplexes, e_i. Let $VS^e = \{\Sigma\alpha_i e_i | \alpha_i \in \mathbb{R}\}$ be the vector space of all linear combinations of $(k-1)$-tuples.

$M = [x_{ij}]$ is a *matrix representation* of G_k^n whose rows are indexed by the $(k-1)$-simplexes e_i, and whose columns are indexed by the k-simplexes t_i.

A vector of VS^e is a column of the matrix when it has non-zero support in the coordinates that represent $(k-1)$-simplexes on its boundary. The ith coordinate of a column corresponds to the coefficient $\alpha_i \in \{\pm 1\}$ from the linear expression $\Sigma\alpha_i e_i$. In this way we see the direct correspondence between points of the geometry and columns of the matrix.

With the matrix M we also define a second vector space, $VS^t = \{\Sigma\alpha_i t_i | \alpha_i \in \mathbb{R}\}$ over the columns of M. The column space of M we think of as the collection of all linear combinations of k-tuples, which we in turn have defined as linear combinations of $(k-1)$-tuples.

The matrix representation serves as a vector space interpretation of the structure of our simplicial geometry. This is evident when we compare the rank function r as defined for subsets of points of G_k^n to the algebraic notion of rank for a vector space. We defined the rank of a set of points A to be $r(A) = |A| - \beta_{k-1}(A)$. In a vector space, the rank of a set of

vectors is essentially the same: the size of the set minus the number of distinct (up to scalar multiples) linear combinations that sum to the zero vector. Because our coordinates in the vector space represent boundary elements, a linear sum equalling zero in the geometry will have a corresponding linear sum equalling zero in columns of the matrix.

1.5 The Matroid of G_k^n

Furnished with a vector space associated with G_k^n, we can interpret the geometry as a matroid, making use of constructs like dependence relations, bases, and rank, to describe the structure of our geometry.

Definition 1.5.1 (The Matroid $\underline{M}(G_k^n)$ of a Simplicial Geometry)**.** Let E denote the set of columns of a matrix representation of G_k^n, and let $C \subseteq E$ be a set whose corresponding columns are a minimal linearly dependent set. Then $\mathfrak{C}$, the collection of all such subsets C, is the set of circuits of the *matroid* $\underline{M}(G_k^n)$. The rank of the matroid is $r(\underline{M}(G_k^n)) = r(G_k^n)$.

We discuss matroid properties by way of collections of column vectors from the matrix. Maximal linearly independent sets of columns from the matrix are bases of the matroid. Minimal linearly dependent sets from the matrix are circuits of the matroid, and so forth.

From the literature [3], we have the result that for $k \leq 2$, the geometries G_k^n and G_{n-k}^n have binary matroids. That is, they are representable as matrices with entries in $\mathbb{Z}_2$. This implies that G_2^5 is binary, and so is G_3^5. In Chapter 2, we give a number of theorems regarding the fact that G_3^6 is not binary, and thus, G_3^n, when $n \geq 6$, by containing G_3^6 as a minor, is not binary. We prove that G_3^6 actually has a ternary matroid: uniquely representable as a matrix with entries in $\mathbb{Z}_3$.

From Definition 1.4.1 and the above definition, we know that the rank function r for our simplicial geometry gives the same values as the rank function defined on the matroid of the geometry's matrix representation.

As defined in [5], the geometry G_k^n has an associated dual geometry, G_{n-k}^n, a geometry on the same ground set, for which the points are the set-complements of the k-sets in G_k^n. The dual geometry is defined by a dual rank function, r_G^*.

Definition 1.5.2 (Nullity)**.** The *nullity* of a set of points of G_k^n is defined, for $A \subseteq V_k$, as $n(A) = |A| - r(A)$, the difference between the size of the set and its rank.

Definition 1.5.3 (Dual Rank Function (Geometry))**.** We define a new rank function r_G^* on the sets of points of G_k^n. For a set A of points, and V_k the collection of all points of G_k^n, the *dual rank function* is $r_G^*(A) = n(V_k) - n(V_k \setminus A)$.

Crapo and Rota in [5] prove that not only is r_G^* a rank function of a simplicial geometry, but in particular that the result of applying r_G^* to the points of G_k^n is a geometry isomorphic to G_{n-k}^n.

Definition 1.5.4 (Dual Rank Function (Matroid))**.** In matroid theory, a *dual rank function of a matroid* M with set of elements E is defined on a set $A \subseteq E$ this way:

$r_M^*(A) = r(E \setminus A) + |A| - r(E)$, where r is the familiar rank function (the same for geometry, matrix and matroid).

Theorem 1.5.5. G_k^n and G_{n-k}^n have dual matroids and dual oriented matroids.

Proof. We show that the dual rank function r_M^* for matroid, when applied to $\underline{M}(G_k^n)$, yields the matroid $\underline{M}(G_{n-k}^n)$. As before, let V_k denote the set of points of G_k^n, and $A \subseteq V_k$. We can rewrite the definition of r_G^* equivalently like this:

$r_G^*(A) = n(V_k) - n(V_k \setminus A)$

$= |S| - r(V_k) - |V_k \setminus A| + r(V_k \setminus A)$

$= |A| - r(V_k) + r(V_k \setminus A)$.

By construction, the points of V_k are in bijective correspondence with columns of the matrix representation, and therefore with the set E of elements of the matroid $\underline{M}(G_k^n)$. Let

$\phi : V_k \mapsto E$ map a point of G_k^n to its corresponding matroid element. Via the equality shown above, we see that $r_G^*(A) = r_M^*(\phi(A))$. Thus, the rank of a set of points of G_{n-k}^n is the same as the rank of a set of elements of $\underline{M}(G_{n-k}^n)$, making the matroids of G_k^n and G_{n-k}^n dual. Because rank is preserved by passing from matroid to oriented matroid, we have $M(G_k^n)$ and $M(G_{n-k}^n)$ are dual oriented matroids. $\square$

Continuing with our example of G_3^6, it is evident from the matrix in Figure 1.1 that subsets of two or three columns are all linearly independent, but several subsets of four columns have minimal dependence relations. When we view the members of such a 4-set as triangles in the complex in K_6, they can be seen to form the closed surface of a tetrahedron. We can represent this dependence as a linear combination of column vectors that sum to zero with the appropriate non-zero coefficients. We interpret this linear algebraic operation in the geometry via the topological boundary map. When all edges "cancel," this corresponds to a linear expression summing to zero. Suppose we take the columns corresponding to triangles abc, abd, acd and bcd. We show how to express the fact that these elements all form a circuit of the matroid, using a linear expression:

$$\delta(abc) - \delta(abd) + \delta(acd) - \delta(bcd)$$
$$= +ab - ac + bc - ab + ad - bd + ac - ad + cd - bc + bd - cd$$
$$= \vec{0}$$

All elements have non-zero coefficients, and all boundary elements cancel after applying the boundary map.

If we wrote our original expression this way:

$\delta(abc) - \delta(abd) + \delta(acd) = \delta(bcd)$

$+ab - ac + bc - ab + ad - bd + ac - ad + cd = +bc - bd + cd$

then we would have a linear expression for one element, bcd, in terms of three others. When we have a set of vectors that form a basis for the matroid, then every element of the matroid is either in the basis, or expressible as a minimal linear combination of elements from the basis.

Theorem 1.5.6. All elements sharing a particular vertex in common form a basis of the matroid of G_k^n.

Proof. Let B_a be the set of all points of G_k^n containing vertex a. In the matrix representation, because the columns are indexed by k-simplexes and the rows are indexed by $(k-1)$-simplexes, the column vector for each member of B_a contains a non-zero entry in some coordinate for which every other vector in the set is zero (this corresponds to the unshared edge from each triangle in G_3^n), making the set linearly independent. From Crapo and Rota [5], we know that the rank of the matrix must be be $\binom{n-1}{k-1}$. Not coincidentally, this is the number of elements containing any one vertex. More generally, any set of elements of G_k^n containing a vertex in common is an independent set. $\square$

1.6 The Oriented Matroid $M(G_k^n)$

The matrix representation also gives rise to an *oriented* matroid, by considering all columns of the matrix, as well as their negations. That is, elements of the oriented matroid are gotten from the set of all elements with both possible orientations. From [1], we know that every oriented matroid realizable as a matrix over $\mathbb{R}$ is also representable as an arrangement of hyperplanes. The arrangement is gotten by taking the collection of hyperplanes orthogonal to each column of the matrix. Formally, we now regard the matrix representation of G_k^n as a real-valued matrix over $\mathbb{R}^{\binom{n-1}{k-1}}$.

Definition 1.6.1 (Hyperplane Arrangement of a Realizable Matroid)**.** $\mathfrak{A} = \{H_i | 1 \leq i \leq \binom{n}{k}$

} is the *arrangement of hyperplanes* associated with $M(G_k^n)$, where $H_i = \{x \in \mathbb{R}^{\binom{n-1}{k-1}} | x \cdot c_i = 0\}$, and c_i is the ith column of the matrix representation of G_k^n. H_i is the hyperplane orthogonal to the ith column of M, and is uniquely associated with the element t_i of the underlying (unoriented) matroid $\underline{M}(G_3^6)$. Then $H_i^+ = \{x \cdot c_i > 0\}$ is the positive side of H_i, associated with the positively oriented element $+t_i$ in the oriented matroid $M(G_k^n)$, and $H_i^- = \{x \cdot c_i < 0\}$ is the negative side of H_i, associated with $-t_i \in M(G_k^n)$.

Note that $\bigcap H_i = \overrightarrow{0}$, and that $\mathfrak{A}$ decomposes $\mathbb{R}^{\binom{n-1}{k-1}}$ into open cells. We have a bijective correspondence between $H_i^+ \to +t_i$ and $H_i^- \to -t_i$. A point x in one of the full-dimensional cells can be said to be on one side or the other of every hyperplane in the arrangement. We can represent the location of x with an ordered listing of $+$ or $-$ for each hyperplane, to denote which side the point x is on. In fact, such an ordered list of signs would be the same for every point on the interior of the cell. We find these sign vectors, which have $\binom{n}{k}$ coordinates (one for every hyperplane), are a unique description of the full-dimensional cells of the arrangement. From [1], we know these sign vectors correspond to maximal covectors of the oriented matroid, and are called topes. Topes are particularly noteworthy in our research, as they designate the choices of element orientations that yield acyclic oriented matroids.

Definition 1.6.2 (Tope)**.** $X = \langle x_1, ..., x_{\binom{n}{k}} \rangle$, with $x_i \in \{+, -\}$, is a *tope* of $M(G_k^n)$ if there is an open full-dimensional cell of $\mathfrak{A}$ so that a point in the interior of that cell is in H_i^+ if $x_i = +$ and in H_i^- if $x_i = -$.

Definition 1.6.3 (Acyclic Orientation)**.** An oriented matroid has an *acyclic orientation* when the given orientation of each element is such that no circuits can be written as linear expressions summing to zero without negative coefficients (we say "there are no positive circuits").

For a given acyclic orientation, often we can find some subset S of the signed elements so that each element not in S "must" have a specific sign lest it upset our acyclic orientation. In other words, the subset S has, in a sense, "enough" oriented elements to give us only one way to orient the rest for an acyclic matroid. A minimal such set corresponds to the set of hyperplanes that form the facets of the cell associated with that particular tope as a signing of the elements.

When the elements of S are columns of the matrix representation of $M(G_k^n)$, we call the basis a *simplicial basis*. We often blur the distinction between the tope as a sign vector and the open cell which it represents, and speak of the cell itself as a tope.

Definition 1.6.4 (Simplicial Tope and Simplicial Basis)**.** In $\mathbb{R}^d$, a subset of an arrangement $\mathfrak{A}$ of hyperplanes consisting of d hyperplanes that forms the set of facets of a full-dimensional cell is a *simplicial tope*. These d-many hyperplanes correspond to a set of vectors orthogonal to them which form a basis for $\mathbb{R}^d$, which we call a *simplicial basis*.

We can get a perspective of the matroid of G_k^n as a convex polytope. A zonotope is the convex polytope that results from taking a Minkowski sum of segments.

Definition 1.6.5 (Zonotope)**.** Let $\{[-t_i, t_i] | 1 \leq i \leq p\}, p \in \mathbb{N}$ be a set of intervals in through the origin in $\mathbb{R}^d$. A *zonotope* Z is defined to be

$$Z = \Sigma_{i=1}^{p} [-t_i, t_i],$$

a Minkowski sum of all points of every interval $[-t_i, t_i]$.

We write $[-t_i, +t_i]$ to denote intervals using the columns of our matrix M. Using the zonotope definition above we obtain a special polytope associated with G_k^n. According to [1], the simplicial topes of the arrangement correspond to simple vertices of this zonotope.

1.7 Circuits

In a matroid, a *circuit* is a minimal dependent set (the removal of any element leaves an independent set), so the rank of a circuit is always one less than the number of elements it contains. In our example of G_3^6, it is possible upon inspection of the matrix representation in Figure 1.1 to observe that the smallest dependent set of columns contains four elements. The simplicial interpretation of this fact has a nice picture.

Because a triangle can share at most one edge with any other, a set of three triangles, however arranged, cannot form a relation for which all edges will cancel. There must be an edge on each triangle which is not shared with either of the other two.

Once again consider the elements abc, abd, acd and bcd. In the equations

$$\delta(abc)-\delta(abd)+\delta(acd) = \delta(bcd) \Leftrightarrow +ab-ac+bc-ab+ad-bc+ac-ad+cd = +bc-bd+cd,$$

we observe that after applying the boundary map, the left-hand side of each equation shows that the three elements sharing vertex a contain three uncanceled edges among them. These are precisely the edges that form bcd. These four triangles can be arranged as the facets of a tetrahedron, forming a closed surface in $\mathbb{R}^3$; many circuits of $M(G_3^n)$ form such nice closed surfaces when viewed as simplicial complexes. The more interesting circuits do not.

For any basis of $M(G_3^6)$, any element not in the basis is expressible as a linear combination of the basis elements. Alternatively, we can say that any element is in some circuit all of whose elements except one are in the basis. The set of circuits, each of which contains one unique non-basis element, associated with a particular basis are known as the *fundamental circuits* of that basis.

Recall the basis B_a, the set of all elements with vertex a in common. What are the fundamental circuits associated with this basis? Any element not in B_a can be thought of as the "base" of a tetrahedron with apex vertex a. This means that each such element has a linear expression in terms of three members of B_a, similar to the example above. The

fundamental circuits of $M(G_3^n)$ associated with the basis B_a are all the tetrahedra, or rank 3 circuits, which contain the vertex a.

Definition 1.7.1 (Binary Circuit)**.** A circuit is *binary* when we can obtain a zero-boundary expression by an appropriate choice of orientation of each element alone. That is, the unique linear combination equal to the zero vector can be written with coefficients all in $\{\pm 1\}$.

Once we consider large enough sets of minimal dependences, a curious object appears. It is possible to find a set of eleven elements of $M(G_3^n)$ which form a circuit that is not binary. For example,

$$\{abc, abd, abe, acd, acf, aef, bce, bcf, bdf, cde, def\}$$

is such a set. Any linear combination of this set that sums to the zero vector requires that one element (in this example, the triangle abc), have a coefficient other than ± 1.

We demonstrate:

$$\begin{aligned}
&\delta(-2abc + abd + abe - acd - acf + aef + bce + bcf - bdf + cde - def) \\
&= -2ab + 2ac - 2bc + ab - ad + bd + ab - ae + be - ac + ad - cd - ac + af - cf + ae \\
&-af + ef + bc - be + ce + bc - bf + cf - bd + bf - df + cd - ce + de - de + df - ef \\
&= \vec{0}
\end{aligned}$$

An inspection of this sum reveals that in the triangle complex, the boundary edges of abc appear three times among the circuit's elements, and we therefore must insert an "extra" copy of these three edges in order for there to be the same number of positively as negatively oriented boundary edges. A non-binary circuit is one for which orientations of elements alone cannot give us zero on the boundary. Lemmas 2.3.17, 2.3.19, and 2.3.20 in

Chapter 2 prove that circuits like the one just demonstrated have no orientation-only linear combination giving a zero-sum of edges.

Chapter 2: Matroid Properties

2.1 Connectedness of $M(G_3^n)$

Connectedness in a matroid is a generalization of the notion of connectedness in graphs.

Definition 2.1.1 (p-Separation)**.** Let M be a matroid, with elements E. Let $X \subseteq E(M)$, $Y = E \setminus X$ and r denote the rank function of M. Then (X, Y) is a *p-separation* of M if the following conditions are satisfied:

$min\{|X|, |Y|\} \geq p$

$r(X) + r(Y) - r(M) \leq p - 1$

Definition 2.1.2 (p-Connected)**.** We say a matroid M is *p-connected* if p is the least integer for which M can be p-separated.

Theorem 2.1.3. $M(G_3^n)$ is 4-connected, for $n \geq 6$.

Let $n \geq 6$. The proof is by a series of Lemmas.

Lemma 2.1.4. For any subset $S \subset E$, with $|S| = 3$, we have rank $r(E \setminus S) = r(E)$, the rank of the matroid $M(G_3^n)$.

Proof. Let $S = \{t_i, t_j, t_k\}$ be an arbitrary set of three elements from E. S is a linearly independent set: since two distinct triangles can share at most one edge, any complex of three triangles will have each member sharing at most two of its edges. In the matrix representation, this is a set of columns for which each column has one non-zero coordinate where all the other columns in the set have zeroes.

Let us extend our linearly independent set S to a basis B_S of the matroid, denoting $B_S = \{t_i, t_j, t_k, b_1, ..., b_{r-3}\}$.

With $n \geq 6$, every element of S is in at least three distinct rank 3 circuits. Each of these circuits contains some element not in the basis B_S.

Let one circuit containing t_i be $C_i = \{t_i, s_1, t_m, t_n\}$, where s_1 is not in B_S. We can express $s_1 = \pm t_i \pm t_m \pm t_n$, where t_m and t_n are other elements of the matroid, to demonstrate that s_1 is not independent of a set containing t_i.

There are three rank 3 circuits also containing t_j, and each of those has at least one element not in B_S. So we can pick one non-basis element from among these which is distinct from s_1. Name this element s_2, and write $s_2 = \pm t_j \pm t_p \pm t_q$.

Likewise, t_k is in three rank 3 circuits, so that we can choose s_3 distinct from s_1 and s_2 to be an element not in B_S, and expressible as $s_3 = \pm t_k \pm t_r \pm t_s$, not independent of a set containing t_k.

Now we have a new set $S' = \{s_1, s_2, s_3\}$ of distinct elements, not from B_S, and each of which is in a distinct dependence relation with t_i, t_j and t_k, respectively. So we can replace t_i, t_j and t_k in B_S and have $B'_S = \{s_1, s_2, s_3, b_1, ..., b_{r-3}\}$, a basis for the matroid not containing any elements of S. We conclude that $E \setminus S$ has rank $r(E)$. □

Lemma 2.1.5. $M(G_3^n)$ has no 1-, 2- or 3-separation.

Proof. It has been previously established that for every $t_i \in E$, $r(t_i) = 1$, and that pairwise, $r(t_i, t_j) = 2$. Because any t_i or any pair of elements t_i, t_j can be thought of as contained in some set of size 3, we use the previous lemma to observe that if any 3-set $\{t_i, t_j, t_k\}$ satisfies $r(E \setminus \{t_i, t_j, t_k\}) = r(E)$, then $r(E \setminus \{t_i, t_j\}) \geq r(E)$, and likewise for $r(E \setminus t_i)$. Let us denote $r(E) = r(M(G_3^n)) = r$.

Then, if $X = t_i$, and $Y = E \setminus t_i$, we have:

$$r(X) + r(Y) - r(M(G_3^n)) = 1 + r - r = 1 \nleq 0,$$

showing that $M(G_3^n)$ has no 1-separation (we also say $M(G_3^n)$ is "connected").

If $X = \{t_i, t_j\}$, and $Y = E \setminus \{t_i, t_j\}$, then

$$r(X) + r(Y) - r(M(G_3^n)) = 2 + r - r = 2 \nleq 1,$$

so G_3^n has no 2-separation.

If $X = \{t_i, t_j, t_k\}$, from Lemma 2, we know that $r(E \setminus X) = r$, and

$$r(X) + r(Y) - r(G_3^n) = 3 + r - r = 3 \nleq 3 - 1 = 2,$$

so $M(G_3^n)$ has no 3-separation.

□

Lemma 2.1.6. $M(G_3^n)$ has a 4-separation.

Proof. Let a set $X = \{t_i, t_j, t_k, t_m\}$ be chosen so that X forms a rank 3 circuit of the matroid.

In the geometry, viewed as the triangle complex on K_n, we can see that such a rank 3 circuit, which forms the surface of a tetrahedron, contains exactly four of the vertices from the ground set. Thus, for any $n > 4$, there is a vertex v not contained in any elements from X. We recall from the Introduction that all elements on this vertex v form a basis for

the matroid, and it is clear that they are all in the complement of the set X. So the set $Y = E \setminus X$ has rank r.

So we have:

$$r(X) + r(Y) - r(M(G_3^n)) = 3 + r - r = 3 \leq 4 - 1 = 3,$$

showing X to be a 4-separation of $M(G_3^n)$.

□

Proof. (of Theorem 2.1.3)

We conclude that the matroid of G_3^n is 4-connected for $n \geq 6$. □

2.2 When $M(G_3^n)$ Is Not Regular

Definition 2.2.1 (Regular Matroid)**.** A matroid is *regular* if it can be represented with a totally unimodular matrix.

Definition 2.2.2 (Binary Matroid)**.** A matroid is *binary* if it can be represented by a matrix with coefficients in $GF(2)$.

Theorem 2.2.3. For $n \geq 6$, $M(G_3^n)$ is not regular.

Proof. It is known that the class of regular matroids is properly contained in the class of binary matroids. We show that $M(G_3^n)$ is not binary, and thus cannot be regular. Let C_1 be a rank 10 circuit. In Theorem 2.3.12 we establish the existence of such a circuit by constructing one. In Lemmas 2.3.19 and 2.3.20 we prove that C_1 contains a type of basis B that forms a rank 10 circuit with every other element of $E \setminus B$.

Let t_1 be the point $B \setminus C_1$. Choose another element t_2, with $t_2 \notin C_1$, and let $C_2 = B \cup t_2$.

A property equivalent with being binary (see [12]) is that the symmetric difference of any two circuits contains a circuit. Clearly this is not the case here, since $C_1 \triangle C_2 = \{t_1, t_2\}$, and all circuits of $M(G_3^6)$ contain four or more elements. It follows that $M(G_3^6)$ is not binary, so we conclude that $M(G_3^6)$ is not regular. Furthermore, $M(G_3^6)$ is a minor of $M(G_3^n)$ for every $n \geq 6$, so that no matroid containing $M(G_3^6)$ is binary and hence not regular.

□

Definition 2.2.4 (Uniform Matroid)**.** Let E be a set with n elements. Let B be the collection of all m-element subsets of E, for $m \leq n$. Then B is the set of bases on a *uniform matroid* denoted $U_{m,n}$.

It is known (see [12]) that the uniform matroid $U_{2,4}$ is an excluded minor of any binary matroid. We show that $M(G_3^6)$ contains $U_{2,4}$ as a minor, and thus so does $M(G_3^n)$ for every $n \geq 6$.

Let $N_0 \subset M$ be a minor by deletion: $E(N_0) = B \cup \{t_1, t_2\}$. Since B is a basis, it has rank 10, so that N_0 has rank 10, and therefore ranks of sets in N_0 are the same as they are in M.

Any rank 2 set containing both t_1 and t_2 cannot contain more than two elements from B, so we know there are $b_1, ..., b_8 \in B$ which are not in the closure $\overline{(t_1 \cup t_2)}$. Contract N_0 by these eight elements of B, so that $E(N_1) = E(N_0) \setminus \{b_1, ..., b_8\}$. Four points remain in this contraction: t_1, t_2, b_9, b_{10}. Let r_{N_1} denote the rank function for the minor N_1, and r the rank function for N_0. We have:

$$r_{N_1}(t_1, t_2, b_9, b_{10}) = r(t_1, t_2, b_9, b_{10}, b_1, ..., b_8) - r(b_1, ..., b_8)$$

$$= 10 - 8$$

$$= 2.$$

By construction, each of t_1, t_2, b_9, b_{10} is pairwise independent of each of $b_1, ..., b_8$, so that:

$$r_{N_1}(t_1) = r(t_1, b_1, ..., b_8) - r(b_1, ..., b_8) = 1,$$

and likewise for each of t_2, b_9, b_{10}.

Since $C_1 = B \cup \{t_1\}$ is a minimal dependence in N_0, then $C_1 \setminus \{b_1, ..., b_8\} = \{t_1, b_9, b_{10}\}$ is a circuit in N_1, and likewise for $C_2 \setminus \{b_1, ..., b_8\} = \{t_2, b_9, b_{10}\}$. It remains to show that the sets $\{t_1, t_2, b_9\}$ and $\{t_1, t_2, b_{10}\}$ are also minimal dependences of N_1.

Recall that in $M(G_3^6)$, the set $\{t_1, t_2\}$ doesn't contain a circuit, so that its rank in $M(G_3^6)$ (and in N_0) is 2. Also recall that none of $\{b_1, ..., b_8\}$ is in the closure $\overline{(t_1 \cup t_2)}$, so $r(t_1, t_2, b_1, ..., b_8) = 10$. Then,

$$\begin{aligned} r_{N_1}(t_1, t_2) &= r(t_1, t_2, b_1, ..., b_8) - r(b_1, ..., b_8) \\ &= 10 - 8 \\ &= 2. \end{aligned}$$

Thus, any set subset of $E(N_1)$ containing both t_1 and t_2 has rank 2.

We have shown that every three elements of $E(N_1)$ form a circuit, implying that every pair of elements is an independent set in N_1. These properties describe N_1 as the matroid $U_{2,4}$. Since N_0 is a minor of $M(G_3^6)$, and N_1 is a minor of N_0, so that this instance of $U_{2,4}$ is a minor of $M(G_3^6)$.

It is important to note that demonstrating the existence of $U_{2,4}$ within $M(G_3^6)$ requires the existence of what we term a non-binary circuit. In the construction, we begin with a particular type of basis, B, with the property that $B \cup t_1$ and $B \cup t_2$ are minimal dependent sets. If we had used a basis without this property, then there would be no way to contract down to four elements and find that every three are minimal dependent sets. This type of basis, which has non-binary circuits as its associated fundamental circuits, is the special

impediment in $M(G_3^6)$ to the property of being regular. This is discussed in detail in the following section.

2.3 Some Large Regular Minors $M(G_3^6)$

In order to give some regular minors, we must first consider the size and construction of the only type of circuit which fails to satisfy a necessary and sufficient condition for a matroid to be binary, as set forth by Tuma in [14]:

> "C is a cycle if and only if any $(k-1)$-subset of [the vertex set] is contained in an even number of k-sets of C. The set $\mathcal{C}$ of minimum non-empty cycles is the set of [circuits] of [a] binary simplicial matroid."

Definition 2.3.1 (Tuma's Edge-evenness Condition)**.** The matroid of G_3^6 satifies *Tuma's Edge-evenness Condition* if every edge contained in a circuit is in the circuit an even number of times.

Here, our vertex set is $V = \{a, b, c, d, e, f\}$, and $k = 3$, so that for a binary matroid, each pair of vertices (which we think of as an edge) found in the circuit must be contained in an even number of elements (or triangles) of the circuit. When this condition fails for some circuit, we say the circuit is non-binary. Hence, so is the matroid. We show, by a series of lemmas, that if there are *any* edges in a circuit an odd number of times, then there are exactly three such edges, and that there is only one type of non-binary circuit in $M(G_3^6)$. For C a circuit, every edge of C is in at least twice. Since the matroid contains only four elements with any edge in common, we take "odd" to mean "three."

Theorem 2.3.2. Let C be a circuit of $M(G_3^6)$. If C contains some edge an odd number of times, then it contains exactly three such odd edges.

Proof. (*Sketch*) The proof goes as follows. We assume C has some number of odd edges, so that it is non-binary. In the unique linear dependency for C, some triangles will have even coefficients, some will have odd coefficients. An edge that is in C an odd number of times must be in both types of triangles: one with an even coefficient (such as -2), and two with odd coefficients (such as +1). The set of odd edges in C forms a closed perimeter around the collection of even-coefficient triangles, and we use this perimeter, named L, to determine what kind of set we must have among the odd coefficients in order to have two odd-coefficient triangles on each edge of L, whilst somehow in the end forming a circuit. Since L is limited by the fact that we have only six vertices to work with, we show that L cannot have four, five, or six edges, and if it has any at all, it must have exactly three.

Lemma 2.3.3. Let C be a circuit of $M(G_3^6)$ containing at least one edge an odd number of times. The unique linear dependency for C contains a term with an even coefficient.

Proof. Let the three elements of C containing the odd edge be t_1, t_2 and t_3. Up to scalar multiples, there is a unique linear expression for C. Let the coefficients associated with t_1, t_2 and t_3 in the expression be α, β and γ, respectively. Since the linear expression sums to zero under the boundary map, it must be that $\alpha + \beta + \gamma = 0$. The sum of two odd numbers is even, so either all three coefficients are even, or without loss of generality, we can say α and β are odd, and γ is even. □

Let X be the set of elements in C that have even coefficient $\pm\gamma$ in the linear dependency. Consider the partial sum of edges of X under the boundary map. It is non-zero (C cannot properly contain a circuit). Let L be the set of remainder edges from X. All edges of X which are not in L are in an even number of triangles of X. This is evident from the fact that they are not in the remainder set L - that is, they canceled out in the partial sum of edges from X. So it must be that $|L| \geq 3$.

We take the simplicial complex of triangles of X to be one edge-connected component,

with every element sharing one or more edge with another, so that L forms the perimeter of the complex, and is a closed loop in the graph of edges of X. If X had more than one edge-disjoint component, because G_3^6 has only six vertices, at least one component would be all three edges of some triangle of X. In such a case, Lemma 2.3.14 establishes the only way for a circuit to contain all three edges of a triangle, and these are the *only* odd edges of C.

The set $C \setminus X$ is the collection of all triangles with coefficients α or β. In case $\alpha \neq \beta$, $C \setminus X$ must contain one triangle with the coefficient α and one with the coefficient β for each edge of L, in order to obtain zero in the boundary sum for C. In case $\alpha = \beta$, we simply require $C \setminus X$ to have two triangles containing each edge of L. Evidently, L consists of the set of odd edges of C. X contains one copy of each of the edges of L, all with even coefficients, and $C \setminus X$ contains two of copies each edge of L having odd coefficients.

Lemma 2.3.4. $\alpha = \beta$

Proof. Suppose $\alpha \neq \beta$. Let A be the set of triangles in $C \setminus X$ having coefficient α. Since A is disjoint from X, any edge in A that is not in L is in an even number of triangles of A. Consider the set $X \cup A$. By construction, every edge is in this set an even number of times. By Tuma's edge-evenness property, $X \cup A$ is a circuit of $M(G_3^6)$. But C is a circuit, and does not properly contain a circuit, so $A = C \setminus X$, and $\alpha = \beta$. □

Lemma 2.3.5. $\alpha = \beta = 1$, and $\gamma = -2$.

Proof. Necessarily, $\alpha + \beta + \gamma = 0$, and γ is even. The linear expression for C is unique up to scalar multiples, so we conlcude that the most reduced integer form for the expression has coefficients all in $\{\pm 1, \pm 2\}$. □

Lemma 2.3.6. (Triangulation Lemma) $C \setminus X$ does not properly contain a set of triangles that form a triangulation of the polygon formed by L.

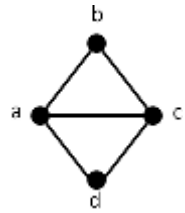

Figure 2.1: Four Odd Edges

Proof. Let $T \subseteq C \setminus X$ be a set of triangles that triangulate the polygon with perimeter L. The edges of $T \cup X$ are each in an even number of triangles. Then, by the previously quoted Tuma's edge-evenness property, $T \cup X$ is a circuit. C cannot properly contain a circuit, so $C \setminus X$ does not contain a triangulation of L. $\square$

Lemma 2.3.7. C cannot contain four odd edges.

Proof. Suppose $|L| = 4$. Then X must contain a minimum of two triangles. We show that even with this minimal requirement for X, a construction of $C \setminus X$ requires too many triangles. Recall that the rank of the matroid $M(G_3^6)$ is ten, so that the largest possible circuit can have no more than eleven elements.

Suppose $X = \{abc, acd\}$. These two triangles share edge ac, so L consists of edges ab, bc, cd and ad. Refer to Figure 2.1. Since X is the set of triangles with even coefficients, the edges of L must appear in $C \setminus X$ twice each. Here is a listing of our possible choices for each edge of L:

$ab : abd, abe, abf$

$bc : bcd, bce, bcf$

$cd : bcd, cde, cdf$

$ad : abd, ade, adf$.

We need to choose two for each edge of L, and in particular we cannot choose both of abd and bcd. These violate the Triangulation Lemma (Lemma 2.3.6), and combined

with X would give us the four elements of a rank 3 circuit, and C cannot properly contain another circuit. On the other hand, if we choose for each edge only the pair containing vertices e and f, we find that C contains the rank 5 circuits $\{abc, acd, abf, bcf, cdf, adf\}$ and $\{abc, acd, abe, bce, cde, ade\}$. So we must have one of abd or bcd in $C \setminus X$. We can without loss of generality suppose that $C \setminus X$ contains abd and not bcd. A symmetric result happens when we choose bcd and not abd.

Then our choices for edges bc and cd are determined for us - we need two triangles for each, and can't choose bcd. Having chosen abd, which contains both ab and ad, we need to make one more triangle choice for each of these edges. Note that if we choose abe, we must have adf, or choosing abf means we must have ade, in order to avoid including the rank 5 circuits previously mentioned. Again, we get symmetric results from either choice.

Thus far, we have the following seven triangles in $C \setminus X$: $\{abd, abe, bce, bcf, cde, cdf, adf\}$. Each edge of L is now in the set twice, but we have some unpaired edges which must be in $C \setminus X$ twice to complete C. These edges are ae, bd, af, be and de. On the other hand, we can only include two more triangles in our set, based on our size constraint for C (X has two elements, so $C \setminus X$ can contain at most nine). No three of our remaining edges are in any one triangle together, so that we would need at least three new triangles to pair them all.

We conclude that it is not possible to have a circuit with four odd edges. □

Lemma 2.3.8. C cannot contain five odd edges.

Proof. Suppose $|L| = 5$. Then X must contain at least three triangles. Up to a permutation of vertices, there is only one configuration of three triangles that gives a perimeter of five edges.

Let $X = \{abe, bcd, bde\}$. By our maximum circuit size constraint, $C \setminus X$ must contain no more than eight triangles. But to have each of five edges in two triangles of $C \setminus X$ implies

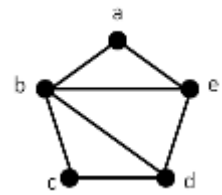

Figure 2.2: Five Odd Edges

ten triangles, so we must have two triangles of $C \setminus X$ containing two edges of L. Since X and $C \setminus X$ are disjoint, there are three such possibilities: abc, cde and ade.

Here is our list of choices for each edge of L:

$ab : abc, abd, abf$

$bc : abc, bce, bcf$

$cd : acd, cde, cdf$

$de : ade, cde, def$

$ae : ace, ade, aef$.

If we use all three of abc, cde and ade, observe that the Triangulation Lemma implies that we must exclude acd and ace from $C \setminus X$ so as not to include the set $\{abc, ade, acd\}$ or the set $\{abc, ace, cde\}$, both triangulations of L.

Case 1: The choice of all three of abc, cde and ade gives us our choice of two triangles on edge de, and one on each of the other edges L. However, excluding acd forces us to include cdf as the second triangle on edge cd, and excluding ace forces the choice of aef on edge ae. We must choose one more triangle for edge ab, and one more for edge bc. So far, $C \setminus X$ contains the five elements abc, cde, ade, cdf and aef. We must use only three triangles to pair edges ab, bc, cf, df, af and ef, which are all in $C \setminus X$ once so far. Each triangle we choose must contain two of these edges (there are none that contain three). Our choices for ab and bc would be abf and bcf. But def is the only choice for the remaining two edges - giving us a copy of edge de in $C \setminus X$. This edge would not be zero in the boundary sum for

C.

We conclude from this case that we cannot include all three of abc, cde and ade in $C \setminus X$.

Case 2: We consider the choice of triangle abc with either cde or ade and not both. Recall that including abc and cde meant we must exclude ace from $C \setminus X$. When we do this, we find that there are only two remaining choices for edge ae: ade and aef. In other words, choosing cde means we must have ade anyway, and we have *Case 1* again. Similarly, when we choose abc and ade, we exclude acd, so that we are forced to choose cde and cdf for edge cd. We see that triangle abc must be the bad choice.

Case 3: Suppose we choose ade and cde to be in $C \setminus X$. From the first two cases, we learned that abc must be excluded. Then for edge ab, we must have abd and abf, and for edge bc we must have bce and bcf. Now $C \setminus X$ contains six triangles, and we can only have eight total in this set. The unpaired edges so far are ae, cd, af, cf and be. No three are in a triangle together, so we still cannot construct $C \setminus X$ with eight or fewer triangles.

Now suppose we still have $|L| = 5$, but $|X| > 3$. The size of X cannot be four - all edges must be paired in X, except for the five in L. X would have $4 \cdot 3 = 12$ edges, with 7 of them required to be paired in X. In general, for $|L| = 5$, $|X|$ must be odd.

If $|X| = 5$, we would be required to build $C \setminus X$ with at most six triangles. This means that four triangles must have two edges of L in them. Here are the five possible triangles from which we must pick four: abc, bcd, cde, ade and abe. Suppose we choose the first four. Then there will be four edges in the set once so far: ac, bd, ce and ad, and we still need second triangles for edges ab and ae in L. Only one pair of triangles can pair all of these edges - any other choice would require three or more triangles. This pair is ace and abd. But including these means that $C \setminus X$ contains $\{abc, ace, cde\}$, a triangulation of L.

Clearly, for any odd number larger than 5, X is too big for us to find enough triangles for $C \setminus X$.

We conclude that a circuit C of $M(G_3^6)$ cannot contain five odd edges. $\square$

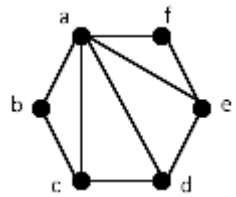

Figure 2.3: Six Odd Edges

Lemma 2.3.9. C cannot contain six odd edges.

Proof. Suppose $|L| = 6$. Then X must contain at least four triangles, so $C \setminus X$ contains at most seven triangles, implying that at least five triangles of $C \setminus X$ contain two edges of L.

Suppose $|X| = 4$. Then as a simplicial complex, X has $4 \cdot 3 = 12$ edges, six of which are on L, and the others all paired. Then at least two triangles in X have two edges in L. Since there are only six total triangles that contain two edges of L, we do not have five of them available to construct $C \setminus X$.

If $|X| > 4$, even if we had all six triangles having two edges in L to build $C \setminus X$ with, we are constrained by the size of C, which has at most eleven triangles, that the other edges of these six triangles cannot be paired.

We conclude that C cannot have six odd edges.

□

We have shown that C cannot contain four, five or six odd edges. If we consider $|L| > 6$, we find that, having used all six vertices, L is no longer a simple loop. This case is subsumed by the foregoing cases, since L is now composed of two or more loops of sizes three, four, five or six.

We conclude that C contains exactly three odd edges. □

Lemma 2.3.10. If L contains exactly three edges, then $C \setminus X$ is the special basis described in Definition 2.3.18.

Proof. $C \setminus X$ contains two copies of each of three edges that form a closed loop. Then there is some triangle t whose edges are the edges of L. By construction, $(C \setminus X) \cup t$ forms a circuit for which all edges of t are in three times. This is the circuit described in subsequent Lemma 2.3.14 and Theorem 2.3.12. □

Lemma 2.3.11. If $C \setminus X$ is a special basis, then $|X| = 1$.

Proof. $C \setminus X$ contains ten triangles and is a linearly independent set. C contains at most eleven elements, so $|X| = 1$. □

Now we show that a non-binary circuit exists.

Theorem 2.3.12. The matroid $M(G_3^6)$ contains a non-binary circuit.

The proof is by a series of lemmas.

Lemma 2.3.13. If a circuit C contains exactly one element whose edges are all in the set an odd number of times, then C has an odd number of elements.

Proof. Let C be a circuit with m elements. Then there are $3m$ boundary edges in the set. There are $3(m-1)$ edges in the set an even number of times. Then $2|3(m-1)$, so $2|(m-1)$. Thus, $m - 1 + 1 = m$ is odd. □

Lemma 2.3.14. If a circuit C contains an element whose edges are all in the set an odd number of times, then C must contain at least ten elements.

Proof. Let T be the element whose edges are in the circuit three times, and let its vertices be $\{a, b, c\}$. Since our elements are triangles, and any two triangles share at most one edge, we only have three copies of T's edges in the circuit if two new elements of C are appended to each edge of T.

Let these six new elements be $\{t_1, ..., t_6\}$, where t_1 and t_2 share edge ab with T; t_3 and t_4 share edge bc with T; and t_5 and t_6 share edge ac with T. Figure 2.4 depicts such a construction.

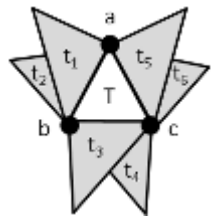

Figure 2.4: Edges in a Circuit An Odd Number of Times

Observe that with only six vertices, the set $\{t_1, ..., t_6\}$ will not all be edge-disjoint. The element T uses three vertices; t_1 and t_2 have two of the other three vertices at their apexes, say d and e. Elements t_3 and t_4 must use the sixth vertex, f.

To show why, suppose we "reuse" vertices d and e to form t_3 and t_4. Then,

$$T = abc$$

$$t_1 = abd$$

$$t_2 = abe$$

$$t_3 = bcd$$

$$t_4 = bce$$

Now we must form t_5 and t_6 using edge ac. At least one of these two elements must use d or e as well, so that t_5 or t_6 is acd or ace. But the elements $\{T = abc, t_1 = abd, t_3 = bcd, acd\}$ form a rank 3 circuit, $abcd$. And $T = abc, t_2 = abe, t_4 = bce, ace$ comprise the rank 3 circuit $abce$. Since circuits cannot properly contain other circuits, such an arrangement is impossible.

We avoid this problem if when forming t_1 and t_2 with vertices d and e, then for edge bc we form t_3 and t_4 with vertices e and f, then for edge ac we form t_5 and t_6 with vertices d and f, we have:

$$T = abc$$

$$t_1 = abd$$

$$t_2 = abe$$

$$t_3 = bce$$

$$t_4 = bcf$$

$$t_5 = acd$$

$$t_6 = acf$$

and find we still have an independent set. Such an assignation is unique up to a permutation of our arbitrary labeling. The property that no two edges of T can have new elements using only five of the six vertices among them is necessary.

It also means that each t_i shares only one of its edges with another t_j but not the other. In all, six edges still appear in our partially constructed circuit only one time. In order to be able to obtain zero on the boundary of the circuit, these edges must all be present in the set more than once.

New elements are required, sharing these six edges, in order to complete the circuit.

Since the edges of T are already present in the circuit three times, we see each new element must contain exactly one of the vertices $\{a, b, c\}$, and two from the set $\{d, e, f\}$. Thus, any new element can share at most two of its edges among $t_1, ..., t_6$. In the optimal situation, a minimum of three new elements are required to share two edges each with $t_1, ..., t_6$. Then, along with T, these three new elements bring our minimum circuit size to ten.

$\square$

Proof. (of Theorem 2.3.12)

We complete the construction from the previous lemma.

Consider the last (at least three) elements $\{s_i\}$ added to the circuit in Lemma 2.3.14. Together with Lemma 2.3.13, we conclude that the circuit we wish to build must have at least 11 elements, and be of rank at least 10. Since $M(G_3^6)$ has rank 10, our circuit must have exactly eleven elements. The elements $\{s_i\}$ that we add to share edges with the t_j must have one vertex among $\{a, b, c\}$ and one edge among $\{de, df, ef\}$. We observe that none of $\{de, df, ef\}$ has yet been present in the set. Then it is necessary for $|\{s_i\}| = 3$, each having one of $\{a, b, c\}$ for an apex and one of $\{de, df, ef\}$ for a base. Then an eleventh element, def ensures that all three base edges of s_1, s_2 and s_3 are in the set twice.

Such an arrangement is possible, and forms a circuit. Recall that up to relabeling, the arrangement of $T, t_1, ..., t_6$ is unique.

If $T = abc$, we can say without loss of generality that:

$$t_1 = abd$$

$$t_2 = abe$$

$$t_3 = bce$$

$$t_4 = bcf$$

$$t_5 = acd$$

$$t_6 = acf,$$

whereby the edges of T are all in the set three times. This will cause three edges to appear twice: ad, be, cf; and six edges to appear once each: bd, ae, ce, bf, cd, af.

By construction, the six edges that appear only once will be uniquely paired by their common vertex of a, b or c. Thus we have the pairs $\{ae, af\}$, $\{bd, bf\}$, and $\{ce, cd\}$. Then

our choice of elements $\{s_1, s_2, s_3\}$ to share these six edges is dictated for us: aef, bdf, cde. Observe that edges $\{de, df, ef\}$ now are the only remaining edges in the set at most once. The eleventh element must be def.

Such an arrangement forms a circuit, with the following linear combination of elements yielding zero under the boundary map:

$$\delta(2abc - abd - abe - bce - bcf + acd + acf - aef + bdf - cde + def) = 0.$$

□

Every circuit C like the one we just constructed is said to be *non-binary*, due to the fact that one of its elements does not have its edges in the set an even number of times. Any choice of orientation of the elements will yield under the boundary map, for each of these edges, either two positive and one negative copy of the edge, or two negative and one positive. So we cannot obtain zero on the boundary through orientation (i.e. choices of ± 1 as coefficients in the linear combination) alone.

Definition 2.3.15 (Boundary Set)**.** For a fixed k, the boundary set of G_k^n is the collection of all $(k-1)$-simplexes on n vertices. We denote this set by E^δ.

Lemma 2.3.16. If a circuit C of $M(G_3^6)$ has an odd-edged element, then C contains every edge from every member of E^δ at least twice.

Proof. C has rank 10, and so contains a basis, which must contain every edge. C is a circuit, and so we only obtain a linear combination of boundary edges equal to zero if each edge is in the circuit more than once.

□

Lemma 2.3.17. In $M(G_3^6)$, a circuit can have at most one odd-edged element, and such a circuit must have rank 10.

Proof. The construction from Lemmas 2.3.14 and 2.3.12 demonstrate a minimum requirement for one odd-edged element. At the same time, it is evident that new elements are required for any more edges to be tripled. The rank of $M(G_3^6)$ constrains us from building any larger circuit.

□

Now we look at the independent set responsible for the non-binary circuits. Its characteristics suggest methods for finding regular minors.

Definition 2.3.18 (Special Basis)**.** We define a basis of $M(G_3^6$ for which every member of E^δ of the matroid elements appears exactly twice to be a *special basis*. In Lemma 2.3.14 and Theorem 2.3.12, C minus its distinguished triple-edged element, T, is a special basis. $C \setminus T$ has rank 10 and is linearly independent and therefore a basis. Every basis must contain every edge from every element. By construction, this basis $C \setminus T$ has each edge exactly twice.

We make the observation that for any special basis B, and any element $t_i \notin B$, the set $B \cup t_i$ is a non-binary circuit. Such a set contains eleven elements, making it linearly dependent, and so it contains a circuit. But the element t_i now has all of its edges in the set three times, which forces us to conclude that $B \cup t_i$ is a non-binary circuit.

Lemma 2.3.19. Every circuit C of $M(G_3^6)$ with an odd-edged element contains a unique special basis.

Proof. This is evident by choosing the ten elements $C \setminus T$, where T is the one element whose edges are in the circuit three times. □

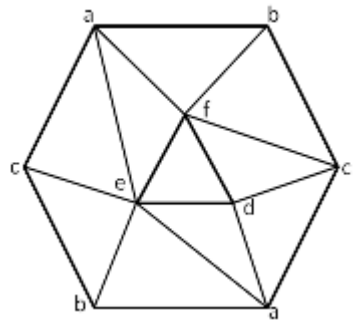

Figure 2.5: Special Basis as a Triangulation of $\mathbb{R}P^2$

Lemma 2.3.20. A special basis B forms a non-binary circuit with each distinct element of $E \setminus B$.

Proof. The basis B contains every edge twice, so for $t_i \in E \setminus B$, $B \cup t_1$ contains the edges of t_i three times each. Thus, $B \cup t_i$ is a non-binary circuit. □

A special basis and the circuits that can be formed from it have a direct relationship with the real projective plane, $\mathbb{R}P^2$, a non-orientable manifold. The simplicial complex of ten triangles with six vertices forming a special basis forms a well-known triangulation of $\mathbb{R}P^2$. Figure 2.5 shows this triangulation with one possible labeling of the vertices. The ten triangles whose interiors are contained within the perimeter of the triangulation are a special basis. Observe that the perimeter of the image contains two copies each of vertices a, b and c. We identify each of the same-named vertices, and the edges containing identical endpoints, to form a closed surface. The triangle abc, with edges all along the perimeter, is not a member of the triangulation, and is seen "twice" to indicate that its edges all appear twice among the elements of the special basis. This image is not unique in its depiction of a particular special basis. It is possible to rearrange some of the vertices, and find that the same ten triangles form the triangulation. We show such a rearrangement in Figure 2.6.

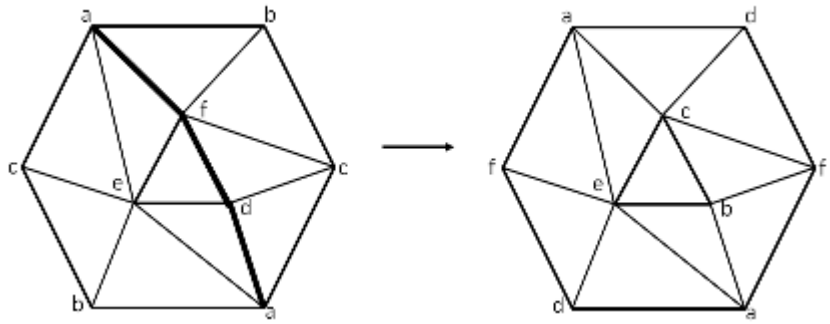

Figure 2.6: A Perimeter Transformation

We think of this vertex rearrangement in terms of a symmetry of the manifold. We describe what we term a perimeter transformation of $\mathbb{T}$. From our original image, if we identify the perimeter edges as indicated, and cut along the edge-path forming the boundary of adf, we form the new image with perimeter adf by unfolding $\mathbb{R}P^2$ along this new cut.

Observe the set of triangles in the triangulation are the same. This is the case for any t_i that, like adf, is not in the triangulation.

Lemma 2.3.21. Regarded as a simplicial complex, a special basis is a triangulation of $\mathbb{R}P^2$.

Proof. Let B be a special basis. Since it is a basis, all edges and vertices are in B, and B has ten triangles. We have $V - E + F = 6 - 15 + 10 = 1$, and $2E = 30 = 3F$. The construction of the non-binary circuit showed the structure of B to be unique up to symmetry, so that the depiction of a special basis as in Figure 2.6 is also unique up to symmetry.

It is also clear that in the above representation of the triangulation, we can surround any vertex with a neighborhood homeomorphic to a disc. Three of the vertices are always in the center, and we can perform a perimeter transformations to view any of them this way. We conclude that a special basis triangulates the non-orientable 2-manifold $\mathbb{R}P^2$. $\square$

Lemma 2.3.22. All circuits of $M(G_3^6)$ with rank less than ten satisfy Tuma's edge-evenness property.

Proof. Suppose C contains one member of E^δ three times. Then it is not possible to obtain zero with a linear combination of elements from E^δ only by choosing orientations of the elements of C. Orientation alone can do this only when, after applying the boundary map, every edge is directed one way the same number of times as it is directed the opposite way. We must conclude from this that C is not binary. □

Using the characteristics of the special basis, we can show how to delete very few elements to define large regular minors of $M(G_3^6)$. Let M_1 be the deleted minor of $M(G_3^6)$ from which three elements all sharing a common edge have been removed.

Lemma 2.3.23. The deleted minor M_1 contains no special bases.

Proof. Since M_1 contains only one element containing a particular edge, it is not possible to have any set of elements in which this edge appears more than once. Thus, we are unable to form a special basis in M_1.

□

Theorem 2.3.24. M_1 is a regular matroid.

Proof. Since there are no special bases in M_1, there can be no non-binary circuits. In other words, every circuit of M_1 satisfies Tuma's edge-evenness condition, and there is no $U_{2,4}$ minor. We conclude that M_1 is a regular matroid. □

Definition 2.3.25 (Complementary Pair)**.** We define a complementary pair of elements to be two elements of G_3^6 which are vertex-disjoint. For example, $\{abc, def\}$ and $\{bdf, ace\}$ are two such pairs.

Let M_2 be the deleted minor of $M(G_3^6)$ from which one complementary pair of elements has been removed.

Theorem 2.3.26. M_2 is a regular matroid.

Lemma 2.3.27. A special basis contains at most one of a complementary pair.

Proof. Let B be a special basis of $M(G_3^6)$. Then B contains every boundary edge of each matroid element exactly twice. Suppose B contains two complementary elements. Every other element of G_3^6 shares an edge with one member of the pair and a vertex with the other.

Let abc, def label our complementary pair. Element abc has three edges, so we can add three elements that share one edge each with abc. Likewise, we can add three elements that share one edge with each in def. This gives us eight elements so far. Any ninth and tenth element will share a third copy of some edge from abc or def.

This contradiction proves the Lemma.

□

Lemma 2.3.28. A special basis contains ten linearly independent elements, each being exactly one of any complementary pair.

Proof. Since there are ten such pairs in our set of matroid elements, we see that a special basis is comprised of exactly one of every complementary pair. □

Proof. (of Theorem)

Let M_2 be the deleted minor defined above. No basis of M_2 can contain ten elements without two of the same complementary pair. Thus, we can no longer form a special basis, and so M_2 contains only circuits which satisfy Tuma's edge-evenness property, and does not contain $U_{2,4}$ as a minor. M_2 is therefore regular. □

2.4 The Special Bases of $M(G_3^6)$

Theorem 2.4.1. The matroid of G_3^6 has twelve special bases.

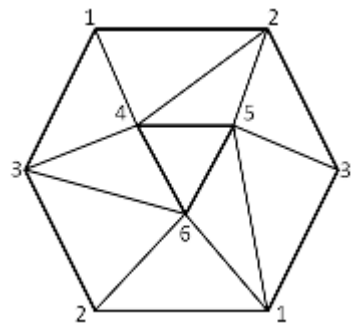

Figure 2.7: A Triangulation of $\mathbb{R}P^2$

Proof. In Lemma 2.3.21, we showed a special basis to be a triangulation of $\mathbb{R}P^2$, so it suffices to show that there are twelve inequivalent triangulations of the non-orientable surface $\mathbb{R}P^2$ with six vertices. We consider permutations of the six vertices in the ground set. There are a total of $6! = 720$ permutations of six vertices.

The perimeter transformation we showed in Figure 2.6 shows that any triangulation of $\mathbb{R}P^2$ has ten transformations which yield the same set of triangles in the triangulation, one for each of the ten elements of the matroid which are not in the triangulation. There are six more equivalent permutations of vertices, all of which give us the same triangulation. In Figure 2.7, we show another triangulation of $\mathbb{R}P^2$, with vertices named $\{1, 2, 3, 4, 5, 6\}$ in order to demonstrate the permutations.

The ten triangles in the triangulation are 124, 245, 235, 135, 156, 126, 236, 346, 134, and 456.

By considering each possible permutation of vertices 1, 2 and 3, we find there is a companion permutation of 4, 5 and 6 preserving the triangulation:

(123)(456)

(21)(64)(3)(5)

(31)(54)(2)(6)

(32)(65)(1)(4)

(312)(564)

(231)(645)

The above permutations are also easy to see by looking at the visible rotational symmetry in the image: in the top left of the image, the vertex 1 is attached by a single edge to vertex 4 in the central triangle. There are similar circumstances for vertices 2 and 3. The other vertex identified with 1 is attached to the central triangle by two edges, going to vertex 5 and vertex 6. If we move vertex 1, we can move 4, 5, and 6 in order to preserve the connections just described. The permutations above are six ways to do this. We have shown that every triangulation of $\mathbb{R}P^2$ with six vertices and ten triangles has 60 equivalent vertex permutations that preserve the triangulation. Thus there are at most $\frac{720}{60} = 12$ distinct triangulations.

Below, we list twelve known special bases, using our usual naming scheme of $\{a, b, c, d, e, f\}$ for the vertices, to conclude that there are exactly twelve of them:

$\{abc, abd, acf, ade, aef, bce, bdf, bef, cde, cdf\}, \{abe, abf, acd, ace, adf, bcd, bcf, bde, cef, def\},$

$\{abc, abe, acd, adf, aef, bcf, bde, bdf, cde, cef\}, \{abd, abf, ace, acf, ade, bcd, bce, bef, cdf, def\},$

$\{abc, abf, ace, ade, adf, bcd, bde, bef, cdf, cef\}, \{abd, abe, acd, acf, aef, bce, bcf, bdf, cde, def\},$

$\{abc, abf, acd, ade, aef, bce, bde, bdf, cdf, cef\}, \{abd, abe, ace, acf, adf, bcd, bdf, bef, cde, def\},$

$\{abc, abe, acf, ade, adf, bcd, bdf, bef, cde, cef\}, \{abd, abf, acd, ace, aef, bce, bcf, bde, cdf, def\},$

$\{abc, abd, ace, adf, aef, bdf, bde, bef, cde, cdf\}, \{abe, abf, acd, acf, ade, bcd, bce, bdf, cef, def\}.$

□

2.5 A Classification of Circuits on Six Vertices

Theorem 2.5.1. The smallest circuit of $M(G_k^n)$, for $k < n$, contains $k + 1$ elements.

Proof. Viewing elements of $M(G_k^n)$ as k-simplexes, we know that a k-simplex can share at most one $(k-1)$-face with another k-simplex. If a k-simplex is in a circuit, all of its

$(k-1)$-faces must be in the circuit at least twice, since we must be able to obtain $\vec{0}$ with the δ map. This means we have a minimum of $k+1$ k-simplexes in a circuit. There are $(k+1)$-many k-simplexes forming the boundary of every $(k+1)$-simplex, a k-polytope with $k+1$ vertices. Since the $(k+1)$-simplex is orientable, we can write a linear combination of these k-simplexes with coefficients all ± 1 that, under the δ map, sums to $\vec{0}$. Thus, there exists a circuit of $M(G_k^n)$ containing exactly $k+1$ elements.

□

A tetrahedron, which contains four triangles, represents the smallest circuit of $M(G_3^n)$. Every four vertices give a unique tetrahedron, so we can say that every matroid of G_3^n has $\binom{n}{4}$ rank 3 circuits.

Theorem 2.5.2. $M(G_3^6)$ has $10 \cdot 12 = 120$ non-binary circuits.

Proof. In Definition 2.3.18 we observed that the non-binary circuits are those for which we have $B \cup t_i$, for B a special basis and t_i and element not in B. There are always ten such t_i not in a basis. In Theorem 2.4.1 we showed that there are twelve such special bases. □

Theorem 2.5.3. In $M(G_3^n)$, there are $\binom{n}{6} \cdot 12 \cdot 10$ non-binary circuits.

Proof. From Theorem 2.5.2, we know that are 120 non-binary circuits on six vertices. Then for n vertices, there are $\binom{n}{6} \cdot 12 \cdot 10$ rank 10 non-binary circuits. □

Recalling Definition 1.7.1 of a binary circuit from the Introduction, we prove the following property.

Theorem 2.5.4. Every binary circuit of G_3^n contains an even number of elements.

Proof. Let $C \subseteq E$ be a binary circuit of $M(G_3^n)$ containing m elements. Each element of C has three edges, so C contains a total of $3m$ edges (not all distinct).

Let $\Sigma\delta(\pm t_i) = 0$ be a linear expression witnessing a choice of orientation for each of the elements of C so that boundaries sum to zero. This expression exists because the circuit is binary; orientation alone (coefficients of ± 1) is enough to obtain zero. Then each edge appears in C an even number of times, since the orientations direct each edge positively the same number of times as negatively. Thus $2|3m$, so that m is even.

□

Example 2. Binary circuits of other even sizes include:

(1) $\{abc, abd, acd, bcd, bde, cde\}$

(2) $\{abf, bcf, acf, abd, acd, bcd, bde, cde\}$

(3) $\{abc, abd, abe, abf, bcd, bef, ace, adf, cde, def\}$.

Observe that a circuit of rank 5 requires five vertices (with four vertices, we obtain only four possible triangles). A circuit of rank 7 requires six vertices: with only five vertices, we obtain the geometry G_3^5, which has rank $\binom{5-1}{3-1} = 6$. Thus, there is no circuit with eight elements (which would have rank 7) with fewer than six vertices.

The third item in the above example is a rank 9 circuit, with 10 elements. Since the rank of this circuit is less than ten, by Lemma 2.3.22, we conclude that it is binary. This type of circuit is unique in that it contains all four elements sharing a common edge (in this case, edge ab). This type of circuit is the smallest possible having this characteristic.

Theorem 2.5.5. The smallest circuit in which an edge is used four times has ten elements.

Proof. Let the elements sharing an edge be named abc, abd, abe, and abf. Each has two other edges that must also be shared with another element in the circuit in order to be able to obtain zero in a linear sum. These eight other edges are $\{ac, ad, ae, af, bc, bd, bd, bf\}$.

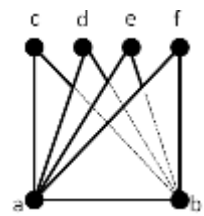

Figure 2.8: Four Triangles Sharing a Common Edge

Since every pair of our original four elements contains exactly four vertices, a unique four element circuit (a tetrahedron) contains each pair. Any element added to the first four can share at most two vertices with any of the four already in the set, so our construction already requires that a fifth element share at most two of its edges with two elements in the set. We seek to minimize the size of this circuit, so we wish to add elements that share as many edges already in the circuit as possible. Four new elements that each share two of their edges with some pair from abc, abc, abe and abf must be added.

However we choose four new elements, we must be sure that they themselves do not share an edge among them - this would complete a four element circuit with a pair of the original four elements. From this we see that eight elements are insufficient to construct this circuit, and, needing an even number of elements for a binary circuit, we conclude that at least ten elements are necessary.

□

It also turns out that when we have exactly six vertices with which to construct circuits, then if there is an edge for which all four elements containing that edge are in a circuit, there is only one such edge in that circuit. That is, every other edge in the circuit is shared by exactly two elements in the circuit.

Theorem 2.5.6. A circuit C of $M(G_3^6)$ contains at most one set of all four triangles sharing

an edge.

Proof. Suppose C contains two edges for which all four triangles containing each edge are in C.

Case 1: The two edges are vertex disjoint.

Let one edge be ab, and the other cd. With six vertices, each edge is contained in exactly four triangles, so when C has four copies of some edge, this means *every* triangle of the matroid with that edge is in the circuit. However, now our construction properly contains a circuit: two triangles containing ab are abc and abd; two triangles that contain cd are acd and bcd. These four triangles form a rank 3 circuit within our set. By the circuit axioms for matroids, a circuit cannot properly contain another circuit. We cannot create a rank 9 circuit with such a set.

Case 2: The two edges share a vertex.

Let the two edges be ab and bc. Then C contains these seven elements so far: abc, abd, abe, abf, bcd, bce, and bcf. We know C is binary, so every edge that occurs in C occurs an even number of times. The edges from the seven elements listed above which are still unpaired are $ac, ad, ae, af, cd, ce, cf$. But we observe now that every element which is not yet in C that contains these edges will create a rank 3 circuit within C. For example, edge ac is in three other elements of the matroid: acd, ace, acf. Any one of these in C will result in a rank 3 circuit: $\{abc, abc, acd, bcd\}, \{abc, abe, ace, bce\}$ or $\{abc, abf, acf, bcf\}$. We are also unable to create a circuit with this set.

We conclude that it is not possible to have more than one edge that has all four copies in $M(G_3^6)$.

$\square$

Theorem 2.5.7. Any rank 9 circuit C on six vertices contains all four elements that share a common edge.

Proof. Let C be a rank 9 circuit. Suppose C does not contain all four elements sharing an edge in common. First we observe that there are no triangulations of the 2-sphere with six vertices and ten triangles. C has ten triangles and six vertices. To triangulate the 2-sphere, its Euler characteristic would have to be 2. The $V - E + F = 6 - E + 10 = 2$, implying that $E = 6$, and C has 14 edges. But a triangulation also satisfies the relation $2E = 3F$, and in this case $2 \cdot 14 \neq 3 \cdot 10$.

But C is binary, and thus every edge must be in the circuit twice (since we've already characterized the case where an edge is in four times). So $2E = 3F$ forces us to conclude that all 15 edges of E^δ are in C.

If there are ten elements in C, then there are ten elements excluded from C. We choose one excluded element, T. While T is not in C, each of its edges is in C twice. In fact, we can refer to the construction of the non-binary circuit to see how to build an object for which all edges are in the circuit exactly twice. In Figure 2.9, let us consider T to be an element excluded from the circuit. The construction of the object in Lemma 2.3.14, which is shown to be unique up to permutation of vertices, turns out to be a ten-element *independent* set (a special basis), and only forms a circuit with the addition of an eleventh element. Since this is the only result of our condition that we use every one of the 15 edges exactly twice each, we are forced to conclude that it is not possible to construct a rank 9 circuit with this property. Thus, the only rank 9 circuits in $M(G_3^6)$ are those which contain all four elements sharing an edge in common.

□

Theorem 2.5.8. In G_3^n there are $\binom{n}{6} \cdot 180$ rank 9 circuits.

Proof. Let us begin the construction of C again. Suppose the four triangles with a common

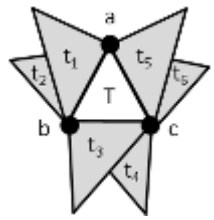

Figure 2.9: Attempting to Construct Another Binary Rank 9 Circuit

$Vertex\ a\ Choice$	$Vertex\ b\ Choice\ 1$	$Vertex\ b\ Choice\ 2$
acd, acf	bce, bdf (1.1)	bcf, bde (1.2)
ade, adf	bcd, bef (2.1)	bce, bdf (2.2)
ace, adf	bcd, bef (3.1)	bcf, bde (3.2)

Figure 2.10: First Set of Choices In Constructing A Rank 9 Circuit

edge in C are abc, abd, abe and abf. Eight edges (two from each of these four) must be paired, and as we saw in the previous theorems, this implies a need for at least four more elements, no two of which share an edge.

Three remaining elements which contain edge ac are acd, ace and acf. If we choose acd, then we are forbidden from choosing bcd in order to pair edge bc. There are many such restrictions, giving us a methodical way to count inequivalent formulations of the rank 9 circuit.

Here is an enumeration of our choices for the next four elements in C. We begin by picking a pair of elements that cover the four edges containing vertex a, and list our remaining choices for the edges on vertex b. Refer to Figure 2.10.

This gives us six possible configurations so far for C, given our original four elements on edge ab. Now for each of the choices we make for the first eight elements, we enumerate the options for completing C with a final pair of elements.

$Configuration$	$Final\ Pair\ 1$	$Final\ Pair\ 2$
(1.1)	cdf, cef	cde, def
(1.2)	cde, cef	cdf, def
(2.1)	cde, cef	cdf, def
(2.2)	cde, cdf	cef, def
(3.1)	cde, cef	cde, def
(3.2)	cde, cdf	cef, def

Figure 2.11: Second Set of Choices In Constructing A Rank 9 Circuit

Suppose we choose configuration (1.1) from Figure 2.10. Then C contains these eight elements: abc, abd, abe, abf, acd, aef, bce, and bdf. Unpaired edges in this set are cd, ce, df and ef. Two elements which pair these edges and complete C are cdf and cef. We could also choose cde and def. No other two elements contain these four edges among them, while completing C.

An analogous thing happens with each configuration from Figure 2.10. In Figure 2.11 we enumerate the "final pair" options for each of the six configurations above.

Having begun with the four elements containing edge ab, we count twelve distinct circuits containing them. Recognizing that any of the 15 edges on the elements of our geometry could serve as the unique one in a circuit four times, we count $12 \cdot 15 = 180$ such circuits containing any six vertices. For general G_3^n, there are $\binom{n}{6} \cdot 180$ such circuits.

$\square$

The circuits of smaller ranks are easier to describe, classify and count, because we can picture them as 3-dimensional polytopes, which are triangulations of the two-sphere S^2. The rank 3 circuit regarded as a polytope is a tetrahedron. With this type of circuit in mind, we can construct a rank 5 circuit that also triangulates S^2. Beginning with the tetrahedron, we place a fifth vertex in the barycenter of one facet, and join it to the three

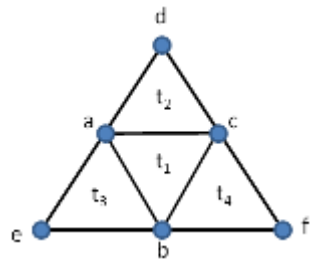

Figure 2.12: Beginning to construct the rank 5 circuit

vertices of the facet with edges. In the circuit, we replace the subdivided facet with the three smaller triangles of the barycentric subdivision, to make a total of six facets. One more such barycentric subdivision substituting a facet of this set will give us a rank 7 circuit also.

Obviously we can continue in this fashion to create binary circuits of arbitrary even size, given an unlimited supply of vertices. Not every circuit of G_3^n can be constructed in this manner, the rank 9 circuit described above being one example.

Lemma 2.5.9. A rank 5 circuit in the matroid of G_3^n has exactly five vertices.

Proof. We know that a rank 5 circuit with five vertices exists (see Example 2). We show that a rank 5 circuit has no more than five vertices.

A rank 5 circuit C must contain six elements. From Lemma 2.3.17 and the above theorem regarding circuits with four copies of an edge, we know that C is binary, and each edge of an element in C occurs exactly twice.

Then six elements have eighteen edges, which are all paired, giving us nine distinct edges in C. Let one element of C be $t_1 = abc$. We must also have t_2, t_3 and t_4 each sharing an edge with t_1. If each of t_2, t_3 and t_4 has one new vertex, then we must add at least three new elements to pair all of these edges, requiring this circuit to contain at least seven elements.

C must contain more than four vertices, since four vertices can determine at most four

triangles. So we suppose without loss of generality that $t_2 = acd$ and $t_3 = abd$ share a fourth vertex, d, and $t_4 = bce$ contains a fifth vertex, e. Now we have the following eight edges in our complex: $ab, ac, ad, bd, cd, be, ce$. In order for us to complete C and include a sixth vertex, there must be a t_5 or t_6 containing a vertex f. Then there will be two edges containing this vertex, increasing our number of edges thus far to ten. This makes our construction of a circuit containing six triangles on six vertices impossible.

□

Theorem 2.5.10. Circuits of rank 5 and rank 7 form triangulations of the 2-sphere.

Proof.

Claim 2.5.11. *There is exactly one type of rank 5 circuit, and it has a planar one-skeleton.*

Let C be a rank 5 circuit. From Lemma 2.3.22 and Theorem 2.5.5 we know that C is binary, and each edge in the circuit appears exactly twice. Then, because C has six elements, there are $\frac{6 \cdot 3}{2} = 9$ distinct edges in C. Let us consider the one-skeleton formed by these nine edges of C. Kuratowski's Theorem [12] tells us that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as minors. Since K_5 has ten edges, clearly C's one skeleton does not contain it. And since $K_{3,3}$ has six vertices, it is no less immediate that the one-skeleton of C does not contain it either. Then we apply Steinitz' Theorem [6], which says that a simple, planar, 3-connected graph is the one-skeleton of a 3-dimensional polytope. In this case, the one-skeleton is 3-connected because it is a triangulation, so the polytope is actually simplicial. There is a unique simplicial 3-polytope with five vertices [6], and we thus know that C represents the only type of rank 5 circuit.

Since C is a simplicial 3-polytope, it is a triangulation of S^2.

Claim 2.5.12. *There are exactly two types of rank 7 circuits, and they both have planar one-skeletons.*

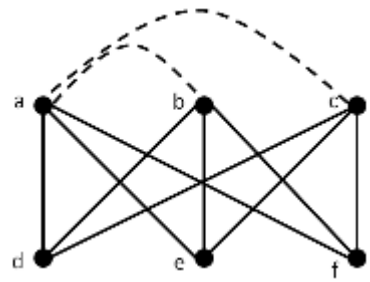

Figure 2.13: Looking For A $K_{3,3}$ Subgraph

Let D be a rank 7 circuit. As with rank 5, we know that a rank 7 circuit must be binary and that each edge is in D exactly twice. Thus, D has $\frac{8 \cdot 3}{2} = 12$ distinct edges. We also must assume that D has six vertices: every circuit with only 5 vertices is a circuit of $M(G_3^5)$, which only has rank 6. Referring again to Kuratowski's Theorem, suppose D has a one-skeleton that contains K_5 as a subgraph. K_5 has ten edges, so five vertices of D are contained in four edges each, while a sixth vertex is contained in only two edges. If this is true, then the sixth vertex is contained in only one element of the circuit. In such a case, the two edges containing the sixth vertex are in D only once, contradicting the fact that D is a circuit. We can conclude that the one-skeleton of D does not contain K_5 as a subgraph.

Suppose the one-skeleton of D contains $K_{3,3}$ as a subgraph. Then among the twelve edges of D's one-skeleton, we can find the complete bipartite graph on six vertices. When we consider any three edges added to $K_{3,3}$ in configurations that might represent D, we find that however we choose three edges, two of them will contain a common vertex.

We arbitrarily choose two such edges for demonstration (dotted lines ab and ac in Figure 2.13), recognizing that because of the symmetry of the graph, any other choice of two edges sharing a vertex would yield equivalent results. So far, we can discern the following elements whose boundaries are in the graph, as possible elements of D: abc, abd, abe, abf, acd, ace, acf, bcd, bce and bcf. But we recall that every edge is in the circuit exactly two times, and we see upon inspection of the list that there is no way to pare it down to eight elements with

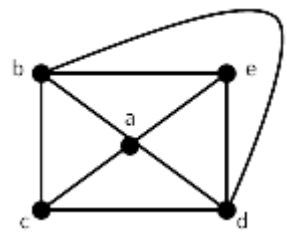

Figure 2.14: One-skeleton of the rank 5 circuit

that property. Adding a twelfth edge will only increase this list and yield more duplicate edges. We conclude that the one-skeleton of D does not contain $K_{3,3}$ as a subgraph either.

Thus, D has a planar one-skeleton. It is also 3-connected as a triangulation, and by Steinitz is also the graph of a simplicial 3-polytope. From [6], we know there are two 3-dimensional simplicial polytopes, the cross-polytope and the cyclic polytope.

Thus, every rank 7 circuit is a triangulation of S^2. □

In fact, regarding low-rank circuits as simplicial 3-polytopes makes them easy to visualize and count.

Theorem 2.5.13. The matroid of G_3^n contains $\binom{n}{5} \cdot 10$ rank 5 circuits.

Proof. There is a unique three-dimensional simplicial polytope P with six facets and five vertices (refer to [6]). We can easily count the triangulations of P by permuting the vertices and then looking for the distinct ones (we obtain many identical ones due to symmetry). Since a 3-polytope always has a planar graph that represents its one-skeleton, we can use such a graph to understand the symmetries of the polytope. Figure 2.14 shows the one-skeleton of P. For each choice of the center vertex labeled a there are two triangulations: the one pictured, and one in which we switch b with c and d with e. Since any one of the five vertices can be put in the center, there are $5 \cdot 2 = 10$ distinct triangulations of P.

□

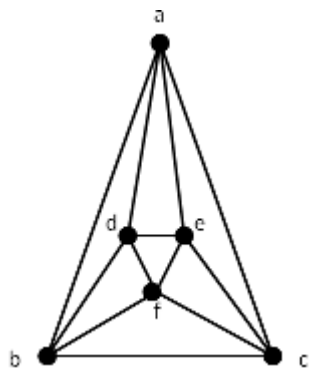

Figure 2.15: P_1, the one-skeleton of the 3-dimensional crosspolytope

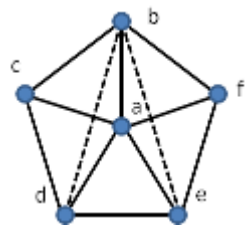

Figure 2.16: P_2, the one-skeleton of the 3-dimensional cyclic polytope

From [5], we know that G_3^5, which has five vertices and has rank 6, so there are no rank 7 circuits on five vertices. We have also shown that the smallest non-binary circuit has rank 10, so we know that there are no circuits of rank 6. We use these facts to see that rank 7 circuits must use six or more vertices.

Theorem 2.5.14. The matroid of G_3^n contains $\binom{n}{6} \cdot 45$ rank 7 circuits, for $n \geq 6$.

Proof. There are exactly two three-dimensional simplicial polytopes (refer to [6]), P_1 and P_2, having six vertices and eight faces. P_1 is a crosspolytope (the octahedron), which is known to have 48 symmetries [4]. Figure 2.15 shows its one-skeleton.

Since there are 6! permutations of six vertices, we have $\frac{6!}{48} = 15$ distinct triangulations of P_1. Thus, 15 circuits with eight faces on six vertices form octahedra.

P_2 is a cyclic 3-polytope with the one-skeleton shown in Figure 2.16.

For each of the six choices of center vertex, there are five distinct triangulations. We can think of the vertex labeled a as the apex of P_2, having a base that is a triangulated pentagon. There are five ways to triangulate the pentagon, so we have $6 \cdot 5 = 30$ circuits forming this type of polytope on six vertices.

Thus, for n vertices, there are $\binom{n}{6} \cdot (15 + 30)$ rank 7 circuits.

□

Theorem 2.5.15. Every circuit of $M(G_3^6)$ can be expressed as a linear sum of tetrahedra.

Proof. We define a linear sum of tetrahedra by way of recursive applications of the boundary map. Just as δ applied to a triangle yields a sum of edges (its boundary), δ applied to a tetrahedron yields a sum of the triangles forming its boundary.

For circuits which are triangulations of S^2, we demonstrate labelings of the polytopes' vertices which can be shown to be the result of linear sums of tetrahedra.

(1) Rank 3 circuits, already described as tetrahedra, are simply expressed by an ordered listing of their four vertices (with either sign). For example, we write $abcd$ to denote the tetrahedron on those four vertices. Then the boundary map gives $\delta(+abcd) = +abc - abd + acd - bcd$. This tells us what the triangles are forming the boundary of the tetrahedron, as well as giving us a signing that will, with one more application of δ, yield a zero sum. Note that the sign reversal $-abcd$ results in the complete sign reversal of the boundary also, so that $\delta(-abcd) = -abc + abd - acd + bcd$.

(2) Rank 5 circuits we see as the 3-polytope with one-skeleton in Figure 2.14. A linear sum of tetrahedra that result in the set of triangles in this polytope is $+abcd + bcde$, where

$\delta(+abcd + bcde)$

$= +abc - abd + acd - bce + bcd - bce + bde - cde$

$= +abc - abd + ace - bce + bde - cde.$

(3) Rank 7 circuits have two types: a crosspolytope (octahedron) and a cyclic polytope. Refer to Figures 2.15 and 2.16. The crosspolytope, which we named P_1, can be written as the sum $+abce + acde + bcef + cdef$, where

$\delta(+abce + acde + bcef + cdef)$

$= +abc - abe + ace - bce + acd - ace + ade - cde + bce - bcf + bef - cef + def - cdf + cef - def$

$= +abc - abe + acd - ace + ade - bcf + bef - def.$

The cyclic polytope can be written $+abcd + abde + abef$, so that

$\delta(+abcd + abde + abef)$

$= +abc - abd + acd - bcd + abd - abe + ade - bde + abe - abf + aef - bef$

$= +abc + acd - bcd + ade - bde - abf + aef - bef.$

(4) Rank 9 circuits we classified as those for which all four elements containing a common edge are found in the circuit. Here is a linear expression of tetrahedra that represents the rank 9 circuit from our example (3): $+abcd - acde + adef - abef$, with the boundary map giving us

$\delta(+abcd - acde + adef - abef)$

$= +abc - abd + abd - bcd - acd + ace - ade + cde + ade - adf + aef - def - abe + abf - aef + bef$

$= +abc - abd - bcd + ace + cde - adf - def - abe + abf + bef.$

(5) We can even construct the one type of non-binary circuit from $M(G_3^6)$ with a linear sum of tetrahedra. Here is one that represents the non-binary circuit we gave at the end of chapter 1: $+abcd + bcdf - cdef - acef + abce$. With the δ map, we have

$\delta(+abcd + bcdf - cdef - acef + abce)$

$= +abc - abd + acd - bcd + bcd - bcf + bdf - cdf - cde + cdf - cef + def - ace + acf - aef + cef + abc - abe + ace - bce$

$= 2abc - abd + acd - bcf + bdf - cde + acf - aef - abe - bce.$

□

2.6 $M(G_3^6)$ Is Ternary

Recall that a ternary matroid is representable over the field $GF(3)$.

Theorem 2.6.1. $M(G_3^6)$ is a ternary matroid.

Proof. Let B be a basis for $M(G_3^6)$, and t_i an element not in B. The determinant of B is non-zero, but suppose $3|\det(B)$.

Let $\vec{x}$ be the unique solution for the equation $B\vec{x} = t_i$. If $\vec{x}$ contains zeroes, then $B \cup t_i$ properly contains a unique circuit, C, which must have rank less than 10 in this case, and by Lemma 2.3.22 is binary. This means that there is a linear combination of elements from $B \cup t_i$ which sums to $\vec{0}$ using only ± 1 for coefficients. Thus, $\vec{x}$ has entries all in $\{0, \pm 1\}$.

If $\vec{x}$ contains no zeroes, then $B \cup t_i = C$ is a rank 10 circuit.

Lemma 2.3.17 and Theorem 2.5.4 allow us to assume that if $B \cup t_i = C$ is a rank 10 circuit containing eleven elements, then it is non-binary. Such a circuit has a linear combination summing to $\vec{0}$ with coefficients in $\{\pm 1, \pm 2\}$, so that $\vec{x}$ has entries in $\{\pm 1, \pm 2\}$.

Let $\det(B) = 3m$, for some $m \in \mathbb{Z}$. Cramer's Rule tells us that for each entry x_j of $\vec{x}$, we have $x_j = \frac{\det(M_j)}{\det(B)}$, where M_j is the matrix B with the column for t_i replacing the jth column of B. Since $x_j \in \{0, \pm 1, \pm 2\}$, it must be the case that $3 | \det(M_j)$. We observe that since B is a basis, we can write the matrix $[B|t_i]$ in a row-reduced form, so that $[B|t_i] \sim [I|r]$, where I is the identity matrix, and r has non-zero entries from $\{\pm 1, \pm 2\}$ in the coordinates corresponding to the elements of B which form the circuit C with t_i. Then $M_j \sim [i_1, i_2, ..., i_{k-1}, r, i_{k+1}, ..., i_{10}]$, the modified identity matrix into which the vector r has replaced the kth column. We can then pivot along a column with only one nonzero entry, so that the determinant of M_j is always determined by a 2×2 matrix with at most three non-zero entries from $\{\pm 1, \pm 2\}$. Such a matrix can never have non-zero determinant divisible by 3.

This contradiction forces us to conclude that $3 \nmid \det(B)$. $\square$

A result from Lee and Scobee [10] tells us that, because $M(G_3^6)$ is ternary, it is dyadic. That is, it has a matrix representation whose coefficients are rational, such that all non-zero sub-determinants are in $\{2^k,\ k \in \mathbb{Z}\}$.

Chapter 3: Zonotope Properties

In the introduction, we defined a zonotope as the convex polytope obtained by a Minkowski sum of segments in a real vector space. The dimension of the zonotope is the same as the rank of the set of vectors forming the segments. The zonotope obtained from the matrix representation of G_k^n is directly related to the hyperplane arrangement. Their face lattices are inversely isomorphic, so that vertices of the zonotope correspond to the full-dimensional cells, which we call topes, of the hyperplane arrangement. It is worthwhile to describe various features of the zonotope (or arrangement), and to look for generalizable methods of identifying them.

In Definition 1.6.4, we described a simplicial tope of a hyperplane arrangement in $\mathbb{R}^d$ as a full-dimensional cell formed by d-many hyperplanes. That is, it has exactly the same number of facets as the dimension of the space. Such a tope corresponds to a simple vertex of the zonotope, a vertex where a minimal number of facets intersect. The set of columns corresponding to the simplicial tope in the matrix forms an oriented basis: the size of the set must be at least full-rank to make a full-dimensional cell, and the signs of the columns are the signs (positive or negative) that indicate the interior points of the cone with respect to each facet; the set is linearly independent because any acyclic minimal dependence relation can be determined by all but one element (recall from Definitions 1.6.3 and 1.6.2 the relationship between a tope and an acyclic orientation). It is equivalent to think of a simplicial basis as one for which there is a fixed signing of the elements of the basis such that every distinct linear expression in the basis for an element of E is positive (that is, there is a choice of sign for each basis member so that we can write any linear

expression in terms of addition only).

In this chapter, we discuss some methods for finding topes and simplicial topes.

3.1 Simplicial Bases From Special Bases

Let B be a special basis of $M(G_3^6)$. For any element $t_1 \in B$ and $t_0 \in E \setminus B$, the set $B' = (B \setminus t_1) \cup t_0$ is a simplicial basis. We demonstrate how this works with an example, then restate this claim as a theorem and give a proof.

First, we convince ourselves that the special basis itself is *not* simplicial. For example, suppose $B = \{abf, bcf, cdf, acd, ade, abe, bce, ace, acf, def\}$. The element abc is not in B, so it has a linear expression using elements in B. Below, (1) is one of them (the other would be the same except for a complete reversal of signs).

(1) $2(-abc) = (-abf) + (-bcf) + (-cdf) + (+acd) + (+ade) + (-abe) + (-bce) + (+ace) + (+aef) + (-def)$

If B were simplicial, we would be able to use B to write a linear expression for any *other* element not in B without changing the signs on any of these elements, using only addition. The element adf is not in B, so let us check its linear expression in B:

(2) $2(-adf) = (-abf) + (-bcf) + (-cdf) + (+acd) + (-ade) + (+abe) + (+bce) + (-ace) + (-aef) + (+def)$.

Notice that some of the signs (like $-abf$ and $-bcf$) are the same as in (1), and some are not ($+aef$ changed to $-aef$, and $-def$ changed to $+def$). So even an expression for $+adf$ would require us to change some of the basis signs we see in (1). Clearly, B is not simplicial.

However, we can make a single change to B to obtain a basis which is simplicial. Suppose we put abc in the basis, and remove ade. Let $B' = (B \setminus ade) \cup abc$. B' is a basis: the set $B \cup abc$ is a (non-binary) circuit, which has eleven elements, and rank 10. We can remove any element from a circuit and be left with a linearly independent set, which in this case is

a linearly independent set containing ten elements - a basis.

We first rewrite (1):

(1)' $0 = (-abf) + (-bcf) + (-cdf) + (+acd) + (+ade) + (-abe) + (-bce) + (+ace) + (+aef) + (-def) + 2(+abc)$.

This takes our expression for abc in the basis B, and displays it as a minimal dependent set. Observe that the right hand side contains all the elements of B, as well as all the elements of B'. Let's fix the signs on the elements of B' (that is everything on the right hand side except for ade), and see what happens.

There is an expression in B' with these signs for ade:

(3) $(-ade) = (-abf)+(-bcf)+(-cdf)+(+acd)+(-abe)+(-bce)+(+ace)+(+aef)+(-def) + 2(+abc)$.

We got this just by moving $+ade$ to the left hand side. Can we use these signs of B' to express any other elements from $E \setminus B'$? Choose adf: notice that if we take (2) for adf in the basis B, and add it to equation (1)', we will be left with

(4) $2(-adf) = 2[(-abf) + (-bcf) + (-cdf) + (acd)] + 2(+abc)$,

an equation with adf on one side, and elements from B' on the other. The term with ade, an element of B but not of B', canceled, giving us an expression for adf in the basis B'. What's more, the signs of elements in B' are the same in (4) as they were in (1)'. This is not a coincidence. The proof of the theorem describes why this works in general. Any element of $E \setminus B'$ that we picked would have a positive expression using the same signs as in equation (1)'.

Theorem 3.1.1. Let B be a special basis, $t_1 \in B$, and $t_0 \in E \setminus B$. Then $B' = (B \setminus t_1) \cup t_0$ is a simplicial basis.

Proof. Let B, t_1, t_0 and B' be as stated. Consider (5), the equation expressing the fundamental circuit of B associated with the element t_0.

(5) $0 = \Sigma_i \alpha_i t_i + (+2t_0)$

Denote $\sigma(t_1) \in \{+, -\}$ to be the sign of t_1's coefficient in (5). Fix the signs of elements in B' to be those attributed to them in equation (5).

Choose $t_j \in E \setminus B'$. If $t_j = t_1$, then we take the linear expression in (5) and subtract the term containing t_1 from both sides, so that we have a linear expression for t_1 in the basis B'. If $t_j \neq t_1$, then it has a linear expression in the special basis B - in fact it has two, one that is equal to $+2t_j$, and one that is equal to $-2t_j$. Both of these expressions contain t_1 with a non-zero coefficient (because B is a special basis, and so every basis element is in every such linear expression), but in one of these two t_1 is positive, and in the other t_1 is negative. Let (6) give the expression in B for $\pm t_j$ that has a sign for t_1 equal to $-\sigma(t_1)$ (that is, so that t_1 has a sign in (6) *opposite* to its sign in (5)).

(6) $\pm 2t_j = \Sigma_i \beta_i t_i$.

We add equations (5) and (6), see that (as was our intent) the term t_1 cancels, and we have a positive expression for t_j in terms of B' with its fixed signing. Thus, there is a signing of B' such that for every element $t_j \in E \setminus B'$ we can express t_j as a positive linear expression of the elements of B' with the signing fixed. B' is a simplicial basis.

□

3.2 The Lexicographic Method

We describe a method for orienting the triangle elements of $M(G_3^n)$ which results in an acyclic orientation (a tope). We continue to use our usual ordering on vertices, edges and triangles.

To use the *lexicographic method*, we orient triangles by choosing appropriate weights for the edges, so that when the sum of the weights is positive, the orientation of the triangle is positive, and when the sum is negative, the triangle's orientation is negative. Obviously, it is important to choose weights that cannot sum to zero. Figure 3.1 gives two examples.

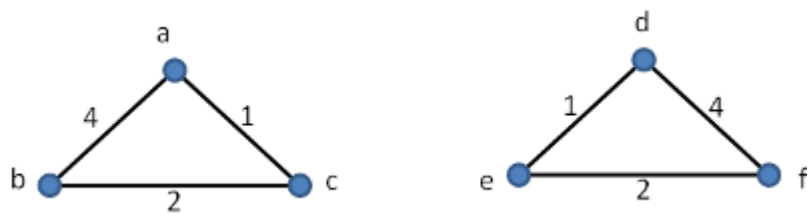

Figure 3.1: Two Different Orientations of G_3^3

In the example on the left, edge ab is weighted 4, bc is 2, and ac is 1. Then the sum $+ab - ac + bc = 4 - 1 + 2 = 5$ is positive. We write $+abc$. The example on the right has $+de - df + ef = 1 - 4 + 2 = -1$, and we write $-def$.

To proceed to orienting a complex of triangles, we need a systematic way to choose weights. Because one edge may be shared among several triangles, we must have a way to ensure we will always be able to obtain an orientation - that is, that no sum of edge weights will ever be zero. We will make use of an infinite sequence $\{a_n\}$ having the property that $a_n > a_{n-1} + a_{n-2}$. One obvious sequence is $a_n = 2^n$. Another, which we use here, is $a_n = a_{n-1} + a_{n-2} + 1$. If we start with $a_n = 1$, our first several terms are $1, 2, 4, 7, 12, 20, 33$. But since the actual values are less important than their specified property, we will often denote them with suggestive placeholder notation (to be defined).

Theorem 3.2.1. The orientation of $M(G_3^6)$ resulting from using the lexicographic method is acyclic.

Proof. Let $\psi : E^\delta \mapsto \{a_i\}_{i=1}^{\binom{n}{2}}$ be an assignment of weights to the edges E^δ of elements of $M(G_3^6)$. Let $A = [\psi(e_1), \psi(e_2), \ldots, \psi(e_{\binom{n}{2}})]$, with $e_i \in E^\delta$, be the row vector of all the assigned edge weights. Then the sum of edge weights for a particular triangle t_i can be seen as the inner product of A with the column vector associated with t_i in the matrix representation for $M(G_3^n)$.

Let $C \subseteq E$ be a set of elements of $M(G_3^6)$ that form a circuit. Then some edge e_j has

$\psi(e_j) = a_{max}$, where a_{max} is the highest weight value among all the edges of elements of C. The value a_{max} must determine the orientation of every element of C that contains edge e_j, by being the dominant value in every sum. If a column vector associated with an element of C has $+1$ in the e_j coordinate, then the inner product will have a positive value, and we give the element a positive orientation. If there is -1 in the e_j coordinate, the inner product will have a negative value, and the associated element is given a negative orientation. This means that every non-zero value for coordinate e_j in the newly oriented columns will be positive. No linear combination of these columns can result in the zero vector unless we have at least one negative coefficient, so that the values in the e_j coordinate can sum to zero.

This proves that orientations of elements given by the lexicographic method are acyclic.

□

We have shown that we can obtain an acyclic orientation for all triangles by arbitrarily assigning values from the sequence to all edges in a triangle complex. However, we proceed in an even more methodical way. In what we call a *strict* lexicographic method, we orient all the elements of $M(G_3^m)$ for each $m \leq n$, beginning with $m = 3$, building the complex of triangles up one vertex at a time.

Giving edge ab weight 1, the first term of our sequence, we then give edges bc and ac weights corresponding to the next two terms of the sequence. Which edge gets which weight will determine if abc is positively or negatively oriented. In adding a fourth vertex, d, we attach it with three new edges, and weight them with the next three values in the sequence in some order, and continue in this way with the next vertex.

We can record these weight assignments efficiently in a matrix, with rows and columns labeled by the vertices. Each entry x_{ij} denotes the weight of the edge with endpoints i and j, where $i < j$. Since there are no loops in our complex, and since the weight of x_{ij} is

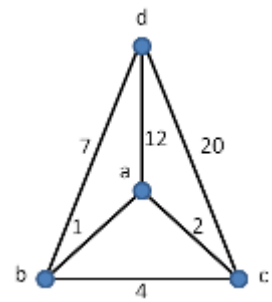

Figure 3.2: $M(G_3^4)$ With Weighted Edges

$$\left(\begin{array}{c|cccc} & a & b & c & d \\ \hline a| & 0 & 1 & 2 & 12 \\ b| & - & 0 & 4 & 7 \\ c| & - & - & 0 & 20 \\ d| & - & - & - & 0 \end{array}\right).$$

Figure 3.3: A Lexicographic Matrix for $M(G_3^4)$

equal to -1 times the weight of x_{ji}, we need only consider the part of the matrix above the diagonal.

Figure 3.2 gives an example of $M(G_3^4)$ with weighted edges, and figure 3.3 gives its corresponding strict lexicographic matrix.

From the diagram or from the matrix we can derive the orientations of the four triangles by observing inequalities in the weights, such as ac being weighted less than bc, or, with an abuse of notation, $ac < bc$. This tells us that abc is oriented positively, so we write $+abc$. The final column gives us inequalities like $bd < ad$, so that we get $-abd$; $ad < cd$ so we write $+acd$; and $bd < cd$, so we write $+bcd$.

This is easily seen to be an acyclic orientation of $M(G_3^4)$. In $M(G_3^4)$ there is only one circuit, involving all four triangles (a tetrahedron). Of all possible orientations of the four triangles, only two are cyclic, or what we call "good" orientations of the circuit (where we can obtain zero on the boundary without changing any signs). The two good signings of

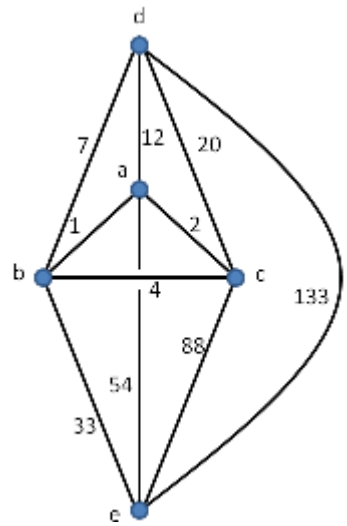

Figure 3.4: A Lexicographic Orientation of $M(G_3^5)$

$$\left(\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a| & 0 & 1 & 2 & 12 & 54 \\ b| & - & 0 & 4 & 7 & 33 \\ c| & - & - & 0 & 20 & 88 \\ d| & - & - & - & 0 & 133 \\ e| & - & - & - & - & 0 \end{array}\right)$$

Figure 3.5: A Lexicographic Matrix for $M(G_3^5)$

the above tetrahedron are $+abc, -abd, +acd, -bcd$, and its reverse, $-abc, +abd, -acd, +bcd$. Our construction gave us neither of these.

Let us continue our example by extending the complex with one more vertex.

Observe that the final column of the matrix in Figure 3.5 gives us relations for orienting the new triangles containing the vertex e. We get $ae < be$, so triangle abe is negative; $ae < ce$, so we have $+ace$; $ac < de$ so we have $+ade$; $be < ce$ and $be < de$, so we get $+bce$ and $+bde$; $cd < de$ so we have $+cde$.

The strict lexicographic method gives us acyclic orientations of all the elements of $M(G_3^n)$, although not every such acyclic orientation, or tope, of these matroids arises this

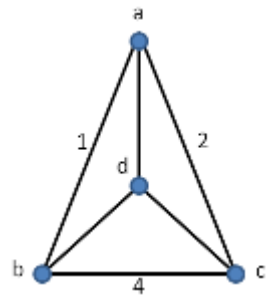

Figure 3.6: A Non-Lexicographic Orientation of $M(G_3^4)$

way. In fact, here is a very small example of a tope which is not obtainable using the strict lexicographic method: $\{+abc, +abd, -acd, +bcd\}$. Figure 3.6 illustrates the problem.

In order to weight ad, it must be greater than cd to obtain $-acd$. Then to get $+bcd$, we must have $bd < cd$, and $+abd$ implies that $ad < bd$. So we require the (impossible) inequality $ad < bd < cd < ad$.

We noted earlier that, while it was important to have a sequence of weights with the property that $a_n > a_{n-1} + a_{n-2}$, the actual values in the sequence are insignificant. In subsequent discussions involving the strict lexicographic method, of primary importance (besides the above property) is that in the matrix, all values in a column are strictly ordered, and all are larger than every value in the previous columns. We then improve readability with the following notation for the weights in a matrix.

For nth vertex v, the weights associated with v's column in the matrix will be $v_1, v_2, ..., v_{n-1}$, where $v_1 < v_2 < \cdots < v_{n-1}$. So, for example, the previous 5-vertex matrix looks like Figure 3.7 in our new notation.

3.3 Chirotopes

It will be useful to define a new perspective of the triangles of our complex on the complete graph. Going back to our original view of triangles as 3-sets on a ground set of vertices

$$\left(\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a| & 0 & b_1 & c_2 & d_2 & e_2 \\ b| & - & 0 & c_1 & d_1 & e_1 \\ c| & - & - & 0 & d_3 & e_3 \\ d| & - & - & - & 0 & e_4 \\ e| & - & - & - & - & 0 \end{array}\right)$$

Figure 3.7: Relative Weights in the Lexicographic Matrix

$\{v_1, ..., v_n\}$, we define a function $\chi : (v_i v_j v_k) \mapsto \{+, -\}$ that sends each 3-set to one of two signs. This corresponds nicely with our notion of orienting the triangles of $M(G_3^n)$.

If χ satisfies the two properties below, then we actually have defined a new oriented matroid, which we denote $G_{3,n}$, on the ground set of vertices. Each 3-set (triangle) is a basis for this matroid, so that $G_{3,n}$ is a uniform matroid, and we call χ a *chirotope*.

Definition 3.3.1 (Chirotope)**.** The function $\chi : (v_i v_j v_k) \in \{+, -\}$ is a *chirotope* if it satisfies the following two properties.

Property 1: χ must be alternating. That is, $\chi(v_{\pi(1)} v_{\pi(2)} v_{\pi(3)}) = sign(\pi)\chi(v_1 v_2 v_3)$ for any permutation π of the vertices of a triangle. This is particularly easy to see for triangles: if $\chi(abc) = +$, then $\chi(acb) = \chi(-abc) = -$, for example. Such changes in orientation are already natural to us under our consistent use of lexicographic orderings. In fact, any orientation of the triangles automatically satisfies this condition.

Property 2: χ must satisfy the Grassmann-Plücker relation (a reference for this definition is [1]). For any five of the $n \geq 5$ vertices $v_1, ..., v_5$, we must be able to find positive scalars α, β, γ so that the following equality holds:

$$\alpha\chi(v_1 v_2 v_3)\chi(v_1 v_4 v_5) - \beta\chi(v_1 v_2 v_4)\chi(v_1 v_3 v_5) + \gamma\chi(v_1 v_2 v_5)\chi(v_1 v_3 v_4) = 0.$$

For readability, we denote $\chi(v_i v_j v_k)$ with brackets $[v_i v_j v_k]$.

$$\left(\begin{array}{c|ccc} & a & b & c \\ \hline a| & 0 & b_1 & c_2 \\ b| & - & 0 & c_1 \\ c| & - & - & 0 \end{array}\right)$$

Figure 3.8: $M(G_3^3)$

With this new perspective of the triangle orientations of $M(G_3^n)$, we can ask questions like "when are chirotopes also topes of $M(G_3^n)$?" and "can a simplicial tope of $M(G_3^n)$ represent a chirotope?"

Theorem 3.3.2. A simplicial tope of $M(G_3^n)$, where $n \geq 5$, built with the strict lexicographic method is never a chirotope.

Proof. We start as indicated by our discussion of the lexicographical method with triangle abc. At this stage, the orientation of the first triangle is arbitrary: a simplicial tope in which abc is positive implies the existence of a simplicial tope in which abc is negative through a complete reversal of all other signs.

So without loss of generality we write $+abc$, or rather, we weight the edges according to the matrix (see Figure 3.8).

In extending to a fourth vertex we must recall what makes a tope simplicial. An acyclic tope for $M(G_3^4)$ that is simplicial has a basis all of whose elements correspond to a "good" signing (one for which the boundary elements shared among all the basis elements cancel each other) of the one circuit in the matroid - the tetrahedron containing all four elements. The fourth element, not in the basis, is the only element whose sign makes the circuit acyclic. Additionally, any simplicial tope we use must now have $+abc$, based on our choice of starting orientation.

Suppose we choose the simplicial tope $\{+abc, -abd, +acd, +bcd\}$. The simplicial basis

$$\left(\begin{array}{c|cccc} & a & b & c & d \\ \hline a| & 0 & b_1 & c_1 & d_2 \\ b| & - & 0 & c_2 & d_1 \\ c| & - & - & 0 & d_3 \\ d| & - & - & - & 0 \end{array}\right)$$

Figure 3.9: Simplicial Orientation for $M(G_3^4)$

for this tope is $\{+abc, -abd, +acd\}$. We continue our proof using this example, and show at the end why we would obtain an analogous result regardless of which simplicial tope of $M(G_3^4)$ we had chosen.

We extend our matrix to include the fourth vertex. Knowing what signs the elements must have tells us how to order the fifth column (refer to Figure 3.9).

This matrix tells us, for example, that the weights on the edges of the vertex e have the inequality $cd > ad > bd$, so that bcd, the fourth triangle (not in the basis) must be positively oriented.

We must now see if we can extend our simplicial tope in $M(G_3^4)$ to a simplicial tope in $M(G_3^5)$ while satisfying the Grassmann-Plücker relation. So let us label a fifth vertex e, and let our vertices a through e correspond directly to the numbers 1 through 5 as given in the Grassmann-Plücker relation. Then we must sign the elements to get $[abc][ade]-[abd][ace]+[abe][acd]=0$. Three signs were fixed in $M(G_3^4)$, and we fill those in: $[+][ade]-[-][ace]+[abe][+]=0$. We need to give signs $+$ or $-$ to ade, ace, and abe so that it is possible for the equation to hold. The table in Figure 3.10 gives the acceptable signs for each of the three terms on the left hand side of the equality, and the signs that are necessarily implied for the remaining three triangles. In the Term Sequence column, the first term is the sign of $[+][ade]$, the second term is the sign of $-[-][ace]$, and the third term is the sign of $[abe][+]$. This column lists all the ways that 0 can be obtained with an appropriate choice of scalars.

$TermSequence$	$[ade]$	$[ace]$	$[abe]$
$+-+$	$+$	$-$	$+$
$-+-$	$-$	$+$	$-$
$++-$	$+$	$+$	$-$
$+--$	$+$	$-$	$-$
$--+$	$-$	$-$	$+$
$-++$	$-$	$+$	$+$

Figure 3.10: Signings Satisfying Grassmann-Plücker Relations on $M(G_3^5)$

$$\left(\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a| & 0 & b_1 & c_2 & d_2 & e_2 \\ b| & - & 0 & c_1 & d_1 & e_4 \\ c| & - & - & 0 & d_3 & e_1 \\ d| & - & - & - & 0 & e_3 \\ e| & - & - & - & - & 0 \end{array}\right) \; OR \; \left(\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a| & 0 & b_1 & c_2 & d_2 & e_2 \\ b| & - & 0 & c_1 & d_1 & e_3 \\ c| & - & - & 0 & d_3 & e_1 \\ d| & - & - & - & 0 & e_4 \\ e| & - & - & - & - & 0 \end{array}\right)$$

Figure 3.11: Lexicographic Matrices for Two Chirotopes

As we can see, we must have one positive and two negative, or two positive and one negative among the orientations of the three indicated elements on the fifth vertex in order to satisfy the Grassmann-Plücker relation. The signs of ade, ace, and abe indicate the inequalities we use to determine the weights of the edges on the fifth vertex. For example, if the term sequence is $+-+$, this forces the three remaining triangles to be signed as $+ade, -ace, +abe$. When ade is positive, this means that the weight on edge de is greater than the weight on edge ae, or, for short, $de > ae$. From the set of inequalities we obtain this way, we find that either $be > de > ae > ce$ or $de > be > ae > ce$. Then the term sequence $+-+$ corresponds to one of the two matrices in Figure 3.11, where the fifth columns satisfy the inequalities.

Unfortunately, the following Lemma shows why none of the signings given in the table allow for a tope to be simplicial.

Lemma 3.3.3. To extend a simplicial tope in $M(G_3^4)$ to one in $M(G_3^5)$ using the strict

lexicographic method, all three new triangles on the first and fifth vertex (sharing edge ae) must have the same orientation.

Proof. Suppose two of the triangles, t_1 and t_2, sharing edge ae have different signs, with t_3 having the same sign as t_1. Then edge ae is differently directed under the boundary map. We require that a simplicial tope contain a simplicial basis - a basis that can give positive linear expressions for every element not in the basis. This means that no element of the basis can "change sign" between two different fundamental circuits. So triangle t_2 cannot have a positive linear expression containing either t_1 or t_3. This means that t_2 must be in a simplicial basis. Likewise, neither t_1 nor t_3 can have a positive linear expression containing t_2, so one of these must be in a simplicial basis also.

Now we observe that there are five triangles in our basis for $M(G_3^5)$ that contain the same vertex a. We demonstrate why this is not simplicial. Suppose we have some assignment of the vertices a through e to the numbers 1 through 5, and have a basis for which some vertex appears five times. Up to a reassignment of the vertices, we can show what happens by choosing the first five elements of the basis to be $\{124, 124, 125, 134, 135\}$. From the remaining elements, we have three choices to extend this set to a basis (two of the elements give us dependent sets). Our options are to complete the basis with 145, 245 or 345.

Suppose we put 145 in the basis. Now we attempt to create positive linear expressions for the remaining four elements. Our first choice of signs is arbitrary, but the rest necessarily follow. Since $\{123, 124, 134, 234\}$ is an unoriented fundamental circuit, we write the positive linear expression:

+123 -124 +134 = +234.

There is another fundamental circuit containing the basis element 123, so we must write:

+123 -125 +135 = +235.

There is another fundamental circuit containing basis elements 124 and 125, but we

have a problem:

-124 - (-125) -145 = -245 is not a positive linear expression for 245. One of 124 or 125 must change sign. Such a change then obviously disagrees with one of the previous fundamental circuits.

Suppose instead we include 245 in the basis, rather than 145. The first two positive linear expressions are the same, but then we need a positive linear expression for 145:

-124 - (-125) -245 = -145 is not a positive linear expression for 145. Again, one or 124 or 125 must change sign. An analogous result occurs when we use 345 for our sixth basis element.

We conclude that any basis for which some vertex occurs five times cannot be simplicial.

□

Because all of the signings which satisfy the Grassmann-Plücker relation have this property, we find that for our example to be simplicial, it cannot satisfy the Grassmann-Plücker condition, and so cannot be a chirotope.

Now recall that we began with an arbitrary example of a simplicial tope for $M(G_3^4)$, and showed that with the basis of this tope having signs $+ - +$ (recall the first basis was $\{+abc, -abd, +acd\}$), and where vertices a, b, c, d, e mapped directly to $1, 2, 3, 4, 5$, we couldn't extend to a simplicial basis for $M(G_3^5)$ while simultaneously satisfying the Grassmann-Plücker condition.

If we had chosen a different simplicial tope, we can show similarly that we have the same problem satsifying the condition. Assuming all our results could be obtained using full sign reversals, we leave those trivial cases aside.

Suppose we chose the simplicial tope $\{+abc, +abd, +acd, -bcd\}$. This has simplicial basis $\{+abc, +acd, -bcd\}$. We would like to know what happens with the Grassmann-Plücker condition, preferably without having to recompute the whole sign table. This is as

$TermSequence$	$[cde]$	$[cbe]$	$[cae]$
$+-+$	$+$	$-$	$+$
$-+-$	$-$	$+$	$-$
$++-$	$+$	$+$	$-$
$+--$	$+$	$-$	$-$
$--+$	$-$	$-$	$+$
$-++$	$-$	$+$	$+$

Figure 3.12: Signings Satisfying the Grassmann-Plücker Relation on a Permutation of the Vertices

easy as changing our mapping of the vertices to the numbers. Since the condition to be satisfied is a condition on six triangles containing a vertex in common (for the first example this was vertex a), we can check the condition on the vertex in common for our new choice of basis, which is c. If we map $c \mapsto 1, a \mapsto 2, b \mapsto 3, d \mapsto 4, e \mapsto 5$, then our basis maps this way: $+abc \mapsto +cab, +acd \mapsto -cad, -bcd \mapsto +cbd$, and the basis now has the same three ordered signs as in our first example. The three new triangles on c are cae, cbe and cde. The Grassmann-Plücker condition now looks like this: $[+][cde] - [-][cbe] + [cae][+] = 0$, giving us the same sign table, but with new column labels.

This fails to be simplicial for the same reason as our previous example. In other words, the proof for our first example holds for any permutation of the vertices, therefore for any simplicial tope of $M(G_3^4)$. $\square$

Bibliography

[1] A.Björner, M.LasVergnas, B.Sturmfels, N.White, G.Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and Its Applications, 46, Cambridge University Press, Cambridge, UK, 1993.

[2] Raul Cordovil, *On Reid's 3-Simplicial Matroid Theorem*, Combinatorica, 2, pp.135-141, 1982.

[3] Raul Cordovil and Bernt Lindström, *Simplicial Matroids*, Encyclopedia of Mathematics, v.29, Cambridge University Press, Cambridge, UK, 1987.

[4] H.Coxeter, *Regular Polytopes*, Dover Publications, Inc., New York, NY, 1973.

[5] Henry H. Crapo and Gian-Carlo Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, M.I.T.Press, Cambridge, MA, 1970.

[6] Branko Grünbaum, *Convex Polytopes*, Springer-Verlag, NY, 2003.

[7] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, UK, 2001.

[8] Gil Kalai, *Enumeration of Q-Acyclic Simplicial Complexes*, Israel Journal of Mathematics, volume 45, no.4, pp.337-351, 1983.

[9] Caroline Klivans, Art M. Duval, Jeremy L. Martin, *Simplicial Matrix-Tree Theorems*, Transactions of the American Mathematical Society, 361 (2009), no. 11, 6073-6114.

[10] Jon Lee and Matt Scobee, *A Characterization of the Orientations of Ternary Matroids*, Journal of Combinatorial Theory, Series B, Volume 77, Issue 2, November 1999, pp. 263-291.

[11] Bernt Lindström, *Non-Regular Simplicial Matroids*, Discrete Mathematics, 28, pp.101-102, 1979.

[12] James G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, UK, 1992.

[13] Michael J. Todd, *Characterizing Binary Simplicial Matroids*, Discrete Mathematics, 16, 1976.

[14] Jiri Tuma, *Note on Binary Simplicial Matroids*, Discrete Mathematics, 49, pp.105-107, 1983.

[15] Roger Webster, *Convexity*, Oxford University Press, Oxford, UK, 1994.

Curriculum Vitae

The author lives with her family in Fairfax County, Virginia.