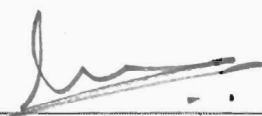


MULTI-FIDELITY STOCHASTIC COLLOCATION METHODS
USING MODEL REDUCTION TECHNIQUES

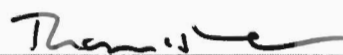
by

Maziar Raissi
A Dissertation
Submitted to the
Graduate Faculty
of
George Mason University
In Partial fulfillment of
The Requirements for the Degree
of
Doctor of Philosophy
Mathematics

Committee:



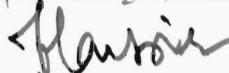
Dr. Padmanabhan Seshaiyer,
Dissertation Director



Dr. Thomas Wanner,
Committee Member



Dr. David Walnut,
Committee Member



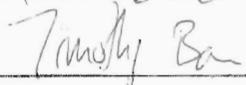
Dr. Harbir Antil,
Committee Member



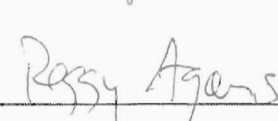
Dr. Andrei Draganescu,
Committee Member



Dr. David Walnut,
Department Chair



Dr. Timothy L. Born, Associate
Dean for Student and Academic Affairs,
College of Science



Dr. Peggy Agouris,
Interim Dean, College of Science

Date: 07/24/2013

Summer Semester 2013
George Mason University
Fairfax, VA

Multi-fidelity Stochastic Collocation

A dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy at George Mason University

By

Maziar Raissi
Master of Science
Isfahan University of Technology, 2011
Bachelor of Science
University of Isfahan, 2008

Director: Dr. Padmanabhan Seshaiyer, Professor
Department of Mathematical Sciences

Summer Semester 2013
George Mason University
Fairfax, VA

Copyright © 2013 by Maziar Raissi
All Rights Reserved

Dedication

I dedicate this to my loving mother Parvaneh, my patient father Sirous, and my supportive brother Mehdi.

Acknowledgments

I would like to thank the many friends and supporters who have made this happen. First and foremost, I thank my advisor, Dr. Padmanabhan Seshaiyer, for his guidance and fatherly supports. My committee members, Drs. Thomas Wanner, David Walnut, Harbir Antil, and Andrei Draganescu, were of invaluable help. Finally, thanks go out to my fellow graduate students, specially Matthew Gerhart for his cute coding tips.

Table of Contents

	Page
List of Tables	vii
List of Figures	viii
Abstract	ix
Introduction	1
1 Multi-fidelity Stochastic Collocation applied to Stochastic Burgers Equation . .	5
1.1 Stochastic Burgers - A tractable while serious test	5
1.2 GFE as a high-fidelity deterministic algorithm	7
1.3 Group POD as a low-fidelity deterministic algorithm	8
1.4 Multi-fidelity Stochastic Collocation	12
1.4.1 Generalization to Sparse Grids	13
1.5 Numerical results	13
2 Multi-fidelity Stochastic Collocation applied to Parabolic PDEs	18
2.1 Problem definition	18
2.2 Multi-fidelity Collocation method	21
2.2.1 Proper Orthogonal Decomposition	22
2.3 Error analysis	30
2.3.1 Analyticity with respect to random parameters	32
2.3.2 Convergence analysis	34
2.4 Numerical experiments	39
3 Stochastic Burgers revisited	43
3.1 Group POD as a low-fidelity deterministic algorithm	44
3.2 Local improvements to POD basis functions	45
3.2.1 Sensitivity analysis of the Burgers equation	47
3.2.2 Improving POD bases	47
3.3 Multi-fidelity Stochastic Collocation	48
3.3.1 Generalization to Sparse Grids	50
3.4 Numerical experiments	50
Conclusions	57
References	58

List of Tables

Table		Page
1.1	Comparison of the errors of the two methods.	15
1.2	Comparison of the execution times of the two methods.	16
1.3	Execution time and convergence pattern of the proposed method as the sparse grid level changes from 7 to 9.	16
2.1	Relative errors when $\eta = 0.1$	42
2.2	Relative errors and the number of times that the finite element code is em- ployed for different values of η	42
3.1	Comparison of the errors of the two methods.	51
3.2	Relative errors and the number of times that the finite element code is em- ployed for different values of η	56

List of Figures

Figure		Page
1.1	Expectation and Second Moment of the solution given at the final time $T = 1.0$ when $d = 2$ for sparse grid level 6.	14
1.2	Standard Deviation of the solution given at final time $T = 1.0$ when $d = 2$ for sparse grid level 6.	15
1.3	Convergence pattern as the sparse grid level changes from 7 to 9.	17
2.1	Comparison of expected values (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right). . .	40
2.2	Comparison of variances of solutions (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right). . .	40
2.3	Convergence pattern of expected values of solutions with respect to η	41
2.4	Convergence pattern of variances of solutions with respect to η	41
2.5	Number of times that the finite element code is employed as a function of η	41
3.1	Standard Deviation of the solution given at final time $T = 1.0$ when $d = 2$ for sparse grid level 6.	51
3.2	Comparison of the effects of extrapolated and expanded local improvements to POD basis functions	52
3.3	Expectation and Second Moment of the solution given at the final time $T = 0.8$ when $d = 3$, for sparse grid level 8 and $\eta = 0.5$	54
3.4	$L_2([0, 1])$ -norm convergence pattern of Expectation and Second Moment of the solution given at the final time $T = 0.8$ when $d = 3$, for sparse grid level 8.	55
3.5	Number of times that the high fidelity algorithm (GFE method) is called as a function of η	56

Abstract

MULTI-FIDELITY STOCHASTIC COLLOCATION

Maziar Raissi, PhD

George Mason University, 2013

Dissertation Director: Dr. Padmanabhan Seshaiyer

Over the last few years there have been dramatic advances in our understanding of mathematical and computational models of complex systems in the presence of uncertainty. This has led to a growth in the area of uncertainty quantification as well as the need to develop efficient, scalable, stable and convergent computational methods for solving differential equations with random inputs. Stochastic Galerkin methods based on polynomial chaos expansions have shown superiority to other non-sampling and many sampling techniques. However, for complicated governing equations numerical implementations of stochastic Galerkin methods can become non-trivial. On the other hand, Monte Carlo and other traditional sampling methods, are straightforward to implement. However, they do not offer as fast convergence rates as stochastic Galerkin. Other numerical approaches are the stochastic collocation (SC) methods, which inherit both, the ease of implementation of Monte Carlo and the robustness of stochastic Galerkin to a great deal. However, stochastic collocation and its powerful extensions, e.g. sparse grid stochastic collocation, can simply fail to handle more levels of complication. The seemingly innocent Burgers equation driven by Brownian motion is such an example. In this work we propose a novel enhancement to stochastic collocation methods using deterministic model reduction techniques that can handle this pathological example and hopefully other more complicated equations like

Stochastic Navier Stokes. Our numerical results show the efficiency of the proposed technique. We also perform a mathematically rigorous study of linear parabolic partial differential equations with random forcing terms. Justified by the truncated Karhunen-Loève expansions, the input data are assumed to be represented by a finite number of random variables. A rigorous convergence analysis of our method applied to parabolic partial differential equations with random forcing terms, supported by numerical results, shows that the proposed technique is not only reliable and robust but also very efficient.

Introduction

The effectiveness of stochastic partial differential equations (SPDEs) in modelling complicated phenomena is a well-known fact. One can name wave propagation [Pap71], diffusion through heterogeneous random media [Pap95], randomly forced Burgers and NavierStokes equations (see e.g [BT73, DPD03, KMS⁺97, MR04] and the references therein) as a couple of examples. Currently, Monte Carlo is by far the most widely used tool in simulating models driven by SPDEs. However, Monte Carlo simulations are generally very expensive. To meet this concern, methods based on the Fourier analysis with respect to the Gaussian (rather than Lebesgue) measure, have been investigated in recent decades. More specifically, Cameron–Martin version of the Wiener Chaos expansion (see, e.g. [CM47, HKPS93] and the references therein) is among the earlier efforts. Sometimes, the Wiener Chaos expansion (WCE for short) is also referred to as the Hermite polynomial chaos expansion. The term polynomial chaos was coined by Nobert Wiener [Wie38]. In Wiener's work, Hermite polynomials served as an orthogonal basis. The validity of the approach was then proved in [CM47]. There is a long history of using WCE as well as other polynomial chaos expansions in problems in physics and engineering. See, e.g. [CC70, OB67, Cho71, Cho74], etc. Applications of the polynomial chaos to stochastic PDEs considered in the literature typically deal with stochastic input generated by a finite number of random variables (see, e.g. [SG02, GS03, XK03, ZL04]). This assumption is usually introduced either directly or via a representation of the stochastic input by a truncated Karhunen–Loève expansion. Stochastic finite element methods based on the Karhunen–Loève expansion and Hermite polynomial chaos expansion [GS03, SG02] have been developed by Ghanem and other authors. Karniadakis et al. generalized this idea to other types of randomness and

polynomials [JSK02, XK03, XK02]. The stochastic finite element procedure often results in a set of coupled deterministic equations which requires additional effort to be solved. To resolve this issue, stochastic collocation (SC) method was introduced. In this method one repeatedly executes an established deterministic code on a prescribed node in the random space defined by the random inputs. The idea can be found in early works such as [MH03, TPPM97]. In these works mostly tensor products of one-dimensional nodes (e.g., Gauss quadrature) are employed. Tensor product construction despite making mathematical analysis more accessible (cf. [BNT07]) leads to the curse of dimensionality since the total number of nodes grows exponentially fast as the number of random parameters increases. In recent years we are experiencing a surge of interest in the high-order stochastic collocation approach following [XH05]. The use of sparse grids from multivariate interpolation analysis, is a distinct feature of the work in [XH05]. A sparse grid, being a subset of the full tensor grid, can retain many of the accuracy properties of the tensor grid. While keeping high-order accuracy, it can significantly reduce the number of nodes in higher random dimensions. Further reduction in the number of nodes was pursued in [AA09, MZ09, NTW08b, NTW08a]. Applications of these numerical methods take a wide range. Here we mention some of the more representative works. It includes Burgers equation [HLRZ06, XK04], fluid dynamics [KLM06, KNG⁺01, LMRN⁺02, LWSK07, XK03], flow-structure interactions [XLSK02], hyperbolic problems [CGH05, GX08, LSK06], model construction and reduction [DGRH07, GMPW05, GD06], random domains with rough boundaries [CK07, LSK07, TX06, XT06], etc.

Along with an attempt to reduce the number of nodes used by sparse grid stochastic collocation, one can try to employ more efficient deterministic algorithms. The current trend is to repeatedly execute a full-scale underlying deterministic simulation on prescribed nodes in the random space. However, model reduction techniques can be employed to create a computationally cheap deterministic algorithm that can be used for most of the grid points. This way one can limit the employment of an established while computationally expensive algorithm to only a relatively small number of points. A related method is being used by K.

Willcox and her team but in the context of optimization [REWH08]. “Multifidelity”, which we also adopt, is the term they employed in their work. For a tractable demonstration of our method we consider linear parabolic partial differential equations with random forcing terms. We show that our method dramatically decreases the computational cost of sparse grid stochastic collocation methods. The idea of the method is very simple. For each point in the stochastic parameter domain we search to see if the resulting deterministic problem is already solved using a high fidelity (accurate but computationally expensive) algorithm, e.g. Backward Euler Finite Elements method, for a sufficiently close problem. If yes, we use the solution to the nearby problem to create POD (Proper Orthogonal Decomposition) basis functions. These are highly relevant global basis functions. We then employ POD-Galerkin method, a low fidelity algorithm (less accurate but computationally cheap), to solve the original problem. We provide a rigorous convergence analysis for our proposed method applied to linear parabolic partial differential equations. It is shown by numerical examples that the results of numerical computation are consistent with theoretical conclusions.

We also consider the Stochastic Burgers equation studied in [HLRZ06] to test our method against a serious while still tractable problem. We believe that the non-linear nature of the Burgers equation with the extra complexity of Brownian motion, makes this equation a proper test for our method. The regular repetitive execution of the full-scale underlying high-fidelity deterministic algorithm for the sparse grid stochastic collocation simply fails for this equation. This equation is also studied in [HLRZ06] using Wiener Chaos expansion. For the deterministic Burgers equation resulting from stochastic collocation samples, we are going to use a high fidelity algorithm called “group finite element” (GFE) and a low fidelity algorithm called “group proper orthogonal decomposition” (Group POD). These two methods are discussed in [DS10]. The GFE method, also known as product approximation, is a finite element (FE) technique for certain types of non-linear partial differential equations. It expresses the non-linear terms of a PDE in a grouped form. As a result, spatial discretization of non-linear terms is computed only once before integration. Therefore, a substantial reduction in computational cost is achieved [CGMSS81, Fle83, Roa98]. Experiments with

the GFE method have indicated an increase in economy and a slight increase in the nodal accuracy compared to FE solutions of the unsteady Burgers equations and many other problems [CGMSS81, Fle83, SF92]. Although theoretical results exist for other problems [CGMSS81, SSA84, CLZ89, DD75, LTZW89, Mur86, TOU90], we are unaware of convergence theory for the GFE applied to Burgers equation. The computational advantage of the GFE method over the conventional FE method for two and three dimensional Burgers equations and viscous compressible flows is demonstrated in [Fle83, SF92]. As for the group POD method, we use the projection of grouped non-linear terms onto a set of global basis functions. This projection onto global basis functions further reduces the cost of simulation due to symmetry in the non-linear terms. Reduced order modelling, using proper orthogonal decompositions (POD) along with Galerkin projection, for fluid flows has seen extensive applications studied in [Sir87, CAM⁺88, HLB98, Fah01, ILD00, KV01, KV02, HY02, RCM04, Cam05]. Proper orthogonal decomposition (POD) was introduced in Pearson [Pea01] and Hotelling [Hot33]. Since the work of Pearson and Hotelling, many have studied or used POD in a range of fields such as oceanography [BV97], fluid mechanics [Sir87, HLB98], system feedback control [Rav00, ABK01, AK01, KBTB02, AK04, LT05], and system modeling [Fah01, KV02, RCM04, HY05].

In an attempt to increase the accuracy of our method, we employ local improvement techniques for reduced-order models using sensitivity analysis of the proper orthogonal decomposition (see [HBP09, BB97]). A serious limitation of proper orthogonal decomposition basis is that while it accurately represents the flow data used to generate it, it may not be as accurate when applied off-design. To mitigate this issue, one should update the basis before applying it to solve a nearby problem.

Chapter 1: Multi-fidelity Stochastic Collocation Method applied to Stochastic Burgers Equation

Our objective in this work is to demonstrate how sparse grid stochastic collocation method can be enhanced using deterministic model reduction techniques. To simplify the presentation while employing a serious test, we study the stochastic Burgers equation. It is a well known fact that the deterministic Burger's equation is not a good model of turbulence. It does not display chaotic behaviour; all solutions converge to a unique stationary solution as time goes to infinity, even in the presence of a forcing term in the right hand side. However, several authors (see e.g. [CAM⁺88,CTMK93,Jen69,HY75,KPZ86]) have suggested to use the Stochastic Burgers equation with random forcing term as a simple model for turbulence and to study the dynamics of interferences.

1.1 Stochastic Burgers - A tractable while serious test

The Stochastic Burgers equation studied in this work is given by,

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + \sigma(x)\dot{W}(t), \quad (1.1)$$

$(t, x) \in (0, T] \times [0, 1]$, $u(0, x) = u_0(x)$, $u(t, 0) = u(t, 1) = 0$, where $W(t)$ is a Brownian motion and $u_0 \in L_2([0, 1])$ is a deterministic initial condition. If $\|u_0\|_{L_2} < \infty$ and $\|\sigma\|_{L_2} < \infty$, it is known (see, e.g [DPDT94]) that (1.1) has a unique square integrable solution. If u is the solution of equation (1.1), then u is not only a function of t and x , but it is also a function of the Brownian motion path $W_0^t = \{W(s), 0 \leq s \leq t\}$. In order to approximate the Brownian motion with a finite number of independent and identically distributed (iid)

standard normal random variables, we let $\{h_k, k = 1, 2, \dots\}$ be an arbitrary orthonormal basis in $L_2([0, t])$ and define $\xi_k := \int_0^t h_k(s) dW(s)$, for $k \in \{1, 2, \dots\}$. It can be shown that ξ_k are iid Gaussian random variables. It is a standard fact that we can expand $W(s)$ as

$$W(s) = \int_0^t \chi_{[0,s]}(\tau) dW(\tau) = \sum_{k=1}^{\infty} \xi_k \int_0^s h_k(\tau) d\tau, \quad (1.2)$$

where $\chi_{[0,s]}(\tau)$ is the characteristic function of the interval $[0, s]$. Note that

$$\chi_{[0,s]}(\tau) = \sum_{k=1}^{\infty} c_k h_k(\tau),$$

where $c_k = \int_0^t \chi_{[0,s]}(\tau) h_k(\tau) d\tau = \int_0^s h_k(\tau) d\tau$. If $\{h_k, k = 1, 2, \dots\}$ are chosen as Haar wavelets, then expansion (1.2) is exactly the Levy-Ciesielski construction [McK69] of Brownian motion. Therefore we can view a solution u of (1.1) as a function of x, t and $\xi = (\xi_1, \xi_2, \dots)$. We also know that expansion (1.2) converges in the mean square sense (i.e., $E[W(s) - \sum_{k=1}^d \xi_k \int_0^s h_k(\tau) d\tau]^2$ goes to zero as $d \rightarrow \infty$ uniformly for $s \leq t$). Our first step is to truncate expansion (1.2) at some point d to get

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + \sigma(x) \sum_{k=1}^d \xi_k h_k(t). \quad (1.3)$$

In the following, we are seeking an approximation to a random field

$$u_d(\hat{\xi}; t, x) \in C^\infty(\mathbb{R}^d; L_2([0, T]; W([0, 1])))$$

which satisfies equation (1.3) along with the initial and boundary conditions of problem (1.1), where $\hat{\xi} = (\xi_1, \xi_2, \dots, \xi_d)$ and $W([0, 1])$ is a Banach space of functions $v : [0, 1] \rightarrow \mathbb{R}$

to be later specified. Let us assume that the random vector $\hat{\xi}$ has $\rho(\hat{\xi})$ (which is simply the multiplication of probability densities of d standard normal variables) as its joint probability density. We are also interested in approximating expectation and higher moments $E[u_d^k] \in L_2([0, T]; W([0, 1]))$, $k = 1, 2, \dots$ of the solution which are given by

$$E[u_d^k](t, x) = \int_{\mathbb{R}^d} u_d^k(\hat{\xi}; t, x) \rho(\hat{\xi}) d\hat{\xi}, \quad k \in \{1, 2, \dots\}. \quad (1.4)$$

For simplicity, we are overloading the symbol $\hat{\xi}$ to denote both a random vector and its realizations. For each fixed $\hat{\xi}$, equation (1.3) can be written as,

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + f_d(t, x), \quad (1.5)$$

where $f_d(t, x) = \sigma(x) \sum_{k=1}^d \xi_k h_k(t)$. Whether we are using Monte Carlo or Stochastic Collocation, in order to solve the stochastic equation (1.3), we need to solve the deterministic equation (1.5) many times for different values of $\hat{\xi}$. In the following, we specify the high fidelity and the low fidelity algorithms that we will employ to solve equation (1.5).

1.2 GFE as a high-fidelity deterministic algorithm

A standard finite element approximation to the solution $u_d(t, x)$ of equation (1.5) can be written as $u_{d,N}(t, x) = \sum_{j=1}^N \alpha_j(t) \beta_j(x)$, where $\{\beta_j(x), j = 1, \dots, N\}$ are N piecewise linear finite element basis functions, and each $\alpha_j(t)$ is an unknown function of time. We also define $W_N([0, 1]) := \text{span}\{\beta_j, j = 1, \dots, N\}$. The basis functions are chosen according to a computational grid on the domain $[0, 1]$ with $\{x_n, n = 0, \dots, N + 1\}$ as its set of nodes, in a way that $\beta_j(x_n) = \delta_{j,n}$, where $\delta_{j,n}$ is the Kronecker delta and $j, n = 1, \dots, N$. Note that $u_{d,N}^2(t, x_n) = \sum_{j=1}^N \alpha_j^2(t) \beta_j(x_n)$. This motivates us to approximate $u_d^2(t, x)$ by

$\sum_{j=1}^N \alpha_j^2(t) \beta_j(x)$. Use of the weak form of (1.5) and the approximations for u_d and u_d^2 results in the following differential equations.

$$\begin{aligned} \mathbf{M} \dot{\boldsymbol{\alpha}} &= -\mu \mathbf{A} \boldsymbol{\alpha} - \frac{1}{2} \mathbf{G}(\boldsymbol{\alpha}) + \mathbf{V}(t), \\ \boldsymbol{\alpha}(0) &= \boldsymbol{\alpha}_0 = [(u_0, \beta_i)]_{i=1}^N, \end{aligned} \tag{1.6}$$

where $\mathbf{G}(\boldsymbol{\alpha}) = \mathbf{N}[\text{diag}(\boldsymbol{\alpha})]\boldsymbol{\alpha}$, $[\mathbf{N}]_{ij} = (\beta_j', \beta_i)$, $[\mathbf{M}]_{ij} = (\beta_j, \beta_i)$, $[\mathbf{A}]_{ij} = (\beta_j', \beta_i')$, $[\mathbf{V}(t)]_i = (f_d(t, \cdot), \beta_i)$ and $(f, g) = \int_0^1 f(x)g(x)dx$ is the standard $L_2([0, 1])$ inner product. We are using $\dot{y}$ and y' to denote the derivatives with respect to time and space, respectively. This algorithm is called group finite elements (GFE). A detailed study of this method can be found in [DS10]. We employ this algorithm only once for $\hat{\boldsymbol{\xi}} = \mathbf{0}$ which results in $f_d \equiv 0$. Note that if more accurate results are needed, this algorithm can be employed at more points in the stochastic space. Details of this improvement will be provided later in the next chapters. For now, it suffices to use it only once.

1.3 Group POD as a low-fidelity deterministic algorithm

Solution to equations (1.6), with $\mathbf{V}(t) \equiv \mathbf{0}$, gives an approximate solution $w(t, x)$ to (1.5), with $f_d(t, x) \equiv 0$. Let $\{w(t_i, \cdot), i = 1, \dots, S\}$ be a set of S “snapshots”, where

$$t_1 < t_2 < \dots < t_S$$

are equally spaced points of time in the interval $[0, T]$. The correlation matrix $\mathbf{K}$ of the data set $\{w(t_i, \cdot), i = 1, \dots, S\}$ can be defined as

$$\mathbf{K} := \left(\frac{1}{S} (w_i, w_j) \right)_{i,j=1}^S \tag{1.7}$$

where $w_i = w(t_i, \cdot)$, $i = 1, \dots, S$. Let $\{\lambda_k, \mathcal{Z}_k\}$ denote the eigenvalues and the corresponding normalized eigenvectors of $\mathbf{K}$. Define $\mathcal{Z}$ to be the matrix $[\mathcal{Z}_1 | \dots | \mathcal{Z}_S]$. The POD basis functions $\{\psi_k\}_{k=1}^S$ are given as

$$\psi_k = \frac{1}{\sqrt{S\lambda_k}} \sum_{i=1}^S [\mathcal{Z}]_{i,k} w_i, \quad k = 1, \dots, S. \quad (1.8)$$

Let the solution u_d to (1.5) be written as

$$u_d(t, x) = U(x) + v_d(t, x) \quad (1.9)$$

where $U(x) := \frac{1}{S} \sum_{i=1}^S w(t_i, x)$. Therefore, using (1.5), we obtain that v_d satisfies

$$v_t + \frac{1}{2}(v^2)_x - \mu(v_{xx} + U'') + UU' + Uv_x + vU' = f_d, \quad (1.10)$$

$(t, x) \in (0, T] \times [0, 1]$, $v(0, x) = v_0(x) = u_0(x) - U(x)$, $v(t, 0) = v(t, 1) = 0$. Let the POD approximation of $v_d(t, x)$ be given by $v_{d,p}(t, x) = \sum_{j=1}^M a_j(t) \psi_j(x)$, where ψ_j is a POD basis function, and $a_j(t)$ is an unknown function of time. Note that M is the dimension of the POD basis. Furthermore, let $v_d^2(t, x)$ be approximated by $\sum_{j=1}^M F_j(\mathbf{a}) \psi_j(x)$, with $F_j(\mathbf{a})$ as an unknown function of $\mathbf{a}(t) = [a_j(t)]_{j=1}^M$. The projection of equation (1.10) onto the POD set $\Psi = \{\psi_k\}_{k=1}^M$ results in the variational problem of finding $v \in L_2([0, T]; W_{N,M}([0, 1]))$, where $W_{N,M}([0, 1]) := \text{span}\{\Psi\}$, such that

$$(v_t, \psi_i) + \frac{1}{2}((v^2)_x, \psi_i) + (UU', \psi_i) + \dots \quad (1.11)$$

$$\mu(v_x + U', \psi'_i) + (Uv_x + U'v, \psi_i) = (f_d, \psi_i),$$

and $(v(0, \cdot), \psi_i) = (v_0, \psi_i)$, for $i = 1, \dots, M$. When the POD approximations for v_d and v_d^2 are substituted into equation (1.11), we obtain the system of ordinary differential equations

$$\mathbf{M}\dot{\mathbf{a}} = -\mathbf{A}\mathbf{a} - \frac{1}{2}\mathbf{N}_p\mathbf{F}(\mathbf{a}) - \mathbf{V}(t), \quad (1.12)$$

$$\mathbf{a}(0) = \mathbf{a}_0 = [(v_0, \psi_i)]_{i=1}^M,$$

where $[\mathbf{N}_p]_{ij} = (\psi'_j, \psi_i)$, $[\mathbf{M}]_{ij} = (\psi_j, \psi_i)$, $[\mathbf{A}]_{ij} = \mu(\psi'_j, \psi'_i) + (U\psi'_j + U'\psi_j, \psi_i)$, $[\mathbf{V}]_i = (-f_d + UU', \psi_i) + \mu(U', \psi'_i)$, and $\mathbf{F}(\mathbf{a}) = [F_i]_{i=1}^M$ is determined as given below. At the grid points x_n , $n = 1, \dots, N$ we want to have

$$\sum_{j=1}^M F_j(\mathbf{a})\psi_j(x_n) = \left(\sum_{j=1}^M a_j(t)\psi_j(x_n) \right)^2. \quad (1.13)$$

Let $\gamma_{nj} = \psi_j(x_n)$, then,

$$\sum_{j=1}^M F_j(\mathbf{a})\gamma_{nj} = \left(\sum_{j=1}^M a_j(t)\gamma_{nj} \right)^2 = \sum_{j,\ell=1}^M \gamma_{nj}\gamma_{n\ell}a_ja_\ell \quad (1.14)$$

Since $\gamma_{nj}\gamma_{n\ell} = \gamma_{n\ell}\gamma_{nj}$, for $j \neq \ell$ we can avoid computing $\gamma_{nj}\gamma_{n\ell}$ twice by writing

$$\sum_{j,\ell=1}^M \gamma_{nj}\gamma_{n\ell}a_ja_\ell = \hat{\gamma}_n \hat{\mathbf{a}} \quad (1.15)$$

where $\hat{\gamma}_n$ is a $1 \times \frac{1}{2}(M^2 + M)$ vector given as

$$\hat{\gamma}_n := [\gamma_{n1}\gamma_{n1}, 2\gamma_{n1}\gamma_{n2}, \dots, 2\gamma_{n1}\gamma_{nM}, \gamma_{n2}\gamma_{n2}, 2\gamma_{n2}\gamma_{n3}, \dots, 2\gamma_{n2}\gamma_{nM}, \dots, \gamma_{nM}\gamma_{nM}]$$

and the $\frac{1}{2}(M^2 + M) \times 1$ vector $\hat{\mathbf{a}}$ is

$$\hat{\mathbf{a}} := [a_1 a_1, a_1 a_2, \dots, a_1 a_M, a_2 a_2, a_2 a_3, \dots, a_2 a_M, \dots, a_M a_M]^T.$$

We may write (1.14) in the matrix form as

$$\mathbf{\Gamma} \mathbf{F}(\mathbf{a}) = \hat{\mathbf{\Gamma}} \hat{\mathbf{a}} \quad (1.16)$$

where the $N \times M$ matrix $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1M} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NM} \end{bmatrix} \quad (1.17)$$

and $\hat{\mathbf{\Gamma}}$ is the $N \times \frac{1}{2}(M^2 + M)$ matrix $[\hat{\gamma}_i]_{i=1}^N$. Equation (1.16) normally has no solution, since M is generally chosen to be less than N . However, similar to the method used in [DS10], one can directly approximate the product $\mathbf{N}_p \mathbf{F}(\mathbf{a})$ by $\mathbf{\Gamma}^T \mathbf{N} \mathbf{\Gamma} \mathbf{F}(\mathbf{a})$, where $\mathbf{N}$ is the matrix used in (1.6). We can finally approximate $\mathbf{N}_p \mathbf{F}(\mathbf{a})$ by

$$\mathbf{N}_p \mathbf{F}(\mathbf{a}) \approx \hat{\mathbf{N}} \hat{\mathbf{a}}, \quad (1.18)$$

where $\hat{\mathbf{N}}$ is the $M \times \frac{1}{2}(M^2 + M)$ matrix given by $\hat{\mathbf{N}} = \mathbf{\Gamma}^T \mathbf{N} \mathbf{\Gamma}$. With this, problem (1.12) takes the form

$$\mathbf{M} \dot{\mathbf{a}} = -\mathbf{A} \mathbf{a} - \hat{\mathbf{N}} \hat{\mathbf{a}} - \mathbf{V}(t), \quad (1.19)$$

$$\mathbf{a}(0) = \mathbf{a}_0 = [(v_0, \psi_i)]_{i=1}^M.$$

This model is then solved for $\mathbf{a}(t)$ to give an approximation $v_{d,p}(t, x) = \sum_{j=1}^M a_j(t) \psi_j(x)$ of $v_d(t, x)$. Therefore, according to (1.9), $u_d(t, x)$ is approximated by $U(x) + v_{d,p}(t, x)$. The above algorithm called group proper orthogonal decomposition (group POD) is studied in detail in [DS10]. We will employ this algorithm for other sample values of $\hat{\boldsymbol{\xi}}$ in the stochastic space.

1.4 Multi-fidelity Stochastic Collocation

We are finally in a position to demonstrate the multi-fidelity stochastic collocation method. Our aim is to approximate the solution of (1.3) in the space $V_{N,\mathbf{p}} = L_2([0, T]; W_N([0, 1])) \otimes \mathcal{P}_{\mathbf{p}}(\mathbb{R}^d)$, where $W_N([0, 1]) = \text{span}\{\beta_j\}$ is the finite element space, and $\mathcal{P}_{\mathbf{p}}(\mathbb{R}^d)$ is the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \dots, p_d)$. The procedure for approximating the solution of (1.3) is divided into two parts:

1. Use the Group POD method to solve problem (1.5) at $\hat{\boldsymbol{\xi}} \in \mathbb{R}^d$ and get the solution $u_d(\hat{\boldsymbol{\xi}}, t, x)$.
2. Collocate on zeros of suitable orthogonal polynomials and build the interpolated solution $u_{d,\mathbf{p}} \in V_{N,\mathbf{p}}$ using

$$u_{d,\mathbf{p}}(\hat{\boldsymbol{\xi}}, t, x) = \mathcal{I}_{\mathbf{p}} u_d(\hat{\boldsymbol{\xi}}, t, x) = \tag{1.20}$$

$$\sum_{j_1=1}^{p_1+1} \cdots \sum_{j_d=1}^{p_d+1} u_d(\xi_1, \dots, \xi_d, t, x) (l_{j_1}(\hat{\boldsymbol{\xi}}) \otimes \cdots \otimes l_{j_d}(\hat{\boldsymbol{\xi}}),$$

where the functions $\{l_{j_k}\}_{k=1}^d$ can be taken as Lagrange polynomials. Using this formula, as described in [BNT07], mean value and variance of u_d can also be easily approximated.

1.4.1 Generalization to Sparse Grids

Here, we give a short description of the isotropic Smolyak algorithm. More detailed information can be found in [BNR00,NTW08b]. Assume $p_1 = p_2 = \dots = p_d = p$. For $d = 1$, let $\{\mathcal{I}_{1,i}\}_{i=1,2,\dots}$ be a sequence of interpolation operators given by equation (1.20). Define $\Delta_0 = \mathcal{I}_{1,0} = 0$ and $\Delta_i = \mathcal{I}_{1,i} - \mathcal{I}_{1,i-1}$. Now for $d > 1$, let

$$\mathcal{A}(q, d) = \sum_{0 \leq i_1 + i_2 + \dots + i_d \leq q} \Delta_{i_1} \otimes \dots \otimes \Delta_{i_d} \quad (1.21)$$

where q is a non-negative integer. $\mathcal{A}(q, d)$ is the Smolyak operator, and q is known as the sparse grid *level*. Now instead of (1.20), $\mathcal{A}(q, d)u_d(\hat{\xi}, t, x)$ can be used to approximate the solution u_d of (1.3). This way one reduces the number of grid points on which the deterministic algorithms should be employed.

1.5 Numerical results

In the following numerical experiment, we let $T = 1.0$, $u_0(x) = (e^{\cos(5\pi x)} - \frac{3}{2})\sin(\pi x)$, $\sigma(x) = 0.01$, and $\mu = \frac{1}{200}$. We discretize time into 50 steps, and divide the spatial domain into 64 intervals. We solve the problem by the Monte Carlo method and compare the results with the solution of our algorithm. We use the Clenshaw-Curtis abscissas (see [CC60]) as collocation points. These abscissas are the extrema of Chebyshev polynomials. We project the Brownian motion in $[0, T]$ on the trigonometric basis functions $h_k(t)$ in $L_2([0, T])$ given by

$$h_1(t) = \frac{1}{\sqrt{T}}, \quad h_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{(k-1)\pi t}{T}\right), \quad k \in \{2, 3, \dots\}. \quad (1.22)$$

For small values of d in (1.3), say $d = 2$, and sparse grid level 6, sparse grid stochastic collocation with the full employment of the expensive high-fidelity GFE provides similar results to when sparse grid stochastic collocation is used with the cheap low-fidelity group

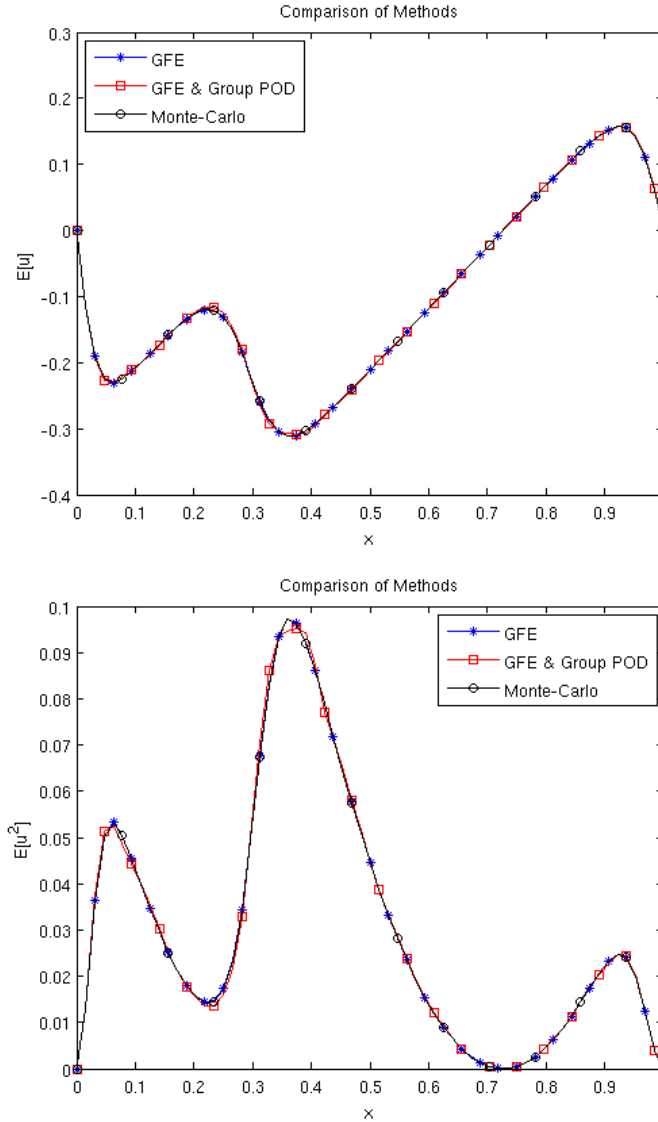


Figure 1.1: Expectation and Second Moment of the solution given at the final time $T = 1.0$ when $d = 2$ for sparse grid level 6.

POD algorithm. The results, using 10 POD basis functions, are summarized in Figure 1.1 and in Table 1.1. In this table Monte Carlo method is the reference point.

Standard deviations of the two methods is shown in Figure 1.2. Note that we are only using a single point in the stochastic parameter space to generate POD basis functions. We are also performing no types of improvement to the POD basis functions. The issue brought

Table 1.1: Comparison of the errors of the two methods.

$L_2([0, 1])$ Rel. Error	$E[u]$	$E[u^2]$	Std. Dev.
POD & Sparse Grid	0.0109	0.0207	0.4125
GFE & Sparse Grid	0.0017	0.0028	0.0183

up in Figure 1.2 can be simply dealt with by either increasing the number of times that the high fidelity algorithm is employed, or by local improvements to POD basis functions (see Chapter 3), or both. Numerical results section of Chapter 3 illustrates the effect these improvements.

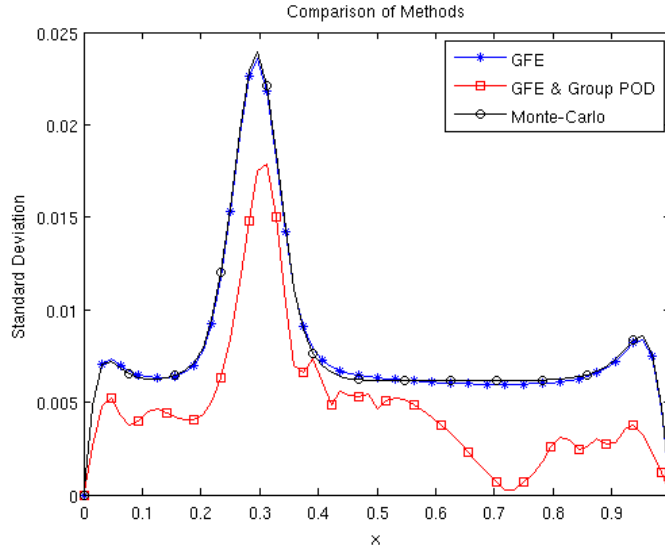


Figure 1.2: Standard Deviation of the solution given at final time $T = 1.0$ when $d = 2$ for sparse grid level 6.

Even for such a low dimension of $d = 2$, the enhanced version of the algorithm using the POD method shows an increase in speed of over 10 times when the number of spatial points is increased to 512. The results, using 10 POD basis functions, are summarized in Table 1.2.

Table 1.2: Comparison of the execution times of the two methods.

	Time
POD & Sparse Grid	10.35 sec
GFE & Sparse Grid	101.58 sec

For larger values of d , say $d = 4$, however, the regular sparse grid stochastic collocation algorithm is not at all efficient due to the huge amount of computations it needs to perform. The enhanced version of the sparse grid stochastic collocation seems to provide a cure in this case. Using 10 POD basis functions, as the sparse grid level changes from 7 to 9, we get the convergence pattern presented in Figure 1.3. These results compared with the solution of Monte-Carlo are illustrated in this figure. This time we are discretizing the spacial domain into 128 intervals. The stochastic collocation with POD for levels 8 and 9 compares well with the Monte Carlo solution. Level 7 of the sparse grid is where convergence is first taking effect.

Execution times and the errors in expected values and second moments, when the sparse grid level changes from 7 to 9, are given in Table 1.3. For the sparse grid part of our code we modified the Matlab toolbox developed in [KW05, Kli07] to benefit from the parallel features of Matlab programming language.

Table 1.3: Execution time and convergence pattern of the proposed method as the sparse grid level changes from 7 to 9.

	level 7	level 8	level 9
Execution time in seconds	238.30	609.11	1489.31
Error in Expectation	0.2190	0.0184	0.0144
Error in Second Moment	0.2167	0.0293	0.0289

Remark 1.1. *It may be noted that stochastic collocation methods and consequently our multi-fidelity method are most effective in small noise regimes. This remark holds true for*

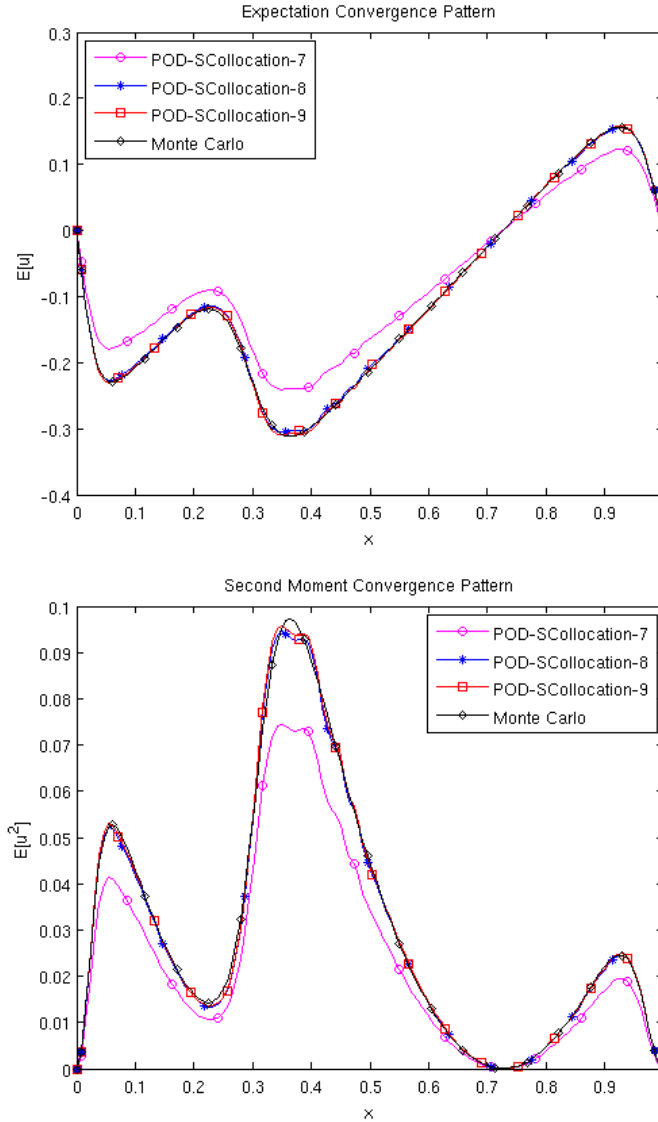


Figure 1.3: Convergence pattern as the sparse grid level changes from 7 to 9.

next chapters as well.

Remark 1.2. For the eigenvalues and eigenvectors computations in (1.8) we are using Matlab's "eigs" command. This is efficient if the size of the correlation matrix introduced in equation (1.7) is relatively small. For more complicated systems, Matlab's "svds" command might be employed as well. This remark holds true for next chapters as well.

Chapter 2: Multi-fidelity Stochastic Collocation Method applied to Parabolic PDEs with Random Input Data

Up to this point we were employing the high fidelity algorithm only once. The idea is to utilize this algorithm more often and consequently increase the accuracy. In order to be able to provide rigorous analysis of our proposed method we start the study of linear parabolic partial differential equations in this chapter.

2.1 Problem definition

Let $D \subset \mathbb{R}^2$ be a bounded, connected and polygonal domain and $(\Omega, \mathcal{F}, P)$ denote a complete probability space with sample space Ω , which corresponds to the set of all possible outcomes. $\mathcal{F}$ is the σ -algebra of events, and $P : \mathcal{F} \rightarrow [0, 1]$ is the probability measure. In this section, we consider the stochastic linear parabolic initial-boundary value problem: find a random field $u : [0, T] \times \overline{D} \times \Omega \rightarrow \mathbb{R}$, such that P -almost surely the following equations hold:

$$\begin{aligned} \partial_t u(t, \mathbf{x}, \omega) - \Delta u(t, \mathbf{x}, \omega) &= f(t, \mathbf{x}, \omega) & \text{in} & \quad (0, T] \times D \times \Omega, \\ u(t, \mathbf{x}, \omega) &= 0 & \text{on} & \quad (0, T] \times \partial D \times \Omega, \\ u(0, \mathbf{x}, \omega) &= 0 & \text{on} & \quad D \times \Omega. \end{aligned} \tag{2.1}$$

In order to guarantee the existence and uniqueness of the solution of (2.1), we assume that the random forcing field $f : [0, T] \times \overline{D} \times \Omega \ni (t, \mathbf{x}, \omega) \mapsto f(t, \mathbf{x}, \omega) \in \mathbb{R}$ satisfies:

$$\int_0^T \int_D f^2(t, \mathbf{x}, \omega) d\mathbf{x} dt < +\infty \quad P\text{-a.e. in } \Omega. \tag{2.2}$$

Following [BNT07] and inspired by the truncated KL expansion [Loe77], we make the assumption that the random field f depends on a finite number of independent random variables. More specifically,

$$f(t, \mathbf{x}, \omega) = f(t, \mathbf{x}, \mathbf{y}(\omega)) \quad \text{on} \quad [0, T] \times \overline{D} \times \Omega, \quad (2.3)$$

where $\mathbf{y}(\omega) = (y_1(\omega), \dots, y_r(\omega))$ and $r \in \mathbb{N}_+$. Let us define the space,

$$L_P^2(\Omega) := \{\mathbf{y} = (y_1, y_2, \dots, y_r)^T : \sum_{n=1}^r \int_{\Omega} |y_n(\omega)|^2 dP(\omega) < \infty\},$$

where $\mathbf{y}$ denotes an r -dimensional random vector over $(\Omega, \mathcal{F}, P)$. We also define the Hilbert space,

$$V := L^2(0, T; H_0^1(D)) \otimes L_P^2(\Omega),$$

with the inner product $(\cdot, \cdot)_V : V \times V \rightarrow \mathbb{R}$ given by:

$$(u, v)_V = \int_0^T \int_D \mathbb{E}[\nabla u(t, \mathbf{x}, \omega) \cdot \nabla v(t, \mathbf{x}, \omega)] d\mathbf{x} dt.$$

A function $u \in V$ is called a *weak solution* of problem (2.1) if:

$$\begin{aligned} \int_D \mathbb{E}[\partial_t uv] d\mathbf{x} + \int_D \mathbb{E}[\nabla u \cdot \nabla v] d\mathbf{x} &= \int_D \mathbb{E}[fv] d\mathbf{x}, \\ \forall v \in H_0^1(D) \otimes L_P^2(\Omega) \text{ and } \forall t \in (0, T], \end{aligned} \quad (2.4)$$

and P -almost surely $u(0, \mathbf{x}, \omega) = 0$. The existence and uniqueness of the solution of problem (2.4) is a direct consequence of assumption (2.2) on f ; see [Eva].

Let $\Gamma_n = y_n(\Omega)$ denote the image of the random variable y_n , for $n = 1, \dots, r$, and

$\Gamma = \prod_{n=1}^r \Gamma_n$. We also assume that the distribution measure of $\mathbf{y}(\omega)$ is absolutely continuous with respect to the Lebesgue measure. Thus, there exists a joint density function $\rho : \Gamma \rightarrow \mathbb{R}_+$ for $\mathbf{y} = (y_1, \dots, y_r)$. Hence, we can use $(\Gamma, \mathcal{B}^r, \rho d\mathbf{y})$ instead of $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}^r$ is the r -dimensional Borel space. Analogous to the definitions of $L_P^2(\Omega)$ and V we can define

$$L_\rho^2(\Gamma) := \{\mathbf{y} \in \Gamma : \int_\Gamma \|\mathbf{y}\|^2 \rho d\mathbf{y} < \infty\},$$

and

$$V_\rho = L^2(0, T; H_0^1(D)) \otimes L_\rho^2(\Gamma),$$

with inner product

$$(u, v)_{V_\rho} = \int_\Gamma (u(\mathbf{y}), v(\mathbf{y}))_{L^2(0, T; H_0^1(D))} \rho d\mathbf{y}.$$

where

$$(u(\mathbf{y}), v(\mathbf{y}))_{L^2(0, T; H_0^1(D))} = \int_0^T \int_D \nabla u(t, \mathbf{x}, \mathbf{y}) \cdot \nabla v(t, \mathbf{x}, \mathbf{y}) d\mathbf{x} dt.$$

The weak solution $u \in V$ of problem (2.1), using the finite dimensional noise assumption (2.3), is of the form $u(t, \mathbf{x}, \omega) = u(t, \mathbf{x}, y_1(\omega), \dots, y_r(\omega))$. Therefore, the weak formulation (2.4) can be equivalently expressed as finding $u \in V_\rho$ such that ρ -almost everywhere in Γ , $u(0, \mathbf{x}, \mathbf{y}) = 0$, and

$$\begin{aligned} \int_\Gamma \int_D \partial_t u v d\mathbf{x} \rho d\mathbf{y} + \int_\Gamma \int_D \nabla u \cdot \nabla v d\mathbf{x} \rho d\mathbf{y} &= \int_\Gamma \int_D f v d\mathbf{x} \rho d\mathbf{y}, \\ \forall v \in H_0^1(D) \otimes L_\rho^2(\Gamma) \text{ and } \forall t \in (0, T]. \end{aligned} \tag{2.5}$$

For each fixed $t \in (0, T]$, the solution u to (2.5) can be viewed as a mapping $u : \Gamma \rightarrow H_0^1(D)$. In order to emphasize the dependence on the variable $\mathbf{y}$, we use the notations $u(\mathbf{y})$ and $f(\mathbf{y})$. Hence, we achieve the following equivalent settings: find $u(\mathbf{y}) \in H_0^1(D)$ such that ρ -almost

everywhere in Γ , $u(0, \mathbf{x}, \mathbf{y}) = 0$, and

$$\int_D \partial_t u(\mathbf{y}) v d\mathbf{x} + \int_D \nabla u(\mathbf{y}) \cdot \nabla v d\mathbf{x} = \int_D f(\mathbf{y}) v d\mathbf{x}, \quad (2.6)$$

$$\forall v \in H_0^1(D) \text{ and } \forall t \in (0, T], \text{ } \rho\text{-a.e. in } \Gamma.$$

Note that there may exist a $\rho d\mathbf{y}$ -zero measure set $\mathcal{N}_\rho \subset \Gamma$ in which (2.6) is not satisfied. Therefore, from a computational perspective, if a point $\mathbf{y} \in \mathcal{N}_\rho$ is chosen, the resulting solution of (2.6) is not the true solution of the original equation. However, the computation of the moments of the solution does not suffer from this disadvantage.

2.2 Multi-fidelity Collocation method

In this section, we apply our multi-fidelity stochastic collocation method to the weak form (2.6). Let $V_{\rho,h}$ be a finite dimensional subspace of V_ρ given by $V_{\rho,h} = L^2(0, T; H_h(D)) \otimes \mathcal{P}_{\mathbf{p}}(\Gamma)$, where $H_h(D) \subset H_0^1(D)$ is a standard finite element space and $\mathcal{P}_{\mathbf{p}}(\Gamma) \subset L_\rho^2(\Gamma)$ is the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \dots, p_r)$. The goal is to find a numerical approximation to the solution of (2.6) in the finite dimensional subspace $V_{\rho,h}$. Choose $\eta > 0$ to be a small real number. The procedure for solving (2.6) is divided into two parts:

1. Fix $\mathbf{y} \in \Gamma$, and search the η -neighbourhood $B_\eta(\mathbf{y}) \subset \Gamma$ of $\mathbf{y}$. If problem (2.6) is not already solved by the finite element method for any nearby problem with $\mathbf{y}' \in B_\eta(\mathbf{y})$, solve problem (2.6) using a regular backward Euler finite element method at $\mathbf{y}$ and let $\mathbf{y}' = \mathbf{y}$. In contrast, if equation (2.6) is already solved using the finite element method for some points in $B_\eta(\mathbf{y})$, choose the closest one to $\mathbf{y}$ and call it $\mathbf{y}'$. In either case, use the solution at $\mathbf{y}' \in B_\eta(\mathbf{y})$ to find a small number $d \in \mathbb{N}_+$ of suitable orthonormal basis functions $\{\psi_j(\mathbf{y}')\}_{j=1}^d \subset H_h(D)$ using proper orthogonal decomposition (POD) method. Now use Galerkin projection on to the subspace $X^d(\mathbf{y}') = \text{span}\{\psi_j(\mathbf{y}')\}_{j=1}^d$

to find

$$\{u_d^m(\mathbf{y})\}_{m=1}^N \subset X^d(\mathbf{y}') \subset H_h(D),$$

such that

$$(u_d^m, v_d) + k(\nabla u_d^m, \nabla v_d) = k(f^m(\mathbf{y}), v_d) + (u_d^{m-1}, v_d), \quad (2.7)$$

$$\forall v_d \in X^d(\mathbf{y}'), \quad m = 1, \dots, N,$$

and $u_d^0 = 0$, where $N \in \mathbb{N}_+$ is the number of time steps, and $k = T/N$ denotes the time step increments. It is worth mentioning that $(\cdot, \cdot)$ denotes the L^2 -inner product. Note that we are employing a backward Euler scheme to discretize time.

2. Collocate (2.7) on zeros of suitable orthogonal polynomials and build the interpolated discrete solution

$$\{u_{d,\mathbf{p}}^m\}_{m=1}^N \subset H_h(D) \otimes \mathcal{P}_{\mathbf{p}}(\Gamma), \quad (2.8)$$

using

$$u_{d,\mathbf{p}}^m(x, \mathbf{y}) = \mathcal{I}_{\mathbf{p}} u_d^m(x, \mathbf{y}) = \quad (2.9)$$

$$\sum_{j_1=1}^{p_1+1} \cdots \sum_{j_r=1}^{p_r+1} u_d^m(x, y_{j_1}, \dots, y_{j_r}) (l_{j_1}(\mathbf{y}) \otimes \cdots \otimes l_{j_r}(\mathbf{y})), \quad m = 1, \dots, N,$$

where the functions $\{l_{j_k}\}_{k=1}^r$ can be taken as Lagrange polynomials. Using this formula, as described in [BNT07], mean value and variance of u can also be easily approximated.

2.2.1 Proper Orthogonal Decomposition

In this section, we choose a fixed $\mathbf{y}' \in B_\eta(\mathbf{y}) \subset \Gamma$ and drop the dependence of equation (2.6) on $\mathbf{y}'$, for notational conveniences. Therefore, we consider the problem of finding

$w \in H_0^1(D)$ such that:

$$(w_t, v) + (\nabla w, \nabla v) = (g, v), \quad \forall v \in H_0^1(D), \quad (2.10)$$

and $w(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$. Note that $g = f(\mathbf{y}')$. Let $t_m = mk, k = 0, \dots, N$, where k denotes the time step increments. Assume $\mathfrak{T}_h$ to be a uniformly regular family of triangulation of $\overline{D}$ (see [Tho97, Cia78]). The finite element space is taken as

$$H_h(D) = \{v_h \in H_0^1(D) \cap C^0(D) : v_h|_K \in P_s(K), \quad \forall K \in \mathfrak{T}_h\},$$

where $s \in \mathbb{N}_+$ and $P_s(K)$ is the space of polynomials of degree $\leq s$ on K . Write $w^m(\mathbf{x}) = w(\mathbf{x}, t_m)$, and let w_h^m denote the fully discrete approximation of w resulting from solving the problem of finding $w_h^m \in H_h(D)$ such that $w_h^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(w_h^m, v_h) + k(\nabla w_h^m, \nabla v_h) = k(g^m, v_h) + (w_h^{m-1}, v_h), \quad (2.11)$$

$$\forall v_h \in H_h(D), \quad m = 1, \dots, N.$$

It is easy to prove that problem (2.11) has a unique solution $w_h^m \in H_h(D)$, provided that $g^m \in L^2(D)$ (see [Tho97]). One can also show that if $w_t \in H^{s+1}(D)$ and $w_{tt} \in L^2(D)$, the following error estimates hold:

$$\|w^m - w_h^m\|_0 \leq Ch^{s+1} \int_0^{t_m} \|w_t\|_{s+1} dt + Ck \int_0^{t_m} \|w_{tt}\|_0 dt, \quad m = 1, \dots, N, \quad (2.12)$$

where $\|\cdot\|_s$ denotes the $H^s(D)$ -norm and C indicates a positive constant independent of the spatial and temporal mesh sizes, possibly different at distinct occurrences.

For the so-called snapshots $U_i := w_h^{m_i} \in H_h(D)$, $i = 1, \dots, \ell$, where

$$1 \leq m_1 < m_2 < \dots < m_\ell \leq N,$$

let

$$\mathcal{V} = \text{span}\{U_1, \dots, U_\ell\}.$$

Assume at least one of U_i is non-zero, and let $\{\psi_j\}_{j=1}^l$ be an orthonormal basis of $\mathcal{V}$ with $l = \dim \mathcal{V}$. Therefore, for each $U_i \in \mathcal{V}$ we will have:

$$U_i = \sum_{j=1}^l (U_i, \psi_j)_{H_0^1(D)} \psi_j, \quad (2.13)$$

where $(U_i, \psi_j)_{H_0^1(D)} = (\nabla u_h^{m_i}, \nabla \psi_j)$.

Definition 2.1. *The POD method consists of finding an orthonormal basis ψ_j ($j = 1, 2, \dots, d$) such that for every $d = 1, \dots, l$, the following problem is solved*

$$\min_{\{\psi_j\}_{j=1}^d} \frac{1}{\ell} \sum_{i=1}^{\ell} \|U_i - \sum_{j=1}^d (U_i, \psi_j)_{H_0^1(D)} \psi_j\|_{H_0^1(D)}^2. \quad (2.14)$$

A solution $\{\psi_j\}_{j=1}^d$ of this minimization problem is known as a POD basis of rank d .

Let us introduce the correlation matrix $K = (K_{ij})_{i,j=1}^{\ell} \in \mathbb{R}^{\ell \times \ell}$ given by

$$K_{ij} = \frac{1}{\ell} (U_i, U_j)_{H_0^1(D)}. \quad (2.15)$$

The following proposition (see [Sir87, KV01, KV02]) solves problem (2.14).

Proposition 2.2. *Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ denote the positive eigenvalues of K and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_l$ the associated orthonormal eigenvectors. Then a POD basis of rank $d \leq l$ is*

given by

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{\ell} (\mathbf{v}_i)_j U_j, \quad i = 1, \dots, d, \quad (2.16)$$

where $(\mathbf{v}_i)_j$ denotes the j -th component of the eigenvector $\mathbf{v}_i$. Furthermore, the following error formula holds:

$$\frac{1}{\ell} \sum_{i=1}^{\ell} \|U_i - \sum_{j=1}^d (U_i, \psi_j)_{H_0^1(D)} \psi_j\|_{H_0^1(D)}^2 = \sum_{j=d+1}^{\ell} \lambda_j. \quad (2.17)$$

One can immediately notice the similarities between equation (2.17) and a truncated singular value decomposition.

Let $X^d := \text{span}\{\psi_1, \psi_2, \dots, \psi_d\}$, and consider the problem of finding $w_d^m \in X^d \subset H_h(D)$ such that $w_d^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(w_d^m, v_d) + k(\nabla w_d^m, \nabla v_d) = k(g^m, v_d) + (w_d^{m-1}, v_d), \quad (2.18)$$

$$\forall v_d \in X^d \subset H_h(D), \quad m = 1, \dots, N.$$

Remark 2.1. If $\mathfrak{T}_h$ is a uniformly regular triangulation and $H_h(D)$ is the space of piecewise linear functions, the total degrees of freedom for problem (2.11) is N_h , where N_h is the number of vertices of triangles in $\mathfrak{T}_h$, while the total of degrees of freedom for problem (2.18) is d (where $d \ll l \ll \ell \ll N_h$).

The following proposition, proved in [LCSY09], gives us an error estimate on the solution of problem (2.18).

Proposition 2.3. If $w_h^m \in H_h(D)$ is the solution of problem (2.11), $w_d^m \in X^d \subset H_h(D)$ is the solution of problem (2.18), $k = O(h)$, $\ell^2 = O(N)$, and snapshots are equably taken,

then for $m = 1, 2, \dots, N$, the following estimates hold:

$$\|w_h^m - w_d^m\|_0 + \frac{1}{\ell} \sum_{j=1}^{\ell} \|\nabla(w_h^{m_j} - w_d^{m_j})\|_0 \leq \quad (2.19)$$

$$C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2}, \quad m = m_i, \quad i = 1, \dots, \ell;$$

$$\|w_h^m - w_d^m\|_0 + \frac{1}{\ell} \left[\|\nabla(w_h^m - w_d^m)\|_0 + \sum_{j=1}^{\ell-1} \|\nabla(w_h^{m_j} - w_d^{m_j})\|_0 \right] \leq$$

$$C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2} + Ck, \quad m \neq m_i.$$

Combining (2.12) and (2.19) we get the following result.

Proposition 2.4. *Under assumptions of proposition 2.3, the error estimate between the solutions of problems (2.10) and (2.18), for $m = 1, 2, \dots, N$, is given by:*

$$\|w^m - w_d^m\|_0 \leq Ch^{s+1} + Ck + C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2} \quad (2.20)$$

Now, with a slight overloading of notation, we assume that the function f is given by $f = f(\mathbf{y})$, where $f \in C(\Gamma; C(0, T; L^2(D)))$ is the function employed in equation (2.6), and consider the following problem: find $u \in H_0^1(D)$ such that $u(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$, and

$$(u_t, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(D). \quad (2.21)$$

Remark 2.2. *Note that since $\|\mathbf{y} - \mathbf{y}'\| < \eta$ and under the assumption that f as a member of*

$C(\Gamma; C(0, T; L^2(D)))$ is Lipschitz continuous on Γ , we get that $\|f(\mathbf{y}) - f(\mathbf{y}')\|_{C(0, T; L^2(D))} = \|f - g\|_{C(0, T; L^2(D))} \leq L_f \|\mathbf{y} - \mathbf{y}'\|$, where L_f is the Lipschitz constant. Also, note that we are slightly overloading the symbol f to denote both the function $f \in C(\Gamma; C(0, T; L^2(D)))$ employed in equation (2.6) and the function $f = f(\mathbf{y}') \in C(0, T; L^2(D))$ used in equation (2.10).

Let us also consider the following problem: find $u_d^m \in X^d \subset H_h(D)$ such that $u_d^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(u_d^m, v_d) + k(\nabla u_d^m, \nabla v_d) = k(f^m, v_d) + (u_d^{m-1}, v_d), \quad \forall v_d \in X^d \subset H_h(D). \quad (2.22)$$

Note that equations (2.22) and (2.7) are identical, according to the fact that we are using $f = f(\mathbf{y})$. Our aim is to find an estimate for $\|u^m - u_d^m\|_0$. First we need to prove two lemmas.

Lemma 2.5. *Let u be the solution of problem (2.21) and let w be the solution of problem (2.10), then we have:*

$$\|u^m - w^m\|_0 \leq C \|f - g\|_{C(0, T; L^2(D))}. \quad (2.23)$$

Proof. let $z = u - w$ and subtract equations (2.10) and (2.21) to get:

$$(z_t, v) + (\nabla z, \nabla v) = (f - g, v), \quad \forall v \in H_0^1(D), \quad (2.24)$$

with $z(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$. Letting $v = z$ and integrating equation (2.24) from 0 to t_m , we get:

$$\frac{1}{2} \int_0^{t_m} \frac{d}{dt} \|z\|_0^2 dt + \int_0^{t_m} (\nabla z, \nabla z) dt = \int_0^{t_m} (f - g, z) dt.$$

This results in

$$\frac{1}{2}\|z^m\|_0^2 \leq \int_0^{t_m} \|f - g\|_0 \|z\|_0 dt \leq \frac{1}{2} \int_0^T \|f - g\|_0^2 dt + \frac{1}{2} \int_0^T \|z\|_0^2 dt.$$

Therefore,

$$\|z^m\|_0^2 \leq T\|f - g\|_{C(0,T;L^2(D))}^2 + \int_0^T \|z\|_0^2 dt. \quad (2.25)$$

Now we need to bound $\int_0^T \|z\|_0^2 dt$. For this, we integrate (2.24) once again but this time upto T , and use the Poincaré inequality $\|v\|_0 \leq C_p \|\nabla v\|_0$, for each $v \in H_0^1(D)$, to get:

$$\frac{1}{2}\|z(T)\|_0^2 + \frac{1}{C_p^2} \int_0^T \|z\|_0^2 dt \leq \int_0^T \|f - g\|_0 \|z\|_0 dt.$$

Therefore,

$$\int_0^T \|z\|_0^2 dt \leq C_p^2 \left(\frac{1}{2\delta} \int_0^T \|f - g\|_0^2 dt + \frac{\delta}{2} \int_0^T \|z\|_0^2 dt \right).$$

Thus,

$$\left(1 - \frac{C_p^2}{2}\delta\right) \int_0^T \|z\|_0^2 dt \leq \frac{C_p^2}{2\delta} T\|f - g\|_{C(0,T;L^2(D))}^2.$$

Choose $\delta > 0$ such that $1 - \frac{C_p^2}{2}\delta > 0$, and let

$$C = \sqrt{T \left(1 + \frac{C_p^2}{2\delta - C_p^2\delta^2}\right)}.$$

Now, equation (2.25) implies (2.23). □

Lemma 2.6. *Let u_d^m be the solution of problem (2.22) and w_d^m be the solution of problem*

(2.18), then we have:

$$\|u_d^m - w_d^m\|_0 \leq C \|f - g\|_{C(0,T;L^2(D))} \quad (2.26)$$

Proof. let $z_d^m = u_d^m - w_d^m$ and subtract equations (2.18) and (2.22) to get:

$$(z_d^m, v_d) + k(\nabla z_d^m, \nabla v_d) = k(f^m - g^m, v_d) + (z_d^{m-1}, v_d), \quad (2.27)$$

$$\forall v_d \in X^d \subset H_h(D), \quad m = 1, \dots, N,$$

with $z_d^0(\mathbf{x}) = 0$. Let $v_d = z_d^m$ in equation (2.27) and use Poincaré inequality $\|v\|_0 \leq C_p \|\nabla v\|_0$, for each $v \in H_0^1(D)$, to achieve:

$$\|z_d^m\|_0^2 + k \frac{1}{C_p^2} \|z_d^m\|_0^2 \leq k \|f^m - g^m\|_0 \|z_d^m\|_0 + \|z_d^{m-1}\|_0 \|z_d^m\|_0.$$

Therefore,

$$\left(1 + k \frac{1}{C_p^2}\right) \|z_d^m\|_0 \leq k \|f^m - g^m\|_0 + \|z_d^{m-1}\|_0,$$

which upon summation yields,

$$\|z_d^m\|_0 \leq k \|f - g\|_{C(0,T;L^2(D))} \sum_{j=1}^m \left(\frac{1}{1 + \frac{k}{C_p^2}} \right)^j.$$

Let $\gamma = \frac{1}{C_p^2}$ and note that $(1 + \gamma k)^m \leq e^{\gamma k m}$. Moreover, setting $\zeta = 1/(1 + \gamma k)$ we find:

$$k \sum_{j=1}^m \left(\frac{1}{1 + \frac{k}{C_p^2}} \right)^j = k \frac{1 - \zeta^m}{\zeta^{-1} - 1} = \frac{1 - \zeta^m}{\gamma} \leq \frac{1 - e^{-\gamma k m}}{\gamma}.$$

Letting $C = (1 - e^{-\gamma km})/\gamma$, we get (2.26). $\square$

Now using estimates (2.20), (2.23) and (2.26) and remark 2.2, we get the following error estimate.

Theorem 2.7. *Let u be the solution of problem (2.21), and u_d^m be the solution of problem (2.22), for $m = 1, \dots, N$, we have*

$$\|u^m - u_d^m\|_0 \leq C\eta + Ch^{s+1} + Ck + C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2}, \quad (2.28)$$

where the eigenvalues λ_j depend on $\mathbf{y}' \in B_\eta(\mathbf{y}) \subset \Gamma$, and the constants C depend on $\mathbf{y}$ and $\mathbf{y}'$, but are independent of h , k and η .

2.3 Error analysis

In this section, we carry out an error analysis for the multi-fidelity collocation method introduced in section 2.2 for problem (2.6). In [BNT07], the authors showed that if the solution of (2.6) is analytic with respect to the random parameters, then the collocation scheme (2.9) attains an exponential error decay for $u_d^m - u_{d,\mathbf{p}}^m$ with respect to each p_n . The convergence proof in [BNT07] applies directly to our case. Therefore, our main task is to prove the analyticity property of the POD solution u_d^m with respect to each random variable y_n . We will then only state the corresponding convergence result. In the following we impose similar restrictions on f as in [BNT07, ZG12], i.e., f is continuous with respect to each element $\mathbf{y} \in \Gamma$ and that it has at most exponential growth at infinity, whenever the domain Γ is unbounded. Moreover, we assume that joint density function ρ behaves like a Gaussian kernel at infinity. In order to make it precise, we introduce the weight function

$\sigma(\mathbf{y}) = \prod_{n=1}^r \sigma_n(y_n) \leq 1$, where

$$\sigma_n(y_n) = \begin{cases} 1 & \text{if } \Gamma_n \text{ is bounded,} \\ e^{-\alpha_n |y_n|} \text{ for some } \alpha_n > 0 & \text{if } \Gamma_n \text{ is unbounded,} \end{cases}$$

and the space

$$C_{\sigma}^0(\Gamma; V) = \{v : \Gamma \rightarrow V : v \text{ is continuous in } \mathbf{y} \text{ and } \max_{\mathbf{y} \in \Gamma} \{\sigma(\mathbf{y}) \|v(\mathbf{y})\|_V\} < +\infty\},$$

where V is a Banach space. In what follows, we assume that $f \in C_{\sigma}^0(\Gamma; C([0, T]; L^2(D)))$ and the joint probability density ρ satisfies

$$\rho(\mathbf{y}) \leq C_M e^{-\sum_{n=1}^r (\delta_n y_n)^2}, \quad \forall \mathbf{y} \in \Gamma, \quad (2.29)$$

for some constant $C_M > 0$, with δ_n being strictly positive if Γ_n is unbounded and zero otherwise. Under these assumptions, the following proposition is immediate; see [BNT07].

Proposition 2.8. *The solution of problem (2.6) satisfies $u \in C_{\sigma}^0(\Gamma; C(0, T; H_0^1(D)))$ and correspondingly, the approximate solution u_d^m resulting from (2.22) or equivalently (2.7), satisfies $u_d^m \in C_{\sigma}^0(\Gamma; H_h(D))$, for $m = 1, \dots, N$.*

Furthermore, we have the following regularity result.

Lemma 2.9. *The following energy estimate holds:*

$$\|u_d^m\|_{L^2(D) \otimes L_{\rho}^2(\Gamma)} \leq C_p^2 (1 - e^{-\frac{km}{C_p^2}}) \|f\|_{C(0, T; L^2(D)) \otimes L_{\rho}^2(\Gamma)},$$

where C_p is the Poincaré Constant.

Proof. Similar to the proof of Lemma 2.6. □

2.3.1 Analyticity with respect to random parameters

We prove that the solution u_d^m of equation (2.22) is analytic with respect to each random parameter $y_n \in \Gamma$, whenever $f(\mathbf{y})$ is analytic and is infinitely differentiable with respect to each component of $\mathbf{y}$. To do this, we introduce the following notations as in [BNT07,ZG12]:

$$\mathbf{y}_n^* \in \Gamma_n^* = \prod_{j=1, j \neq n}^r \Gamma_j \quad \text{and} \quad \sigma_n^* = \prod_{j=1, j \neq n}^r \sigma_j.$$

We first make the additional assumption that for every $\mathbf{y} = (y_n, \mathbf{y}_n^*) \in \Gamma$, there exists $\gamma_n < +\infty$ such that

$$\frac{\|\partial_{y_n}^j f(\mathbf{y})\|_{C(0,T;L^2(D))}}{1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}} \leq \gamma_n^j j!. \quad (2.30)$$

Remark 2.3. Under the finite dimensional noise assumption (2.3), $f(t, \mathbf{x}, \omega)$ is represented by a truncated linear or nonlinear expansion so that assumption (2.30) holds. For example, consider a truncated KL expansion for random forcing term $f(t, \mathbf{x}, \omega)$ given by

$$f(t, \mathbf{x}, \omega) = f(t, \mathbf{x}, \mathbf{y}(\omega)) = \mathbb{E}[f](t, \mathbf{x}) + \sum_{n=1}^r \sqrt{\mu_n} c_n(t, \mathbf{x}) y_n(\omega). \quad (2.31)$$

We have

$$\frac{\|\partial_{y_n}^j f(\mathbf{y})\|_{C(0,T;L^2(D))}}{1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}} \leq \begin{cases} \sqrt{\mu_n} \|c_n\|_{C(0,T;L^2(D))}, & j = 1, \\ 0, & j > 1. \end{cases}$$

Therefore, we can set $\gamma_n = \sqrt{\mu_n} \|c_n\|_{C(0,T;L^2(D))}$, and observe that definition (2.31) satisfies assumption (2.30). Moreover, the random forcing $f(t, \mathbf{x}, \mathbf{y})$ defined in (2.31), satisfies the Lipschitz continuity assumption of remark 2.2.

Lemma 2.10. *Under assumption (2.30), if the solution $u_d^m(\mathbf{x}, y_n, \mathbf{y}_n^*)$ is considered as a function of y_n , i.e., $u_d^m : \Gamma_n \rightarrow C_{\sigma_n^*}^0(\Gamma_n^*, L^2(D))$, then the j -th derivative of $u_d^m(\mathbf{x}, \mathbf{y})$ with respect to y_n satisfies*

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_{L^2(D)} \leq C j! \gamma_n^j, \quad m = 1, \dots, N, \quad (2.32)$$

where C depends on $\|f(\mathbf{y})\|_{C(0,T;L^2(D))}$, and the Poincaré constant C_p .

Proof. Take the j -th derivative of formulation (2.22) or equivalently (2.7) with respect to y_n , and let $v_d = \partial_{y_n}^j u_d^m(\mathbf{y})$ to get

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0^2 + k \|\partial_{y_n}^j \nabla u_d^m(\mathbf{y})\|_0^2 = k(\partial_{y_n}^j f^m(\mathbf{y}), \partial_{y_n}^j u_d^m(\mathbf{y})) + (\partial_{y_n}^j u_d^{m-1}(\mathbf{y}), \partial_{y_n}^j u_d^m(\mathbf{y})).$$

Therefore,

$$(1 + \frac{k}{C_p^2}) \|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq k \|\partial_{y_n}^j f^m(\mathbf{y})\|_0 + \|\partial_{y_n}^j u_d^{m-1}(\mathbf{y})\|_0,$$

which upon summation yields

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq k \|\partial_{y_n}^j f(\mathbf{y})\|_{C(0,T;L^2(D))} \sum_{i=1}^m \left(\frac{1}{1 + \frac{k}{C_p^2}} \right)^i.$$

Thus,

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq C_p^2 (1 - e^{-\frac{km}{C_p^2}}) [1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}] \gamma_n^j j!.$$

Letting $C = C_p^2 (1 - e^{-\frac{km}{C_p^2}}) [1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}]$ we get (2.32). $\square$

We will immediately obtain the following theorem, whose proof closely follows the proof of Theorem 4.4 in [ZG12].

Theorem 2.11. *Under assumption (2.30), the solution $u_d^m(\mathbf{x}, y_n, \mathbf{y}_n^*)$ considered as a function of y_n , admits an analytic extension $u_d^m(\mathbf{x}, z, \mathbf{y}_n^*)$, $z \in \mathbb{C}$, in the region of complex plane*

$$\Sigma(\Gamma_n, \tau_n) := \{z \in \mathbb{C} : \text{dist}(z, \Gamma_n) \leq \tau_n\},$$

where $0 < \tau_n < 1/\gamma_n$.

Proof. For each $y_n \in \Gamma_n$ we define the power series $u_d^m : \mathbb{C} \rightarrow C_{\sigma_n^*}^0(\Gamma_n^*; L^2(D))$ as

$$u_d^m(\mathbf{x}, z, \mathbf{y}_n^*) = \sum_{j=0}^{\infty} \frac{(z - y_n)^j}{j!} \partial_{y_n}^j u_d^m(\mathbf{x}, y_n, \mathbf{y}_n^*)$$

Thus,

$$\begin{aligned} \sigma_n(y_n) \|u_d^m(z)\|_{C_{\sigma_n^*}^0(\Gamma_n^*; L^2(D))} &\leq \sum_{j=0}^{\infty} \frac{|z - y_n|^j}{j!} \|\partial_{y_n}^j u_d^m(y_n)\|_{C_{\sigma_n^*}^0(\Gamma_n^*; L^2(D))} \\ &\leq \sigma_n(y_n) C(y_n) \sum_{j=0}^{\infty} (|z - y_n| \gamma_n)^j \leq \hat{C} \sum_{j=0}^{\infty} (|z - y_n| \gamma_n)^j, \end{aligned}$$

where $C(y_n)$ is a function of $\|f(y_n)\|_{C_{\sigma_n^*}^0(\Gamma_n^*; C(0, T; L^2(D)))}$, and the constant $\hat{C}$ is a function of $\|f\|_{C_{\sigma}^0(\Gamma; C(0, T; L^2(D)))}$. The series is convergent for all $z \in \mathbb{C}$, provided that $|z - y_n| \leq \tau_n < 1/\gamma_n$. Therefore, the function u_d^m admits an analytic extension in the region $\Sigma(\Gamma_n; \tau_n)$. $\square$

2.3.2 Convergence analysis

Our goal is to provide an estimate for the total error $e^m = u^m - u_{d, \mathbf{p}}^m$ in the norm $L^2(D) \otimes L^2_\rho(\Gamma)$, for each $m = 1, \dots, N$. The error splits naturally into $e^m = (u^m - u_d^m) + (u_d^m - u_{d, \mathbf{p}}^m)$. Recall that $u_{d, \mathbf{p}}^m = \mathcal{I}_{\mathbf{p}} u_d^m$ and is given by (2.9). We can estimate the interpolation error $(u_d^m - u_{d, \mathbf{p}}^m)$ by repeating the same procedure as in [BNT07], using the analyticity result

of theorem 2.11. All details about the estimates of the interpolation error can be found in section 4 of [BNT07] and the references cited therein. Therefore we state the following theorem without proof.

Theorem 2.12. *Under assumption (2.30), there exist positive constants $b_n, n = 1, \dots, r$, and C that are independent of h, d , and $\mathbf{p}$ such that*

$$\|u_d^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_p^2(\Gamma)} \leq C \sum_{n=1}^r \beta_n(p_n) \exp(-b_n p_n^{\theta_n}), \quad (2.33)$$

where

$$\theta_n = \beta_n = 1 \quad \text{and} \quad b_n = \log \left[\frac{2\tau_n}{|\Gamma_n|} \left(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2}} \right) \right] \quad \text{if } \Gamma_n \text{ is bounded,}$$

and

$$\theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n}), \quad \text{and} \quad b_n = \tau_n \delta_n \quad \text{if } \Gamma_n \text{ is unbounded,}$$

where τ_n is the minimum distance between Γ_n and the nearest singularity in the complex plane, as defined in theorem 2.11, and δ_n is defined in assumption (2.29).

Remark 2.4 (Convergence with respect to the number of collocation points). *For an isotropic full tensor-product approximation, i.e., $p_1 = p_2 = \dots = p_r = p$, the number of collocation points Θ is given by $\Theta = (1 + p)^r$. Thus, one can easily obtain the following error bound with respect to Θ ; see [ZG12].*

$$\|u_d^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_p^2(\Gamma)} \leq \begin{cases} C \Theta^{-b_{\min}/r}, & \text{if } \Gamma \text{ is bounded,} \\ C \Theta^{-b_{\min}/2r}, & \text{if } \Gamma \text{ is unbounded,} \end{cases} \quad (2.34)$$

where $b_{\min} = \min\{b_1, b_2, \dots, b_r\}$ as in theorem 2.12. The constant C does not depend on r

and $b_{\min}$.

Remark 2.5 (Extensions to sparse grid stochastic collocation methods). *Note that the convergence as shown in (2.34) becomes slower as the dimension r increases. This slow-down effect as a result of increase in dimension is called the curse of dimensionality. For large values of r , sparse grid stochastic collocation methods [NT09, NTW08b], specially adaptive and anisotropic ones, e.g., [MZ09, NTW08a] are more effective in dealing with this problem. Our analyticity result (theorem 2.11) combined with the analysis in [NT09, NTW08b, MZ09, NTW08a], can easily lead to the derivation of error bounds for sparse grid approximations. For instance, for an isotropic Smolyak approximation [NT09, NTW08b] with a total of Θ sparse grid points, the error can be bounded by*

$$C\Theta^{-b_{\min}/(1+\log(2r))}.$$

Here, we will give a short description of the isotropic Smolyak algorithm. More detailed information can be found in [BNR00, NTW08b]. Assume $p_1 = p_2 = \dots = p_r = p$. For $r = 1$, let $\{\mathcal{I}_{1,i}\}_{i=1,2,\dots}$ be a sequence of interpolation operators given by equation (2.9). Define $\Delta_0 = \mathcal{I}_{1,0} = 0$ and $\Delta_i = \mathcal{I}_{1,i} - \mathcal{I}_{1,i-1}$. Now for $r > 1$, let

$$\mathcal{A}(q, r) = \sum_{0 \leq i_1 + i_2 + \dots + i_r \leq q} \Delta_{i_1} \otimes \dots \otimes \Delta_{i_r} \quad (2.35)$$

where q is a non-negative integer. $\mathcal{A}(q, r)$ is the Smolyak operator, and q is known as the sparse grid level.

Now we need to find error bounds for the deterministic part of our algorithm in the $L^2(D) \otimes L^2_\rho(\Gamma)$ norm, i.e., $u^m - u_d^m$. First, note that according to (2.29), the joint density function ρ behaves like a Gaussian kernel at infinity. Therefore, in practice we are literally dealing with a compact random parameter set Γ , since we can approximate Γ with a large enough compact set. So from now on we assume that Γ is compact. We know that $\Gamma \subset$

$\bigcup_{\mathbf{y}' \in \Gamma} B_\eta(\mathbf{y}')$. Thus, using the compactness assumption on Γ , there exist $\Upsilon \in \mathbb{N}_+$ and $\{\mathbf{y}'\}_{i=1}^\Upsilon \subset \Gamma$ such that $\Gamma = \bigcup_{i=1}^\Upsilon B_\eta(\mathbf{y}') \cap \Gamma$. Letting ${}^i\Gamma = B_\eta(\mathbf{y}') \cap \Gamma$, we can write $\Gamma = \bigcup_{i=1}^\Upsilon {}^i\Gamma$.

Theorem 2.13. *Under the Lipschitz continuity (see Remark 2.2) assumption, there exist constants C and Λ such that*

$$\|u^m - u_d^m\|_{L^2(D) \otimes L^2_\rho(\Gamma)} \leq C\eta + Ch^{s+1} + Ck + Ck^{1/4}\Lambda. \quad (2.36)$$

Proof. Let us first integrate the the last term in estimate (2.28). Thus, we have

$$\int_\Gamma \left\{ C(\mathbf{y}, \mathbf{y}'(\mathbf{y})) \left(k^{1/2} \sum_{j=d(\mathbf{y}, \mathbf{y}'(\mathbf{y}))+1}^{l(\mathbf{y}, \mathbf{y}'(\mathbf{y}))} \lambda_j(\mathbf{y}'(\mathbf{y})) \right)^{1/2} \right\}^2 \rho(\mathbf{y}) d\mathbf{y} =$$

$$k^{1/2} \int_\Gamma C(\mathbf{y}, \mathbf{y}'(\mathbf{y}))^2 \sum_{j=d(\mathbf{y}, \mathbf{y}'(\mathbf{y}))+1}^{l(\mathbf{y}, \mathbf{y}'(\mathbf{y}))} \lambda_j(\mathbf{y}'(\mathbf{y})) \rho(\mathbf{y}) d\mathbf{y} =$$

$$k^{1/2} \sum_{i=1}^\Upsilon \left(\sum_{j=d({}^i\mathbf{y}')+1}^{l({}^i\mathbf{y}')} \lambda_j({}^i\mathbf{y}') \right) \int_{{}^i\Gamma} C(\mathbf{y}, {}^i\mathbf{y}')^2 \rho(\mathbf{y}) d\mathbf{y}$$

Now letting $\Lambda_i = \sum_{j=d({}^i\mathbf{y}')+1}^{l({}^i\mathbf{y}')} \lambda_j({}^i\mathbf{y}')$, and assuming $\Lambda^2 = \max_{i=1, \dots, \Upsilon} \{\Lambda_i\}$, we get the following upper bound for the above expression:

$$k^{1/2} \Lambda^2 \sum_{i=1}^\Upsilon \int_{{}^i\Gamma} C(\mathbf{y}, {}^i\mathbf{y}')^2 \rho(\mathbf{y}) d\mathbf{y} = k^{1/2} \Lambda^2 \int_\Gamma C(\mathbf{y}, \mathbf{y}'(\mathbf{y}))^2 \rho(\mathbf{y}) d\mathbf{y}.$$

Letting $C^2 = \int_\Gamma C(\mathbf{y}, \mathbf{y}'(\mathbf{y}))^2 \rho(\mathbf{y}) d\mathbf{y}$, we get the last term in (2.36). The first three terms of (2.36) can also be easily computed by integrating the first three terms of (2.28). We will

get the same expressions for the constants C as above. $\square$

Remark 2.6. *Due to the way that the POD method works, the constant Λ is so small that the $k^{1/4}$ term has a very little effect on the error. The fact that the largest eigenvalues of the correlation matrix (2.15) capture most of the information is a distinct feature of ellipticity conditions. Addition of advective terms or non-linearities might affect this feature.*

Combining (2.33) and (2.36), we will finally get the following total error estimate.

Theorem 2.14. *Under assumption (2.30) and the Lipschitz continuity (see remark 2.2) assumption, there exist positive constants C and Λ that are independent of h, k, η and $\mathbf{p}$, and there exist constants $b_n, n = 1, \dots, r$, such that*

$$\|u^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L^2_\rho(\Gamma)} \leq C\eta + Ch^{s+1} + Ck + Ck^{1/4}\Lambda + C \sum_{n=1}^r \beta_n(p_n) \exp(-b_n p_n^{\theta_n}), \quad (2.37)$$

where θ_n, β_n and b_n are the same as the ones in theorem 2.12.

Remark 2.7. *In some cases, one might be interested in estimating the expectation error, i.e., $\|\mathbb{E}[u^m - u_{d,\mathbf{p}}^m]\|_{L^2(D)}$. This can be easily achieved by observing that:*

$$\begin{aligned} \|\mathbb{E}[u^m - u_{d,\mathbf{p}}^m]\|_{L^2(D)}^2 &= \int_D \left[\int_\Gamma [u^m(\mathbf{x}, \mathbf{y}) - u_{d,\mathbf{p}}^m(\mathbf{x}, \mathbf{y})] \rho(\mathbf{y}) d\mathbf{y} \right]^2 d\mathbf{x} \\ &\leq \int_D \left[\int_\Gamma [u^m(\mathbf{x}, \mathbf{y}) - u_{d,\mathbf{p}}^m(\mathbf{x}, \mathbf{y})]^2 \rho(\mathbf{y}) d\mathbf{y} \int_\Gamma \rho(\mathbf{y}) d\mathbf{y} \right] d\mathbf{x} \\ &= \int_\Gamma \left[\int_D [u^m(\mathbf{x}, \mathbf{y}) - u_{d,\mathbf{p}}^m(\mathbf{x}, \mathbf{y})]^2 d\mathbf{x} \right] \rho(\mathbf{y}) d\mathbf{y} \\ &= \|u^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L^2_\rho(\Gamma)}^2. \end{aligned} \quad (2.38)$$

2.4 Numerical experiments

In this section, we provide a computational example to illustrate the advantages of multi-fidelity stochastic collocation method. Specifically, we consider problem (2.1) with $D = (0, 1)^2 \subset \mathbb{R}^2$, $T = 1$, and the forcing term being given by:

$$f(t, \mathbf{x}, \omega) = 10 + e^t \sum_{n=1}^r \mathbf{y}_n(\omega) \sin(n\pi x).$$

The real-valued random variables $y_n, n = 1, \dots, r$, are supposed to be independent and have uniform distributions $U(0, 1)$. In the following, we let $r = 4$. We employ the sparse grid stochastic collocation method introduced in Remark 2.5 with sparse grid level $q = 8$. We use the Clenshaw-Curtis abscissas (see [CC60]) as collocation points. These abscissas are the extrema of Chebyshev polynomials. We divide the spatial domain D into 32×32 small squares with side length $\Delta x = \Delta y = 1/32$, and then we connect the diagonals of the squares to divide each square into two triangles. These triangles constitute the triangulation $\mathfrak{T}_h$, with $h = \sqrt{2}/32$. Take $k = 0.1$ as the time step increment. We use all of the time steps to form the snapshots. We employ 6 POD basis functions.

Remark 2.8. *In order to form the correlation matrix (2.15) needed to find POD basis functions we are in practice using the $L_2(D)$ norm. This way we can avoid the computation of derivatives of the finite element solutions.*

In the following, we compare the solution resulting from a regular isotropic sparse grid stochastic collocation method which only uses the finite element method, with the hybrid multi-fidelity method proposed in this work which employs both finite element and POD methods. In Figure 2.1, we compare the expected values resulting from the multi-fidelity method and a regular sparse grid stochastic collocation method. We take $\eta = 0.1$. Recall that for each $\mathbf{y} \in \Gamma$ our method searches the η neighbourhood of $\mathbf{y}$ to check whether for some $\mathbf{y}' \in B_\eta(\mathbf{y})$ problem (2.6) is already solved. If a nearby problem (at $\mathbf{y}'$) is found to be

solved by finite element method, our algorithm uses this information to create POD basis functions and solves problem (2.6) at $\mathbf{y}$ using Galerkin-POD method which is computationally much cheaper than finite element. Moreover, Figure 2.2 compares variances of solutions resulting from the two methods.

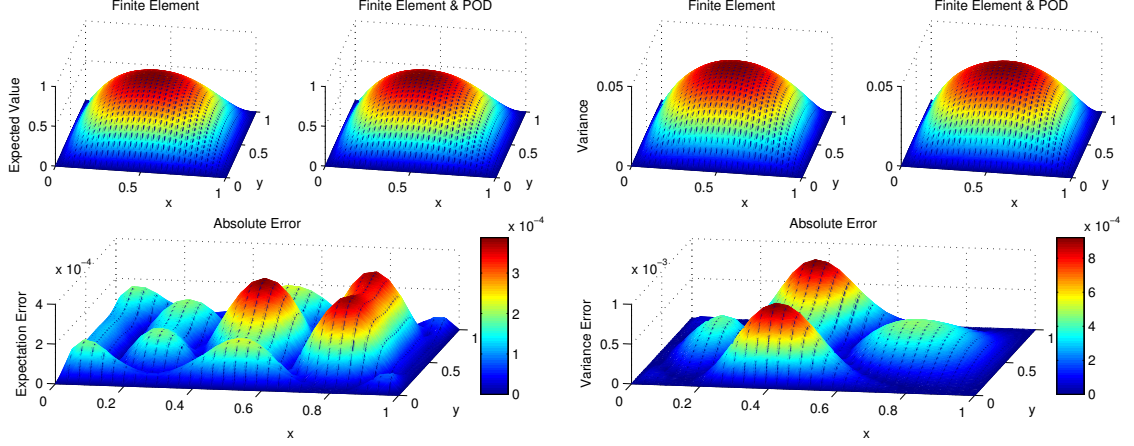


Figure 2.1: Comparison of expected values (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right).

Figure 2.2: Comparison of variances of solutions (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right).

Figures 2.3 and 2.4, show the convergence patterns of expectations and variances of solutions with regard to η , respectively. These results validate our theoretical estimates of previous sections. We are actually comparing our multi-fidelity method with a regular sparse grid stochastic method. Note that for small enough η (less than the shortest distance between the collocation points) we get the regular sparse grid method back. Therefore the error is zero for such a small η .

Figure 2.5 demonstrates how the number of times that the finite element code is employed increases with respect to a decrease in η .

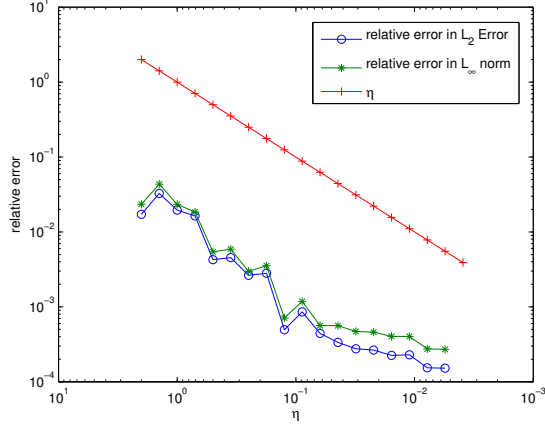


Figure 2.3: Convergence pattern of expected values of solutions with respect to η .

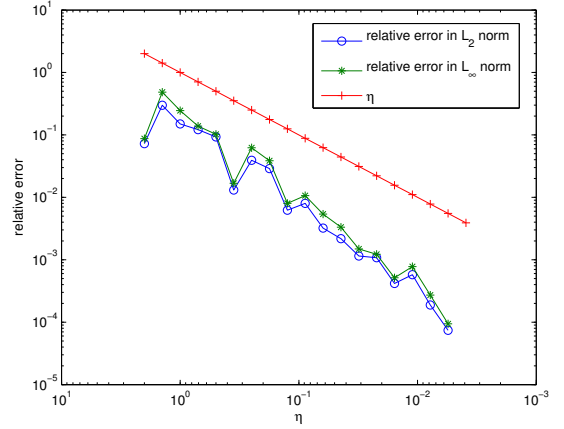


Figure 2.4: Convergence pattern of variances of solutions with respect to η .

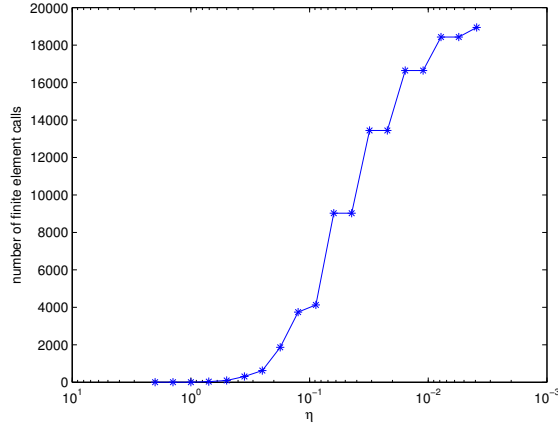


Figure 2.5: Number of times that the finite element code is employed as a function of η .

Table 2.1, summarizes the results when $\eta = 0.1$. In this case, the number of times that the finite element code is utilized by the multi-fidelity method is 3745. Compared it to 18946, the number of times that a regular sparse grid calls the finite element code.

Table 2.2 is just another way of presenting the data depicted in Figures 2.3, 2.4, and 2.5.

Table 2.1: Relative errors when $\eta = 0.1$.

	Relative error in L_2 norm	Relative error in L_∞ norm
Expected value	3.6×10^{-4}	4.8×10^{-4}
Variance	1.2×10^{-2}	2.0×10^{-2}

Table 2.2: Relative errors and the number of times that the finite element code is employed for different values of η .

η	# FE calls	Exp. L_2 error	Exp. L_∞ error	Var. L_2 error	Var. L_∞ error
4	1	1.72E-02	2.34E-02	7.25E-02	8.72E-02
2	3	3.27E-02	4.35E-02	2.99E-01	4.84E-01
1	5	1.95E-02	2.33E-02	1.50E-01	2.45E-01
1/2	36	1.63E-02	1.85E-02	1.21E-01	1.38E-01
$(1/2)^2$	92	4.26E-03	5.43E-03	9.27E-02	1.02E-01
$(1/2)^3$	306	4.55E-03	5.89E-03	1.31E-02	1.68E-02
$(1/2)^4$	621	2.64E-03	2.98E-03	3.91E-02	6.21E-02
$(1/2)^5$	1866	2.81E-03	3.55E-03	2.88E-02	3.86E-02
$(1/2)^6$	3743	4.96E-04	7.09E-04	6.23E-03	8.00E-03
$(1/2)^7$	4129	8.58E-04	1.19E-03	8.00E-03	1.07E-02
$(1/2)^8$	9026	4.42E-04	5.63E-04	3.22E-03	5.39E-03
$(1/2)^9$	9026	3.35E-04	5.61E-04	2.18E-03	3.33E-03
$(1/2)^{10}$	13442	2.76E-04	4.69E-04	1.15E-03	1.49E-03
$(1/2)^{11}$	13442	2.65E-04	4.59E-04	1.08E-03	1.22E-03
$(1/2)^{12}$	16642	2.25E-04	4.04E-04	4.18E-04	5.15E-04
$(1/2)^{13}$	16642	2.29E-04	4.02E-04	5.75E-04	7.78E-04
$(1/2)^{14}$	18434	1.54E-04	2.75E-04	1.89E-04	2.71E-04
$(1/2)^{15}$	18434	1.52E-04	2.71E-04	7.42E-05	9.43E-05
$(1/2)^{16}$	18946	0	0	0	0

Remark 2.9. *Note that this chapter closely follows our paper [RSa].*

Chapter 3: Stochastic Burgers revisited

In this chapter we once again consider the stochastic Burgers equation (1.1). Recall that after truncating the expansion (1.2) at some point d we were able to get

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + \sigma(x) \sum_{k=1}^d \xi_k h_k(t). \quad (3.1)$$

For each fixed $\hat{\xi} = (\xi_1, \dots, \xi_d)$, equation (3.1) can be written as,

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + f_d(t, x), \quad (3.2)$$

where $f_d(t, x) = \sigma(x) \sum_{k=1}^d \xi_k h_k(t)$. Recall that we needed to solve the deterministic equation (3.2) many times for different values of $\hat{\xi}$. In the following, we revisit the low fidelity algorithm of Group POD and apply it on

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + g_d(t, x), \quad (3.3)$$

where $g_d(t, x) = \sigma(x) \sum_{k=1}^d \zeta_k h_k(t)$, which is practically equation (3.2) evaluated at the new parameter $\hat{\zeta} = (\zeta_1, \dots, \zeta_d)$. Note that $\hat{\xi}$ and $\hat{\zeta}$ are not necessarily equal. Consequently, f_d and g_d are generally different. We solve equation (3.2) using the GFE method (1.6).

3.1 Group POD as a low-fidelity deterministic algorithm

Solution to equations (1.6), with $\mathbf{V}(t) = [(f_d(t, \cdot), \beta_i)]_{i=1}^N$, gives an approximate solution to (3.2) given by $w(t, x) = \sum_{j=1}^N \alpha_j(t) \beta_j(x)$. Let $\{w(t_i, \cdot), i = 1, \dots, S\}$ be a set of S “snapshots”, where $t_1 < t_2 < \dots < t_S$ are equally spaced points of time in the interval $[0, T]$. Furthermore, define $\mathbf{Y} = [\boldsymbol{\alpha}(t_1) \cdots \boldsymbol{\alpha}(t_S)]$ to be the snapshots matrix. Let $\mathbf{L}$ be a lower triangular matrix resulting from the Cholesky decomposition of the mass matrix $\mathbf{M}$, i.e., $\mathbf{M} = \mathbf{L}\mathbf{L}^T$. Define $\tilde{\mathbf{Y}} := \mathbf{L}^T \mathbf{Y}$ to be the weighted snapshots matrix. The correlation matrix $\mathbf{K}$ of the data set $\{w(t_i, \cdot), i = 1, \dots, S\}$ is given by

$$\mathbf{K} := \left(\frac{1}{S} (w_i, w_j) \right)_{i,j=1}^S = \frac{1}{S} \mathbf{Y}^T \mathbf{M} \mathbf{Y} = \frac{1}{S} \tilde{\mathbf{Y}}^T \tilde{\mathbf{Y}} \quad (3.4)$$

where $w_i = w(t_i, \cdot), i = 1, \dots, S$. Let $\{\lambda_k, \mathbf{Z}_k\}$ denote the eigenvalues and the corresponding normalized eigenvectors of $\mathbf{K}$. Define $\mathbf{Z}$ to be the matrix $[\mathbf{Z}_1 | \dots | \mathbf{Z}_S]$. The POD basis functions $\{\psi_k\}_{k=1}^S$ are given as

$$\psi_k = \frac{1}{\sqrt{S\lambda_k}} \sum_{i=1}^S [\mathbf{Z}]_{i,k} w_i, \quad k = 1, \dots, S. \quad (3.5)$$

Note that

$$\frac{1}{S} \tilde{\mathbf{Y}}^T \tilde{\mathbf{Y}} \mathbf{Z} = \mathbf{Z} \boldsymbol{\Lambda}, \quad (3.6)$$

where $\boldsymbol{\Lambda} := \text{diag}(\lambda_k, k = 1, \dots, S)$. Let us express the POD basis function ψ_j as $\psi_j(x) = \sum_{i=1}^N \psi_{i,j} \beta_i(x)$ and let $\boldsymbol{\psi} = (\psi_{i,j})$ for $i = 1, \dots, N$ and $j = 1, \dots, S$. Using (3.5) we get

$$\boldsymbol{\psi} = \mathbf{Y} \mathbf{Z} (\mathbf{S} \boldsymbol{\Lambda})^{-\frac{1}{2}}. \quad (3.7)$$

Using exactly the same procedure as the one in Group POD introduced in section 1.3, with f_d being replaced by g_d , we can get an approximate solution $u_d(t, x)$ to (3.3).

3.2 Local improvements to POD basis functions

Note that we want to use the POD basis functions found at $\hat{\xi}$ to solve the deterministic problem (3.3) at a nearby point $\hat{\zeta}$. It is a well known fact that if $\hat{\zeta} = \hat{\xi}$ we get sufficiently accurate approximations of the solution of problem (3.3) or equivalently (3.2). However, for $\hat{\zeta} \neq \hat{\xi}$, we usually experience a drop in accuracy. In order to deal with issue, we use the sensitivity analysis of POD basis functions to improve the accuracy of the solution found at $\hat{\zeta}$. Note that the POD bases (3.5) are a function of $\hat{\xi}$. We are specifically interested in the sensitivity of these POD bases in the direction of $\hat{\zeta} - \hat{\xi}$. To do so, we introduce a new parameter $\theta \in [0, 1]$ and consider the mapping $\hat{\xi} + \theta(\hat{\zeta} - \hat{\xi})$. Therefore, the POD bases can be considered to be a function of θ . We start by noting that:

$$\mathbf{K}\mathbf{Z}_k = \lambda_k\mathbf{Z}_k, \quad (3.8)$$

where $\mathbf{K}$ is defined by (3.4), and $\{\lambda_k, \mathbf{Z}_k\}$ denote the eigenvalues and the corresponding normalized eigenvectors of $\mathbf{K}$. We assume that the entries of $\mathbf{K}$, $\mathbf{Z}$, and $\mathbf{\Lambda}$ are smooth functions of the parameter θ so that (3.8) can be differentiated with respect to θ . In what follows, partial derivative of any matrix or vector is denoted using the superscript (θ) . Therefore, by implicit differentiation of (3.8) with respect to θ we get:

$$(\mathbf{K} - \lambda_k\mathbf{I})\mathbf{Z}_k^{(\theta)} = -(\mathbf{K}^{(\theta)} - \lambda_k^{(\theta)}\mathbf{I})\mathbf{Z}_k. \quad (3.9)$$

Equation (3.9) has a solution only if the right-hand side vector belongs to the range of $\mathbf{K} - \lambda_k\mathbf{I}$ and thus must be orthogonal to $\ker(\mathbf{K} - \lambda_k\mathbf{I})$ which is spanned by $\mathbf{Z}_k$. Therefore,

we should have that

$$\mathcal{Z}_k^T (\mathbf{K}^{(\theta)} - \lambda_k^{(\theta)} \mathbf{I}) \mathcal{Z}_k = 0. \quad (3.10)$$

Since $\mathcal{Z}_k$ has unit norm, the sensitivity of the eigenvalues is obtained by (see e.g. [Lan64, FKS⁺68, MH88, SLO94])

$$\lambda_k^{(\theta)} = \mathcal{Z}_k^T \mathbf{K}^{(\theta)} \mathcal{Z}_k. \quad (3.11)$$

Note that $\mathbf{K} = \frac{1}{S} \hat{\mathbf{Y}}^T \hat{\mathbf{Y}}$ and $\hat{\mathbf{Y}} = \mathbf{L}^T \mathbf{Y}$. Therefore, $\mathbf{K}^{(\theta)} = \frac{1}{S} (\hat{\mathbf{Y}}^{(\theta)T} \hat{\mathbf{Y}} + \hat{\mathbf{Y}}^T \hat{\mathbf{Y}}^{(\theta)})$ and $\hat{\mathbf{Y}}^{(\theta)} = \mathbf{L}^T \mathbf{Y}^{(\theta)}$. Provided that we know what $\mathbf{Y}^{(\theta)}$ is, we can now fully characterize the solution $\mathcal{Z}_k^{(\theta)}$ of equation (3.9). We find a particular solution $\mathcal{S}_k$ of (3.9) in the least-square sense (obtaining the minimum norm solution). Since λ_k is simple, for all $\varrho \in \mathbb{R}$, $\mathcal{S}_k + \varrho \mathcal{Z}_k$ is the general expression for the solutions of (3.9). To determine the particular solution of (3.9) which corresponds to the sensitivity $\mathcal{Z}_k^{(\theta)}$ of $\mathcal{Z}_k$, we need an additional condition. This comes naturally from the normalization condition $\mathcal{Z}_k^T \mathcal{Z}_k = 1$ which was employed to specify $\mathcal{Z}_k$. Differentiating the normalization condition we get $\mathcal{Z}_k^T \mathcal{Z}_k^{(\theta)} = 0$ and consequently $\varrho = -\mathcal{Z}_k^T \mathcal{S}_k$. Finally,

$$\mathcal{Z}_k^{(\theta)} = \mathcal{S}_k - (\mathcal{Z}_k^T \mathcal{S}_k) \mathcal{Z}_k. \quad (3.12)$$

Once the sensitivity of matrices $\mathcal{Z}$ and $\mathbf{\Lambda}$ are determined, the sensitivity of POD basis modes $\boldsymbol{\psi}$ and POD bases ψ_k are straightforward to be computed by differentiating (3.7). More specifically,

$$\begin{aligned} \boldsymbol{\psi}^{(\theta)} &= \mathbf{Y}^{(\theta)} \mathcal{Z} (S \mathbf{\Lambda})^{-1/2} + \mathbf{Y} \mathcal{Z}^{(\theta)} (S \mathbf{\Lambda})^{-1/2} + \mathbf{Y} \mathcal{Z} S^{-1/2} (\mathbf{\Lambda}^{-1/2})^{(\theta)} \\ &= \mathbf{Y}^{(\theta)} \mathcal{Z} (S \mathbf{\Lambda})^{-1/2} + \mathbf{Y} \mathcal{Z}^{(\theta)} (S \mathbf{\Lambda})^{-1/2} - \frac{1}{2} \mathbf{Y} \mathcal{Z} S^{-1/2} (\mathbf{\Lambda}^{-1/2} \mathbf{\Lambda}^{(\theta)} \mathbf{\Lambda}^{-1}) \\ &= \mathbf{Y}^{(\theta)} \mathcal{Z} (S \mathbf{\Lambda})^{-1/2} + \mathbf{Y} \mathcal{Z}^{(\theta)} (S \mathbf{\Lambda})^{-1/2} - \frac{1}{2} \boldsymbol{\psi} \mathbf{\Lambda}^{(\theta)} \mathbf{\Lambda}^{-1}. \end{aligned} \quad (3.13)$$

To complete the sensitivity analysis of the POD bases, we still need to find the sensitivity of the snapshot data matrix $\mathbf{Y}^{(\theta)}$. This can be done by the sensitivity analysis of equation (3.2).

3.2.1 Sensitivity analysis of the Burgers equation

Let u be the solution of equation (3.2) and let $z = \partial_\theta u$ be the partial derivative of u with respect to the parameter θ . Taking the derivative of equation (3.2) with respect to θ , we get that z should satisfy

$$z_t + (zu)_x = \mu z_{xx} + \partial_\theta f_d(t, x), \quad (3.14)$$

where $\partial_\theta f_d(t, x) = \partial_\theta [\sigma(x) \sum_{k=1}^d (\xi_k + \theta(\zeta_k - \xi_k)) h_k(t)] = \sigma(x) \sum_{k=1}^d (\zeta_k - \xi_k) h_k(t)$, with zero boundary and initial conditions. Note that equation (3.14) is no longer non-linear and can be efficiently solved using a regular Finite Element method in a negligible amount of time. The finite element basis functions are assumed to be as before and we are seeking the solution in the space $W_N([0, 1]) = \text{span}\{\beta_j, j = 1, \dots, N\}$ defined earlier. Let $z(t, x) = \sum_{i=1}^N z_i(t) \beta_i(x)$ be the solution resulting from solving (3.14) with the Finite Element method. Then the sensitivity of the snapshot matrix $\mathbf{Y}$ is given by $\mathbf{Y}^{(\theta)} = (z_i(t_j))$, $i = 1, \dots, N$ and $j = 1, \dots, S$. This completes the sensitivity analysis of POD basis functions in the directions of $\hat{\boldsymbol{\zeta}} - \hat{\boldsymbol{\xi}}$.

3.2.2 Improving POD bases

Following [HBP09], we state two ideas for constructing improved reduced bases.

1. *Extrapolated basis:* Note that POD bases introduced in (3.5) are functions of $\hat{\boldsymbol{\xi}}$ and consequently functions of θ used in the transformation $\hat{\boldsymbol{\xi}} + \theta(\hat{\boldsymbol{\zeta}} - \hat{\boldsymbol{\xi}})$. Let us use $\psi_k(\theta)$ to emphasize this dependence. Note that when $\theta = 0$ we are considering the POD

basis functions at $\hat{\xi}$ and when $\theta = 1$ we are considering them at $\hat{\zeta}$. Now let us use

$$\psi_k(\theta) \simeq \psi_k(0) + \Delta\theta \frac{\partial \psi_k}{\partial \theta}(0) \quad (3.15)$$

to approximate $\psi_k(\theta)$. The capability of this extrapolation obviously depends on the assumption that POD modes behave nearly linear with respect to the parameter θ . However, using this method, the dimension of the reduced basis is preserved. Other approaches based on this idea can be found in [LLN⁺05, LFL06, MSN⁺07]. These papers motivate the extrapolation approach based on mode sensitivity, by showing an increase in robustness of the derived POD models with respect to parameter change.

2. *Expanded basis:* The sensitivity of the modes generally seem to span a different subspace than the POD modes. Therefore, it seems plausible to expect that if we seek the approximate solution in the space spanned by the union of these two sets, we can represent a broader range of solutions. Therefore, we use $\{\partial_\theta \psi_k\}$ to expand the original POD basis functions $\{\psi_k\}$. By a misuse of notation, we are still using

$$W_{N,M}([0, 1]) = \text{span}_{k=1, \dots, M} \{\psi_k, \partial_\theta \psi_k\}$$

to denote the space where we seek the solution of equation (1.11). The underlying assumption of this approach is that $W_{N,M}([0, 1])$ is well suited to address the change in the solution induced by a change in parameter. This indeed is a legitimate assumption since the sensitivities represent changes in the parameter space. However, the dimension of the reduced basis is now doubled.

3.3 Multi-fidelity Stochastic Collocation

We are finally in a position to demonstrate the multi-fidelity stochastic collocation method. Our aim to approximate the solution of (3.1) in the space $V_{N,\mathbf{p}} = L_2([0, T]; W_N([0, 1])) \otimes$

$\mathcal{P}_{\mathbf{p}}(\mathbb{R}^d)$, where $W_N([0, 1]) = \text{span}\{\beta_j\}$ is the finite element space, and $\mathcal{P}_{\mathbf{p}}(\mathbb{R}^d)$ is the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \dots, p_d)$. Choose $\eta > 0$ to be a small real number. The procedure for approximating the solution of (3.1) is divided into two parts:

1. Fix $\hat{\zeta} \in \mathbb{R}^d$, and search the η -neighbourhood $B_\eta(\hat{\zeta})$ of $\hat{\zeta}$. We use $B_\eta(\hat{\zeta}) = \{\hat{\xi} \in \mathbb{R}^d : |\xi_k - \zeta_k| < \eta, \forall k = 1, \dots, d\}$. If problem (3.2) is not already solved by the GFE method for any nearby problem with $\hat{\xi} \in B_\eta(\hat{\zeta})$, let $\hat{\xi} = \hat{\zeta}$ and solve problem (3.2) using the GFE method. In contrast, if equation (3.2) is already solved for some points in $B_\eta(\hat{\zeta})$, pick the closest one to $\hat{\zeta}$ and call it $\hat{\xi}$. In either case, use the solution at $\hat{\xi} \in B_\eta(\hat{\zeta})$ to find a small number of suitable basis functions resulting from local improvements to POD bases using sensitivity analysis. Let $W_{N,M}([0, 1]) \subset W_N([0, 1])$ be the span of these basis functions. Now use the Group POD method to solve problem (3.3) at $\hat{\zeta} \in \mathbb{R}^d$ and get the solution $u_d(\hat{\zeta}, t, x)$.
2. Collocate on zeros of suitable orthogonal polynomials and build the interpolated solution $u_{d,\mathbf{p}} \in V_{N,\mathbf{p}}$ using

$$u_{d,\mathbf{p}}(\hat{\zeta}, t, x) = \mathcal{I}_{\mathbf{p}} u_d(\hat{\zeta}, t, x) = \quad (3.16)$$

$$\sum_{j_1=1}^{p_1+1} \cdots \sum_{j_d=1}^{p_d+1} u_d(\zeta_1, \dots, \zeta_d, t, x) (l_{j_1}(\hat{\zeta}) \otimes \cdots \otimes l_{j_d}(\hat{\zeta})),$$

where the functions $\{l_{j_k}\}_{k=1}^d$ can be taken as Lagrange polynomials. Using this formula, as described in [BNT07], mean value and variance of u_d can also be easily approximated.

3.3.1 Generalization to Sparse Grids

Here, we give a short description of the isotropic Smolyak algorithm. More detailed information can be found in [BNR00,NTW08b]. Assume $p_1 = p_2 = \dots = p_d = p$. For $d = 1$, let $\{\mathcal{I}_{1,i}\}_{i=1,2,\dots}$ be a sequence of interpolation operators given by equation (3.16). Define $\Delta_0 = \mathcal{I}_{1,0} = 0$ and $\Delta_i = \mathcal{I}_{1,i} - \mathcal{I}_{1,i-1}$. Now for $d > 1$, let

$$\mathcal{A}(q, d) = \sum_{0 \leq i_1 + i_2 + \dots + i_d \leq q} \Delta_{i_1} \otimes \dots \otimes \Delta_{i_d} \quad (3.17)$$

where q is a non-negative integer. $\mathcal{A}(q, d)$ is the Smolyak operator, and q is known as the sparse grid *level*. Now instead of (3.16), $\mathcal{A}(q, d)u_d(\hat{\xi}, t, x)$ can be used to approximate the solution u_d of (3.1). This way one reduces the number of grid points on which the deterministic algorithms should be employed.

3.4 Numerical experiments

In this section, we consider equation (3.1) given once again below for conveniences.

$$u_t + \frac{1}{2}(u^2)_x = \mu u_{xx} + \sigma(x) \sum_{k=1}^d \xi_k h_k(t), \quad (3.18)$$

$(t, x) \in (0, T] \times [0, 1], u(0, x) = u_0(x), u(t, 0) = u(t, 1) = 0$, where $u_0 \in L_2([0, 1])$ is a deterministic initial condition. In the following numerical experiments, before employing the full-blown version of the multi-fidelity algorithm, we first consider the case when we are still utilizing a single point $\xi = \mathbf{0}$ (similar to Chapter 1) in the stochastic parameter domain to generate the POD bases, however, this time we are performing local improvements using the extrapolated basis technique. We let $T = 1.0, u_0(x) = (e^{\cos(5\pi x)} - \frac{3}{2}) \sin(\pi x), \sigma(x) = 0.01$ and $\mu = 1/200$. In Figure 3.1 we observe that the mere usage of improved basis functions without any increase in the number of times that the high fidelity algorithm is employed

fixes the issue brought up in Figure 1.2. Table 3.1 compares the corresponding errors of the two methods with the Monte Carlo method being the point of reference.

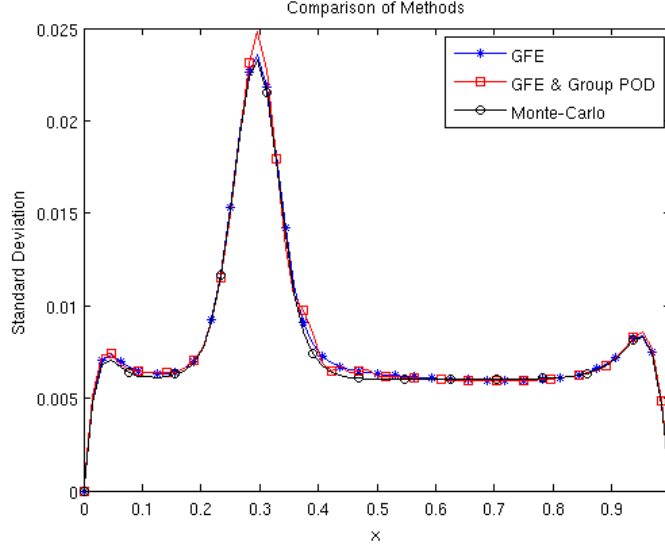


Figure 3.1: Standard Deviation of the solution given at final time $T = 1.0$ when $d = 2$ for sparse grid level 6.

Table 3.1: Comparison of the errors of the two methods.

$L_2([0, 1])$ Rel. Error	$E[u]$	$E[u^2]$	Std. Dev.
POD & Sparse Grid	0.0105	0.0208	0.0408
GFE & Sparse Grid	0.0024	0.0046	0.0248

For the rest of this section, unless otherwise stated, we assume that $\sigma(x) = 0.1 \cos(4\pi x)$ and $\mu = 1/100$. We let $T = 0.8$ and $u_0(x) = (e^{\cos(5\pi x)} - \frac{3}{2}) \sin(\pi x)$. We are still projecting the Brownian motion in $[0, T]$ on the trigonometric basis functions $h_k(t)$ in $L_2([0, T])$ given by

$$h_1(t) = \frac{1}{\sqrt{T}}, \quad h_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{(k-1)\pi t}{T}\right), \quad k \in \{2, 3, \dots\}. \quad (3.19)$$

Let us first study the effect of local improvements to POD basis functions. Choose $\hat{\xi}$ and $\hat{\zeta}$ to be two d dimensional vectors with Gaussian distributed randomly chosen entries. We again consider the transformation $\hat{\xi} + \theta(\hat{\zeta} - \hat{\xi})/||\hat{\zeta} - \hat{\xi}||$. We let θ to change in the interval $[-1/2, 1/2]$. We compute the POD basis functions at $\hat{\xi}$ and use them, along with their extrapolated and expanded local improvements, to solve the nearby problem at $\hat{\xi} + \theta(\hat{\zeta} - \hat{\xi})/||\hat{\zeta} - \hat{\xi}||$ using the Group POD method. We then compare the results with the ones from the GFE method. For the finite element code we partition the spatial domain $[0, 1]$ into 64 intervals. We also divide the time domain $[0, T]$ into 200 time steps. For the reduced order models, we use 10 POD basis functions which results in 10 extrapolated basis functions and 10+10 extended basis functions. We also consider the case when 20 POD basis functions and consequently 20 extrapolated bases are employed, for comparison. We let $d = 10$. Figure 3.2 shows the typical behaviour of Group POD method when

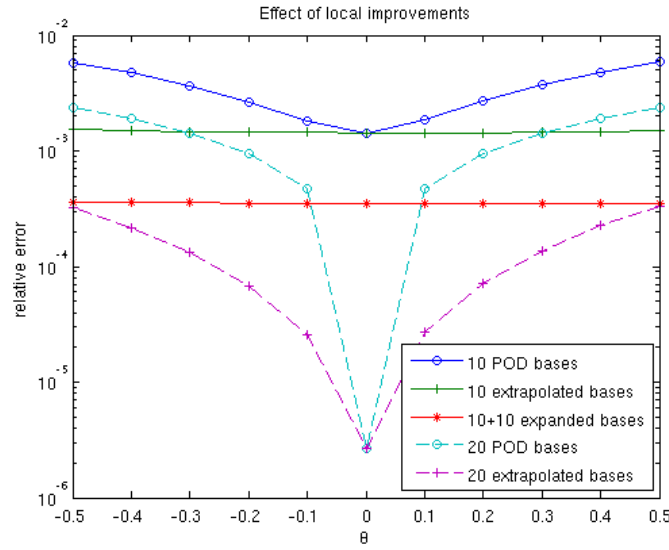


Figure 3.2: Comparison of the effects of extrapolated and expanded local improvements to POD basis functions

POD, Extrapolated, and Expanded basis functions are employed. The authors believe that the improvement in accuracy that is achieved by employing Expanded basis functions is not worth the increase in the number of bases. Therefore, in the multi-fidelity collocation method proposed earlier we use Extrapolated basis functions to locally improve the performance of Group POD method. Now we apply the multi-fidelity collocation method. Let us assume that $\eta = 1/2$ and $d = 3$. For the finite element code we partition the spatial domain $[0, 1]$ into 32 intervals. We also divide the time domain $[0, T]$ into 20 time steps. We use 10 extrapolated POD basis functions. We employ the Smolyak algorithm with sparse grid level $q = 8$. We use the Clenshaw-Curtis abscissas (see [CC60]) as collocation points. These abscissas are the extrema of Chebyshev polynomials. In Figure 3.3 we compare expectations and second moments of solutions resulting from our multi-fidelity method (GFE & Group POD) and the ones resulting from a regular sparse grid method with the full employment of GFE as the high fidelity algorithm. In this figure, we are also including the solutions coming out of the Monte-Carlo method with the full employment of the high fidelity algorithm for reference.

For this value of $\eta = 1/2$, the number of times that our algorithm calls the high fidelity (GFE) code is reduced to 2030. Compare it to 6018, the number of times that the GFE code is called when sparse grid stochastic collocation with full employment of the high-fidelity algorithm is utilized. Comparing the two methods, assuming that sparse grid stochastic collocation with full employment of the high-fidelity algorithm is accurate enough, we get that the error in expectation is given by 8.2×10^{-4} and the one in the second moment is 1.4×10^{-3} .

Figure 3.4, shows the convergence patterns of expectations and second moments of solutions with regard to η . We are in fact comparing our multi-fidelity method with a regular sparse grid stochastic collocation method. Note that for small enough η (less than the shortest distance between the collocation points) we get the regular sparse grid method back. Therefore the error is zero for such a small η .

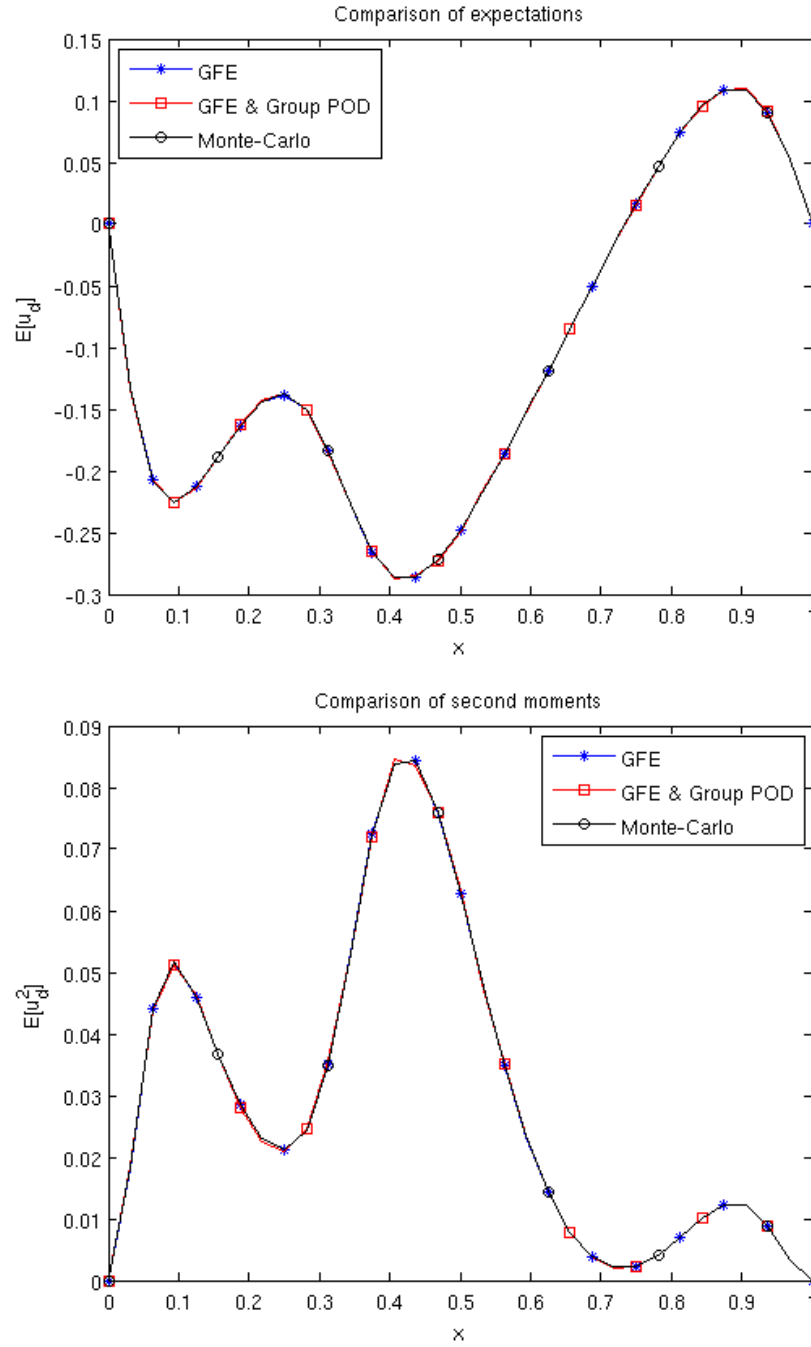


Figure 3.3: Expectation and Second Moment of the solution given at the final time $T = 0.8$ when $d = 3$, for sparse grid level 8 and $\eta = 0.5$.

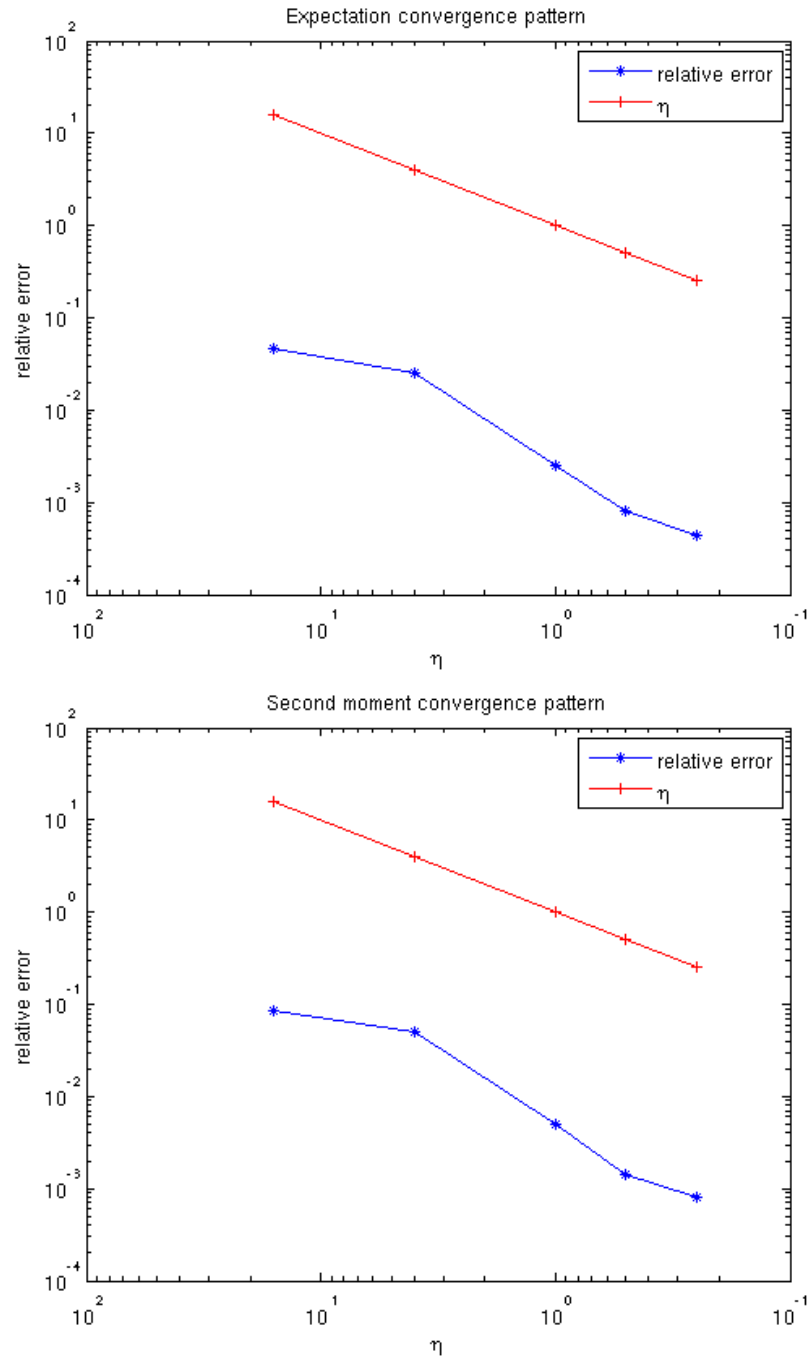


Figure 3.4: $L_2([0, 1])$ -norm convergence pattern of Expectation and Second Moment of the solution given at the final time $T = 0.8$ when $d = 3$, for sparse grid level 8.

Figure 3.5 demonstrates how the number of times that the finite element code is employed increases with respect to a decrease in η . Table 3.2 is just another way of presenting the data depicted in Figures 3.4 and 3.5.

Remark 3.1. *Note that this chapter closely follows our paper [RSb].*

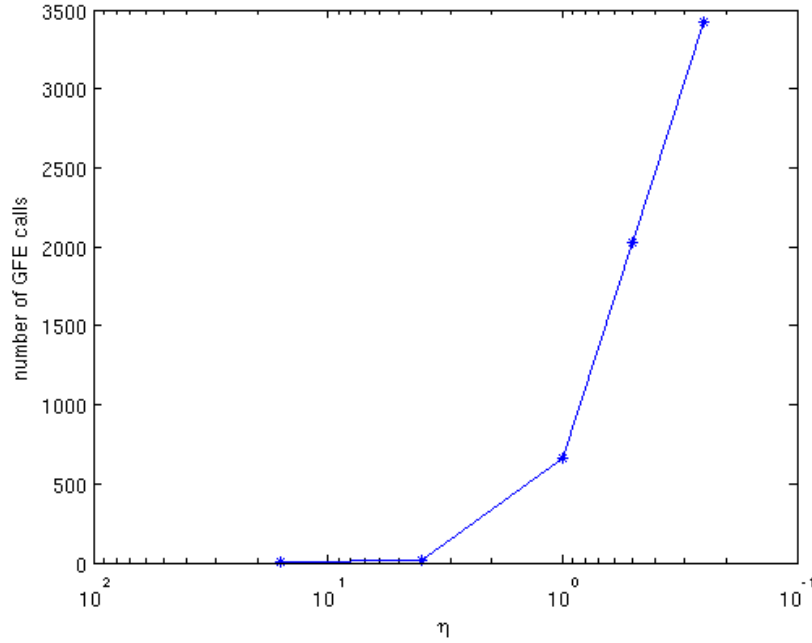


Figure 3.5: Number of times that the high fidelity algorithm (GFE method) is called as a function of η .

Table 3.2: Relative errors and the number of times that the finite element code is employed for different values of η .

η	# FE calls	Expectation L_2 error	Second Moment L_2 error
16	1	4.56E-02	8.60E-02
4	20	2.53E-02	4.94E-02
1	659	2.55E-03	4.95E-03
1/2	2030	8.16E-04	1.43E-03
(1/4)	3424	4.45E-04	8.05E-04

Conclusions

In this work, we have proposed a method to enhance the performance of stochastic collocation methods using proper orthogonal decomposition. We have carried out detailed error analyses of the proposed multi-fidelity stochastic collocation methods for parabolic partial differential equations with random forcing terms. We illustrated and supported our theoretical analyses with a numerical example. The analysis of this work can simply be generalized to parabolic partial differential equations with random initial conditions and random coefficients. Our method only requires a well-posedness argument of the corresponding deterministic equations. We have also chosen to apply our method to the stochastic Burgers equation because of its tractability and because lots of powerful model reduction techniques such as group POD existed for this equation. Our method enhances the power of stochastic collocation methods in handling more stochastic differential equations. Our work provides another reason for the importance of research in Model reduction techniques. We believe that more research can follow our work and apply our method to more interesting problems. Such problems can be in fluid dynamics and fluid structure interactions. Navier-Stokes equations under uncertainty can for instance be a reasonable next step.

References

- [AA09] N. Agarwal and NR Aluru. A domain adaptive stochastic collocation approach for analysis of mems under uncertainties. *Journal of Computational Physics*, 228(20):7662–7688, 2009.
- [ABK01] J.A. Atwell, J.T. Borggaard, and B.B. King. Reduced order controllers for burgers’ equation with a nonlinear observer. *Applied Mathematics And Computer Science*, 11(6):1311–1330, 2001.
- [AK01] J.A. Atwell and B.B. King. Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations. *Mathematical and computer modelling*, 33(1):1–19, 2001.
- [AK04] J.A. Atwell and B.B. King. Reduced order controllers for spatially distributed systems via proper orthogonal decomposition. *SIAM Journal on Scientific Computing*, 26(1):128–151, 2004.
- [BB97] Jeff Borggaard and John Burns. A pde sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, 136(2):366–384, 1997.
- [BNR00] Volker Barthelmann, Erich Novak, and Klaus Ritter. High dimensional polynomial interpolation on sparse grids. *Advances in Computational Mathematics*, 12(4):273–288, 2000.
- [BNT07] Ivo Babuška, Fabio Nobile, and Raúl Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 45(3):1005–1034, 2007.

- [BT73] A. Bensoussan and R. Temam. Equations stochastiques du type navier-stokes. *Journal of Functional Analysis*, 13(2):195–222, 1973.
- [BV97] H. Björnsson and SA Venegas. A manual for eof and svd analyses of climatic data. *CCGCR Report*, 97(1), 1997.
- [CAM⁺88] DH Chambers, RJ Adrian, P. Moin, DS Stewart, and HJ Sung. Karhunen–loève expansion of burgers model of turbulence. *Physics of Fluids*, 31(9):2573–2582, 1988.
- [Cam05] R.C. Camphouse. Boundary feedback control using proper orthogonal decomposition models. *Journal of guidance, control, and dynamics*, 28(5):931–938, 2005.
- [CC60] Charles W Clenshaw and Alan R Curtis. A method for numerical integration on an automatic computer. *Numerische Mathematik*, 2(1):197–205, 1960.
- [CC70] SC Crow and GH Canavan. Relationship between a wiener-hermite expansion and an energy cascade. *J. Fluid Mech*, 41(2):387–403, 1970.
- [CGH05] Q.Y. Chen, D. Gottlieb, and J.S. Hesthaven. Uncertainty analysis for the steady-state flows in a dual throat nozzle. *Journal of Computational Physics*, 204(1):378–398, 2005.
- [CGMSS81] I. Christie, D.F. Griffiths, A.R. Mitchell, and J.M. Sanz-Serna. Product approximation for non-linear problems in the finite element method. *IMA Journal of Numerical Analysis*, 1(3):253–266, 1981.
- [Cho71] A.J. Chorin. Hermite expansions in monte-carlo computation. *Journal of Computational Physics*, 8(3):472–482, 1971.
- [Cho74] A.J. Chorin. Gaussian fields and random flow. *Journal of Fluid Mechanics*, 63:21–32, 1974.

- [Cia78] Philippe G Ciarlet. *The finite element method for elliptic problems*, volume 4. North Holland, 1978.
- [CK07] C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of pdes in stochastic domains. *Numerische mathematik*, 107(2):257–293, 2007.
- [CLZ89] C.M. Chen, S. Larsson, and N.Y. Zhang. Error estimates of optimal order for finite element methods with interpolated coefficients for the nonlinear heat equation. *IMA journal of numerical analysis*, 9(4):507–524, 1989.
- [CM47] R.H. Cameron and W.T. Martin. The orthogonal development of non-linear functionals in series of fourier-hermite functionals. *The Annals of Mathematics*, 48(2):385–392, 1947.
- [CTMK93] Haecheon Choi, Roger Temam, Parviz Moin, and John Kim. Feedback control for unsteady flow and its application to the stochastic burgers equation. *Journal of Fluid Mechanics*, 253:509–543, 1993.
- [DD75] J. Douglas and T. Dupont. The effect of interpolating the coefficients in nonlinear parabolic galerkin procedures. *Mathematics of Computation*, 29:360–389, 1975.
- [DGRH07] A. Doostan, R.G. Ghanem, and J. Red-Horse. Stochastic model reduction for chaos representations. *Computer Methods in Applied Mechanics and Engineering*, 196(37):3951–3966, 2007.
- [DPD03] G. Da Prato and A. Debussche. Ergodicity for the 3d stochastic navier–stokes equations. *Journal de mathématiques pures et appliquées*, 82(8):877–947, 2003.
- [DPDT94] G. Da Prato, A. Debussche, and R. Temam. Stochastic burgers’ equation. *NoDEA: Nonlinear Differential Equations and Applications*, 1(4):389–402, 1994.

- [DS10] B.T. Dickinson and J.R. Singler. Nonlinear model reduction using group proper orthogonal decomposition. *Int. J. Numer. Anal. Model*, 7(2):356–372, 2010.
- [Eva] LC Evans. Partial differential equations. 2002. *American Mathematical Society, Providence, Rhode Island*.
- [Fah01] M. Fahl. Computation of pod basis functions for fluid flows with lanczos methods. *Mathematical and computer modelling*, 34(1):91–107, 2001.
- [FKS⁺68] RL Fox, MP Kapoor, Jong-Yeob Shin, Gary J Balas, and M Alpay Kaya. Rates of change of eigenvalues and eigenvectors. *AIAA journal*, 6(12):2426–2429, 1968.
- [Fle83] CAJ Fletcher. The group finite element formulation. *Computer Methods in Applied Mechanics and Engineering*, 37(2):225–244, 1983.
- [GD06] R.G. Ghanem and A. Doostan. On the construction and analysis of stochastic models: characterization and propagation of the errors associated with limited data. *Journal of Computational Physics*, 217(1):63–81, 2006.
- [GMPW05] R. Ghanem, S. Masri, M. Pellissetti, and R. Wolfe. Identification and prediction of stochastic dynamical systems in a polynomial chaos basis. *Computer methods in applied mechanics and engineering*, 194(12):1641–1654, 2005.
- [GS03] R.G. Ghanem and P.D. Spanos. *Stochastic finite elements: a spectral approach*. Dover publications, 2003.
- [GX08] D. Gottlieb and D. Xiu. Galerkin method for wave equations with uncertain coefficients. *Commun. Comput. Phys*, 3(2):505–518, 2008.
- [HBP09] Alexander Hay, Jeffrey T Borggaard, and Dominique Pelletier. Local improvements to reduced-order models using sensitivity analysis of the proper orthogonal decomposition. *Journal of Fluid Mechanics*, 629(1):41–72, 2009.

- [HKPS93] T. Hida, H.H. Kuo, J. Potthoff, and L. Streit. *White noise: an infinite dimensional calculus*, volume 253. Springer, 1993.
- [HLB98] P. Holmes, J.L. Lumley, and G. Berkooz. *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge University Press, 1998.
- [HLRZ06] T.Y. Hou, W. Luo, B. Rozovskii, and H.M. Zhou. Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics. *Journal of Computational Physics*, 216(2):687–706, 2006.
- [Hot33] H. Hotelling. Analysis of a complex of statistical variables into principal components. *Journal of educational psychology*, 24(6):417, 1933.
- [HY75] I Hosokawa and K Yamamoto. Turbulence in the randomly forced, one-dimensional burgers flow. *Journal of Statistical Physics*, 13(3):245–272, 1975.
- [HY02] T. Henri and J.P. Yvon. *Stability of the POD and convergence of the POD Galerkin method for parabolic problems*. Université de Rennes, 2002.
- [HY05] T. Henri and J.P. Yvon. Convergence estimates of pod-galerkin methods for parabolic problems. *System Modeling and Optimization*, pages 295–306, 2005.
- [ILD00] A. Iollo, S. Lanteri, and J.A. Désidéri. Stability properties of pod-galerkin approximations for the compressible navier–stokes equations. *Theoretical and Computational Fluid Dynamics*, 13(6):377–396, 2000.
- [Jen69] Dah-Teng Jeng. Forced model equation for turbulence. *Physics of Fluids*, 12:2006, 1969.
- [JSK02] M. Jardak, C.H. Su, and G.E. Karniadakis. Spectral polynomial chaos solutions of the stochastic advection equation. *Journal of Scientific Computing*, 17(1):319–338, 2002.

- [KBTB02] GM Kepler, HT Banks, HT Tran, and SC Beeler. Reduced order modeling and control of thin film growth in an hpcvd reactor. *SIAM Journal on Applied Mathematics*, 62(4):1251–1280, 2002.
- [Kli07] Andreas Klimke. Sparse Grid Interpolation Toolbox – user’s guide. Technical Report IANS report 2007/017, University of Stuttgart, 2007.
- [KLM06] OM Knio and OP Le Maitre. Uncertainty propagation in cfd using polynomial chaos decomposition. *Fluid Dynamics Research*, 38(9):616–640, 2006.
- [KMS⁺97] K. Khanin, A. Mazel, Y. Sinai, et al. Probability distribution functions for the random forced burgers equation. *Physical Review Letters*, 78:1904–1907, 1997.
- [KNG⁺01] O.M. Knio, H.N. Najm, R.G. Ghanem, et al. A stochastic projection method for fluid flow: I. basic formulation. *Journal of Computational Physics*, 173(2):481–511, 2001.
- [KPZ86] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889–892, 1986.
- [KV01] K Kunisch and S Volkwein. Galerkin proper orthogonal decomposition methods for parabolic problems. *Numerische Mathematik*, 90(1):117–148, 2001.
- [KV02] K Kunisch and S Volkwein. Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. *SIAM Journal on Numerical analysis*, 40(2):492–515, 2002.
- [KW05] Andreas Klimke and Barbara Wohlmuth. Algorithm 847: spinterp: Piecewise multilinear hierarchical sparse grid interpolation in MATLAB. *ACM Transactions on Mathematical Software*, 31(4), 2005.
- [Lan64] Peter Lancaster. On eigenvalues of matrices dependent on a parameter. *Numerische Mathematik*, 6(1):377–387, 1964.

- [LCSY09] ZhenDong Luo, Jing Chen, Ping Sun, and XiaoZhong Yang. Finite element formulation based on proper orthogonal decomposition for parabolic equations. *Science in China Series A: Mathematics*, 52(3):585–596, 2009.
- [LFL06] T Lieu, C Farhat, and M Lesoinne. Reduced-order fluid/structure modeling of a complete aircraft configuration. *Computer methods in applied mechanics and engineering*, 195(41):5730–5742, 2006.
- [LLN⁺05] OLIVER Lehmann, MARK Luchtenburg, BERND R Noack, RUDIBERT King, M Morzynski, and GILEAD Tadmor. Wake stabilization using pod galerkin models with interpolated modes. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 500–505. IEEE, 2005.
- [LMRN⁺02] O.P. Le Maitre, M.T. Reagan, H.N. Najm, R.G. Ghanem, and O.M. Knio. A stochastic projection method for fluid flow: Ii. random process. *Journal of computational Physics*, 181(1):9–44, 2002.
- [Loe77] Michel Loeve. Probability theory, volume i. *Graduate Texts in Mathematics*, Springer-Verlag, New York (fourth edition 1977), 1977.
- [LSK06] G. Lin, C.H. Su, and GE Karniadakis. Predicting shock dynamics in the presence of uncertainties. *Journal of Computational Physics*, 217(1):260–276, 2006.
- [LSK07] G. Lin, C.H. Su, and G.E. Karniadakis. Random roughness enhances lift in supersonic flow. *Physical review letters*, 99(10):104501, 2007.
- [LT05] CH Lee and HT Tran. Reduced-order-based feedback control of the kuramoto–sivashinsky equation. *Journal of computational and applied mathematics*, 173(1):1–19, 2005.

- [LTZW89] S. Larsson, V. Thomée, N.Y. Zhang, and W. Wendland. Interpolation of coefficients and transformation of the dependent variable in finite element methods for the non-linear heat equation. *Mathematical Methods in the Applied Sciences*, 11(1):105–124, 1989.
- [LWSK07] G. Lin, X. Wan, C.H. Su, and G.E. Karniadakis. Stochastic computational fluid mechanics. *Computing in Science & Engineering*, 9(2):21–29, 2007.
- [McK69] H.P. McKean. *Stochastic integrals*. American Mathematical Society, 1969.
- [MH88] Durbha V Murthy and Raphael T Haftka. Derivatives of eigenvalues and eigenvectors of a general complex matrix. *International Journal for Numerical Methods in Engineering*, 26(2):293–311, 1988.
- [MH03] L. Mathelin and M.Y. Hussaini. *A stochastic collocation algorithm for uncertainty analysis*. Citeseer, 2003.
- [MR04] R. Mikulevicius and BL Rozovskii. Stochastic navier–stokes equations for turbulent flows. *SIAM Journal on Mathematical Analysis*, 35(5):1250–1310, 2004.
- [MSN⁺07] Marek Morzyński, Witold Stankiewicz, Bernd R Noack, Rudibert King, Frank Thiele, and Gilead Tadmor. Continuous mode interpolation for control-oriented models of fluid flow. In *Active Flow Control*, pages 260–278. Springer, 2007.
- [Mur86] T. Murdoch. An error bound for the product approximation of nonlinear problems in the finite element method. numerical analysis group research report 86/17, 1986.
- [MZ09] Xiang Ma and Nicholas Zabaras. An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations. *Journal of Computational Physics*, 228(8):3084–3113, 2009.

- [NT09] Fabio Nobile and Raul Tempone. Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients. *International journal for numerical methods in engineering*, 80(6-7):979–1006, 2009.
- [NTW08a] Fabio Nobile, Raul Tempone, and Clayton G Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2411–2442, 2008.
- [NTW08b] Fabio Nobile, Raúl Tempone, and Clayton G Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2309–2345, 2008.
- [OB67] S.A. Orszag and LR Bissonnette. Dynamical properties of truncated wiener-hermite expansions. *Physics of Fluids*, 10:2603, 1967.
- [Pap71] G.C. Papanicolaou. Wave propagation in a one-dimensional random medium. *SIAM Journal on Applied Mathematics*, 21(1):13–18, 1971.
- [Pap95] G.C. Papanicolaou. Diffusion in random media. *Surveys in applied mathematics*, 1:205–253, 1995.
- [Pea01] K. Pearson. Liii. on lines and planes of closest fit to systems of points in space. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 2(11):559–572, 1901.
- [Rav00] SS Ravindran. A reduced-order approach for optimal control of fluids using proper orthogonal decomposition. *International Journal for Numerical Methods in Fluids*, 34(5):425–448, 2000.
- [RCM04] C.W. Rowley, T. Colonius, and R.M. Murray. Model reduction for compressible flows using pod and galerkin projection. *Physica D: Nonlinear Phenomena*, 189(1):115–129, 2004.

- [REWH08] TD Robinson, MS Eldred, KE Willcox, and R. Haimes. Surrogate-based optimization using multifidelity models with variable parameterization and corrected space mapping. *Aiaa Journal*, 46(11):2814–2822, 2008.
- [Roa98] P.J. Roache. *Verification and validation in computational science and engineering*. Hermosa Albuquerque, NM, 1998.
- [RSa] M Raissi and P Seshaiyer. A multi-fidelity stochastic collocation method for parabolic pdes with random input data. *In Review*.
- [RSb] M Raissi and P Seshaiyer. A multi-fidelity stochastic collocation method using locally improved reduced-order models. *In Review*.
- [SF92] K. Srinivas and C.A.J. Fletcher. *Computational techniques for fluid dynamics: a solutions manual*. Springer, 1992.
- [SG02] S. Sakamoto and R. Ghanem. Simulation of multi-dimensional non-gaussian non-stationary random fields. *Probabilistic Engineering Mechanics*, 17(2):167–176, 2002.
- [Sir87] Lawrence Sirovich. Turbulence and the dynamics of coherent structures. i-coherent structures. ii-symmetries and transformations. iii-dynamics and scaling. *Quarterly of applied mathematics*, 45:561–571, 1987.
- [SLO94] Alexander P Seyranian, Erik Lund, and Niels Olhoff. Multiple eigenvalues in structural optimization problems. *Structural Optimization*, 8(4):207–227, 1994.
- [SSA84] JM Sanz-Serna and L. Abia. Interpolation of the coefficients in nonlinear elliptic galerkin procedures. *SIAM journal on numerical analysis*, 21(1):77–83, 1984.
- [Tho97] Vidar Thomée. *Galerkin finite element methods for parabolic problems*, volume 25. Springer Verlag, 1997.

- [TOU90] Y. TOURIGNY. Product approximation for nonlinear klein-gordon equations. *IMA journal of Numerical Analysis*, 10(3):449–462, 1990.
- [TPPM97] M.A. Tatang, W. Pan, R.G. Prinn, and G.J. McRae. An efficient method for parametric uncertainty analysis of numerical geophysical models. *Journal of Geophysical Research*, 102(D18):21925–21, 1997.
- [TX06] D.M. Tartakovsky and D. Xiu. Stochastic analysis of transport in tubes with rough walls. *Journal of Computational Physics*, 217(1):248–259, 2006.
- [Wie38] N. Wiener. The homogeneous chaos. *Amer. J. Math*, 60(4):897–936, 1938.
- [XH05] D. Xiu and J.S. Hesthaven. High-order collocation methods for differential equations with random inputs. *SIAM Journal on Scientific Computing*, 27(3):1118–1139, 2005.
- [XK02] D. Xiu and G.E. Karniadakis. The wiener–askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2):619–644, 2002.
- [XK03] D. Xiu and G.E. Karniadakis. Modeling uncertainty in flow simulations via generalized polynomial chaos. *Journal of Computational Physics*, 187(1):137–167, 2003.
- [XK04] D. Xiu and G.E. Karniadakis. Supersensitivity due to uncertain boundary conditions. *International journal for numerical methods in engineering*, 61(12):2114–2138, 2004.
- [XLSK02] D. Xiu, D. Lucor, C.H. Su, and G.E. Karniadakis. Stochastic modeling of flow-structure interactions using generalized polynomial chaos. *Journal of Fluids Engineering*, 124:51, 2002.

- [XT06] D. Xiu and D.M. Tartakovsky. Numerical methods for differential equations in random domains. *SIAM Journal on Scientific Computing*, 28(3):1167–1185, 2006.
- [ZG12] Guannan Zhang and Max Gunzburger. Error analysis of a stochastic collocation method for parabolic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 50(4):1922–1940, 2012.
- [ZL04] D. Zhang and Z. Lu. An efficient, high-order perturbation approach for flow in random porous media via karhunen–loeve and polynomial expansions. *Journal of Computational Physics*, 194(2):773–794, 2004.

Biography

Maziar Raissi grew up in Isfahan, a large, beautiful, and historical city at the heart of Iran. He attended University of Isfahan, where he received his Bachelor of Science in Applied Mathematics in 2008. He went on to receive his Master of Science in Applied Mathematics from Isfahan University of Technology in 2011. The same year, he joined the Doctor of Philosophy program in Mathematics at George Mason University.