HELLY-TYPE THEOREMS ON SUPPORT LINES FOR FAMILIES OF CONGRUENT DISKS IN THE PLANE

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Abstract

HELLY-TYPE THEOREMS ON SUPPORT LINES FOR FAMILIES OF CONGRUENT DISKS IN THE PLANE

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In this dissertation, we consider the problem to determine Helly-type numbers for support lines of nonoverlapping families of congruent disks in the plane. This problem, originally posed by R. Dawson for the case of disjoint families of convex bodies and by V. Soltan for the case of disjoint families of unit disks, has been recently solved. This research generalizes to the case of non-overlapping families of congruent disks. An essential part of the argument is based on the study of "critical" families of congruent disks.

Chapter 1: Introduction and Main Results

1.1 Helly-Type Theorems in n Dimensions

In 1913, Edward Helly discovered the following result.

Theorem 1.1. *Suppose $\mathcal{K}$ is a family of at least $n+1$ convex sets in the n-dimensional vector space $\mathbb{R}^n$ such that $\mathcal{K}$ is finite or each member of $\mathcal{K}$ is compact. If each $n+1$ members of $\mathcal{K}$ have a common point, then all members of $\mathcal{K}$ share a point p in common.*

That same year, Edward Helly communicated this result to his colleague, Johann Radon, who published a proof of it in 1921.

Since then, various results of a similar spirit have been discovered. Nowadays, these related ideas and results form a well-established field of combinatorial geometry, called Helly-type theorems. Detailed descriptions of results in this field are given in the classical surveys of Danzer, Grünbaum, and Klee [5] and of Eckhoff [8]. Various books and proceedings contain individual chapters dedicated to Helly-type results (see, e.g. [3] and [9]).

Helly's theorem can be generalized in a variety of directions by reinterpreting the point p of Helly's theorem as a set of various kinds or by relaxing the condition that the members of $\mathcal{K}$ be compact. We list below some notable generalizations of Helly's theorem. Everywhere below, $\mathcal{K}$ means a finite family of convex sets in $\mathbb{R}^n$.

The first generalization is a theorem of Grünbaum [11] which provides conditions guaranteeing the existence of a j-dimensional convex set contained in each member of $\mathcal{K}$.

Theorem 1.2. *Let $g(n,0) = n+1$, $g(n,1) = 2n$, $g(n,j) = 2n-j$ for $1 < j < n$, and let $g(n,n) = n+1$. If $\mathcal{K}$ is a finite family of at least $g(n,j)$ convex sets in $\mathbb{R}^n$ and each* subfamily of $g(n,j)$ members of $\mathcal{K}$ has an intersection of dimension no less than j, then the intersection $\bigcap \mathcal{K}$ is at least j-dimensional.

Observe that in the plane with $j = 0$, we obtain the original statement of Helly's theorem. Similar observations apply to the subsequent generalizations.

Another generalization, due to De Santis [7], relaxes the condition that the members of $\mathcal{K}$ be compact, and focuses on convex subsets of $\mathbb{R}^n$ whose intersection contains a plane of specified dimension.

Theorem 1.3. If $\mathcal{K}$ is a finite family of at least $n+1-j$ convex subsets of $\mathbb{R}^n$ and the intersection of each $n+1-j$ members of $\mathcal{K}$ contains a j-dimensional plane, then $\bigcap \mathcal{K}$ contains a j-dimensional plane.

We will say that a plane $L \subset \mathbb{R}^n$ of dimension m is an *m-transversal* of a given family $\mathcal{K}$ of convex sets in $\mathbb{R}^n$ if L meets every member of $\mathcal{K}$. A generalization due to Horn [15] and Klee [16] provides conditions which guarantee that a j-dimensional plane is a transversal of $\mathcal{K}$.

Theorem 1.4. For integers $1 \leq j \leq n+1$ and a family $\mathcal{K}$ of at least j compact convex sets in $\mathbb{R}^n$, the following statements are equivalent:

(a) each j members of $\mathcal{K}$ have a common point;

(b) every plane of deficiency $j-1$ in $\mathbb{R}^n$ admits a translate which intersects each member of $\mathcal{K}$;

(c) every plane of deficiency j in $\mathbb{R}^n$ lies in a plane of deficiency $j-1$ which intersects each member of $\mathcal{K}$.

Another result of a similar spirit is due to Santaló [22].

Theorem 1.5. If $\mathcal{P}$ is a family of parallelotopes in $\mathbb{R}^n$ with edges parallel to the coordinate axes, and an $(n-1)$-transversal is admitted by each subfamily $\mathcal{Q} \subset \mathcal{P}$ of at most $2^{n-1}(n+1)$ members, then $\mathcal{P}$ itself admits an $(n-1)$-transversal.

The following theorem of Bohnenblust-Karlin-Shapley [2] has its roots in game theory and represents an important application of Helly's theorem.

Theorem 1.6. Suppose C is a compact convex set in $\mathbb{R}^n$ and Φ is a finite family of continuous convex functions on C such that for each $x \in C$ there exists $\phi \in \Phi$ with $\phi(x) > 0$. Then there are positive numbers $\alpha_0, \ldots, \alpha_j$ with $j \leq n$, and members $\phi_0, \ldots, \phi_j$ of Φ such that $\sum_0^j \alpha_i \phi_i(x) > 0$ for all $x \in C$.

1.2 Transversal and Support Lines in the Plane

An actively developing subfield of Helly-type theorems is devoted to *transversal lines* of convex sets in the plane. We will say that a family $\mathcal{K}$ of convex sets in the plane has the (transversal) property T provided a line meets every member of $\mathcal{K}$. Similarly, $\mathcal{K}$ has the (transversal) property $T(n)$ provided every subfamily of n members from $\mathcal{K}$ admits a common transversal line.

The geometric nature of Helly-type theorems on transversal lines is rather complex even for the case of disjoint families of compact convex sets in the plane. The book of Hadwiger and Debrunner [12] and the survey of Eckhoff [8] describe various statements and counter-examples on the existence of line transversals for such families. In particular, arbitrarily large families of pairwise disjoint convex bodies in the plane exist that have the property $T(5)$ but not T.

Danzer [4] proved that $T(5) \implies T$ for any disjoint family of congruent disks in the plane, and Grünbaum [10] claimed that $T(4) \implies T$ for any disjoint family $\mathcal{F}$ of at least six congruent disks. It turns out, however, that Grünbaum's assertion is incorrect. As shown in Aronov et al. [1], arbitrarily large disjoint families $\mathcal{F}$ of unit disks exist that have the property $T(4)$ but not T.

A special class of transversals, namely *support lines*, was introduced by Dawson [6]. Given a family $\mathcal{K}$ of convex bodies in the plane, Dawson says that $\mathcal{K}$ has the (support) property S provided a line supports every member of $\mathcal{K}$ (compare Figure 1.1). Similarly,

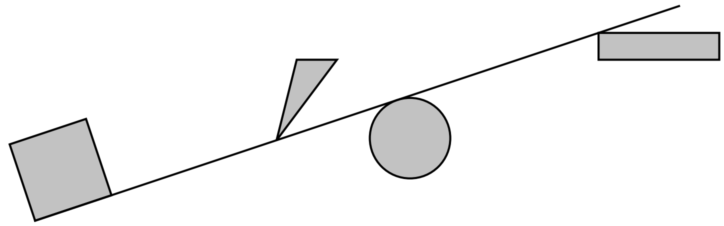


Figure 1.1: Support line for a family of convex bodies in the plane.

the family $\mathcal{K}$ has the (support) property $S(n)$ if every subfamily of size n has property S. Dawson [6] proved a number of assertions which are summarized in the following theorem.

Theorem 1.7. For a finite disjoint family $\mathcal{K}$ of convex bodies in the plane, one has

$$S(5) \implies S, \quad S(4) \implies S \text{ if } \operatorname{card}\mathcal{K} \geq 7, \quad S(3) \implies S \text{ if } \operatorname{card}\mathcal{K} \geq 237,$$

where $\operatorname{card}\mathcal{K}$ stands for the cardinality of $\mathcal{K}$.

A primary method of proof in Dawson [6] is a combinatorial approach exemplified in his proof of the statement $S(4) \implies S$ if $\operatorname{card}\mathcal{K} \geq 7$ of Theorem 1.7. To convert this problem to a combinatorial setting, a set of combinatorial objects (symbols) is chosen, each symbol standing for a convex body. Next, the relevant geometric constraints on the bodies in the plane are translated into incidence properties on these symbols in a one-to-one correspondence. Three incidence properties are relevant for this proof, and we state them together with their corresponding geometric interpretation, given parenthetically, in the following: (i) No pair of symbols is contained in more than four words. (Two disjoint bodies have precisely four support lines.) (ii) No triple of symbols is contained in more than three words. (Three disjoint bodies have at most three support lines.) (iii) Every quadruple of symbols is contained in at least one word. (This corresponds to the property $S(4)$ for the family.)

Since the theorem specifies a threshold of seven bodies, it suffices to designate a ground set of seven symbols $\{A, B, \ldots, G\}$. The author exhaustively denumerates all possible sets of words, each respective set called a *code*, that satisfies the given incidence properties. Three combinatorial configurations are possible, two of which avoid the word $ABCDEFG$ which corresponds to the property S for the seven bodies. Each respective incidence structure is then translated back into the geometric setting where the specific support relations implicit in the incidence relations represented by each code are expressed. It is then demonstrated that the codes avoiding the property S are not geometrically realizable by showing that their respective support relations result in impossible configurations.

Revenko and Soltan [18] improved the last assertion of Theorem 1.7 by proving that $S(3) \implies S$ for any disjoint (possibly infinite) family $\mathcal{K}$ of convex bodies, with card $\mathcal{K} \geq 143$. Figure 1.2, reproduced from [18], shows a disjoint family of 16 convex bodies with property $S(3)$ but not S. This construction together with the above statement places the exact *threshold number*, the minimum required cardinality that guarantees the family satisfies the property, between 17 and 143, inclusive. The following problem, formulated in [18], is still open.

Problem 1.8. Find the smallest value of the natural number n such that $S(3) \implies S$ for any disjoint family of n or more convex bodies in the plane.

Revenko and Soltan [19] (see also [20]) relaxed the disjointness condition and examined Helly-type theorems for the case of families of pairwise nonoverlapping convex bodies in the plane. First, they observed that $S(n)$ does not imply S for any integer $n \geq 1$, as illustrated by the construction depicted in Figure 1.3. Next, they proved that if a nonoverlapping family of convex bodies has no point in common, then $S(6) \implies S$. Figure 1.4 below shows that, generally, $S(5)$ does not imply S.

Various generalizations of these results to the case of k-disjoint families of convex bodies in the plane can be found in the paper of Revenko and Soltan [21].

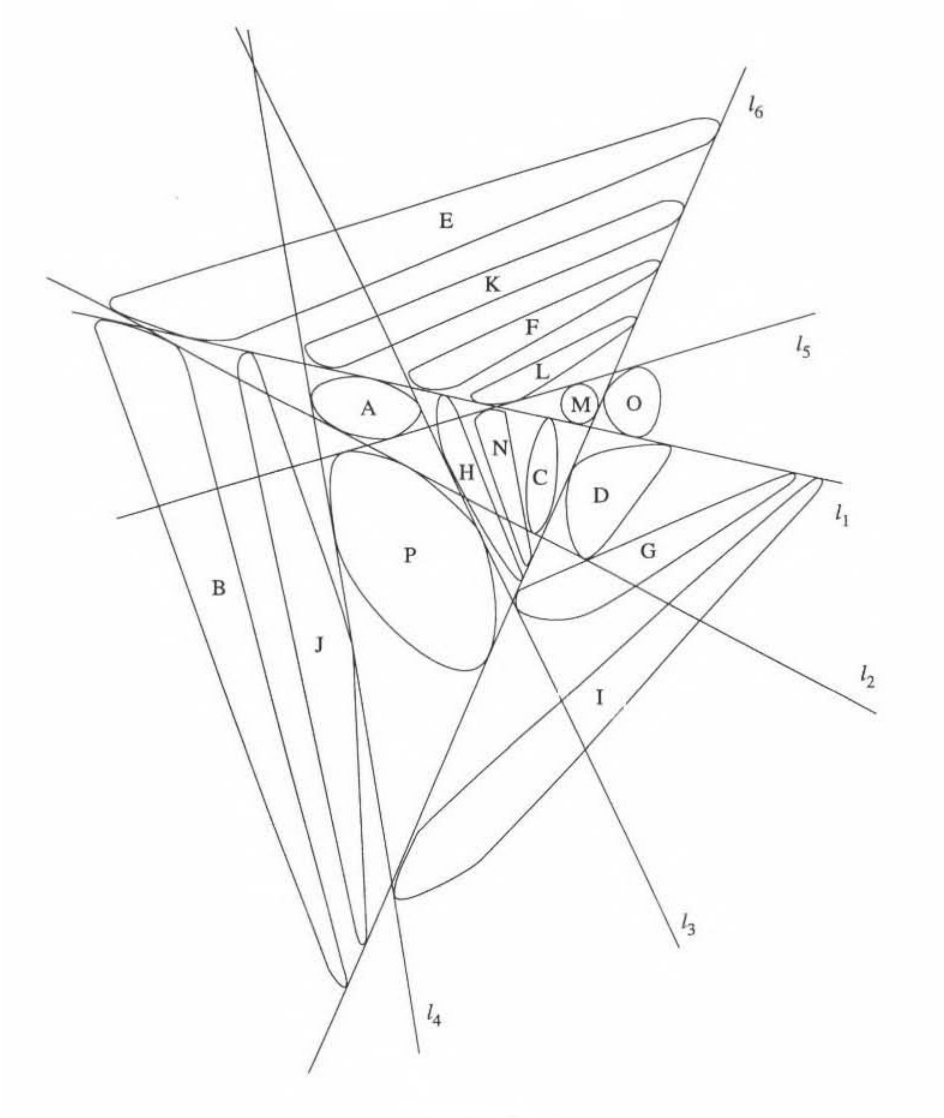


Figure 1.2: Sixteen convex bodies with the property $S(3)$ but not S.

1.3 Main Results of the Dissertation

In view of the rather challenging combinatorial nature of Problem 1.8, V. Soltan [23, 24] studied a similar problem for the case of congruent disks in the plane, the results of which are summarized together in the following theorem.

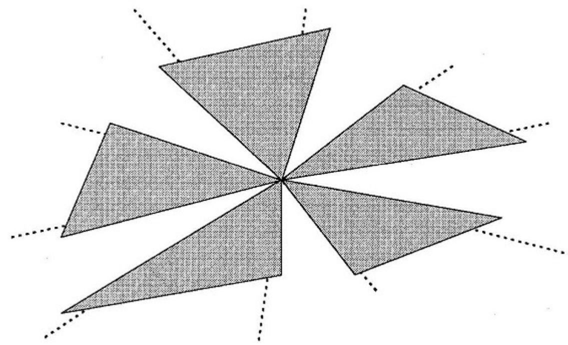


Figure 1.3: Nonoverlapping triangles, with the property $S(n)$ but not S.

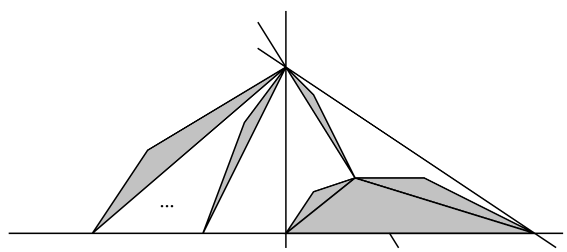


Figure 1.4: Nonoverlapping triangles with the property $S(5)$ but not S.

Theorem 1.9. If $\mathcal{F}$ is a disjoint family (possibly infinite) of unit disks in the plane, then $S(4) \implies S$. Furthermore, $S(3) \implies S$ provided $\operatorname{card} \mathcal{F} \geq 7$.

Figure 1.5 below reproduced from [24] shows that, generally, for disjoint families of six congruent disks, the property $S(3)$ does not imply S.

The results detailed in Section 1.2 above illustrate a number of the essential respective differences in establishing various Helly-type results for disjoint versus for nonoverlapping families of convex bodies. These differences motivated the study of the following problem.

Problem 1.10. Let $\mathcal{F}$ be a nonoverlapping family of congruent disks in the plane.

1. Determine the conditions, including the threshold number, which guarantee the existence of a natural number k satisfying the implication $S(k) \implies S$.

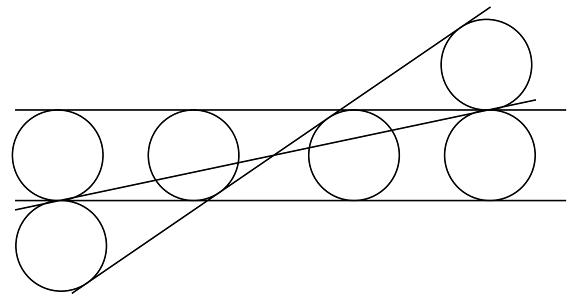


Figure 1.5: Disjoint family of six unit disks with the property $S(3)$ but not S.

2. Find the respective minimum values for k satisfying each set of relevant conditions in the preceding statement.

This problem is completely solved in the present dissertation. The methods used here are primarily geometric and constructive in contrast to the combinatorial methods used in Dawson [6] and in Revenko and Soltan [18]. The Helly-type results obtained here derive from the exhaustive study of an express series of well-posed combinatorial problems of a geometric nature. These problems are studied and solved in order of the steps described below.

We need some terminology for their description. We will say that a line ℓ is a *common support* for the family $\mathcal{F}$ provided every disk from $\mathcal{F}$ is supported by ℓ. If the family $\mathcal{F}$ has precisely n members, then we will denote it by $\mathcal{F}_n$.

<u>STEP 1.</u> Given an arbitrary family $\mathcal{F}_n$ of n congruent disks in the plane, we establish bounds on the possible values of the number $s(\mathcal{F}_n)$ of common support lines for $\mathcal{F}_n$ (see Chapter 2). We prove the inequalities

$$3 \leq s(\mathcal{F}_2) \leq 4, \quad s(\mathcal{F}_3) \leq 3, \quad s(\mathcal{F}_n) \leq 2 \text{ if } n \geq 4$$

,

and describe all combinatorial types of the families $\mathcal{F}_n$ for all maximum values of $s(\mathcal{F}_n)$.

STEP 2. Utilizing the above bounds on the numbers $s(\mathcal{F}_n)$ of common support lines, we prove our first Helly-type theorem regarding any nonoverlapping family $\mathcal{F}$ of congruent disks in the plane: $S(4) \implies S$ (see Theorem 2.15).

STEP 3. In Chapter 3, we describe all combinatorially distinct configurations of touching critical families $\mathcal{F}_4$. This description is given separately for the following two cases:

1) No three disks from $\mathcal{F}_4$ have their centers on a line (see Section 3.1).

2) The centers of three disks in $\mathcal{F}_4$ belong to a line (see Section 3.2).

STEP 4. Based on the description of all touching critical families $\mathcal{F}_4$, we prove in Chapter 4 that any touching critical family $\mathcal{F}$ contains at most seven disks (see Theorem 4.6, Lemma 4.20, and Figure 4.7).

STEP 5. Our second Helly-type theorem regarding any nonoverlapping family $\mathcal{F}$ of congruent disks in the plane follows from Theorem 4.6: $S(3) \implies S$ provided $\operatorname{card} \mathcal{F} \geq 8$ (see Theorem 4.25).

Chapter 2: Common Support Lines for Finite Families of Congruent Disks

2.1 Preliminaries

In this chapter we determine the number of common support lines for an arbitrary family $\mathcal{F}_n$ ($n \geq 2$) of congruent disks of radius $r > 0$ in the plane. We also characterize those families $\mathcal{F}_n$ which allow the maximum possible number of such lines.

In what follows, we denote the members of $\mathcal{F}_n$ by the labels $C_1, C_2, \ldots, C_n$. With this notation, we will assume that the disks C_1 and C_2 have their centers o_1 and o_2 on the x-axis of the plane with respective coordinates $(-\delta, 0)$ and $(\delta, 0)$, so that 2δ is the distance between their centers (see Figure 2.1).

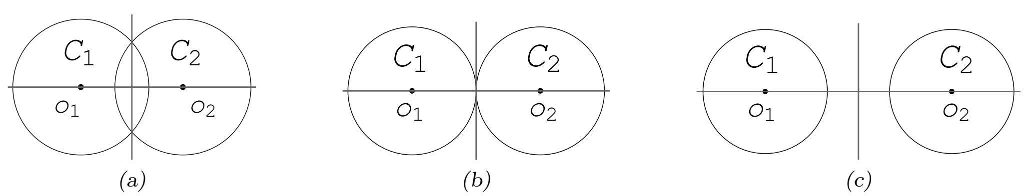


Figure 2.1: Disks centered at $o_1\,(-\delta, 0)\,, o_2\,(\delta, 0)$ either overlap ($\delta < r$), touch ($\delta = r$), or are disjoint ($\delta > r$).

Remark 2.1. Observe that each family $\{C_1, C_2\}$ together with the x- and y-axes has the following four symmetries: reflection over the y-axis, reflection over the x-axis, rotation of 180° about the origin, and identity symmetry. These symmetries form a group under composition and correspond to the group of symmetries of the rectangle which we know from elementary group theory coincides with the Klein four-group V (Kleinsche Vierergruppe).

We use the following terminology and notation.

Definition 2.2. A family $\mathcal{F}$ of congruent disks of positive radius r in the plane is called

1) disjoint if every pair of disks taken from $\mathcal{F}$ is disjoint,

2) nonoverlapping if every pair of disks taken from $\mathcal{F}$ does not overlap,

3) touching if it is nonoverlapping and a pair of disks in $\mathcal{F}$ touch,

4) overlapping if a pair of disks in $\mathcal{F}$ overlap.

The line containing points a and b is denoted $\langle a, b\rangle$. The closed line segment with endpoints a and b is denoted either ab or $[a, b]$. The open line segment with endpoints a and b is denoted (a, b). The length of a line segment $[a, b]$ is denoted $\|a - b\|$. A triangle with vertices at points a, b, c in the plane is denoted $\triangle abc$. The acute angle formed by lines ℓ and ℓ' is denoted $\left(\widehat{\ell, \ell'}\right)$.

2.2 Common Support Lines for a Family $\mathcal{F}_2$

Theorem 2.3. For a family $\mathcal{F}_2 = \{C_1, C_2\}$ of congruent disks of positive radius r in the plane, the following statements hold.

(a) If $\mathcal{F}_2$ is overlapping, then it has precisely two support lines. These are the horizontal lines $\{y = \pm r\}$ (see Figure 2.2).

(b) If $\mathcal{F}_2$ is touching, then it has precisely three support lines. Two of these are the horizontal lines $\{y = \pm r\}$, and one is the vertical line $\{x = 0\}$ (see Figure 2.3).

(c) If $\mathcal{F}_2$ is disjoint, then it has precisely four support lines. Two of these are the horizontal lines $\{y = \pm r\}$, and two are the (slant) lines $\left\{y = \dfrac{\pm r}{\sqrt{\delta^2 - r^2}}\,x\right\}$ (see Figure 2.4).

Proof of Theorem 2.3

Part (a). Observe that the origin is interior to both disks (see Figure 2.2).

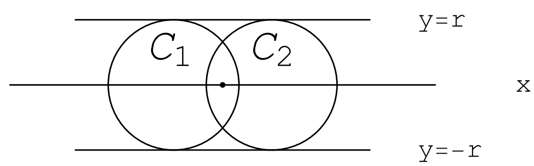


Figure 2.2: Overlapping family $\mathcal{F}_2$ and its two support lines.

Part (b). If the family $\mathcal{F}_2$ is touching, then the respective centers of C_1 and C_2 are $o_1\,(-r, 0)$ and $o_2\,(r, 0)$ and the disks share a common boundary point at the origin $o\,(0, 0)$. The vertical line ℓ_v through the origin is the only line that separates the disks since it is the unique tangent line for each disk at their common boundary point. The remaining supports are the horizontal lines $\{y = \pm r\}$ (see Figure 2.3).

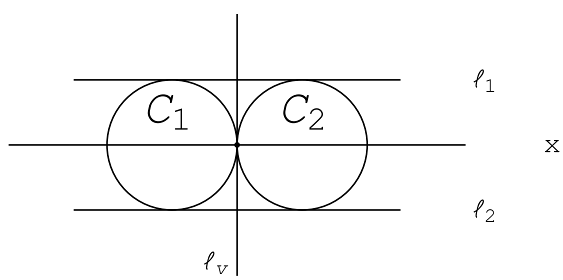


Figure 2.3: Touching family $\mathcal{F}_2$ and its three support lines.

Remark 2.4. This is a continuation of Remark 2.1. We observe that the collection consisting of the touching family $\mathcal{F}_2 = \{C_1, C_2\}$ together with its support lines ℓ_1, ℓ_2, ℓ_v and the x-axis has a set of symmetries whose structure under composition coincides with the Klein four-group V.

We will need the following basic lemma from plane analytic geometry.

Lemma 2.5. The distance from a line ℓ in the coordinate (x, y)-plane, given by the equation $Ax + By + C = 0$, to a point $p\,(x_1, y_1)$ equals

$$\mathrm{d}\ (\ell, p) = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

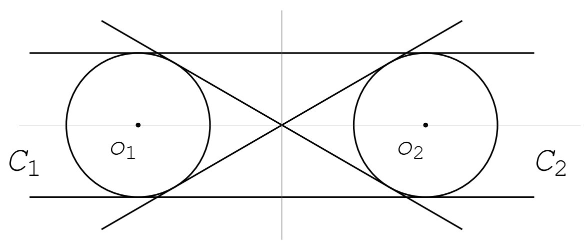


Figure 2.4: Disjoint family $\mathcal{F}_2$ and its four support lines.

Part (c). If C_1 and C_2 are disjoint, then $2\delta > 2r$. Observe that any common support line of the two disks is either horizontal, or is a slant line that crosses the x-axis in the origin o and separates the disks. The horizontal lines $\{y = \pm r\}$ support $\mathcal{F}_2$ (compare Figure 2.4). In the following, we refer to Figure 2.5 repeatedly. Consider the segment op from the origin o to the boundary ∂C_2 of C_2 in the first quadrant, which belongs to the slant line of positive slope supporting both disks. This segment forms the hypotenuse of a right triangle with legs of length x_0 and y_0 as depicted in Figure 2.5. This segment is also a leg in the right triangle with hypotenuse oo_2 of length δ whose second leg po_2 has length r so that $\|p - o\| = \sqrt{\delta^2 - r^2}$ and $x_0^2 + y_0^2 = \delta^2 - r^2$ (see Figure 2.5). In particular, the support line ℓ with positive slope $k = y_0/x_0$ has standard form $y_0x - x_0y = 0$, and supports the disk C_2 at the point $p\,(x_0, y_0)$ at a distance r to its center o_2. Applying Lemma 2.5, we have

$$\mathrm{d}\;(\ell, o_2) = \frac{|y_0\,\delta + (-x_0)\cdot 0|}{\sqrt{x_0^2 + y_0^2}} = \frac{|y_0\,\delta|}{\sqrt{\delta^2 - r^2}} = r,$$

which yields

$$(y_0\,\delta)^2 = r^2\left(\delta^2 - r^2\right) \iff y_0 = \frac{r}{\delta}\sqrt{\delta^2 - r^2}.$$

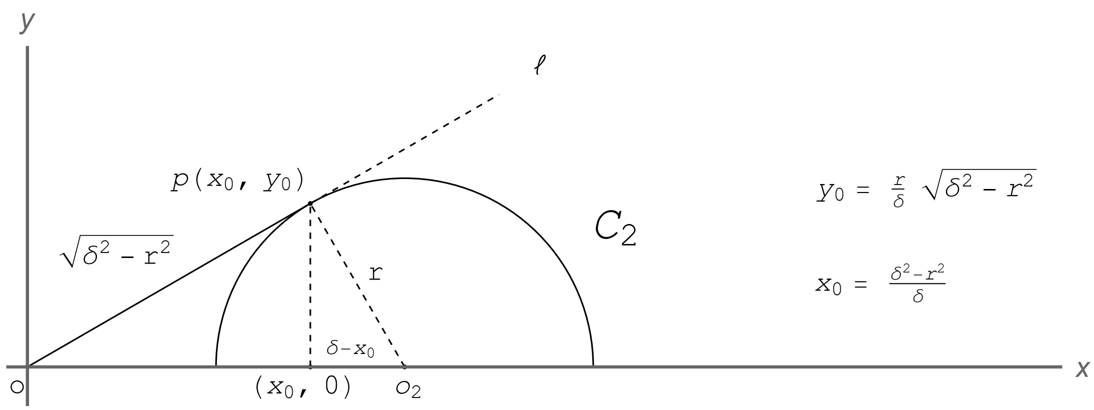


Figure 2.5: Slant line ℓ through the origin o supporting the family $\mathcal{F}_2$.

The Pythagorean relation $(\delta - x_0)^2 + y_0^2 = r^2$, observed in Figure 2.5, together with the value for y_0, yields the identity

$$\delta - x_0 = \sqrt{r^2 - y_0^2} = \sqrt{r^2 - \left(\frac{r}{\delta}\sqrt{\delta^2 - r^2}\right)^2} = \sqrt{\frac{r^2\delta^2 - r^2\left(\delta^2 - r^2\right)}{\delta^2}} = \sqrt{\frac{r^4}{\delta^2}} = \frac{r^2}{\delta}.$$

This in turn determines

$$x_0 = \delta - \frac{r^2}{\delta} = \frac{\delta^2 - r^2}{\delta}.$$

The positive slope k of the slant line supporting $\mathcal{F}_2$ is given by the ratio

$$k = \frac{y_0}{x_0} = \frac{\dfrac{r\sqrt{\delta^2 - r^2}}{\delta}}{\dfrac{\delta^2 - r^2}{\delta}} = \frac{r\sqrt{\delta^2 - r^2}}{\delta} \cdot \frac{\delta}{\delta^2 - r^2} = r\sqrt{\frac{\delta^2 - r^2}{(\delta^2 - r^2)^2}} = \frac{r}{\sqrt{\delta^2 - r^2}}.$$

By symmetry, the slant line with negative slope supporting C_1 and C_2 has the equation $y = -kx$, where k is given above. $\square$

2.3 Common Support Lines for a Family $\mathcal{F}_3$

Let $\mathcal{F}_3 = \{C_1, C_2, C_3\}$ be a family of pairwise distinct congruent disks of positive radius r. As above, we parameterize the disks C_1, C_2 by their respective centers $o_1(-\delta, 0)$, $o_2(\delta, 0)$.

Theorem 2.6. For a family $\mathcal{F}_3 = \{C_1, C_2, C_3\}$ of congruent disks of radius r in the plane, the following statements hold.

(a) If $\mathcal{F}_3$ is overlapping, then it has at most two support lines. $\mathcal{F}_3$ has precisely two support lines if and only if it lies in the slab between two parallel support lines (see Figure 2.6). Explicitly, disk C_3 has center $(x_0, 0)$ where $x_0 \neq \pm\delta$.

(b) If $\mathcal{F}_3$ is touching, then it has at most two support lines. $\mathcal{F}_3$ has precisely two support lines if and only if its configuration is equivalent up to a symmetry in the Klein four-group V to one of the configurations depicted in Figure 2.8, or it lies in a slab (see Figure 2.9). That is, the center of C_3 is one of the following:

(i) $o_3(\pm r, \pm 2r)$, or

(ii) $o_3(x_0, 0)$, where $|x_0| \geq 3r$.

(c) If $\mathcal{F}_3$ is disjoint, then it has at most three support lines (see Figure 2.12a). $\mathcal{F}_3$ has precisely three support lines if and only if (*i*) C_3 has center $(0, \pm 2r)$ and $\delta = \dfrac{2r}{\sqrt{3}}$.

Additionally, $\mathcal{F}_3$ has precisely two support lines if and only if it has one of the configurations depicted in Figures 2.12b, 2.12c, 2.12d, or 2.12e. That is, $\mathcal{F}_3$ has precisely two support lines if and only if one of the following holds: the center of C_3 is

(*ii*) $o_3\left(\pm\left(\delta + 2\sqrt{\delta^2 - r^2}\right), \pm 2r\right)$,

(*iii*) $o_3\left(\pm\left(\delta - 2\sqrt{\delta^2 - r^2}\right), \pm 2r\right)$, where $\delta \neq \dfrac{2r}{\sqrt{3}}$,

(*iv*) $o_3\left(0, \pm\dfrac{r\delta}{\sqrt{\delta^2 - r^2}}\right)$, where $\delta \neq \dfrac{2r}{\sqrt{3}}$, or

(*v*) $o_3(\pm x_0, 0)$, where $|x_0| > \delta + 2r$, or $|x_0| < \delta - 2r$, whenever $\delta > 2r$.

Preliminary Discussion for the Proof of Theorem 2.6. We construct each arbitrary family $\mathcal{F}_3$ (up to symmetries in the Klein four-group V) from a suitable arbitrary family $\mathcal{F}_2$, by adjoining to it a disk C_3. The subfamilies $\mathcal{F}_2 = \{C_1, C_2\}$ are described in Theorem 2.3. In later proofs, we will make use of the fact that an arbitrary family $\mathcal{F}_n = \{C_1, \ldots, C_n\}$ with m support lines can be constructed by extending a suitable (not necessarily unique) family $\mathcal{F}_{n-1}$ with $k \geq m$ support lines by adjoining a disk C_n. This holds in general, and the number of support lines of a finite family $\mathcal{F}_{n-1}$ forms a natural upper bound on the number of support lines of any family $\mathcal{F}_n$ constructed from it. In particular, when we construct a family $\mathcal{F}_3$ by adjoining a disk C_3 to a family $\mathcal{F}_2$, we tighten the constraints on the placement of lines supporting the resulting extended family. The number of supports of any subfamily $\mathcal{F}_2 \subset \mathcal{F}_3$ provides a natural upper bound on the number of support lines of the family $\mathcal{F}_3$ containing it.

Proof of Theorem 2.6

Part (a). If a family $\mathcal{F}_3$ is overlapping, then it has an overlapping subfamily. Re-index, if needed, so that C_1 and C_2 overlap. The subfamily $\{C_1, C_2\}$ has precisely two

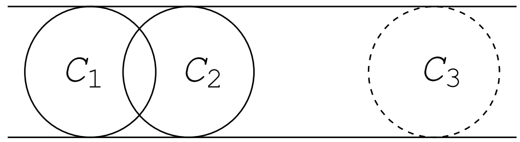


Figure 2.6: An overlapping family $\mathcal{F}_3$ within a slab.

support lines ℓ_1 and ℓ_2 (see Theorem 2.3, Part (a), and Figure 2.2). Hence the family $\mathcal{F}_3$ has at most two support lines by the preliminary discussion above.

If the family $\mathcal{F}_3$ has two support lines, then C_3 must be in the slab between the lines ℓ_1, ℓ_2, the sole supports of $\mathcal{F}_2$. So, $\mathcal{F}_3$ lies entirely in the slab (see Figure 2.6). The converse assertion is simply if $\mathcal{F}_3$ lies in a slab, then the family has at least two supports and no more than two, so it has precisely the two supports ℓ_1, ℓ_2.

Part (b). If a family $\mathcal{F}_3$ is touching, it has a touching subfamily $\mathcal{F}_2$. Reparametrize if needed so that C_1 and C_2 are touching. This subfamily has three support lines $\{\ell_1, \ell_2, \ell_v\}$ (Theorem 2.3, Part (b)), and these lines divide the plane into 6 regions (see Figure 2.7). Two lines support $\mathcal{F}_3$ only if two of these lines support C_3, which happens only if C_3 is placed optimally in a corner of one of the six labeled regions. Regions $1, 3, 5$ are equivalent to regions $2, 4, 6$ by reflection symmetry over the y-axis. Furthermore, since regions 1 and 5 are equivalent by reflection symmetry over the x-axis, up to symmetries in V we have only two cases to consider: place C_3 in region 1 or in region 3. With C_3 in region 1, the centers of the disks are not collinear. With C_3 in region 3, the centers of the disks are collinear.

Case 1 (Touching family $\mathcal{F}_3$ with two support lines, members with noncollinear centers)**.** Let $\mathcal{F}_3$ be a nonoverlapping family with touching subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ as in Theorem 2.3, Part (b), so that $\delta = r$. Then $\mathcal{F}_3$ has two support lines, and its members have noncollinear centers, if and only if the center of C_3 is one of $o_3\,(\pm r, \pm 2r)$. Furthermore, disk C_3 is optimally placed in one of the four identical regions labeled $1, 2, 5, 6$ in Figure 2.7

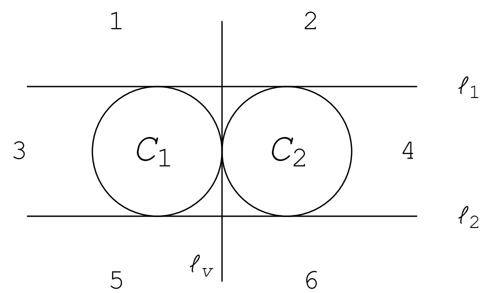


Figure 2.7: The support lines divide the plane into 6 regions.

and touches $\mathcal{F}_2$ at a point in $\{(\pm r, \pm r)\}$, and $\mathcal{F}_3$ is supported by the lines in one of the sets $\{\ell_1, \ell_v\}, \{\ell_2, \ell_v\}$ (see Figure 2.8).

Proof. Let a nonoverlapping family $\mathcal{F}_3$ contain the touching subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ supported by the lines ℓ_1, ℓ_2, ℓ_v (Theorem 2.3, Part (b)). Suppose $\mathcal{F}_3$ has two support lines and the centers of its members do not lie on a line. Then C_3 does not lie in a slab with $\mathcal{F}_2$ and is not supported by both of its parallel support lines ℓ_1, ℓ_2. The family $\mathcal{F}_3$ has two support lines only if the lines in one of the sets $\{\ell_1, \ell_v\}, \{\ell_2, \ell_v\}$ support C_3. This obtains when C_3 is placed optimally in one of the four identical regions labeled $1, 2, 5, 6$ of Figure 2.7 (up to symmetries in V), and its center is one of $o_3\,(\pm r, \pm 2r)$. Conversely, if disk C_3 has its center in $\{o_3\,(\pm r, \pm 2r)\}$, then the centers of the members of $\mathcal{F}_3$ do not lie on a line, and C_3 is positioned optimally in a corner of one of the four identical regions labeled $1, 2, 5, 6$, so that the lines in one of the sets $\{\ell_1, \ell_v\}, \{\ell_2, \ell_v\}$ support C_3, and consequently $\mathcal{F}_3$ has two support lines. Furthermore, C_3 is a translate of C_1, so any point of contact occurs in the lines $y = \pm r$ so that $\{C_3\} \cap \mathcal{F}_2 \in \{(\pm r, \pm r)\}$. $\square$

Case 2 (Touching subfamily $\mathcal{F}_2$ with two parallel support lines, members with collinear centers)**.** Let $\mathcal{F}_3$ be a nonoverlapping family of congruent disks with touching subfamily $\mathcal{F}_2 = \{C_1, C_2\}$, so that $\delta = r$. Then $\mathcal{F}_3$ has two parallel support lines if and only if the centers of its members lie on a line. Explicitly, C_3 has center $o_3\,(x_0, 0)$ where $|x_0| \geq 3r$. Furthermore, C_3 has point of contact $\{C_3\} \cap \mathcal{F}_2 \in \{(\pm(\delta + r), 0)\}$ when $x_0 = \pm 3r$.

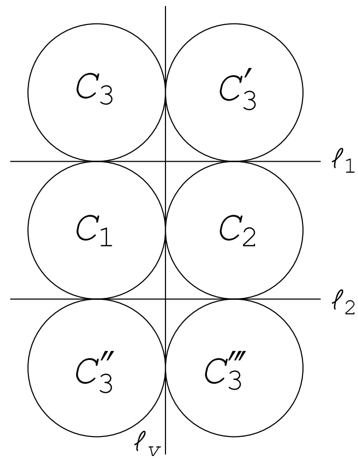


Figure 2.8: Positions for disk C_3 of touching family $\mathcal{F}_3$ as in Case 1 with two common support lines.

Otherwise, disk C_3 lies in the slab in one of the two regions labeled $3, 4$ in Figure 2.7, disjoint from $\mathcal{F}_2$, and $\mathcal{F}_3$ is supported by both of ℓ_1, ℓ_2 (see Figure 2.9).

Proof. Let a nonoverlapping family $\mathcal{F}_3$ contain the touching subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ supported by ℓ_1, ℓ_2, ℓ_v (Theorem 2.3, Part (b)), so that the centers of the disks in $\mathcal{F}_2$ lie on the x-axis. Suppose $\mathcal{F}_3$ has two parallel support lines which necessarily coincide with ℓ_1, ℓ_2. Since both of ℓ_1, ℓ_2 support C_3 and the disks are congruent, the center of C_3 necessarily lies on the x-axis, and the centers of the disks are collinear. Conversely, suppose the centers of the disks in $\mathcal{F}_3$ lie on a line. The family $\mathcal{F}_2$ lies in a slab between ℓ_1, ℓ_2 and C_3 is congruent to these disks, so C_3 lies in the slab between ℓ_1, ℓ_2, and $\mathcal{F}_3$ is supported by this pair of parallel lines. Furthermore, the centers of the disks lie on the x-axis, and C_3 has center $(x_0, 0)$ with $|x_0| \geq 3r$ since the family is nonoverlapping. When $x_0 = \mp 3r$, the disk C_3 touches either C_1 or C_2 at one of $(\mp(\delta + r), 0)$, respective of order. Otherwise, C_3 lies in the slab in one of the two regions labeled $3, 4$ in Figure 2.7, disjoint from $\mathcal{F}_2$ (see Figure 2.9). $\square$

If the nonoverlapping family $\mathcal{F}_3$ has the touching property, reparametrize the disks if needed, so that its subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ touches, and has the three common support lines ℓ_1, ℓ_2, ℓ_v (Theorem 2.3, Part (b)). Disk C_3 cannot be supported by all three lines since the disks are distinct. Either the members of $\mathcal{F}_3$ have their centers on a line or they avoid this property. When their centers do not lie on a line, two lines support $\mathcal{F}_3$ only when C_3 is placed optimally, as documented in Case 1. Otherwise, the disks have collinear centers and exactly two parallel lines support $\mathcal{F}_3$, as documented in Case 2. Any touching family $\mathcal{F}_3$ has a maximum of two support lines. $\square$

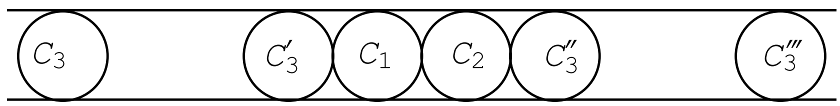


Figure 2.9: Positions for disk C_3 for a touching family $\mathcal{F}_3$ within a slab.

Part (c). Let the family $\mathcal{F}_3$ be disjoint. Then its subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ is disjoint and therefore has four support lines (Theorem 2.3, Part (c)). It follows that the natural upper bound on the number of support lines of $\mathcal{F}_3$ is four. To show that any geometrically realizable family $\mathcal{F}_3$ has at most three common support lines, we suppose for the moment that four lines support $\mathcal{F}_3$ in order to induce a contradiction. We provisionally label these support lines for reference:

$$\ell_1 = \{y = r\}, \quad \ell_2 = \{y = -r\}, \quad \ell_3 = \left\{ y = \frac{rx}{\sqrt{\delta^2 - r^2}} \right\}, \quad \ell_4 = \left\{ y = \frac{-rx}{\sqrt{\delta^2 - r^2}} \right\} \tag{2.1}$$

The lines $\ell_1, \ell_2, \ell_3, \ell_4$, together with the boundaries of the disks of disjoint $\mathcal{F}_2$, partition the plane into 12 nontrivial regions which are labeled in Figure 2.10. The remaining 4 regions do not affect our analysis. To construct $\mathcal{F}_3$ from $\mathcal{F}_2$, we adjoin a congruent disk C_3 to $\{C_1, C_2\}$. In particular, in order that $\mathcal{F}_3$ has four support lines, both of ℓ_1, ℓ_2 necessarily support C_3, so C_3 must lie in the slab between ℓ_1 and ℓ_2. None of the regions $5, 6, 7$ and 9 of Figure 2.10 can individually contain C_3: by symmetry, if C_3 were disjoint from $\mathcal{F}_2$ and

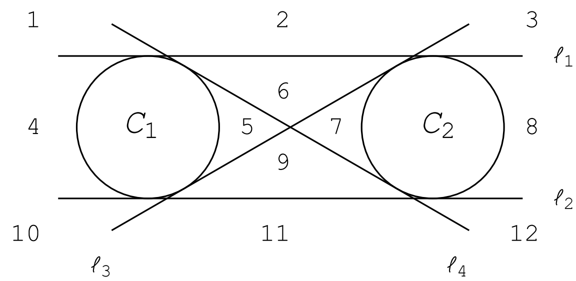


Figure 2.10: The 12 nontrivial regions formed by the lines $\ell_1, \ell_2, \ell_3, \ell_4$ and curves $\partial C_1, \partial C_2$.

wholly contained in region 5, it would not contain the origin in its interior, which would force $\delta > 3r$. Since C_3 must be disjoint from C_1, it would lie to the right of the vertical line $x = -\delta + r$. Within this triangular region, the maximum vertical height between the slant lines ℓ_3, ℓ_4 is less than $2r$, so these lines cut C_3 and the region cannot contain the disk (see Figure 2.11). Up to reflection symmetry over the y-axis, this entails that C_3 is

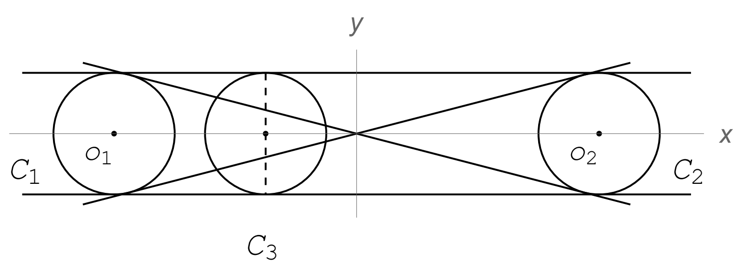


Figure 2.11: The slant support lines of any disjoint family $\mathcal{F}_2$ cut any disk situated between its members.

in region 4. If line ℓ_3 supports C_3, then C_3 coincides with C_1, but by convention the disks must be distinct and by assumption disjoint. No placement for C_3 satisfies the constraints, so no family of three disks has four support lines.

The disjoint families $\mathcal{F}_3$ with two or more support lines are depicted in Figure 2.12. The

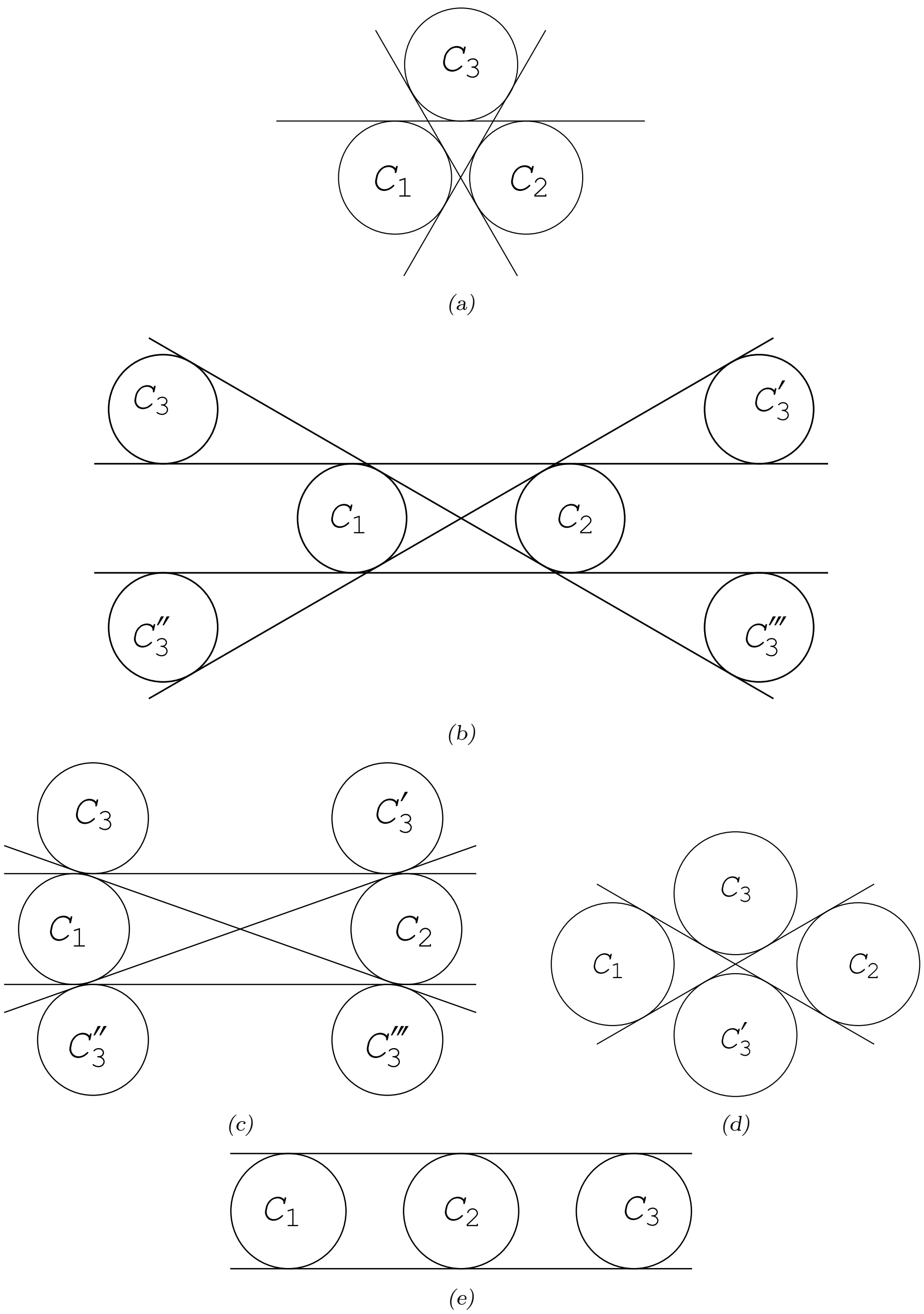


Figure 2.12: Configurations of disjoint families $\mathcal{F}_3$ with two or more support lines.

following case establishes that a configuration $\mathcal{F}_3$ in which three lines support each of the congruent disks exists, and provides explicit calculations for the placement of those disks.

Case 3 (Disjoint family $\mathcal{F}_3$ supported by three lines)**.** Let $\mathcal{F}_3$ be a disjoint family of three congruent disks in the plane. The family $\mathcal{F}_3$ is supported by three lines if and only if its disjoint subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ is parameterized by $\delta = 2r/\sqrt{3}$, and disk C_3 has its center in $\{o_3\,(0, \pm 2r)\}$. Furthermore, the lines $\{y = r\}$ and $\{y = \pm\sqrt{3}x\}$ support the disjoint family $\mathcal{F}_3$. This is the only configuration of three disks with three support lines up to symmetries in the Klein four-group V.

Proof. Suppose $\mathcal{F}_3$ is a family of three disks in the plane supported by three lines. Then $\mathcal{F}_3$ is necessarily disjoint since Parts (a) and (b) of this theorem (Theorem 2.6) show that any $\mathcal{F}_3$ that contains a nondisjoint subfamily $\mathcal{F}_2$ has a maximum of two support lines. In particular, the subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ is disjoint, and we adopt the parametrization $o_1\,(-\delta, 0)$, and $o_2\,(\delta, 0)$ for the respective centers of C_1, C_2 as in Theorem 2.3, Part (c).

The family $\mathcal{F}_3$ has three support lines if and only if C_3 and $\mathcal{F}_2$ share three of the four support lines listed in Equation (2.1). The $\binom{4}{3} = 4$ ways to select three of these support lines are listed here:

$$\{\ell_1, \ell_2, \ell_3\}, \{\ell_1, \ell_2, \ell_4\}, \{\ell_1, \ell_3, \ell_4\}, \{\ell_2, \ell_3, \ell_4\}$$

If both of ℓ_1, ℓ_2 support C_3, then the family $\mathcal{F}_3$ lies in a slab, and has exactly two support lines since no slant line supports three distinct disks in a slab. Since three lines support C_3, precisely one of ℓ_1, ℓ_2 supports C_3, so that one of the sets $\{\ell_1, \ell_3, \ell_4\}, \{\ell_2, \ell_3, \ell_4\}$ contains the supports of C_3. By symmetry, we may assume ℓ_1 supports C_3, so that C_3 has its center on the horizontal line $\{y = 2r\}$ and is supported by the lines in the set $\{\ell_1, \ell_3, \ell_4\}$. Since ℓ_1 supports C_3 from below, the disk lies in some combination of the regions $1, 2, 3$ of Figure 2.10. If C_3 is placed optimally in region 1 or 3, precisely two lines support C_3, so C_3 necessarily overlaps with region 2 of Figure 2.10.

Lines ℓ_3 and ℓ_4, are symmetric about the y-axis, and form part of the boundary of region 2. By symmetry, both of ℓ_3, ℓ_4 support C_3 if and only if disk C_3 lies entirely in region 2 with its center on the line $\{x = 0\}$. This determines the coordinates of the center of C_3 as $o_3\,(0, 2r)$. It remains to determine δ. Line ℓ_3 supports C_3 if and only if the distance from line ℓ_3 to the point $o_3\,(0, 2r)$ is r. By Lemma 2.5, this holds whenever

$$\mathrm{d}\,(\ell_3, o_3) = \frac{\left|r \cdot 0 + (-\sqrt{\delta^2 - r^2}) \cdot 2r\right|}{\sqrt{r^2 + \left(-\sqrt{\delta^2 - r^2}\right)^2}} = \frac{2r\sqrt{\delta^2 - r^2}}{\sqrt{r^2 + \delta^2 - r^2}} = r,$$

which reduces to

$$4\left(\delta^2 - r^2\right) = \pm\delta^2.$$

Since $\delta > r > 0$, we have

$$4\left(\delta^2 - r^2\right) = \delta^2 \iff \delta = \frac{2r}{\sqrt{3}}.$$

Conversely, suppose the family $\mathcal{F}_3$ has a disjoint subfamily $\mathcal{F}_2$ parameterized by $\delta = 2r/\sqrt{3}$, and disk C_3 has center $o_3\,(0, 2r)$. Then line ℓ_1 supports C_3, and the slant lines ℓ_3 and ℓ_4, with respective slopes $\pm r/\sqrt{\delta^2 - r^2} = \pm\sqrt{3}$, are at a distance r to point o_3 and, consequently support C_3, so that disjoint family $\mathcal{F}_3$ has three support lines. In the case that each line in the set $\{\ell_2, \ell_3, \ell_4\}$ supports C_3, similar arguments show that $\delta = 2r/\sqrt{3}$ and disk C_3 has center $o_3\,(0, -2r)$. $\square$

Case 3 above establishes the unique disjoint family $\mathcal{F}_3$ of congruent disks supported by three lines up to symmetries in V. The extension of a disjoint family $\mathcal{F}_2$ to an overlapping family $\mathcal{F}_3$ with exactly two support lines is equivalent by a reparameterization to the family described in Part (a) above. The extension of a disjoint family $\mathcal{F}_2$ to a touching family $\mathcal{F}_3$ with exactly two support lines is equivalent by a reparametrization to one of the families described in Part (b) of the present theorem. Otherwise, a disjoint family $\mathcal{F}_2$ is extended to the disjoint family $\mathcal{F}_3 = \mathcal{F}_2 \cup \{C_3\}$. The remaining cases detail the explicit placement of the disk C_3 for each respective disjoint family $\mathcal{F}_3$ with precisely two common support lines.

If precisely one of ℓ_1, ℓ_2 supports C_3, then the disk lies in one of the angular regions labeled $1, 3, 10, 12$ in Figure 2.10, or overlaps with one of the regions bounded by three lines (regions $2, 11$). If neither of ℓ_1, ℓ_2 supports C_3 and the family has precisely two support lines, then C_3 is necessarily supported by both slant lines of $\mathcal{F}_2$ and has its center on the line $\{x = 0\}$. Disk C_3 lies in the union of regions 2 and 6 or in the union of regions 9 and 11, and the center of C_3 is not permitted to lie on either of the lines $\{y = \pm 2r\}$. If both parallel support lines support C_3, then the family lies in a slab. By symmetry, we have the following four cases to consider. We can place C_3 either in an angular region (e.g. region 1), or with support from one horizontal line such that C_3 overlaps with a region bounded by three lines (e.g. region 2). Or, we can place C_3 in an angular region bounded by the two slant lines (e.g. the union of region 9 and 11) with center $o_3(0, y_3)$ where $y_3 \neq \pm 2r$, or within the slab determined by ℓ_1 and ℓ_2 (e.g. region 4) of Figure 2.10.

Case 4 (Disjoint family $\mathcal{F}_3$ with its third disk in an angular region such as region 1)**.** Let the family $\mathcal{F}_3$ have disjoint subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ whose respective members have centers $o_1(-\delta, 0)$ and $o_2(\delta, 0)$ (Theorem 2.3, Part (c)). $\mathcal{F}_3$ has exactly two support lines and a disk in an angular region of Figure 2.10 if and only if C_3 has its center in $\{o_3(\pm x_0, \pm 2r)\}$, where $|x_0| = \delta + 2\sqrt{\delta^2 - r^2}$. Furthermore, the disjoint family $\mathcal{F}_3$ is supported by exactly one horizontal support line (ℓ_1 or ℓ_2) and one slant line (ℓ_3 or ℓ_4) (see Figure 2.12b).

Proof. Let $\mathcal{F}_3$ contain the disjoint subfamily $\mathcal{F}_2$. Suppose $\mathcal{F}_3$ is supported by two lines, one of which is horizontal, and C_3 is wholly contained in a wedge, one of the the regions labeled $1, 3, 10, 12$ in Figure 2.10. By symmetry, assume the disk is in region 1 so that the lines ℓ_1, ℓ_4 support C_3. Since ℓ_1 supports C_3 from below, we provisionally assign its center the label $o_3(x_0, 2r)$ with $x_0 < 0$. Line ℓ_4 supports C_3 if and only if the distance from ℓ_4 to the point o_3 is exactly r (see Equation (2.1) for ℓ_4). Lemma 2.5 determines the following relation:

$$\mathrm{d}\left(-rx-\sqrt{\delta^2-r^2}y=0,o_3\right)=r \iff \frac{\left|-rx_0-2r\sqrt{\delta^2-r^2}\right|}{\sqrt{r^2+(\sqrt{\delta^2-r^2})^2}}=r$$

$$\iff \left|-x_0-2\sqrt{\delta^2-r^2}\right|=\delta$$

Disk C_3 is supported by ℓ_4 from above, which means $\ell_4(x_0)>2r$. This inequality shows that the expression above is positive which permits us to write

$$-x_0-2\sqrt{\delta^2-r^2}=\delta \iff -x_0=\delta+2\sqrt{\delta^2-r^2},$$

where $x_0<0$. Conversely, if C_3 has center $o_3(x_0,2r)$ with the value stated above, then C_3 lies in region 1 supported by both of ℓ_1,ℓ_4, so that the family $\mathcal{F}_3$ has two support lines. By symmetry disk C_3 lies in an angular region for each choice in $\{o_3(\pm x_0,\pm 2r)\}$ for the center of C_3. $\square$

Case 5 (Disjoint family $\mathcal{F}_3$ with one horizontal support, and a disk that overlaps with a region bounded by three lines such as region 2)**.** Let the family $\mathcal{F}_3$ have disjoint subfamily $\mathcal{F}_2=\{C_1,C_2\}$ whose respective members have centers $o_1(-\delta,0),o_2(\delta,0)$ (Theorem 2.3, Part (c)). $\mathcal{F}_3$ has exactly two support lines, one of which is horizontal, and a disk overlapping with a region bounded by three lines of Figure 2.10 if and only if the center of C_3 is in $\{o_3(\pm x_0,\pm 2r)\}$ with $|x_0|=2\sqrt{\delta^2-r^2}-\delta$ and $\delta\neq 2r/\sqrt{3}$. Furthermore, the disjoint family $\mathcal{F}_3$ is supported by exactly one horizontal support line (ℓ_1 or ℓ_2) and one slant support line (ℓ_3 or ℓ_4) (see Figure 2.12c).

Proof. If $\mathcal{F}_3$ is supported by two lines, one of which is horizontal, and C_3 is not wholly contained in a wedge (regions $1,3,10,12$ of Figure 2.10), then C_3 overlaps with a region bounded by three lines such as region 2, and is supported by the lines in one of the sets $\{\ell_1,\ell_3\},\{\ell_1,\ell_4\}$.

By symmetry, assume the lines in $\{\ell_1, \ell_4\}$ support C_3 centered at $o_3(x_0, 2r)$ and $\delta \neq 2r/\sqrt{3}$. In the proof of Case 4 above, we determined the placement of C_3 in region 1 supported by ℓ_1 and ℓ_4 where we derived the relation

$$\left|-2\sqrt{\delta^2 - r^2} - x_0\right| = \delta.$$

If C_3 lies on the opposite side of ℓ_4 as calculated in Case 4 above, then C_3 overlaps with region 2 and the other branch of the solution applies for x_0 so that

$$-2\left(-\sqrt{\delta^2 - r^2}\right) - x_0 = \delta \iff x_0 = 2\sqrt{\delta^2 - r^2} - \delta.$$

Observe that if $\delta = 2r/\sqrt{3}$, then

$$x_0 = 2\sqrt{\delta^2 - r^2} - \delta = 2\sqrt{\left(\frac{2r}{\sqrt{3}}\right)^2 - r^2} - \left(\frac{2r}{\sqrt{3}}\right) = 2\sqrt{\frac{r^2}{3}} - \frac{2r}{\sqrt{3}} = 0,$$

and the family satisfies the conditions of Case 3, implying it has three support lines. Conversely, suppose disk C_3 has center $o_3\left(2\sqrt{\delta^2 - r^2} - \delta, 2r\right)$ and $\delta \neq 2r/\sqrt{3}$. Then, the lines ℓ_1 and ℓ_4 both support C_3 from below, so that C_3 overlaps with region 2. For δ small, disk C_3 may overlap with region 3. Furthermore, disjoint $\mathcal{F}_3$ is supported by exactly one horizontal support line (ℓ_1) and one slant support line (ℓ_4). $\square$

Case 6 (Disjoint family $\mathcal{F}_3$ with its third disk supported by both slant support lines of $\mathcal{F}_2$)**.** Let $\mathcal{F}_3$ have disjoint subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ whose respective members have centers $o_1(-\delta, 0)$ and $o_2(\delta, 0)$ as in Theorem 2.3, Part (c). $\mathcal{F}_3$ has exactly two slant support lines ℓ_3, ℓ_4 if and only if C_3 has its center in $\{o_3(0, \pm\gamma)\}$ with $\gamma = r\delta/\sqrt{\delta^2 - r^2}$, and $\delta \neq 2r/\sqrt{3}$ (see Figure 2.13). Furthermore, C_3 is entirely contained in region 2 or 11, respectively, whenever $0 < \delta < 2r/\sqrt{3}$, and partially contained in one of these regions whenever $\delta > 2r/\sqrt{3}$, in which case it is either in the union of regions 9 and 11, or in the union of regions 2 and 6.

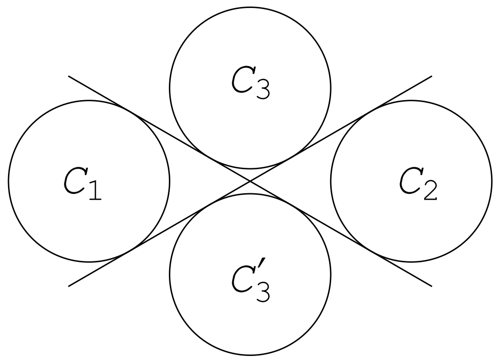


Figure 2.13: A disk C_3 supported by the slant lines ℓ_3, ℓ_4 of $\mathcal{F}_2$.

Proof. $\mathcal{F}_2 = \{C_1, C_2\}$ is a disjoint subfamily of $\mathcal{F}_3$. If $\mathcal{F}_3$ has exactly two support lines none of which is horizontal, then the slant lines ℓ_3, ℓ_4 necessarily support C_3. Disk C_3 is either in the union of regions 9 and 11 or in the union of regions 2 and 6 of Figure 2.10. By symmetry, let C_3 lie in the union of regions 9 and 11. Lines ℓ_3 and ℓ_4 intersect at the origin, and their symmetry about the y-axis implies both of ℓ_3, ℓ_4 support the congruent disk C_3 if and only if the center o_3 of C_3 has x-coordinate equal to zero. Each of C_2, C_3 has a radius orthogonal to ℓ_4. These radii are legs in similar right triangles each of which has its hypotenuse on one of the coordinate axes (see Figure 2.14). From similar triangles in the figure, inspection verifies the equivalent proportions between the triangles

$$\frac{r}{\sqrt{\delta^2 - r^2}} = \frac{\gamma}{\delta} \iff \gamma = \frac{r\delta}{\sqrt{\delta^2 - r^2}}.$$

Family $\mathcal{F}_3$ has exactly two support lines if $\delta \neq 2r/\sqrt{3}$. Otherwise, we have

$$\gamma = \frac{r\delta}{\sqrt{\delta^2 - r^2}} = \frac{r \cdot \dfrac{2r}{\sqrt{3}}}{\sqrt{\left(\dfrac{2r}{\sqrt{3}}\right)^2 - r^2}} = \frac{\dfrac{2r^2}{\sqrt{3}}}{\sqrt{\dfrac{r^2}{3}}} = \frac{2r^2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{r} = 2r,$$

implying C_3 has center $(0, -2r)$, and the family satisfies the conditions of Case 3, so that the three lines ℓ_2, ℓ_3, ℓ_4 support C_3, contrary to supposition. Conversely, suppose C_3 has its center in $\{o_3\,(0, \pm\gamma)\}$ where $\gamma = r\delta/\sqrt{\delta^2 - r^2}$, and $\delta \neq 2r/\sqrt{3}$. By symmetry, let its center be $o_3\,(0, \gamma)$. Then, by direct calculation using Lemma 2.5, lines ℓ_3 and ℓ_4 are both at distance r to the point o_3, which implies the lines support C_3. And with $\delta \neq 2r/\sqrt{3}$, the disjoint family $\mathcal{F}_3$ has exactly two support lines. Furthermore, the boundary condition $\delta = 2r/\sqrt{3}$ forces an optimally placed disk C_3 entirely in region 2 or 11 (Figure 2.10), respectively, supported by three lines. Whenever $0 < \delta < 2r/\sqrt{3}$, the acute vertical angles $\left(\widehat{\ell_3, \ell_4}\right)$ formed by ℓ_3, ℓ_4 narrow, forcing disk C_3 entirely into region 2 or 11 supported by the slant lines ℓ_3, ℓ_4. When $\delta > 2r/\sqrt{3}$, an optimally placed disk C_3 is supported by both of ℓ_3, ℓ_4 and is contained either in the union of regions 9 and 11, or in the union of regions 2 and 6. $\square$

The centers of the members of $\mathcal{F}_3$ lie on a line if and only if C_3 lies in a slab with C_1, C_2 if and only if $\mathcal{F}_3$ is supported by the parallel support lines ℓ_1, ℓ_2.

Case 7 (Disjoint family $\mathcal{F}_3$ lies in a slab)**.** Let $\mathcal{F}_3$ have disjoint subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ as in Theorem 2.3, Part (c). $\mathcal{F}_3$ is disjoint and contained in a slab if and only if C_3 has center $(x_0, 0)$ with either $|x_0| > \delta + 2r$, or $|x_0| < \delta - 2r$ only if $\delta > 2r$. Furthermore, such a disk C_3 is in region 4, region 8, or in the union of regions $5, 6, 7, 9$ of Figure 2.10. The centers of C_1, C_2, C_3 are collinear, and the slab is determined by the parallel support lines ℓ_1, ℓ_2 of $\mathcal{F}_3$ (see Figure 2.12e).

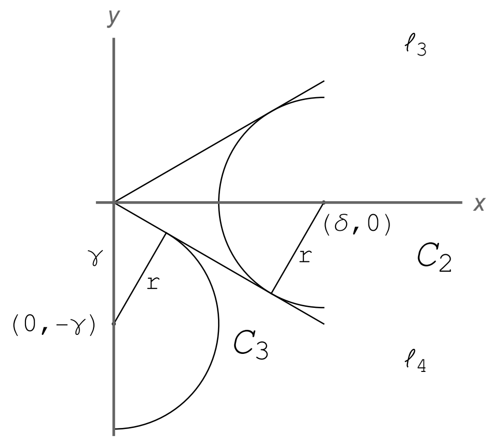


Figure 2.14: Calculating the center of a region 9 disk supported by ℓ_3 and ℓ_4.

Proof. $\mathcal{F}_2 = \{C_1, C_2\}$ is a disjoint subfamily of $\mathcal{F}_3$, which is contained in a slab. The parallel lines ℓ_1, ℓ_2 determine the boundary of the closed slab containing $\mathcal{F}_3$. Disks C_1, C_2, C_3 are congruent, so their centers are equidistant to the boundary of the slab, which places their centers on the x-axis. We provisionally label the center of C_3 as $o_3\,(x_0, 0)$, and since $\mathcal{F}_3$ is disjoint, we have either $|x_0| > \delta + 2r$, or $|x_0| < \delta - 2r$ only if $\delta > 2r$. The first condition places C_3 in region 4 or 8 of Figure 2.10. The second condition places C_3 in the union of regions $5, 6, 7, 9$ between C_1 and C_2 only if $\delta > 2r$. Conversely, suppose C_3 has center $o_3\,(x_0, 0)$ and $|x_0| > \delta + 2r$, or $|x_0| < \delta - 2r$ only if $\delta > 2r$. Since each disk has radius r and the center of C_3 is collinear with the centers of C_1, C_2, disk C_3 lies in the slab whose boundary is determined by ℓ_1, ℓ_2, both of which support $\mathcal{F}_3$. The stated conditions on x_0 guarantee that $\mathcal{F}_3$ is disjoint. $\square$

In the Preliminary Discussion to this theorem, we observed that the number of supports of $\mathcal{F}_2 \subset \mathcal{F}_3$ forms a natural upper bound on the number of supports of $\mathcal{F}_3$. In the introduction to Part (c) we determined that the upper bound for disjoint families of size three is 4 and then showed that this upper bound is not realizable, so that any family $\mathcal{F}_3$ has fewer than four support lines. This bound is sharp. Case 3 (Theorem 2.6, Part (c)) above describes the particular configuration of three disks supported by three lines. Cases 4 through 7

(Theorem 2.6, Part (c)) above show that in all other cases, a family of three disks has a maximum of two support lines. It follows that any family $\mathcal{F}_3$ of congruent disks in the plane has at most three support lines. □

Corollary 2.7. (Corollary to Case 4 of Theorem 2.6, Part (c).) Let the family $\mathcal{F}_2$ be disjoint. Disjoint $\mathcal{F}_3 = \mathcal{F}_2 \cup \{C_3\}$ with its third disk in region 1 of Figure 2.10 has precisely two support lines ℓ_1, ℓ_4 if and only if C_3 has center

$$o_3\left(-\gamma, 2r\right) = o_3\left(-\left(\delta + 2\sqrt{\delta^2 - r^2}\right), 2r\right).$$

Proof. $\mathcal{F}_2 = \{C_1, C_2\}$ is a disjoint subfamily of $\mathcal{F}_3$. Suppose $\mathcal{F}_3$ has two support lines and C_3 is in region 1 of Figure 2.10, then the pair of lines ℓ_1, ℓ_4 support disk C_3. Since line ℓ_1 supports any disk of radius r whose center lies on the horizontal line $\{y = 2r\}$, we tentatively label the center of C_3 as $o_3\left(-\gamma, 2r\right)$. Observe that we use γ here in place of x_0 which was used in Case 4. The distance from ℓ_4 to o_4 must be r, which happens if and only if C_3 is placed optimally in region 1, as depicted in Figure 2.15. Lines ℓ_1 and ℓ_4 intersect at the point p, forming four cones in the plane.

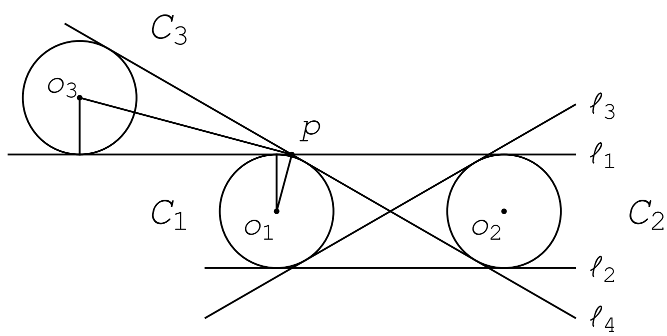


Figure 2.15: Disk C_3 in region 1 supported by lines ℓ_1 and ℓ_4.

As seen in Figure 2.16 (though true in general) the perpendicular bisectors of the adjacent cones created by lines ℓ_1 and ℓ_4 form a right angle. The center o_3 of disk C_3 lies on the angle bisector of the acute angle $\left(\widehat{\ell_1, \ell_4}\right)$, so line segment $[o_3, p]$ bisects this angle. This implies $\alpha = (1/2) \cdot \left(\widehat{\ell_1, \ell_4}\right)$ for angle α as labeled in the figure. Similarly, line segment $[o_1, p]$ bisects the obtuse angle formed by lines ℓ_1 and ℓ_4. It follows that line segment $[o_1, p]$ is perpendicular to line segment $[o_3, p]$, so that angles α and β as labeled are complementary. Line segment $[o_3, p]$ forms the hypotenuse of a right triangle, that is similar by the angle-angle (AA) similarity theorem to the right triangle with hypotenuse $[o_1, p]$ as seen in Figure 2.16. Since point $\{p\,(x_1, y_1)\}$ belongs to ℓ_1, we have $y_1 = r$. Its x-coordinate x_1 is given by the following (compare Equation (2.1)).

$$\ell_1\,(x_1) = \ell_4\,(x_1) \iff \frac{-r x_1}{\sqrt{\delta^2 - r^2}} = r \iff x_1 = -\sqrt{\delta^2 - r^2}.$$

The point p has coordinates $(x_1, y_1) = \left(-\sqrt{\delta^2 - r^2}, r\right)$. The right triangle $\triangle o_3 p q$ (point

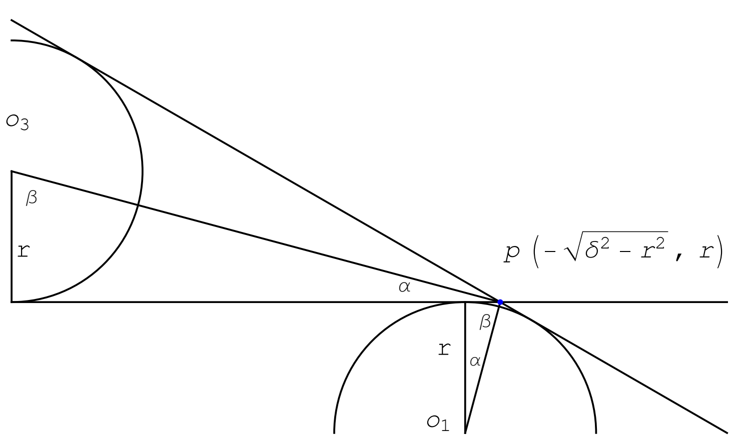


Figure 2.16: Disk C_3 in region 1 supported by lines ℓ_1 and ℓ_4.

$q := \{(-\gamma, r)\}$) has a leg of length r and a (horizontal) leg with endpoints p and $q\,(-\gamma, r)$ of length

$$-\sqrt{\delta^2 - r^2} - (-\gamma) = \gamma - \sqrt{\delta^2 - r^2}.$$

The right triangle $\triangle o_1 p q'$ (point $q' := \{(-\delta, r)\}$) with vertex $o_1\,(-\delta, 0)$ has a leg of length r, and a leg that is a horizontal line segment with endpoints $q'\,(-\delta, r)$ and $p\left(-\sqrt{\delta^2 - r^2}, r\right)$. The length of this horizontal leg is given by the difference

$$-\sqrt{\delta^2 - r^2} - (-\delta) = \delta - \sqrt{\delta^2 - r^2}.$$

The side lengths of these similar triangles, yields a proportion involving γ that leads to

$$\begin{aligned}
\frac{r}{\gamma - \sqrt{\delta^2 - r^2}} = \frac{\delta - \sqrt{\delta^2 - r^2}}{r} \iff \gamma &= \frac{2r^2 + \delta\left(\sqrt{\delta^2 - r^2} - \delta\right)}{\left(\delta - \sqrt{\delta^2 - r^2}\right)} \\
\iff \gamma &= \frac{2r^2}{\delta - \sqrt{\delta^2 - r^2}} - \delta.
\end{aligned}$$

Conversely, if $\mathcal{F}_3$ has disjoint subfamily $\mathcal{F}_2$ parameterized by convention and has disk C_3 with center $o_3\,(-\gamma, 2r)$ with γ as derived above, then the lines ℓ_1, ℓ_4 support C_3 in region 1. Furthermore, disjoint $\mathcal{F}_3$ has two support lines. $\square$

The following remark provides an equivalent expression for x_0 to that given in Case 4.

Remark 2.8. The values $x_0 = \delta + 2\sqrt{\delta^2 - r^2}$ and $x_0 = \dfrac{2r^2}{\delta - \sqrt{\delta^2 - r^2}} - \delta$ are equal.

Proof. Either rewrite the second expression, multiplying by the conjugate of the expression in its denominator, or observe the following string of equivalences:

$$\begin{aligned}\delta + 2\sqrt{\delta^2 - r^2} = \frac{2r^2}{\delta - \sqrt{\delta^2 - r^2}} - \delta &\iff 2\delta + 2\sqrt{\delta^2 - r^2} = \frac{2r^2}{\delta - \sqrt{\delta^2 - r^2}} \\ &\iff (\delta + \sqrt{\delta^2 - r^2})(\delta - \sqrt{\delta^2 - r^2}) = r^2 \\ &\iff \delta^2 - (\delta^2 - r^2) = r^2\end{aligned}$$

□

Corollary 2.9. (Corollary to Case 5 of Theorem 2.6, Part (c).) Let $\mathcal{F}_2$ be disjoint. From the proof of Case 5 of Theorem 2.6, Part (c), we derive the values

$$x_0 = 2\sqrt{\delta^2 - r^2} - \delta, \quad x_{p'} = 2\sqrt{\delta^2 - r^2} - \delta + \frac{r^2}{\delta} = x_0 + \frac{r^2}{\delta}, \quad \text{and} \quad y_{p'} = \frac{r}{\delta}\left(2\delta - \sqrt{\delta^2 - r^2}\right).$$

A third disk C_3 in region 2 with center $o_3\left(2\sqrt{\delta^2 - r^2} - \delta, 2r\right)$ is supported by lines ℓ_1 and ℓ_3 at the respective points (x_0, r) and $\left(x_{p'}, y_{p'}\right)$. Similarly, a disk C_3 in region 2 with center $o_3\left(-2\sqrt{\delta^2 - r^2} + \delta, 2r\right)$ is supported by lines ℓ_1 and ℓ_4 at the respective points $(-x_0, r)$ and $\left(-x_{p'}, y_{p'}\right)$. □

Remark 2.10. (Remark on Case 5 of Theorem 2.6, Part (c).) Let $\mathcal{F}_2$ be disjoint, so that $\delta > r > 0$. The expression

$$|x_0| = \sqrt{\left(2\delta - \sqrt{\delta^2 - r^2}\right)^2 - 3r^2},$$

is identical to the expression derived in Theorem 2.6, Part (c).

Proof. Either solve $\sqrt{\left(2\delta-\sqrt{\delta^2-r^2}\right)^2-3r^2}=p\sqrt{\delta^2-r^2}-q$ for p,q, or observe the following string of equivalences:

$$\sqrt{\left(2\delta-\sqrt{\delta^2-r^2}\right)^2-3r^2}=2\sqrt{\delta^2-r^2}-\delta$$

$$\iff 4\delta^2-4\delta\sqrt{\delta^2-r^2}+\left(\delta^2-r^2\right)-3r^2=4\left(\delta^2-r^2\right)-4\delta\sqrt{\delta^2-r^2}+\delta^2$$

$$\iff \delta^2-4r^2=-4r^2+\delta^2$$

□

Remark 2.11. (Remark on Case 5 of Theorem 2.6, Part (c).) Let the family $\mathcal{F}_2$ be disjoint, so that $\delta>r>0$, then

$$\sqrt{\left(2\delta-\sqrt{\delta^2-r^2}\right)^2-3r^2}<\delta.$$

Proof. By the preceding Remark,

$$\sqrt{\left(2\delta-\sqrt{\delta^2-r^2}\right)^2-3r^2}=2\sqrt{\delta^2-r^2}-\delta.$$

Furthermore,

$$2\sqrt{\delta^2-r^2}-\delta<\delta \iff 4\delta^2-4r^2<4\delta^2 \iff -4r^2<0.$$

□

2.4 Common Support Lines for a Family $\mathcal{F}_4$

Let $\mathcal{F}_4=\{C_1,C_2,C_3,C_4\}$ be a planar family of pairwise distinct congruent disks of positive radius r. As in Theorem 2.3, we parameterize disks C_1,C_2 by their respective centers $o_1\left(-\delta,0\right),o_2\left(\delta,0\right)$.

Theorem 2.12. Any family $\mathcal{F}_4=\{C_1,C_2,C_3,C_4\}$ of congruent disks of radius r in the plane has at most two support lines.

(a) If $\mathcal{F}_4$ is overlapping, then it has at most two support lines. $\mathcal{F}_4$ has precisely two support lines if and only if it lies entirely in a slab.

(b) If $\mathcal{F}_4$ is touching, then it is the extension of a nonoverlapping family $\mathcal{F}_3$, and has at most two support lines. $\mathcal{F}_4$ has precisely two support lines if and only if the family lies in a slab, or if it has the configuration in Figure 2.17a. Furthermore, if $\mathcal{F}_4$ lies in a slab then it is an extension of a disjoint $\mathcal{F}_3$ or of a touching $\mathcal{F}_3$ as depicted in Figure 2.9 or Figure 2.12e (see Figure 2.17b). If $\mathcal{F}_4$ has the configuration in Figure 2.17a then it is the extension of a disjoint $\mathcal{F}_3$ or of the touching $\mathcal{F}_3$ depicted in Figure 2.8.

(c) If $\mathcal{F}_4$ is disjoint, then it is the extension of a disjoint $\mathcal{F}_3$, and has at most two support lines. It has precisely two support lines if and only if it lies entirely in a slab, or if it has the configuration in Figure 2.18 (up to symmetries in the Klein four-group V).

Proof of Theorem 2.12 To show that any $\mathcal{F}_4$ has at most two support lines, we induce a contradiction by supposing that the extension $\mathcal{F}_4$ of some $\mathcal{F}_3$ is supported by at least three lines, including, say, ℓ_1, ℓ_2, ℓ_3. Since $\mathcal{F}_4$ contains $\mathcal{F}_3 = \{C_1, C_2, C_3\}$ as a subfamily, each of ℓ_1, ℓ_2, ℓ_3 supports $\mathcal{F}_3$. Considering Theorem 2.6, Part (c), $\mathcal{F}_3$ must have the configuration in Figure 2.12a where ℓ_1, ℓ_2, ℓ_3 correspond to the depicted lines. In the subfigure, three (unbounded) polyhedral regions support a disk of radius r with three lines, and each of these three regions contains a disk. If C_4 is placed in one of these polyhedral regions, then the Dirichlet principle guarantees the disks are not distinct.

Disk C_4 must be placed in the remaining region bounded by the three lines ℓ_1, ℓ_2, ℓ_3, which is a closed regular polygon with side length $4r/\sqrt{3}$. This region is bounded above by the line $\{y = r\}$, and has $(0, 0)$ as a vertex. Calculating, we find its incenter to be the point $I\,(0, 2r/3)$. This implies the inradius of the incircle is $r/3$. Since this region cannot support a disk of radius r, either C_4 is not supported by ℓ_1, ℓ_2, ℓ_3 or the disks in the family are not distinct, contrary to supposition. No construction resolves the contradiction. It follows that $\mathcal{F}_n$ $(n \geq 4)$ cannot have more than two support lines. The disjoint families $\mathcal{F}_4$ with precisely two support lines are depicted in Figures 2.18 and 2.19.

Part (a) If $\mathcal{F}_4$ is overlapping then it has an overlapping subfamily $\mathcal{F}_3$. Reparametrize the disks if needed, so that C_1 and C_2 of $\mathcal{F}_3$ overlap. By Theorem 2.6, Part (a), any overlapping $\mathcal{F}_3$ has at most two support lines, so the family $\mathcal{F}_4$ has at most two support lines. In particular, the overlapping family $\mathcal{F}_3$ with two support lines lies in a slab which by convention is determined by the supports ℓ_1, ℓ_2, and the subfamily $\{C_1, C_2, C_4\}$ is supported by two lines only if C_4 lies in the slab. It follows that $\mathcal{F}_4$ necessarily lies in the slab. Conversely, suppose overlapping $\mathcal{F}_4$ lies entirely in a slab. No slant line supports the family since C_1, C_2 are not separable. Since the disks are congruent and distinct, their centers lie on a line and each disk is supported by precisely the two parallel support lines ℓ_1, ℓ_2. $\square$

Part (b) If $\mathcal{F}_4$ is touching, it has a touching subfamily $\mathcal{F}_3$. Reparametrize the disks if needed, so that $C_1, C_2 \in \mathcal{F}_3$ are touching. If $\mathcal{F}_4$ has two support lines, then it is the extension of a touching $\mathcal{F}_3$ supported by at least two lines. The two choices for $\mathcal{F}_3$ satisfying these conditions are depicted in Figures 2.8, and 2.9 (Cases 1 and 2 of Theorem 2.6, Part (b)). Disk C_3 of the family depicted in Figure 2.8 has center $o_3\,(-r, 2r)$ and $\mathcal{F}_3$ is supported by both of ℓ_v, ℓ_2. In an extension of the family, disk C_4 must have center $o_4\,(r, 2r)$, so that both of ℓ_v, ℓ_2 support C_4, and the family $\mathcal{F}_4$ has the configuration in Figure 2.17a. If we extend the family $\mathcal{F}_3$ depicted in Figure 2.9 which lies entirely in a slab, then the family $\mathcal{F}_4$ is supported by the two lines ℓ_1, ℓ_2 only if C_4 is supported by these lines. This places C_4 in the slab containing $\mathcal{F}_3$, so that $\mathcal{F}_4$ lies entirely in the slab.

Conversely, suppose touching $\mathcal{F}_4$ lies in a slab, or $\mathcal{F}_4$ has the configuration in Figure 2.17a. If $\mathcal{F}_4$ lies in a slab, then it has precisely two support lines. If $\mathcal{F}_4$ has the configuration in Figure 2.17a, then it is touching and has two support lines. Furthermore, if touching $\mathcal{F}_4$ lies in a slab, and $C_1 \cap C_2 \neq \emptyset$, then various placements for C_4 with center $o_4\,(x_0, 0)$ are permitted. Let $o_3(\gamma, 0)$ denote the center of C_3. The family is nonoverlapping

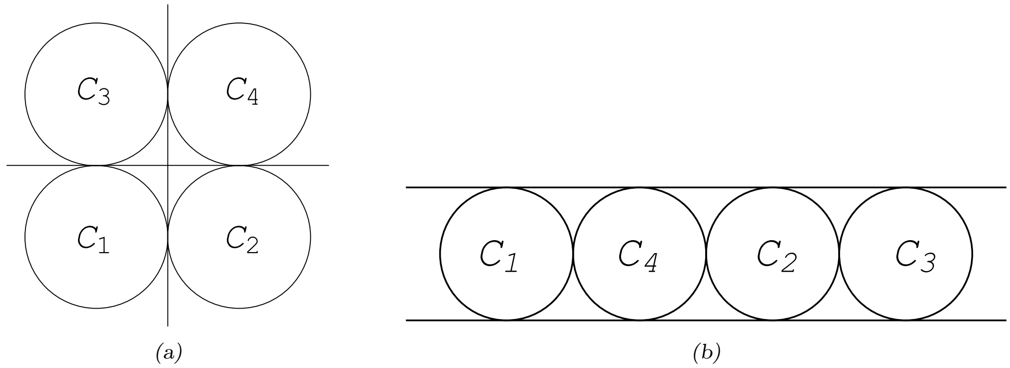


Figure 2.17: Touching families $\mathcal{F}_4$ with two support lines.

when the following conditions hold:

$$|\gamma| \geq 3r \qquad \text{and} \qquad |x_0| \geq 3r \qquad \text{and} \qquad x_0 \notin (\gamma - 2r, \gamma + 2r)$$

Disk C_4 touches C_1 or C_2, respective of order, for the two values $x_0 = \mp 3r$. Disk C_4 touches C_2 and C_3 when $|x_0| = 3r$ and $\gamma = 5r$. And, disk C_4 touches C_3 when $x_0 = \gamma \pm 2r$. $\square$

Part (c) If $\mathcal{F}_4$ is disjoint, then each subfamily $\mathcal{F}_3$ is disjoint. If $\mathcal{F}_4$ has two support lines, then it is necessarily the extension of a disjoint family $\mathcal{F}_3$ with at least two support lines. Five families have this property as documented in Theorem 2.6, Part (c) and depicted in Figure 2.12.

Extension 1: Disjoint $\mathcal{F}_3$ described in Case 3 of Theorem 2.6, Part (c) and depicted in Figure 2.12a has three support lines. Recall that no disk of radius r can be contained in the bounded triangular region supported by the three lines since the incircle of the triangular region has radius $r/3$ (as stated in the proof above). If any two of these lines support C_4, then it must overlap with some disk of $\mathcal{F}_3$. In particular, since the disks are congruent, the center of C_4 must coincide with the center of one of these disks, so that the disks are not distinct, a contradiction. The configuration of lines determines three angular (polyhedral) regions not containing disks, and it follows that C_4 is distinct from the disks of $\mathcal{F}_3$ and

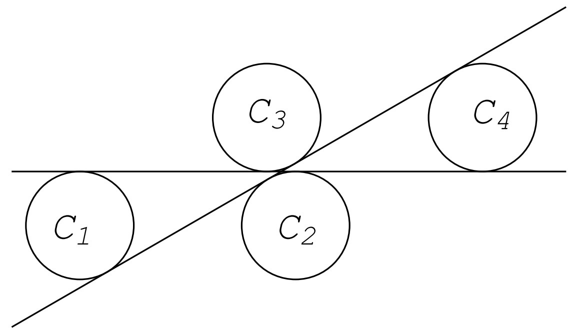


Figure 2.18: A disjoint family $\mathcal{F}_4$ with two support lines.

supported by two lines only if it is optimally placed in one of these three empty angular regions. This leads to a configuration with incidence relations equivalent to those of the family depicted in Figure 2.18. Conversely, a family with this configuration meets the stated conditions.

Extension 2: Disjoint $\mathcal{F}_3$ with two common supports described in Case 4 of Theorem 2.6, Part (c) and depicted in Figure 2.12b has its third disk in an angular region, which we take to be region 1 of Figure 2.10 so that both of ℓ_1, ℓ_4 support $\mathcal{F}_3$ up to symmetry. Since the lines ℓ_1, ℓ_4 create four angular regions in the plane, three of which are occupied by C_1, C_2, C_3, the only construction that preserves these two support lines is to place the fourth disk optimally in region 2, which yields a configuration whose incidence relations are equivalent to those of the family depicted in Figure 2.18. Conversely, a family with this configuration meets the stated conditions.

Extension 3: Disjoint $\mathcal{F}_3$ described in Case 5 of Theorem 2.6, Part (c) and depicted in Figure 2.12c with two support lines has its third disk in a region bounded by three lines. Up to symmetry, we take this to be the union of regions $2, 6$ of Figure 2.10 bounded by the lines ℓ_1, ℓ_3, ℓ_4. Let both of ℓ_1, ℓ_4 support disk C_4. Since the lines ℓ_1, ℓ_4 create four angular regions in the plane, three of which are occupied by C_1, C_2, C_3, the only construction that preserves its two support lines is to place the fourth disk optimally in region 1 (see Figure 2.10), so that the family has a configuration whose incidence relations are equivalent to those of the family depicted in Figure 2.18. Conversely, a family with this configuration meets the stated conditions.

Extension 4: Disjoint $\mathcal{F}_3$ described in Case 6 of Theorem 2.6, Part (c) and depicted in Figure 2.12d with two support lines has its third disk supported by the slant lines of $\mathcal{F}_2$, and we take C_3 to be in the region formed by the union of regions 9 and 11 of Figure 2.10 up to symmetry. Since the transverse lines ℓ_3, ℓ_4 create four angular regions in the plane, three of which are occupied respectively by C_1, C_2, C_3, the only construction that preserves its two support lines is to place the fourth disk optimally in the union of regions $2, 6$ of Figure 2.10 supported by both of ℓ_3, ℓ_4, which yields a configuration whose incidence relations are equivalent to those of the family depicted in Figure 2.18. Conversely, a family with this configuration meets the stated conditions.

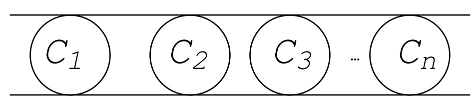


Figure 2.19: A disjoint family $\mathcal{F}_n$ with two support lines.

Extension 5: Disjoint $\mathcal{F}_3$ described in Case 7 of Theorem 2.6, Part (c) and depicted in Figure 2.12e lies entirely in a slab between two horizontal lines. We take its third disk to be in region 8 of Figure 2.10 and note that no slant line supports the family. We place disk C_4 disjoint from the disks of $\mathcal{F}_3$, and maintain its two support lines only if both of ℓ_1, ℓ_2 support C_4 which entails that the disk is placed in the slab. The resulting disjoint $\mathcal{F}_4$ lies in a slab as in Figure 2.19. Conversely, suppose disjoint $\mathcal{F}_4$ lies in a slab (compare Figure 2.19) or has the configuration depicted in Figure 2.18, then $\mathcal{F}_4$ is disjoint and has precisely two support lines. $\square$

2.5 Common Support Lines for a Family $\mathcal{F}_n$ $(n \geq 5)$

Let $\mathcal{F}_n$ $(n \geq 5)$ be a family of pairwise distinct congruent disks of positive radius r. As in Theorem 2.3, we parameterize disks C_1, C_2 by their respective centers $o_1(-\delta, 0), o_2(\delta, 0)$.

Theorem 2.13. Any family $\mathcal{F}_n$ $(n \geq 5)$ of congruent disks of radius r in the plane has at most two support lines.

(a) If $\mathcal{F}_n$ is overlapping, then it has at most two support lines. $\mathcal{F}_n$ has precisely two support lines if and only if it lies entirely in a slab.

(b) If $\mathcal{F}_n$ is touching, then it has at most two support lines. $\mathcal{F}_n$ has precisely two support lines if and only if it lies entirely in a slab.

(c) If $\mathcal{F}_n$ is disjoint, then it has at most two support lines. $\mathcal{F}_n$ has precisely two support lines if and only if it lies entirely in a slab.

Proof of Theorem 2.13 Theorem 2.12 lists exhaustively the families with four members supported by exactly two lines. Some of these families are extendable to a family with two support lines, and if two lines support an extension of one of these families, then it lies in a slab as we prove in the following.

Part (a) Overlapping $\mathcal{F}_2$ lies in a slab (compare Theorem 2.3, Part (a)). For $n = 3, 4$, any overlapping $\mathcal{F}_n$ with two support lines lies in a slab by Theorem 2.6, Part (a), and Theorem 2.12, Part (a), respectively. For $n \geq 5$, suppose overlapping $\mathcal{F}_n$ with two support lines does not lie entirely in a slab. If the disk C_i $(i \in \{3, 4, 5, \ldots, n\})$ is not in the slab with $\mathcal{F}_2$, then the subfamily $\{C_1, C_2, C_i\} \subset \mathcal{F}_n$ is not supported by two lines, and since $\mathcal{F}_n$ contains this subfamily, it has at most one support line, a contradiction. To resolve the contradiction, disk C_i $(i \in \{3, 4, 5, \ldots, n\})$ must lie in the slab between the support lines ℓ_1, ℓ_2 of $\mathcal{F}_2$, so that the entire famlily lies in the slab. Conversely, if overlapping $\mathcal{F}_n$ $(n > 2)$ lies in a slab, then it has precisely two support lines. Overlapping $\mathcal{F}_n$ $(n \geq 5)$ has two support lines if and only if the family lies in a slab.

Part (b) If $\mathcal{F}_5$ is touching, then it has a touching subfamily $\mathcal{F}_4$ where C_1 touches C_2. If precisely two lines support $\mathcal{F}_5$, then at least two lines support $\mathcal{F}_4$. The two choices (up to symmetries in the Klein four-group V) for $\mathcal{F}_4 \subset \mathcal{F}_5$ are depicted in the two subfigures of Figure 2.17 (Theorem 2.12, Part (b)). If we extend the family depicted in Figure 2.17a to a family with five congruent members retaining its two support lines, then both of ℓ_1, ℓ_v must support the adjoined disk C_5. Since the two intersecting lines form exactly four angular regions in the plane, each of which supports precisely one optimally placed distinct congruent disk, the Dirichlet principle guarantees that a fifth disk placed optimally in one of these angular regions coincides with one of the four original disks. The extension has at most one common support line contradicting the claim $\mathcal{F}_4 \subset \mathcal{F}_5$ with two common supports.

Consider the family depicted in Figure 2.17b, which lies entirely in a slab. An extension $\mathcal{F}_5$ of the family retains two support lines only if C_5 is supported by both of ℓ_1, ℓ_2. This entails that the disk lies in the slab with $\mathcal{F}_4$, so that $\mathcal{F}_5$ lies entirely in the slab. Since no other construction results in a touching family $\mathcal{F}_5$ with two support lines, any touching $\mathcal{F}_5$ with two support lines lies in a slab.

Furthermore, if two lines support touching $\mathcal{F}_n$ $(n > 5)$, then each of its touching subfamilies of size five necessarily lies in a slab. Since $n > 5$ the family $\mathcal{F}_n$ has at least six subfamilies of size five. Suppose the distinct subfamilies $\mathcal{H}$ and $\mathcal{H}'$ of size five are supported by the respective pairs of lines ℓ, ℓ' and m, m'. Since two lines support $\mathcal{F}_n$, it has the support property S, so that $\{\ell, \ell'\} \cap \{m, m'\} \neq \emptyset$. If the lines are not identical, then precisely one line supports $\mathcal{F}_n$ contrary to supposition. It follows that both subfamilies $\mathcal{H}$ and $\mathcal{H}'$ lie in a slab between a single pair of common support lines. This holds for every pair of subfamilies of size 5 and it follows transitively that the entire family lies in a slab. Conversely, if $\mathcal{F}_n$ $(n \geq 5)$ lies in a slab, then the family is supported by exactly two parallel support lines since the disks are congruent and distinct. Any touching $\mathcal{F}_n$ $(n \geq 5)$ has precisely two support lines if and only if the family lies entirely in a slab.

Part (c) Let disjoint $\mathcal{F}_4 \subset \mathcal{F}_5$. If $\mathcal{F}_5$ has precisely two support lines, then $\mathcal{F}_4$ has at least two common supports. The two choices (up to symmetries in V) for $\mathcal{F}_4 \subset \mathcal{F}_5$ are depicted in Figures 2.18 and 2.19 (Theorem 2.12, Part (c)). If we adjoin a congruent disk C_5 to the family depicted in Figure 2.18, both of ℓ_1, ℓ_3 must support C_5 to retain two support lines. Since the intersecting lines ℓ_1, ℓ_3 form exactly four angular regions in the plane, each of which supports precisely one of the optimally placed distinct congruent disks of $\mathcal{F}_4$, the Dirichlet principle guarantees that a fifth disk placed optimally in one of these angular regions coincides with one of the four original disks. It follows that any extension $\mathcal{F}_5$ of the family has at most one common support contrary to supposition.

Consider the family depicted in Figure 2.19, which lies entirely in a slab supported by both of ℓ_1, ℓ_2. In order for an extension $\mathcal{F}_5$ to have two support lines, both of ℓ_1, ℓ_2 must support the adjoined disk C_5. This places C_5 in the slab containing $\mathcal{F}_4$, so that $\mathcal{F}_5$ lies entirely in the slab. The remainder of the proof is identical to the last paragraph of the preceding part, and we summarize it here: no other construction results in a disjoint $\mathcal{F}_5$ with two support lines, so any disjoint $\mathcal{F}_5$ with two support lines lies in a slab. If two lines support disjoint $\mathcal{F}_n$ $(n > 5)$, then each of its touching subfamilies of size five necessarily lies in a slab. The family $\mathcal{F}_n$ has at least six subfamilies of size five. Any two subfamilies of size five lie in a slab between a single pair of common support lines, otherwise only one line supports $\mathcal{F}_n$ contrary to supposition. It follows transitively that the entire family lies in a slab. Conversely, if $\mathcal{F}_n$ $(n \geq 5)$ lies in a slab, then the family is supported by exactly two parallel support lines since the disks are congruent and distinct. Any disjoint $\mathcal{F}_n$ $(n \geq 5)$ has precisely two support lines if and only if the family lies entirely in a slab. $\square$

We often have the explicit coordinates of the point where a support line contacts the boundary of a disk. To describe an extension of a family of disks, it is often convenient to calculate the image of a point reflected over a given support line. The following lemma, whose proof immediately follows from standard facts of analytic geometry, provides this result.

Lemma 2.14. (a) Given a horizontal line $\{y = m\}$, and a point $p\,(x_0, y_0)$, the image of its reflection over the line is $p'\,(x'_0, y'_0) = p'\,(x_0, 2m - y_0)$.

(b) Given a vertical line $\{x = c\}$, and a point $p\,(x_0, y_0)$, the image of its reflection over the line is $p'\,(x'_0, y'_0) = p'\,(2c - x_0, y_0)\,.$

(c) Given a line of direct variation $\{y = kx\}$, with $k \neq 0$, and a point $p\,(x_0, y_0)$, the image of its reflection over the line is $p'\,(x'_0, y'_0)$, where

$$x'_0 = \frac{2ky_0 - \left(k^2 - 1\right) x_0}{k^2 + 1} \quad \text{and} \quad y'_0 = \frac{\left(k^2 - 1\right) y_0 + 2kx_0}{k^2 + 1}.$$

(d) Given a line $\{y = kx + m\}$ with $k \neq 0$, and a point $p\,(x_0, y_0)$, the image of its reflection over the line is $p'\,(x_0 - 2\Delta x, y_0 - 2\Delta y)$, where

$$\Delta x = \frac{k\,(kx_0 - y_0 + m)}{k^2 + 1} \quad \text{and} \quad -\,\Delta y = \frac{kx_0 - y_0 + m}{k^2 + 1}.$$ $\square$

2.6 First Helly-Type Theorem on Support Lines

This section is devoted to the proof of our first Helly-type result. It extends the assertion of Theorem 1 from [23], proved there for the case of disjoint families.

Theorem 2.15. For any nonoverlapping family $\mathcal{F}$ of congruent disks in the plane, one has $S(4) \implies S$.

Geometric Proof of Theorem 2.15. Assume, for contradiction, the existence of a nonoverlapping family $\mathcal{F}$ of congruent disks in the plane, with the property $S(4)$ but not the property S. Necessarily, $|\mathcal{F}| \geq 5$. Since $S(4) \implies S$ for any disjoint family of congruent disks in the plane (see the paper [23]), it follows that the family $\mathcal{F}$ is touching. Let r denote the common radius of the disks in $\mathcal{F}$.

Choose a touching pair of disks from $\mathcal{F}$ and denote them by C_1 and C_2. As established above by convention, we assume that the disks C_1 and C_2 have their respective centers o_1 and o_2 on the x-axis of the plane with respective coordinates $(-r, 0)$ and $(r, 0)$. As established in Theorem 2.3, the subfamily $\{C_1, C_2\}$ has precisely three support lines: two of them are the horizontal lines ℓ_1, ℓ_2, given, respectively, by the equations $y = \pm r$, and the third is the vertical line ℓ_v, given by the equation $x = 0$ (see Figure 2.20). We let $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$.

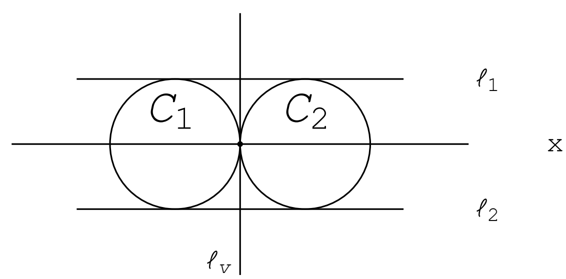


Figure 2.20: Touching family $\{C_1, C_2\}$ and its three support lines.

With this notation, we prove the following auxiliary lemma.

Lemma 2.16. Every disk $C \in \mathcal{F} \setminus \{C_1, C_2\}$ is supported by at least two lines from the family $\mathcal{L}_{12}$.

Proof. Assume for the moment the existence of a disk $C \in \mathcal{F} \setminus \{C_1, C_2\}$ which is supported by exactly one line, say ℓ, from the family $\mathcal{L}_{12}$ (the existence of ℓ is guaranteed by the property $S(4)$). Choose any other disk $C' \in \mathcal{F} \setminus \{C_1, C_2, C\}$ (this is possible since $|\mathcal{F}| \geq 5$). By the condition $S(4)$, a line ℓ' supports the family $\{C_1, C_2, C, C'\}$. Because ℓ' supports $\{C_1, C_2, C\}$, the choice of C implies that $\ell' = \ell$. Since the disk C' was chosen arbitrarily in $\mathcal{F} \setminus \{C_1, C_2, C\}$, we conclude that ℓ is a common support line for the entire family $\mathcal{F}$. The latter contradicts the assumption on $\mathcal{F}$. $\square$

We continue with the proof of Theorem 2.15. The assumption that $\mathcal{F}$ does not have the property S implies that no line from the family $\mathcal{L}_{12}$ supports $\mathcal{F}$. Denote by C_3, C_4, and C_5 the disks from $\mathcal{F}$ which are not supported by the lines ℓ_1, ℓ_2, and ℓ_v, respectively. We observe that the disks C_3, C_4, and C_5 are pairwise distinct. Indeed, if, for instance, $C_3 = C_4$, then ℓ_v would be the only line from $\mathcal{L}_{12}$ which supports C_3. The latter contradicts Lemma 2.16.

Hence each of the disks C_3, C_4, C_5 is supported by precisely two lines from $\mathcal{L}_{12}$, and no line from $\mathcal{L}_{12}$ supports $\{C_3, C_4, C_5\}$. Renumbering, if necessary, the disks C_3, C_4, C_5, we obtain the only possible configuration of disks from $\mathcal{F}$ and lines from $\mathcal{L}_{12}$:

1. ℓ_1 supports $\{C_1, C_2, C_4, C_5\}$
2. ℓ_2 supports $\{C_1, C_2, C_3, C_5\}$
3. ℓ_v supports $\{C_1, C_2, C_3, C_4\}$

Analysis of this configuration shows that C_5 is supported by the lines ℓ_1 and ℓ_2, and thus is contained in the slab between these lines, which is the union of regions 3 and 4 in Figure 2.21. Similarly, C_4 is supported by the lines ℓ_1 and ℓ_v (but not by ℓ_2), and thus is contained in the corner of one of the regions 1 and 2 in Figure 2.21. Finally, C_3 is supported by the lines ℓ_2 and ℓ_v (but not by ℓ_1), and thus is contained in the corner of one of the regions 5 and 6 in Figure 2.21.

A straightforward geometric argument shows that the family $\{C_1, C_3, C_4, C_5\}$ has no common support line, in contradiction with the assumption that $\mathcal{F}$ has the property $S(4)$. The obtained contradiction shows that $\mathcal{F}$ has the property S. $\square$

Combinatorial Proof of Theorem 2.15. Since $\mathcal{F}$ is touching, its tangent subfamily $\mathcal{F}_2 = \{C_1, C_2\}$ has supports $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$. If $\mathcal{F}$ has $S(4)$ and not S, then at least two lines in $\mathcal{L}_{12}$ support each disk $C_k \in \mathcal{F}$ $(k \geq 3)$. If a single line $\ell \in \mathcal{L}_{12}$ supports a particular disk C_k $(k \geq 3)$, then a line supports each of $\{C_1, C_2, C_k, C_4\}, \{C_1, C_2, C_k, C_5\}, \ldots, \{C_1, C_2, C_k, C_n\}$ consistent with $S(4)$ since $|\mathcal{F}| = n \geq 5$. Since ℓ alone supports $\{C_1, C_2, C_k\}$, it necessarily supports the union of the listed families which is $\{C_1, C_2, C_3, C_4, C_5, \ldots, C_n\} = \mathcal{F}$, a contradiction. So at least two lines in $\mathcal{L}_{12}$ support each disk $C_k \in \mathcal{F}$ ($k \geq 3$). We prove the following lemma before concluding the proof.

Lemma 2.17. If a finite touching family $\mathcal{F}$ of congruent disks in the plane of size $|\mathcal{F}| \geq 5$ has property $S(4)$, then a line supports each subfamily $\{C_1, C_2, C_3, C_4, C_k\}$ ($k \geq 5$).

Proof. We repeat Figure 2.7 as Figure 2.21 below. The family $\mathcal{F}$ has property $S(4)$, and

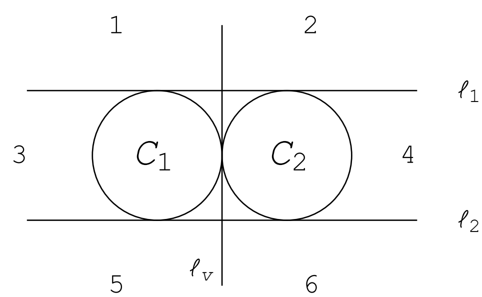


Figure 2.21: Originally Figure 2.7.

from the paragraph preceding this lemma, two lines in $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$ necessarily support each of the disks C_3, C_4 and each $C_k \in \mathcal{F}$ for each fixed $k \geq 5$. Precisely one arrangement of the supports in $\mathcal{L}_{12}$ avoids the property $S(5)$ for each subfamily $\mathcal{F}_4 \cup \{C_k\} = \mathcal{G}_k \subset \mathcal{F}$ ($k \geq 5$). Combinatorially, to avoid $S(5)$ we assign two distinct labels from the multiset $\{\ell_1, \ell_2, \ell_1, \ell_v, \ell_2, \ell_v\}$ to each of the three disks, stipulating up to labels that both of ℓ_1, ℓ_2 support C_3 (in the slab), both of ℓ_1, ℓ_v support C_4, and both of ℓ_2, ℓ_v support C_k (with k fixed) (see Figure 2.21). Since ℓ_v is disjoint from C_3, no line supports one of $\{C_1, C_3, C_4, C_k\}$, $\{C_2, C_3, C_4, C_k\}$, a contradiction. To avoid contradiction, one of the labels necessarily appears with higher frequency in the multiset. Up to labels, replacing one copy of ℓ_v with ℓ_1 yields $\{\ell_1, \ell_2, \ell_1, \ell_v, \ell_2, \ell_1\} = \{\ell_1, \ell_1, \ell_1\} \sqcup \{\ell_2, \ell_v, \ell_2\}$ and by the Dirichlet principle, line ℓ_1 supports $\{C_3, C_4, C_k\}$, so that necessarily a line supports each subfamily $\{C_1, C_2, C_3, C_4, C_k\} \subset \mathcal{F}$ $(k \geq 5)$. $\square$

We conclude the proof, assuming that the family $\mathcal{F}$ does not have property S. As established in the preceding, since at least two lines in $\mathcal{L}_{12}$ support each disk $C_k \in \mathcal{F}$ $(k \geq 3)$, the subfamily $\mathcal{F}_4 \subset \mathcal{F}$ necessarily has two support lines. According to Theorem 2.12, Part (b), either $\mathcal{F}_4$ lies in a slab as in Subfigure 2.17b, or it has the square configuration shown in Subfigure 2.17a.

If $\mathcal{F}_4 \subset \mathcal{F}$ lies in a slab, then at least one of ℓ_1, ℓ_2 supports each disk $C, D \in \mathcal{F}$ by Lemma 2.17. If precisely one of ℓ_1, ℓ_2 supports one of $C, D \in \mathcal{F}$, then a line supports each disk of $\mathcal{F}$ by the argument given in the paragraph preceding Lemma 2.17. Otherwise both of ℓ_1, ℓ_2 support both of C, D, and the family has property S (compare Figure 2.21 and Subfigure 2.17b).

If $\mathcal{F}_4 \subset \mathcal{F}$ has the square configuration depicted in Subfigure 2.17a, the two supports of $\mathcal{F}_4$ are $\ell_1, \ell_v \in \mathcal{L}_{12}$. At least one of ℓ_1, ℓ_v supports each disk $C, D \in \mathcal{F}$ by Lemma 2.17, and it is impossible for both of ℓ_1, ℓ_v to support disk C in any position, so that precisely one of ℓ_1, ℓ_v supports it. It follows from the argument given in the paragraph preceding Lemma 2.17 that the family $\mathcal{F}$ has property S. $\square$

Chapter 3: Nonoverlapping Critical Families of Four Congruent Disks

We rely on the following definitions.

Definition 3.1. A nonoverlapping (touching, disjoint) family $\mathcal{F}$ of congruent disks in the plane with the property $S\,(3)$ but not $S\,(4)$ is called *critical*.

Any critical family $\mathcal{F}$ necessarily has a minimum of four members.

Remark 3.2. As noted earlier, the subfamily $\{C_1, C_2\}$ together with its family of support lines $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$ has the symmetries in the Klein four-group V which consist of reflection over line ℓ_v, reflection over the x-axis, rotation of 180° about the origin, and identity symmetry. Since the condition on support lines of critical families prohibits a line from supporting the entire family, adjoining additional disks cannot result in a family with the symmetries of the square, since this constraint implicitly prevents 90° rotational symmetry. Furthermore, it is impossible for any family to develop an additional line of symmetry through the origin since that line of symmetry would cut each disk C_1, C_2 and each adjoined disk is nonoverlapping. For this reason, we do not need to concern ourselves with the symmetries represented by the various dihedral groups. Restricting to the symmetries in the Klein four-group V is sufficient to determine whether any two touching critical families $\mathcal{F}_4$ containing the touching subfamily $\{C_1, C_2\}$ are distinct.

This chapter is devoted to the description of nonoverlapping critical families $\mathcal{F}_4$ of congruent disks in the plane of positive radius r. In particular, we retain the use of the parameter r with the understanding that a generalization of this method to disks with two or more distinct radii would require a similar explicit parametrization. Since the case of *disjoint* families $\mathcal{F}_4$ is studied in the paper of Soltan [23], it remains to consider the case of *touching* families $\mathcal{F}_4$. We first describe all combinatorially distinct families $\mathcal{F}_4$ with the property that no three disks from $\mathcal{F}_4$ have their centers on a line (Section 3.1). We then describe all combinatorially distinct families $\mathcal{F}_4$ with the property that three disks from $\mathcal{F}_4$ have their centers on a line (Section 3.2).

We need some definitions and terminology. Given a critical family $\mathcal{F}_4$, we will say that a line supporting at least three disks is a *critical support line* or a *critical support* of the family. A subfamily of size three of a critical family is a *critical subfamily.* These definitions are meaningful since if a single subfamily of size three of $\mathcal{F}_4$ is not supported, then the family is not critical; whereas, it is not necessary that every subfamily of size three have a distinct support line. For critical subfamilies of size $k > 3$, we will say *critical subfamily of size k* or *critical subfamily* $\mathcal{F}_k$. The expressions *critical subfamily of size* 3 and *critical subfamily* are interchangeable.

A non-horizontal *definite critical support line* of a family of congruent disks either supports the disks on the left or on the right, while a horizontal *definite critical support line* of the family either supports the disks from above or from below. A non-horizontal line that supports a subfamily $\{C_i, C_j\}$ $(i < j)$ on the left (respectively, on the right) will be denoted $\ell_{def\mathrm{ij}L}$ (respectively, $\ell_{def\mathrm{ij}R}$). If a separating support of $\{C_i, C_j\}$ $(i < j)$ meets C_i on the left and C_j on the right then it will be denoted $\ell_{sep\mathrm{ij}LR}$. Similarly, a separating support of $\{C_i, C_j\}$ $(i < j)$ which meets C_i on the right and C_j on the left will be denoted $\ell_{sep\mathrm{ij}RL}$. A class of configurations can be explicitly characterized where this labeling scheme is potentially ambiguous, and in the few places where those configurations arise in this text their descriptions are elaborated for clarity.

3.1 Touching Critical Families $\mathcal{F}_4$ Avoiding Three Disks in a Slab

In what follows, we assume that no three disks of the touching critical family $\mathcal{F}_4$ have their centers on a line. For a touching family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$ we adhere to the convention that disks C_1 and C_2 have their respective centers $o_1\,(-r,0)$ and $o_2\,(r,0)$ on the x-axis of the plane and they share a common point at the origin $o\,(0,0)$. The vertical line ℓ_v through the origin is the only common support of C_1 and C_2 that separates the disks, and the remaining supports are the horizontal lines ℓ_1 and ℓ_2 given by the equations $y = \pm r$ (see Figure 3.1).

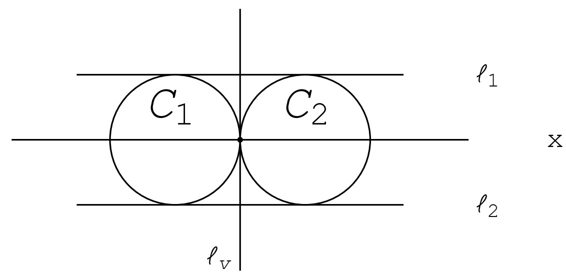


Figure 3.1: Touching family $\{C_1, C_2\}$ and its set of supports $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$.

Consequently, neither of C_3, C_4 lies between ℓ_1 and ℓ_2 supported by both lines. And since $\mathcal{F}_4$ is critical neither of ℓ_1, ℓ_2 supports both C_3 and C_4. By symmetries in the Klein four-group V, we may assume that either ℓ_1 or ℓ_v supports C_3 and that its center $o_3(\gamma, y_3)$ has parameter $\gamma \geq 0$.

We organize our description of the touching critical families $\mathcal{F}_4$ by the magnitude of the parameter γ. For γ small ($0 \leq \gamma < r$), line ℓ_1 necessarily supports $\{C_1, C_2, C_3\}$. When $\gamma = r$, line ℓ_v necessarily supports C_3. When $\gamma > r$, line ℓ_1 necessarily supports C_3. We document all touching critical families $\mathcal{F}_4$ avoiding three disks in a slab with γ in the three respective ranges $0 \leq \gamma < r$, the assignment $\gamma = r$, and $\gamma > r$.

3.1.1 Case 1: $0 \leq \gamma < r$

With γ in this range, ℓ_1 necessarily supports C_3. Since $\mathcal{F}_4$ is critical, a line supports $\{C_1, C_2, C_4\}$, so that one of ℓ_2, ℓ_v necessarily supports C_4.

a). We begin by documenting critical families where line ℓ_2 supports disk C_4, which avoids support from line ℓ_v. These configurations are documented in Lemmas 3.3 and 3.4. A line must support subfamily $\{C_1, C_3, C_4\}$, which requires that a support of $\{C_1, C_3\}$ supports C_4. The subfamily $\{C_1, C_3\}$ is disjoint since line ℓ_1 separates C_1 from C_3 and $\gamma \neq -r$. By Theorem 2.3, Part (c), this subfamily has two definite and two separating supports which are listed in the set

$$\mathcal{L}_{13} = \{\ell_{def13L}, \ell_{def13R}, \ell_{sep13LR} = \ell_1, \ell_{sep13RL}\}.$$

Since line ℓ_1 is disjoint from C_4, three candidate lines remain. In the following lemma line ℓ_{def13L} supports C_4.

Lemma 3.3. If ℓ_2 supports C_4, and C_3 has center $o_3(\gamma, 2r)$ with $\gamma = (r/3) \cdot \left(9 - 4\sqrt{3}\right)$, then the touching family $\mathcal{F}_4$ is critical. Furthermore, the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ supports C_4 on the right (from below), and the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ supports C_4 on the left (from above). Furthermore, $\gamma < r$ and C_4 avoids ℓ_v.

Proof. Let $\delta = r$ so that $\{C_1, C_2\}$ is a touching subfamily. Let $0 \leq \gamma < r$, so that ℓ_1 necessarily supports C_3 and consequently $\{C_1, C_2, C_3\}$. Since C_4 avoids support from ℓ_v by supposition, line ℓ_2 necessarily supports it, and consequently $\{C_1, C_2, C_4\}$. Let ℓ_{def13L} support C_4, and consequently $\{C_1, C_3, C_4\}$. Since γ is nonnegative, disk C_4 has center $o_4(x_4, -2r)$ with $x_4 < 0$.

Explicitly, line ℓ_{def13L} is parallel to $\langle o_1, o_3 \rangle$ with slope $2r/(\gamma + r)$. Since this line supports C_1, applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(\ell_{def13L}(x) - y = 0, o_1(-r, 0)\right)$ leads to the following:

$$\mathrm{d}\left(\frac{2rx}{\gamma + r} - y + m = 0, (-r, 0)\right) = r \iff \left|\frac{2r \cdot (-r)}{\gamma + r} - (0) + m\right| = r\sqrt{\left(\frac{2r}{\gamma + r}\right)^2 + (-1)^2}$$

Since the expression $\ell_{def13L}(-r) = -2r^2/(\gamma + r) + m > 0$ is positive by construction (see Figure 3.2), we lift the absolute value and solve for m to derive the following equation:

$$\ell_{def13L}(x) = \frac{2rx}{\gamma + r} + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}$$

Since $\mathcal{F}_4$ avoids three disks in a slab, line ℓ_{def13L} separates C_4 from $\{C_1, C_3\}$, necessarily supporting C_4 on the right (from below) at a distance r to the point $o_4(x_4, -2r)$. Applying Lemma 2.5 to line ℓ_{def13L} and point o_4 yields the following equation:

$$\left| \frac{2r}{\gamma + r}(x_4) - (1)(-2r) + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r} \right| = r\frac{\sqrt{4r^2 + (\gamma + r)^2}}{\gamma + r}$$

Since the point $(x_4, \ell_{def13L}(x_4))$ lies on the line (see Figure 3.2), the facts $|\ell_{def13L}(x_4)| > 2r$ and $\ell_{def13L}(x_4) < 0$ together imply

$$|\ell_{def13L}(x_4)| - 2r = -\ell_{def13L}(x_4) - 2r > 0$$

is equivalent to $|\ell_{def13L}(x_4) + 2r|$. This leads to the expression

$$x_4 = -\left(\gamma + 2r + \sqrt{5r^2 + 2r\gamma + \gamma^2}\right),$$

which guarantees ℓ_{def13L} supports $\{C_1, C_3, C_4\}$.

To ensure property $S(3)$, a line necessarily supports $\{C_2, C_3, C_4\}$. This line must be the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ since ℓ_1 (its associated separating support) and the definite supports of $\{C_2, C_3\}$ are disjoint from C_4 whenever ℓ_{def13L} supports C_4 on the right (see Figure 3.2). The line $\ell_{sep23LR}$ either supports C_4 on the left or on the right. In Lemma 3.4 below, we describe the configuration with support on the right. Here, we describe the family in which $\ell_{sep23LR}$ supports C_4 on the left.

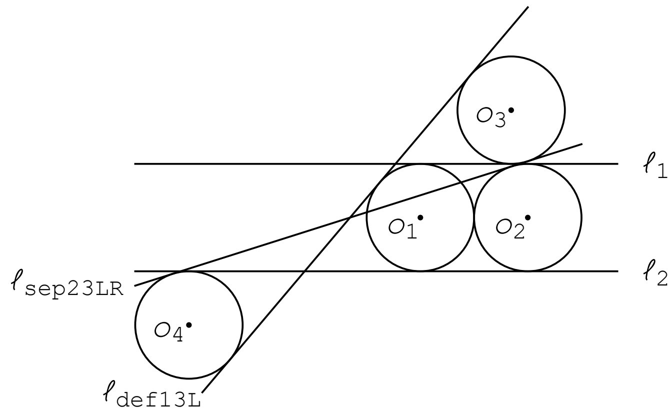


Figure 3.2: Touching critical $\mathcal{F}_4$ with $r = 1$ and $\gamma = (1/3) \cdot \left(9 - 4\sqrt{3}\right) \approx 0.6906$. Line $\ell_{sep23LR}$ supports C_4 on the left.

Let $\ell_{sep23LR}$ support C_4 on the left, so that it is also a definite support of $\{C_2, C_4\}$ parallel to $\langle o_2, o_4 \rangle$ which has the following slope:

$$k_{sep23LR} = \frac{0 - (-2r)}{r - \left[-\left(\gamma + 2r + \sqrt{4r^2 + \gamma^2 + 2r\gamma + r^2}\right)\right]} = \frac{2r}{\gamma + 3r + \sqrt{5r^2 + 2r\gamma + \gamma^2}}$$

We derive an expression for the slope of this line by a second method. Observe that line $\ell_{sep23LR}$ contains the midpoint $((\gamma + r)/2, r)$ of segment $[o_2, o_3]$ by symmetry. Applying Lemma 2.5 to line $\ell_{sep23LR}$ with slope k and the point o_2 yields the following equivalence:

$$\mathrm{d}\left(kx - y - k\frac{\gamma + r}{2} + r = 0, (r, 0)\right) = r \iff \left|kr - 1 \cdot (0) - k\frac{\gamma + r}{2} + r\right| = r\sqrt{k^2 + 1}$$

Since $\ell_{sep23LR}(r) > 0$, we lift the absolute value and solve to derive the expression k for its slope. Equating the two expressions $k_{sep23LR} = k$, as in

$$\frac{2r}{\gamma + 3r + \sqrt{4r^2 + \gamma^2 + 2r\gamma + r^2}} = \frac{4r(r - \gamma)}{3r^2 + 2r\gamma - \gamma^2},$$

leads to the following equation in the indeterminate γ with parameter r:

$$(3)\gamma^4 - (12r)\gamma^3 - (11r^2)\gamma^2 + (4r^3)\gamma + 11r^4 = 0$$

In terms of r, the affiliated real root is given by nonnegative $\gamma = \frac{r}{3}\left(9 - 4\sqrt{3}\right)$ where $\gamma < r$ since $\left(9 - 4\sqrt{3}\right) < 3$. With $\delta = r > 0$, the subfamily $\{C_1, C_2\}$ is touching, and with γ as given above, the touching family $\mathcal{F}_4$ is critical. $\square$

Lemma 3.4. If $\gamma = (r/3) \cdot (\beta^+ + \beta^- - 1)$ with $\beta^\pm = \sqrt[3]{2}\sqrt[3]{13 \pm 3\sqrt{33}}$, then the touching family $\mathcal{F}_4$ is critical. Furthermore, the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ and the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ both support C_4 on the right (from below). Additionally, $\gamma < r$ and C_4 avoids ℓ_v.

Proof. Suppose $\mathcal{F}_4$ has property $S\,(3)$ and γ is nonnegative. Let the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ support C_4 and consequently $\{C_1, C_3, C_4\}$. The equation of this line derived in Lemma 3.3 is reproduced here:

$$\ell_{def13L}(x) = \frac{2r}{\gamma + r}x + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}$$

As documented in Lemma 3.3, the following expression for the parameter x_4 guarantees that ℓ_{def13L} supports C_4 on the right:

$$x_4 = -\left(\gamma + 2r + \sqrt{5r^2 + 2r\gamma + \gamma^2}\right)$$

As discussed in Lemma 3.3, a line supports $\{C_2, C_3, C_4\}$ only if $\ell_{sep23LR}$ supports C_4. If $\ell_{sep23LR}$ supports C_4 on the right, then it is a definite support of $\{C_3, C_4\}$, parallel to

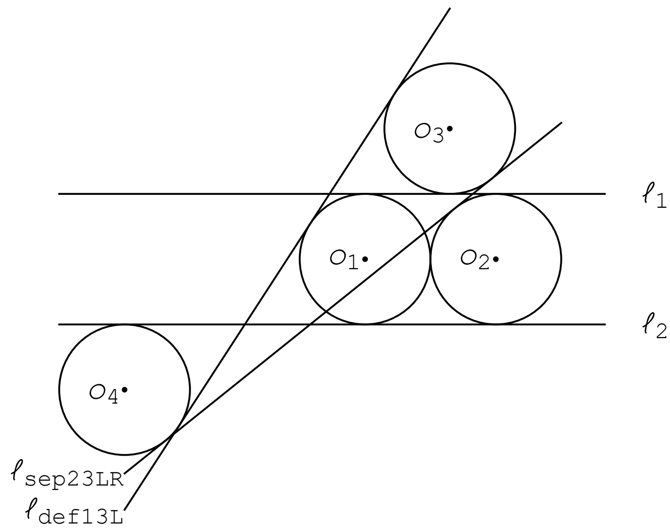


Figure 3.3: Touching critical family $\mathcal{F}_4$ with $r = 1$ and $\gamma \approx 0.2956$. Line $\ell_{sep23LR}$ supports C_4 on the right.

$\langle o_3, o_4 \rangle$, with slope

$$k_{sep23LR} = \frac{4r}{2\gamma + 2r + \sqrt{5r^2 + 2r\gamma + \gamma^2}}.$$

We obtain a second expression k for the slope of $\ell_{sep23LR}$. Since the line contains the midpoint $((\gamma + r)/2, r)$ of $[o_2, o_3]$, and passes at a distance r to the point $o_2\,(r, 0)$, applying Lemma 2.5 to line $\ell_{sep23LR}$ and the point o_2 yields the following string of equivalences:

$$\begin{aligned} \mathrm{d}\left(kx - y - \frac{\gamma + r}{2}k + r = 0, o_2\,(r, 0)\right) = r \iff & \left|k(r) - 1 \cdot (0) - \frac{\gamma + r}{2}k + r\right| = r\sqrt{k^2 + 1} \\ \iff & k(r) - \frac{\gamma + r}{2}k + r = r\sqrt{k^2 + 1} \end{aligned}$$

Since $\ell_{sep23LR}\,(r) > 0$ (see Figure 3.3), we lift the absolute value to solve for k. Equating the two expressions for the slope $k_{sep23LR} = k$, as in

$$\frac{4r}{2\gamma + 2r + \sqrt{5r^2 + 2r\gamma + \gamma^2}} = \frac{4r(r - \gamma)}{3r^2 + 2r\gamma - \gamma^2},$$

leads to the following equation in the indeterminate γ parameterized by r:

$$\gamma^3 + (r)\gamma^2 + (3r^2)\gamma - r^3 = 0$$

In terms of r, the nonnegative real root is given by

$$\gamma = \frac{r}{3}\left(\beta^+ + \beta^- - 1\right),$$

with $\beta^\pm = \sqrt[3]{2}\sqrt[3]{13 \pm 3\sqrt{33}}$. Since $\beta^+ + \beta^- - 1 < 3$, the bound $\gamma < r$ holds. With $\delta = r > 0$, and γ equal to the value derived above, the touching family as described is critical. $\square$

b). Lemmas 3.3 and 3.4 account for the touching critical families $\mathcal{F}_4$, where both ℓ_2 and ℓ_{def13L} support C_4, and C_4 avoids support from ℓ_v. We now show that no further critical families ($\gamma < r$) avoid three disks in a slab where C_4 avoids support from ℓ_v. Recall that the lines in $\mathcal{L}_{13}$ comprise the supports of $\{C_1, C_3\}$. Of these supports, line ℓ_1 is disjoint from C_4 by the definition of critical family, and the two preceding lemmas exhaust the configurations where line ℓ_{def13L} supports C_4.

We proceed with the two remaining supports $\ell_{def13R}, \ell_{sep13RL}$ of $\mathcal{L}_{13}$ in order. In the following, we describe each relevant critical configuration of disks and support lines, and explicitly show that each one is not geometrically realizable.

In no touching critical $\mathcal{F}_4$ that avoids three disks in a slab do both of ℓ_2, ℓ_{def13R} support C_4 with $y_4 = -2r$. If ℓ_{def13R} supports C_4 it necessarily separates C_4 from $\{C_1, C_3\}$ to avoid three disks in a slab, supporting C_4 on the left. A line in

$$\mathcal{L}_{23} = \{\ell_{def23L}, \ell_{def23R}, \ell_{sep23LR}, \ell_{sep23RL} = \ell_1\}$$

necessarily supports $\{C_2, C_3, C_4\}$ (compare Figure 3.4). Of these lines, line ℓ_1 is disjoint from C_4, and the definite supports of $\{C_2, C_3\}$ do not support C_4: by symmetry, the supports $\ell_{def13R}, \ell_{def23L}$ of C_3 support both of C_3, C_4 if and only if ℓ_{def13R} supports C_4

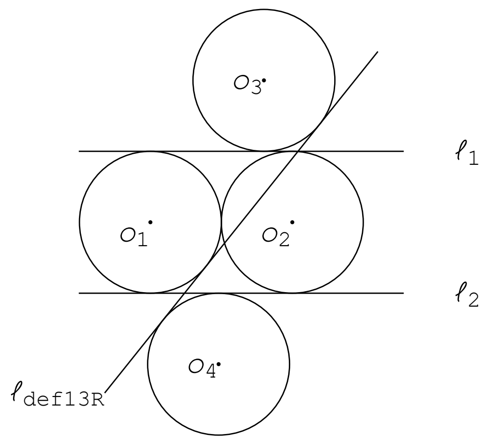


Figure 3.4: The depicted family $\mathcal{F}_4$ with $\gamma < r$ is not critical.

on the left and their point of intersection p lies on the line $\{y = 0\}$. With $\gamma < r$, line ℓ_{def23L} has negative slope and cuts C_1, exiting its boundary ∂C_1 below the line $\{y = 0\}$ since C_1, C_2 are touching. Since the support ℓ_{def13R} also meets ∂C_1 below the line $\{y = 0\}$, the supports $\ell_{def13R}, \ell_{def23L}$ of C_3 necessarily intersect at a point p below the x-axis. Since ℓ_{def13R} supports C_4 on the left and p lies below the x-axis, the line ℓ_{def23L} necessarily cuts C_4.

Furthermore, since ℓ_{def23L} cuts C_4, the parallel definite support ℓ_{def23R} is disjoint from C_4 by symmetry (compare Figure 3.4). Finally, line $\ell_{sep23LR}$ ($\neq \ell_1$) is disjoint from C_4: rotate line ℓ_{def13R}, which supports C_4 and cuts C_2, clockwise away from C_4 dynamically maintaining contact with the boundary of C_3 until it supports C_2 on the left in the position of support $\ell_{sep23LR}$ disjoint from C_4. No line supports the subfamily.

In no touching critical family $\mathcal{F}_4$ that avoids three disks in a slab does ℓ_2 support C_4 with $y_4 = -2r$, and line $\ell_{sep13RL}$ (negative slope) support C_4 on the right. Since $\mathcal{F}_4$ is critical, a line supports $\{C_2, C_3, C_4\}$. Line ℓ_{def23L} cuts C_4: rotate line $\ell_{sep13RL}$, which supports C_4 (and cuts C_2), clockwise into C_4 dynamically maintaining contact with the boundary of C_3 until it supports C_2 on the left. This line is in the position of support ℓ_{def23L} and it cuts C_4 for $\gamma < r$. Since the disk has radius r, the parallel right definite support ℓ_{def23R} of $\{C_2, C_3\}$

is disjoint from C_4. By symmetry, the supports $\ell_{sep13RL}$ and $\ell_{sep23LR}$ (positive slope) of C_3 support both of C_3, C_4 if and only if $\ell_{sep13RL}$ supports C_4 on the right and their point of intersection p lies on the x-axis (so that $\ell_{sep23LR}$ supports C_4 on the left). Line $\ell_{sep13RL}$ (negative slope) meets the boundary ∂C_1 of C_1 above the x-axis, and line $\ell_{sep23LR}$ (positive slope) meets the boundary ∂C_2 of C_2 above the x-axis, so these two lines meet at a point p above the x-axis, exterior to both of C_1, C_2. Since $\ell_{sep13RL}$ supports C_4 on the right and p lies above the x-axis, the line $\ell_{sep23LR}$ is disjoint from C_4. No line supports the subfamily.

In no touching critical family $\mathcal{F}_4$ that avoids three disks in a slab does $\ell_{sep13RL}$ support C_4 on the left, and line ℓ_2 support C_4 with $y_4 = -2r$. If $\ell_{sep13RL}$ supports C_4 on the left, then $x_4 > r$: in the limit $\gamma \to r$, the line $\ell_{sep13RL} \to \ell_v$, and ℓ_v supports C_4 on the left if and only if $x_4 = r$. Perturbing disk C_3 from this position a positive distance $\varepsilon > 0$ so that $\gamma = r - \varepsilon < r$ induces a negative slope in line $\ell_{sep13RL}$ which then cuts any congruent disk with center $o(r, -2r)$, forcing $x_4 > r$ since $y_3 = -2r$. Consequently C_4 avoids ℓ_v. Since $\mathcal{F}_4$ is critical, a line in $\mathcal{L}_{23}$ supports $\{C_2, C_3, C_4\}$. The left definite support ℓ_{def23L} of $\{C_2, C_3\}$ is disjoint from C_4 by construction since it lies to the left of $\ell_{sep13RL}$ below ℓ_2.

Continuing with this configuration (lines $\ell_2, \ell_{sep13RL}$ support C_4), line ℓ_{def23R} cuts C_4 whenever $0 \leq \gamma < r$: suppose $\gamma > 0$ and ℓ_{def23R} supports C_4. Line ℓ_{def23R} supports C_4 on the right precisely when $\gamma = r$ which is not permitted, so the line necessarily supports C_4 on the left. Reflecting the family over the vertical line ℓ_v (a symmetry in V) preserves the labels on C_3, C_4, interchanges C_1 with C_2, and reverses left-right orientation. So the critical supports that support C_4 on the left map to $\ell_{sep13RL} \mapsto \ell_{sep23LR}$ and $\ell_{def23R} \mapsto \ell_{def13L}$ both of which support C_4 on the right. The family coincides with the configuration documented in Lemma 3.4 with $\gamma > 0$. Since this is the unique family with this configuration of supports, it follows that the family where $\ell_{sep13RL}$ and ℓ_{def23R} support C_4 on the left (by reflection over line ℓ_v) is critical only if $\gamma < 0$, a contradiction. As a consequence, whenever $0 \leq \gamma < r$, line ℓ_{def23R} cuts C_4. Finally, in the slightly expanded range $-r < \gamma \leq r$ the line $\ell_{sep23LR}$ (with nonnegative slope) never enters the cone in the fourth quadrant determined by ℓ_2 and

ℓ_v, and is therefore necessarily disjoint from C_4 in the restricted range $0 \leq \gamma < r$. No line supports $\{C_2, C_3, C_4\}$.

The preceding arguments exhaust the critical configurations for touching families $\mathcal{F}_4$ avoiding three disks in a slab, in which C_4 is not supported by ℓ_v.

c). We proceed documenting critical families $\mathcal{F}_4$ avoiding three disks in a slab, where ℓ_1 supports C_3 with center $o_3(\gamma, 2r)$ (nonnegative $\gamma < r$). In the following, we lift the restriction on ℓ_v and require C_4 to have its center in $\{o_4(\pm r, y_4)\}$ so that ℓ_v supports C_4 and consequently $\{C_1, C_2, C_4\}$. We place no restrictions on support from ℓ_2, so that $y_4 = \pm 2r$ is permitted and the bound $|y_4| \geq 2r$ ensues since $\mathcal{F}_4$ is nonoverlapping. Line $\ell_1 \in \mathcal{L}_{13}$ is not permitted to support $\{C_1, C_3, C_4\}$.

If $y_4 \leq -2r$, then the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ is disjoint from C_4: line ℓ_{def13L}, which has positive slope and supports C_1 on the left, is disjoint from disk C_4 which lies directly below C_1 supported by ℓ_v. In no critical family $\mathcal{F}_4$ does ℓ_{def13L} support C_4 with $y_4 < 0$. In the following lemma, we document the touching critical family $\mathcal{F}_4$, where line ℓ_{def13L} supports C_4 on the right and $y_4 > 0$.

Lemma 3.5. Let $\delta = x_4 = r$, and let ℓ_1 support C_3 with center $o_3(\gamma, 2r)$. If nonnegative $\gamma < r$ is a real solution of the equation

$$\gamma^4 - (4r)\gamma^3 + \left(9r^2\right)\gamma^2 - \left(14r^3\right)\gamma + 4r^4 = 0,$$

then the touching family $\mathcal{F}_4$ is critical. Furthermore, the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ supports C_4 on the right (from below), and the right definite support ℓ_{def23R} of $\{C_2, C_3\}$ supports C_4 on the left (from below).

Proof. Let ℓ_1 support $\{C_1, C_2, C_3\}$ which then supports C_3 with center $o_3(\gamma, 2r)$ where $0 \leq \gamma < r$. Let $\delta = x_4 = r$, so that ℓ_v supports $\{C_1, C_2, C_4\}$. Let ℓ_{def13L} support $\{C_1, C_3, C_4\}$, separating C_4 from $\{C_1, C_3\}$ to avoid three disks in a slab. An equation for line ℓ_{def13L} is derived in the preceding lemma, and we reproduce it here:

$$\ell_{def13L}(x) = \frac{2r}{\gamma + r}x + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}$$

Since $y_4 < 0$ implies C_4 is disjoint from the line, we must have $y_4 > 0$, and consequently ℓ_{def13L} supports C_4 from below (on the right). Applying Lemma 2.5 to calculate the distance

$$\mathrm{d}\left(\ell_{def13L}(x) - y = 0, o_4(r, y_4)\right) = r$$

yields the following equation:

$$\left|\frac{2r}{\gamma + r}(r) + (-1)(y_4) + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}\right| = r\sqrt{\left(\frac{2r}{\gamma + r}\right)^2 + (-1)^2}$$

Here, since $\ell_{def13L}(r) < y_4$, we evaluate the norm using the inequality $y_4 - \ell_{def13L}(r) > 0$, and simplify to obtain

$$y_4 = \frac{4r^2 + 2r\sqrt{5r^2 + 2r\gamma + \gamma^2}}{\gamma + r}.$$

A line must support $\{C_2, C_3, C_4\}$. The left definite support ℓ_{def23L} of $\{C_2, C_3\}$ is disjoint from C_4 since it avoids the first quadrant. The separating support $\ell_{sep23LR}$ ($\neq \ell_v$) of $\{C_2, C_3\}$ is disjoint from C_4: rotate ℓ_{def13R}, which is disjoint from C_4 and cuts C_2, clockwise away from C_4 a positive magnitude, dynamically maintaining contact with the boundary of C_3 until it supports C_2. The resulting line, in the position of $\ell_{sep23LR}$, remains disjoint from C_4.

Finally, line ℓ_{def23R} necessarily supports C_4, on the left in order to avoid three disks in a slab and the support property S. Line ℓ_{def23R} is parallel to $\langle o_2, o_3\rangle$ with slope $-2r/(r - \gamma)$ and we denote its y-intercept by m. Applying Lemma 2.5 to calculate the distance

$\mathrm{d}\left(\ell_{def23R}(x) - y = 0, o_2\left(r, 0\right)\right) = r$ yields the following equation:

$$\left|\frac{-2r}{r-\gamma}(r) + (-1)(0) + m\right| = r\sqrt{\left(\frac{-2r}{r-\gamma}\right)^2 + 1}$$

Since $\ell_{def23R}(r) > 0$, we lift the absolute value to solve for the intercept $m > 0$ (see Figure 3.5), which corresponds to the positive branch of the solution, and an equation for the line follows:

$$\ell_{def23R}(x) = -\frac{2r}{r-\gamma}x + \frac{2r^2 + r\sqrt{5r^2 - 2r\gamma + \gamma^2}}{r-\gamma}$$

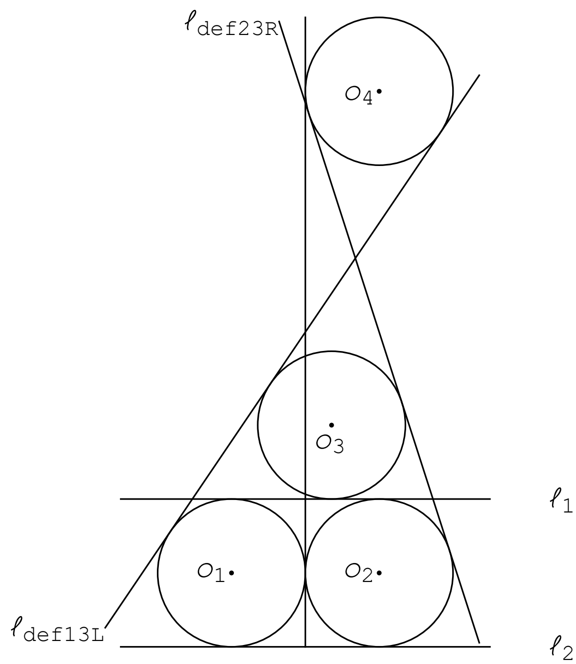


Figure 3.5: Touching critical family $\mathcal{F}_4$ with $r = 1$ and $\gamma \approx 0.3551$. Line ℓ_{def23R} supports C_4 on the left.

Since the line supports $\{C_2, C_3, C_4\}$, Lemma 2.5 provides the following equation for the distance $\mathrm{d}\left(\ell_{def23R}(x) - y = 0, o_4(r, y_4)\right) = r$:

$$\left|\frac{-2r}{r-\gamma}(r) - \frac{4r^2 + 2r\sqrt{5r^2 + 2r\gamma + \gamma^2}}{\gamma + r} + \frac{2r^2 + r\sqrt{r^2 - 2r\gamma + \gamma^2}}{r - \gamma}\right| = r\sqrt{\left(\frac{-2r}{r-\gamma}\right)^2 + (-1)^2}$$

Since $\ell_{def23R}(r) < y_4$, we evaluate the norm with $y_4 - \ell_{def23R}(r) > 0$, and simplify to arrive at the following equation in γ with parameter r, which determines the critical family:

$$(\gamma + r)^2 \left(\gamma^4 - (4r)\gamma^3 + \left(9r^2\right)\gamma^2 - \left(14r^3\right)\gamma + 4r^4\right) = 0$$

Since the fourth degree polynomial above evaluated at $\gamma = 0$ yields $4r^4$ and evaluated at $\gamma = r$ yields $-4r^4$, the intermediate value theorem guarantees a solution $0 < \gamma < r$ which places the family in the critical configuration described. $\square$

We recall that the lines in $\mathcal{L}_{13} = \{\ell_{def13L}, \ell_{def13R}, \ell_{sep13LR} = \ell_1, \ell_{sep13RL}\}$ comprise the supports of $\{C_1, C_3\}$. The preceding lemma describes critical families $\mathcal{F}_4$ where ℓ_1 supports C_3 and both of ℓ_{def13L}, ℓ_v support C_4. Since line ℓ_1 supports C_3, it is disjoint from C_4. We show that the remaining supports in $\mathcal{L}_{13}$ do not induce any further critical families $\mathcal{F}_4$ that are geometrically realizable.

In no touching critical family $\mathcal{F}_4$ that avoids three disks in a slab do both of ℓ_v, ℓ_{def13L} support C_4 with center $o_4(-r, y_4)$. Since line ℓ_{def13L} is disjoint from C_4 whenever it lies below the x-axis with $x_4 = -r$, this forces C_4 into the second quadrant with $y_4 > 0$. A line must support $\{C_2, C_3, C_4\}$. To avoid three disks in a slab, any definite support in $\mathcal{L}_{23}$ must separate C_4 from the subfamily. Line ℓ_{def23L} supports C_4 on the right only if $\gamma = r$, which is beyond the bound given. Since line ℓ_{def13L} supports C_4 with $x_4 = -r$, this forces the center of C_4 below the line $\{y = 4r\}$, so that the definite support ℓ_{def23R} cannot support C_4 on the left. Line $\ell_{sep23LR}$ ($\neq \ell_1$) is disjoint from C_4 since it does not enter the cone bounded by ℓ_1 and ℓ_v. No critical configuration is possible.

In no touching critical family $\mathcal{F}_4$ with γ small ($\gamma < r$ so ℓ_1 supports C_3) do both of ℓ_v, ℓ_{def13R} support C_4 with center $o_4(\pm r, y_4)$. The line ℓ_{def13R} must support C_4 on the left to avoid three disks in a slab. If $y_4 > 0$ then C_4 with $x_4 = -r$ is disjoint from the line (for all $\gamma \geq 0$). If $x_4 = r$ (with $y_4 > 0$), then ℓ_{def13R} supports C_4 on the left precisely when $\gamma = -r$ (and not when $\gamma > 0$). The alternative, $y_4 < 0$, requires $y_4 \leq -2r$ since $\mathcal{F}_4$ is nonoverlapping. If $x_4 = r$, then ℓ_{def13R} is disjoint from C_4 since it doesn't enter the cone bounded by ℓ_2 and ℓ_v.

Finally, if $x_4 = -r$ (with $y_4 \leq -2r$), it is possible for ℓ_{def13R} to support C_4 on the left. A line must support $\{C_2, C_3, C_4\}$. Both definite supports of $\{C_2, C_3\}$ have negative slope (since $\gamma < r$) and are disjoint from the cone bounded by ℓ_2 and ℓ_v that contains C_4. The remaining support $\ell_{sep23LR}$ ($\neq \ell_v$) is disjoint from C_4: rotate ℓ_{def13R}, which supports C_4 and cuts C_2, clockwise a positive magnitude away from C_4 dynamically maintaining contact with the boundary of C_3 until it supports C_2 on the left. This line, disjoint from C_4, is in the position of $\ell_{sep23LR}$. No critical family fits this description.

In no touching critical family $\mathcal{F}_4$ where γ is small ($\gamma < r$) do both of $\ell_v, \ell_{sep13RL}$ support C_4 with center $o_4(\pm r, y_4)$. If either $y_4 > 0$ and $x_4 = r$, or alternatively if $y_4 < 0$ and $x_4 = -r$, then the line $\ell_{sep13RL}$ (negative slope) is disjoint from the respective cone containing C_4 (bounded respectively by ℓ_1, ℓ_v or ℓ_2, ℓ_v). If $x_4 = -r$ and $y_4 > 0$, then support on the left is possible. A line must support $\{C_2, C_3, C_4\}$. Line ℓ_{def23L} necessarily cuts C_4: rotate $\ell_{sep13RL}$, which supports C_4, clockwise into C_4 dynamically maintaining contact with the boundary of C_3 until it supports C_2. This line is in the position of ℓ_{def23L} and necessarily cuts C_4 (since $\gamma < r$). By symmetry, line ℓ_{def23R} (parallel at a distance $2r$) is disjoint from C_4. The separating support ($\neq \ell_1$) of $\{C_2, C_3\}$ has positive slope and is disjoint from the cone bounded by ℓ_1, ℓ_v that contains C_4. The family is not geometrically realizable.

Similarly, if $x_4 = r$ and $y_4 < 0$, then $\ell_{sep13RL}$ (for a restricted range of γ) supports C_4 on the right. In this position, the left definite support ℓ_{def23L}, which separates $\{C_2, C_3\}$ from $\{C_4\}$, necessarily cuts C_4: rotate $\ell_{sep13RL}$, which supports C_4 and cuts C_2, clockwise into C_4 dynamically maintaining contact with the boundary of C_3, until it supports C_2 (on the left). The line is in the position of ℓ_{def23L} and necessarily cuts C_4 (since it has negative slope for $\gamma < r$). Since ℓ_{def23L} cuts C_4, the parallel support ℓ_{def23R} is disjoint from C_4 by symmetry. Finally, line $\ell_{sep23LR}$ (positive slope) is disjoint from the cone containing C_4. No critical family fits this description.

3.1.2 Case 2: $\gamma = r$

When $\gamma = r$, line ℓ_v necessarily supports C_3 and though ℓ_1 is permitted to support it, no critical family has this configuration of supports: if both ℓ_1, ℓ_v support C_3, then disk C_3 has center $o_3(r, 2r)$, and the touching subfamily $\{C_2, C_3\}$ has precisely three support lines two of which coincide with ℓ_1, ℓ_v. Line ℓ_2 must support $\{C_1, C_2, C_4\}$ to secure $S(3)$ and avoid $S(4) = S$, and the vertical line ℓ_{def23R} must support $\{C_2, C_3, C_4\}$ since lines ℓ_1, ℓ_v are not permitted to support C_4. This forces C_4 to have center $o_4(3r, -2r)$, and inspection confirms that no line in $\mathcal{L}_{13}$ supports subfamily $\{C_1, C_3, C_4\}$ with C_4 in this position.

Since $\mathcal{F}_4$ is nonoverlapping, the bound $|y_3| \geq 2r$ holds. By reflection symmetry over the x-axis (in the Klein four-group V) any configuration with $y_3 \leq -2r$ maps to an equivalent one with $y_3 \geq 2r$, so we stipulate that $y_3 \geq 2r$. By the argument given above, no critical family avoids three disks in a slab where C_3 has center $o_3(r, 2r)$. We proceed with critical families where C_3 has center $o_3(r, y_3)$ with $y_3 > 2r$ so that ℓ_v supports C_3 and ℓ_1 may support C_4. One of ℓ_1, ℓ_2 must support C_4. The critical families with $\gamma = r$ where line ℓ_1 supports C_4 are described in Lemmas 3.8 through 3.15. We proceed with the configuration where both of ℓ_2, ℓ_{def13L} support C_4 so that C_4 has center $o_4(x_4, -2r)$. In the following lemma with parameter $\gamma = r$, the family avoids three disks in a slab, line ℓ_2 supports C_4, and line ℓ_{def13L} supports C_4 on the right.

Lemma 3.6. Let $\delta = r$, coordinate $y_4 = -2r$, and C_3 have center $o_3(r, y_3)$. If $y_3 > 2r$ is a real solution of the equation

$$(3)y_3^4 + (4r)y_3^3 - \left(20r^2\right) y_3^2 + \left(32r^3\right) y_3 - 128r^4 = 0,$$

then the touching family $\mathcal{F}_4$ is critical. Furthermore, the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ and the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ both support C_4 on the right.

Proof. Let $\delta = r = \gamma$, so that ℓ_v supports C_3, and consequently $\{C_1, C_2, C_3\}$. Let C_3 have center $o_3(r, y_3)$ where we stipulate that $y_3 > 2r$. Let ℓ_{def13L} support C_4 on the right so that it is parallel to $\langle o_1, o_3\rangle$ with slope $y_3/2r$. Denote its y-intercept by m, and observe that this line supports C_1. Applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(\ell_{def13L}(x) - y = 0, o_1(-r, 0)\right) = r$ leads to the following equation:

$$\left|\frac{y_3}{2r}(-r) + (-1)(0) + m\right| = r\sqrt{\left(\frac{y_3}{2r}\right)^2 + (-1)^2}$$

By construction $y_3, m > 0$ (compare Figure 3.6). Furthermore, $m = y_3$ implies line ℓ_{def13L} is a vertical support of C_3, so $m > y_3 > \dfrac{y_3}{2} > 0$ which allows us to lift the norm and derive

$$\ell_{def13L}(x) = \frac{y_3}{2r}x + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2}.$$

Applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(\ell_{def13L}(x) - y = 0, o_4(x_4, -2r)\right) = r$ yields the following equation:

$$\left|\frac{y_3}{2r}x_4 + (-1)(-2r) + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2}\right| = r\sqrt{\left(\frac{y_3}{2r}\right)^2 + (-1)^2}$$

Since $\ell_{def13L}(x_4) < -3r$, we have $-(\ell_{def13L}(x_4) + 2r) > 0$, which provides the following expression for x_4 consistent with support from this line:

$$x_4 = -\frac{2r}{y_3}\left(\sqrt{y_3^2 + 4r^2} + \frac{y_3}{2} + 2r\right)$$

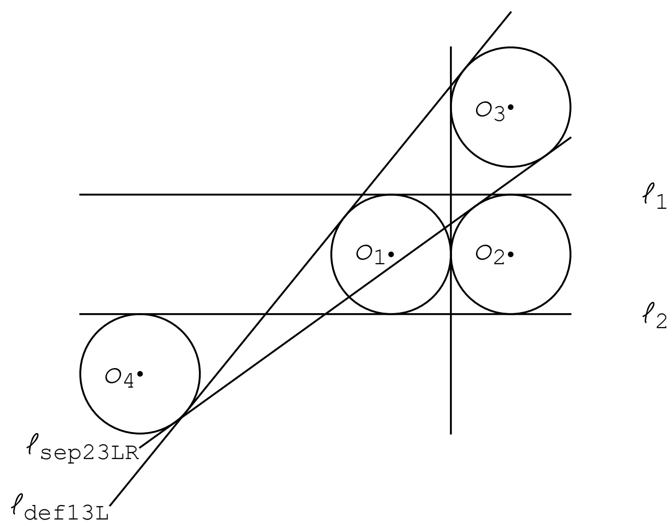


Figure 3.6: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $x_4 \approx -5.1984$ and $y_3 \approx 2.4648$. Line $\ell_{sep23LR}$ supports C_4 on the right.

A line must support $\{C_2, C_3, C_4\}$. Since $x_4 < -4$ by construction, both definite supports of $\{C_2, C_3\}$ are disjoint from C_4 (observe that $\ell_{def23L} = \ell_v$). So, the separating support $\ell_{sep23LR}$ (with positive slope) must support C_4. The line supports C_4 on the left or on the right. In Lemma 3.7 below, we describe the family with support on the left.

Let $\ell_{sep23LR}$ support C_4 on the right. This line contains the midpoint $(r, y_3/2)$ of the interval $[o_2, o_3]$, and the resulting expression for its slope as a definite support of $\{C_3, C_4\}$

leads to the following equation for the line:

$$\ell_{sep23LR}(x) = \frac{2y_3(y_3+2r)}{4ry_3+8r^2+4r\sqrt{y_3^2+4r^2}}(x-r)+\frac{y_3}{2}$$

Applying Lemma 2.5 to line $\ell_{sep23LR}$ with slope k and the point $o_4(x_4,-2r)$ leads to the following equation:

$$\frac{2r}{y_3}\left(\sqrt{y_3^2+4r^2}+\frac{y_3}{2}+2r\right)k-2r+kr-\frac{y_3}{2}=r\sqrt{k^2+1}$$

Substituting the expression for the slope of line $\ell_{sep23LR}$ from its equation above in place of the parameter k in the equation given directly above and rewriting leads to the following equation in the indeterminate y_3 parameterized by r:

$$(y_3+2r)^2\left((3)y_3^4+(4r)y_3^3-\left(20r^2\right)y_3^2+\left(32r^3\right)y_3-128r^4\right)=0$$

Since the fourth degree polynomial evaluated at $y_3=2r$ yields $64r^4$ and evaluated at $y_3=3r$ yields $-139r^4$, the intermediate value theorem guarantees a solution $2r<y_3<3r$ which places the family in the critical configuration described. □

Lemma 3.7. Let $\delta=r$, disk C_4 have center $o_4(x_4,-2r)$, and C_3 have center $o_3(r,y_3)$. If $y_3>2r$ is a positive real solution of the equation

$$(2)y_3^3+(7r)y_3^2-\left(8r^2\right)y_3-32r^3=0,$$

then the touching family $\mathcal{F}_4$ is critical. Furthermore, the left definite support ℓ_{def13L} of $\{C_1,C_3\}$ supports C_4 on the right and the separating support $\ell_{sep23LR}$ of $\{C_2,C_3\}$ supports C_4 on the left.

Proof. Let $\delta = \gamma = r$ and let $y_3 > 2r$ so that C_3 with center $o_3(r, y_3)$ is disjoint from ℓ_1. Let ℓ_{def13L} of $\mathcal{L}_{13}$ support C_4, whose equation we reproduce from the preceding lemma:

$$\ell_{def13L}(x) = \frac{y_3}{2r}x + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2}$$

An expression for x_4 consistent with support from this line is found by applying Lemma 2.5 to line ℓ_{def13L} and the point o_4:

$$x_4 = -\frac{2r}{y_3}\left(\sqrt{y_3^2 + 4r^2} + \frac{y_3}{2} + 2r\right)$$

The expression for x_4 follows from the inequality $\ell_{def13L}(x_4) + 2r < 0$.

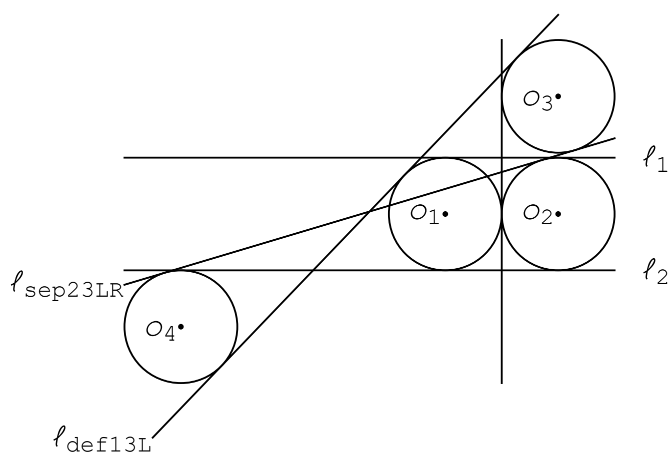


Figure 3.7: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 2.0876$ and $x_4 \approx -5.6859$. Line $\ell_{sep23LR}$ supports C_4 on the left.

As in the preceding lemma, the definite supports of $\{C_2, C_3\}$ are disjoint from C_4, and here we describe the family where $\ell_{sep23LR}$ supports C_4 on the left. Line $\ell_{sep23LR}$ contains the point $\left(r, \dfrac{y_3}{2}\right)$ and is a definite support of $\{C_2, C_4\}$, so its equation is the following:

$$\ell_{sep23LR}(x) = \frac{y_3}{2r + y_3 + \sqrt{4r^2 + y_3^2}}(x - r) + \frac{y_3}{2}$$

Alternatively, since the line supports C_2, Lemma 2.5 applied to line $\ell_{sep23LR}$ and the point o_2 yields a second expression k for its slope. Equating the two expressions, as in

$$\frac{\sqrt{y_3^2 - 4r^2}}{2r} = \frac{y_3}{2r + y_3 + \sqrt{4r^2 + y_3^2}},$$

leads to the following equation:

$$(2)y_3^3 + (7r)\, y_3^2 - \left(8r^2\right) y_3 - 32r^3 = 0$$

An expression for the solution of this polynomial equation is

$$y_3 = \frac{r}{6}\left(\beta^+ + \beta^- - 7\right)$$

where $\beta^{\pm} = \sqrt[3]{881 \pm 24i\sqrt{237}}$. Using a suitable reference triangle, this expression can be rewritten as

$$y_3 = \frac{r}{6}\left(-7 + 2\sqrt{97}\cos(\phi/3)\right) \in \mathbb{R}$$

with $\phi = \tan^{-1}(24\sqrt{237}/881)$ which verifies the solution is real. Since the polynomial evaluated at $y_3 = 2r$ yields $-4r^3$ and evaluated at $y_3 = 3r$ yields $61r^3$, the intermediate value theorem guarantees a solution $2r < y_3 < 3r$ which places the family in the critical configuration described. $\square$

With $\gamma = r$ and $y_3 > 2r$, the preceding proof shows that in no further critical families do both of ℓ_2, ℓ_{def13L} support C_4 since no other lines in $\mathcal{L}_{23}$ are capable of supporting C_4.

Since $y_3 > 2r$, line ℓ_1 is permitted to support C_4, and we describe critical configurations of disks where both of ℓ_1, ℓ_{def13L} support C_4.

Lemma 3.8. Let $\delta = r$, coordinate $y_4 = 2r$, and C_3 have center $o_3(r, y_3)$. If $y_3 > 2r$ is the smallest positive real solution of the equation

$$(2)y_3^3 - (7r)y_3^2 - \left(8r^2\right) y_3 + \left(32r^3\right) = 0,$$

then the touching family $\mathcal{F}_4$ is critical. Furthermore, both the left definite support ℓ_{def13L} of $\{C_1, C_3\}$ and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ support C_4 on the right.

Proof. Let $\delta = r = \gamma$, so that ℓ_v supports C_3 and consequently $\{C_1, C_2, C_3\}$. Let C_3 have center $o_3(r, y_3)$ with $y_3 > 0$ which forces $y_3 > 2r$. Let ℓ_1 support $\{C_1, C_2, C_4\}$, so that C_4 has center $o_4(x_4, 2r)$. Let ℓ_{def13L} support $\{C_1, C_3, C_4\}$, so that the following equation from the previous lemma describes the line:

$$\ell_{def13L}(x) = \frac{y_3}{2r}x + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2}$$

We find a commensurate expression for x_4 by applying Lemma 2.5 to evaluate the distance $\mathrm{d}\left(\ell_{def13L}(x) - y = 0, o_4(x_4, 2r)\right) = r$ which results in the following equation:

$$\left| \frac{y_3}{2r}x_4 + (-1)(-2r) + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2} \right| = r\sqrt{\left(\frac{y_3}{2r}\right)^2 + (-1)^2}$$

Since line ℓ_{def13L} necessarily supports C_4 on the left to avoid three disks in a slab, the positive value $\ell_{def13L}(x_4) < r$ implies $|\ell_{def13L}(x_4) - 2r| = 2r - \ell_{def13L}(x_4) > 0$ which permits us to rewrite the preceding as

$$x_4 = -\frac{r}{y_3}\left(2\sqrt{y_3^2 + 4r^2} + y_3 - 4r\right).$$

The definite supports of $\{C_2, C_3\}$ and the separating support $\ell_{sep23LR}$ are disjoint from C_4, so the associated separating support $\ell_{sep23RL}$ (with negative slope) must support C_4, necessarily on the right hand side. This line contains the midpoint $(r, y_3/2)$ of $[o_2, o_3]$ and is a definite support of $\{C_2, C_4\}$ parallel to $\langle o_2, o_4 \rangle$, so an equation for the line is given by

$$\ell_{sep23RL}(x) = \frac{y_3}{2r - y_3 - \sqrt{4r^2 + y_3^2}}(x - r) + \frac{y_3}{2}.$$

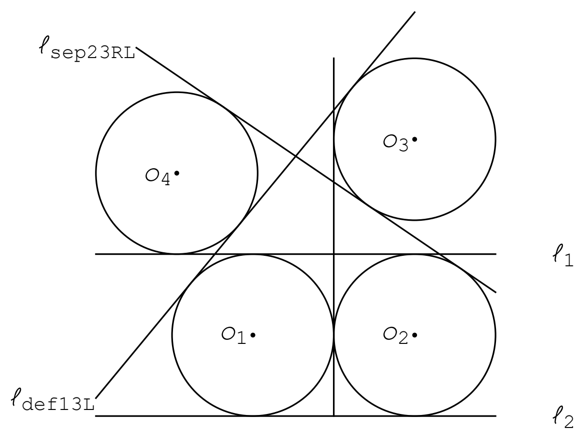


Figure 3.8: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 2.4185$ and $x_4 \approx -1.9414$. Line $\ell_{sep23RL}$ supports C_4 on the right.

Since this line supports C_4, Lemma 2.5 applied to line $\ell_{sep23RL}$ with slope k and the point o_4 leads to the following equation:

$$-\frac{2r}{y_3}\left(\sqrt{y_3^2 + 4r^2} + \frac{y_3}{2} - 2r\right)k - 2r - kr + \frac{y_3}{2} = r\sqrt{k^2 + 1}$$

The preceding expression without the norm is correct since $\ell_{sep23RL}(x_4) > 2r$. Substituting the expression for the slope of line $\ell_{sep23RL}(x)$ from its equation given above in place of k

in the equation directly above, leads to the following equation in the indeterminate y_3 with parameter r:

$$(2)y_3^3 - (7r)y_3^2 - \left(8r^2\right)y_3 + \left(32r^3\right) = 0$$

If y_3 is the smallest positive real solution of this equation, then the family $\mathcal{F}_4$ is critical. Specifically, the real solution can be written as

$$y_3 = \frac{r}{6}\left(7 + \sin\left(\frac{\phi}{3}\right)\left(\sqrt{97} + \frac{97}{\sqrt{97}}\right)\right),$$

where $\phi = \tan^{-1}\left(881/24\sqrt{237}\right)$. The solution is real since its imaginary component (not expressed above) contains the conjugate factor $97/\sqrt{97} - \sqrt{97}$ which is identically zero. Since the polynomial evaluated at $y_3 = 2r$ yields $4r^3$ and evaluated at $y_3 = 3r$ yields $-r^3$, the intermediate value theorem guarantees a solution $2r < y_3 < 3r$ which places the family in the critical configuration described. □

The preceding contains exhaustive descriptions of the critical families with $\gamma = r$ where lines ℓ_{def13L} and ℓ_1 both support C_4. We proceed with the description of critical configurations ($\gamma = r$), where $y_3 > 2r$ (by symmetry) and line ℓ_{def13R} supports C_4, necessarily on the left to avoid three disks in a slab. By the comments leading up to Lemma 3.3, in no critical family does ℓ_{def13R} support C_4, and C_3 have center $o_3(r, 2r)$.

In no critical family with $y_3 > 2r$ do both of ℓ_2, ℓ_{def13R} support C_4. Since ℓ_{def13R} necessarily supports C_4 on the left, no line supports $\{C_2, C_3, C_4\}$: by construction line $\ell_{def23L} = \ell_v$ cuts C_4 for all $2r \leq y_3 < \infty$. By symmetry, the right definite support ℓ_{def23R} (at distance $2r$) is disjoint from C_4 for all $2r \leq y_3 < \infty$. Line $\ell_{sep23LR}$ (with positive slope) is disjoint from C_4: rotate ℓ_{def13R}, which supports C_4, clockwise, away from C_4, dynamically maintaining contact with the boundary of C_3, until it supports C_2. This line, in the position of $\ell_{sep23LR}$, is disjoint from C_4. The associated separating support $\ell_{sep23RL}$ is also disjoint. The family is not geometrically realizable.

We proceed with descriptions of critical families where both of ℓ_1, ℓ_{def13R} support C_4.

Lemma 3.9. Let $\delta = r$, line ℓ_1 support C_4, and C_3 have center $o_3(r, y_3)$. If $y_3 > 4r$ is a positive real solution of the equation

$$(3)y_3^4 - (4r)y_3^3 - \left(20r^2\right) y_3^2 - \left(32r^3\right) y_3 - 128r^4 = 0,$$

then the touching family $\mathcal{F}_4$ is critical. Furthermore, the right definite support ℓ_{def13R} of $\{C_1, C_3\}$ and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (with negative slope) both support C_4 on the left.

Proof. Let $\gamma = \delta = r$, and let ℓ_1 support C_4 and consequently $\{C_1, C_2, C_4\}$. Let line ℓ_{def13R} support C_4 (on the left) and consequently $\{C_1, C_3, C_4\}$. The derivation of the equation for ℓ_{def13L} in Lemma 3.7 differs from that of the parallel line ℓ_{def13R} only in the sign on the radical, which leads to

$$\ell_{def13R}(x) = \frac{y_3}{2r}x + \frac{-\sqrt{y_3^2 + 4r^2} + y_3}{2}.$$

Since this line supports C_4, a commensurate expression for x_4 results from applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(\ell_{def13R}(x) - y = 0, o_4(x_4, 2r)\right) = r$ by the following equation:

$$\left|\frac{y_3}{2r}x_4 + (-1)(2r) + \frac{-\sqrt{y_3^2 + 4r^2} + y_3}{2}\right| = r\sqrt{\left(\frac{y_3}{2r}\right)^2 + (-1)^2}$$

Since $\ell_{def13R}(x_4) - 2r > 0$, we lift the absolute value, and rewrite the preceding as

$$x_4 = \frac{2r}{y_3}\left(\sqrt{y_3^2 + 4r^2} + 2r - \frac{y_3}{2}\right).$$

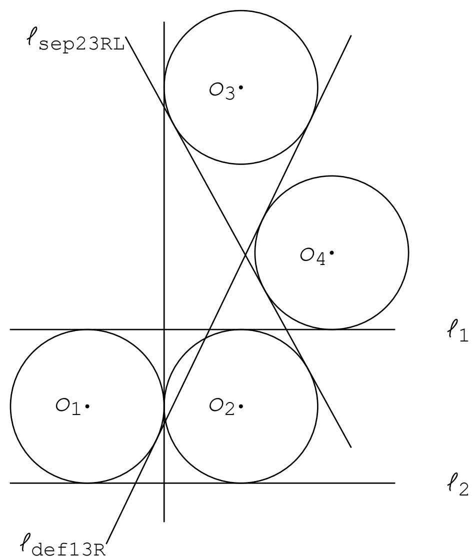


Figure 3.9: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 4.1529$ and $x_4 \approx 2.1830$. Line $\ell_{sep23RL}$ supports C_4 on the left.

A line must support $\{C_2, C_3, C_4\}$. The left definite support ℓ_v of $\{C_2, C_3\}$ is disjoint from C_4. The separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ with positive slope is disjoint from C_4: rotate ℓ_{def13R} which supports C_4, clockwise, away from C_4, dynamically maintaining contact with the boundary of C_3 until it supports C_2 in the position of $\ell_{sep23LR}$ disjoint from C_4. The remaining lines to consider are $\ell_{sep23RL}$ (with negative slope) and ℓ_{def23R}. We document the family with support from ℓ_{def23R} in Lemma 3.10.

Let the line $\ell_{sep23RL}$ support C_4. This requires $y_3 > 4r$: the two oblique separating supports of $\{C_2, C_3\}$ meet at the midpoint $p\,(r, y_3/2)$ of segment $[o_2, o_3]$ by symmetry. If the center of C_4 lies on the horizontal line containing p, then $y_3/2 = y_4 = 2r$, so that $y_3 = 4r$, and both separating supports of $\{C_2, C_3\}$ support C_4 by symmetry. But then ℓ_{def13R} cuts C_4 contrary to the construction. Furthermore, translating disk C_3 vertically downward from this position ($y_3 = 4r$), dynamically maintaining ℓ_{def13R} in contact with ∂C_4, forces C_4 disjoint from $\ell_{sep23RL}$ by geometric inference (compare Figure 3.9). This means disk C_3 must be translated vertically upward in order for both of $\ell_{sep23RL}, \ell_{def13R}$ to support C_4, forcing $y_3 > 4r$. Furthermore, this line can only support C_4 on the left in this configuration.

Since $\ell_{sep23RL}$ contains the midpoint $(r, y_3/2)$ of $[o_2, o_3]$ and supports C_2, applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(k(x-r) - y + \frac{y_3}{2} = 0, o_2(r,0)\right) = r$ results in the following equation:

$$\left|kr + (-1)(0) - kr + \frac{y_3}{2}\right| = \left|\frac{y_3}{2}\right| = \frac{y_3}{2} = r\sqrt{k^2+1}$$

Solving for k, we choose the negative branch, and an equation for the line follows:

$$\ell_{sep23RL}(x) = -\frac{\sqrt{y_3^2 - 4r^2}}{2r}x + \frac{\sqrt{y_3^2 - 4r^2} + y_3}{2}$$

Alternatively, the separating support $\ell_{sep23RL} = \ell_{def34L}$ is a definite support of $\{C_3, C_4\}$, parallel to $\langle o_3, o_4 \rangle$, which provides a second expression k for its slope. Equating the two expressions for the slope, as in

$$\frac{y_3(2r - y_3)}{2r\left(\sqrt{y_3^2 + 4r^2} + 2r - \frac{y_3}{2}\right) - \gamma \cdot y_3} = -\frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$(y_3 - 2r)^2\left((3)y_3^4 - (4r)y_3^3 - \left(20r^2\right)y_3^2 - \left(32r^3\right)y_3 - 128r^4\right) = 0$$

Since the fourth degree polynomial above evaluated at $y_3 = 4r$ yields $64r^4$ and evaluated at $y_3 = 5r$ yields $-587r^4$, the intermediate value theorem guarantees a solution $4r < y_3 < 5r$ which places the family in the critical configuration described. □

We proceed with the description of critical configurations with $\gamma = r$, where ℓ_{def13R} supports C_4. As stated in the preceding, we document the configuration in which ℓ_{def23R} supports C_4 in the following lemma.

Lemma 3.10. If $\delta = \gamma = r$, disk C_3 has center $o_3(r, 8r/3)$, and C_4 has center $o_4(3r, 2r)$, then the touching family $\mathcal{F}_4$ is critical. Furthermore, line ℓ_1 supports C_4, and the right definite support ℓ_{def13R} of $\{C_1, C_3\}$ and the (vertical) right definite support ℓ_{def23R} of $\{C_2, C_3\}$ both support C_4 on the left.

Proof. Let $\delta = \gamma = r$. Let the lines ℓ_1, ℓ_{def13R} and ℓ_{def23R} support C_4, so that the three lines support the respective subfamilies $\{C_1, C_2, C_4\}$, $\{C_1, C_3, C_4\}$ and $\{C_2, C_3, C_4\}$. Since both of ℓ_1, ℓ_{def23R} support C_4, its center is necessarily $o_4(3r, 2r)$. Since line ℓ_{def13R} is a

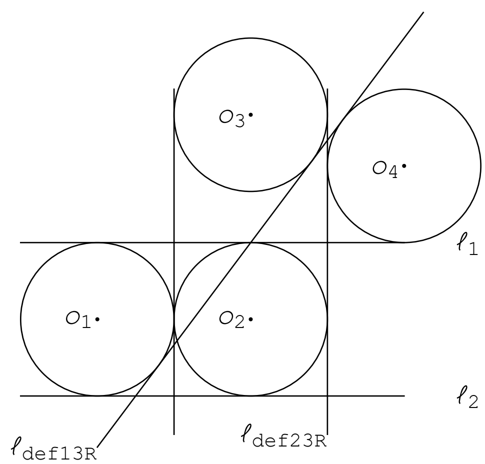


Figure 3.10: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $x_4 = 3$ and $y_3 = 8/3$. Line ℓ_{def23R} supports C_4 on the left.

separating support of $\{C_1, C_4\}$, it contains the midpoint (r, r) of segment $[o_1, o_4]$. The line supports C_1, and applying Lemma 2.5 to line ℓ_{def13R} with slope k and the point o_1 results in the following equation:

$$|k(-r) + (-1)(0) - kr + r| = r\sqrt{k^2 + 1}$$

Solving yields $k = \dfrac{4}{3}$. Since ℓ_{def13R} is parallel to $\langle o_1, o_3 \rangle$, the equation $\dfrac{y_3 - 0}{r - (-r)} = \dfrac{4}{3}$ reduces to $y_3 = 8r/3$, and the line has equation

$$\ell_{def13R}(x) = \frac{4}{3}(x - r) + r = \frac{4}{3}x - \frac{r}{3}.$$

The touching family $\mathcal{F}_4$ as described is critical. $\square$

The preceding exhausts the critical configurations with $\gamma = r$ where a definite support in $\mathcal{L}_{13}$ supports $\{C_1, C_3, C_4\}$. We now describe critical configurations with $\gamma = r$ where a separating support in $\mathcal{L}_{13}$ supports $\{C_1, C_3, C_4\}$. When $\gamma = r$, the (vertical) separating support $\ell_{sep13RL} = \ell_v$ of $\{C_1, C_3\}$ supports $\{C_1, C_2, C_3\}$, and is not permitted to support C_4 to avoid the property S. The only separating support to consider is $\ell_{sep13LR}$.

In no critical family with disk C_3 centered at $o_3(r, y_3)$ ($y_3 > 0$) does line $\ell_{sep13LR}$ support C_4 ($y_4 = -2r$). The bound $y_3 > 0$ requires $y_3 \geq 2r$. With the assignment $y_3 = 2r$, the separating support $\ell_{sep13LR} = \ell_1$ is not permitted to support C_4 to avoid property S. We therefore require $y_3 > 2r$. The separating support $\ell_{sep13LR}$ ($\neq \ell_v$) is permitted to support C_4 on the left or on the right. In either configuration, no line supports $\{C_2, C_3, C_4\}$: note that $\ell_{def23L} = \ell_v$ is not permitted to support C_4. Since C_4 is in the cone bounded by ℓ_2 and ℓ_v, the definite support ℓ_{def23R} and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (with negative slope) are disjoint from C_4 in either configuration. Line $\ell_{sep23LR}$ in $\mathcal{L}_{23}$ is the only viable support that remains. If $\ell_{sep13LR}$ supports C_4 on the right, then the separating support $\ell_{sep23LR}$ (positive slope) is disjoint from C_4. One configuration remains which we describe in the following paragraph.

In no critical family does ℓ_v support C_3 ($y_3 > 0$), line ℓ_2 support C_4, and $\ell_{sep13LR}$ ($\neq \ell_v$) support C_4 on the left. As shown in the preceding paragraph, the definite supports of $\{C_2, C_3\}$ and the separating support $\ell_{sep23RL}$ are disjoint from C_4. So line $\ell_{sep23LR}$ supports C_4, necessarily on the right. The family described is not constructible which we prove analytically in the following proof environment.

Proof. Let $\delta = \gamma = r$, line ℓ_2 support C_4 ($y_4 = -2r$), and let C_3 have center $o_3(r, y_3)$ with $y_3 > 2r$. Here we prove that in no critical family does line $\ell_{sep13LR}$ support C_4 on the left and line $\ell_{sep23LR}$ support C_4 on the right (compare Figure 3.11).

The separating support $\ell_{sep13LR}$ ($\neq \ell_v$) of $\{C_1, C_3\}$ with slope k contains the midpoint $(0, y_3/2)$ of $[o_1, o_3]$. Applying Lemma 2.5 to calculate the distance $\mathrm{d}\left(\ell_{sep13}(x) - y = 0, o_1(-r, 0)\right) = r$ yields the equation

$$\left| k(-r) + (-1)(0) + \frac{y_3}{2} \right| = r\sqrt{k^2 + 1}.$$

By construction, $\ell_{sep13LR}(-r) > r > 0$, permitting us to drop the norm in the preceding, and derive an equation for the line as follows:

$$\ell_{sep13LR}(x) = \frac{y_3^2 - 4r^2}{4ry_3} x + \frac{y_3}{2}$$

Since the line supports C_4 with center $o_4(x_4, -2r)$, Lemma 2.5 provides a commensurate expression for x_4 by the following equation:

$$\left| \frac{y_3^2 - 4r^2}{4ry_3}(x_4) + (-1)(-2r) + \frac{y_3}{2} \right| = r\sqrt{\left(\frac{y_3^2 - 4r^2}{4ry_3}\right)^2 + 1}$$

Since the line supports C_4 from above, the inequality $\ell_{sep13}(x_4)+2r > 0$ allows us to remove the norm and write

$$x_4 = \frac{r\left(4r^2 - 8ry_3 - y_3^2\right)}{y_3^2 - 4r^2}.$$

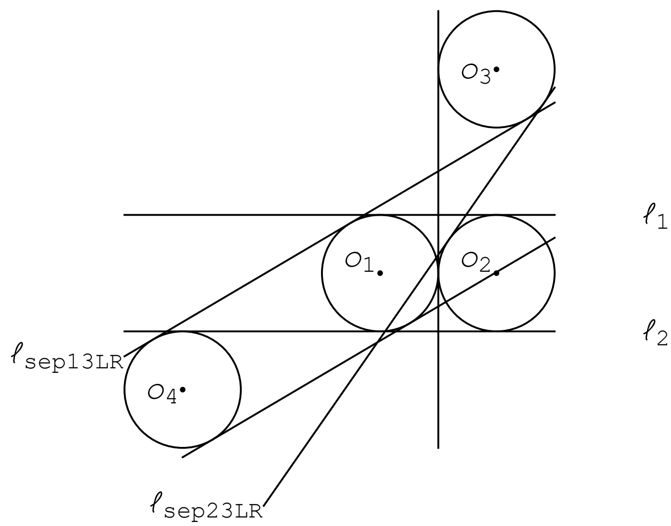


Figure 3.11: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 = 7/2$ and $x_4 \approx -4.3940$. Line $\ell_{sep23LR}$ approaches ∂C_4 from the right as $y_3 \to \infty$.

Lemma 3.7 provides one expression for the slope of $\ell_{sep23LR}$, and we derive a second expression for its slope by viewing the line as a definite support of C_3, C_4. Equating the two expressions, as in

$$\frac{\sqrt{y_3^2 - 4r^2}}{2r} = \frac{(y_3 + 2r)^2(y_3 - 2r)}{2ry_3^2 + 8r^2y_3 - 8r^3},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^3 + (2r)y_3^2 - \left(4r^2\right)y_3 + 8r^3 = 0$$

The equation has a pair of complex conjugate solutions and one real solution which can be expressed in the form

$$y_3 = -\frac{2r}{3}\left(\beta^+ + \beta^- + 1\right),$$

where $\beta^\pm = \sqrt[3]{19 \pm 3\sqrt{33}} > 0$. The negative expression for y_3 above contradicts the requirement $y_3 > 2r > 0$, so that the family as described is not critical. In the limit $y_3 \to \infty$, the lines $\ell_{sep13LR}$ and $\ell_{sep23LR}$ converge to the boundary of C_4 from the left and right, respectively (compare Figure 3.11). An explicit quantification of this convergence is given in Remark 3.11 following this proof environment.

We provide a second verification that the configuration as described is not constructible. To force an inconsistency, let $y_3 = 2r + \varepsilon$ for some $\varepsilon > 0$ since $y_3 > 2r$ by construction. Substituting $r = 1$ and $y_3 = 2r + \varepsilon$ into the equation in the indeterminate y_3 given above leads to the equation

$$(\varepsilon^2 + 8\varepsilon + 8)^2 = \varepsilon(4 + \varepsilon)^3.$$

Solving for ε, the solution is either complex ($\varepsilon = -1.1607 \pm 1.2126i$) or negative ($\varepsilon = -5.6786$). Since no positive value for ε appears, the configuration described is not constructible. $\square$

Remark 3.11. As noted in the preceding proof, lines $\ell_{sep13LR}$ and $\ell_{sep23LR}$ converge to the boundary of C_4 as $y_3 \to \infty$ (compare Figure 3.11). For any $\varepsilon > 0$ (independent of ε in the preceding proof), a range of choices for y_3 guarantees that the separating support $\ell_{sep23LR}$ (positive slope) and disk C_4 are ε-close. Since the limit $y_3 \to \infty$ entails $x_4 \to r$, the boundary ∂C_4 of disk C_4 approaches ℓ_v from the left in the limit. Using this property, we provide a loose analytic bound that guarantees the desired convergence.

Let $\varepsilon > 0$ be given with the requirement that $\mathrm{d}\left(C_4, \ell_{sep23LR} - y = 0\right) < \varepsilon$. If C_4 is ε-close to ℓ_v then the distance from C_4 to $\ell_{sep23LR}$ is less than ε since the separating support $\ell_{sep23LR}$ passes between the disk and the vertical line (compare Figure 3.11). By inspection, disk C_4 is ε-close to ℓ_v if and only if x_4 is ε-close to $-r$ since C_4 has radius r. That is $\mathrm{d}(C_4, \ell_v) \le \varepsilon$ if and only if $|x_4 - (-r)| \le \varepsilon$ with the expression for x_4 from the preceding proof. Solving the equation $|x_4 + r| = \varepsilon$ leads to the expression $y_3 = \dfrac{2r}{\varepsilon}[2r + \sqrt{4r^2 + \varepsilon^2}]$. If $y_3 \ge \dfrac{2r}{\varepsilon}[2r + \sqrt{4r^2 + \varepsilon^2}]$ then the distance from C_4 to the line $\ell_{sep23LR}$ is less than ε. In the sense used above, the family is then ε-close to the critical configuration described.

The preceding shows that in no critical family with $\gamma = r$ that avoids three disks in a slab do both of $\ell_2, \ell_{sep13LR}$ support C_4. Up to this point, we have exhausted both definite supports of $\mathcal{L}_{13}$ and we recall that line $\ell_v = \ell_{sep13RL}$ cannot support C_4. The line $\ell_{sep13LR} \in \mathcal{L}_{13}$ remains. This line may support C_4 on the left or on the right. Since we have exhausted the configurations where ℓ_2 supports C_4, the configuration where ℓ_1 supports C_4 remains ($y_3 > 2r$). In the following, we examine families where both of $\ell_1, \ell_{sep13LR}$ support C_4, so that $y_4 = 2r$. If $\ell_{sep13LR}$ supports C_4 on the right, then both definite supports and the separating support $\ell_{sep23LR}$ (positive slope) of $\{C_2, C_3\}$ are disjoint from C_4. The following lemma describes the critical configuration where line ℓ_1 supports C_4 and line $\ell_{sep23RL}$ supports C_4.

Lemma 3.12. Let $\gamma = r$ and C_4 have center $o_4(x_4, 2r)$ with $x_4 < 0$ expressed below. If $y_3 > 2r$ is a solution of the equation

$$y_3^4 - \left(5r^2\right) y_3^2 - \left(4r^3\right) y_3 - 4r^4 = 0,$$

then the family $\mathcal{F}_4$ is critical. Furthermore, the separating support $\ell_{sep13LR}$ of $\{C_1, C_3\}$, and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ both support C_4 on the right.

Proof. Let $\gamma = r$, and let C_4 have center $o_4(x_4, 2r)$ so that ℓ_1 supports $\{C_1, C_2, C_4\}$. Let $\ell_{sep13LR}$ support C_4 on the right so that $x_4 < 0$. Since $\ell_{sep13LR}$ contains the midpoint $(0, y_3/2)$ of $[o_1, o_3]$, applying Lemma 2.5 to $\ell_{sep13LR}$ with slope k and the point o_1 yields the following equation for the line:

$$\ell_{sep13LR}(x) = \frac{y_3^2 - 4r^2}{4ry_3}x + \frac{y_3}{2}$$

This line supports C_4, and applying Lemma 2.5 to line $\ell_{sep13LR}$ and the point o_4 yields the following commensurate expression for x_4:

$$x_4 = \frac{r(4r^2 - 8ry_3 + 3y_3^2)}{4r^2 - y_3^2}$$

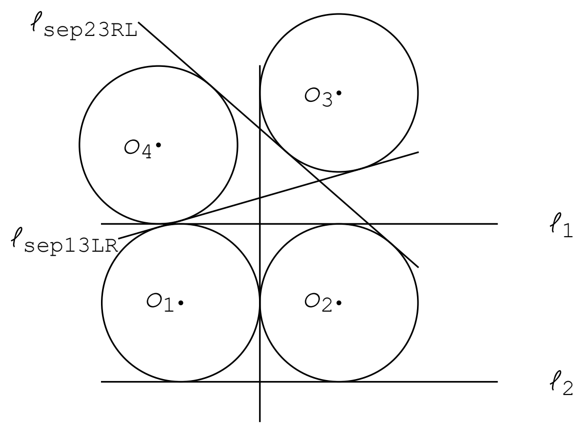


Figure 3.12: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 2.6590$ and $x_4 \approx -1.2829$. Line $\ell_{sep23RL}$ supports C_4 on the right.

As stated in the paragraph preceding the lemma, line $\ell_{sep23RL}$ supports C_4. Furthermore, it must support C_4 on the right, otherwise the subfamily $\{C_2, C_3\}$ is touching and the family has property S. An equation for line $\ell_{sep23RL}$ derived in Lemma 3.9 is reproduced here:

$$\ell_{sep23RL}(x) = -\frac{\sqrt{y_3^2 - 4r^2}}{2r}x + \frac{\sqrt{y_3^2 - 4r^2} + y_3}{2}$$

Viewing this line as a definite support of $\{C_2, C_4\}$ parallel to $\langle o_2, o_4 \rangle$ provides a second expression for its slope. Equating the two expressions for the slope, as in

$$\frac{2r + y_3}{2y_3} = -\frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^4 - \left(5r^2\right) y_3^2 - \left(4r^3\right) y_3 - 4r^4 = 0$$

Since the polynomial above evaluated at $y_3 = 2r$ yields $16r^4$ and evaluated at $y_3 = 3r$ yields $-20r^4$, the intermediate value theorem guarantees a solution $2r < y_3 < 3r$ which places the family in the critical configuration described. $\square$

Lemma 3.12 describes the only configuration where ℓ_v supports C_3, line ℓ_1 supports C_4, and $\ell_{sep13LR}$ supports C_4 on the right. We proceed with configurations where ℓ_v supports C_3 ($\gamma = r$), line ℓ_1 supports C_4, and $\ell_{sep13LR}$ supports C_4 on the left. In this configuration, $\ell_{def23L} = \ell_v$ is not permitted to support C_4. In the following lemma, the right definite support ℓ_{def23R} of $\{C_2, C_3\}$ is a critical support of $\mathcal{F}_4$.

Lemma 3.13. Let $\delta = \gamma = r$, and let C_4 have center $o_4(3r, 2r)$. If $y_3 = r\left(1 + \sqrt{5}\right)$, then the family is critical. Furthermore, the separating support $\ell_{sep13LR}$ of $\{C_1, C_3\}$, and the (vertical) right definite support ℓ_{def23R} of $\{C_2, C_3\}$ both support C_4 on the left.

Proof. Let $\delta = \gamma = r$ so that line ℓ_v supports C_3 on the left. Let C_4 have center $o_4(3r, 2r)$, so that ℓ_1 supports $\{C_1, C_2, C_4\}$, and the (vertical) right definite support $\ell_{def23R} = \{x = 2r\}$ of $\{C_2, C_3\}$ supports C_4 on the left. Necessarily $y_3 > 2r$ since C_3 is disjoint from ℓ_1. Let $\ell_{sep13LR}$ support C_4 on the left. By observation, line $\ell_{sep13LR}$ is a definite support of $\{C_1, C_4\}$, and its slope is $k = 1/2$. By symmetry, the line contains the midpoint $(0, y_3/2)$

of $[o_1, o_3]$, so one equation for the line is

$$\ell_{sep13LR}(x) = \frac{1}{2}x + \frac{y_3}{2}.$$

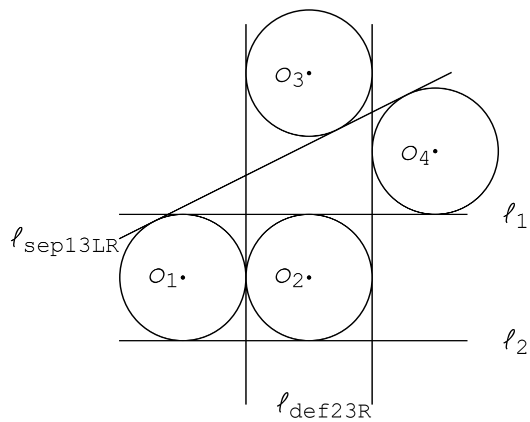


Figure 3.13: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 = 1 + \sqrt{5}$ and $x_4 = 3$. Line ℓ_{def23R} supports C_4 on the left.

Since line $\ell_{sep13LR}$ supports C_3, Lemma 2.5 applied to line $\ell_{sep13LR}$ and the point o_3 leads to the following equation:

$$\left|\frac{1}{2}r + (-1)y_3 + \frac{y_3}{2}\right| = r\sqrt{\left(\frac{1}{2}\right)^2 + 1}$$

Since positive $\ell_{sep13LR}(r) < y_3$, we lift the absolute value by negating the expression on the left hand side in the preceding equation, and solve to find $y_3 = r\left(1 + \sqrt{5}\right)$. These values for the parameters and lines completely determine the critical family. $\square$

We continue with the case $\gamma = r$ so that ℓ_v supports C_3 on the left. The separating supports of $\{C_2, C_3\}$ are permitted to support C_4 on the left or on the right since either configuration for each line avoids three disks in a slab. In the following lemma, line $\ell_{sep23RL}$ supports C_4 on the left.

Lemma 3.14. Let $\gamma = r$, and let C_4 have center $o_4(x_4, 2r)$ with $x_4 > 0$ as expressed below. If $y_3 > 2r$ is a solution of the equation

$$y_3^3 - (2r)y_3^2 - \left(4r^2\right) y_3 - 8r^3 = 0,$$

then the family is critical. Furthermore, the separating support $\ell_{sep13LR}$ of $\{C_1, C_3\}$ and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ both support C_4 on the left, and the subfamily $\{C_3, C_4\}$ is touching.

Proof. Let $\gamma = r$, and let C_4 have center $o_4(x_4, 2r)$ so that ℓ_1 supports C_4 with $x_4 > 0$. Let $\ell_{sep13LR}$ support C_4 on the left. We reproduce its equation, derived in Lemma 3.12:

$$\ell_{sep13LR}(x) = \frac{y_3^2 - 4r^2}{4ry_3}x + \frac{y_3}{2}$$

This line supports C_4, and a commensurate expression for $x_4 > 0$ is the negative of that obtained in Lemma 3.12 since the respective disks C_4 are on opposite sides of the line:

$$x_4 = -\frac{r(4r^2 + 8ry_3 - y_3^2)}{4r^2 - y_3^2}$$

As stated in the paragraph preceding the lemma, line $\ell_{sep23RL}$ is permitted to support C_4 on the left or on the right. Furthermore, if line $\ell_{sep23RL}$ supports C_4 on the right, then line $\ell_{sep13LR}$ coincides with ℓ_{def14L}, and line $\ell_{sep23RL}$ coincides with ℓ_{def24R}. Interchanging the labels on C_3 and C_4 also changes the labels on line $\ell_{def14L} \mapsto \ell_{def13L}$ and on line $\ell_{def24R} \mapsto \ell_{def23R}$ so that the disks and lines coincide with the family described in Lemma 3.5.

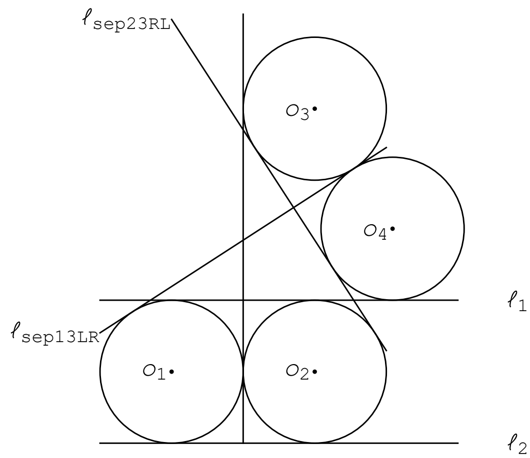


Figure 3.14: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 3.6786$ and $x_4 \approx 2.0874$. Line $\ell_{sep23RL}$ supports C_4 on the left.

Let $\ell_{sep23RL}$ (negative slope) support C_4 on the left. The line supports C_2 and contains the midpoint $(r, y_3/2)$ of $[o_2, o_3]$, so Lemma 2.5 applied to line $\ell_{sep23RL}$ with slope k and the point o_2 yields the following:

$$\left| k(r) - kr + (-1)(0) + \frac{y_3}{2} \right| = r\sqrt{k^2 + 1}$$

Since $y_3/2 > 0$, we lift the absolute value in the preceding and solve for the (negative) slope of $\ell_{sep23RL}$:

$$k = -\frac{\sqrt{y_3^2 - 4r^2}}{2r}$$

Viewing the line $\ell_{sep23RL}$ as a definite support of $\{C_3, C_4\}$ parallel to $\langle o_3, o_4 \rangle$ provides a second expression for its slope. Equating the two expressions for the slope, as in

$$-\frac{4r^2 - y_3^2}{4ry_3} = -\frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^3 - (2r)y_3^2 - \left(4r^2\right) y_3 - 8r^3 = 0$$

Two of its roots form a complex conjugate pair, and its real root is expressed by

$$y_3 = \frac{2r}{3}\left(\beta^+ + \beta^- + 1\right),$$

where $\beta^\pm = \sqrt[3]{19 \pm 3\sqrt{33}} > 0$. Either a numerical or an explicit algebraic calculation demonstrating $\mathrm{d}\left(o_3, o_4\right) = 2r$ verifies that $\{C_3, C_4\}$ is touching. The calculation is omitted for brevity. These values for the parameters and lines completely determine the critical family. □

This exhausts the configurations where $\ell_{sep23RL}$ supports $\{C_2, C_3, C_4\}$. We next consider its associated separating support $\ell_{sep23LR}$. This line is permitted to support C_4 on the left or on the right since either configuration avoids three disks in a slab. In the following lemma line $\ell_{sep23LR}$ supports C_4 on the left.

Lemma 3.15. Let $\delta = r = \gamma$, and let C_4 have center $o_4(x_4, 2r)$ with $x_4 > 0$ as expressed below so that ℓ_1 supports C_4. If $y_3 > 2r$ is the smaller of the two positive real solutions of the equation

$$y_3^4 - (8r)y_3^3 + \left(4r^2\right) y_3^2 + \left(32r^3\right) y_3 + 32r^4 = 0,$$

then the family is critical. Furthermore, the separating support $\ell_{sep13LR}$ of $\{C_1, C_3\}$ and the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ both support C_4 on the left.

Proof. The equation for line $\ell_{sep13LR}$ given in the previous lemma is reproduced here:

$$\ell_{sep13LR}(x) = \frac{y_3^2 - 4r^2}{4ry_3}x + \frac{y_3}{2}$$

A commensurate expression for x_4 from the previous lemma follows:

$$x_4 = -\frac{r\left(4r^2 + 8ry_3 - y_3^2\right)}{4r^2 - y_3^2}$$

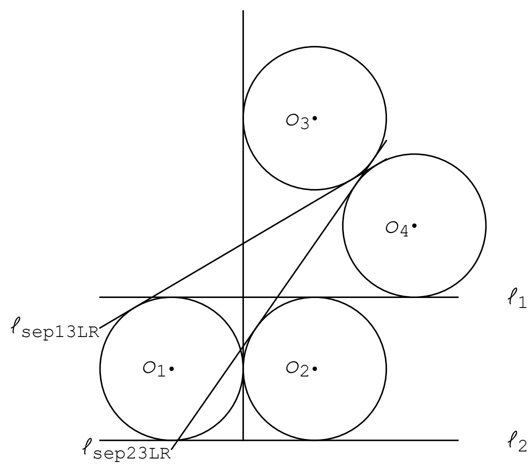


Figure 3.15: Touching critical family $\mathcal{F}_4$ with $r = \gamma = 1$ and coordinates $y_3 \approx 3.5010$ and $x_4 \approx 2.3920$. Line $\ell_{sep23LR}$ supports C_4 on the left.

The equation for line $\ell_{sep23LR}$ differs from that of line $\ell_{sep23RL}$ given in the previous lemma in the sign on the term for its slope which is positive here. An equation for the line follows:

$$\ell_{sep23LR}(x) = \frac{\sqrt{y_3^2 - 4r^2}}{2r}(x - r) + \frac{y_3}{2}$$

As a left definite support of $\{C_2, C_4\}$, we derive a second expression for its slope. Equating the two expressions for its slope, as in

$$\frac{4r^2 - y_3^2}{y_3^2 - 4ry_3 - 4r^2} = \frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^4 - (8r)y_3^3 + \left(4r^2\right) y_3^2 + \left(32r^3\right) y_3 + 32r^4 = 0$$

Two roots of this equation form a complex conjugate pair, and the affiliated root for parameter y_3 is the smaller of its two positive real roots. Since the polynomial evaluated at $y_3 = 3r$ yields $29r^4$ and evaluated at $y_3 = 4r$ yields $-32r^4$, the intermediate value theorem guarantees a solution $3r < y_3 < 4r$ which places the family in the critical configuration described. □

The preceding documents the configuration where $\ell_{sep13LR}$ and $\ell_{sep23LR}$ support C_4 on the left. We proceed to describe configurations where $\ell_{sep13LR}$ supports C_4 on the left, and $\ell_{sep23LR}$ supports C_4 on the right. In the description of these configurations, line $\ell_{sep23LR}$ coincides with line $\ell_{sep24LR}$ (positive slope), and line $\ell_{sep13LR}$ coincides with line ℓ_{def14L}. A line in $\mathcal{L}_{12} \setminus \{\ell_v\} = \{\ell_1, \ell_2\}$ must support $\{C_1, C_2, C_4\}$. Precisely two placements for C_4 avoid three disks in a slab.

In no critical family of congruent disks with $\delta = \gamma = r$ and $y_3 > 2r$ does line $\ell_{sep13LR}$ support C_4 on the left, line $\ell_{sep23LR}$ support C_4 on the right, and line ℓ_1 support C_4 with $y_4 = 2r$. In a configuration of this description, line $\ell_{sep23LR}$ necessarily coincides with $\ell_{sep24LR}$, but this line is disjoint from C_3 as we show in the following. Since $\ell_{sep13LR} = \ell_{def14L}$ supports C_3 on the right, the parallel definite support ℓ_{def14R} (positive slope) supports C_4 and is disjoint from C_3. Rotate ℓ_{def14R}, which cuts C_2 and is disjoint from C_3, clockwise away from C_3, dynamically maintaining contact with the boundary ∂C_4 of C_4 until it supports C_2 on the left in the position of $\ell_{sep24LR}$ (with positive slope). The line is disjoint from C_3, a contradiction since $\ell_{sep23LR} = \ell_{sep24LR}$ supports C_3. The family as described is not geometrically realizable.

In no critical family with $\delta = \gamma = r$ and $y_3 > 2r$ does line $\ell_{sep13LR}$ support C_4 on the left, line $\ell_{sep23LR}$ support C_4 on the right, and line ℓ_2 support C_4 with $y_4 = -2r$. Observe that the family described coincides with the configuration documented in the proof preceding Lemma 3.12 by interchanging the positions of C_3 and C_4 and relabeling as needed. The family was shown to have no geometric realization, so no corresponding critical family exists.

Remark 3.16. For the configurations described in the preceding paragraphs, each family approaches the configuration of a critical family. For the configuration with $y_4 = -2r$, Remark 3.11 shows that the configuration is ε-close to a critical configuration with y_3 large enough; namely, one disk is separated a positive distance ε from its intended critical support. In the configuration with $y_4 = 2r$, observe that the line $\ell_{sep23LR}$ cuts C_4. However, as $y_3 \to \infty$, the line approaches the boundary of C_4. That is, for any $\varepsilon > 0$, a sufficiently large value for y_3 forces the line $\ell_{sep23LR}$ within an ε-distance of the boundary ∂C_4 of C_4, so that the family is ε-close to the critical configuration described in the following sense. This one line fails to support C_4 and is instead secant to C_4, demarcating an arbitrarily small segment of the disk.

The preceding exhausts the configurations for touching critical families $\mathcal{F}_4$ with $\gamma = r$ that avoid three disks in a slab.

3.1.3 Case 3: $\gamma > r$

We now show that no additional critical configurations avoid three disks in a slab. A handful of observations condense our analysis.

Observe that the description of any critical family with $\gamma > r$ where ℓ_v supports C_4 is necessarily equivalent to one of the preceding descriptions of critical families by symmetries in the Klein four-group V. In the descriptions of those families, line ℓ_v supports one of C_3, C_4, and one of ℓ_1, ℓ_2 supports the other disk. Since we permitted C_4 to have any position along ℓ_1, ℓ_2 in each configuration where ℓ_v supports C_3 ($y_3 \neq 2r$), some symmetry in V necessarily transforms the family to one of the previously documented configurations.

Observe further that any configuration with $\gamma > r$ where ℓ_2 supports C_4 with $-r < x_4 < r$, is identical by a symmetry in V to a configuration where ℓ_1 supports C_3 with $0 \leq \gamma < r$. Consider for the moment configurations avoiding three disks in a slab where ℓ_1 supports C_3

with $\gamma > r$, and ℓ_2 supports C_4. If line ℓ_{def13R} supports C_4, then the corresponding range for x_4 is $-r < x_4 < r$ since the line necessarily supports C_4 on the left. Also, if line $\ell_{sep13RL}$ supports C_4 on the left hand side, then the corresponding range for x_4 is $-r < x_4 < r$. In these configurations, a symmetry in V maps the family so that disk C_4 coincides with C_3 in a configuration with $0 \leq \gamma < r$, which have been exhaustively documented. Since line ℓ_1 $(= \ell_{sep13LR})$ is necessarily disjoint from C_4, two lines remain to consider. With $\gamma > r$, if ℓ_{def13L} supports C_4, then the supports of $\{C_2, C_3\}$ are disjoint from C_4 since the line necessarily supports C_4 on the right.

Finally, the configuration where $\ell_{sep13RL}$ supports C_4 on the right forces $x_4 < -r$, so that no preceding configuration is equivalent to a family with this description. Consider the lines in $\mathcal{L}_{23} = \{\ell_{def23L}, \ell_{def23R}, \ell_{sep13LR}, \ell_{sep13RL}\}$ separately. Line ℓ_{def23L} necessarily cuts C_4 by construction since the line lies to the left of $\ell_{sep13RL}$ (which supports C_4) below line ℓ_2. This implies the parallel line ℓ_{def23R} is disjoint from C_4. Finally, the separating supports $\ell_{sep23RL}$ and $\ell_{sep23LR} = \ell_1$ are disjoint from C_4 by construction. No critical family fits this description. The condition $\gamma > r$ induces no new critical configurations that avoid three disks in a slab.

3.2 Touching Critical Families $\mathcal{F}_4$ with Three Disks in a Slab

In the preceding, we examined touching critical families $\mathcal{F}_4$ avoiding three disks in a slab. In the following, we consider touching critical families $\mathcal{F}_4$ permitting three disks in a slab. Following our convention, disks $C_1, C_2 \in \mathcal{F}_4$ have their centers on the x-axis with $\delta = r$. Since $\{C_1, C_2\}$ is touching, Theorem 2.3, Part (b) guarantees the subfamily has three support lines.

With three disks in a slab, either a third disk lies in the slab with touching subfamily $\{C_1, C_2\}$, or no other disk lies in the slab determined by ℓ_1, ℓ_2. If $\{C_1, C_2\}$ is in the slab with three disks, then one of C_3, C_4 is in the slab. Relabel as needed so that C_3 is in the slab with center $o_3(\gamma, 0)$, observing the bound $|\gamma| \geq 3r$ since $\mathcal{F}_4$ is nonoverlapping. Since any configuration with $\gamma \leq 0$ is identical by reflection symmetry about ℓ_v (in V) to one with $\gamma > 0$, we stipulate $\gamma \geq 3r$.

If C_3 lies in the slab with $\{C_1, C_2\}$ both of ℓ_1, ℓ_2 support $\{C_1, C_2, C_3\}$, so line ℓ_v must support $\{C_1, C_2, C_4\}$ to secure $S(3)$ and avoid property S. This means $x_4 = \pm r$ and the parameter y_4 associated with the center $o_4(x_4, y_4)$ of C_4 must avoid the values in $\{-2r, 0, 2r\}$ in particular, and necessarily $|y_4| \geq 2r$ since $\mathcal{F}_4$ is nonoverlapping. Any configuration with $y_4 < 0$ is identical to a configuration with $y_4 > 0$ by reflection symmetry over the x-axis (in V), so we stipulate $y_4 > 2r$.

In no critical family with touching $\{C_1, C_2, C_3\}$ in a slab does C_3 have center $o_3(3r, 0)$, and C_4 have center $o_4(r, y_4)$ with $y_4 > 2r$. If $\gamma = 3r$, then touching $\{C_2, C_3\}$ has three support lines by Theorem 2.3, Part (b). Since $\mathcal{F}_4$ is critical, a line must support $\{C_2, C_3, C_4\}$. The definite supports $\ell_1, \ell_2 \in \mathcal{L}_{23}$ are necessarily disjoint from C_4, so the vertical separating support $\ell_{sep23RL}$ must support C_4 which entails $x_4 = r$. The vertical lines ℓ_v and $\ell_{sep23RL}$ are parallel and support C_4 on the left and on the right, respectively. A line must support $\{C_1, C_3, C_4\}$. Since the definite supports ℓ_1, ℓ_2 of $\{C_1, C_3\}$ are disjoint from C_4, a separating support of $\{C_1, C_3\}$ must support C_4. But disk C_4 lies in the slab determined by the vertical lines $\ell_v, \ell_{sep23RL}$ above line ℓ_1 and the separating supports of $\{C_1, C_3\}$ don't enter this region of the plane. No line supports $\{C_1, C_3, C_4\}$, so the critical family as described is not geometrically realizable.

Since no critical family $\mathcal{F}_4$ has its subfamily $\{C_1, C_2, C_3\}$ in a slab with $\gamma = 3r$, we proceed with $\gamma > 3r$. Line ℓ_v necessarily supports C_4 with center $o_4(\pm r, y_4)$ where $y_4 > 2r$ by symmetry.

In no critical family does C_3 have center $o_3(\gamma, 0)$ with $\gamma > 3r$, and C_4 have center $o_4(r, y_4)$ with $y_4 > 2r$. Since ℓ_v supports C_4 on the left, no line supports $\{C_1, C_3, C_4\}$: the definite supports ℓ_1, ℓ_2 are necessarily disjoint from C_4, and as in the case with $\gamma = 3r$, the separating supports of $\{C_1, C_3\}$ are disjoint from C_4 for analogous reasons.

Up to symmetries in V, precisely one family $\mathcal{F}_4$ has $\mathcal{F}_3 = \{C_1, C_2\} \cup \{C_3\}$ in a slab $(x_4 = -r)$, which is documented in the following lemma.

Lemma 3.17. Let $\delta = r$, disk C_4 have center $o_4(-r, y_4)$, and C_3 have center $o_3(\gamma, 0)$, so that the subfamily $\{C_1, C_2, C_3\}$ lies in the slab determined by ℓ_1, ℓ_2. If $\gamma > 3r$ is a solution of the equation

$$\gamma^4 - \left(10r^2\right)\gamma^2 - \left(16r^3\right)\gamma + 9r^4 = 0,$$

then the family $\mathcal{F}_4$ is critical. Furthermore, the separating support $\ell_{sep13RL}$ of $\{C_1, C_3\}$, and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (both with negative slope) support C_4 from below and above, respectively.

Proof. Let $\delta = r$, so that the touching subfamily $\{C_1, C_2\}$ has the set of support lines $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$ following Theorem 2.3, Part (b). Let disk C_3 have center $o_3(\gamma, 0)$ with $\gamma > 3r$, so that both lines ℓ_1, ℓ_2 support $\{C_1, C_2, C_3\}$ (see Figure 3.16).

A line must support $\{C_1, C_2, C_4\}$, so line ℓ_v necessarily supports C_4. As detailed in the paragraphs preceding the lemma, disk C_4 necessarily has center $o_4\left(-r, y_4\right)$ with $|y_4| > 2r$.

A separating support of $\{C_1, C_3\}$ necessarily supports $\{C_1, C_3, C_4\}$. With $x_4 = -r$, the separating support of $\{C_1, C_3\}$ with positive slope is disjoint from C_4, so the separating support $\ell_{sep13RL}$ with negative slope supports C_4. This line supports C_4 from below by construction since ℓ_v supports disk C_4. The line $\ell_{sep13RL}$ with slope k contains the midpoint $((\gamma - r)/2, 0)$ of $[o_1, o_3]$ and supports C_1. Applying Lemma 2.5 to calculate the distance

$$\mathrm{d}\left(kx - y - k\frac{\gamma - r}{2} + 0 = 0,\, o_1(-r, 0)\right) = r,$$

leads to the following equation in the indeterminate k

$$-kr - k\frac{\gamma - r}{2} = r\sqrt{k^2 + 1},$$

since $\ell_{sep13RL}(-r) > 0$. An equation for the line follows:

$$\ell_{sep13RL}(x) = -\frac{2r}{\sqrt{\gamma^2 + 2r\gamma - 3r^2}}\left(x - \frac{\gamma - r}{2}\right)$$

This line supports C_4, and Lemma 2.5 applied to the line and the point o_4 leads to the following expression:

$$y_4 = \frac{2r(r+\gamma)}{\sqrt{\gamma^2 + 2r\gamma - 3r^2}}$$

When C_4 has center $o_4\,(-r, y_4)$ with y_4 expressed above, line $\ell_{sep13RL}$ supports subfamily $\{C_1, C_3, C_4\}$.

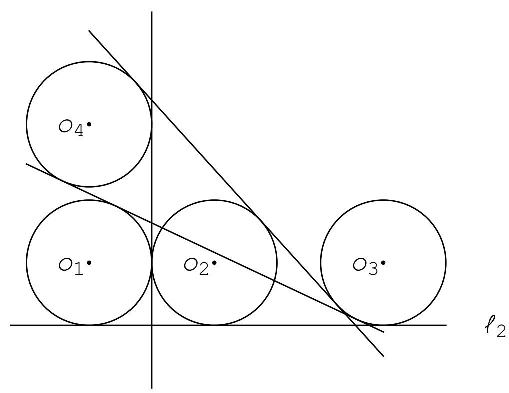


Figure 3.16: Touching critical family $\mathcal{F}_4$ with $r = 1$ and $\gamma \approx 3.6972$.

A line must support $\{C_2, C_3, C_4\}$. Since ℓ_1, ℓ_2, and the separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ (positive slope) are disjoint from C_4, the associated separating support $\ell_{sep23RL}$ (negative slope) supports C_4. This line supports C_4 from above, otherwise $\ell_{sep13RL}$ is disjoint from C_4. The line $\ell_{sep23RL}$ contains the midpoint $((r+\gamma)/2, 0)$ of $[o_2, o_3]$ and supports C_2.

Lemma 2.5 applied to the line and the point o_2 to calculate the distance

$$\mathrm{d}\left(kx - y - k\frac{r+\gamma}{2} + 0 = 0, o_2(r,0)\right) = r$$

leads to the derived equation $k\frac{r-\gamma}{2} = r\sqrt{k^2+1}$ since $\ell_{sep23RL}(r) > 0$. Solving for k yields the following equation for the line:

$$\ell_{sep23RL}(x) = -\frac{2r}{\sqrt{\gamma^2 - 2r\gamma - 3r^2}}\left(x - \frac{r+\gamma}{2}\right)$$

Since $\ell_{sep23RL}$ is a definite support of $\{C_2, C_4\}$ parallel to $\langle o_2, o_4\rangle$, we derive a second expression k_{def24} for the slope. Equating the two expressions $k_{sep23RL} = k_{def24}$ for the slope, as in

$$-\frac{(r+\gamma)}{\sqrt{\gamma^2 + 2r\gamma - 3r^2}} = -\frac{2r}{\sqrt{\gamma^2 - 2r\gamma - 3r^2}},$$

leads to the following equation in γ with parameter r:

$$\gamma^4 - \left(10r^2\right)\gamma^2 - \left(16r^3\right)\gamma + 9r^4 = 0$$

An expression for the affiliated solution is

$$\gamma = \frac{r}{\sqrt{3}}\left(\sqrt{\beta^+ + \beta^- + 5} + \sqrt{10 - (\beta^+ + \beta^-) + \frac{12\sqrt{3}}{\sqrt{\beta^+ + \beta^- + 5}}}\right)$$

with $\beta^\pm = \sqrt[3]{89 \pm 6\sqrt{159}}$. The equation has two positive real solutions, one of which is less than r, forcing $\{C_1, C_2\}$ to overlap with C_3. Since the polynomial evaluated at $\gamma = 3r$ yields $-48r^4$ and evaluated at $\gamma = 4r$ yields $41r^4$, the intermediate value theorem guarantees a solution $3r < \gamma < 4r$ which places the family in the critical configuration described. $\square$

The preceding contains exhaustive descriptions of the touching critical families $\mathcal{F}_4$ with a touching subfamily $\mathcal{F}_3$ in a slab, and Lemma 3.17 documents the one constructible critical family with this property up to symmetries in V. Since only one critical family $\mathcal{F}_4$ has a touching critical subfamily in a slab, up to symmetries in V, any other configuration must have a disjoint critical subfamily in a slab. We observe that the touching critical families $\mathcal{F}_4$ with three disks in a slab avoids two pairs of touching disks in contrast to the critical families $\mathcal{F}_4$ avoiding three disks in a slab (see Lemma 3.14).

The preceding implies that either $\mathcal{F}_4 \setminus \{C_1\} = \{C_2, C_3, C_4\}$ or $\mathcal{F}_4 \setminus \{C_2\} = \{C_1, C_3, C_4\}$ lies in a slab. Since either/both configuration(s) is/are identical under reflection over ℓ_v (in V), we stipulate that disjoint $\{C_1, C_3, C_4\}$ lies in a slab. By symmetry, one of ℓ_1, ℓ_v supports C_3 with center $o_3(\gamma, y_3)$. We document these configurations sequentially beginning with the case that ℓ_v supports C_3. Observe that any configuration where ℓ_v supports either of C_3, C_4 maps to a configuration (by symmetries in V) where ℓ_v supports the disk C_3, so it suffices to restrict our attention to configurations where line ℓ_v supports C_3. In particular, $\gamma \neq x_4$ since the assignment $\gamma = -r$ implies ℓ_v supports $\mathcal{F}_4$. This confirms our statement above that any critical configuration with a touching critical family in a slab coincides with the family described in Lemma 3.17. So it suffices to consider $\gamma = r$ and $y_3 \geq 2r$ since $\mathcal{F}_4$ is nonoverlapping. Subfamily $\{C_1, C_3, C_4\}$ lies in a slab supported by the definite supports of $\{C_1, C_3\}$ by construction. Line ℓ_v is not permitted to support C_4, so precisely one of ℓ_1, ℓ_2 supports $\{C_1, C_2, C_4\}$.

In no critical touching family $\mathcal{F}_4$ with three disjoint disks in a slab does C_3 have center $o_3(r, 2r)$. As noted above, any configuration with a touching subfamily of size three in a slab must coincide with the family described in Lemma 3.17. Explicitly, since C_3 touches C_2 in this configuration, both of ℓ_1, ℓ_v support C_3, and line ℓ_2 necessarily supports C_4 with center $o_4(x_4, -2r)$. A line must support $\{C_2, C_3, C_4\}$ and line $\ell_1 = \ell_{sep23}$ must remain disjoint from C_4. The final confirmation that only one family has a touching critical subfamily in a slab is that C_4 touches C_1 only if ℓ_v supports the disk. Since the definite supports of

$\{C_2, C_3\}$ (one of which is ℓ_v) are disjoint from C_4 by construction, the subfamily has no support and the family $\mathcal{F}_4$ described is not geometrically realizable.

Since no critical configuration has $y_3 = 2r$, we proceed with $y_3 > 2r$. A line must support $\{C_1, C_2, C_4\}$, and since line ℓ_v supports C_3 ($\gamma = r$), precisely one of ℓ_1, ℓ_2 supports C_4 since no disk lies in the slab with $\{C_1, C_2\}$.

We first document the critical families $\mathcal{F}_4$ where ℓ_1 supports C_4. Following this, we document the configurations where ℓ_2 supports C_4. In the following lemma, line ℓ_1 supports C_4.

Lemma 3.18. Let $\gamma = r$, and let C_4 have center $o_4(x_4, 2r)$ with $x_4 < 0$ as determined below. If $y_3 = r\left(\sqrt{5 + 4\sqrt{2}} + 1\right)$, then the family $\mathcal{F}_4$ is critical. Furthermore, subfamily $\{C_1, C_3, C_4\}$ lies in a slab supported by both definite supports of $\{C_2, C_3\}$, and the separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (with negative slope) supports C_4 on the right.

Proof. Let $\gamma = r$ and let ℓ_1 support C_4. Since C_4 lies in a slab with $\{C_1, C_3\}$, both definite supports $\ell_{def13L}, \ell_{def13R}$ support C_4 and consequently $\{C_1, C_3, C_4\}$. With $\gamma = r$, line ℓ_v supports C_3 on the left, and the equation of the left definite support of $\{C_1, C_3\}$ is identical to that derived in Lemma 3.6:

$$\ell_{def13L}(x) = \frac{y_3}{2r}x + \frac{\sqrt{y_3^2 + 4r^2} + y_3}{2}$$

Lemma 2.5 applied to line ℓ_{def13L} and the point o_4 leads to the following expression for x_4 consistent with support from the line:

$$x_4 = \frac{r}{y_3}\left(4r - y_3\right)$$

A line must support $\{C_2, C_3, C_4\}$. Line ℓ_{def23L} necessarily cuts C_4 and line ℓ_{def23R} is disjoint from C_4 by symmetry. A separating support of $\{C_2, C_3\}$ must support C_4. The

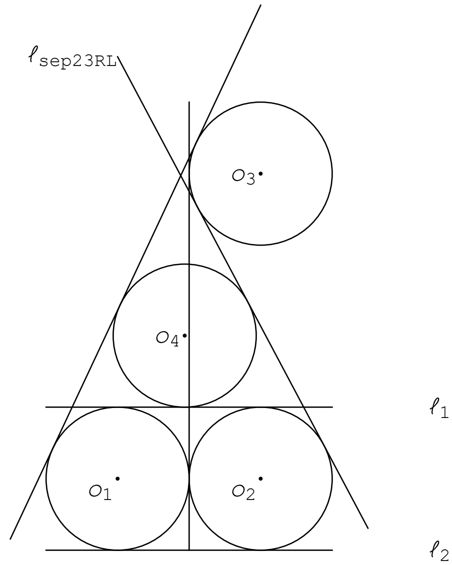


Figure 3.17: Touching critical family $\mathcal{F}_4$ with $r = 1 = \gamma$ and coordinates $y_3 = \sqrt{5+4\sqrt{2}}+1$ and $x_4 = \sqrt{10+8\sqrt{2}} - \sqrt{5+4\sqrt{2}} - \sqrt{2}$.

separating support $\ell_{sep23LR}$ (with positive slope) cuts C_4: rotate ℓ_{def13R}, which supports C_4 on the right, clockwise, dynamically maintaining contact with the boundary of C_3, until it supports C_2 on the left. This line cuts C_4. The separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (with negative slope) necessarily supports C_4. As in Lemma 3.7 with $\gamma = r$, line $\ell_{sep23RL}$ contains the midpoint $(r, y_3/2)$ of $[o_2, o_3]$, and an equation for the line follows:

$$\ell_{sep23RL}(x) = -\frac{\sqrt{y_3^2 - 4r^2}}{2r}(x - r) + \frac{y_3}{2}$$

Additionally, $\ell_{sep23RL}$ as a definite support of $\{C_2, C_4\}$ is parallel to $\langle o_2, o_4\rangle$ providing a second expression for its slope. Equating the expressions for the slope of $\ell_{sep23RL}$, as in

$$\frac{y_3}{2r - y_3} = -\frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^4 - (4r)y_3^3 - \left(4r^2\right)y_3^2 + \left(16r^3\right)y_3 - 16r^4 = 0$$

With coordinate $y_3 = r\left(\sqrt{5+4\sqrt{2}}+1\right)$, the affiliated solution of the above equation, the family $\mathcal{F}_4$ as described is critical. □

We proceed with the configuration where disjoint $\{C_1, C_3, C_4\}$ lies in a slab and ℓ_v supports C_3 centered at $o_3(r, y_3)$ with $y_3 > 2r$. A line must support $\{C_1, C_2, C_4\}$. Line ℓ_v is not permitted to support it, and the one configuration where line ℓ_1 supports C_4 has been documented. In the following lemma, line ℓ_2 supports $\{C_1, C_2, C_4\}$.

Lemma 3.19. Let $\delta = r = \gamma$, and let C_4 have center $o_4(x_4, -2r)$ with $x_4 < 0$, as documented below. If $y_3 = r\left(\sqrt{5+4\sqrt{2}}-1\right)$, then the family $\mathcal{F}_4$ is critical. Furthermore, subfamily $\{C_1, C_3, C_4\}$ lies in a slab supported by the definite supports of $\{C_1, C_3\}$. The separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ (with positive slope) supports C_4 on the left (from above).

Proof. Let $\gamma = r$ so that line ℓ_v supports C_3, and let C_4 lie in a slab with $\{C_1, C_3\}$. The equation of the left definite support, derived in Lemma 3.6, is reproduced here:

$$\ell_{def13L}(x) = \frac{y_3}{2r}x + \frac{\sqrt{y_3^2+4r^2}+y_3}{2}$$

This line supports C_4 with center $o_4(x_4, -2r)$. Lemma 2.5 applied to line ℓ_{def13L} and the point o_4 leads to the following expression for x_4 consistent with support from the line:

$$x_4 = -\frac{r}{y_3}\left(4r + y_3\right)$$

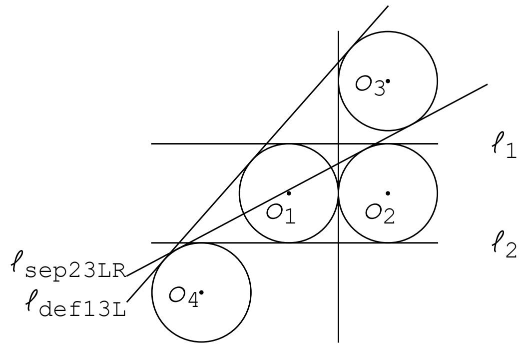


Figure 3.18: Touching critical family $\mathcal{F}_4$ with $r = 1$ and $y_3 = \sqrt{5 + 4\sqrt{2}} - 1$.

A line must support $\{C_2, C_3, C_4\}$. Since the definite supports of $\{C_2, C_3\}$ are disjoint from C_4, a separating support of $\{C_2, C_3\}$ must support it. Since the separating support with negative slope is disjoint from C_4, the associated separating support $\ell_{sep23LR}$ necessarily supports C_4. This support is necessarily on the left: if $\ell_{sep23LR}$ supports C_4 on the right, then it coincides with the definite supports ℓ_{def34R} and ℓ_{def13R} since the subfamily $\{C_1, C_3, C_4\}$ lies in a slab. Line $\ell_{sep23LR}$ then supports C_1 on the right and C_2 on the left, so it coincides with $\ell_{sep12RL} = \ell_v$ which is disjoint from C_4, a contradiction.

Line $\ell_{sep23LR}$ (positive slope) supports C_4 on the left, and its equation, derived in Lemma 3.7, is reproduced here:

$$\ell_{sep23LR}(x) = \frac{\sqrt{y_3^2 - 4r^2}}{2r}(x - r) + \frac{y_3}{2}$$

As a definite support of $\{C_2, C_4\}$, we have a second expression k for its slope. Equating the expressions $k = k_{sep23LR}$ for the slope, as in

$$\frac{y_3}{2r + y_3} = \frac{\sqrt{y_3^2 - 4r^2}}{2r},$$

leads to the following equation in the indeterminate y_3 with parameter r:

$$y_3^4 + (4r)y_3^3 - \left(4r^2\right) y_3^2 - \left(16r^3\right) y_3 - 16r^4 = 0$$

With coordinate $y_3 = r\left(\sqrt{5+4\sqrt{2}} - 1\right)$, the affiliated solution of the above equation, the family $\mathcal{F}_4$ as described is critical. $\square$

The preceding lemmas and counterexamples exhaust the configurations for critical $\mathcal{F}_4$ where ℓ_v supports C_3. In all other configurations we stipulate that ℓ_1 supports C_3 and ℓ_2 supports C_4 by symmetry. The disjoint subfamily $\{C_1, C_3, C_4\}$ lies in a slab, and a line in $\mathcal{L}_{12} \setminus \{\ell_1\} = \{\ell_2, \ell_v\}$ must support $\{C_1, C_2, C_4\}$. As stated earlier, any configuration where ℓ_v supports C_4 is identical to one where ℓ_v supports C_3 (by symmetries in V), so it is sufficient to consider when ℓ_2 supports C_4 which entails $y_4 = -2r$.

We proceed to identify a bound on γ that ensures we include every describable critical configuration of disks and avoid the description of duplicate families. The condition $\gamma \geq 0$ where $\{C_1, C_3, C_4\}$ lies in a slab does not produce every possible configuration of disks. No value for nonnegative γ places $\{C_1, C_3, C_4\}$ in a vertical slab, so the range $\gamma \in [0, \infty)$ is not exhaustive. Allowing γ in the unrestricted range $\gamma \in (-\infty, \infty)$ generates duplicate families by symmetry as we show in the following paragraph.

Observe that the assignment $\gamma = 0$, induces $x_4 = -2r$: line ℓ_v supports C_1 on the right and the center o_3 of C_3 lies on the line. The subfamily $\{C_1, C_3, C_4\}$ has rotation symmetry of $180°$ about the center o_1 of disk C_1. By rotational symmetry about o_1, since line ℓ_v contains the center o_3 of C_3, it follows that the vertical tangent line $\{x = -2r\}$ to C_1 contains the center o_4 of C_4. This means that for γ in the range $\gamma \geq 0$, the corresponding range for x_4 is $x_4 \leq -2r$. Furthermore, reflecting any family with $\gamma \geq 0$ over the x-axis (a symmetry in V) maps the family to an equivalent configuration with $\gamma \leq -2r$ by the map $x_4 \mapsto \gamma$ since it preserves the placement of C_1 and C_2 and it preserves subfamily $\{C_1, C_3, C_4\}$ in a slab. This reflection interchanges C_3 with C_4 and by interchanging the labels on C_3 and C_4 we have an equivalent configuration for $\mathcal{F}_4$ with $\gamma \leq -2r$ in which line $\ell_1 = \{y = r\}$ supports C_3 and line $\ell_2 = \{y = -r\}$ supports C_4. It follows that the description of the families with $\gamma \geq 0$ accounts for the configurations with $\gamma \leq -2r$.

With $\gamma \geq 0$, the only subrange of the reals $(-\infty, \infty)$ not accounted for is $\gamma \in (-\infty, 0) \setminus (-\infty, -2r] = (-2r, 0)$. It follows that every critical configuration with $\gamma \in (-\infty, \infty)$ has a representative with γ in the restricted range $\gamma \in (-2r, \infty)$. We consider the two ranges $(-2r, 0)$ and $[0, \infty)$ separately.

For $\gamma \in (-2r, 0)$, the value $\gamma = -r$ is excluded since ℓ_v supports $\mathcal{F}_4$ in this configuration. So line ℓ_1 necessarily supports C_3 in this range. It is sufficient to examine the two subranges $-2r < \gamma < -r$ and $-r < \gamma < 0$. As detailed above, we stipulate that ℓ_2 supports C_4. A line in $\mathcal{L}_{23}$ must support C_4, and the lines $\ell_1 = \ell_{sep23RL}$ and ℓ_{def23R} are disjoint from C_4 by construction. Two candidate lines $\ell_{def23L}, \ell_{sep23LR}$ remain. For each range, we show that no line supports $\{C_2, C_3, C_4\}$. Recall that disjoint $\{C_1, C_3, C_4\}$ lies in a slab.

Let γ be in the range $-2r < \gamma < -r$. In the limit $\gamma \to -r$, the line $\ell_{sep23LR} \to \ell_v$ supports C_4 on the right. A perturbation $\gamma = -r - \varepsilon < -r$ by a positive distance ε causes line $\ell_{sep23LR}$ to cut C_4: rotate ℓ_{def13R} which supports C_4 and cuts C_2, into C_4 dynamically maintaining contact with ∂C_3 until the line supports C_2, so that the resulting line is in the position of $\ell_{sep23LR}$ (which lies to the right of ℓ_v below ℓ_2) and cuts C_4. To verify that $\ell_{sep23LR}$ cuts C_4 over the interval $-2r < \gamma < -r$, observe that in the configuration with $\gamma = -2r$, the line does not avoid the disk, and cuts it. Since the line cuts the disk, no critical configuration ensues.

Finally, line ℓ_{def23L} is disjoint from C_4. If line ℓ_{def23L} supports C_4, then it necessarily supports the disk on the right. This configuration coincides with the family documented in Lemma 3.20 by a reflection over the x-axis as we show in the following. This reflection preserves left-right orientation and the labels on C_1, C_2 and interchanges C_3 with C_4. Under the reflection, the subfamily $\{C_1, C_3, C_4\}$ remains in a slab preserving the correspondences of the critical support $\ell_{def13L} = \ell_{def34L} = \ell_{def14L}$. The other critical support ℓ_{def23L} has the respective correspondences $\ell_{def23L} = \ell_{sep24LR} = \ell_{sep34LR}$. Each of these labels is mapped according to $\ell_{def23L} \mapsto \ell_{def24L}$, $\ell_{sep24LR} \mapsto \ell_{sep23LR}$, and $\ell_{sep34LR} \mapsto \ell_{sep43LR} = \ell_{sep34RL}$. This results in the correspondence $\ell_{def24L} = \ell_{sep23LR} = \ell_{sep34RL}$. These correspondences coincide with the family documented in Lemma 3.20. In particular, since $x_4 = r(1 - 2\sqrt{3}) < -2r$, the corresponding value for γ ($x_4 \mapsto \gamma$ reflected over the x-axis) is outside the prescribed bound. So no critical family of this description has γ in this range.

The final subrange to consider is $-r < \gamma < 0$. Since $\{C_1, C_3, C_4\}$ is in a slab, both lines $\ell_{def13L}, \ell_{def13R} \in \mathcal{L}_{13}$ support C_4. No line supports $\{C_2, C_3, C_4\}$: In this range, line ℓ_{def23L} with negative slope is disjoint from C_4 by the remarks in the preceding paragraph. The only support not disjoint from C_4 is $\ell_{sep23LR}$ which cuts C_4: rotate ℓ_{def13R}, which supports C_4 and cuts C_2, into C_4 maintaining contact with ∂C_3 until the line supports C_2 in the position of $\ell_{sep23LR}$, cutting C_4 for γ in this narrow range. This exhausts the possible critical configurations for $-2r < \gamma < 0$.

To complete the documentation of the critical configurations with disjoint $\{C_1, C_3, C_4\}$ in a slab, we proceed to describe critical configurations with γ in the range $\gamma \geq 0$ where ℓ_1 supports C_3, and ℓ_2 supports C_4. We begin with the interval $0 \leq \gamma < r$ and adjoin disk C_3 with center $o_3(\gamma, 2r)$ to the family $\{C_1, C_2\}$. A line must support $\{C_2, C_3, C_4\}$, and the definite supports of $\{C_2, C_3\}$ and its separating support ℓ_1 are disjoint from C_4 since $0 \leq \gamma < r$. The separating support $\ell_{sep23LR}$ ($\neq \ell_1$) of $\{C_2, C_3\}$ necessarily supports C_4. In the following lemma line $\ell_{sep23LR}$ supports C_4 on the left.

Lemma 3.20. Let $\delta = r$, disk C_4 have center $o_4(x_4, -2r)$, and C_3 have center $o_3(\gamma, 2r)$. If $\gamma = r\left(2\sqrt{3} - 3\right)$, then the family $\mathcal{F}_4$ is critical. Furthermore, subfamily $\{C_1, C_3, C_4\}$ lies in a slab supported by the definite supports of $\{C_1, C_3\}$. The separating support $\ell_{sep23LR}$ of $\{C_2, C_3\}$ (with positive slope) supports C_4 on the left (from above).

Proof. Let $\delta = r$, and let C_3 have center $o_3(\gamma, 2r)$ with $0 < \gamma < r$ where $\{C_1, C_3, C_4\}$ lies in a slab. The equation for the left definite support is identical to that given in Lemma 3.3

which we reproduce here:

$$\ell_{def13L}(x) = \frac{2r}{\gamma + r}x + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}$$

Applying Lemma 2.5 to line ℓ_{def13L} and the point o_4 yields the following equation:

$$\left|\frac{2r}{\gamma + r}(x_4) - (-2r) + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}\right| = r\sqrt{\left(\frac{2r}{\gamma + r}\right)^2 + 1}$$

This leads immediately to

$$x_4 = -(2r + \gamma),$$

since $0 < \ell_{def13L}(x_4) + r < \ell_{def13L}(x_4) + 2r$ (compare Figure 3.19).

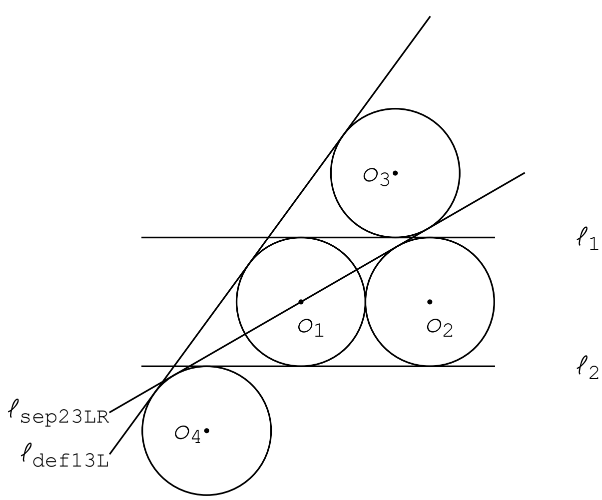


Figure 3.19: Touching critical family $\mathcal{F}_4$ with $r = 1$ and coordinates $\gamma = 2\sqrt{3} - 3$ and $x_4 = 1 - 2\sqrt{3}$.

A line must support $\{C_2, C_3, C_4\}$, and as in the paragraph preceding this lemma, line $\ell_{sep23LR}$ must support C_4. Furthermore, this support must be on the left: if line $\ell_{sep23LR}$ supports C_4 on the right, then the line is a definite support of $\{C_3, C_4\}$, so that the string $\ell_{sep23LR} = \ell_{def34R} = \ell_{def13R}$ implies that line $\ell_{def13R} = \ell_{sep12LR} = \ell_v$, contrary to supposition. The expression for $k_{sep23LR}$ from Lemma 3.3, together with the fact that the line contains the midpoint $\left(\frac{r+\gamma}{2}, r\right)$ of $[o_2, o_3]$, leads to the following equation for the line:

$$\ell_{sep23LR}(x) = \frac{4r(r-\gamma)}{3r^2 + 2r\gamma - \gamma^2}\left(x - \frac{r+\gamma}{2}\right) + r$$

As a definite support of $\{C_2, C_4\}$, we derive a second expression for its slope. Equating the two expressions for the slope, as in

$$\frac{0-(-2r)}{r-x_4} = \frac{2r}{3r+\gamma} = \frac{4r(r-\gamma)}{3r^2+2r\gamma-\gamma^2},$$

simplifies to the following equation in the indeterminate γ with parameter r

$$\gamma^2 + (6r)\gamma - 3r^2 = 0.$$

With $\gamma = r\left(2\sqrt{3}-3\right)$, the affiliated solution of the equation, and consequently $x_4 = r\left(1-2\sqrt{3}\right)$, the family as described is critical. $\square$

The preceding lemmas and nonconstructibility proofs exhaust the configurations for touching critical families $\mathcal{F}_4$ where $\{C_1, C_3, C_4\}$ lies in a slab and C_3 has center $o_3(\gamma, 2r)$ with $0 \leq \gamma \leq r$. We proceed documenting configurations with $\gamma > r$, where line ℓ_1 supports C_3, line ℓ_2 supports C_4, and disjoint $\{C_1, C_3, C_4\}$ lies in a slab.

A line must support $\{C_2, C_3, C_4\}$. The right definite support ℓ_{def23R} of $\{C_2, C_3\}$ is disjoint from C_4 by construction, and $\ell_1 \in \mathcal{L}_{23}$ is prohibited. Two lines remain. The separating support $\ell_{sep23RL}$ of $\{C_2, C_3\}$ (with positive slope) is disjoint from C_4: rotate ℓ_{def23R}, which is disjoint from C_4, through a positive angle (counterclockwise) away from C_4, dynamically maintaining contact with the boundary of C_2, until it supports C_3 on the left. The line is in the position of $\ell_{sep23RL}$ ($\neq \ell_1$), disjoint from C_4. The line ℓ_{def23L} remains. Since the line does not coincide with ℓ_{def13L}, it cannot support C_4 on the left. In the following lemma the line supports C_4 on the right.

Lemma 3.21. Let $\delta = r$, disk C_4 have center $o_4(x_4, -2r)$, and C_3 have center $o_3(\gamma, 2r)$. If

$$\gamma = r\left(2\sqrt{3}+1\right) \quad \text{and} \quad x_4 = -r\left(3+2\sqrt{3}\right),$$

then the family $\mathcal{F}_4$ is critical. Furthermore, subfamily $\{C_1, C_3, C_4\}$ lies in a slab supported by the definite supports of $\{C_1, C_3\}$, and the left definite support ℓ_{def23L} of $\{C_2, C_3\}$ supports C_4 on the right.

Proof. Let $\delta = r$, and let subfamily $\{C_1, C_3, C_4\}$ lie in a slab where C_3 has center $o_3(\gamma, 2r)$ with $\gamma > r$. Let ℓ_2 support $\{C_1, C_2, C_4\}$ so that C_4 has center $o_4(x_4, -2r)$. Since $\{C_1, C_3, C_4\}$ lies in a slab, the definite supports of $\{C_1, C_3\}$ support C_4, and the equation of the left definite support from Lemma 3.20 is reproduced here:

$$\ell_{def13L}(x) = \frac{2r}{\gamma + r}x + \frac{r\sqrt{5r^2 + 2r\gamma + \gamma^2} + 2r^2}{\gamma + r}$$

We reproduce the commensurate expression $x_4 = -(2r + \gamma)$ from the previous lemma.

A line must support $\{C_2, C_3, C_4\}$, and as shown in the paragraph preceding the lemma, line ℓ_{def23L} must support C_4 on the right. The definite support ℓ_{def23L} has slope parallel to $\langle o_2, o_3\rangle$ and we denote its y-intercept by m. Applying Lemma 2.5 to line ℓ_{def23L} and the point o_2 yields

$$\left|\frac{2r}{\gamma + r}(r) + (-1)(0) + m\right| = r\sqrt{\left(\frac{2r}{\gamma - r}\right)^2 + 1},$$

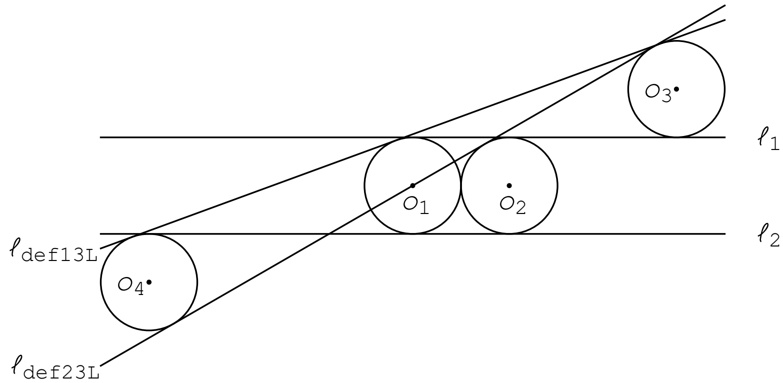


Figure 3.20: Touching critical family $\mathcal{F}_4$ with $r = 1$ and $\gamma = 2\sqrt{3} + 1$.

which leads to the following equation:

$$\ell_{def23L}(x) = \frac{2r}{\gamma - r}x + \frac{r\sqrt{5r^2 - 2r\gamma + \gamma^2} - 2r^2}{\gamma - r}$$

This line also supports C_4, and Lemma 2.5 applied to the line and the point o_4 leads to the equation

$$\gamma^2 - (2r)\gamma - 11r^2 = 0.$$

With the affiliated solution $\gamma = r\left(2\sqrt{3} + 1\right)$, we rewrite the expression $x_4 = -r\left(3 + 2\sqrt{3}\right)$, and these assignments guarantee the family is critical. $\square$

This exhausts all describable configurations for touching critical families of size four with three disks in a slab.

3.3 Touching Critical Families $\mathcal{F}_4$: Summary of Results

In the preceding section, the touching critical families $\mathcal{F}_4$ were constructed by exhaustion. This collection contains a representative of every touching critical family $\mathcal{F}_4$.

Lemma 3.22. The number of touching critical families $\mathcal{F}_4$ is no more than 17, and at least one representative of each distinct type is depicted in the collection of Figures 3.21, 3.22, and 3.23.

We proceed to show that precisely 17 touching families of size four are critical up to symmetries in the Klein four-group V.

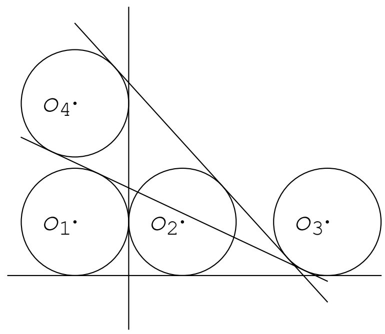
Theorem 3.23. The number of combinatorially distinct touching critical families $\mathcal{F}_4$ is precisely 17 and a representative for each family is depicted in the collection of Figures 3.21, 3.22, and 3.23.

Proof. Denote by $\mathfrak{F}$ the collection of $|\mathfrak{F}| = 17$ touching critical families depicted in Figures 3.21, 3.22, and 3.23. Let $\mathfrak{F}_3$ denote the set of disks C_3 (distinguished by their centers o_3) of the families in $\mathfrak{F}$, and denote by $\mathfrak{F}_4$ the set of disks C_4 (distinguished by their centers o_4) of these families.

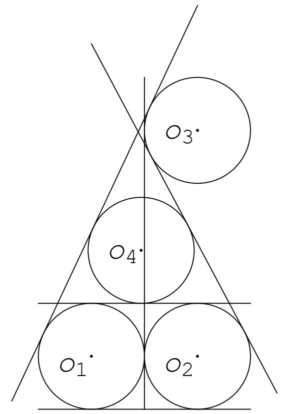
Table 3.1 contains standardized data with $r = 1$ for the coordinates of the center o_3 of disk C_3 for each respective touching critical family $\mathcal{F}_4$ in $\mathfrak{F}$. Direct observation of the rows in Table 3.1 confirms that the center o_3 of disk C_3 in each of the 17 families in $\mathfrak{F}$ is distinct since each pair of coordinates is distinct. Explicitly, each respective coordinate pair (γ, y_3) associated with some center o_3 has either $\gamma = 1$ or $y_3 = 2$ with the one exception listed in the row for Figure 3.21a. Observe that any pair of coordinates o_3, o_3' listed in Table 3.1 with $\gamma = r = 1$ in their first coordinate differs in value in their respective second coordinate, y_3. Similarly, any two coordinates o_3, o_3' with $y_3 = 2r = 2$ differ in value in their respective γ-coordinate. Since no pair $\mathcal{F}$, $\mathcal{G}$ in $\mathfrak{F}$ of touching critical $\mathcal{F}_4$ has disk C_3 in the same position, the $|\mathfrak{F}_3| = 17$ disks C_3 in these families are distinct so that $|\mathfrak{F}_3| = |\mathfrak{F}|$.

A similar direct comparison of the values in Table 3.2 confirms that precisely one pair of families in $\mathfrak{F}$ has their respective disks C_4 in the same position with common center $o_4(3, 2)$. The two families are depicted in the respective Figures 3.23b and 3.23d. Furthermore, an

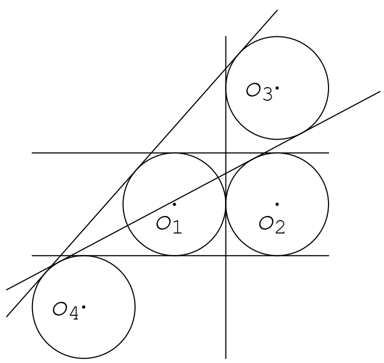
[1]The exact value $\gamma = (1/3) \cdot \big(\beta^+ + \beta^- - 1\big)$ with $\beta^\pm = \sqrt[3]{2}\,\sqrt[3]{13 \pm 3\sqrt{33}}$ for the coordinate in $o_3(\gamma, y_3)$ does not fit conveniently in Table 3.1, so its decimal approximation $\gamma \approx 0.2956$ is listed there.



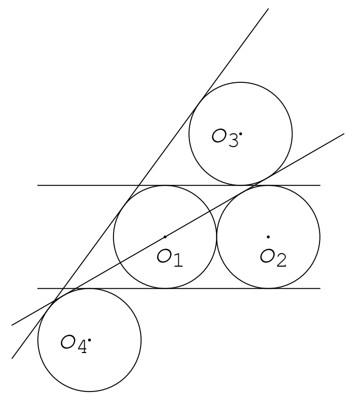
(a)



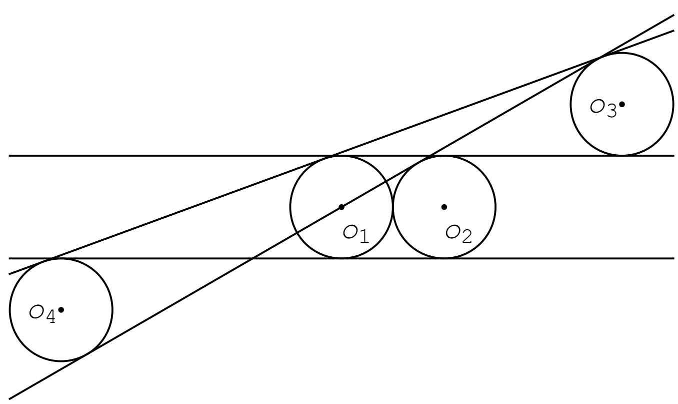
(b)



(c)



(d)



(e)

Figure 3.21: Critical $\mathcal{F}_4$ with three disks in a slab.

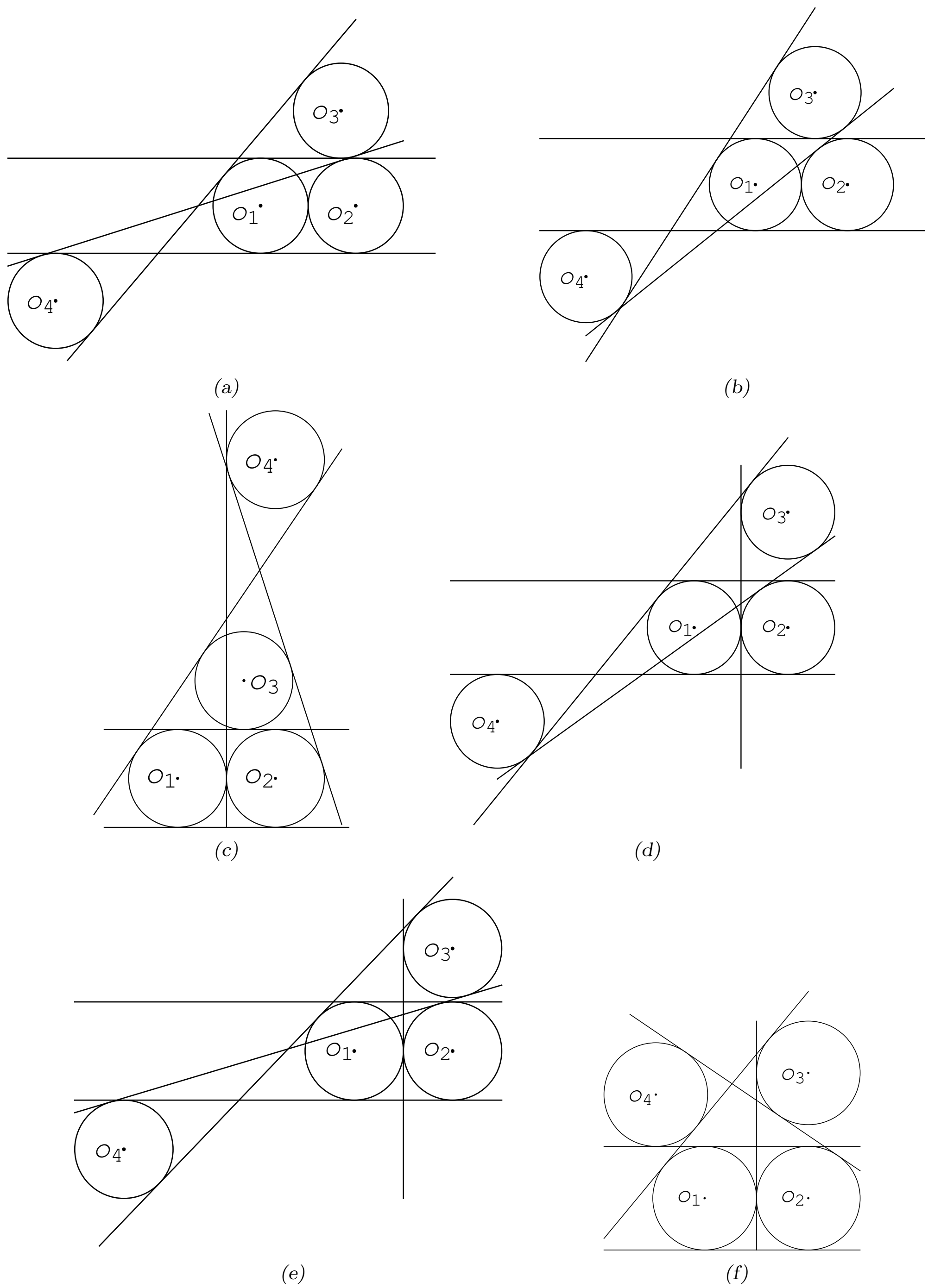


Figure 3.22: Six touching critical $\mathcal{F}_4$ avoiding three disks in a slab.

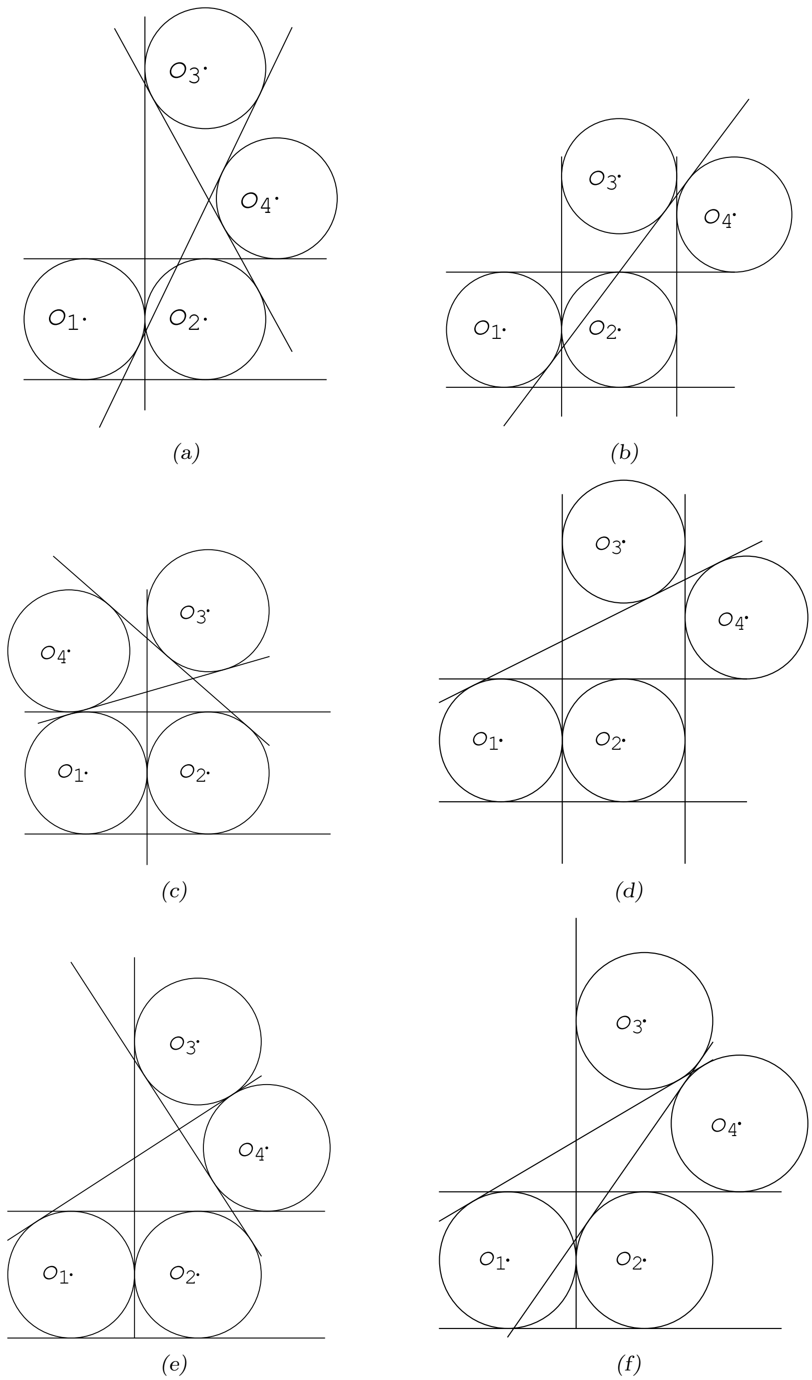


Figure 3.23: Remaining six touching critical $\mathcal{F}_4$ avoiding three disks in a slab.

Table 3.1: Centers $o_3(\gamma, y_3)$ of disks C_3 in each touching critical family $\mathcal{F}_4$ $(r = 1)$

Three disks in a slab	Figure 3.21a	$o_3\,(3.6972, 0)$
	Figure 3.21b	$o_3\left(1, \sqrt{5+4\sqrt{2}}+1\right) \approx o_3\,(1, 4.2645)$
	Figure 3.21c	$o_3\left(1, \sqrt{5+4\sqrt{2}}-1\right) \approx o_3\,(1, 2.2645)$
	Figure 3.21d	$o_3\left(2\sqrt{3}-3, 2\right) \approx o_3\,(0.4641, 2)$
	Figure 3.21e	$o_3\left(2\sqrt{3}+1, 2\right) \approx o_3\,(4.4641, 2)$
Avoiding 3 disks in a slab	Figure 3.22a	$o_3\left((1/3)\cdot\left(9-4\sqrt{3}\right), 2\right) \approx o_3\,(0.6906, 2)$
	Figure 3.22b	$o_3\,(0.2956, 2)^1$
	Figure 3.22c	$o_3\,(0.3551, 2)$
	Figure 3.22d	$o_3\,(1, 2.4648)$
	Figure 3.22e	$o_3\,(1, 2.0876)$
	Figure 3.22f	$o_3\,(1, 2.4185)$
	Figure 3.23a	$o_3\,(1, 4.1529)$
	Figure 3.23b	$o_3\,(1, 8/3) \approx o_3\,(1, 2.6667)$
	Figure 3.23c	$o_3\,(1, 2.6590)$
	Figure 3.23d	$o_3\left(1, 1+\sqrt{5}\right) \approx o_3\,(1, 3.2361)$
	Figure 3.23e	$o_3\,(1, 3.6786)$
	Figure 3.23f	$o_3\,(1, 3.5010)$

exhaustive comparison of the values for o_3 in Table 3.1 and the values for o_4 in Table 3.2 reveals that no coordinate pair is common to both tables.

To avoid repeated values among the respective centers o_4, and to avoid multiple placements for C_4 induced by symmetries in V, we define the restriction $\mathfrak{F}'$ of $\mathfrak{F}$ to contain the touching critical families of $\mathfrak{F}$ depicted in Figures 3.21, 3.22, and 3.23 excluding the families depicted in Figures 3.21a and 3.23d, so that $|\mathfrak{F}'| = 15$. In order for a critical subfamily $\{C_1, C_2, C_3\} \subset \mathcal{F}_4 \in \mathfrak{F}'$ to have a symmetry in V its respective disk C_3 must have its center o_3 on the x-axis or on line ℓ_v. No family in $\mathfrak{F}'$ has this property since the family depicted in Figure 3.21a does not belong to the collection. We note here that the family depicted in Figure 3.21a uniquely has a touching critical subfamily $\mathcal{F}_3$ in a slab, and is necessarily distinct from all other families $\mathcal{F}_4$. It suffices to consider the remaining 16 families.

We define the corresponding restrictions $\mathfrak{F}_3'$ and $\mathfrak{F}_4'$ to denote the respective restrictions of $\mathfrak{F}_3$ and $\mathfrak{F}_4$ that exclude the respective disks C_3, C_4 of the families depicted in Figures 3.21a and 3.23d. Since the respective coordinates for the center $o_3(\gamma, y_3)$ of each disk C_3 in $\mathfrak{F}_3$

Table 3.2: Centers $o_4(x_4, y_4)$ of disks C_4 in each touching critical family $\mathcal{F}_4$ $(r = 1)$

Three disks in a slab	Figure 3.21a	$o_4\,(-1, 2.2104)$
	Figure 3.21b	$o_4\,(-0.0620, 2)$
	Figure 3.21c	$o_4\,(-2.7664, -2)$
	Figure 3.21d	$o_4\left(1 - 2\sqrt{3}, -2\right) \approx o_4\,(-2.4641, -2)$
	Figure 3.21e	$o_4\left(-(3 + 2\sqrt{3}), -2\right) \approx o_4\,(-6.4641, -2)$
Avoiding 3 disks in a slab	Figure 3.22a	$o_4\left(-(1/3) \cdot \left(9 + 4\sqrt{3}\right), -2\right) \approx o_4\,(-5.3094, -2)$
	Figure 3.22b	$o_4\,(-4.6786, -2)$
	Figure 3.22c	$o_4\,(1, 6.5173)$
	Figure 3.22d	$o_4\,(-5.1984, -2)$
	Figure 3.22e	$o_4\,(-5.6859, -2)$
	Figure 3.22f	$o_4\,(-1.9414, 2)$
	Figure 3.23a	$o_4\,(2.1830, 2)$
	Figure 3.23b	$o_4\,(3, 2)$
	Figure 3.23c	$o_4\,(-1.2829, 2)$
	Figure 3.23d	$o_4\,(3, 2)$
	Figure 3.23e	$o_4\,(2.0874, 2)$
	Figure 3.23f	$o_4\,(2.3920, 2)$

are distinct as confirmed by inspecting Table 3.1, the same holds for the subset $\mathfrak{F}_3' \subset \mathfrak{F}_3$. To clarify, the respective sizes of the sets are $|\mathfrak{F}_3| = 17$, $|\mathfrak{F}_4| = 16$, and $|\mathfrak{F}_3'| = 15 = |\mathfrak{F}_4'|$.

For each family $\mathcal{F} \in \mathfrak{F}'$, define the rule denoted by ϕ, that associates the disk $C_3 \in (\mathfrak{F}_3' \cap \mathcal{F})$ with the disk $C_4 \in (\mathfrak{F}_4' \cap \mathcal{F})$. The mapping $\phi : \mathfrak{F}_3' \to \mathfrak{F}_4'$ is a bijection by the following: exhaustive construction of the set $\mathfrak{F}$ of the touching critical families $\mathcal{F}_4$ reveals that adjoining each respective disk $C_3 \in \mathfrak{F}_3' \subset \mathfrak{F}_3$ to touching $\{C_1, C_2\}$ induces precisely one critical family $\mathcal{F}_4 \in \mathfrak{F}'$. This holds since with $C_3 \in \mathfrak{F}_3'$ fixed, only one position for C_4 induces the property $S(3)$ without inducing S, and this pair of disks belongs to the same family $\{C_3, C_4\} \subset \mathcal{F} \in \mathfrak{F}'$. Since ϕ associates each disk C_3 with precisely one disk C_4, the rule determined by $\phi : \mathfrak{F}_3' \to \mathfrak{F}_4'$ is a function. Furthermore, since each disk C_4 in the restricted set $\mathfrak{F}_4'$ is in a family in $\mathfrak{F}'$ by construction, the mapping ϕ is onto. Since the sizes of $\mathfrak{F}_3$ and $\mathfrak{F}_4$ are identical ($|\mathfrak{F}_4'| = 15 = |\mathfrak{F}'|$), the mapping ϕ is one-to-one. It follows that $\phi : \mathfrak{F}_3' \to \mathfrak{F}_4'$ is a bijection, and its inverse $\phi^{-1} : \mathfrak{F}_4' \to \mathfrak{F}_3'$ is well-defined.

Equipped with this notation, we now verify that the families in $\mathfrak{F}$ are distinct. To verify that no two families are duplicates, Remark 3.2 confirms that it is sufficient to restrict our attention to the symmetries in the Klein four-group V since $\{C_1, C_2\} \subset \mathcal{F}$. Furthermore, any critical family $\mathcal{F}$ of size four has a representative $\mathcal{H}$ in $\mathfrak{F}$ since the collection was constructed by exhaustion. Since $\mathcal{F}$ maps onto its representative $\mathcal{H}$ by some symmetry in V, we stipulate without a loss of generality that each critical family $\mathcal{F}$ is oriented in accordance with Figures 3.21, 3.22, and 3.23, and their respective parameters are listed in Tables 3.1 and 3.2.

If the mapping $\mathrm{id} : \mathcal{F} \mapsto \mathcal{G}$ with $\mathrm{id} \in V$ preserves labels on the disks, then $\mathrm{id} : (C_3 \in \mathcal{F}) \mapsto (C_3 \in \mathcal{G})$. If $\mathcal{F}, \mathcal{G} \in \mathfrak{F}$ are distinct, then the map is impossible since $\mathcal{F}, \mathcal{G}$ are oriented as in the figures and tables, and their respective disks labeled C_3 are distinct. If such an identification exists, then necessarily id maps the center o_3 of $C_3 \in \mathcal{F} \in \mathfrak{F}$ to the center o_4 of $C_4 \in \mathcal{G} \in \mathfrak{F}$ so that $\mathrm{id} \in V$ interchanges the labels on C_3, C_4. However, as noted above, no coordinate pair for o_3 listed in Table 3.1 appears in the list of coordinates for o_4 in Table 3.2, and since $\mathrm{id} \in V$ preserves the coordinates together with their signs as listed, this outcome is impossible.

For some pair $\mathcal{F}, \mathcal{G} \in \mathfrak{F}'$, the possibility remains that $C_3 \in \mathcal{F}$ corresponds to $C_4 \in \mathcal{G}$ by a nonidentity symmetry in V. Since the nontrivial symmetries in V correspond to reflections over the x- and y-axes and rotation of 180° about the origin, the order of the coordinates and their respective numerical values up to sign are preserved, so that $(\gamma, y_3) \mapsto (\pm\gamma, \pm y_3)$ and $(x_4, y_4) \mapsto (\pm x_4, \pm y_4)$ under these transformations. If a pair of families $\mathcal{F}, \mathcal{G} \in \mathfrak{F}'$ are identical under symmetry, then a necessary condition on the coordinates for the center o_4 of $C_4 \in \mathcal{F}$ is that $|x_4| = |\gamma| = \gamma$ and $|y_4| = |y_3| = y_3$ for $C_3 \in \mathcal{G}$ with coordinate $o_3(\gamma, y_3)$ since $(o_4 \in \mathcal{F}) \mapsto (o_3 \in \mathcal{G})$. No value of the parameter γ in any pair of coordinates (γ, y_3) for o_3 listed in Table 3.1 appears as a value for $\pm x_4$ in any coordinate (x_4, y_4) for o_4 listed in Table 3.2. Since this necessary condition is not met, the map described is impossible. No disk $C_4 \in \mathfrak{F}'_4$ can be mapped by nonidentity elements of V (reflection or rotation) onto a disk $C_3 \in \mathfrak{F}'_3$, so the families in $\mathfrak{F}'$ are distinct. This accounts for the 15 families of $\mathfrak{F}'$ which are pairwise distinct.

Observe that if two families $\mathcal{F}, \mathcal{G} \in \mathfrak{F}$ are identical under a symmetry not contained in V, then two conditions must be met. Namely, both families must have the same symmetry (not in V) and the respective subfamilies $\{C_3, C_4\}$ must be touching since the proposed line of symmetry cannot cut $\{C_1, C_2\}$. Since the subfamily $\{C_3, C_4\}$ touches in only one family of $\mathfrak{F}$, both of these necessary conditions are not met by any pair of families $\mathcal{F}_4$. The symmetries in V are sufficient.

The preceding accounts for the 15 pairwise distinct families in $\mathfrak{F}'$ and the family depicted in Figure 3.21a, which in total comprises 16 of the 17 families. The single family depicted in Figure 3.23d remains, which we denote for the remainder of this proof by $\mathcal{F}$. Since the 17 disks of $\mathfrak{F}_3$ are distinct, the mapping $\mathrm{id} : \mathcal{F} \mapsto \mathcal{G}$ that carries $C_3 \in \mathcal{F}$ to $C_3 \in \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{F}$ is impossible, so that some symmetry $\psi \in V$ necessarily maps $\psi : (C_4 \in \mathcal{F}) \mapsto (C_3 \in \mathcal{G})$, interchanging disks C_3, C_4. In $\mathcal{F}$, the coordinate pair for o_4 is $(3, 2)$. However, since $\psi : \mathcal{F} \mapsto \mathcal{G}$ carries $\psi : o_4 \mapsto o_3$ and since no coordinate o_3 in Table 3.1 is in the set $\{(\pm 3, \pm 2)\}$, no such mapping $\psi \in V$ exists.

The center $o_4(3, 2)$ of $C_4 \in \mathcal{F}$ coincides with the center of C_4 belonging to the family depicted in Figure 3.23b. The touching critical subfamily $\{C_1, C_2, C_4\}$ with C_4 centered at $o_4(3, 2)$ induces two distinct critical families $\mathcal{F}_4$, which are depicted in the respective Figure 3.23b and 3.23d. No symmetry in V aligns the respective disks C_3, so the two families are distinct. It follows that the family depicted in Figure 3.23d is pairwise distinct from its complement of families in $\mathfrak{F}$.

The 17 families represented in $\mathfrak{F}$ are distinct. □

Remark 3.24. Regarding the preceding theorem, disjoint critical families are documented in Soltan [23]. The 17 touching critical families depicted in Figures 3.21, 3.22 and 3.23 were constructed by exhaustion up to the symmetries in the Klein four-group V, and represent every touching critical family of size four. Five families have three disks in a slab, which are depicted in Figure 3.21. In these families, either the three disks form a touching critical subfamily $\mathcal{F}_3$ as in Figure 3.21a, or the critical subfamily $\mathcal{F}_3$ is disjoint. Precisely 12 families avoid three disks in a slab and they are depicted in Figures 3.22 and 3.23.

Chapter 4: Nonextendable Nonoverlapping Critical Families of Disks

In this chapter we determine the threshold number for nonoverlapping critical families. We identify the maximal nonoverlapping critical families which are, equivalently, nonextendable by exhaustively documenting these families. This includes determining which of the 17 critical families $\mathcal{F}_4$ documented in Chapter 3 are extendable and subsequently constructing an explicit representative corresponding to each extension.

4.1 Reduction to Critical Families $\mathcal{F}_4$

In this section we show that any nonoverlapping critical family $\mathcal{F}_n$ $(n \geq 5)$ of congruent disks in the plane can be obtained from a suitable nonoverlapping critical subfamily $\mathcal{F}_4$ by a consecutive extension of critical subfamilies.

Theorem 4.1. Any critical nonoverlapping (disjoint) family $\mathcal{F}_n$ $(n \geq 5)$ of congruent disks in the plane contains a critical nonoverlapping (disjoint) subfamily $\mathcal{F}_4$.

Proof. This follows immediately from Theorem 2.15 since a critical family does not have property S, and the fact that a subfamily of a disjoint family is itself disjoint. □

Corollary 4.2. Any critical nonoverlapping family $\mathcal{F}_n = \{C_1, \ldots, C_n\}$ $(n \geq 5)$ can be renumbered such that every subfamily $\mathcal{F}_k = \{C_1, \ldots, C_k\}$ $(4 \leq k \leq n)$ is critical. □

Corollary 4.3. Any nonoverlapping subfamily $\mathcal{F}_m \subset \mathcal{F}_n$ that contains a critical subfamily $\mathcal{F}_4 \subset \mathcal{F}_n$ is itself critical. □

4.2 Extensions $\mathcal{F}_n$ from disjoint critical subfamilies $\mathcal{F}_4$

Suppose we are given a touching critical family $\mathcal{F}_n$. In accordance with Theorem 4.1, at least one nonoverlapping critical subfamily $\mathcal{F}_4$ belongs to $\mathcal{F}_n$. If this subfamily $\mathcal{F}_4$ is touching, then it is possible to derive $\mathcal{F}_n$ by a consecutive extension of $\mathcal{F}_4$, and these extensions are detailed in Section 4.3. If the family $\mathcal{F}_n$ does not concurrently contain a touching critical subfamily $\mathcal{F}_4$, then every critical subfamily $\mathcal{F}_4$ of $\mathcal{F}_n$ is disjoint. The latter case is considered below.

Theorem 4.4. Let $\mathcal{F}_n$ be a touching critical family of congruent disks in the plane. If $\mathcal{F}_n$ does not contain a touching critical family $\mathcal{F}_4$, then $n = 5$ and the family $\mathcal{F}_5$ has the configuration depicted in Figure 4.3. The family $\mathcal{F}_5$ depicted in Figure 4.3 is nonextendable.

Proof. 1. Assume first that $n = 5$. Suppose a touching critical family $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ is the extension of a disjoint critical family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$. If $\mathcal{F}_5$ is not concurrently the extension of a touching critical subfamily $\mathcal{F} \subset \mathcal{F}_5$ of size four, then each of its touching subfamilies of size four has the support property $S(4)$. Since C_5 touches at least one disk of $\mathcal{F}_4$, we stipulate up to labels that C_5 touches C_4. The subfamilies of size four containing $\{C_4, C_5\}$ are of the form $\{C_4, C_5\} \cup \{C_i, C_j\}$ $(i \neq j \in \{1, 2, 3\})$, so that precisely $1 \cdot \binom{3}{2} = 3$ touching subfamilies of size four in $\mathcal{F}_5$ contain subfamily $\{C_4, C_5\}$. Additionally, the subfamily $\{C_1, C_2, C_3, C_5\}$ is not necessarily disjoint.

For notational convenience, we fix the touching subfamilies by label as in the following:

$$\mathcal{F} = \{C_4, C_5\} \cup \{C_1, C_2\} = \{C_1, C_2, C_4, C_5\}$$

$$\mathcal{G} = \{C_4, C_5\} \cup \{C_1, C_3\} = \{C_1, C_3, C_4, C_5\}$$

$$\mathcal{H} = \{C_4, C_5\} \cup \{C_2, C_3\} = \{C_2, C_3, C_4, C_5\}$$

Since $\{C_4, C_5\}$ is touching, it has three support lines $\mathcal{L}_{45} = \{\ell_1, \ell_2, \ell_3\}$. Reparametrize as needed so that C_4, C_5 have their respective centers o_4, o_5 on the x-axis of the coordinate plane with coordinates $(-r, 0)$ and $(r, 0)$, respective of order. This is depicted in Figure 4.1.

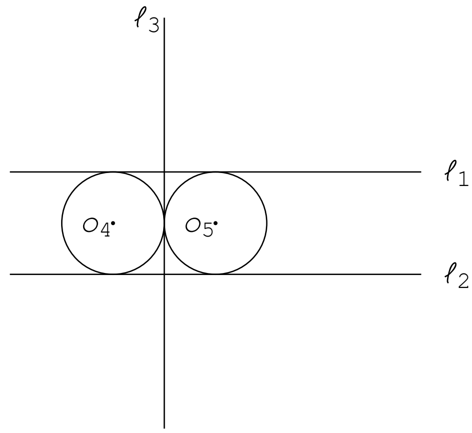


Figure 4.1: Touching subfamily $\{C_4, C_5\}$ of the extension $\mathcal{F}_5$ of disjoint critical $\mathcal{F}_4$.

If none of $\mathcal{F}, \mathcal{G}, \mathcal{H}$ is critical, then a line supports each subfamily, and these lines are necessarily in $\mathcal{L}_{45}$ since $\{C_4, C_5\}$ belongs to each of $\mathcal{F}, \mathcal{G}, \mathcal{H}$. And since disjoint $\mathcal{F}_4$ is critical so that no line supports it, each of the three lines in $\mathcal{L}_{45}$ supports precisely one of the subfamilies $\mathcal{F}, \mathcal{G}, \mathcal{H}$. Up to labels, we stipulate the following:

$$\ell_1 \text{ supports } \mathcal{F} \implies \ell_1 \text{ supports } \{C_1, C_2, C_4\}$$

$$\ell_2 \text{ supports } \mathcal{G} \implies \ell_2 \text{ supports } \{C_1, C_3, C_4\}$$

$$\ell_3 \text{ supports } \mathcal{H} \implies \ell_3 \text{ supports } \{C_2, C_3, C_4\}$$

The preceding implies the following pairs of lines support each respective disk as labeled:

$$\text{Both of } \ell_1, \ell_2 \text{ support } C_1, \quad \text{both of } \ell_1, \ell_3 \text{ support } C_2, \quad \text{and} \quad \text{both of } \ell_2, \ell_3 \text{ support } C_3.$$

These constraints determine the configuration of disjoint $\mathcal{F}_4$. In the following paragraph, recall that $\mathcal{F}_4$ is disjoint and C_4 is centered at $(-r, 0)$ (see Figure 4.1).

Since both of ℓ_1, ℓ_3 support C_2, the disk is centered at either $(-r, 2r)$ or $(r, 2r)$. Since $C_2 \cap C_4 = \emptyset$, the center o_2 of disk C_2 necessarily has coordinates $(r, 2r)$ (compare Figures 4.1 and 4.2). Since both of ℓ_2, ℓ_3 support C_3, the disk is centered at either $(-r, -2r)$ or $(r, -2r)$. Since $C_3 \cap C_4 = \emptyset$, the center o_3 of disk C_3 has coordinates $(r, -2r)$ (compare Figures 4.1 and 4.2). The resulting subfamily $\{C_2, C_3, C_4, C_5\}$ is depicted in Figure 4.2.

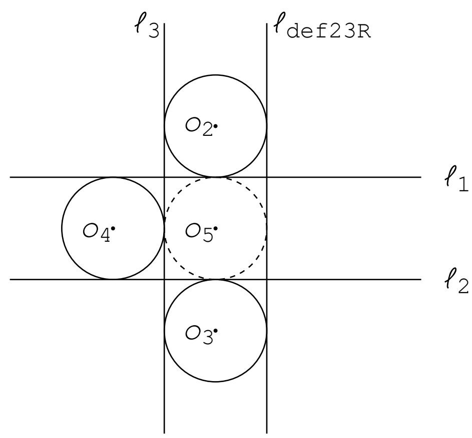


Figure 4.2: Induced placement of C_2, C_3 relative to C_4 given the support relations.

The third condition states that both of ℓ_1, ℓ_2 support C_1, so C_1 lies in the slab with $\{C_4, C_5\}$. Furthermore, since $\mathcal{F}_4$ is critical, a line necessarily supports $\{C_1, C_2, C_3\}$, and this line coincides with a support in $\mathcal{L}_{23}$. Disk C_1 is in the slab, and line $\ell_3 \in \mathcal{L}_{23}$ is not permitted to support C_1 since $\mathcal{F}_4$ is critical. If a separating support in $\mathcal{L}_{23}$ supports C_1, then C_1 overlaps with C_5, and the extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ is overlapping, a contradiction.

The remaining line $\ell_{def23R} \in \mathcal{L}_{23}$ necessarily supports C_1. This entails that the center o_1 of C_1 has coordinates $(3r, 0)$. This configuration coincides with the critical family $\mathcal{F}_4$ depicted in Figure 15 of Soltan [23] with the particular parametrization for the family in which the pairs of support lines are orthogonal, which is permitted for disjoint families of size four. Extending this family by adjoining C_5 centered at $(r, 0)$ as in Figure 4.3, the touching family $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ is critical. Notably, though this family coincides with the extension $\mathcal{F}_5$ depicted in Figure 15 in Soltan [23], this particular parametrization is prohibited for disjoint families.

Explicitly, the family $\mathcal{F}_5$ has $S(3)$ since a line supports each of the four critical subfamilies of its disjoint critical subfamily $\mathcal{F}_4 \subset \mathcal{F}_5$, and line ℓ_1 supports each of $\{C_1, C_2, C_5\}$, $\{C_1, C_4, C_5\}$, $\{C_2, C_4, C_5\}$; line ℓ_2 supports each of $\{C_1, C_3, C_5\}, \{C_3, C_4, C_5\}$; and, ℓ_3 supports $\{C_2, C_3, C_5\}$, so that a line supports each of the six critical subfamilies containing C_5. Since ℓ_1 supports $\mathcal{F}$, line ℓ_2 supports $\mathcal{G}$, line ℓ_3 supports $\mathcal{H}$, and ℓ_{def23R} supports $\{C_1, C_2, C_3, C_5\}$, the family does not contain a touching critical subfamily $\mathcal{F}_4$.

The constraints on the touching subfamily $\{C_4, C_5\}$ force the respective pairs of support lines to be orthogonal. The condition that $\mathcal{F}_4$ is disjoint forces the configuration of the four disjoint disks $\{C_1, C_2, C_3, C_4\}$ to coincide with that depicted in Figure 4.3 which is necessarily the only configuration and parametrization of the disjoint critical subfamily $\mathcal{F}_4$ that induces a touching critical family $\mathcal{F}_5$.

The family $\mathcal{F}_5$ depicted in Figure 4.3 is not extendable. To form the extension $\mathcal{F}_6 = \mathcal{F}_5 \cup \{C_6\}$, we either place disk C_6 with $x_6 < 0$ in the slab with $\{C_4, C_5\}$, or in the convex region of the plane bounded by the lines ℓ_1, ℓ_3 since $\mathcal{F}_5$ has the symmetries of the square. If we place C_6 in the slab with $\{C_4, C_5\}$, then no line supports $\{C_2, C_3, C_6\}$. If we place C_6 in the cone bounded by ℓ_1, ℓ_3, then no line supports $\{C_1, C_3, C_6\}$. The family $\mathcal{F}_5$ is nonextendable and therefore maximal.

The Dirichlet principle implies that if the aforementioned touching subfamilies $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of size four of $\mathcal{F}_5$ are not critical, then precisely two lines in $\mathcal{L}_{45}$ support each of C_1, C_2, C_3. In any other configuration for touching critical family $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ extended from a disjoint critical subfamily $\mathcal{F}_4$ (where C_5 touches C_4), then for at least one disk in $\{C_1, C_2, C_3\}$, precisely one of ℓ_1, ℓ_2, ℓ_3 supports the disk. Up to labels, suppose ℓ_1 supports C_1, and neither of ℓ_2, ℓ_3 supports C_1. Then, since disjoint $\mathcal{F}_4$ is critical, line ℓ_1 is prohibited from

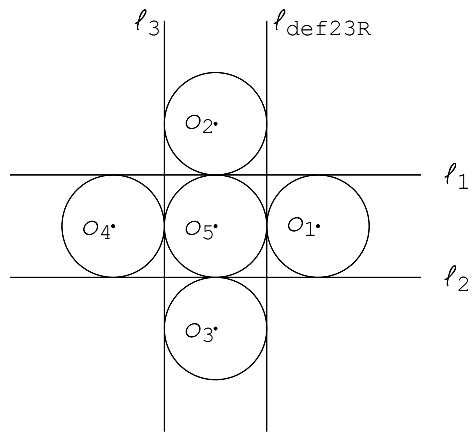


Figure 4.3: Unique touching critical $\mathcal{F}_5$ with no touching critical $\mathcal{F}_4$.

supporting both of C_2, C_3. Suppose ℓ_1 does not support C_2, then we immediately infer that no line supports $\{C_1, C_2, C_4, C_5\}$. Since $\mathcal{F}_5$ has the support property $S(3)$, a line supports each of $\{C_1, C_2, C_4\}$, $\{C_1, C_2, C_5\}$, $\{C_1, C_4, C_5\}$, and $\{C_2, C_4, C_5\}$ so that the subfamily $\mathcal{F}_4' = \{C_1, C_2, C_4, C_5\} \subset \mathcal{F}_5$ is necessarily critical. Since $\mathcal{F}_4' \subset \mathcal{F}_5$ is touching, it is a touching critical family of size four, and this extension $\mathcal{F}_5$ is accounted for in Section 4.3.

2. Consider the case of any touching critical family $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \{C_n\}$ ($n \geq 6$) that does not contain a touching critical subfamily $\mathcal{F}_4$. The family necessarily contains a disjoint critical subfamily $\mathcal{F}_4$ by Theorem 4.1. We proceed to show that this leads to a contradiction.

If any disk $C_5, C_6, \ldots, C_n \in \mathcal{F}_n$ touches the critical disjoint subfamily $\mathcal{F}_4$, then the subfamily $\{C_i\} \cup \mathcal{F}_4 \subset \mathcal{F}_n$ ($i \in \{5, \ldots, n\}$) of size five necessarily takes on the configuration $\mathcal{F}$ depicted in Figure 4.3 by the arguments given above. Since this family $\mathcal{F}_5$ is nonextendable, this contradicts the fact that $\mathcal{F}$ belongs to $\mathcal{F}_n$, so necessarily $\mathcal{F}_4 \cap \{C_5, C_6, \ldots, C_n\} = \emptyset$. Since $\mathcal{F}_n$ is touching, it necessarily contains, up to labels, at least one pair C_m, C_n of touching disks with $m \in \{5, \ldots, n-1\}$. Combinatorially, the family $\mathcal{F}_n$ contains precisely $t := 1 \cdot \binom{k-2}{2} \geq 6$ touching subfamilies $\mathcal{G}_k \subset \mathcal{F}_n$ ($k \in \{1, \ldots, t\}$) of size four of the form $\{C_m, C_n\} \cup \{C_i, C_j\}$ ($i, j \neq m, n$). This includes the following six families:

$$\mathcal{G}_1 = \{C_1, C_2, C_m, C_n\}, \quad \mathcal{G}_2 = \{C_1, C_3, C_m, C_n\}, \quad \mathcal{G}_3 = \{C_1, C_4, C_m, C_n\}$$

$$\mathcal{G}_4 = \{C_2, C_3, C_m, C_n\}, \quad \mathcal{G}_5 = \{C_2, C_4, C_m, C_n\}, \quad \mathcal{G}_6 = \{C_3, C_4, C_m, C_n\}$$

Since a line supports each subfamily $\mathcal{G}_k$, and $\{C_m, C_n\} \subset \mathcal{G}_k$ (for each k), then each $\mathcal{G}_k$ is supported by at least one line in $\mathcal{L}_{mn} = \{\ell_1, \ell_2, \ell_3\}$ (observe that we renew the labels ℓ_1, ℓ_2, ℓ_3 for this part, Part 2., of the current proof). By the Dirichlet principle, one of the lines in $\{\ell_1, \ell_2, \ell_3\}$ supports at least $\left\lceil \dfrac{6}{3} \right\rceil = 2$ subfamilies $\mathcal{G}_{k_\lambda}, \mathcal{G}_{k_\mu}$ with $k_\lambda, k_\mu \in \{1, \ldots, 6\}$ (the subscript k is indexed by the symbols λ, μ and below by ν). If line ℓ supports $\{\mathcal{G}_{k_\lambda}, \mathcal{G}_{k_\mu}\}$ with $\mathcal{G}_{k_\lambda} \cap \mathcal{G}_{k_\mu} = \{C_m, C_n\}$, then line ℓ necessarily supports $\mathcal{F}_4$ since

$$\left|\mathcal{G}_{k_\lambda} \cup \mathcal{G}_{k_\mu}\right| = |\mathcal{G}_{k_\lambda}| + \left|\mathcal{G}_{k_\mu}\right| - \left|\mathcal{G}_{k_\lambda} \cap \mathcal{G}_{k_\mu}\right| = 4 + 4 - 2 = 6,$$

so that $\mathcal{G}_{k_\lambda} \cup \mathcal{G}_{k_\mu} = \mathcal{F}_4 \cup \{C_m, C_n\}$.

And if line $\ell \in \mathcal{L}_{mn}$ supports three or more subfamilies including $\mathcal{G}_{k_\lambda}, \mathcal{G}_{k_\mu}, \mathcal{G}_{k_\nu}$ with $k_\lambda, k_\mu, k_\nu \in \{1, \ldots, 6\}$, then line ℓ supports either $\mathcal{F}_4$ or, up to labels, $\{C_1, C_2, C_3\} \in \mathcal{F}_4$ by the Dirichlet principle since the intersection $\left|\mathcal{G}_{k_\lambda} \cap \mathcal{G}_{k_\mu}\right| \leq 3$ for each pair of disks among the six $\mathcal{G}_k$. Then, by a further application of the Dirichlet principle, the remaining two lines in $\mathcal{L}_{mn}$ support the remaining three G_k, so that a line $\ell' \in \mathcal{L}_{mn}$ supports, up to labels, $\{C_1, C_2\}$. If lines ℓ, ℓ' correspond to the parallel supports ℓ_1, ℓ_2, then ℓ supports $\mathcal{F}_4$, a contradiction. So lines ℓ, ℓ' correspond to ℓ_1, ℓ_3 or ℓ_2, ℓ_3, in which case the intersection $\{C_1, C_2\} \cap \{C_m, C_n\} \neq \emptyset$, a contradiction.

To avoid this contradiction, each line $\ell \in \mathcal{L}_{mn}$ necessarily supports precisely two of the subfamilies $\mathcal{G}_{k_\lambda}, \mathcal{G}_{k_\mu}$ with $k_\lambda, k_\mu \in \{1, \ldots, 6\}$ where $\left|\mathcal{G}_{k_\lambda} \cap \mathcal{G}_{k_\mu}\right| = 3$ which necessarily implies that each line $\ell \in \mathcal{L}_{mn}$ supports three disks of $\mathcal{F}_4$. In particular, lines $\ell_1, \ell_3 \in \mathcal{L}_{mn}$ each respectively support three disks in $\mathcal{F}_4$, which we temporarily label as $\{D_1, D_2, D_3\} \subset \mathcal{F}_4$ and $\{E_1, E_2, E_3\} \subset \mathcal{F}_4$. And since $|\mathcal{F}_4| = 4$, necessarily $|\{D_1, D_2, D_3\} \cap \{E_1, E_2, E_3\}| \geq 2$, so that ℓ_1, ℓ_3 both simultaneously support at least two disks in $\mathcal{F}_4$. Observe that this forces the disks to have respective centers $(-r, 2r)$ and $(r, 2r)$, so that these disks touch C_m, C_n, respective of order, which contradicts the condition $\mathcal{F}_4 \cap \{C_m, C_n\} = \emptyset$.

The obtained contradiction implies that any touching critical family $\mathcal{F}_n$ $(n \geq 6)$ necessarily contains a touching critical subfamily $\mathcal{F}_4$. $\square$

Corollary 4.5. The touching critical families $\mathcal{F}_n$ described in Section 4.3 together with the family $\mathcal{F}_5$ depicted in Figure 4.3 completely describe all touching critical families $\mathcal{F}_n$ with $n \geq 5$. $\square$

4.3 Extendibility of Touching Critical Families $\mathcal{F}_4$

It is not trivial to determine whether a particular critical family $\mathcal{F}_4$ is extendable. No *a priori*, combinatorial criterion prevents any particular extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$. As a representative example, consider for the moment the family depicted in Figure 3.21a. If we adjoin a congruent disk C_5 to $\mathcal{F}_4$ with center $o_5(x_5, y_5) = (x_5, 2r)$, then line ℓ_1 (not shown) supports it and consequently the subfamilies $\{C_1, C_2, C_5\}$, $\{C_1, C_3, C_5\}$, and $\{C_2, C_3, C_5\}$. A subsequent horizontal translation brings C_5 in contact with $\ell_{sep23RL} = \ell_{sep34LR}$ on the left side of the disk, and the line consequently supports the additional subfamilies $\{C_2, C_4, C_5\}$ and $\{C_3, C_4, C_5\}$. In this configuration, lines ℓ_1 and $\ell_{sep23RL}$ support five of the six critical subfamilies of $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ that contain C_5. A line in $\mathcal{L}_{14}$ must support the remaining subfamily $\{C_1, C_4, C_5\}$ to ensure $S(3)$. One possibility is line $\ell_{sep14LR}$ (not pictured in Figure 3.21a), which, together with ℓ_1 and $\ell_{sep23RL}$, forms the boundary of a region in the plane where each line simultaneously supports a disk of nonzero radius which is potentially congruent to the disks in $\mathcal{F}_4$. It remains to determine the radius of the disk.

The description of the region in the preceding paragraph involves two of the five critical supports of $\mathcal{F}_4$ and a third line that supports two disks in $\mathcal{F}_4$. Heuristically, many configurations of lines and the associated bounded regions that they respectively inscribe support a disk of nonzero radius that is potentially congruent to those of $\mathcal{F}_4$. No a priori reason forbids an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ where any number of the critical supports of $\mathcal{F}_4$ support C_5.

Explicitly, if none of the critical supports of $\mathcal{F}_4$ support C_5, then six distinct lines, one each from the respective sets of support lines $\mathcal{L}_{ij}$ ($i \neq j$ and $1 \leq i < j \leq 4$) must support C_5. Such a configuration is unlikely but not impossible. Surveying the 17 critical families $\mathcal{F}_4$ of Chapter 3, one must check at least 384 relevant configurations of lines. Alternatively, if one critical support of $\mathcal{F}_4$ supports C_5, then at least three additional distinct lines among the supports in the sets $\mathcal{L}_{ij}$ ($i \neq j$ and $1 \leq i < j \leq 4$) must also support C_5. Surveying the 17 critical families $\mathcal{F}_4$ of Chapter 3, one must check at least 350 configurations of lines. Alternatively, precisely two critical supports of $\mathcal{F}_4$ may support C_5 in which case one must check at least 221 relevant configurations of lines.[1] Additionally, it is possible for three critical supports of $\mathcal{F}_4$ to support the disk C_5. Furthermore, once we have identified a suitable region bounded by three or more relevant lines that concurrently support a disk of nonzero radius, we must determine whether the disk is congruent to the members of $\mathcal{F}_4$ which unavoidably requires a geometric or analytic justification.

Since this direct approach involves evaluating multiple inscribed regions among the 955 configurations of lines mentioned, we proceed by showing in particular that two critical supports of $\mathcal{F}_4$ necessarily support C_5 in any extension, significantly reducing the scope of our analysis. The primary goals of this section are encapsulated in the following theorem.

Theorem 4.6. Of the 17 families in $\mathfrak{F}$ (introduced in Theorem 3.23), precisely four families are extendable. These correspond to the families depicted in Figures 3.21d, 3.21e, 3.22a, and 3.22b. The number of extensions $\mathcal{F}_k$ ($k \geq 5$) is finite, and the size of the largest maximal extension $\mathcal{F}_k$ contains seven disks.

[1] See Appendix A for explicit calculations.

The proof of Theorem 4.6 (see page 144) is a direct consequence of the results contained in Lemma 4.7 through Corollary 4.16 detailed below. These lemmas and corollaries describe the properties of the touching subfamilies of size four of an extension $\mathcal{F}_5$. In particular, Lemmas 4.7 and 4.8 and Corollaries 4.9 and 4.10 taken together show that any touching critical family $\mathcal{F}_5$ has precisely one critical subfamily $\mathcal{F}_4$. The two remaining touching subfamilies of size four have the support property S. The following lemmas assume $\mathcal{F}_4 \in \mathfrak{F}$ is parametrized by convention as in Tables 3.1 and 3.2.

Lemma 4.7. If a touching critical family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$ is extendable to $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$, then the subfamily $\mathcal{F}_4' = \{C_1, C_2, C_3, C\} \subset \mathcal{F}_5$ is a touching critical family or has the support property S.

Proof. Since $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ is critical by supposition, at least one line supports each of its subfamilies $\{C_1, C_2, C\}$, $\{C_1, C_3, C\}$, and $\{C_2, C_3, C\}$. If no line supports $\mathcal{F}_4' = \{C_1, C_2, C_3, C\}$, then it is a critical subfamily. The subfamily $\mathcal{F}_4' \subset \mathcal{F}_5$ is not disjoint since $\{C_1, C_2\} \subset \mathcal{F}_4'$ is a touching subfamily. Either a line supports $\{C_1, C_2, C_3, C\}$ or $\mathcal{F}_4'$ is a touching critical family with a representative in $\mathfrak{F}$. $\square$

Lemma 4.8. If a touching critical family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$ is extendable to $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$, then a line in $\mathcal{L}_{12}$ supports the subfamily $\mathcal{F}_4' = \{C_1, C_2, C_3, C\} \subset \mathcal{F}_5$, and this line is a critical support of $\mathcal{F}_4$.

Proof. From Lemma 4.7, family $\mathcal{F}_4'$ is either critical or it has the support property S. To induce a contradiction, suppose $\mathcal{F}_4'$ is a touching critical family with a representative in $\mathfrak{F}$. As noted in the proof of Theorem 3.23, each respective disk C_3 of the 17 families depicted in Figures 3.21, 3.22, and 3.23 has distinct coordinates according to Table 3.1. Since the extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ contains the critical subfamily $\mathcal{F}_4' = \{C_1, C_2, C_3, C\} \subset \mathcal{F}_5$, it follows that disk $C \notin \mathfrak{F}_4$ since this would imply $\mathcal{F}_4' = \mathcal{F}_4$. This requires disk C to be in a symmetric position to C_4, which implies the subfamily $\{C_1, C_2, C_3\}$ has at least one symmetry of the Klein four-group V. As in the proof of Theorem 3.23, the only family with this property is that depicted in Figure 3.21a, so that disk $C \notin \mathfrak{F}_4$ is centered at $(-1, -2.2104)$ (compare Table 3.2). However, direct inspection confirms that no line in $\mathcal{L}_{34}$ supports $\{C_3, C_4, C\}$.

Since the subfamily $\{C_1, C_2, C_3\} \subset \mathcal{F}_4$ does not coincide with that depicted in Figure 3.21a, the disk $C_3 \in (\mathfrak{F}_3 \cap \mathcal{F}_4)$ induces a unique placement for $C_4 \in (\mathfrak{F}_4 \cap \mathcal{F}_4)$. Since $\mathcal{F}_4 \neq \mathcal{F}'_4$, the subfamily $\mathcal{F}'_4 \subset \mathcal{F}_5$ is not identical as labeled to a family in $\mathfrak{F}$. Heuristically, disk $C_3 \in \mathcal{F}'_4$ may correspond to disk C_4 in some family $\mathcal{G} \in \mathfrak{F}$. However, the fact that the families in $\mathfrak{F}$ are pairwise distinct up to symmetries in V precludes this possibility.

It is impossible that both $\mathcal{F}_5$ and its subfamily $\mathcal{F}'_4$ are critical, so the touching subfamily $\mathcal{F}'_4 \subset \mathcal{F}_5$ is not critical. A line ℓ supports every member of $\mathcal{F}'_4$ as a direct consequence of Lemma 4.7. Since $\{C_1, C_2\} \subset \mathcal{F}'_4$, the line ℓ is in $\mathcal{L}_{12}$, and ℓ is by definition a critical support of $\mathcal{F}_4$. □

Corollary 4.9. If a touching critical family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$ is extendable to $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$, then a line in $\mathcal{L}_{12}$ supports the subfamily $\mathcal{F}''_4 = \{C_1, C_2, C_4, C\} \subset \mathcal{F}_5$, and this line is a critical support of $\mathcal{F}_4$.

Proof. Compare this with the proof of Lemma 4.8. As a corollary to Lemma 4.7, either $\mathcal{F}''_4$ has a representation in $\mathfrak{F}$ or it has the support property S. Assume $\mathcal{F}''_4 \in \mathfrak{F}$. In particular, assume for the moment that $\mathcal{F}''_4 \in \mathfrak{F}'$ as in the proof of Theorem 3.23, so that $\mathcal{F}''_4$ is one of the families depicted in Figures 3.21, 3.22 and 3.23 excluding those depicted in Figures 3.21a and 3.23d.

Using the function $\phi : \mathfrak{F}'_3 \mapsto \mathfrak{F}'_4$ introduced in Theorem 3.23, we determine the structure of critical $\mathcal{F}''_4 \in \mathfrak{F}'$ by the following method. By construction, the function evaluation $\phi^{-1}(C_4) = C \in \mathfrak{F}'_3$ is unique, so that given disk $C_4 \in \mathcal{F}''_4$, we have $\mathcal{F}''_4 = \{C_1, C_2\} \cup \{C_4, \phi^{-1}(C_4)\} = \{C_1, C_2, C_4, C\} \in \mathfrak{F}'$. Since $\{C_1, C_2, C_4\} \subset (\mathcal{F}''_4 \cap \mathcal{F}_4)$, the containment $\mathcal{F}_4 \in \mathfrak{F}'$ implies $\phi^{-1}(C_4) = C_3 = C$ and $\mathcal{F}''_4 = \{C_1, C_2, C_4, \phi^{-1}(C_4)\} = \mathcal{F}_4$, so that $|\mathcal{F}_5| = 4$, a contradiction.

Since critical $\mathcal{F}_4'' \neq \mathcal{F}_4$, necessarily $\mathcal{F}_4'' \notin \mathfrak{F}'$. Family $\mathcal{F}_4'' \in (\mathfrak{F} \setminus \mathfrak{F}')$ coincides with one of the families depicted in Figures 3.21a and 3.23d. If $\mathcal{F}_4''$ is identical to the family depicted in Figure 3.21a, then the family $\mathcal{F}_4$ does not have this configuration, and $C_3 \neq C$. Since $C_4 \in \mathcal{F}_4$, we infer $C_4 \in \mathfrak{F}_4$, so that C_4 has center $(-1, 2.2104)$ as in Table 3.2 with $r = 1$. From the exhaustive construction of $\mathfrak{F}$, we infer that the subfamily $\{C_1, C_2, C_4\}$ induces precisely one placement for a disk C and necessarily $C \in \mathfrak{F}_3$ since only one point in the set $\{(\pm 1, \pm 2.2104)\}$ is in $\mathfrak{F}_3$ and none of these is in $\mathfrak{F}_4$, as verified by inspecting Tables 3.1 and 3.2. In particular, $C = C_3 \in \mathfrak{F}_3$ so that $\mathcal{F}_4'' = \mathcal{F}$, a contradiction.

Alternatively, if $\mathcal{F}_4''$ is identical to the family depicted in Figure 3.23d, then a disk in $\mathcal{F}_4$ has its center in $\{(\pm 3, \pm 2)\}$. And since $\mathcal{F}_4$ is parametrized by convention, the disk has center $(3, 2)$ and corresponds to $C_4 \in \mathfrak{F}_4$. Since $\mathcal{F}_4'' \neq \mathcal{F}_4$, disk $C_3 \neq C$. Since only two families have disk C_4 centered at $(3, 2)$, the subfamily $\mathcal{F}_4$ corresponds to the family depicted in Figure 3.23b. Superimposing $\mathcal{F}_4$ with $\mathcal{F}_4''$, the disks C and C_3 overlap, so that $\mathcal{F}_5$ containing $\{C_3, C\}$ is overlapping, a contradiction.

The preceding arguments apply equally if instead we consider separately $\mathcal{F}_4 \in \mathfrak{F}'$ and $\mathcal{F}_4 \notin \mathfrak{F}'$. It follows that the touching subfamily $\mathcal{F}_4'' = \{C_1, C_2, C_4, C\} \subset \mathcal{F}_5$ is not critical and some line ℓ in $\mathcal{L}_{12}$ supports each of its members and is a critical support of $\mathcal{F}_4$ by definition. $\square$

Corollary 4.10. Any touching critical family $\mathcal{F}_5$ not depicted in Figure 4.3 contains precisely one touching critical subfamily of size four which corresponds to one of those in $\mathfrak{F}$ up to the symmetries in V.

Proof. The family $\mathcal{F}_5$ does not have the configuration depicted in Figure 4.3, so it contains a touching critical subfamily $\mathcal{F}_4$. The result follows as an immediate consequence of Lemmas 4.7, 4.8 and Corollary 4.9. Of the $1 \cdot 1 \cdot \binom{3}{2} = 3$ subfamilies of size four of $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$, the subfamilies $\mathcal{F}_4'$ and $\mathcal{F}_4''$ have the support property S, whereas $\mathcal{F}_4 \subset \mathcal{F}_5$ is critical and corresponds to one of the families depicted in Figures 3.21, 3.22, and 3.23. $\square$

Corollary 4.11. If a critical family $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ is the extension of a touching critical family $\mathcal{F}_4$, then at least two critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ support C.

Corollary 4.12. If a critical family $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \{C_n\}$ is the extension of a touching critical family $\mathcal{F}_{n-1}$ which itself is the extension of a touching critical family $\mathcal{F}_4$, then at least two critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ support C_n.

Proof. According to Lemma 4.8 and Corollary 4.9, each subfamily $\mathcal{F}_4' = \{C_1, C_2, C_3, C_n\} \subset (\mathcal{F}_4 \cup \{C_n\})$ and $\mathcal{F}_4'' = \{C_1, C_2, C_4, C_n\} \subset (\mathcal{F}_4 \cup \{C_n\})$ has the support property S. Furthermore, the lines that support these subfamilies are critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$, so that at least two critical supports of $\mathcal{F}_4$ support C_n. □

To summarize the preceding, each family in $\mathfrak{F}$ is distinct. If an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ exists, then it has precisely one touching critical subfamily which is $\mathcal{F}_4$. In particular, Lemma 4.8 and Corollary 4.9 require that two distinct critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ support disk C and consequently at least one critical support of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ supports each of the two remaining touching subfamilies of size four.

We now develop criteria that guarantee a touching critical $\mathcal{F}_4$ is extendable. Since the family depicted in Figure 3.21a is unique in having a touching critical subfamily in a slab, we dispense with this family separately.

Lemma 4.13. The family depicted in Figure 3.21a is not extendable.

Proof. Suppose $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ is an extension of $\mathcal{F}_4$ depicted in Figure 3.21a. Lemma 4.8 guarantees that a line in $\mathcal{L}_{12}$ supports the subfamily $\{C_1, C_2, C_3, C\} = \mathcal{F}_4' \subset \mathcal{F}_5$, which must be one of ℓ_1, ℓ_2 since the subfamily $\{C_1, C_2, C_3\}$ lies in a slab (ℓ_1 is not shown). Furthermore, Corollary 4.9 guarantees that a line in $\mathcal{L}_{12}$ supports $\mathcal{F}_4'' = \{C_1, C_2, C_4, C\}$, consequently ℓ_v supports C. Disk C is supported by either both of ℓ_1, ℓ_v or both of ℓ_2, ℓ_v which places the disk at one of the corners determined by these respective pairs of lines (refer to Figure 3.21a). These constraints provide precisely three possible placements for the disk C in the extension since disks are not permitted to overlap or coincide. A line supports each critical subfamily of $\mathcal{F}_5$ containing C except for $\{C_3, C_4, C\}$.

In particular, if line ℓ_2 supports disk C, then C is disjoint from the supports in $\mathcal{L}_{34}$ since line ℓ_2 separates it from the lines in $\mathcal{L}_{34}$. On the other hand if line ℓ_1 (not pictured) supports C, then line ℓ_{def34L} is disjoint from C, and since C overlaps with the slab containing $\{C_3, C_4\}$, the line ℓ_{def34R} cuts C by symmetry. By inspection, line $\ell_{sep34LR}$ (pictured) cuts C and the associated separating support $\ell_{sep34RL}$ (not pictured) cuts C: a rotational shift of ℓ_1 maintaining contact with the boundary of C_3 that brings the line to the position of $\ell_{sep34RL}$ drives the line into C, so that it cuts the disk. □

We are now ready to state the criteria for extendable critical families $\mathcal{F}_4$.

Lemma 4.14. Any touching critical family $\mathcal{F}_4$ is extendable if and only if line ℓ_1 supports C_3, and line ℓ_2 supports C_4.

Proof. By the conventions in this paper, the parameters γ, y_3 associated with the center o_3 of C_3 are nonnegative, so that each line in $\{\ell_1, \ell_2\}$ supports a disk in $\{C_3, C_4\}$ only if line ℓ_1 supports C_3 and line ℓ_2 supports C_4.

($\Longleftarrow$) Suppose line ℓ_1 supports C_3 and line ℓ_2 supports C_4 in a touching critical $\mathcal{F}_4$. Then neither of C_3, C_4 lies in the slab with $\{C_1, C_2\}$ since $\mathcal{F}_4$ does not have property S. This describes the families depicted in Figures 3.21d, 3.21e, 3.22a, and 3.22b. By Corollary 4.11, two critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ necessarily support C, so disk C lies in the slab with $\{C_1, C_2\}$ supported by both of ℓ_1, ℓ_2. These lines support five of the six critical subfamilies, the exception being $\{C_3, C_4, C\}$. By convention, a line in $\mathcal{L}_{34}$ supports C_1 on the left (line ℓ_{def13L} in each of the four respective figures). Place C in the slab opposite C_1, so that it contacts this line. The line supports $\{C_3, C_4, C\}$ and C is disjoint from $\mathcal{F}_4$, so the touching family $\mathcal{F}_5$ is critical.

($\Longrightarrow$) Suppose $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ is a critical extension of $\mathcal{F}_4$. As a consequence of Corollary 4.11, at least two distinct critical supports in $\mathcal{L}_{12}$ support C, so that a line supports $\mathcal{F}_4' = \{C_1, C_2, C_3, C\}$ and a line supports $\mathcal{F}_4'' = \{C_1, C_2, C_4, C\}$. If either line ℓ_1

does not support C_3 or line ℓ_2 does not support C_4, then line ℓ_v supports one of $\mathcal{F}_4', \mathcal{F}_4''$ (and neither of ℓ_1, ℓ_2 supports it). Either disk C_3 lies above ℓ_1, and ℓ_2 supports C_4, or both disks lie above ℓ_1 as confirmed by inspecting Figures 3.21, 3.22, and 3.23 depicting the 17 members of $\mathfrak{F}$. The case when C_3 lies in a slab with $\{C_1, C_2\}$ is accounted for in Lemma 4.13.

In the first of the configurations described above, line ℓ_2 supports C_4 and line ℓ_v supports C_3. By Corollary 4.11 both lines support C. This describes the three families depicted in Figures 3.21c, 3.22d, and 3.22e. The lines ℓ_v and ℓ_2 intersect in a right angle, creating four angular regions capable of supporting a disk tangent to both lines, two of which are occupied respectively by C_1, C_2. If C is centered at $o(r, -2r)$ in the fourth quadrant of the plane, then in each respective family, the line ℓ_{def34R} of the four in $\mathcal{L}_{34}$ approaches nearest to the boundary of disk C but does not enter the quadrant.

Otherwise, under the convention $r = 1$, disk C is centered at $(-r, -2r) = (-1, -2)$ (where ℓ_2 supports C_4 and ℓ_v supports C_3). For the family depicted in Figure 3.21c, disk C overlaps with C_4 since the center o_4 of C_4 has coordinates $(-2.7664, -2)$ with $r = 1$ (see Table 3.2), and the pair C, C_4 would touch precisely if disk C_4 had center o_4 with coordinates $(-3, 2)$. For the respective Figures 3.22d and 3.22e with C centered at $o(-1, -2)$, line ℓ_{def34R} of the four in $\mathcal{L}_{34}$ approaches nearest to the boundary of C. In the family depicted in Figure 3.22d, line ℓ_{def34R} supports C precisely when the line meets the horizontal at an angle of 45° since it supports C_2. The line is disjoint from C since it meets the horizontal at an angle less than 45° since $\gamma - x_4 > y_3 - y_4$ for the family.

For the family depicted in Figure 3.22e, line ℓ_{def34R} does not support C_2. We observe instead that ℓ_{def34R} supports C_5 centered at $(-1, -2)$ with $r = 1$ if and only if a translate C_5' of disk C_5 along the line parallel to ℓ_{def34R} through the point $(-1, -2)$ touches disk C_3. A direct calculation shows that this condition does not hold. Explicitly, we approximate the positive slope of ℓ_{def34R} by

$$k = 0.6212 \approx \frac{4.1}{6.6} > \frac{4.0876\ldots}{6.6859\ldots} \approx 0.6114,$$

which inclines the line slightly toward C_3, so that the translate C_5' of disk C_5 with its center o_5' on the line with slope k through $(-1,-2)$ approaches closer to C_3 than it would along the line parallel to ℓ_{def34R} through $(-1,-2)$. Solving for $\mathrm{d}\left(o_3, o_5'\right) \leq 2$, where o_5' is the center of the translate C_5', leads to a quadratic with negative discriminant whose graph lies above the x-axis, which implies C_3 and the translate C_5' are disjoint for all values of x. The calculation is omitted for brevity. Since disks C_3, C_5' are disjoint for any position of o_5' on the line, this implies disk C_5 is disjoint from ℓ_{def34R}. The families in this first configuration are not extendable.

For a family in the second configuration described above, both of C_3, C_4 lie above line ℓ_1. Up to labels, ℓ_v supports C_3 and ℓ_1 supports C_4 (the one exception is Figure 3.22c). This describes the three families depicted in Figures 3.21b, 3.22c, 3.22f, and the six families depicted in Figure 3.23. As a consequence of Lemma 4.8 and Corollary 4.9, both of ℓ_1, ℓ_v support disk C. These lines intersect in a right angle, creating four angular regions capable of supporting a disk tangent to both lines, two of which are respectively occupied by one of C_1, C_2. With the convention $r = 1$, either C has center $(r, 2r) = (1, 2)$ or center $(-r, 2r) = (-1, 2)$. In Figures 3.21b, 3.22c, 3.22f, and 3.23c, disk C overlaps with one of C_3, C_4 in either position, so no extension is possible.

Five families remain. In Figures 3.23a, 3.23b, 3.23d, 3.23e, and 3.23f, disk C centered at $(r, 2r) = (1, 2)$ overlaps with one of C_3, C_4 by geometric inference. The remaining position centers disk C at $(-r, 2r) = (-1, 2)$. Inspecting Figure 3.23a, the supports in $\mathcal{L}_{34}$ are disjoint from a congruent disk C centered at $(-1, 2)$. In Figures 3.23d and 3.23f, no line in $\mathcal{L}_{34}$ supports disk C centered at $(-1, 2)$. In both families, the definite supports and one separating support of $\{C_3, C_4\}$ are disjoint from C by geometric inference (see the respective figures). The associated separating support $\ell_{sep13LR} = \ell_{sep34RL}$ of each respective family cuts C since the corresponding line in each respective family supports C_1 on the left from above and C touches C_1. The family depicted in Figure 3.23e is similar to those depicted in Figures 3.23d and 3.23f since the definite supports of $\{C_3, C_4\}$ are disjoint from C by observation. Furthermore, the support $\ell_{sep13LR} = \ell_{sep34}$ cuts C since it supports C_1 from above and the pair $\{C_1, C\}$ is touching. Since $\{C_3, C_4\}$ is touching this accounts for the three supports of $\mathcal{L}_{34}$.

For the remaining family depicted in Figure 3.23b, line $\ell_{def34R} \in \mathcal{L}_{34}$ and both separating supports of $\mathcal{L}_{34}$ are disjoint from C centered at $(-1, 2)$ by observation. The remaining line ℓ_{def34L} cuts C: the centers of disks C_4, C lie on the line $\{y = 2r = 2\}$ (standardized with $r = 1$), and by symmetry either both separating supports of $\{C_4, C\}$ support disk C_3 or they do not. The line $\ell_{sep14RL} = \ell_{def13R}$ supports C_3, and rotating $\ell_{sep14RL}$ clockwise into C_3, dynamically maintaining contact with the boundary of C_4, until it supports C from below shows that this separating support of $\{C_4, C\}$ cuts C_3. Since the separating support of $\{C_4, C\}$ that supports C from below cuts C_3, it is impossible that the line ℓ_{def34L} supports C (from above) since this would require both separating supports of $\{C_4, C\}$ to support C_3, a contradiction.

This accounts for the nine families, and it follows that no extension is possible if either line ℓ_1 fails to support C_3 or line ℓ_2 fails to support C_4. And when ℓ_1 supports C_3 and ℓ_2 supports C_4, an extension is possible with C placed in the slab with $\{C_1, C_2\}$ as outlined above. $\square$

The following corollaries are an immediate consequence of the preceding.

Corollary 4.15. The 7 critical families of size four depicted in Figures 3.21a, 3.21b, 3.21c, 3.22c, 3.22d, 3.22e, 3.22f, and the 6 families depicted in Figure 3.23 are nonextendable and therefore maximal.

Corollary 4.16. The critical families of size four depicted in Figures 3.21d, 3.21e, 3.22a, and 3.22b are extendable.

We now explicitly describe the extensions of the families identified in Corollary 4.16 including the maximal extensions of each family. Each family is distinct since the placement of disks C_3, C_4 is distinct as confirmed by Tables 3.1 and 3.2. Furthermore, adjoining disks to each family cannot induce an additional line of symmetry through the origin, so that we do not need to concern ourselves with the symmetries of the various dihedral groups. The symmetries in V are sufficient to verify the various extensions are distinct. Corollary 4.12 guarantees that in any touching extension $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{C_{k+1}\}$ $(k \geq 4)$ the congruent disk C_{k+1} is placed in the slab determined by ℓ_1, ℓ_2 since two critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ must support C_{k+1}.

Lemma 4.17. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.21d has two extensions $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ of size five. Each family $\mathcal{F}_5$ is nonextendable and therefore maximal.

Proof. Corollary 4.16 states that the family depicted in Figure 3.21d is extendable. As a consequence of Corollary 4.11, disk C_5 must lie in the slab determined by ℓ_1, ℓ_2 since two critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ must support C_5. Disk C_5 is not permitted to overlap with $\mathcal{F}_4$, and in particular since C_5 is placed in this slab, it is sufficient to ensure that C_5 does not overlap with the subfamily $\{C_1, C_2\}$.

Since ℓ_1 supports $\{C_1, C_2, C_3, C_5\}$ and ℓ_2 supports $\{C_1, C_2, C_4, C_5\}$, a line supports every critical subfamily except $\{C_3, C_4, C_5\}$. To preserve $S(3)$ in the extension $\mathcal{F}_5$, a support of $\{C_3, C_4\}$ must support C_5. Since the lines ℓ_1, ℓ_2 correspond to the definite supports of each respective subfamily $\{C_1, C_5\}$ and $\{C_2, C_5\}$, any other support of the respective subfamilies separates the respective pair of disks. Furthermore, Theorem 2.3, Part (c) guarantees that the disjoint subfamily $\{C_3, C_4\}$ has four support lines which are listed in the set $\mathcal{L}_{34} = \{\ell_{def34L}, \ell_{def34R}, \ell_{sep34LR}, \ell_{sep34RL}\}$. When line $\ell_{def34L} = \ell_{def13L}$ supports C_5 on the right, it separates C_5 from $\{C_1, C_2\}$ since $\ell_{def34L} = \ell_{sep15LR}$, and the resulting extension $\mathcal{F}_5$ is touching (see Figure 4.4a).

The critical support $\ell_{sep34RL} = \ell_{sep23LR}$ of $\mathcal{F}_4$ separates $\{C_2, C_5\}$ when it supports C_5 (necessarily on the right), so we must only determine whether C_5 overlaps with C_1. Since line $\ell_{sep34RL}$ contains the center o_1 of C_1, rotate the collection of disks and lines C_1, C_2, and $\ell_{sep34RL}$ through an angle of 180° about o_1 so that the image C_2' of C_2 lies in the slab adjacent to C_1. Disks C_3, C_4 remain in place and the rotation about o_1 maps line $\ell_{sep34RL}$ onto itself. The image C_2' of C_2 is in the position of C_5 supported by $\ell_{sep34RL}$. Since $\{C_1, C_2\}$ is touching, the subfamily $\{C_1, C_2'\} \mapsto \{C_1, C_5\}$ is touching as is the resulting extension $\mathcal{F}_5$ (see Figure 4.4b).

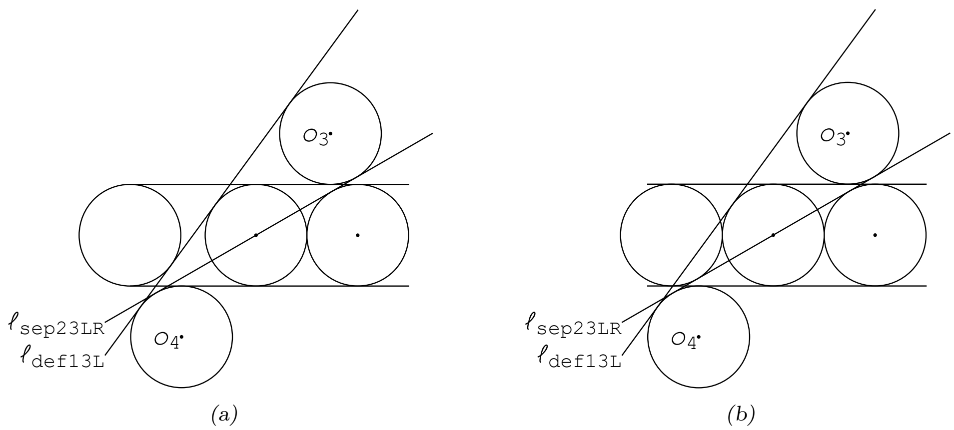


Figure 4.4: Extensions of size 5 of the family depicted in Figure 3.21d.

The remaining lines $\ell_{def34R}, \ell_{sep34LR}$ in $\mathcal{L}_{34}$ induce overlapping extensions which are not permitted. Since line $\ell_{def34R} = \ell_{def13R}$ supports C_1 on the right, it necessarily supports C_5 on the left in an extension, separating $\{C_1, C_5\}$ since $\ell_{def13R} = \ell_{sep15RL}$. We show that disk C_5 overlaps with C_2. Let C_5 with center $(3r, 0)$ touch C_2 and assume that ℓ_{def34R} (positive slope) supports C_5 on the left. Since C_5 lies to the right of C_2, a line with negative slope supports C_3 on the right and C_5 on the left, a contradiction. Since ℓ_{def34R} supports a congruent disk C_5 with $x_5 < 3r$, that disk overlaps with C_2 which is not permitted.

The remaining line $\ell_{sep34LR}$ contains the center o_1 of C_1. As detailed above, its associated separating support $\ell_{sep34RL} = \ell_{sep23LR}$ supports the (two) disks C_2 and C_5 that touch C_1

on opposite sides. Rotating this associated separating support $\ell_{sep34RL}$ counterclockwise about the point o_1 (through disk C_3) to coincide with $\ell_{sep34LR}$ implies the line is disjoint from both of C_2, C_5 since $\gamma \neq x_4$. To the left of $\ell_{sep34LR}$, a disk translated from the position of C_5 centered at $(-3r, 0)$ to the right $C_5 \mapsto C_5'$ meets the line, and overlaps with C_1 which is not permitted. Similarly, to the right of $\ell_{sep34LR}$ a disk translated from the position of C_2 centered at $(r, 0)$ to the left $C_2 \mapsto C_2'$ meets the line and overlaps with C_2 (and C_1) which is not permitted.

The families $\mathcal{F}_5$ depicted in Figure 4.4 cannot be extended since an extension $\mathcal{F}_6$ necessarily incorporates both disks labeled C_5 in the respective extensions $\mathcal{F}_5$ and the pair of disks overlaps which is not permitted. $\square$

Lemma 4.18. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.21e has five extensions $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ of size five. Each of these families is extendable.

Proof. An extension $\mathcal{F}_5$ has congruent disk C_5 in the slab between ℓ_1, ℓ_2. It is nonoverlapping if C_5 does not overlap with $\{C_1, C_2\}$. Since the lines ℓ_1, ℓ_2 coincide with the definite supports of both subfamilies $\{C_1, C_5\}$ and $\{C_2, C_5\}$, the remaining supports of each respective subfamily separates the disks. Since ℓ_1 supports C_3 and ℓ_2 supports C_4, a line supports every critical subfamily except possibly $\{C_3, C_4, C_5\}$, so a line in $\mathcal{L}_{34}$ necessarily supports C_5. By Theorem 2.3, Part (c), the disjoint subfamily $\{C_3, C_4\}$ has the four supports listed in the set $\mathcal{L}_{34} = \{\ell_{def34L}, \ell_{def34R}, \ell_{sep34LR}, \ell_{sep34RL}\}$. When $\ell_{def34L} = \ell_{def13L}$ supports C_5, it separates C_5 from $\{C_1, C_2\}$ (see Figure 4.5a).

The critical support ℓ_{def34R} of $\mathcal{F}_4$ necessarily supports C_5 on the left in any extension. Since the line has positive slope and supports C_3 from below, it supports a disk C_5 in the slab disjoint from C_2 (see Figure 4.5d). The critical support $\ell_{sep34LR} = \ell_{def23L}$ of $\mathcal{F}_4$ necessarily supports C_5 on the right in any extension, separating C_5 from C_2, so we check if C_5 overlaps with C_1. Since line $\ell_{sep34LR}$ contains point o_1, rotating the collection of disks and lines C_1, C_2 and $\ell_{sep34RL}$ through an angle of 180° about o_1 maps disk $C_2 \mapsto C_2' = C_5$

in the slab so that it touches C_1 and line $\ell_{sep34LR}$ supports C_5 on the right (see Figure 4.5c).

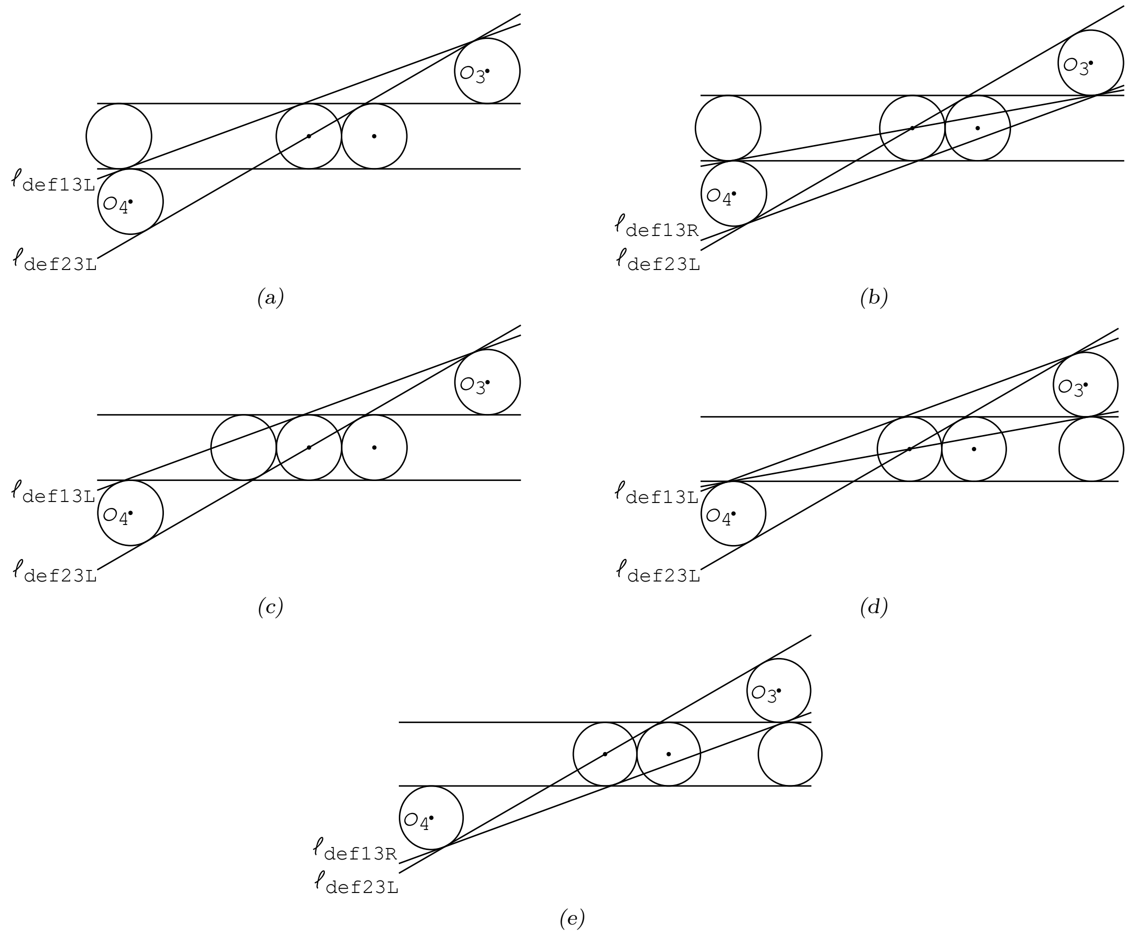


Figure 4.5: Extensions of size 5 of the family depicted in Figure 3.21e.

Line $\ell_{sep34RL}$ contains the point o_1, and cuts C_2 which touches C_1. By symmetry about point o_1, the line cuts any congruent disk in the slab that touches C_1 on its left, so the line supports a disk C_5 disjoint from C_1 (see Figure 4.5b). Additionally, since line $\ell_{sep34RL}$ has positive slope and supports C_3 from below, it supports disk C_5 disjoint from C_2 on the left. Explicitly, since $\gamma = r + 2r\sqrt{3}$, a disk centered at $(\gamma, 0)$ is disjoint from C_2 since

$\gamma - \delta = (r + 2r\sqrt{3}) - r = 2r\sqrt{3} > 2r$, and $x_5 > \gamma$ implies disk C_5 with center $o_5(x_5, 0)$ is disjoint from C_2 (see Figure 4.5d).

The family has 5 extensions which are depicted in Figure 4.5. $\square$

Lemma 4.19. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.21e has eight extensions $\mathcal{F}_6 = \mathcal{F}_5 \cup \{C_6\}$ of size six. Each of these families is extendable.

Proof. The touching critical $\mathcal{F}_4$ depicted in Figure 3.21e has five extensions $\mathcal{F}_5$. The number of extensions of size six is bounded above by $\binom{5}{2} = 10$ since each extension incorporates two of the disks labeled C_5 in the respective extensions $\mathcal{F}_5$. This upper limit is not attained. In the respective families depicted in Figure 4.5a (ℓ_{def34L} supports C_5) and Figure 4.5b ($\ell_{sep34RL}$ supports C_5), the two disks labeled C_5 overlap: each line $\ell_{def34L}, \ell_{sep34RL}$ has positive slope and supports C_4 and its respective disk C_5 on the left, so the respective congruent disks C_5 necessarily overlap by construction. Similar comments show that the two disks labeled C_5 in the respective families depicted in Figures 4.5d and 4.5e also overlap. The two respective pairs of overlapping disks labeled C_5 in their respective extensions prevent two possible extensions $\mathcal{F}_6$. The remaining $10 - 2 = 8$ extensions are depicted in Figure 4.6. $\square$

Lemma 4.20. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.21e has four extensions $\mathcal{F}_7 = \mathcal{F}_6 \cup \{C_7\}$ of size seven. These families are nonextendable and therefore maximal.

Proof. The family depicted in Figure 3.21e has five extensions $\mathcal{F}_5$ as described in Lemma 4.18. Since each respective extension $\mathcal{F}_7$ uses three of the disks labeled C_5, the number of extensions $\mathcal{F}_7$ is bounded above by $\binom{5}{3} = 10$. As noted in Lemma 4.19, two respective pairs of disks labeled C_5 in their respective extensions overlap. Since adjoining an overlapping pair to one of the three remaining disks labeled C_5 forms a family of size 7, we lose three extensions $\mathcal{F}_7$ for each pair of overlapping disks. A total of six possible extensions are lost, and the remaining $10 - 6 = 4$ extensions of size 7 are depicted in Figure 4.7.

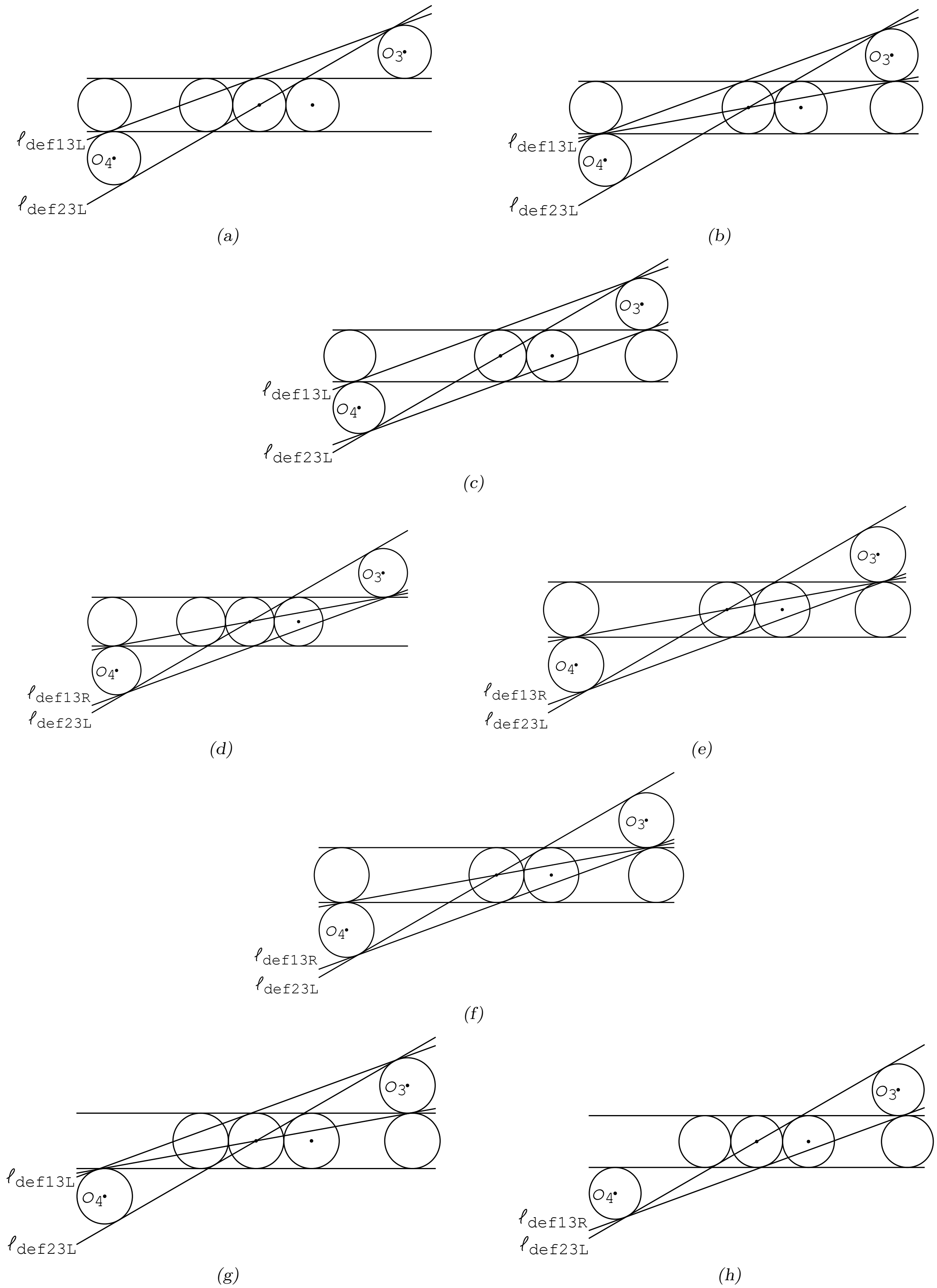


Figure 4.6: Extensions of size 6 of the family depicted in Figure 3.21e.

These families are nonextendable. Each extension of size 7 depicted necessarily contains one disk from each of the overlapping pairs of disks labeled C_5. Any further extension incorporates a pair of overlapping disks in the family. □

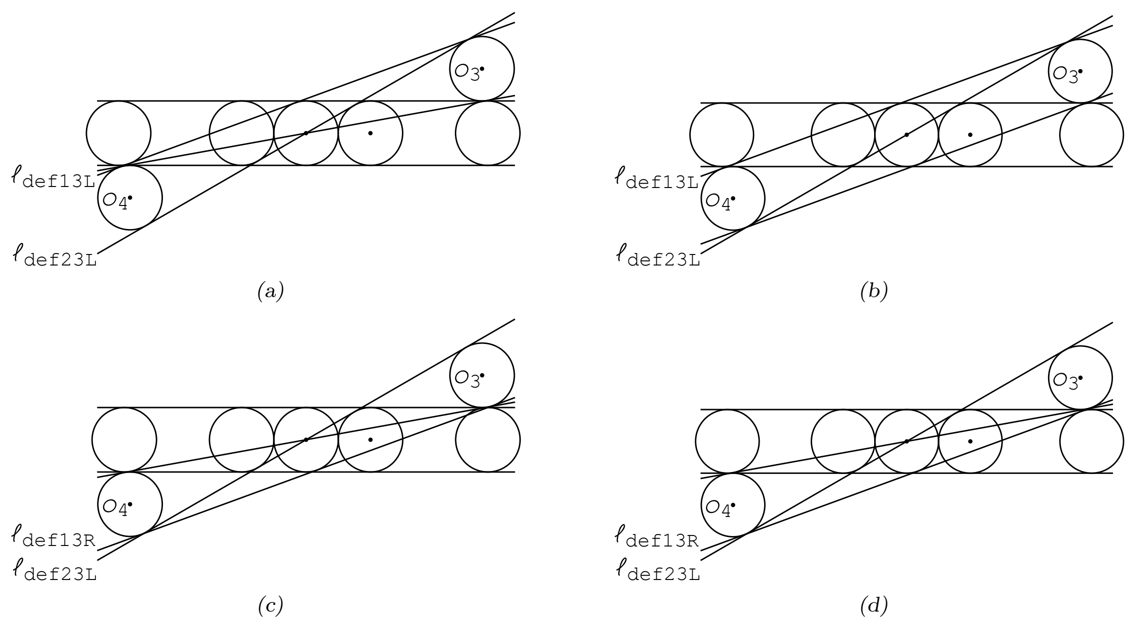


Figure 4.7: Extensions of size 7 of the family depicted in Figure 3.21e.

Lemma 4.21. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.22a has three extensions $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ of size five. Each of these families is extendable.

Proof. Line ℓ_{def34L} supports a disk C_5 on the right that is disjoint from $\mathcal{F}_4$, so the extension is nonoverlapping (Figure 4.8a). If ℓ_{def34L} supports C_5 on the left then ℓ_{def34R} supports C_5 on the right, and the disk overlaps with C_1 since disk C_1 overlaps significantly with the slab determined by lines $\ell_{def34L}, \ell_{def34R}$ (losing two positions). Any disk supported by ℓ_{def34R} on the left overlaps with C_2 since the disks nearly coincide. Any disk C_5 placed in contact with the critical support $\ell_{sep34LR} = \ell_{def13L}$ on its left within the slab determined by ℓ_1, ℓ_2

is nonoverlapping with $\mathcal{F}_4$ (Figure 4.8c). The line $\ell_{sep34RL}$ supports a disk C_5 on the right in the slab that is nonoverlapping with $\mathcal{F}_4$ (Figure 4.8b). The three extensions $\mathcal{F}_5$ with disk C_5 placed to the left of $\ell_{sep34LR} = \ell_{def13L}$ and therefore nonoverlapping with $\mathcal{F}_4$ are depicted in Figure 4.8. $\square$

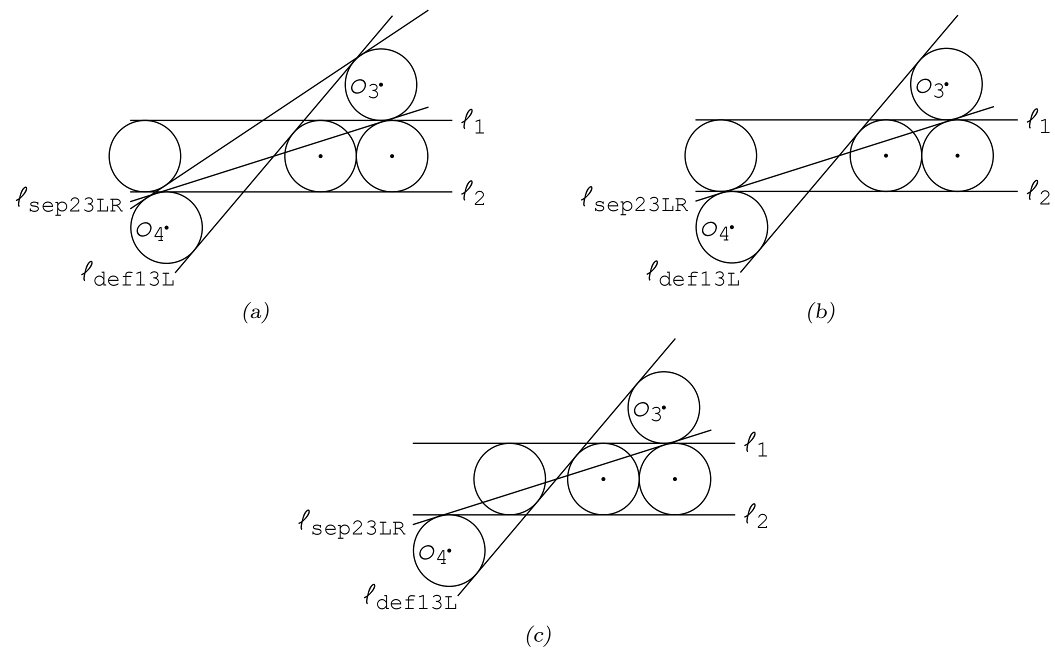


Figure 4.8: Extensions of size 5 of the family depicted in Figure 3.22a.

Lemma 4.22. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.22a has two extensions $\mathcal{F}_6 = \mathcal{F}_5 \cup \{C_6\}$ of size six. These families are nonextendable and therefore maximal.

Proof. The family $\mathcal{F}_4$ depicted in Figure 3.22a has three extensions $\mathcal{F}_5$ which are depicted in Figure 4.8. Any extension of size six selects two of the respective disks labeled C_5, so the number of extensions $\mathcal{F}_6$ is bounded above by $\binom{3}{2} = 3$. By inspection, one of these pairs overlaps, and the remaining $3 - 1 = 2$ extensions $\mathcal{F}_6$ are depicted in Figure 3.22a.

An extension of size 7 requires adjoining all three of the respective disks labeled C_5, two of which overlap, so each family $\mathcal{F}_6$ is nonextendable and therefore maximal. □

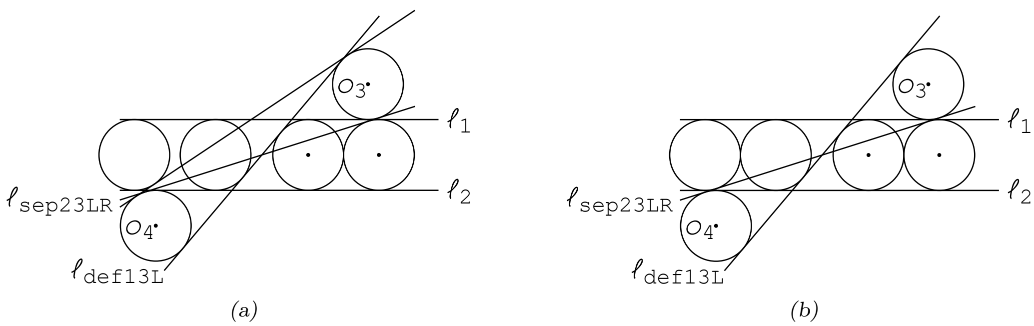


Figure 4.9: Extensions of size 6 of the family depicted in Figure 3.22a.

Lemma 4.23. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.22b has three extensions $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ of size five. Some of these families are extendable.

Proof. The line ℓ_{def34L} supports a disk C_5 on the right that is disjoint from $\mathcal{F}_4$ (Figure 4.10a). If ℓ_{def34L} supports C_5 on the left, then the disk overlaps with C_1 since disk C_1 overlaps significantly with the slab determined by the lines $\ell_{def34L}, \ell_{def34R}$ (losing two positions).

Any disk C_5 within the slab determined by ℓ_1, ℓ_2 that contacts line $\ell_{sep34LR} = \ell_{def13L}$ from the left does not overlap with $\mathcal{F}_4$ (Figure 4.10c). The line $\ell_{sep34RL}$ supports a disk on the right disjoint from $\mathcal{F}_4$ (Figure 4.10b). However, a congruent disk supported on the left by $\ell_{sep34RL}$ nearly coincides with C_2.

The three extensions $\mathcal{F}_5$ with disk C_5 placed to the left of $\ell_{sep34LR} = \ell_{def13L}$ are depicted in Figure 4.10. □

Lemma 4.24. The touching critical family $\mathcal{F}_4$ depicted in Figure 3.22b has one extension $\mathcal{F}_6 = \mathcal{F}_5 \cup \{C_6\}$ of size six. This family is nonextendable and therefore maximal.

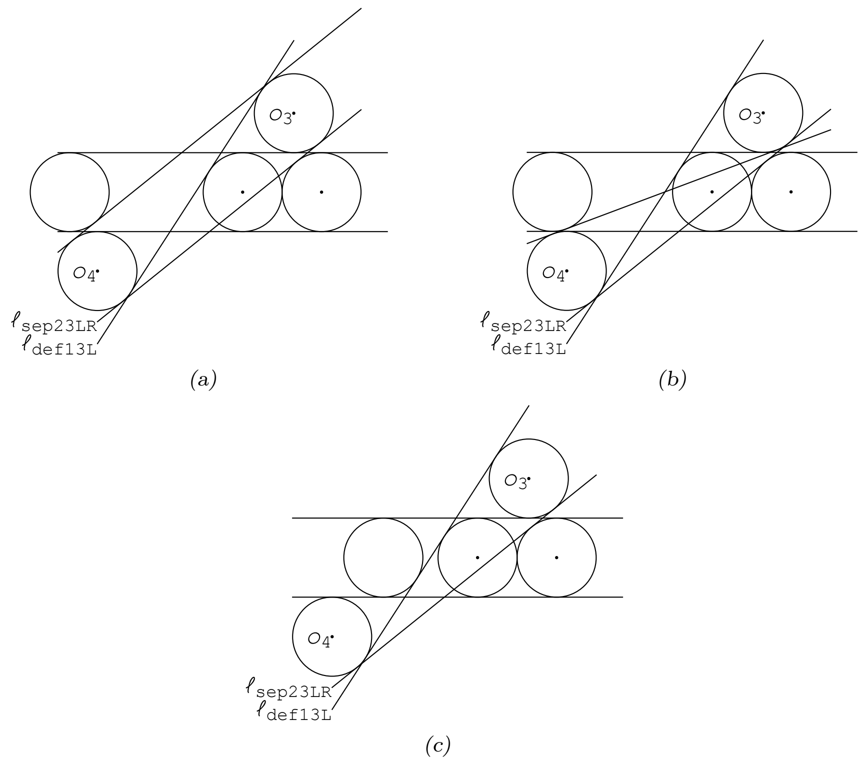


Figure 4.10: Extensions of size 5 of the family depicted in Figure 3.22b.

Proof. The family $\mathcal{F}_4$ depicted in Figure 3.22b has three extensions $\mathcal{F}_5$, and any extension of size six selects two of the respective disks labeled C_5, so the number of extensions of size six is bounded above by $\binom{3}{2} = 3$. Two of these pairs overlap, and the remaining $3 - 2 = 1$ extension of size six is depicted in Figure 4.11. Any further extension (e.g. to size 7) forces adjoining a disk that overlaps with the family, so the family is not extendable and therefore maximal. $\square$

With the exhaustive documentation of the maximal, nonextendable touching critical families complete, we are ready to prove our main results.

Proof of Theorem 4.6. As noted in Remark 3.24, the 17 touching critical families depicted in Figures 3.21, 3.22, and 3.23 are distinct by Theorem 3.23, and they represent all touching critical families $\mathcal{F}_4$ up to symmetries in the Klein four-group V. Lemma 4.7 through

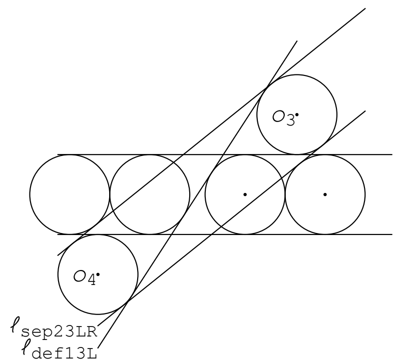


Figure 4.11: Extension of size 6 of the family depicted in Figure 3.22b.

Corollary 4.11 establish subsequent results on extendable and nonextendable families. If an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ exists, then it has precisely one touching critical subfamily which is $\mathcal{F}_4$. In particular, Lemma 4.8 and Corollary 4.9 require that two distinct critical supports of $\mathcal{F}_4$ in $\mathcal{L}_{12}$ support disk C and consequently a line supports each of the two remaining touching subfamilies of size four. Lemma 4.13 shows that the family depicted in Figure 3.21a is not extendable.

Lemma 4.14 establishes specific criteria to identify precisely when a critical family $\mathcal{F}_4$ is extendable. Corollary 4.15 relies on this criteria and states that the 7 families of size four depicted in Figures 3.21a, 3.21b, 3.21c, 3.22c, 3.22d, 3.22e, 3.22f, and the 6 families depicted in Figure 3.23 are nonextendable and therefore maximal. Corollary 4.16 states that the families of size four depicted in Figures 3.21d, 3.21e, 3.22a, and 3.22b are extendable.

In particular, the number of extensions is finite. Lemmas 4.17 through 4.24 explicitly document the extensions of touching critical families of sizes five, six, and seven, and proves that the respective maximal families are nonextendable. And in particular, the size of the largest extension $\mathcal{F}_7$ contains seven disks as depicted in Figure 4.12 below (see also Figure 4.7). $\square$

4.4 Second Helly-Type Theorem on Support Lines

Theorem 4.25. For a nonoverlapping family $\mathcal{G}$ of congruent disks in the plane, $S(3) \implies S$ if the family has eight or more members.

Proof. The threshold number for nonoverlapping critical families is the minimum size of a family that guarantees the implication $S(3) \implies S$ holds. Theorem 4.6 guarantees that 7 is the size of the largest touching family that has property $S(3)$ and not property S. Every touching critical family $\mathcal{F}_7$ is nonextendable and therefore maximal (see Lemma 4.20, and Figure 4.7). The number eight is necessarily a lower bound for the threshold number since a critical touching family of size seven exists as depicted in Figure 4.12.

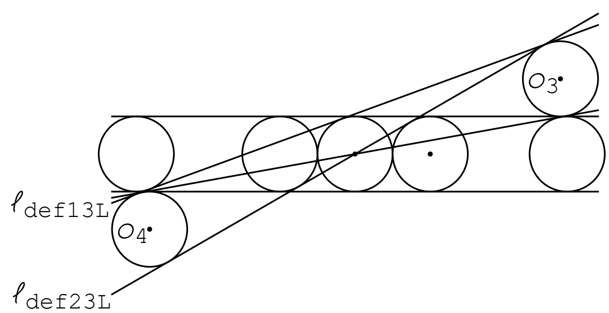


Figure 4.12: A critical touching family $\mathcal{F}_7$ of size 7.

To show that 8 is the threshold number it suffices to show that it also functions as an upper bound. To induce a contradiction, suppose that we can find a nonoverlapping critical family $\mathcal{G}$ that has eight disks. To be explicit, the family $\mathcal{G}$ has the property $S(3)$ and not S, so that at least one line supports each of its critical subfamilies and no line supports all of its members. Since $\mathcal{G}$ is nonoverlapping, it is either disjoint or tangent. If $\mathcal{G}$ is disjoint with more than seven members and the property $S(3)$, then the family has the property S by the results in Soltan [23]. So the family $\mathcal{G}$ is necessarily a touching critical family.

Since G is a touching critical family, Theorem 2.15 of Chapter 2, the first Helly-type result of this paper, implies the family does not have property $S(4)$. So the family $\mathcal{G}$ necessarily contains a nonoverlapping critical subfamily $\mathcal{F} \in \mathfrak{F}$ of size four by Theorem 4.1. This means the subfamily $\mathcal{F} \subset \mathcal{G}$ either corresponds to the disjoint critical $\mathcal{F}_4$ belonging to the touching critical $\mathcal{F}_5$ depicted in Figure 4.3, or it appears among the 17 families depicted in the collection of Figures 3.21, 3.22, and 3.23. Since the family of size 5 depicted in Figure 4.3 is maximal, $\mathcal{G}$ is not an extension of this family. So the family $\mathcal{F} \subset \mathcal{G}$ is touching. The touching critical $\mathcal{F}_4$ are nonextendable except for the families depicted in Figures 3.21d, 3.21e, 3.22a, and 3.22b following Corollary 4.16, so the subfamily $\mathcal{F} \subset \mathcal{G}$ is necessarily one of these four families.

Since the touching critical families $\mathcal{F}_4$ were constructed by exhaustion, no touching critical family outside of the extensions of these families remains as a candidate for $\mathcal{G}$. Theorem 4.6 states that the number of nonoverlapping critical families $\mathcal{F}_k$ with $k \in \mathbb{N}$ is finite, and it follows that the family G necessarily appears among the families documented in Lemmas 4.17 through 4.24 since the maximal extensions of these families are exhaustively documented there. In particular, the largest touching critical families are documented in Lemma 4.20 which describes families $\mathcal{F}_7$ of size 7. Since the subfamily $\mathcal{F} \subset \mathcal{G}$ is in $\mathfrak{F}$, any maximal extension of $\mathcal{F}$ contains at most seven disks, which contradicts the fact that G contains $\mathcal{F}$ as a subfamily.

By exhaustion, no touching critical family of congruent disks has more than 7 members. If a finite touching family $\mathcal{G}$ has more than 7 members, it is not identical to any finite touching critical family, so that $\mathcal{G}$ is not a critical extension of $\mathcal{F} \in \mathfrak{F}$. The supposition that $\mathcal{G}$ is critical leads to a contradiction. If $\mathcal{G}$ has the property $S(3)$ it cannot contain a critical subfamily $\mathcal{F}$ of size four, so that $\mathcal{F} \not\subset \mathcal{G}$. The family $\mathcal{G}$ necessarily has the property $S(4)$ and, by implication, the support property S, so that a line supports each of its members. It follows that no touching family with 8 or more disks has property $S(3)$ and no common support line. We conclude that any nonoverlapping family with at least eight members and the property $S(3)$ is supported by a common line and has the support property S. $\square$

Appendix A: A heuristic counting of 955 minimal support configurations preserving $S(3)$

This appendix provides a heuristic lower bound on the number of minimal support configurations among the lines in $\mathcal{L}_{ij}$ ($i \neq j$ and $1 \leq i < j \leq 4$) for each touching critical family $\mathcal{F}_4$ that preserve $S(3)$ whenever they inscribe a region in the plane that supports a disk C of nonzero radius. By minimal, we mean that only one line in $\mathcal{L}_{ij}$ supports disk C for each $i \neq j$ with the one exception documented in Lemma A.4. To preserve $S(3)$ in an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$, specific subsets of supports among the lines in $\mathcal{L}_{ij}$ of each $\mathcal{F}_4$ necessarily support disk C. For each support configuration it remains to determine whether any disk C in an inscribed region is congruent to those of $\mathcal{F}_4$. The numbers derived here are referenced in the introduction to Section 4.3 which begins on page 125.

For each of the 17 touching critical families $\mathcal{F}_4$, we heuristically count the number of support line configurations when precisely N ($0 \leq N \leq 2$) critical support lines of $\mathcal{F}_4$ support disk C and preserve $S(3)$. Other configurations of lines preserving $S(3)$ are possible and are not counted here. The lower bound provided by this heuristic demonstrates that a direct approach to the problem to determine whether the 17 touching critical families $\mathcal{F}_4$ are extendable requires checking a minimum of 955 support configurations of lines.

The 17 families depicted in Figures 3.21, 3.22 and 3.23 are of four combinatorial types in the distribution of their critical and noncritical support lines. The family depicted in Figure 3.21a is distinguished by the property that it has a touching critical subfamily in a slab (Type 1: see Table A.1). The four families depicted in Figures 3.21b through 3.21e

Table A.1: Distribution of the supports of the one Type 1 family

	$\mathcal{L}_{12}$	$\mathcal{L}_{13}$	$\mathcal{L}_{14}$	$\mathcal{L}_{23}$	$\mathcal{L}_{24}$	$\mathcal{L}_{34}$
critical	3	3	2	3	2	2
noncritical	0	1	2	1	2	2

are distinguished by the property that each family has a disjoint critical subfamily in a slab (Type 2: see Table A.2). The family depicted in Figure 3.23e is distinguished by the property that it contains two pairs of touching disks (Type 3: see Table A.3). The 11 families depicted in Figures 3.22 and 3.23 excluding the family depicted in Figure 3.23e are distinguished by the property that each family avoids a critical subfamily in a slab and each contains a disjoint subfamily of size three (Type 4: see Table A.4).

Table A.2: Distribution of the supports of the Type 2 families

	$\mathcal{L}_{12}$	$\mathcal{L}_{13}$	$\mathcal{L}_{14}$	$\mathcal{L}_{23}$	$\mathcal{L}_{24}$	$\mathcal{L}_{34}$
critical	2	3	3	2	2	3
noncritical	1	1	1	2	2	1

Table A.3: Distribution of the supports of the one Type 3 family

	$\mathcal{L}_{12}$	$\mathcal{L}_{13}$	$\mathcal{L}_{14}$	$\mathcal{L}_{23}$	$\mathcal{L}_{24}$	$\mathcal{L}_{34}$
critical	2	2	2	2	2	2
noncritical	1	2	2	2	2	1

Table A.4: Distribution of the supports of the Type 4 families

	$\mathcal{L}_{12}$	$\mathcal{L}_{13}$	$\mathcal{L}_{14}$	$\mathcal{L}_{23}$	$\mathcal{L}_{24}$	$\mathcal{L}_{34}$
critical	2	2	2	2	2	2
noncritical	1	2	2	2	2	2

Lemma A.1. If $N = 0$ critical lines support disk C_5 then 384 minimal support configurations preserve $S(3)$ in an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ where $\mathcal{F}_4 \in \mathfrak{F}$ is among the 17 touching critical families.

Proof. The Type 1 family depicted in Figure 3.21a has the distribution of critical supports listed in Table A.1. For this family, it is impossible that no critical support ($N = 0$) supports disk C and preserves $S(3)$ in an extension since a line must support $\{C_1, C_2, C_3\}$, and each line in $\mathcal{L}_{12}$ is critical. For the remaining 16 families, if no critical support ($N = 0$) of $\mathcal{F}_4$ supports disk C, then to preserve $S(3)$ a minimum of six lines necessarily support C, one from each set in $\mathcal{L}_{ij}$ ($i \neq j$ and $1 \leq i < j \leq 4$).

Each Type 2 family (Figures 3.21b through 3.21e) has the distribution of critical supports listed in Table A.2. Since we consecutively select one noncritical support from each set $\mathcal{L}_{ij}$, the multiplication principle of counting implies that $1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 = 4$ minimal support configurations preserve $S(3)$. Over the four Type 2 families a total of $4 \cdot 4 = 16$ minimal support configurations preserve $S(3)$.

The Type 3 family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.3. If no critical support ($N = 0$) of $\mathcal{F}_4$ supports disk C, we consecutively select six noncritical supports, one from each $\mathcal{L}_{ij}$, and the multiplication principle confirms that $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 16$ minimal support configurations preserve $S(3)$.

Each of the 11 Type 4 families depicted in Figures 3.22 and 3.23 excluding the family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.4. If no critical support ($N = 0$) of $\mathcal{F}_4$ supports disk C, we consecutively select six noncritical supports, one from each $\mathcal{L}_{ij}$, and the multiplication principle confirms that $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ minimal support configurations preserve $S(3)$. Over the 11 Type 4 families a total of $32 \cdot 11 = 352$ relevant configurations preserve $S(3)$.

The total number of minimal support configurations preserving $S(3)$ where precisely no critical support ($N = 0$) of $\mathcal{F}_4$ supports C is given by the sum $0 + 16 + 16 + 352 = 384$. $\square$

Remark A.2. In Tables A.5 through A.12, the six positions in each product correspond to the supports as listed in Tables A.1 through A.4. The symbol $*$ stands in for a position where a support from the corresponding set $\mathcal{L}_{ij}$ is not needed. To evaluate each product in the tables, either ignore the symbol $*$ and multiply the numbers, or replace each occurrence of $*$ with 1 and multiply.

Lemma A.3. If $N = 1$ critical lines support disk C_5 then 350 minimal support configurations preserve $S(3)$ in an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ where $\mathcal{F}_4 \in \mathfrak{F}$ is among the 17 touching critical families.

Proof. If precisely one ($N = 1$) critical support of $\mathcal{F}_4$ supports disk C, then to preserve $S(3)$ a minimum of three additional lines support C, one support from each of three corresponding sets in $\mathcal{L}_{ij}$ ($i \neq j$ and $1 \leq i < j \leq 4$).

The Type 1 family depicted in Figure 3.21a has the distribution of critical supports listed in Table A.1. For this family, if precisely one ($N = 1$) critical support ℓ supports C, then necessarily $\ell \in \mathcal{L}_{12}$ since a line supports $\{C_1, C_2, C\}$. Of the lines in $\mathcal{L}_{12}$, precisely 2 support $\{C_1, C_2, C_3\}$ and 1 supports $\{C_1, C_2, C_4\}$. Note that when ℓ_1 or ℓ_2 supports disk C, then the disk is not permitted in the slab with C_1, C_2. For the one Type 1 family with

Table A.5: Counting Type 1 configurations with $N = 1$

[123]	$2\,(* \cdot * \cdot 2 \cdot * \cdot 2 \cdot 2) = 16$
[124]	$1\,(* \cdot 1 \cdot * \cdot 1 \cdot * \cdot 2) = 2$

$N = 1$, a total of $16 + 2 = 18$ minimal support configurations preserve $S(3)$ (see Table A.5).

Each of the 4 Type 2 families depicted in Figures 3.21b through 3.21e has the distribution of critical supports listed in Table A.2. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, and $\{C_2, C_3, C_4\}$. Precisely 2 critical supports of $\mathcal{F}_4$ support $\{C_1, C_3, C_4\}$. We consecutively select one critical support from each relevant set $\mathcal{L}_{ij}$. Each

Table A.6: Counting Type 2 configurations with $N = 1$

$$\begin{array}{ll} [123] & 1\,(* \cdot * \cdot 1 \cdot * \cdot 2 \cdot 1) = 2 \\ [124] & 1\,(* \cdot 1 \cdot * \cdot 2 \cdot * \cdot 1) = 2 \\ [134] & 2\,(1 \cdot * \cdot * \cdot 2 \cdot 2 \cdot *) = 8 \\ [234] & 1\,(1 \cdot 1 \cdot 1 \cdot * \cdot * \cdot *) = 1 \end{array}$$

Type 2 family with $N = 1$, has a total of $2 + 2 + 8 + 1 = 13$ relevant configurations (see Table A.6). Over the 4 families, a total of 52 minimal support configurations preserve $S(3)$.

The Type 3 family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.3. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, $\{C_1, C_3, C_4\}$, and $\{C_2, C_3, C_4\}$. We consecutively select one critical support from each relevant family $\mathcal{L}_{ij}$. The one Type 3 family with $N = 1$, has a total of $4 \cdot 4 = 16$ minimal

Table A.7: Counting Type 3 configurations with $N = 1$

$$\begin{array}{ll} [123] & 1\,(* \cdot * \cdot 2 \cdot * \cdot 2 \cdot 1) = 4 \\ [124] & 1\,(* \cdot 2 \cdot * \cdot 2 \cdot * \cdot 1) = 4 \\ [134] & 1\,(1 \cdot * \cdot * \cdot 2 \cdot 2 \cdot *) = 4 \\ [234] & 1\,(1 \cdot 2 \cdot 2 \cdot * \cdot * \cdot *) = 4 \end{array}$$

support configurations that preserve $S(3)$ (see Table A.7).

Each Type 4 family depicted in Figures 3.22 and 3.23 excluding the family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.4. We consecutively select one critical support from each relevant family $\mathcal{L}_{ij}$. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, $\{C_1, C_3, C_4\}$, and $\{C_2, C_3, C_4\}$. Each Type 4 family with $N = 1$, has a total of $2 \cdot 8 + 2 \cdot 4 = 24$ minimal support configurations (see Table A.8). Over the 11 families, a total of $24 \cdot 11 = 264$ minimal support configurations preserve $S(3)$.

Table A.8: Counting Type 4 configurations with $N = 1$

[123]	$1\,(* \cdot * \cdot 2 \cdot * \cdot 2 \cdot 2) = 8$
[124]	$1\,(* \cdot 2 \cdot * \cdot 2 \cdot * \cdot 2) = 8$
[134]	$2\,(1 \cdot * \cdot * \cdot 2 \cdot 2 \cdot *) = 4$
[234]	$1\,(1 \cdot 2 \cdot 2 \cdot * \cdot * \cdot *) = 4$

The total number of minimal support configurations that preserve $S(3)$ where precisely one critical support ($N = 1$) of $\mathcal{F}_4$ supports C is given by the sum $18 + 52 + 16 + 264 = 350$. $\square$

Lemma A.4. If $N = 2$ critical lines support disk C_5 then 221 minimal support configurations preserve $S(3)$ in an extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C\}$ where $\mathcal{F}_4 \in \mathfrak{F}$ is among the 17 touching critical families.

Proof. If precisely two ($N = 2$) critical supports of $\mathcal{F}_4$ support disk C, then at least one noncritical support from among the sets of support lines $\mathcal{L}_{ij}$ ($i \neq j : 1 \leq i < j \leq 4$) must support C. Since our heuristic counts minimal support configurations, we only count those configurations admitting one additional line.

The one Type 1 family depicted in Figure 3.21a has the distribution of critical supports listed in Table A.1. For this family, if precisely two critical supports ($N = 2$) support C, then necessarily one of these lines ℓ belongs to $\mathcal{L}_{12}$ since a line supports $\{C_1, C_2, C\}$ and each line in $\mathcal{L}_{12}$ is critical. Among the lines in $\mathcal{L}_{12}$, precisely 2 support $\{C_1, C_2, C_3\}$ and 1 line supports $\{C_1, C_2, C_4\}$. Following the choice of the first critical support, we then select a second critical support of $\mathcal{F}_4$ not in $\mathcal{L}_{12}$. We select the remaining noncritical support line to support C from the appropriate set $\mathcal{L}_{ij}$. Note that when both critical supports of $\{C_1, C_2, C_3\}$ support disk C then disk C lies in the slab with C_1, C_2 and three additional noncritical supports necessarily support C. Observe that the support configurations ($N = 2$) where one critical support of each of $\{C_1, C_3, C_4\}$ and $\{C_2, C_3, C_4\}$ support C is not viable since a line must support $\{C_1, C_2, C\}$, and this forces three critical supports to support C. The one Type 1

Table A.9: Counting Type 1 configurations with $N = 2$

[123] and [123]	$1 \cdot 1\,(* \cdot * \cdot 2 \cdot * \cdot 2 \cdot 2) = 8$
[123] and [124]	$2 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot * \cdot 2) = 4$
[123] and [134]	$2 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot 2 \cdot *) = 4$
[123] and [234]	$2 \cdot 1\,(* \cdot * \cdot 2 \cdot * \cdot * \cdot *) = 4$
[124] and [134]	$1 \cdot 1\,(* \cdot * \cdot * \cdot 1 \cdot * \cdot *) = 1$
[124] and [234]	$1 \cdot 1\,(* \cdot 1 \cdot * \cdot * \cdot * \cdot *) = 1$

Table A.10: Counting Type 2 configurations with $N = 2$

[123] and [124]	$1 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot * \cdot 1) = 1$
[123] and [134]	$1 \cdot 2\,(* \cdot * \cdot * \cdot * \cdot 2 \cdot *) = 4$
[123] and [234]	$1 \cdot 1\,(* \cdot * \cdot 1 \cdot * \cdot * \cdot *) = 1$
[124] and [134]	$1 \cdot 2\,(* \cdot * \cdot * \cdot 2 \cdot * \cdot *) = 4$
[124] and [234]	$1 \cdot 1\,(* \cdot 1 \cdot * \cdot * \cdot * \cdot *) = 1$
[134] and [134]	$1 \cdot 1\,(1 \cdot * \cdot * \cdot 2 \cdot 2 \cdot *) = 4$
[134] and [234]	$2 \cdot 1\,(1 \cdot * \cdot * \cdot * \cdot * \cdot *) = 2$

family with $N = 1$, has a total of $8 + 3 \cdot 4 + 2 \cdot 1 = 22$ minimal support configurations that preserve $S(3)$ (see Table A.9).

Each of the four Type 2 families depicted in Figures 3.21b through 3.21e has the distribution of critical supports listed in Table A.2. If precisely two critical supports ($N = 2$) support C, then we consecutively select two critical supports from each relevant family $\mathcal{L}_{ij}$, and one additional noncritical line to support C. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, and $\{C_2, C_3, C_4\}$. Precisely 2 critical supports support $\{C_1, C_3, C_4\}$. Each Type 2 family with $N = 2$, has a total of $3 \cdot 4 + 3 \cdot 1 + 2 = 17$ configurations (see Table A.10). Over the 4 families, a total of 68 minimal support configurations preserve $S(3)$.

The Type 3 family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.3. If precisely two critical supports ($N = 2$) support C, then we consecutively select two critical supports from each pair of relevant families $\mathcal{L}_{ij}$. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, $\{C_1, C_3, C_4\}$, and $\{C_2, C_3, C_4\}$.

Table A.11: Counting Type 3 configurations with $N = 2$

[123] and [124]	$1 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot * \cdot 1) = 1$
[123] and [134]	$1 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot 2 \cdot *) = 2$
[123] and [234]	$1 \cdot 1\,(* \cdot * \cdot 2 \cdot * \cdot * \cdot *) = 2$
[124] and [134]	$1 \cdot 1\,(* \cdot * \cdot * \cdot 2 \cdot * \cdot *) = 2$
[124] and [234]	$1 \cdot 1\,(* \cdot 2 \cdot * \cdot * \cdot * \cdot *) = 2$
[134] and [234]	$1 \cdot 1\,(1 \cdot * \cdot * \cdot * \cdot * \cdot *) = 1$

Table A.12: Counting Type 4 configurations with $N = 2$

[123] and [124]	$1 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot * \cdot 2) = 2$
[123] and [134]	$1 \cdot 1\,(* \cdot * \cdot * \cdot * \cdot 2 \cdot *) = 2$
[123] and [234]	$1 \cdot 1\,(* \cdot * \cdot 2 \cdot * \cdot * \cdot *) = 2$
[124] and [134]	$1 \cdot 1\,(* \cdot * \cdot * \cdot 2 \cdot * \cdot *) = 2$
[124] and [234]	$1 \cdot 1\,(* \cdot 2 \cdot * \cdot * \cdot * \cdot *) = 2$
[134] and [234]	$1 \cdot 1\,(1 \cdot * \cdot * \cdot * \cdot * \cdot *) = 1$

The one Type 3 family with $N = 2$, has a total of $4 \cdot 2 + 2 \cdot 1 = 10$ minimal support configurations that preserve $S(3)$ (see Table A.11).

Each of the 11 Type 4 families depicted in Figures 3.22 and 3.23 excluding the family depicted in Figure 3.23e has the distribution of critical supports listed in Table A.4. If precisely two critical supports ($N = 2$) support C, then we consecutively select two critical supports from each pair of relevant families $\mathcal{L}_{ij}$, and one additional noncritical line to support C. Precisely 1 critical support of $\mathcal{F}_4$ supports each of $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$, $\{C_1, C_3, C_4\}$, and $\{C_2, C_3, C_4\}$. Each Type 4 family with $N = 2$, has a total of $5 \cdot 2 + 1 = 11$ configurations (see Table A.12). Over the 11 families, a total of $11 \cdot 11 = 121$ minimal support configurations preserve $S(3)$.

The total number of minimal support configurations that preserve $S(3)$ where two critical supports ($N = 2$) of $\mathcal{F}_4$ support C is given by the sum $22 + 68 + 10 + 121 = 221$. $\square$

The total number of minimal support configurations that preserve $S(3)$ where either none, one, or two $(N = 0, 1, 2)$ critical supports of $\mathcal{F}_4$ support C is given by the sum $384 + 350 + 221 = 955$.

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