SYNTHESIS OF SLOWLY-VARIABLE SYSTEMS

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VITA

John Nelson Warfield was born in Sullivan, Missouri, on November 21, 1925. He entered the University of Missouri in 1942, and after an interruption of two years for service in the United States Army returned to the University of Missouri where he received the A. B. and B. S. in E. E. degrees in 1948 and the M. S. in E. E. degree in 1949. In the years 1949-1951, he served as Instructor of Electrical Engineering at the Pennsylvania State College and as a part-time Research Assistant in the Naval Ordnance Research Laboratory where he performed various theoretical studies of a mathematical and electrical-engineering nature in connection with naval projects.

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SYNTHESIS OF SLOWLY-VARIABLE SYSTEMS

ABSTRACT

A procedure is given for the synthesis of slowly-variable systems by means of active networks. The procedure begins with the choice of a rational function of the complex-frequency s which contains variable parameters. The transmission function which is chosen to meet the system specifications is selected either on the basis of well-known pole-zero configurations or by approximation techniques, often with the aid of the Wiener optimum-mean-square statistical filter theory. Transmission forms which provide the desired transmission are realized by manipulation of signal-flow graphs. One of the transmission forms is selected as the basis for the physical realization of the system. The selection is made on comparison of the relative merits of the various forms with regard to ease of instrumentation, sensitivity to changes in value of the elements of the system, susceptibility to corrupting signals, relative stability and susceptibility to saturation. Dynamic variation in parameters is achieved by means of variable amplifiers, controlled by an external source of voltage.

An illustrative design is presented wherein a multiple-loop feedback system serves as the realization of a filter with an upper cutoff frequency subject to dynamic control over a decade of frequency. The rate of cutoff in the vicinity of cutoff is preset by means of a single potentiometer. Test results compare favorably with theoretical design performance.

SYNTHESIS OF SLOWLY-VARIABLE SYSTEMS

I. INTRODUCTION

A problem of major importance in communication is that of maintaining optimum transmission when the statistical properties of the input message and noise vary. The severity of this filtering problem increases as the power-density spectra of message and noise tend toward equal levels and as the spectra tend toward coincidence along the real-frequency axis. The logical approach to the solution of this filtering problem is to measure suitable statistical parameters associated with the message and noise, and apply the result of the measurement to cause the properties of the transmission system to vary in order to maintain optimum transmission.

A second problem which is of prime importance in the theory and application of transmission systems is the problem of maintaining a prescribed overall transmission when a part of the system has a variable transmission. The severity of the problem of compensating for the variation normally increases as design control over the channel of transmission decreases. In radio communication, for example, the variation in the channel necessitates adaptation of broadcasting conditions and techniques to take account of the variation. In other instances, the problem may be more amenable to a design approach. In the control of the flight of a missile of variable mass, the variable quantity can be measured to obtain a signal which can be used to control a variable-compensation device. It is reasonably evident from the logical aspects of the problem that a general approach to the solution is to measure the variation in the transmission system and apply the result of the measurement to cause a compensating variation in the system.

If it is assumed that suitable measurements of the variation can be made in design situations in which either or both of the two major problems just considered are of interest, the problem of obtaining satisfactory transmission reduces to the problem of synthesis of variable systems which can be controlled by the results of the measurements. The class of variable systems which is considered in this thesis is the class of slowly-variable systems. A slowly-variable system may be described by reference to Figure 1.1 where, for simplicity, a single variable parameter is indicated.

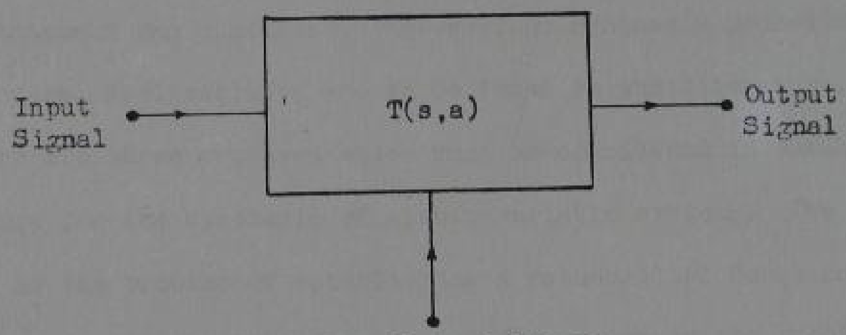

Figure 1.1. A Slowly-Variable System with a Single Variable Parameter

The transmission $T(s,a)$ is a function of the complex-frequency variable $s = \sigma + j\omega$ and of the parameter $\underline{a}$. The value of the parameter $\underline{a}$ is varied by a parameter-control signal injected into the system. The success of the quasi-stationary treatment implied by the use of the complex frequency s is contingent on the rate of variation of the parameter. Physically, the rate should be such that the impulse response of the system is substantially the same whether it is obtained with the parameter fixed or varying throughout the duration of the response. When the rate condition is not met the sense of the analysis is lost and the considerations of this thesis are not generally applicable.

Zadeh[1] states that "the rather limited use of variable networks ... is due largely to lack of practical means for their analysis, synthesis, and mechanization". Zadeh has presented several articles pertinent to the problem of analysis of variable systems since the quoted statement was published. However, no synthesis procedures of any general applicability are to be found in the literature.

There are three problems which must be considered in establishing a procedure for the synthesis of slowly-variable systems. The first of these is the problem of establishing a mathematical function which incorporates the specifications. The second is the problem of realizing a system from the mathematical function. The third problem, closely allied with the second, is that of instrumenting the variable system to achieve the desired variation.

This thesis presents a procedure for the synthesis of slowly-variable systems. The result of the procedure is a system with

1. Zadeh, L. A., "Frequency Analysis of Variable Networks", Proc. I. R. E., 38, 3, March, 1950.

parameters subject to dynamic variation; i.e., a system with variable transmission properties. An important factor in the synthesis procedure is the realization of variable parameters by means of variable (ideal) amplifiers[1]. If variable parameters are to be realized by variable amplifiers, the synthesis procedure must be, basically, a procedure for the synthesis of active systems. If active systems are employed, attention must be given to the practical aspects of the performance of active systems, in order that the systems shall perform in accordance with the linear theory and be not unduly sensitive to changes in element values.

Chapters II, III, and IV are devoted to methods of obtaining a suitable transmission function, realizing the transmission function as an active system, and incorporating practical considerations in the employment of the synthesis procedure. Chapter V consists of a discussion of the adaptation of the synthesis procedures of Chapters II-IV to slowly-variable systems. Chapter VI contains an example of the realization of a variable-bandwidth filter from the synthesis procedure. Test results are included in Chapter VI. The example in Chapter VI illustrates many of the concepts of Chapters II-V. Chapter VII is devoted to a summary of the principal results and conclusions, and suggestions for future investigations in connection with variable systems.

1. An ideal amplifier provides constant (positive or negative) transmission at all frequencies. The term "variable" refers to the amplitude level of the transmission.

II. SPECIFICATIONS FOR SLOWLY-VARIABLE SYSTEMS

If a slowly-variable system is to be more advantageous than a fixed system in a filtering process, the distortion introduced because of the variation must be less than the distortion removed by virtue of the variation. This means that a slowly-variable system must perform essentially as a fixed system over a representative period of time. The representative period of time should equal or exceed the effective duration of the impulse response of the system. In view of this requirement, it is logical to synthesize slowly-variable systems on a quasi-stationary basis. Conversely, the quasi-stationary treatment may be imposed in order that the complex s-plane familiarly associated with fixed systems may be used as a basis for the description and synthesis of slowly-variable systems. Specifications for a slowly-variable system may take the form of a rational function of the complex-frequency s involving parameters, any or all of which may be subject to variation.

One of the most difficult problems associated with a given synthesis problem is the determination of a simple, realizable, rational function of s which incorporates the specifications. Only occasionally is the determination of this function relatively straightforward: e.g., in an application of a slowly-variable system N_1 to cancel the effect of unwanted variation in a slowly-variable system N_2 the required transmission function is established from a knowledge of the transmission function of N_2 and the desired overall transmission. If the slowly-variable system is to separate a message from noise, the determination of a suitable transmission function for the system is a difficult task, involving two major factors. First, the specifications must be determined from the nature of the problem, then a suitable function which incorporates the specifications must be obtained. The function obtained is called the transmission function of the system. In this chapter the determination of a realizable, rational function of s is considered with emphasis on the problem of filtering[1].

2.1. Synthesis Based on a Frequency-Domain Approach

A simple and somewhat trivial solution to the problem of incorporating specifications for a slowly-variable system in a realizable, rational function of s is that obtained by selecting the form of the transmission function directly from a body of well-known pole-zero configurations and appending an appropriate level factor[2]. This solution is called "trivial" because if it is to be employed, it follows that the specifications must be adaptable to fit the well-known characteristics. This solution is certainly not trivial from the standpoint of the relative number of times it is employed (in the synthesis of fixed systems) since specifications for many filtering devices fall into common patterns; e.g., it is not unusual to specify the properties of a servomechanism on the basis of the location of a single pair of conjugate-complex poles[3]. From the standpoint of relative ease

1. Since a servomechanism may be regarded as a filter, servomechanisms are included in the consideration.
2. The term "level factor" is used to describe the amplitude level of transmission, since the term "gain" will later be synonomous with bandwidth, or other parameters.
3. Truxal, J. G., "Servomechanism Synthesis through Pole-Zero Configurations", M.I.T. Tech. Rpt. #162, August 25, 1950.

of application and familiarity with resulting performance, the determination of the rational function on the basis of well-known characteristics is often a desirable one.

As an illustration of how well-known characteristics are employed in connection with slowly-variable systems, consider the realization of a type of low-pass, variable-bandwidth system for which the maximally-flat Butterworth characteristic, defined (except for a level factor) by the location of two pairs of equally-spaced conjugate-complex poles on a circle about the origin at the angular positions indicated in Figure 2.1. The well-known, low-pass transfer function associated with the pole-zero configuration of Figure 2.1 for a passive system is of the form

$$T(s) = \frac{\omega_1^{\,4}}{(s^2 + 2\omega_1 s \cos 22.5^\circ + \omega_1^{\,2})(s^2 + 2\omega_1 s \cos 67.5^\circ + \omega_1^{\,2})} . \tag{2.1}$$

Since the radius of the circle in Figure 2.1 is ω_1, and if the bandwidth of the system represented by (2.1) is defined to be $\omega_1/2\pi$, a slowly-variable system with variable bandwidth may be realized if the poles of the system move along radial lines through the origin in the s-plane in such a way that the poles remain in conjugate-complex pairs, and always lie on a circle about the origin of radius ω_1. This motion is illustrated by the loci shown in Figure 2.2. Stated in another way, if the shape of the amplitude-versus-frequency characteristic is to be maintained, this is assured in the example by the pole motion indicated in Figure 2.2. However, it is not

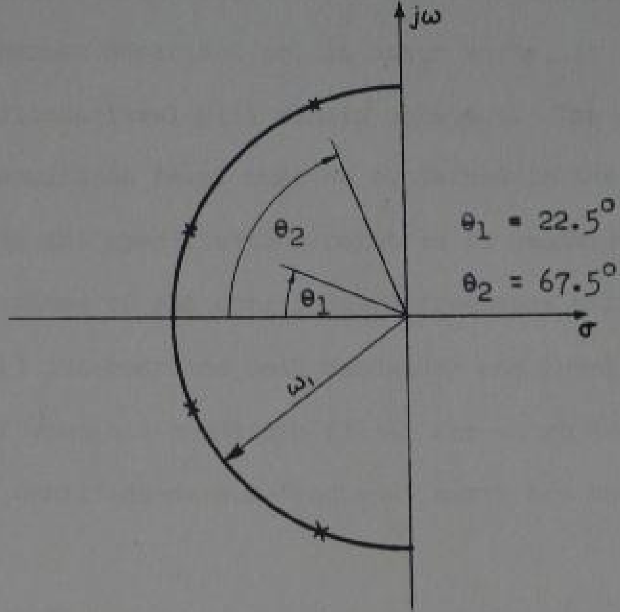

Figure 2.1. A Butterworth Filter Characteristic

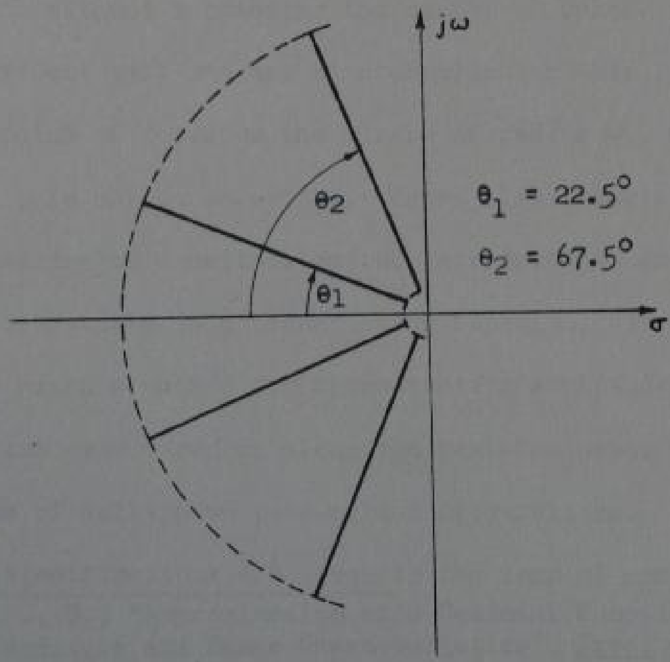

Figure 2.2. Loci of Poles for Variable-Bandwidth Filter

assured from the stated pole-motion that the zero frequency-amplitude of the frequency characteristic will remain constant as the poles move in the manner described or, in other words, it is not assured that the amplitude level will remain constant. The specification in terms of amplitude level must be contained in the transfer function in addition to the specification relative to bandwidth-variation and retention of shape of the amplitude-vs-frequency characteristic. Equation (2.1) incorporates both bandwidth and level specifications for a type of variable-bandwidth filter for which the level and the shape of the amplitude-versus-frequency curve are unchanged by the variation.

In a slightly more general variant of the example just discussed, it might be desired to achieve a higher rate of cutoff in the vicinity of cutoff than that provided by the maximally-flat Butterworth characteristic without increasing the number of poles. It is reasonably evident that one way of accomplishing this is to relocate one or both pairs of poles on the circle of radius ω_1, and then allow radial pole motion as before. Hence slight variations from familiar transmission functions may be introduced to achieve the desired specifications in a transmission function. In this connection, Linvill[1] has given a method for approximating amplitude and phase characteristics over a region along the real-frequency axis by means of variations of well-known pole-zero configurations.

If the specifications are given in the form of normalized

1. Linvill, J. G., "Approximation with Rational Functions of Prescribed Magnitude and Phase Characteristics", Proc. I. R. E. 40,6, June, 1952, pp. 711-721.

characteristics not readily adapted to familiar pole-zero configurations, any or all of the approximation techniques of the synthesis of fixed systems may be applied to determine a suitable pole-zero configuration.

2.2. Synthesis Based on the Wiener Mean-Square-Error Criterion

The choice of a realizable, rational function from specifications on a system is complicated by the fact that the number and types of specifications which may be imposed are countless, albeit in practice only a few are usually imposed. A recent advance in the method of incorporating the specifications in a mathematical function involves the assignment of essentially two specifications in terms of the response in the time domain. The first specification is that the output of the system lead or lag the desired portion of the input by a specified amount of time. The second specification is that the system reproduce the desired portion of the input (i.e., the message) by a linear operation which minimizes the mean-square-error between the message and the output. The separation of message from noise is achieved on a statistical basis. The approach just mentioned is that suggested by Wiener[1] and discussed by Levinson[2], Bode and Shannon[3], Lee[4], Stutt[5], and others.

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1. Wiener, N., The Extrapolation, Interpolation and Smoothing of Stationary Time Series, Technology Press and Wiley, 1949.
2. Levinson, N., "The Wiener RMS Error Criterion in Filter Design and Prediction", Jour. Math. and Phys., 25, 4, Jan., 1947, pp. 261-278
3. Bode, H. W., and Shannon, C. E., "A Simplified Derivation of Linear Least Square Smoothing and Prediction Theory", Proc. I. R. E., 38, 4, April, 1950, pp. 417-425.
4. Lee, Y. W., "Application of Statistical Methods to Communication Problems", MIT Technical Report #181, Sept. 1, 1950
5. Stutt, C. A., "Experimental Study of Optimum Filters", MIT Technical Report #183, May 15, 1951.

If the required statistical information is available (and to obtain such information, approximation from experimental data is often required) Wiener's theory may be applied to obtain a linear operator which meets the specifications. While the linear operator may not be realizable by means of a linear, lumped-parameter system, a suitable approximation may often be found with relative ease if the system is fixed. If, however, the system is to be a slowly-variable system, the approximation to the linear operator is complicated. In a given case the optimum-mean-square (o-m-s) linear operator may reveal that an extremely peculiar variation of system parameters is required in order to maintain optimum performance as the statistical parameters of the input vary slowly. If several parameters of the system are required to vary, the instrumentation of the system may appear impractical, since the relations between the variable parameters of the system and the signals which control the parameter variation may be very complicated. In this connection the direct application of the Wiener theory may be undesirable and is certainly unsatisfying, since there is no way of knowing from the analysis whether or not a much simpler system might perform as well in practice as the complicated one, for in a minimization process there is always the possibility that the minimum is not sharply defined. It follows that the optimum system may be much more complicated than a simple system which could give essentially the same performance. In this connection, Stutt[1] has shown experimentally that in certain special cases "the minimum mean-square-error adjustment is quite non-critical".

1. Stutt, C. A., Op. cit., pp. 63-112.

Perhaps the greatest advantage to be attained by employing the o-m-s theory in its present form is that a measure of system performance is established. If the optimum linear operator is determined, the performance of any linear system may be compared with that of the optimum linear operator on a mean-square-error basis. Another advantage of the o-m-s theory is that it shows what parameters of the input are significant in determining the optimum linear operator. In other words, the o-m-s theory reveals what properties of the input should be utilized to control the dynamic variation of the parameters of the system in order to maintain optimum performance as the statistical parameters of the input change slowly. The o-m-s theory serves, therefore, as a guide to the type of measurements which are necessary in order to permit the properties of the input to control the variable parameters of the slowly-variable system. A third advantage of the o-m-s theory is that it may provide an indication of the practical bounds of a solution or may show that no satisfactory solution may be found without resort to non-linear operations on the input. If there is no satisfactory solution to the approximation problem on a linear basis, it is certainly desirable to recognize the fact rather than to struggle endlessly in an effort to find that which is non-existent.

In order to illustrate in greater detail some of the problems which arise in the utilization of the Wiener o-m-s theory, an application of the theory to an idealized example is presented in Appendix I. The example illustrates many of the remarks of this section. It is hoped that the example may suggest avenues for further investigation.

2.3. Summary and Conclusions

The specifications of a slowly-variable system may be incorporated into a description of the variation of pole-zero positions in the s-plane and the assignment of a variable level factor. Often well-known pole-zero configurations may be selected as a basis for the incorporation of the specifications. Approximation techniques usually associated with the synthesis of fixed systems may be carried over to the approximation of prescribed amplitude and phase characteristics for variable systems.

The Wiener optimum-mean-square theory provides a means for analytical determination of how the variable parameters of the system should be controlled. In addition, the Wiener theory enables comparisons of linear systems with the optimum linear system.

In general, no simple approach to the problem of determination of a suitable rational function is entirely satisfactory. Application of appropriate features of all available tecaniques may often be required. In instances where there is doubt concerning the existence or extent of a practical solution, the Wiener o-m-s theory provides a basis for a scientific consideration of the issue, and enables performance tolerances to be established.

III. SYNTHESIS OF ACTIVE LINEAR SYSTEMS

The need for a method of realizing an active system from a prescribed transmission function with a simple method of portraying the intermediate steps and the final result is basic in the synthesis of slowly-variable systems. Since it is desired to realize slowly-variable systems by means of active systems (because of the relative ease of achieving and controlling the variation), it is necessary to devote attention to an integrated description of the synthesis of active linear systems. In the description it is desirable to dignify certain simple operations by assigning nomenclature, in order that proper emphasis be given to the operations and discussion be made possible.

The choice of active linear systems as a basis for the realization of slowly-variable systems is a natural one to make, since it enables the variable parameters to be instrumented by amplifiers with variable gain. The use of amplifiers with variable gain as instrumentation for systems with slowly-variable parameters is not new[1,2], but the viewpoint taken in this chapter represents a more general approach to the problem. In the literature, variable amplifiers have been employed to realize variable level of transmission (in connection with avc systems) and in reactance-tube circuits to obtain variable impedances.

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1. Scott, H. H., "Dynamic Noise Suppressor", Electronics, December, 1947, p. 96.
2. Yu, Y. P., "Dynamic Impedance Circuit", Tele-Tech, Vol. 7, August, 1948.

It is common practice in the literature of network synthesis to identify elements of the system according to their physical nature; e.g., inductances, capacitances and resistors. In a more general interpretation especially applicable to active-network synthesis, the elements of the transmission system may be regarded as integrators, differentiators, and amplifiers. It should be noted that under this interpretation, constraints are not imposed on the physical identity of the elements at the outset of the synthesis procedure. The advantage of this viewpoint lies in the freedom available in the final physical realization. The outcome of the adoption of this broader view of what constitutes an element is a synthesis procedure which is not constrained to any particular physical form (e.g., a reactance-tube circuit) and is generally applicable to synthesis from a prescribed stable, rational function of the complex-frequency variable, as opposed to synthesis performed on the basis of knowledge of the performance of specific configurations of components.

There are certain well-known disadvantages in the use of active systems. These disadvantages are ignored in this chapter, for expedience and clarity of presentation. Chapter IV is devoted to means for taking certain of these disadvantages into account in the synthesis procedure and in the choice of a system in order that the effects of the disadvantages may be curtailed.

3.1. Realizability Conditions

If a system is to be realizable as a linear, active system consisting of ideal elements herein referred to as integrators, differentiators, and amplifiers, the mathematical function representing the transmission from input to output must satisfy certain requirements called realizability conditions. These conditions are that the transmission function T(s) be a rational function of the complex-frequency variable s and that the transmission function represent a stable system for the envisioned type of transmission; i.e., if the function is to be realized as a transfer function, the poles of the function must lie in the left-half s-plane, while the zeros lie anywhere, but if the function is to be realized as a driving-point function, the poles and zeros all must lie in the left-half s-plane, for stability.

3.2. Signal-Flow Graphs

Mason[1] has described the representation of a set of linear, homogeneous, algebraic equations by means of a signal-flow graph. Since a linear, active system with lumped elements can be described by means of a set of such equations, it can be described alternately (as Mason points out) by means of a signal-flow graph. The use of a signal-flow graph to describe an active system is far superior to the use of a set of equations, since possible instrumentation of the system is much easier to visualize. A signal-flow graph is also superior to a block diagram, since the graph is easier to

1. Mason, S. J., "Signal-Flow Graphs", Paper #38, IRE National Convention, March, 1951.

manipulate and presents a more complete description of the system. The symbols of a signal-flow graph are of two types. Circular areas on the graph are called nodes and may represent voltages or currents (in electrical networks), while directed lines indicate directed transmittances in the network, or relations between the node quantities in the set of defining equations. In establishing rules for manipulation of the graphs it is convenient to note that the value of a node variable is the sum of the signals which enter it and this value is transmitted along each outgoing branch, while the directed lines are branches traversed by signals from one node to another. As an example, consider the signal-flow graph of Figure 3.1.

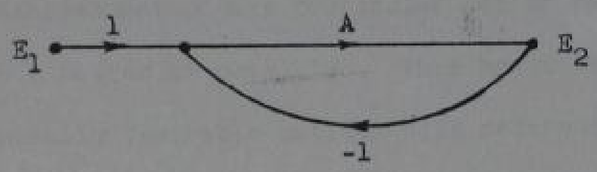


Figure 3.1. A Signal-Flow Graph of Transmission from E_1 to E_2

This signal-flow graph might represent a feedback amplifier with forward gain A, driven from a voltage source E_1, with the output voltage E_2 returned in its entirety to the input where a subtraction takes place. The overall transmission of the system of Figure 3.1 is

$$\frac{E_2}{E_1} = \frac{A}{1 + A} .$$

Additional examples illustrating the use of signal-flow graphs will appear in following sections, where the role of the graphs in the synthesis of active systems will be illustrated.

3.3. The Realization of Transmission Forms

By the realization of a transmission form is meant the determination of a signal-flow graph which provides the desired overall transmission from input to output. A given transmission function can be realized by an infinite variety of transmission forms, depending on how the realization is performed. It is therefore well to keep in mind Bode's general comment on methods for the synthesis of active systems. Bode[1] states that "they are best when they leave the final synthesis in the hands of the designer but stress the development of conceptions and processes which make the establishment of any particular set of relationships as simple and easy a matter as possible". This basic philosophy represents an especially desirable outlook with reference to the methods to be presented, since the synthesis procedure serves to provide a mere shell of a system. The spark of life must be injected by the individual designer. Stated another way, a topological form is provided without an associated instrumentation other than the amplifiers required to realize the variable parameters of the system.

Methods of realizing a transmission form are now considered. Let T(s) be the desired function interposed between an input E_1 and output E_2, where the E's may pertain to typical input or output quantities, such as voltage, current, angle or velocity.

1. Bode, H. W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand Co., Inc., New York, N. Y., 1945, pp. 103-104.

The transmission is represented by the basic signal-flow graph of Figure 3.2.

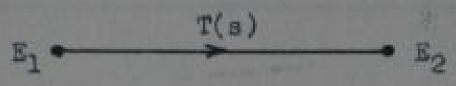


Figure 3.2. The Basic Signal-Flow Graph Representing Transmission From Input to Output

The problem at hand is to obtain transmission forms for the transmission function T(s) by manipulations beginning with the basic graph of Figure 3.2. The following techniques represent a variety of methods which are useful in establishing transmission forms:

1) tandem separation
2) parallel separation
3) simple-loop formation
4) double-loop formation
5) intra-loop shifting .

These methods are explained individually and illustrated symbolically in the following examples.

Example 3.1. Tandem Separation

Tandem separation consists of writing a transmittance T(s) in the form $T_1(s)T_2(s)$ and identifying T_1 and T_2 as separate transmittances. Both T_1 and T_2 must be stable, for physical realizability. Note that this process is reversible, and a tandem recombination can be made when desired. Figure 3.3 shows the representation of tandem separation by a signal-flow graph.

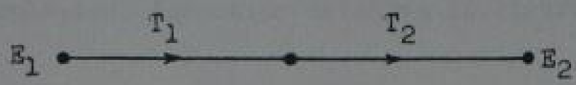


Figure 3.3. The Result of Tandem Separation

Example 3.2. <u>Parallel Separation</u>

Parallel separation consists of writing a transmittance T(s) as $T_1(s) + T_2(s)$ and identifying T_1 and T_2 as separate stable transmittances. The signal-flow graph representing parallel separation is shown in Figure 3.4.

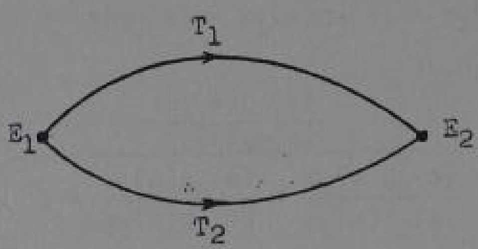


Figure 3.4. The Result of Parallel Separation

Example 3.3. <u>Simple-Loop Formation</u>

Simple-loop formation is performed by writing

$$T(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{q_1(s) + q_2(s)} = \frac{p(s)/q_1(s)}{1 + q_2(s)/q_1(s)} .$$

The p's and q's are polynomials in the complex-frequency s.

Let $\qquad T_L(s) = q_2(s)/q_1(s)$ = the loop transmittance

and $\qquad T_o(s) = p(s)/q_2(s)$ = the external transmittance.

If T_L and T_o are to be realizable, the zeros of q_1 and q_2 should lie in the left half-plane. The resulting signal-flow graph illustrating simple-loop formation is shown in Figure 3.5.

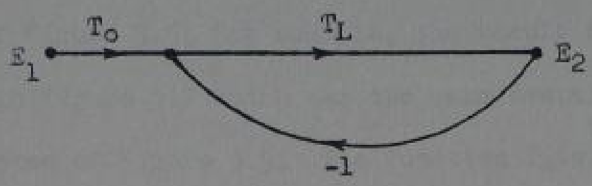


Figure 3.5. The Result of Simple-Loop Formation

Example 3.4. Double-Loop Formation

Double-loop formation is performed by writing

$$T(s) = \frac{p(s)}{q(s)} = \frac{p_1(s)p_2(s)}{q_1(s)q_2(s) + p_1(s)q_3(s) + p_2(s)q_4(s)}$$

$$= \frac{\dfrac{p_1(s)p_2(s)}{q_1(s)q_2(s)}}{1 + \dfrac{p_1(s)q_3(s)}{q_1(s)q_2(s)} + \dfrac{p_2(s)q_4(s)}{q_1(s)q_2(s)}} .$$

Let $T_{F1}(s) = p_1(s)/q_1(s)$, $T_{F2}(s) = p_2(s)/q_2(s)$, $T_{B1}(s) = q_3(s)/q_2(s)$, and $T_{B2}(s) = q_4(s)/q_1(s)$. For stability in either loop, the zeros of the q-polynomials should lie in the left half-plane. The signal-flow graph illustrating a double-loop formation is shown in Figure 3.6.

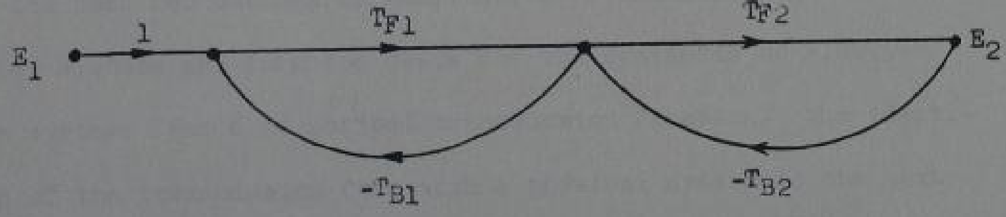


Figure 3.6. The Result of Double-Loop Formation

Example 3.5. Intra-loop Shifting

Intra-loop shifting is essentially a transformation which leaves both forward and loop gains invariant, but shifts the position of elements in the signal-flow graph. If this process is applied to the simple loop of Figure 3.5, for example, the result is a signal-flow graph shown in Figure 3.7 which has the same overall transmission as the system of Figure 3.5. The function $T_A(s)$ may be any stable transmission function.

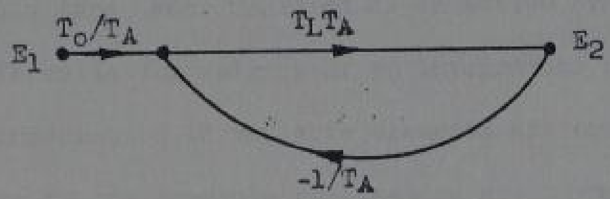


Figure 3.7. Result of Intra-loop Shifting Applied to the Transmission Form of Figure 3.5.

It is not meant to imply that there is any set pattern for applying the techniques just discussed. It is of interest to note that the methods given constitute a sufficient set of techniques for the realization of any stable transmission function in a form reduced to integrators, differentiators and amplifiers. These techniques fulfill the need for methods of realization of transmission forms for active systems and form the basis for the synthesis of slowly-variable systems from a prescribed transmission function. The identification of the transmission form with a physical system is the next item in the synthesis procedure to be considered, but before proceeding to this problem, an example is presented illustrating the use of some of the techniques just discussed.

3.4. An Illustrative Example of the Realization of a Transmission Form

A network function commonly used in the stabilization of velocity servomechanisms has the transmission

$$T(s) = \frac{s + a}{s + b} \quad .$$

A function of this type, when realized as an active system, can serve either as a differentiating network or an integrating network, or as a variable compensation if variable elements are employed. The steps leading to one possible realization are given in Figure 3.8. Step 1 consists of establishing the basic form. Step 2 illustrates the result of a parallel separation, and step 3 indicates the result of loop formation applied in each branch of the graph obtained in Step 2.

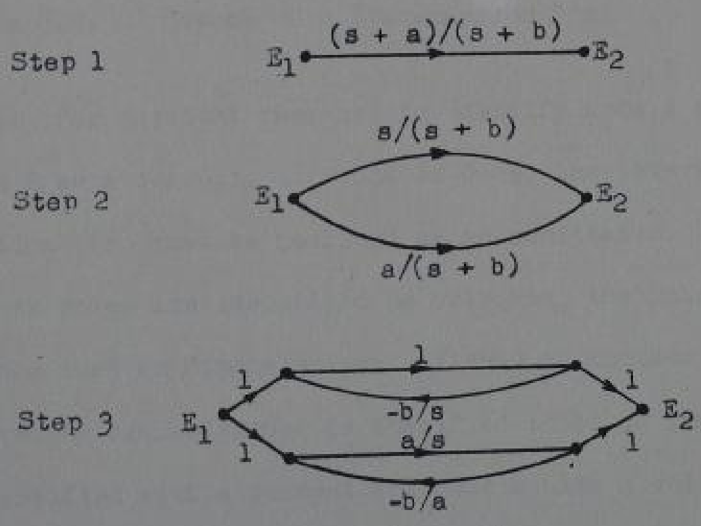


Figure 3.8. Steps in the Formation of a Transmission Form for $T(s) = (s + a)/(s + b)$

3.5. Node Identification

After a transmission form is obtained, the problem remains to identify the form with a physical system. The identification consists of assigning physical quantities to the nodes of the transmission form. No rigid rules can be given relative to what particular node identifications should be made, therefore the discussion emphasizes the significance of the node-identification process.

Node identification is the process of assigning physical variables to the nodes of the transmission form, thereby simultaneously assigning a physical dimension to the transmissions interposed between nodes. (A perfectly satisfactory, alternate viewpoint might be taken wherein the dimensions of the transmissions are assigned, thereby fixing the nature of the node variable.) A typical branch of a transmission form is indicated in Figure 3.9.

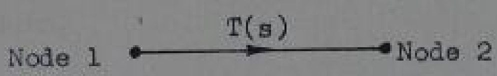


Figure 3.9. A Branch of a Transmission Form

It may be desirable, for physical reasons, to identify node 1 as a voltage and node 2 as a current. If this is done, the intervening transmission function T(s) must be realized as an admittance. Alternately, if both nodes are identified as voltages, the intervening transmittance must be dimensionless. Viewed in another way, if the intervening transmittance is specified to be an impedance node 1 must be identified with a current and node 2 with a voltage.

The most useful set of node identifications will usually be determined by the requirements of the application, together with the knowledge of available components and constraints imposed by the nature of the components. It is therefore desirable, when making node identifications, to visualize the physical system as the node identifications are made, to insure that the process leads to a reasonable system. Once the node identifications are complete, a block diagram of the system may be drawn if desired. The problem of realizing the individual elements of the system exactly or approximately is then in the hands of the designer. (In the synthesis of a system given in Chapter VI, the choice of node identifications and a representative solution to the problem of obtaining approximations to the desired elements of the system are indicated.) Simple examples which illustrate node-identification are given in the following section.

3.6. Illustrative Examples

The node-identification procedure may be illustrated by a consideration of the realization of the transmission function

$$T(s) = \frac{-1}{s + 0.1} .$$

Since a transmission gain of more than unity is required at low values of s, an active system is required. One method of realizing a transmission form for this function is to separate the function into a tandem arrangement, as indicated in the transmission form of Figure 3.10.

Figure 3.10. A Tandem Transmission Form for $T(s) = -1/(s + 0.1)$

In this simple example the physical realization may be achieved by identifying the tandem combination as an amplifier followed by a series resistance-capacitance combination, as indicated in Figure 3.11.

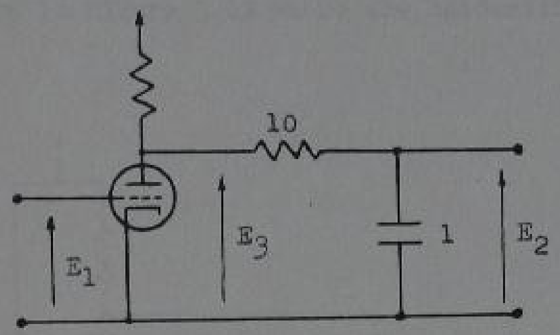


Figure 3.11. A Physical Realization Associated with the Transmission Form of Figure 3.10

The unidentified node in Figure 3.10 is identified as a voltage E_3 in obtaining the realization shown in Figure 3.11.

A second method of realizing the desired transmission consists of forming a simple loop according to the method given in Section 3.3. One possible loop is shown in Figure 3.12.

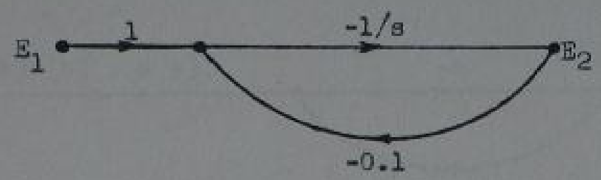


Figure 3.12. A Transmission Form for $T(s) = -1/(s + 0.1)$

A possible approximate physical realization of the form given in Figure 3.12 is shown in Figure 3.13 where the unidentified node in

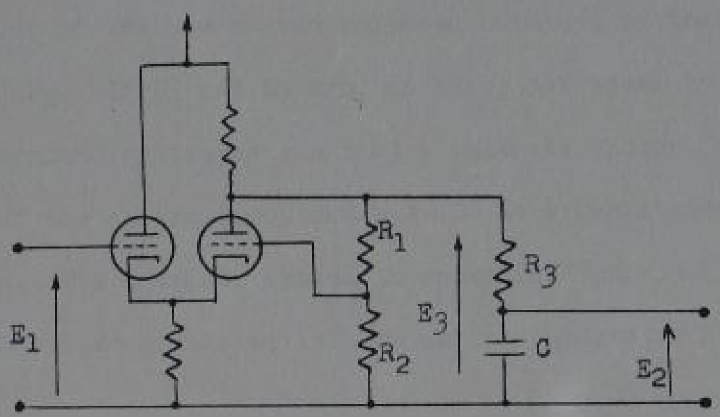


Figure 3.13. A Physical Realization Associated with the Transmission Form of Figure 3.12

Figure 3.12 is designated to be a voltage. The combination of R_3 and C must serve to approximate an ideal integrator, while the voltage divider consisting of R_1 and R_2 must be a high-resistance device.

A third possibility for the realization is indicated by the transmission form of Figure 3.14 where the values a and b satisfy

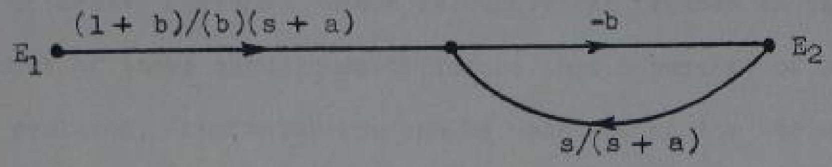
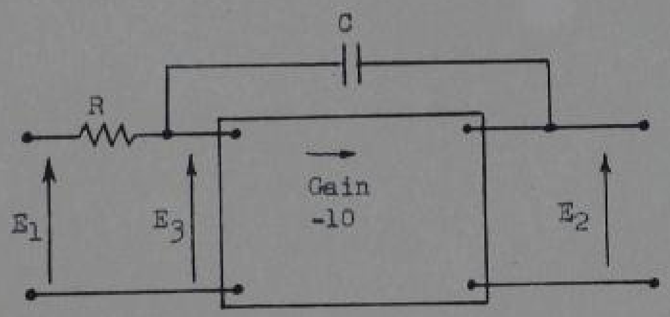


Figure 3.14. A Transmission Form for $T(s) = -1/(s + 0.1)$

the relation $a/(1 + b) = 0.1$. To visualize the instrumentation of the system, let the unidentified node of Figure 3.14 be a voltage node designated as E_3. Let $-\underline{b}$ be an amplifier gain. Then note that E_3 is made up of the sum of two voltages obtained by transmission from E_1 and E_2. If E_1 and E_2 are, in turn, set equal to zero it may be determined by inspection that a suitable choice for the remainder of the system (provided b = 10) is a resistance-capacitance combination. The complete system is shown in Figure 3.15. This system is one form of the well-known Miller integrator circuit.



(1/RC = 1.1)

Figure 3.15. A Physical Realization Associated with the Transmission Form of Figure 3.14

3.7. Summary

In this chapter techniques have been presented for the realization of active systems. There is sufficient freedom in the employment of these techniques to insure that a variety of systems can be realized, from which one can be selected on the basis of certain practical factors. The medium chosen for a graphical portrayal of the transmission form serves to provide the designer with a picture which aids materially in the instrumentation of the associated system. The choice of active systems as a means of realizing transmission functions leaves the way clear for the introduction of variable amplifiers as variable elements in the synthesis of slowly-variable systems.

IV. THE CHOICE OF A SYSTEM

It was stated in Chapter III that the synthesis procedures there developed can provide a number of transmission forms to realize a given transmission function. The essential requirements for identifying transmission forms were outlined. It is now desirable to consider how a logical choice of a form can be made from the available forms, or more specifically, how certain practical factors associated with the use of active systems can be dealt with effectively in the realization or selection of a transmission form. Examples serve to illustrate some of the factors involved, while a discussion of the use of sensitivity to variation in parameters as a guide in the synthesis provides additional helpful information.

4.1. Relative Stability and Saturation

The problems of maintaining suitable relative stability and avoiding saturation are basically peculiar to a given situation, yet certain features of these problems stand out as worthy of special consideration. The problem of maintaining suitable ~~relative~~ stability is fundamentally a problem of preventing the passage of poles in a transfer function from the left half of the s-plane to the right half-plane. The problem of avoiding saturation is one of preventing the signal level from rising beyond the extent of linearity of the system. If the system is realized in such a manner that the individual units in the system can be identified with individual poles, the problems of relative stability and saturation are rather closely related, for they are both outgrowths of pole-locations in the vicinity of the $j\omega$-axis. There are extenuating circumstances to be considered, to be sure, since if the pole-location is relatively insensitive to changes in elements of the system the problem of relative stability is not serious; while if the input signal level to the unit is extremely small, even the occurrence of very high gain in some portion of the spectrum may not result in saturation. As an illustration of how the factors of relative stability and saturation may enter the synthesis procedure, consider the realization of the transmission function

$$T(s) = \frac{1}{s^2 + 2s + 1} \ .$$

Alternate methods of realizing this transmission function are shown in Figure 4.1. Slight variations in the forward transmission of the system of Figure 4.1.a (of the type which may be

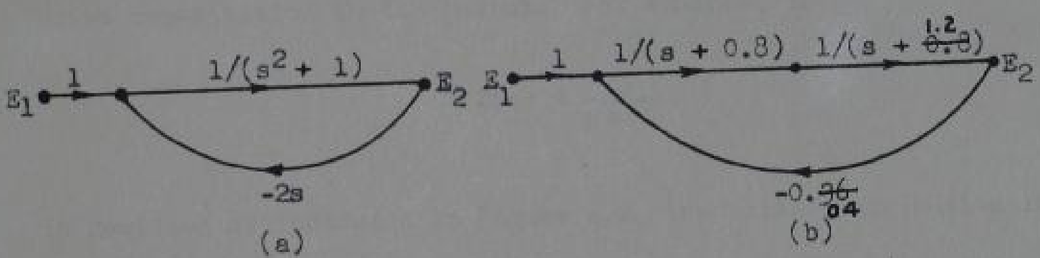


Figure 4.1. Methods of Realizing $T(s) = \dfrac{1}{s^2 + 2s + 1}$

expected to occur in active systems) will result in unstable operation, and even if the system is maintained in a stable condition saturation may well result from the effect of any stray signal introduced in the forward branch. In the system of Figure 4.1.b which gives the same overall transmission as that of Figure 4.1.a, slight variation in the pole-locations along the negative-real axis will not lead to unsuitable relative stability, and the possibility of saturation is much less because of the relatively smooth amplitude-vs-frequency characteristics of the pole-units in the forward transmission.

4.2. Susceptibility to Noise

System susceptibility to noise has been discussed in the literature by many writers, but it seems appropriate to include a brief statement here for completeness. It should be recognized that if noise is introduced at some point in a system, it may be possible by a simple rearrangement of the elements of the system to maintain the overall transmission constant while markedly diminishing the noise contribution in the output. For example, if

$$T(s) = \frac{2}{s+1}$$

is realized as indicated in Figure 4.2, the noise contribution in the output will be much less if the system of Figure 4.2.a is chosen rather than that of Figure 4.2.b, because of the different transmission functions governing transmission from the noise source to the output in the two systems. The system of Figure 4.2.a appears to the noise source as a low-pass filter, while the system of Figure 4.2.b appears to the noise source as a high-pass filter.

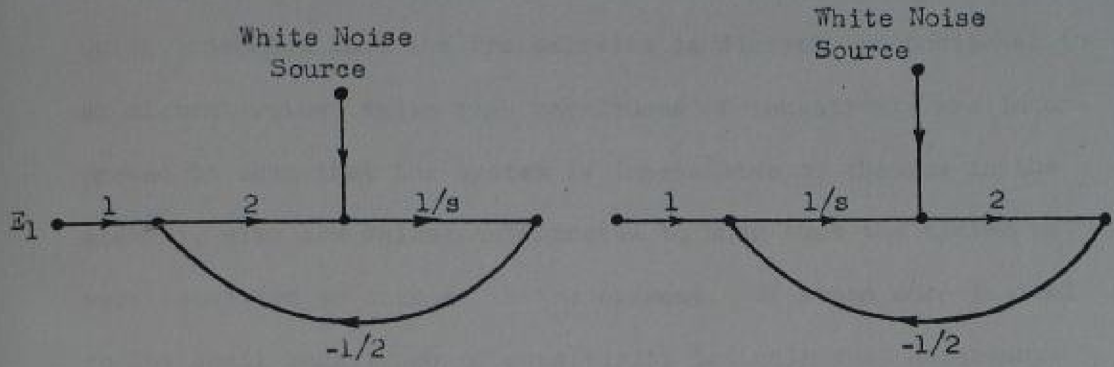


Figure 4.2. Loops Subjected to a Noise Input

The intra-loop shifting techniques described in Section 3.3 may be used to advantage to change the transmission from a noise source to the output, while retaining the desired transmission from input to output.

4.3. Sensitivity to Changes in Element Values

It is well-known that one of the most pressing problems in many applications of electronics is the problem of reliability of vacuum tubes. A great deal of attention is currently being directed to this problem. One aspect of the reliability problem is the difficulty of maintaining low tolerances in the manufacture of tubes and throughout tube life. Since close tolerances cannot be maintained at present, it is essential to have some indication of what effect changes in the properties of vacuum tubes will have on the transmission of any active system containing tubes as active elements. The quantity given by Bode[1] as a measure of the relative influence

1. Bode, H. W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand Company, Inc., New York, N. Y., 1945, pp. 52-53.

of an element of a system on the overall transmission is called sensitivity. Under Bode's definition of sensitivity, a value of unity indicates that the transmission is directly proportional to an element value, while high magnitudes of sensitivity are interpreted to mean that the system is insensitive to changes in the element, with low values interpreted to mean that the system is very sensitive to changes in the element. It seems more logical to let small magnitudes of sensitivity indicate weak dependence and large magnitudes indicate strong dependence, with a value of unity retaining the same meaning as ascribed in Bode's treatment. The reciprocal of Bode's definition will therefore be used in the discussion and referred to as sensitivity.

Bode[1] states that "design methods to give direct control of sensitivity, in cases where it departs materially from the return difference, have not yet been developed". Since it is desirable in the synthesis of slowly-variable systems to have some control over sensitivity, it is of interest to consider how control over sensitivity can be achieved. If it is assumed that a transmission function is given, then a certain set of sensitivities is automatically prescribed. To make this statement clear, consider the transmission function $T(s) = 1/(s + a)$. The sensitivity of the function to changes in a is determined by the form of the function and is given by

$$S_a(T) = \frac{\partial T}{\partial a}\frac{a}{T} = \frac{-a}{s+a}.$$

Note that the sensitivity just obtained is a function of the

1. Bode, H. W., Op. cit., p. 120.

complex-frequency s, as well as the parameter <u>a</u>. Of major interest is the sensitivity of the transmission function to changes in the elements employed to instrument the associated system. The real issue is, therefore, a matter of how the elements of the system combine to form <u>a</u>. In other words, control of the functional relationship between the elements of the system and the quantity <u>a</u> is the basic approach to the control of system sensitivity. The sensitivity of <u>a</u> with respect to the elements which compose it is the determining factor in the attainment of a satisfactory sensitivity of the transmission T to changes in the elements of the system.

In more general terms, it may be said that a transmission function chosen to represent an active system will contain certain parameters in addition to the frequency variable. These parameters may be taken as the poles, zeros and level factor of the function, or the coefficients of the numerator and denominator polynomials, according to individual preference. The sensitivity of the transmission with respect to one of the parameters α, with the other parameters held constant, is given by

$$S_{\alpha}(T) = \frac{\partial T}{\partial \alpha}\frac{\alpha}{T}\,. \tag{4.1}$$

The sensitivity defined in (4.1) is only a part of the consideration of overall sensitivity, yet it is the part over which (as has been shown in the case of a simple pole) there is no design control. It is convenient, therefore, to refer to this sensitivity as the <u>principal partial sensitivity</u> of the transmission T with respect to the parameter α. The principal partial sensitivity will normally be a function of the complex-frequency variable s. However, since the principal partial sensitivities are properties of the original transmission function, it is unnecessary to make a separate computation of these quantities for each transmission form which might be associated with the transmission function.

Once the principal partial sensitivities are computed for a transmission function, comparisons of systems which give the same transmission can be made without repeated extensive computations involving the original transmission function. Interest then centers on the relationships between the parameters of the transmission function and the elements of the system. If β is such an element, the sensitivity of a parameter of the transmission function α with respect to the element β is called a <u>subsidiary partial sensitivity</u> of the transmission function and is

$$S_\beta(\alpha) = \frac{\partial\alpha}{\partial\beta}\frac{\beta}{\alpha} \quad . \tag{4.2}$$

Equation (4.2) expresses the relationship between a change in an element of the system and the resulting change in a parameter which depends on that element. Since both α and β are frequency-independent, the subsidiary partial sensitivities of a system are also, as may be seen from (4.2). These sensitivities are consequently much easier to deal with than the frequency-dependent principal partial sensitivities or the overall system sensitivity.

Nevertheless, the final goal is to obtain a satisfactory overall sensitivity to changes in element values. It is easily shown that the overall sensitivity of the transmission with respect to a change in an element β of a system with parameters α_i is given by

$$S_{\beta}(T) = \sum_i S_{\alpha_i}(T)\, S_{\beta}(\alpha_i)\ . \qquad (4.3)$$

The value of sensitivity computed from (4.3) is found directly from the values of principal and subsidiary partial sensitivities. In applying a numerical result obtained from (4.3) to the understanding of system performance, it should be recognized that the assumption that the elements of the system are independent may often be violated, and in such cases a strict interpretation of performance based on (4.3) is not possible unless the equation can be modified to take into account the interdependence of elements of the system.

How may the discussion be directed toward the aim of control over sensitivity in the design? It has been shown in Chapter III that for a given transmission function there are many different physical realizations. Among this set of physical realizations sensitivities vary widely. The basic problem is then to select from this group a system with a suitable set of sensitivity values. This selection is facilitated by plotting the principal partial sensitivities at the outset of the synthesis, thereafter employing the particular set of subsidiary partial sensitivities associated with a given transmission form to compare forms on the basis of overall sensitivity.[1]

A few additional comments about sensitivity as applied to the consideration of slowly-variable systems may be of interest.

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1. Alternatively, a functional relation between the parameters and the elements may be prescribed at the outset of the synthesis. The full potentialities of this truly "direct control" have not been investigated in detail.

Feedback has traditionally been applied to obtain low values of sensitivity to changes in values of active elements. In the instrumentation of slowly-variable systems, amplifier gains may be used to realize variable parameters. This means that the transmission functions must be fairly sensitive to changes in gains. Admittedly, this is one unfortunate feature of the synthesis, since an active system may not be able to distinguish between external control of the amplification and internal effects which could cause the amplification to change. It follows that high accuracy cannot be expected from slowly-variable systems unless every precaution is taken to insure that the gains of the tubes do not vary with plate, screen and bias voltages (excepting those voltages employed for controlling the variation in gain). If there are amplifiers with fixed gains serving as elements of the system, such amplifiers should be capable of maintaining a constant gain, within reasonable limits. Performance should, of course, be considered in the light of long-term or short-term requirements.

It has been stated that the values of sensitivity to changes in elements depend strongly on the functional relation between the elements of the system and the parameters of the transmission function. This fact is illustrated in Chapter VI in connection with the illustrative design example where an active system is compared with a passive system on a sensitivity basis.

It may be shown from the considerations of this section that one active system can perform much more satisfactorily than another from the standpoint of reliability of performance; while an active system can, if its elements are held to close tolerances, be more reliable than a passive system with tolerances of the same order of magnitude. This statement is made to bring out the fact that while passive systems are usually regarded as more reliable than active systems, it is not because of the relative values of sensitivities but is purely a matter of the relative ability to maintain tolerances in construction and throughout the life of the element.

4.4. Size, Weight and Number of Components

It was stated in Section 3.3 that a wide variety of systems can be found to achieve the same overall transmission. A large number of these systems may be eliminated by consideration of size, weight and number of components. The process of intra-loop shifting is likely to be helpful in making alterations in a system, if it appears that the size, weight or number of the components may be decreased in the process.

In assigning physical components to perform the required integration or differentiation, some control over size, weight and number of components will ordinarily be available.

4.5. Summary

It has been shown that the practical factors of relative stability; saturation; noise-susceptibility; sensitivity to changes in element values; and size, weight and number of components all may be controlled to some extent in the synthesis procedure and choice of a system. The importance of sensitivity has been discussed, and a procedure of comparison of systems suggested which simplifies the comparison and distinguishes effects over which no control can be exercised from effects which are subject to partial control in the choice of a system.

V. THE REALIZATION OF VARIABLE PARAMETERS

In Chapters II-IV methods for the synthesis of active systems were presented. These methods were presented, as previously stated, in order that the synthesis procedure might be adapted with ease to the synthesis of slowly-variable systems. In order to realize slowly-variable systems, the variable parameters of the system are identified with variable amplifiers. In this chapter the identification technique is illustrated and certain aspects of the physical realization of variable parameters are discussed.

5.1. The Variable-Amplifier Identification Technique

It is established in Chapter II that the ratio of polynomials in the complex frequency s is a suitable form for the transmission function of a slowly-variable system. The polynomials contain parameters which, when assigned numerical values, complete the specification of the transmission function. If one or more of the parameters is to be variable, the variable-amplifier identification technique is employed in obtaining the transmission form. The technique consists simply of identifying the variable parameter or parameters with an ideal amplifier or amplifiers in the system.

As an example, consider the realization of a transmission form for the function

$$T(s) = \frac{1}{s + a}$$

where a is the variable parameter of the system. In order to distinguish variable parameters from fixed parameters in a system, let the variable parameters be denoted by the symbols K_1, K_2, etc. In this simple example with only one variable parameter, let $a = K$.

A possible transmission form is indicated in Figure 5.1, where a

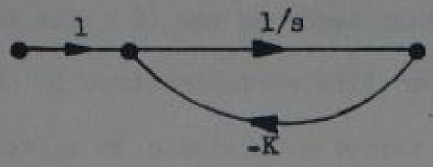

Fig. 5.1 A Transmission Form for a Variable Pole

simple integration is performed in the forward branch, while the reverse transmission consists of a variable amplifier.

5.2. Linear Variation of Gain with Control Voltage

In general, the specifications will include variation of a parameter over a range of values and in some specified way with the parameter-control signal. Variation over a large range of values is not difficult to achieve in practice, provided the upper limit on the range is within reason. However, variation in a prescribed manner with the parameter-control signal is difficult to achieve except over limited ranges. A linear variation of amplification with an external control signal can be achieved (approximately) over a rather wide range. This type of variation is the only type to be considered here.

It is well-known that a dynamically-variable gain can be achieved by applying a direct voltage (control voltage) to a grid of a multi-grid tube. Since the gain is required to vary in a prescribed manner with the control voltage, the problem of obtaining the prescribed variation involves the dynamic characteristics of the vacuum tube. Since dynamic characteristics vary among tubes of a given type, as a rule a prescribed mode of variation cannot be achieved with great accuracy, if at all. It may be hoped that the expanding research in the field of semiconductors will provide devices which overcome the difficulties of obtaining a proper characteristic. Perhaps, in the future, a desired characteristic may be manufactured into the device. Becker[1] states that "because of its relatively long life ... the transistor may well be applied to those industrial fields where reliability is of prime importance". This statement suggests that if suitable characteristics are achieved in semiconductors the problems now associated with tube replacement will be obviated. For the present, probably the most useful type of variation is that which is achieved by employing a direct voltage applied to the control grid of a pentode to achieve a linear variation of gain with control voltage. This type of variation is established in engineering practice and needs little discussion. However, a few comments may be made about this type of variation in connection with slowly-variable systems.

Since the variable parameter is controlled by an external source of direct voltage not under precise design control, operation anywhere along the load line of the pentode must often be anticipated and stability of operation at any point on the load line is required. Furthermore, since the variable amplifier is represented in the synthesis procedure as an ideal amplifier, it is essential that any variation in plate resistance of the amplifier tube have little or no effect on the overall transmission. Either the load line must

1. Becker, J. A., "Transistors", Electrical Engineering 69, 1, January, 1950, pp. 58-64

be selected so that no sizeable change in plate resistance takes place along the load line, or the coupling networks must be selected in such a way that the system is rendered insensitive to changes in the plate resistance. One way to make the transmission insensitive to changes in plate resistance is to employ a low value of plate load resistance with the pentode and a coupling network with high input impedance. The pentode with its associated plate load resistor may then be treated as a linear voltage source. A further advantage of employing a relatively low value of plate load resistance is that no rapid change in the transconductance of the tube at low values of bias voltage results; a factor which is to be expected in operation with a high value of plate load resistance.

While the approximation to linear variation over the range of variation is better if sharp-cutoff pentodes are employed (assuming a wide range of variation is desired), the decrease in the allowed dynamic range of the signal tends to offset the advantage of linear variation of gain with control voltage. If a remote-cutoff pentode is used the allowed dynamic range of the signal is increased, but the linear variation of gain with control voltage is destroyed and the characteristics of the slowly-variable system are insensitive to changes in the control voltage when operating in the range of large bias voltages.

Conditions of operation at either end of the load line are important in establishing the limits of variation in the transmission properties of the system. It is doubtful, however, whether sufficient accuracy in performance at the ends of the load line can be obtained to enable more than rough limits on performance in the vicinity of maximum and minimum gain to be established.

5.3. Summary

Variable parameters can be realized by means of variable amplifiers. A linear variation of a parameter with a dc control voltage can be approximated if the control voltage is applied to the control grid of a pentode. If variation over a wide range is anticipated, special attention must be given to the conditions at low and high limiting values of bias to insure satisfactory performance.

VI. AN ILLUSTRATIVE DESIGN

In this chapter the techniques discussed in Chapters II-V are employed in the realization and test of a variable-bandwidth electronic filter. The chief aim in presenting this chapter is to illustrate the application of the steps in the synthesis procedure to a representative problem and thereby bring the ideas presented in the preceding discussion together in a single cohesive illustration. It is felt that in presenting this illustrative design a unity and depth of treatment is achieved which justifies the attention devoted to the illustration.

The discussion of the details of the methods employed is rather long, hence it is desirable to establish the gist of the discussion before proceeding to the details. The problem of the synthesis is established by means of rather loose specifications on the system[1]. A suitable transmission function is selected which incorporates the specifications. The transmission form is realized from the transmission function. The relationships between the parameters of the transmission function and the elements (loop-gain parameters) of the system are established. Working values of the elements are selected on the basis of the specifications. The sensitivity of the transmission to changes in element values is investigated and a comparison of the sensitivities of the active system (to be employed in the design) with those for a passive system giving the same overall transmission is made. The integrators required in the physical realization are realized approximately subject to the constraints that the coupling

1. By employing rather loose specifications the essence of the synthesis procedure is retained, while sufficient leeway is allowed for exploration of the example.

networks in the system contain blocking capacitors, that the loops of the system attain suitable relative stability, and that pentodes be used in the realization of the variable parameters.

Approximate methods are used to establish overall performance. Variants of the established system are considered and it is shown that the system can be modified slightly to meet more general specifications than those originally proposed. Finally, the results of tests are given and a comparison of experimental and theoretical results is made.

6.1. The Specifications

It is desired to realize a variable-bandwidth filter with an upper cutoff frequency which is slowly-variable over a decade of frequency extending from 10,000 cps down to 1,000 cps. The asymptotic rate of cutoff at high frequencies should be twelve decibels per octave. The amplitude characteristic should be flat to within plus or minus one decibel for any particular value of upper cutoff frequency. Low frequency cutoff should occur below 100 cps, but the rate of low frequency cutoff is not specified.

6.2. Determination of a Transmission Function from the Specifications

The specifications given in Section 6.1 may be incorporated in a familiar transmission function which is commonly written in the form

$$T(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where A is a constant indicative of the level of transmission,

ω_n is the undamped natural angular velocity of oscillation and ζ is the relative-damping ratio. If the relative-damping ratio is set at a value of 0.6, the specification on the amplitude of the frequency characteristic is met and a relatively sharp cutoff is achieved.

Because the system need not transmit down to zero frequency it is not necessary to realize the transmission function exactly. The eventual bandwidth of the system is somewhat uncertain, but for convenience and since attention is centered on high-frequency performance the undamped natural frequency $f_n = \omega_n/2\pi$ will be referred to as the bandwidth of the system. The specifications on the upper cutoff frequency may be taken as synonomous with specifications on f_n and the system may be designated as a variable-bandwidth filter. Now that a suitable transmission function and terminology is established, the next step in the procedure is to realize a suitable transmission form.

6.3. Realization of a Transmission Form

The problem is to realize

$$T(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{6.1}$$

such that ω_n is continuously variable, but ζ is constant.

To proceed, put $2\zeta\omega_n = K$ and $\omega_n = aK$, and write $T(s)$ in the form

$$T(s) = \frac{A\, a^2 K^2}{s^2 + Ks + a^2K^2} \ . \tag{6.2}$$

In (6.2) the variable-amplifier identification is made. It is convenient to normalize the frequency variable with respect to K.

The normalized transmission function is

$$T(s) = \frac{A\,a^2}{s^2 + s + a^2}$$

$$= \frac{A\,a^2}{(s + 0.5)^2 + d^2} \qquad \text{where } d^2 = a^2 - 1/4$$

$$= \frac{\dfrac{A\,a^2}{(s + 0.5)^2}}{1 + \dfrac{d^2}{(s + 0.5)^2}}\,. \tag{6.3}$$

Equation (6.3) is of the form required for a simple-loop formation as discussed in Section 3.3. The transmission form associated with (6.3) is shown in Figure 6.1.

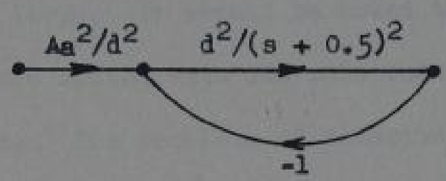

Fig. 6.1 A Transmission Form for the Transmission Function of (6.3)

It is desirable to convert the transmission form shown in Figure 6.1 to another form which is more amenable to the visualization of a physical system. It can be shown that the system of Figure 6.2 (obtained by application of the techniques given in Section 3.3) is equivalent to that of Figure 6.1.

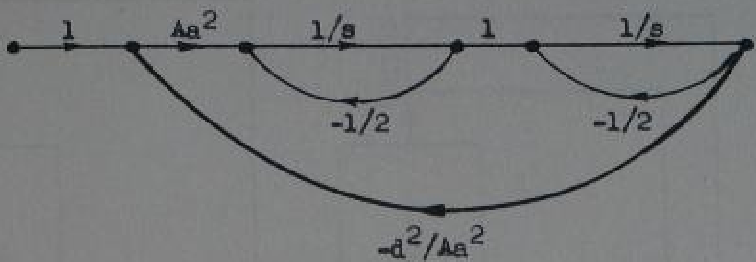

Fig. 6.2 A Transmission Form Equivalent to the Form of Fig. 6.1

The transmission form shown in Figure 6.2 contains two integrating elements. All other elements of the system are independent of frequency. Since the variable parameter is hidden in the integration terms, the effect of realizing the system according to the transmission form of Figure 6.2 is that the variation in the bandwidth of the system is dependent upon variation in the loop gains of the two inner loops. It should be noted that each of the inner loops is a one-pole system with the pole located on the negative-real axis in the s-plane. The problems of relative stability and saturation in the inner loops are consequently monitored to some extent, as discussed in Section 4.1.

6.4. Block Diagram of the System

Since the transmission form of Figure 6.2 is deemed suitable for the realization of the system, it is desirable to determine a block diagram or tentative plan of physical realization. The form can be instrumented as shown in Figure 6.3 if all nodes are identified as voltage variables, and the input impedance of the integrators is sufficiently high.[1]

1. The significance of the requirements is discussed in Section 5.2.

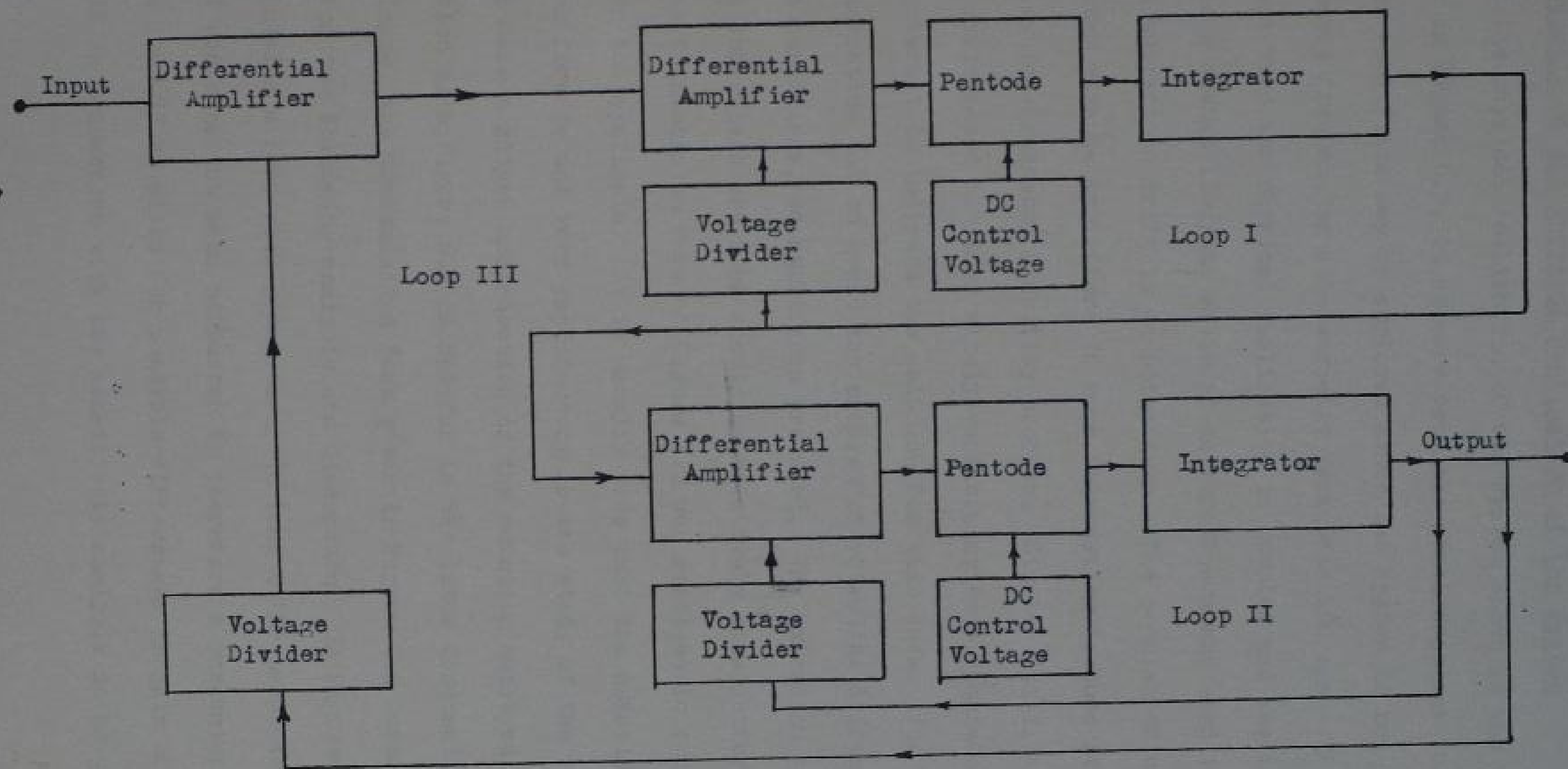

Figure 6.3. Preliminary Block Diagram of Slowly-Variable System

6.5. Details of the Physical Realization of the System

In the physical realization of the system shown in the block diagram in Figure 6.3, it appears from the specifications that either dc or ac amplifiers may be employed since the system is not required to transmit frequencies substantially less than 100 cps. If ac amplifiers are used the physical realization is complex and the low frequency range is somewhat limited, while if dc amplifiers are used the system will be subject to drift in dc potentials. The problem of realizing the system with ac amplifiers is more interesting because it involves a complex low-frequency stability problem. Similar stability problems may be anticipated in other slowly-variable systems, therefore it may be of interest to indicate the solution for this case. For purposes of illustration and to avoid the problem of potential drifts associated with dc amplifiers, ac amplifiers are used. The extension to the case when dc amplifiers are used should be relatively straightforward.

The transmission form of Figure 6.2 was realized to provide a form for the synthesis. It is usually true that the notation of the original form is not very satisfactory in the study of the system. In this case a slight modification of the notation employed in connection with Figure 6.2 is helpful in the later discussion. Accordingly, the transmission form given in Figure 6.4 forms the basis for the following analysis and discussion. In Figure 6.4 each loop contains a loop-gain parameter. In loop I the parameter is K. Loop II contains the same parameter K, therefore a parameter b is used in loop II to allow for possible differences in gain of loops I and II in connection with the sensitivity analysis to be presented.

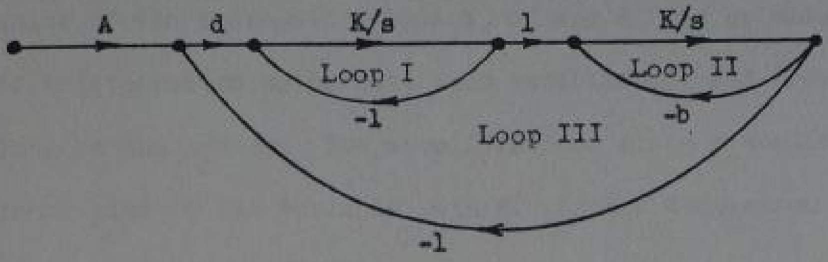

Fig. 6.4 A Transmission Form
to be Realized by a Physical System

The loop-gain parameter of loop III is denoted by the symbol d. In terms of the notation of Figure 6.4, the transfer function of the system is given by

$$T(s) = \frac{A\,d\,K^2}{s^2 + K(b+1)s + K^2(b+d)} \tag{6.4}$$

where $0 \leq b \leq 1$.

The important relationships between the system parameters (ω_n and ζ) and the system element values (loop-gain parameters) are found by comparing (6.4) with the original form of the transmission function given in (6.1). These relationships are

$$\omega_n = K\,(b + d)^{1/2} \tag{6.5}$$

and

$$\zeta = \frac{b + 1}{2(b + d)^{1/2}} \tag{6.6}$$

It is convenient to note, for later reference, that the bandwidth of the system is proportional to the variable gain K, as desired, and that the relative damping ratio is determined by adjusting the loop gain of the outer loop in Figure 6.4 when the two inner loops have equal gains.

6.5.1. Choice of Loop-Gain Values

The choice of the loop-gain values K, b, and d, may be made on the basis of relationships which have been established, and from the specifications on the system. The specifications place a condition on the maximum value of the variable gain K. From a comparison of (6.1) and (6.4), and since $0 \leqslant b \leqslant 1$,

$$\zeta\omega_n \leqslant K_{max} \leqslant 2\zeta\omega_n \quad . \tag{6.7}$$

Putting $\omega_n = 2\pi(10{,}000)$ and $\zeta = 0.6$, it follows from (6.7) that

$$12{,}000\pi \leqslant K_{max} \leqslant 24{,}000\pi \quad . \tag{6.8}$$

The lower limit in (6.8) corresponds to b = 1, while the upper limit corresponds to b = 0. If a value of b = 1 is chosen, the two inner loops of the system are, in theory, identical. If a value of b = 0 is chosen, the gain of loop I assumes the maximum value indicated in (6.8) and the reverse branch of loop II degenerates to an open circuit. The choice of b = 1 would permit K_{max} to be a minimum and would simplify the design and presentation of the example, since any consideration of loop I would apply equally well to loop II. For the reasons just mentioned the design proceeds with

$$\underline{b = 1}.$$

Since the lower limit of the inequality (6.8) corresponds to the condition b = 1, the lower limit constitutes K_{max}. However, it is convenient to allow a small working margin and employ a round number

$$\underline{K_{max} = 40{,}000.}$$

The final loop-gain parameter to be assigned is found by putting b = 1 and $\zeta = 0.6$ in (6.6). This gives the loop-gain parameter of loop III

$$\underline{d = 1.78}.$$

6.5.2. <u>Determination of Sensitivity to Changes in Loop-Gain Parameters</u>

Since working values for the loop-gain parameters are established, it is both possible and desirable to consider the sensitivity of the system to small changes in the loop gains. First the principal partial sensitivities (as defined in Section 4.3) are obtained in order that systems with equivalent transmission functions may be compared. The parameters of interest are ω_n and ζ. From (4.1) and (6.1) it may be shown that

$$S_{\omega_n}(T) = \frac{2s(s+\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and

$$S_{\zeta}(T) = \frac{-2\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \tag{6.9}$$

The amplitude and phase of the sensitivity for $s = j\omega$ is plotted for each parameter as a function of the ratio f/f_n. The curves are given in Figures 6.5 and 6.6, for values of $\zeta = 0.6$ and 0.3. For the moment, interest centers on the curves for $\zeta = 0.6$.

The relationships between the parameters ω_n and ζ, and the elements of the system (loop-gain parameters) have been given in (6.5) and (6.6). From these relationships and the definition of

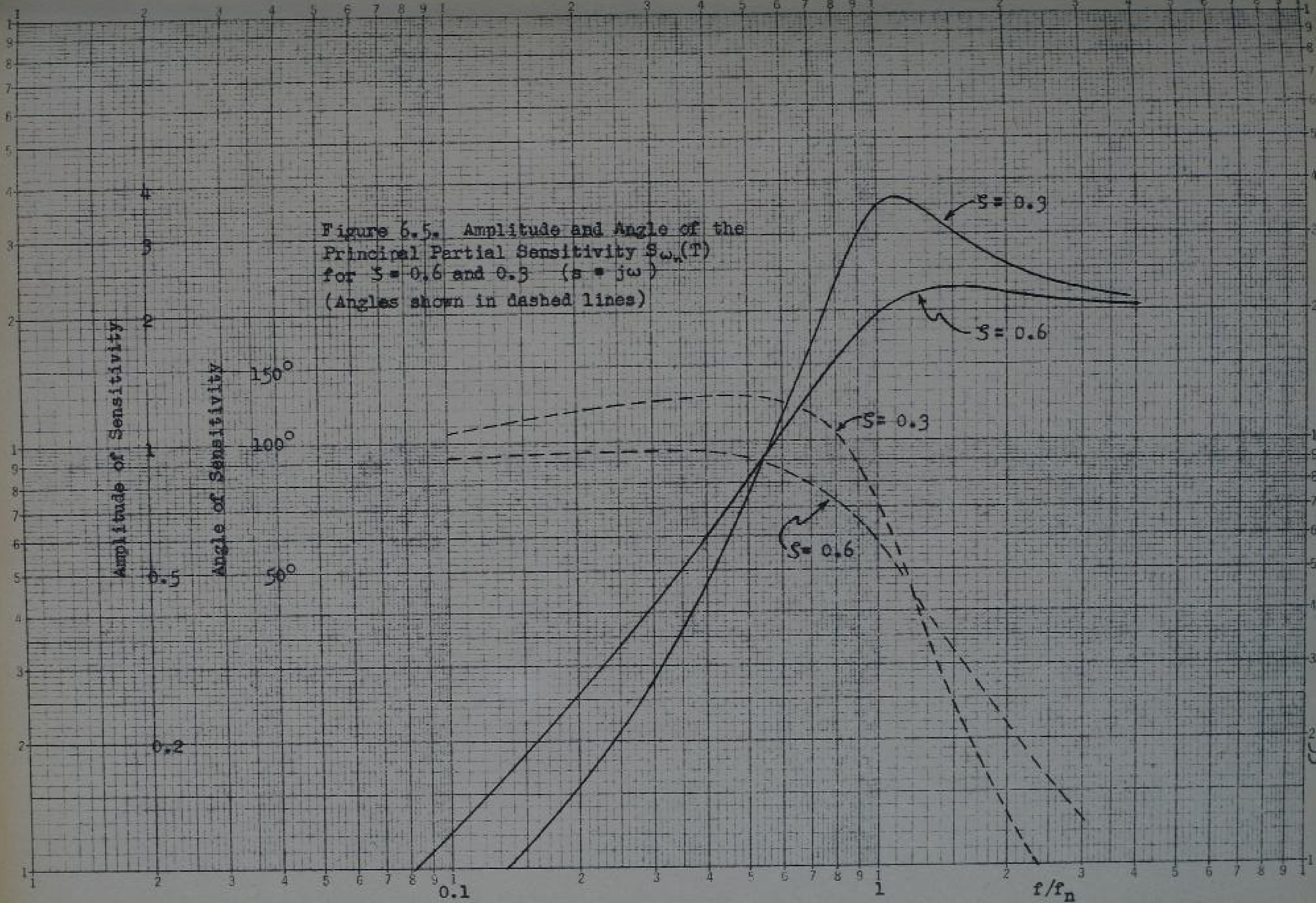

Figure 6.5. Amplitude and Angle of the Principal Partial Sensitivity $S_{\omega_n}(T)$ for $\zeta = 0.6$ and 0.3 $(s = j\omega)$
(Angles shown in dashed lines)

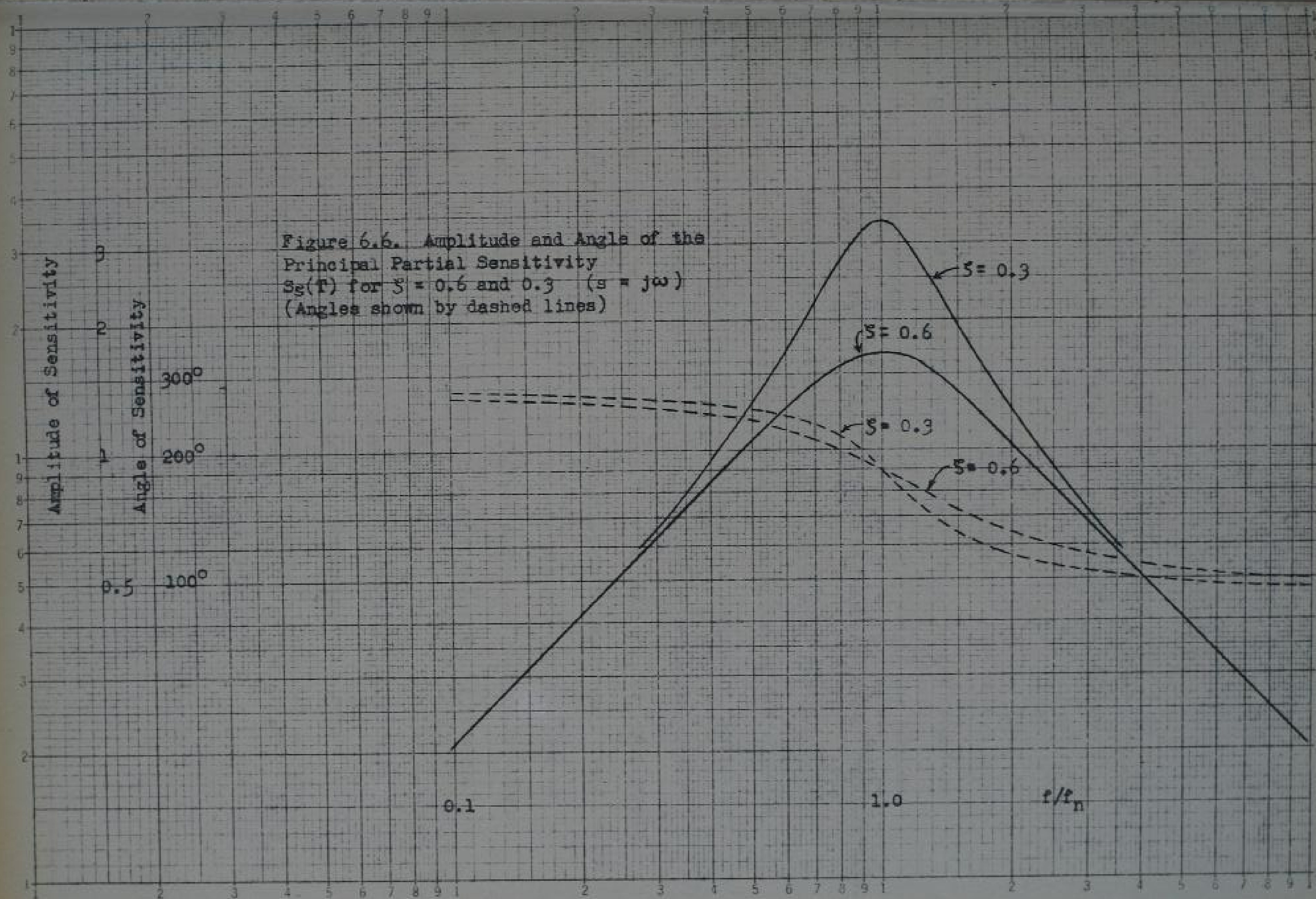

Figure 6.6. Amplitude and Angle of the Principal Partial Sensitivity $S_g(f)$ for $\zeta = 0.6$ and 0.3 $(s = j\omega)$ (Angles shown by dashed lines)

subsidiary partial sensitivity given in (4.2), it can be shown that

$$S_K(\omega_n) = 1.0 \qquad S_K(\zeta) = 0$$

$$S_b(\omega_n) = 0.5b/(b + d) \qquad S_b(\zeta) = \frac{b(b + 2d - 1)}{2(b + d)(b + 1)}$$

$$S_d(\omega_n) = 0.5d/(b + d) \qquad S_d(\zeta) = -d/2(b + d) \tag{6.10}$$

The relations (6.10) illustrate the statement of Section 4.3 to the effect that the subsidiary partial sensitivities do not depend on the frequency variable, but are numerics which weight the complex principal partial sensitivities to give the (overall) system sensitivities.

It is instructive to compare (6.10) with corresponding results obtained for a passive network giving the same form of transmission function. The comparison will illustrate the remarks of Section 4.3 where it was stated that though active elements are normally more susceptible to undesirable changes in value than are passive elements, the undesirable effects of element variation on the overall transmission may be less in an active system yielding the same transmission as a given passive system, for equivalent percentage changes in element values. In short, smaller overall sensitivities may be obtained in connection with an active system.

A passive system which gives the same form of transmission function as the active system under consideration is a series R-L-C circuit shown in Figure 6.7. For this circuit, connected as shown,

$$T(s) = \frac{1/LC}{s^2 + (R/L)s + 1/LC} .$$

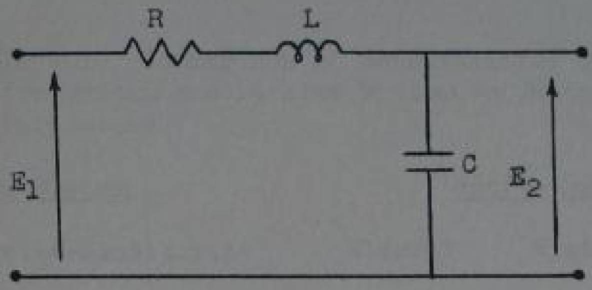

Fig. 6.7 R-L-C Series Circuit

For the passive system of Figure 6.7 the subsidiary partial sensitivities are

$$S_R(\omega_n) = 0 \qquad S_R(\zeta) = 1.0$$
$$S_L(\omega_n) = -0.5 \qquad S_L(\zeta) = -0.5$$
$$S_C(\omega_n) = -0.5 \qquad S_C(\zeta) = 0.5 \quad . \tag{6.11}$$

In contrast to the subsidiary sensitivities for the active system given in (6.10), the subsidiary sensitivities for the passive system do not depend on element values. It follows that there is no control over sensitivity in the passive system of Figure 6.7. This fact places the brunt of the design on choice of materials and manufacturing tolerances.

The overall sensitivities for the transfer function under consideration are computed at the undamped natural frequency f_n (where the principal partial sensitivities are near the maximum absolute value) for ζ = 0.6 with b = 1 and d = 1.78, as selected earlier in this section. The results for the active system and passive system may be compared by reference to Table 6.1.

Table 6.1. Comparison of Sensitivities of Active System and Passive System to Changes in Parameters

Passive System		Active System	
Element	System-Sensitivity	Element	System-Sensitivity
L	$0.91\angle{-67.8^\circ}$	b	$0.44\angle{42.8^\circ}$
C	$1.57\angle{212.4^\circ}$	d	$1.04\angle{31.0^\circ}$
R	$1.66\angle{180.0^\circ}$	K	$1.94\angle{59.8^\circ}$

In obtaining the values in Table 6.1, the loop-gain parameters (elements) of the active system were assumed independent. If this assumption is recognized as a concession to the intricacy of the general problem of sensitivity, and the results of Table 6.1 are taken at face value, the active system seems superior to the passive system[1]. The active system is much less sensitive to changes in b (the loop-gain parameter of loop II) than is the passive system to changes in any element value, while the magnitude of the sensitivity to changes in d (the loop-gain parameter of loop III) is smaller than the average of the magnitudes of the sensitivities in the passive case. The magnitude of the sensitivity of the active system to changes in K is the highest of all in Table 6.1, but this is an outgrowth of the requirements on the system.

The interpretation of the magnitudes of the sensitivities for the active system is of interest. It is desirable that the system be insensitive to changes in b, since it is not anticipated that

1. Admittedly the elements of the active system will probably be subject to much larger variation than the elements of the passive system, so the comparison on the basis of sensitivities tends to be unfair to the passive system.

the loop gains of the two inner loops will be matched exactly in practice. It would be desirable if the sensitivity of the system to changes in the parameter d were less, but some consolation is derived from the fact that the slight (complex) change in the overall transmission is added at an angle to the (complex) transmission, therefore the effect on the amplitude characteristic is not as great as would be expected from a sensitivity of equivalent magnitude at an angle of zero or one hundred-eighty degrees. It should be recalled that the values in Table 6.1 are computed at a comparatively critical point along the real frequency axis and will be lower for frequencies nearer the center of the pass band.

The comparison between the active and passive systems serves, in this example, as an indication of the comparison of systems which constitutes an important part of the synthesis procedure. The next item for consideration is the problem of determining suitable approximations for the required integrating elements.

6.5.3. Loop Transmission Functions for Loops I and II

Consider the loop transmission T_I of loop I. (Since loops I and II are identical, both loops will be disposed of in the discussion of loop I.) The ideal form of the loop transmission of loop I is $-K/s$. Since a-c amplifiers are to be used, the choice of an approximation to the integration characteristic is constrained to take account of the presence of two coupling networks which are suitable for use in a-c amplifiers. Furthermore, the effect of the internal impedance of the tubes on the approximation should be considered in the determination of the approximation. Since the use of twin triodes is contemplated as instrumentation for the differential amplifiers indicated in Figure 6.3, the common assumption that the triode is a linear voltage source isolated from the input to the grid is employed in connection with the differential amplifiers. The variable-gain devices are instrumented by means of pentodes, as described in Chapter V. If the value of plate-load resistance associated with a pentode is sufficiently small compared to the input impedance of the coupling network following the pentode, the pentode and the associated plate-load resistance may be regarded as a linear voltage source, and are so regarded in the remainder of the discussion.

If the vacuum tubes with associated plate-load resistances are treated as linear voltage sources, the approximation to -K/s must take the form of a voltage transfer-ratio. A simple, straight-line approximation to the required loop logarithmic gain is shown in Figure 6.8. This figure is formed by drawing the ideal integration characteristic as the high-frequency asymptote, and drawing the low frequency asymptote to conform to the requirement of blocking capacitors in the two coupling networks. Numerical values in Figure 6.8 show the slopes of the lines in decibels per octave.

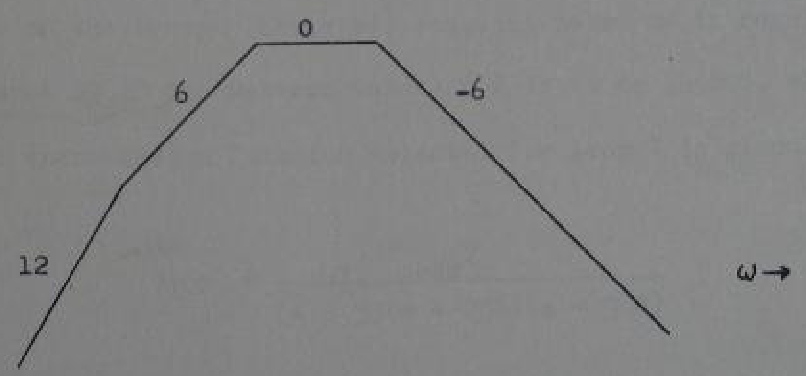

Figure 6.8. Approximation to Integration Characteristic in Loop I

The frequency axis is not drawn in Figure 6.8, since the location of the poles in the transfer function of the two tandem (isolated) coupling networks is as yet undetermined. The determination of the loop transmission of loop I to approximate $-K/s$ is now considered in detail.

Based on the knowledge of the desired transmission and the form of the approximation shown in Figure 6.8, the form of the loop transmission of loop I may be written as

$$T_I(s) = \frac{-Ks^2}{(s - s_1)(s - s_2)(s - s_3)} \tag{6.12}$$

where s_1, s_2, and s_3 are the poles of $T_I(s)$ and lie on the negative-real axis of the s-plane. The factor K is interpreted as the product of (a) the gain of the triode tube with its associated plate-load resistor, (b) the variable gain of the pentode tube with its associated plate-load resistor, and (c) the so-called "gain constant" of the resistance-capacitance coupling networks to be employed. The maximum value of the product of the first two of these three factors is estimated to be of the order of 1,000; therefore, the gain constant of the transfer function of the tandem (isolated) coupling networks is required to be at least 40 if the maximum value of K is to be 40,000, as required.

The transmission function selected for loop I is given by

$$T_I(s) = \frac{-Ks^2}{(s + 3)(s + 250)(s + 500)} \,. \tag{6.13}$$

The required transfer function of the two coupling networks in tandem is, accordingly,

$$\frac{40as^2}{(s + 3)(s + 250)(s + 500)} \tag{6.14}$$

where the constant <u>a</u> should be at least unity, and preferably greater, in order that freedom be available in choice of tubes and adjustment in the laboratory. The explanation of the location of the poles in (6.13) is now given. The poles are located on the negative-real axis in the s-plane in order to permit a simple realization with the use of resistances and capacitances. The pole nearest the origin cannot be chosen too close to the origin or the element sizes will be unduly large. On the other hand, it is desirable that this pole be located close to the origin, since a good approximation to the desired integration characteristic should begin as close to the origin as possible. Another factor is involved in the location of the pole nearest the origin, namely, relative stability of the closed-loop transmission of loop I. Consideration of the latter factor indicates that the pair of poles nearest the origin should be staggered in location. The pair of poles farthest from the origin should also be staggered, in order to provide a gradual transition in the phase characteristic into the region of good approximation.

It is apparent that there is room for any number of choices of the pole locations, based on consideration of the factors mentioned. The choices given in (6.13) represent one compromise solution to the problem of pole location. It can be shown that with this choice of poles the phase margin is of the order of fifty-five degrees at the low-frequency zero-decibel point in the open-loop transmission when the gain K takes on the value of 40,000. An additional loop gain of ten decibels is required to decrease the phase margin to thirty degrees. Under these conditions, the inner loops should be free from low-frequency oscillation. The problem remains to realize the coupling networks, and if this can be accomplished the discussion of the loop transmission of loop I will be essentially complete.

In order to obtain two networks to provide the desired transmission function (6.14), the function is separated into two transfer functions of the forms

$$\frac{40as}{(s - s_k)(s - s_j)}$$

and

$$\frac{s}{s - s_m} \quad .$$

The poles of these functions are the poles of the desired overall transmission, but obviously the poles may be assigned to the individual functions in three ways, according to which pole is assigned to the single-pole transfer function. This means that there are three distinct cases to be considered.

The simple and widely-used form of coupling network represented by the transfer function $s/(s - s_m)$ is determined by inspection and is shown in Figure 6.9.

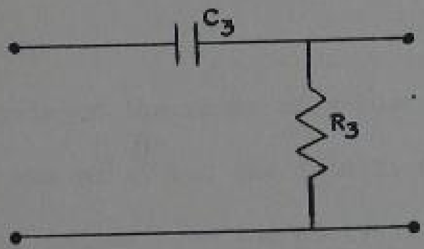

Figure 6.9. Coupling Network "A"

A simple form of the network containing two poles may be determined by applying the principles of R-C network synthesis to a representative function such as

$$\frac{s}{(s + 1)(s + 2)} \, .$$

If the numerator and denominator of this expression are divided by $s(s + 1.5)$, for example, and the resulting numerator is identified as the open-circuit transfer impedance of an R-C network, and the denominator as the open-circuit driving-point impedance of an R-C network, it is readily determined that a possible form of the network is that indicated in Figure 6.10.

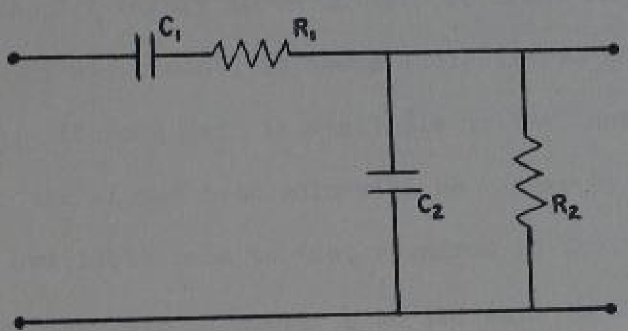

Figure 6.10. Coupling Network "B"

Each of the networks of Figures 6.9 and 6.10 begins with a blocking capacitor, as required.

A thorough analysis of the range of values of the parameters of the networks A and B was made, but the details need not be included here. Typical sets of values from the analysis are given in Table 6.2 which is prepared under the constraint that the largest capacitor in either network be 0.05 microfarads in value.

Table 6.2. Possible Sets of Values for Elements of Coupling Networks in the Loop Transmission of Loop I (in microfarads and megohms)

Row	g_m	R_1	C_1	R_2	C_2	R_3	C_3	a
1	500	0.18	0.05	3.2	0.05	0.20	0.01	3.2
2	500	3.2	0.05	3.2	0.0025	0.20	0.01	3.2
3	250	3.3	0.05	3.3	0.0012	0.40	0.01	6.3
4	250	0.08	0.05	3.3	0.05	0.40	0.01	6.3
5	3	2.38	0.0012	0.28	0.01	6.66	0.05	1.05

The last row of Table 6.2 gives the smallest capacitors, but also gives a value of a barely above unity. To allow for tube aging and laboratory adjustment, it seems desirable to choose a larger value of a. If more gain is available in the loop than the minimum required, the closed loop gain will be higher by the ratio of the amount of available gain to that required in the loop. While the value of the gain parameter a is the same in Rows 3 and 4, the low value of R_1 in Row 4 is not very satisfactory, because the assumption is made in the analysis that the input impedance of the coupling networks is high. Row 3 is deemed most satisfactory of the group, in view of the requirement on large a and the requirement on high input impedance of the coupling networks. The set of values in Row 3 is, therefore, selected for the coupling networks. Figure 6.11 shows the coupling networks with the chosen element values indicated on the figure.

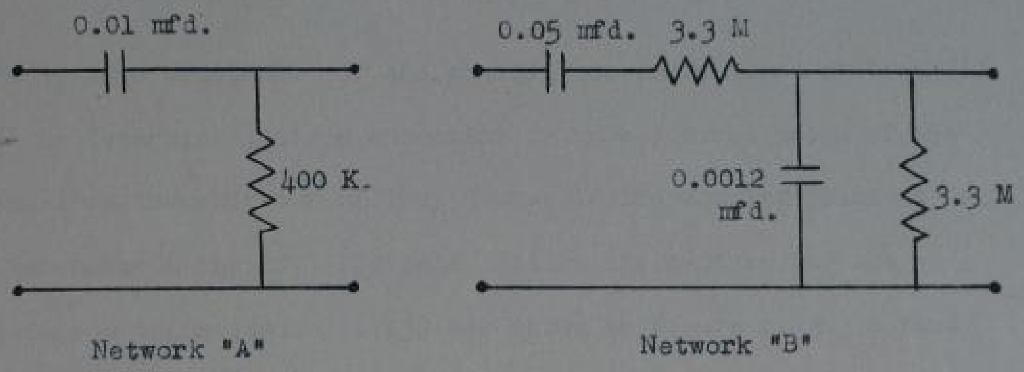

Figure 6.11. Coupling Networks of Loop I

A decision must be made as to the position of the two coupling networks within the loop. Because the resistor in the grid circuit of the twin triodes should not exceed one megohm in value, it is desirable that network A precede a triode, leaving network B to precede the variable-gain pentode. The high resistance in shunt in network B is desirable, since if a dc control voltage is inserted in series with this resistor, the performance of the system remains essentially independent of the impedance of the source of control voltage.

6.5.4. The Performance of Loop III

Having established satisfactory performance in the inner loops, attention is now directed to the performance of the outer loop, loop III. It will be recalled from Figure 6.3 that the performance of loop III depends on the closed-loop performance of loop I. It may be shown from (6.13) that the closed-loop transmission of loop I is

$$T_{Ic} = \frac{G_1 K\, s^2}{s^3 + (K + 753)s^2 + 127{,}150s + 375{,}000} \qquad (6.15)$$

where G_1 is a constant. If the closed-loop performance of loop III is to be determined, it is essential to know how the poles of the closed-loop transmission of loop I move in the complex plane as the parameter K varies. The pole loci in the s-plane for the closed-loop transmission (6.15) are shown in Figure 6.12. A plot of the absolute values of the real poles as a function of K is given in Figure 6.13.

In obtaining Figures 6.12 and 6.13, poles on the negative-real axis are established first, since the values of such poles may be determined very simply. To determine these poles, consider the loop transmission of loop I given in (6.13). Poles of the closed-loop transmission occur only when the phase angle of the loop transmission is an odd multiple of π radians. By assigning negative-real values to s in the loop transmission, the possible loci of poles of the closed-loop transmission on the negative-real axis in the s-plane are readily determined from the condition on the phase angle of the loop transmission. If values from the possible loci are assigned to s in (6.13) and the magnitude of (6.13) is equated to unity (the second condition that the closed-loop transmission have poles), the value of K required to give a pole at the selected value of s is easily established. The method employed is that set forth by Evans[1],

1. Evans, W. R., "Control System Synthesis by Root-Locus Method", Trans. A.I.E.E., V. 69, Part 1, 1950, pp. 66-69.

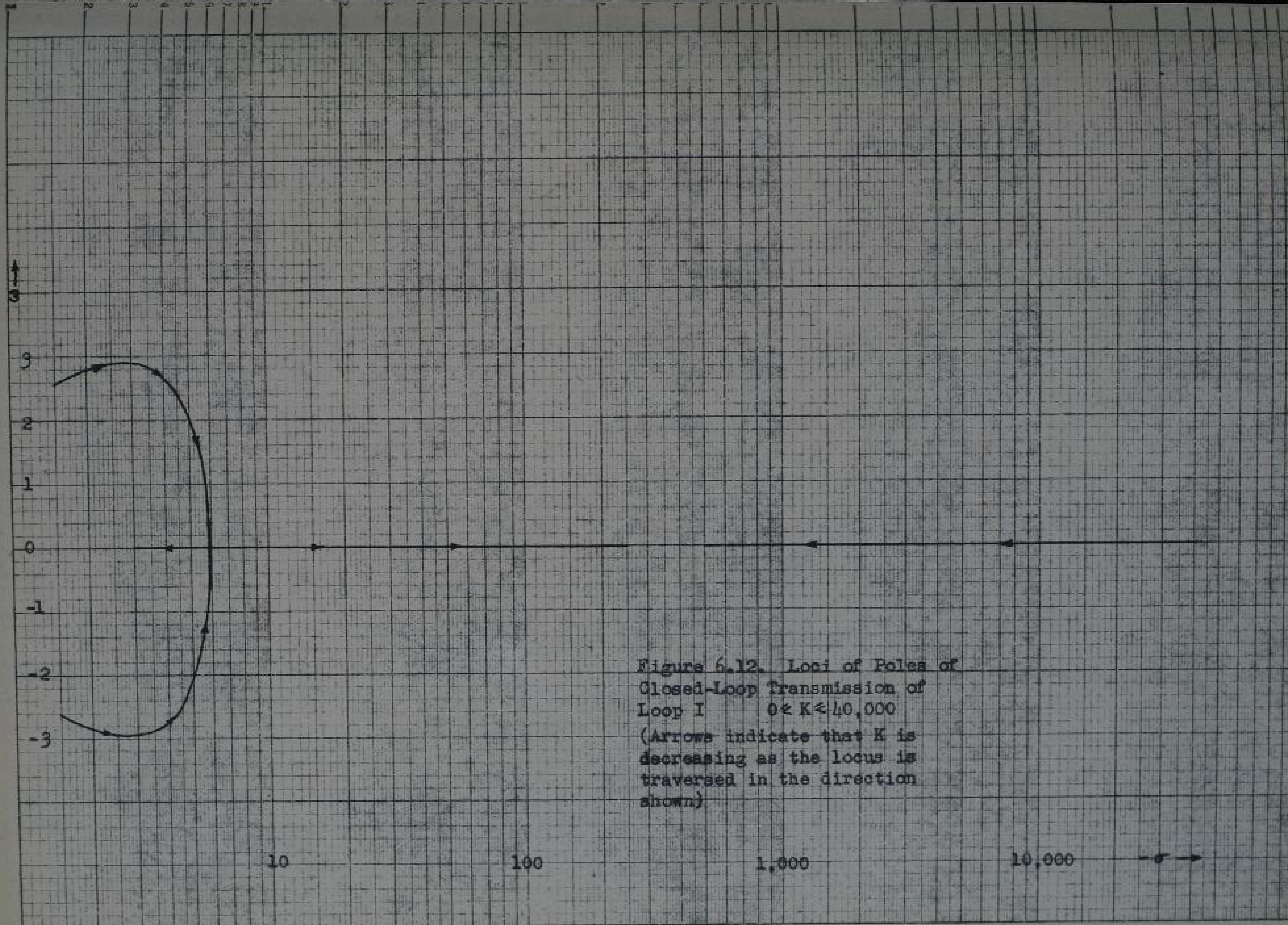

Figure 6.12. Loci of Poles of Closed-Loop Transmission of Loop I $0 \leq K \leq 40{,}000$
(Arrows indicate that K is decreasing as the locus is traversed in the direction shown)

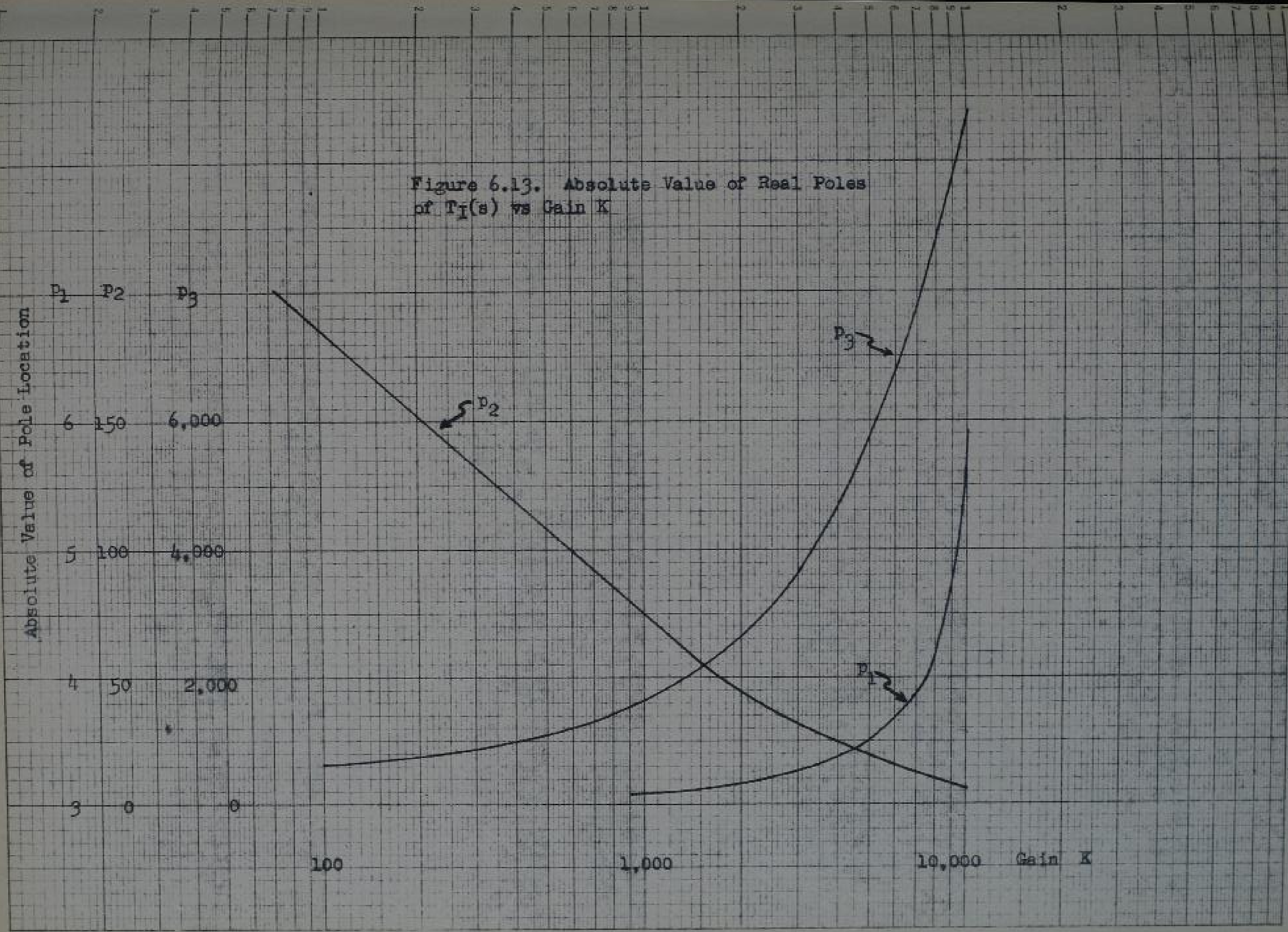

Figure 6.13. Absolute Value of Real Poles of $T_I(s)$ vs Gain K

with the minor exception that analytical rather than graphical methods for obtaining values of K were employed. Actual values for the complex poles over the range of variation are not given, since the pole-positions do not change greatly when the poles are complex and the approximation to the desired transmission is best when K is large; a condition which corresponds to a pair of conjugate-complex poles in the closed-loop transmission.

In order to discuss the effects of the individual poles of the closed-loop transmission of loop I on the system performance, the closed-loop transmission is written symbolically as

$$T_{Ic}(s) = \frac{G_1Ks^2}{(s + p_1)(s + p_2)(s + p_3)} . \tag{6.16}$$

In (6.16) p_1, p_2, and p_3 are the negatives of the pole values written in ascending order of magnitude and labeled accordingly in Figure 6.13. It is desirable at this point to investigate the major features of system performance, based on (6.16) and Figure 6.13.

Certain approximate relations which involve the poles of $T_{Ic}(s)$ may now be established in order to study the degree of success in achieving the desired characteristics, and to determine certain relations between loop-gain constants. These relations comprise (6.17) through (6.23) and lead to Figures 6.14 through 6.16. The reader may wish to omit the numerical details (which are included for completeness) and proceed by noting the results given in Figures 6.14 through 6.16 to the discussion of low-frequency stabilization given later in this sub-section.

A good approximation to (6.16) for the closed-loop transmission of loop I is

$$T_{Ic} \doteq \frac{G_1 K}{s + p_3} \tag{6.17}$$

which holds in the range of real frequencies above the nominal value of 100 cps. For loops I and II in tandem, the transmission is approximated by the square of (6.17), since loops I and II are identical, thus

$$T_{Ic}T_{IIc} = T_{Ic}^2 \doteq \frac{G_1^2 K^2}{(s + p_3)^2} . \tag{6.18}$$

If additional transmission functions added in loop III external to loops I and II do not influence the transmission[1] of loop III for frequencies above the nominal value of 100 cps, the loop transmission of loop III is approximated by

$$T_{III} \doteq \frac{-G_2 K^2}{(s + p_3)^2} \tag{6.19}$$

where G_2 is a positive constant.

From (6.19) the closed-loop transmission of loop III is then approximated by

$$T_{IIIc} \doteq \frac{G_3 K^2}{s^2 + 2p_3 s + (p_3^2 + G_2 K^2)} . \tag{6.20}$$

From (6.20) it may be shown that good approximations for the undamped natural frequency f_n and for the relative damping ratio ζ are

$$f_n \doteq \frac{(p_3^2 + G_2 K^2)^{1/2}}{2\pi} \tag{6.21}$$

1. A change in gain level is, of course, allowed.

and

$$\zeta \doteq \frac{p_3}{(p_3{}^2 + G_2K^2)^{1/2}} \quad . \tag{6.22}$$

The midband gain of the closed loop transmission of loop III is approximated from (6.20) by the expression

$$T_{IIIc(mid.)} \doteq \frac{G_3K^2}{p_3{}^2 + G_2K^2} \quad . \tag{6.23}$$

The design parameter G_2 of (6.19) is almost the equivalent of the loop-gain parameter d, determined at the outset to be 1.78. Because of deviations from the ideal resulting from the imperfect integration the value $G_2 = 1.85$ is used. This value is employed along with values of p_3 from Figure 6.13 to obtain the results of interest. The results are given in Figures 6.14, 6.15 and 6.16. The figures show, respectively, how the undamped natural frequency f_n varies with K, how the midband gain of the system falls off as a function of the undamped natural frequency f_n, and how the relative damping ratio ζ depends on the undamped natural frequency f_n. These figures show imperfections in system performance (resulting from imperfect integration) which may be regarded as the price paid in return for the use of a-c amplifiers in the system. The figures show performance over a wider range of variation than is specified. It will be recalled that the specifications require variation of f_n from 10,000 cps. to 1,000 cps.

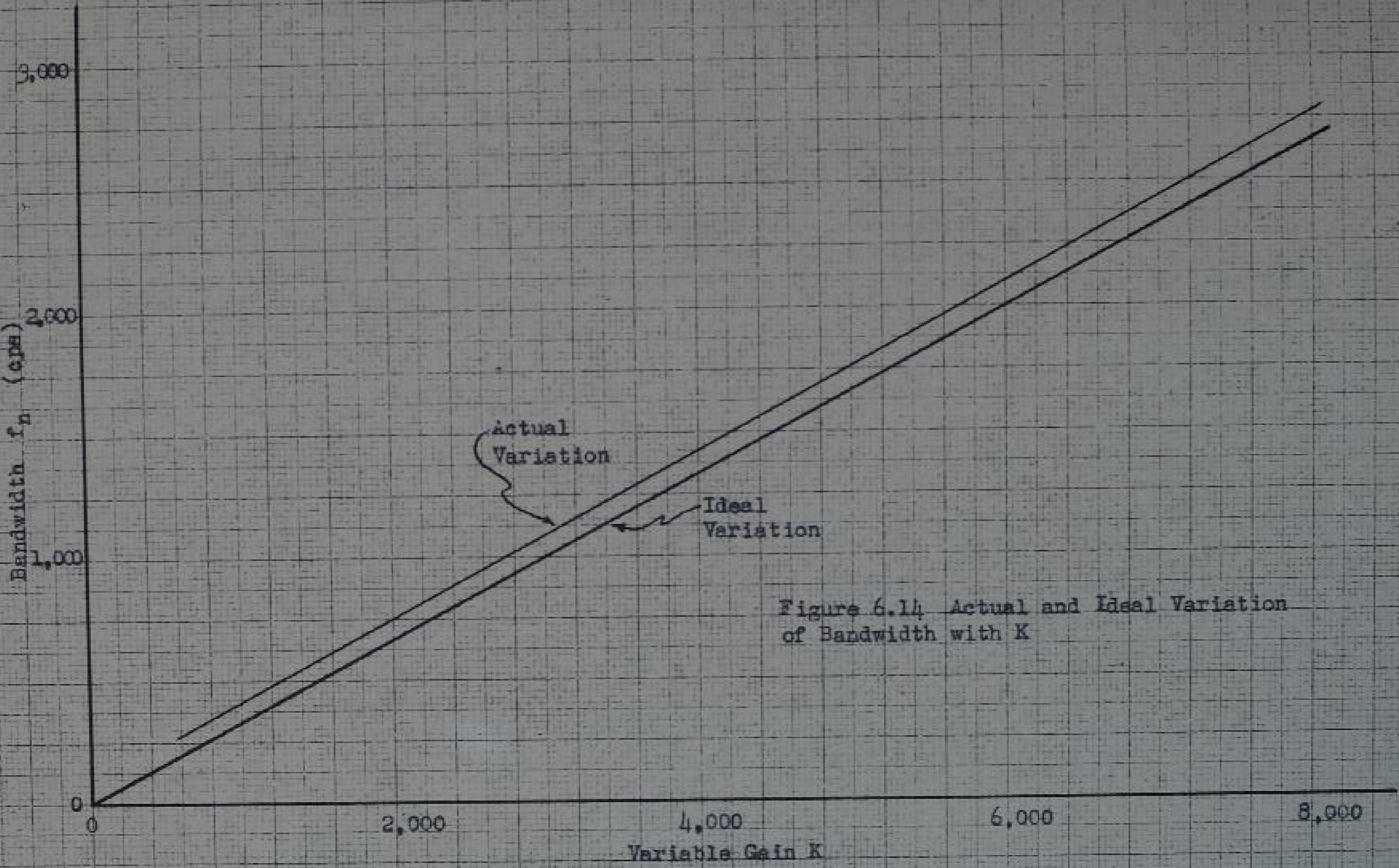

Figure 6.14 Actual and Ideal Variation of Bandwidth with K

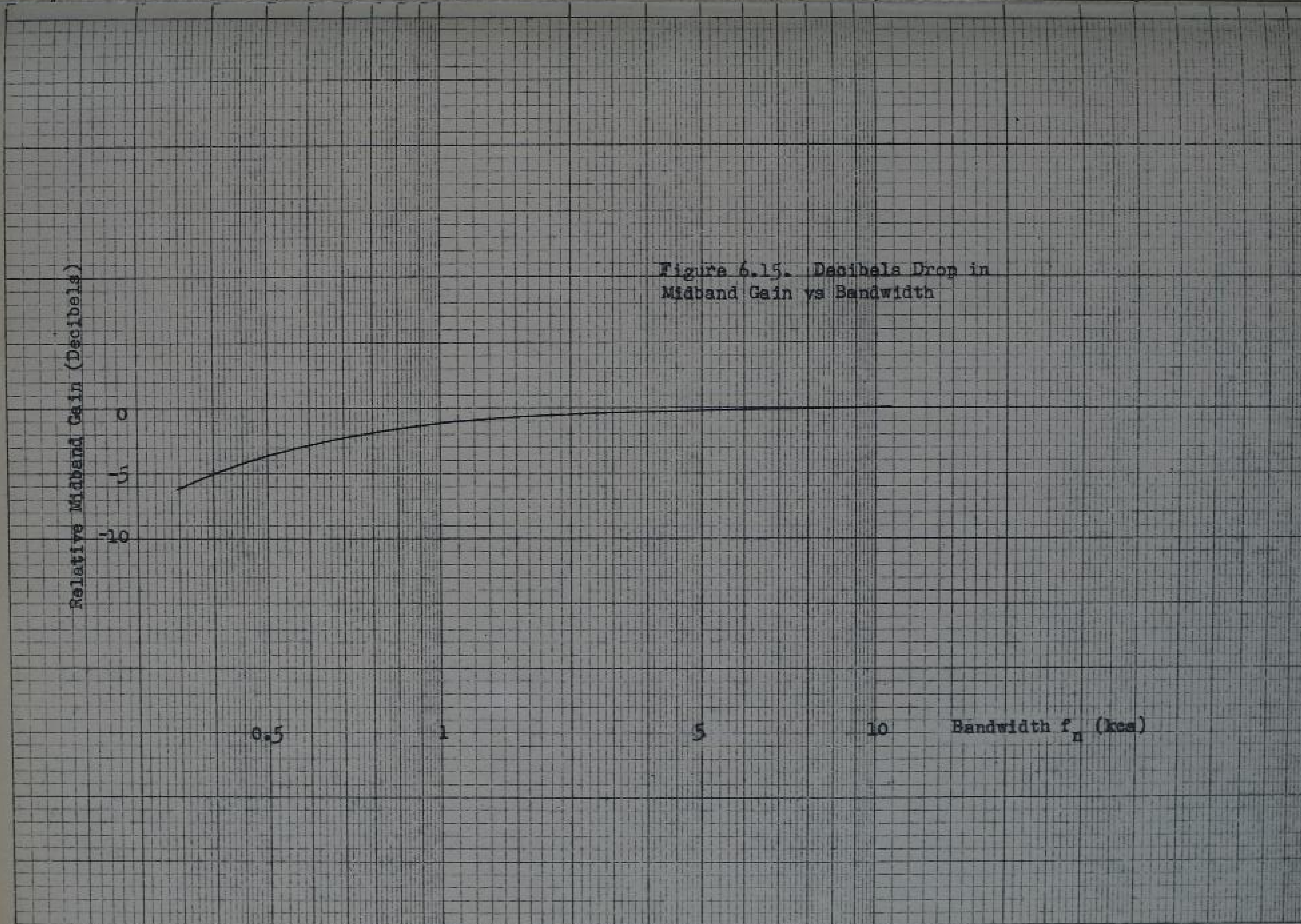

Figure 6.15. Decibels Drop in Midband Gain vs Bandwidth

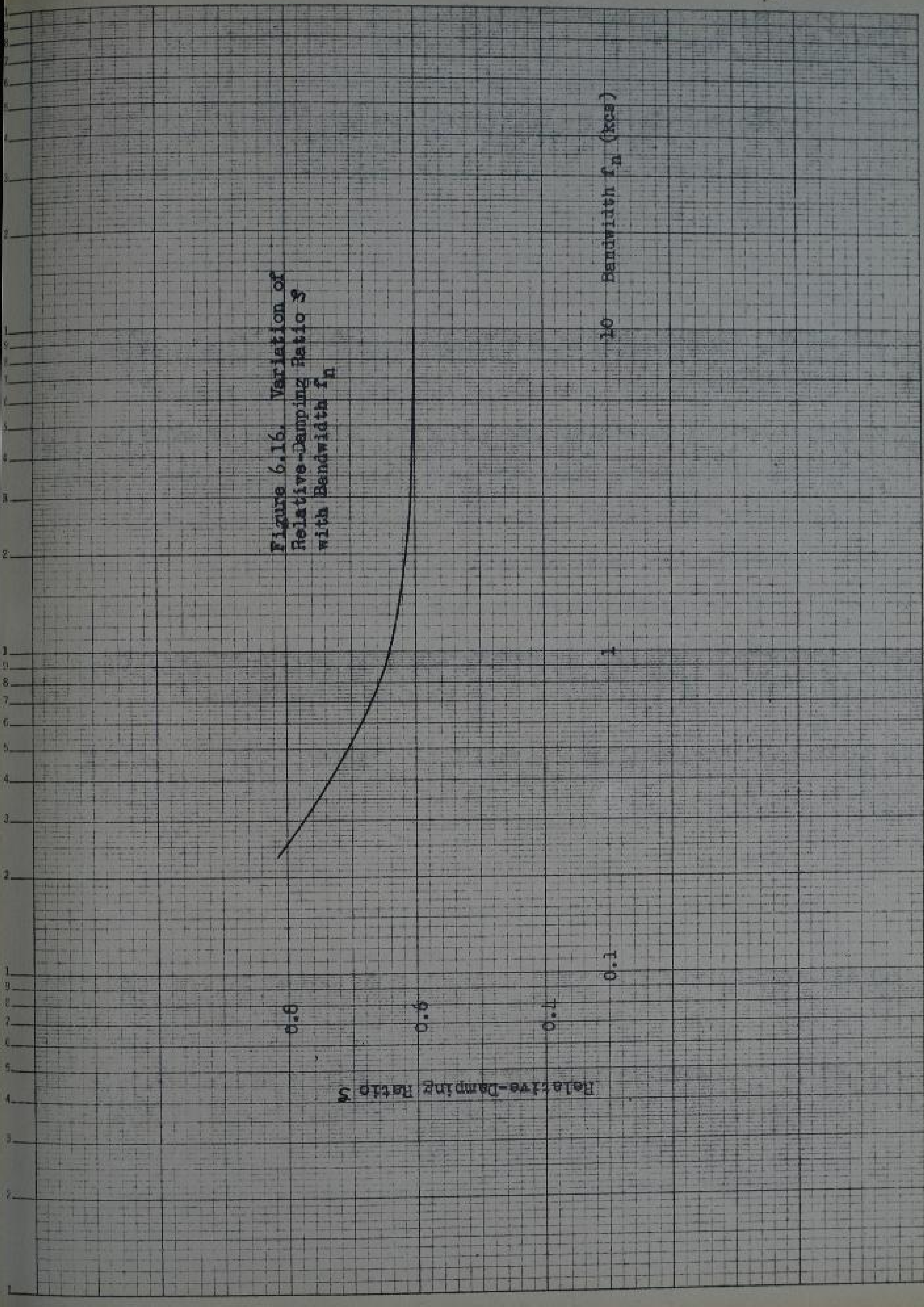

Figure 6.16. Variation of Relative-Damping Ratio ζ with Bandwidth f_n

The next step in the investigation of loop III consists of the determination of a suitable transfer function for the coupling network associated with the differential amplifier external to loops I and II. This function must be chosen according to requirements of suitable relative stability in the loop, the ability to block the passage of direct current, and a negligible effect on the transmission of the system in the frequency band extending from 100 cps. to 10,000 cps.

The problem of stabilization is complicated by the fact that two pairs of double-poles vary along the loci shown in Figure 6.12 in the region near the origin where the singularities of the coupling-network transfer function are to be placed. In addition, the midband gain of the system varies, as the bandwidth is varied dynamically. Two extenuating circumstances make possible the solution of the stabilization problem. The outer loop has a relatively low maximum value of midband gain which falls below zero decibels as the variable gain K diminishes, and the pole-zero configuration tends to be most obtrusive at low values of K.

Again, the choice of a suitable transfer function results from a compromise based on the various factors inherent in the design situation. The requirements imposed by the situation indicate that a transfer function of the form

$$\frac{s}{s - s_1}$$

with the pole at s_1 located within the circle $|s| = 100$ may be satisfactory. The results given in Figures 6.13 and 6.15, showing the variation in midband gain and pole locations with bandwidth enable typical logarithmic-gain and phase plots to be drawn for the loop transmission (exclusive of the coupling-network transmission) from which it can be established that a satisfactory transmission function for the coupling network is

$$\frac{s}{s + 20} \; .$$

With the pole located as indicated, a phase margin of 30° is established as the minimum value to be anticipated for any value of K. The corresponding gain margin is approximately six decibels. It can be shown that deviations of the pole location from the indicated location at (-20) in either direction along the negative-real axis of the s-plane result in a decrease in the phase margin.

To complete the discussion of stability of the closed-loop transmission of loop III, the high-frequency stability is considered at this point. If the effects of stray and interelectrode capacitance are neglected, the high-frequency relative stability of loop III is determined by the double pole at $(-p_3)$ in the open-loop transmission of loop III. If a thirty-degree phase margin is arbitrarily selected as a limiting value for purposes of relative stability, it can be shown that an open-loop midband gain of about twenty-two decibels can be attained before the limit is exceeded. Since the required gain for $\zeta = 0.6$ is only five decibels, there is no problem of relative stability at high frequencies. Viewed in another way, satisfactory high-frequency relative stability is assured in the process of meeting the specifications on the system.

6.5.5. Theoretical Performance of the System

Since the complete system has been determined, it is now possible to establish the performance of the system for any value of the variable gain K. The loop transmission of loop III is given by

$$T_{III}(s) = \frac{1.85K^2s^5}{(s+20)\left[(s+p_1)(s+p_2)(s+p_3)\right]^2} \quad . \tag{6.24}$$

Equation (6.24) is formed by squaring the closed-loop transmission of loop I given in (6.16) and multiplying by the transmission function of the coupling network external to loops I and II (s/s + 20) established in the preceding sub-section. The amplitude level factor 1.85 is the G_2 of (6.19), with the value given in the preceding sub-section. For values of K less than 10,070, the p's of (6.24) are real and positive, while for larger K, p_1 and p_2 are conjugate-complex quantities, with loci indicated by Figure 6.12.

The system performance is shown in Figure 6.17 for the nominal values given in Table 6.3. The procedure for obtaining the results

Table 6.3. Values for which the Amplitude-vs-Frequency Characteristics of the System are Computed

K	f_n
760	275
3,350	1,000
40,600	11,000

given in Figure 6.17 consists of plotting the loop gain and phase characteristics from (6.24) and information contained in

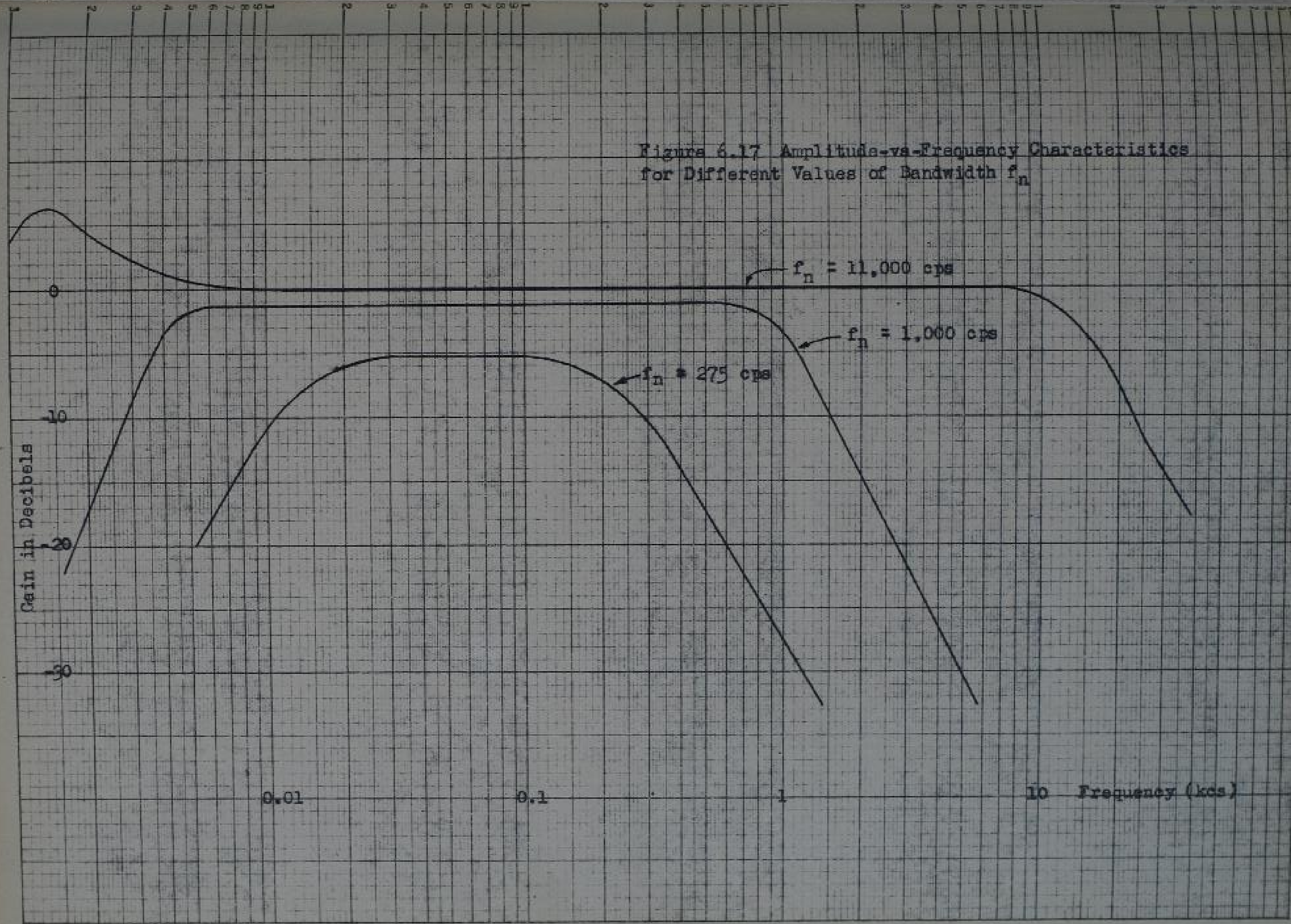

Figure 6.17 Amplitude-vs-Frequency Characteristics for Different Values of Bandwidth f_n

Figures 6.13-6.15 inclusive; then employing the Nichols charts to determine closed-loop performance. The closed-loop frequency characteristics shown in Figure 6.17 do not indicate the actual values of midband gain since the gain is subject to the choice of vacuum tubes, but the relative gains are indicated with the midband gain corresponding to a bandwidth of 11,000 cps arbitrarily taken as zero decibels.

A block diagram of the system is given in Figure 6.18. The diagram incorporates the main results of the synthesis procedure, but the absolute values of the individual amplifier gains and the transmissions in the feedback paths are not indicated, since these values are best determined in the laboratory. Laboratory adjustments may be made on the basis of the theoretical loop-gain requirements given in the preceding discussion.

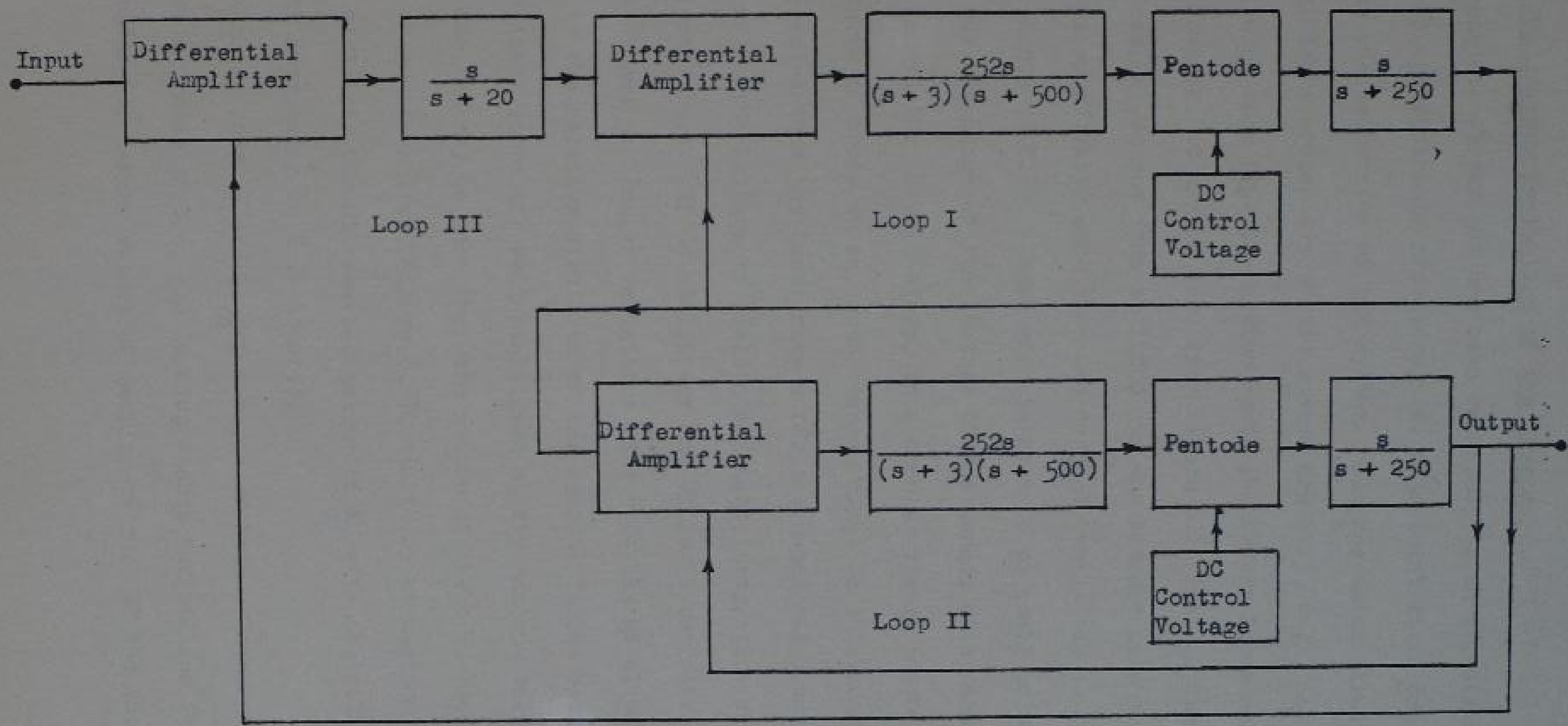

Figure 6.18. Block Diagram of System to Approximate $\frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

6.5.6. Preliminary Evaluation of the System

A complete system has now been realized, as an illustrative example of the synthesis procedure. At this point it is desirable to review the properties of the system. Figures which illustrate the significant properties of the system are 6.15, 6.16, and 6.17. These figures show that the system meets the rather loose specifications. However, it may be seen from Figure 6.15 that as the bandwidth varies from 10,000 cps down to 1,000 cps the midband gain drops to the extent of slightly more than one decibel, while the damping ratio falls from 0.6 to 0.64. These effects seem to be small prices to pay for the advantage of the feature of one decade of dynamic variation in bandwidth. Nevertheless, in a complex system composed of two or three of these units, such changes might be intolerable. It is repeated that the basic reason for the mentioned unwanted variations is the failure to realize an exact integration characteristic in the inner loops. If dc amplifiers and Miller integrators are employed, these particular variations will probably be eliminated.

As to the relative stability of the system, it has been stated that the inner loops are well stabilized at the maximum value of K. Furthermore, as may be seen from the nature of the open-loop transmission of an inner loop (Figure 6.8), stability at maximum gain implies stability for all smaller values of K, with the relative stability increasing as K is lowered.

The outer loop is stable for a value of loop gain corresponding to a damping ratio of 0.6, and is also stable for smaller loop gains corresponding to larger values of damping ratio. If the gain of the outer loop is raised six decibels above the value corresponding to a damping ratio of 0.6, the system will become unstable. The system can be employed to yield any damping ratio between 0.6 and unity if desired, with suitable relative stability.

6.5.7. Extension of the Range of the Relative-Damping Ratio

While the system under discussion is not intended to be a system with dynamic control over both the bandwidth and the damping ratio ζ , such a system could be realized by resort to the basic principles of the synthesis procedure. However, for the system under consideration, it might be desirable to set the damping ratio manually at some value lower than 0.6 in certain applications requiring a sharper high-frequency cutoff, and it is desirable to ascertain whether this can be achieved by some modification of the original design. A value of $\zeta = 0.3$ is chosen as the goal toward which the modification is directed.

The chief problem in the modification is the problem of stability of the outer loop. It can be shown from (6.6) that the required value of the loop-gain parameter d to give a damping ratio of 0.3 is 10.1, which leads to a value for the gain-constant G_2 (of Equation (6.19)) of 10.5. It will be recalled that the corresponding value of G_2 when the damping ratio $\zeta = 0.6$ is 1.85, and that a six-decibel increase in the gain of loop III will result in instability. Since the required increase for the lower damping ratio is of the order of fifteen decibels, the scheme previously employed must be modified, or a new approach developed.

On the basis of experience gained in the synthesis of the system with $\zeta = 0.6$, the only possibility for obtaining satisfactory transmission with a simple coupling network of the type previously used involves relocating the pole of the coupling-network transfer function much farther to the left in the s-plane. This relocation causes the pole to affect the transmission of the open loop to some extent in the midband region. The details of the relocation involve reference to graphical procedures, approximate computations, and to the "feel" for the system performance obtained through previous considerations. It is therefore expedient to state simply that the chosen transfer function for the R-C coupling network in question is

$$\frac{s}{s + 1{,}000}$$

and indicate the effects of this choice by presenting typical system performance curves. The solid lines in Figures 6.19 and 6.20 are typical curves of the frequency response of the system. The loop gain and phase are indicated by dashed lines. It should be noted that the price of moving the pole of the coupling network into a position where it affects the midband open-loop transmission is an increase in the low-frequency cutoff value, although the low-frequency cutoff remains below 100 cps. It may be shown that with this pole location adequate stability margins are maintained throughout the range of variation of K.

It is again of interest to compare sensitivities of the active system and the corresponding R-L-C passive system (Figure 6.7), for the new condition that the damping ratio is 0.3 at $f = f_n$. For this

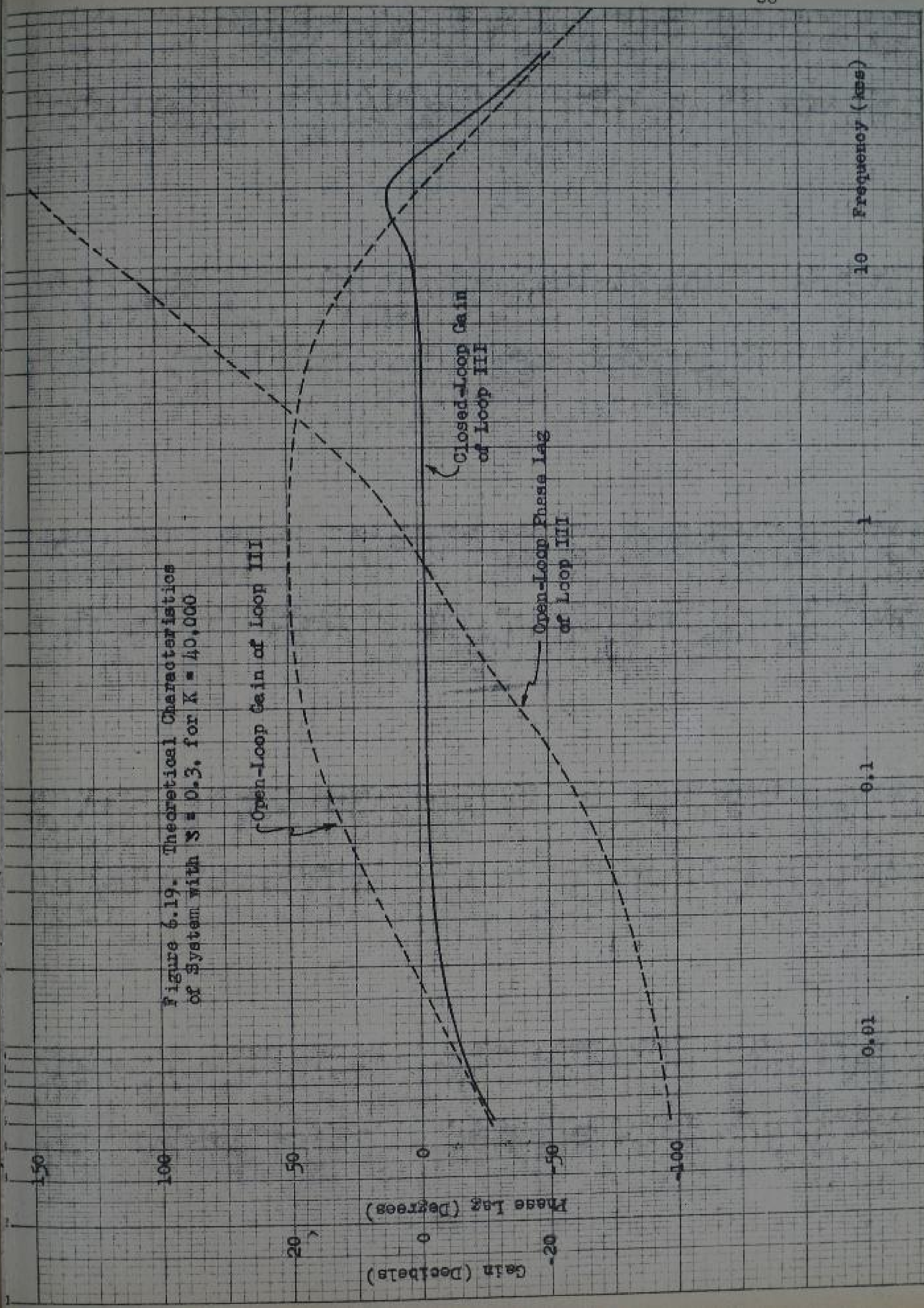

Figure 6.19. Theoretical Characteristics of System with ζ = 0.3, for K = 40,000

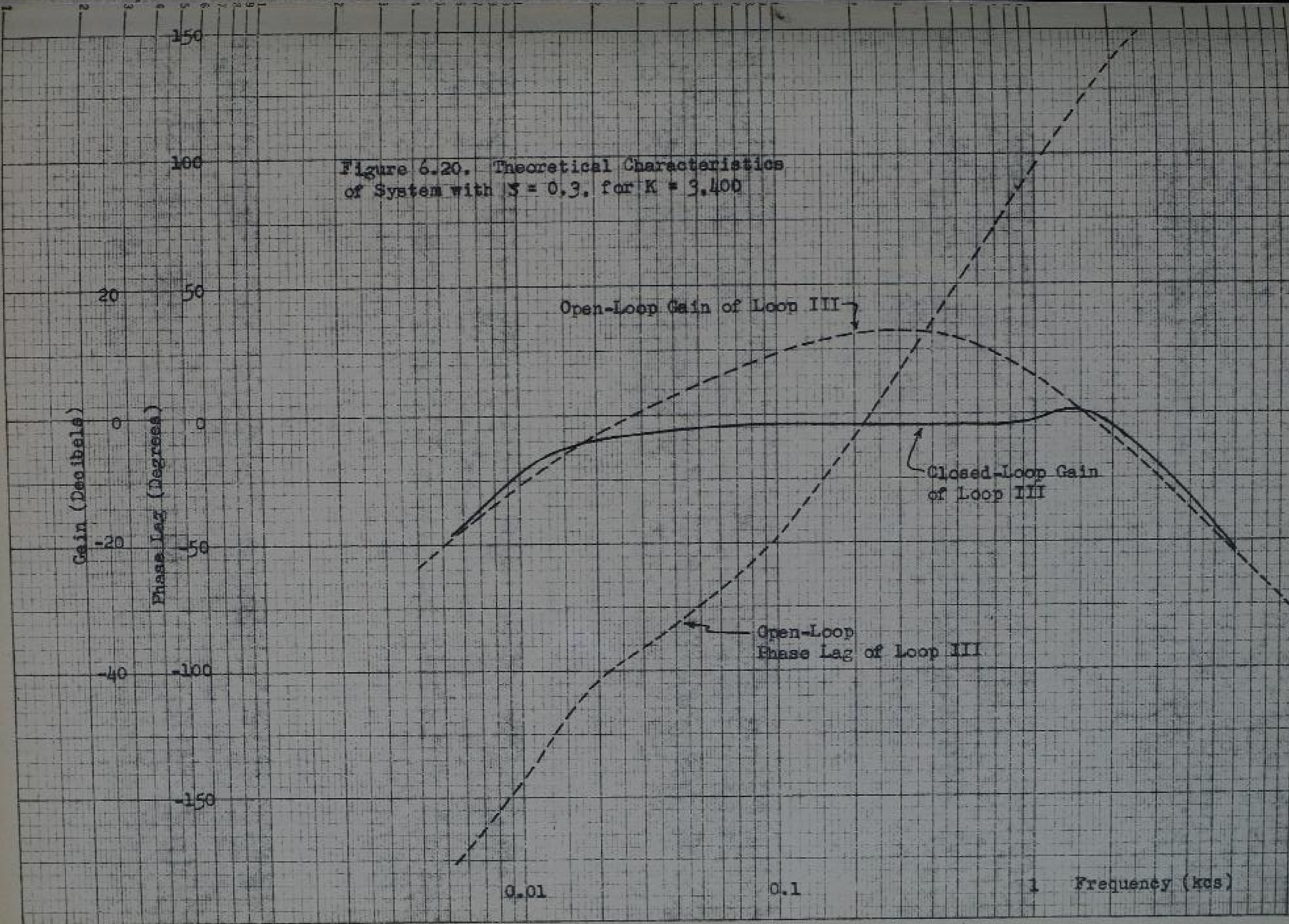

Figure 6.20. Theoretical Characteristics of System with $\zeta = 0.3$, for K = 3,400

purpose (6.10), (6.11) and (4.3) with values from Figures 6.5 and 6.6 are employed to obtain Table 6.4.

Table 6.4. Comparison of Sensitivities of Active System and Passive System to Changes in Element Values for a Damping Ratio of 0.3 at $f = f_n$

Passive System		Active System	
Element	System-Sensitivity	Element	System-Sensitivity
L	$2.04\underline{/-55^\circ}$	b	$1.46\underline{/11.9^\circ}$
C	$2.75\underline{/217^\circ}$	d	$2.50\underline{/37.5^\circ}$
R	$3.35\underline{/180^\circ}$	K	$3.50\underline{/73^\circ}$

It may be noted by comparing Tables 6.1 and 6.4 that the sensitivities are all higher in absolute value for the condition of a lower relative-damping ratio. A simple reason for the fact just stated is not apparent. While the principal partial sensitivities increase in magnitude as the relative-damping ratio is lowered, this fact alone cannot be construed to account for the higher overall sensitivities, since the weighting of the principal partial sensitivities by the subsidiary partial sensitivities is a complicated process (although simply computed).

The absolute values of sensitivity obtained for the active system, although somewhat lower than for the passive system (with the exception of the variable gain K), are higher than might be desirable in long-term applications. The values tend to indicate that tube replacement might necessitate complete recalibration of the system. It should be noted, however, that if the sensitivities computed for K and d were not rather large it would be difficult to obtain a large range of variation of the bandwidth and damping ratio. The absolute value of the sensitivity to change in the loop-gain parameter b indicates that if differences between loop gains of loops I and II occur in practice, a measurable deviation between predicted and experimental results may result in the vicinity of cutoff.

The comparatively lower values of sensitivity in the active system as opposed to those for the passive system again illustrate the remarks of Section 4.3 to the effect that active systems may be (in theory) less sensitive to changes in parameters than passive systems, and illustrate the conclusion that values of sensitivity are influenced in large measure by the functional relationship between system-parameters (such as ω_n and ζ) and the elements of the system.

6.5.8. Resume of Performance of the System with an Extended Range of Variation of the Relative-Damping Ratio.

The system performance, upon relocation of the pole of the coupling network external to loops I and II from (-20) to (-1,000) is improved from the standpoint of versatility. The damping ratio may be set between limits of 0.3 and 1.0 by adjusting the gain of loop III with stable operation anticipated between these limits. Figures 6.19 and 6.20 indicate that a decrease in midband gain and damping ratio accompanies a decrease in bandwidth.

6.5.9. A Circuit Diagram of the System

A circuit diagram of the system is given in Figure 6.21 which shows how the system was instrumented for test purposes. Tubes V_2 and V_3 and the associated coupling networks are identifiable with loop I. Tubes V_4 and V_5 and the associated coupling networks are identifiable with loop II. Tube V_1 serves as the differential amplifier element in loop III. The cathode follower arrangement employing V_6 makes possible a low output impedance which is convenient for test purposes and for isolating the system from a load in the practical application of the system.

The combination of tubes and plate-load resistors in loops I and II is such that no voltage division is required to establish the proper range of loop gain values. Although the entire output of these loops is returned to the differential amplifier, a gain of the order of 2 is available because of the asymmetrical action of the differential amplifier on the two input voltages. In loop III voltage division is required to establish the proper loop gain. This division is achieved at the output of V_5 with the 40,000-ohm potentiometer adjustment serving to determine the loop gain. The capacitor C_t compensates the voltage divider.

The two values indicated for the coupling capacitor following V_1 correspond to pole locations at (-20) and (-1,000) in the transfer function of the coupling network. If the value of 0.05 microfarads is employed, the minimum value of damping ratio is 0.6 and the pass band extends down to about 3 cps., while if the value of 0.001 microfarads is used, the minimum value of damping ratio is 0.3 and the

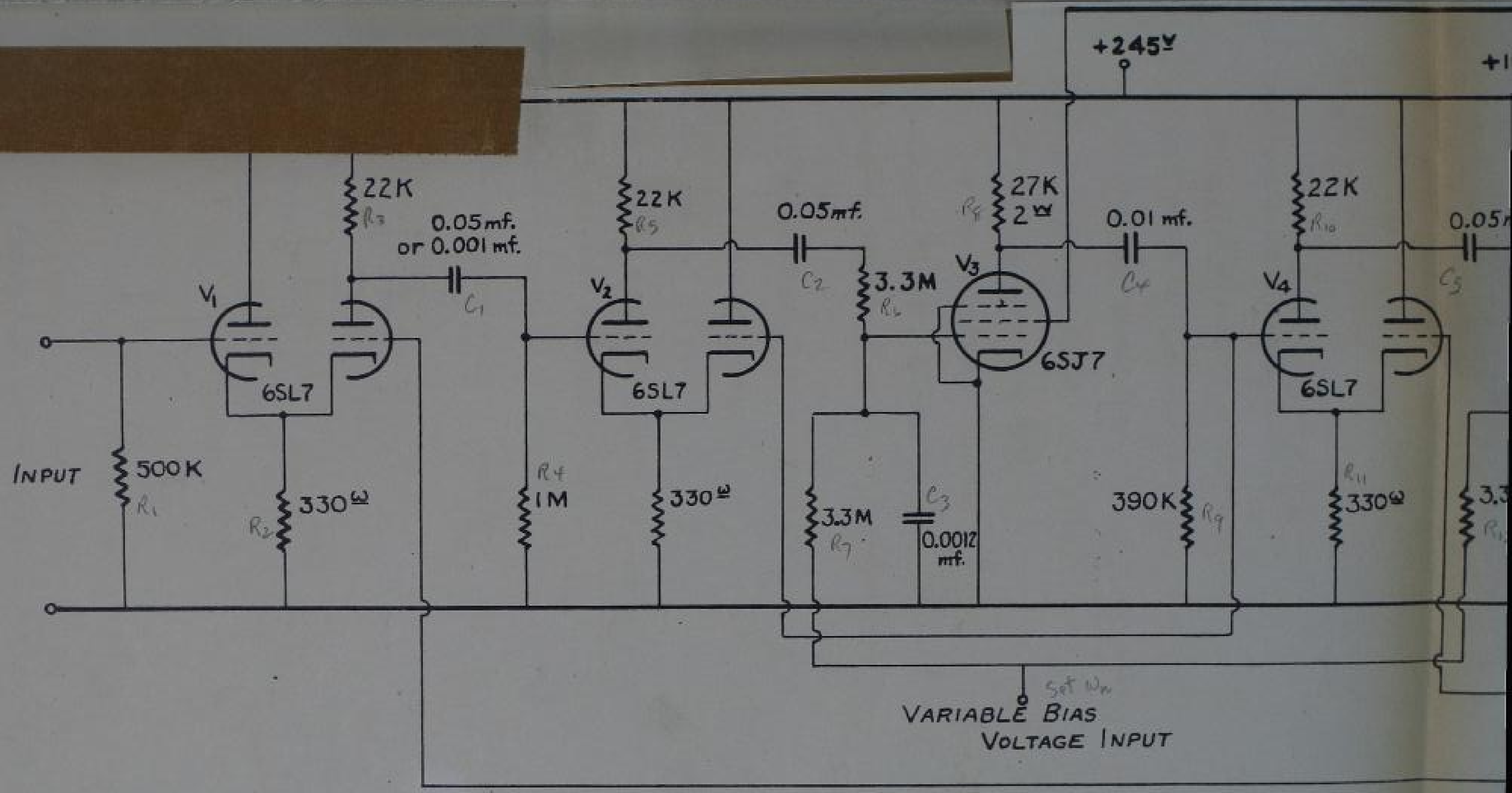

FIG. 6.21. CIRCUIT DIAGRAM OF DYNAMICALLY-VARIABLE ELEC

pass band extends down to about 10 cps.

As a general summary of the philosophy employed to obtain the elements of the circuit diagram, it may be stated that the active-network synthesis procedures of Chapters III and IV were employed, tempered by the result of a sensitivity analysis of the type discussed in section 4.3. Passive-network synthesis techniques were employed when it appeared feasible. Compromise played a significant role throughout. Finally, experimental methods were employed to determine the correct setting of loop gains for the proper loop transmissions.

6.5.10 Experimental Methods and Test Results

The laboratory procedure described in this section serves three purposes: the desired (theoretical) values of loop gains are set within operating limits; the performance of individual loops is compared with the desired performance; and finally, the overall performance of the system is established.

The test arrangement is shown in Figure 6.22. The oscillator

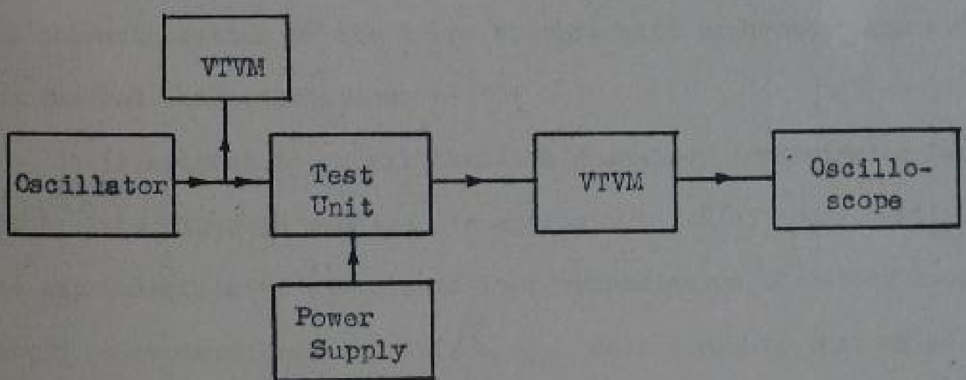

Figure 6.22. Block Diagram of Test Arrangement

supplies a sinusoidal input to the test unit and the rms value of the input is read on the vacuum-tube voltmeter. The rms value of the output of the test unit is read on a second vacuum-tube voltmeter and the waveform of the output is monitored by use of a cathode-ray oscilloscope. The power supply provides plate, screen, and sixty-cycle-per-second heater voltages, as well as a variable bias voltage which permits the gain of the pentodes to be varied. The latter voltage simulates the dc control voltage which can be introduced to give dynamic variation of the bandwidth of the system.

It will be recalled that the two inner loops of the system are identical, in theory, but in practice unavoidable differences exist. The principal difference between the two loops results from the fact that vacuum tubes of the same type (e.g., two 6SJ7's) do not give identical performance under equivalent external operating conditions. If the performance of the system is to be considered in the light of the theory, it is desirable to test loops I and II individually for open- and closed-loop conditions to determine how the characteristics of the loops compare with each other and with the desired characteristics.

It is helpful to recall that the open-loop transmission for either inner loop is required to approximate $-K/s$; consequently, the approximation to the closed-loop transmission of either loop should be proportional to $K/(s + K)$. This suggests that an additional test for purposes of comparison of the inner loops be made to observe the variation of the 3-db bandwidth of each loop as a function of the control voltage (variable bias voltage) applied to the control grid of the pentode in the loop.

Tests were performed on both inner loops to determine the amplitude characteristic of the open-loop frequency response and of the response with the loop closed. Tests were also performed on both loops to determine the variation in the 3-db bandwidth as a function of control voltage. The experimental amplitude characteristics for loops I and II are given in Figures 6.23 and 6.24, respectively. Figure 6.25 shows the variation in 3-db bandwidth with control voltage for loops I and II.

Figures 6.23 and 6.24 serve to establish that the experimental open-loop integration characteristics agree very closely with the theoretical characteristic expressed in analytical form in (6.14). The closed-loop characteristics are also revealed to be of the desired form as indicated by the slope of the high-frequency asymptote, the flat pass band, and the virtual coincidence of the 3-db bandwidth with the open-loop zero-db frequency value.

A comparison of Figures 6.23 and 6.24 reveals that the two loops are very similar in regard to transmission characteristics with the most noticeable discrepancy consisting of a difference of approximately six-tenths of a decibel in midband gain. A more significant difference in the performance of the two loops is revealed in Figure 6.25. The variation in 3-db bandwidth with dc control voltage is noticeably different in the two inner loops. This result is attributed to the fact that the variation of transconductance with control-grid bias is different for the two 6SJ7 tubes.

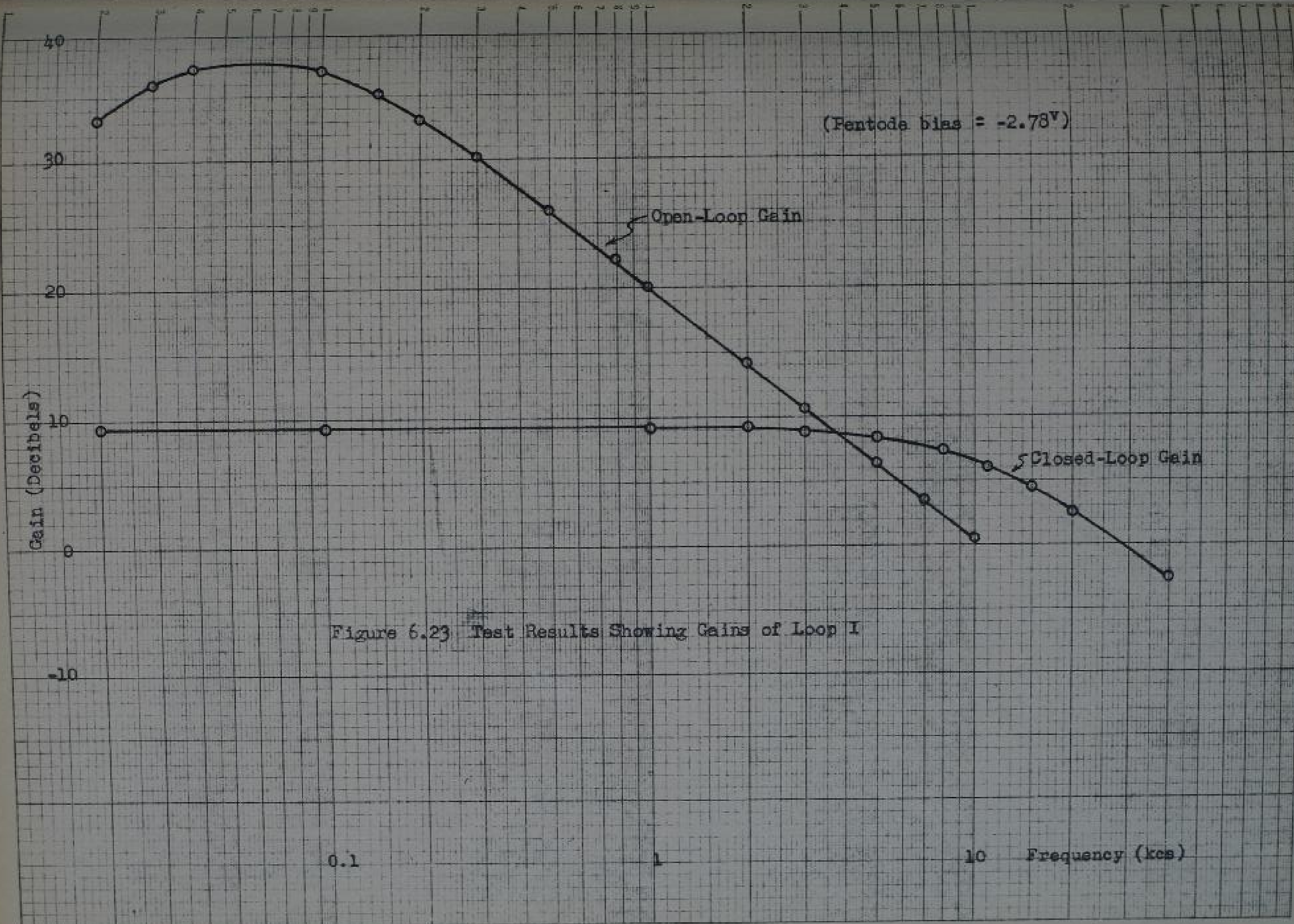

Figure 6.23 Test Results Showing Gains of Loop I

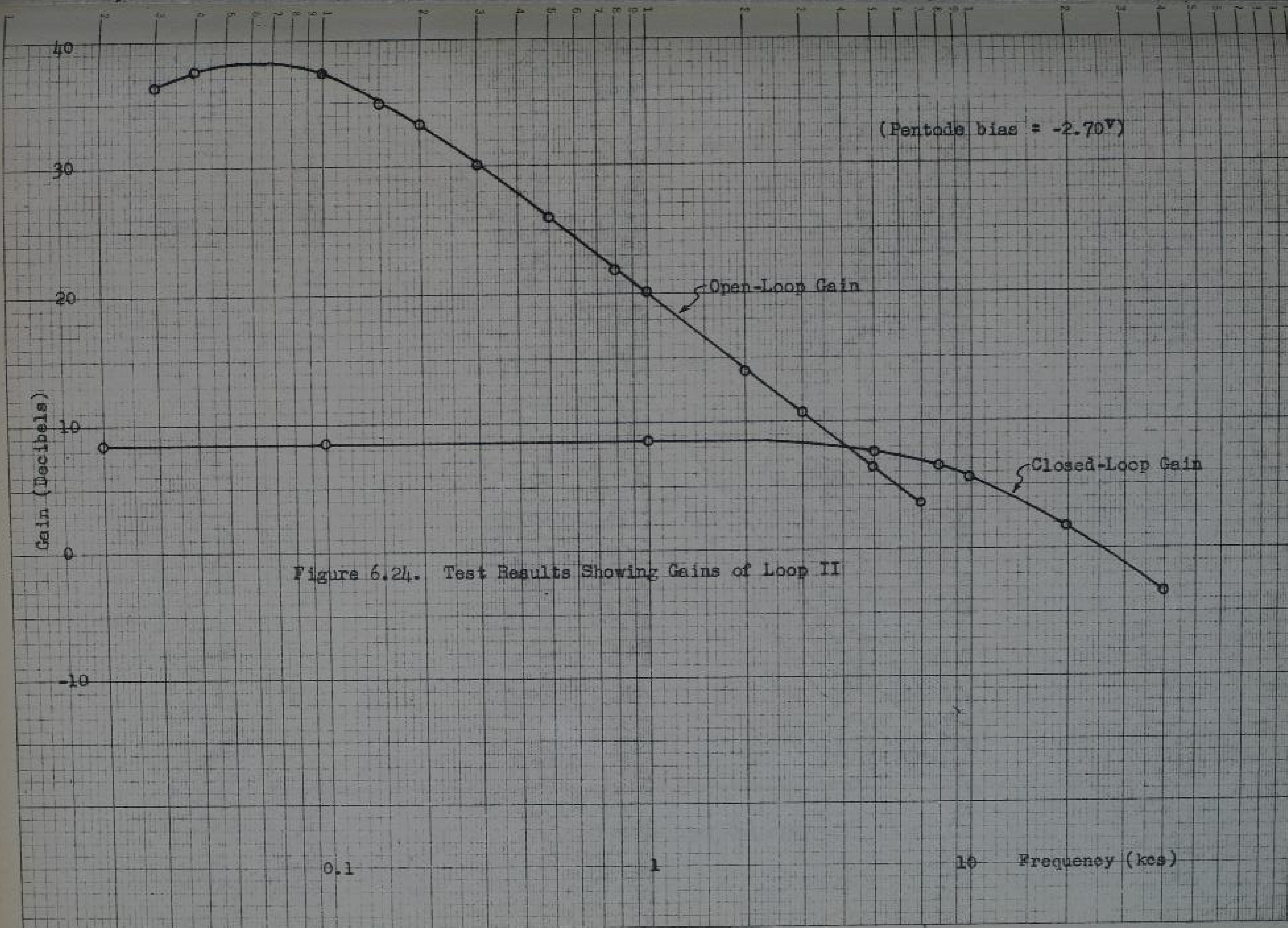

Figure 6.24. Test Results Showing Gains of Loop II

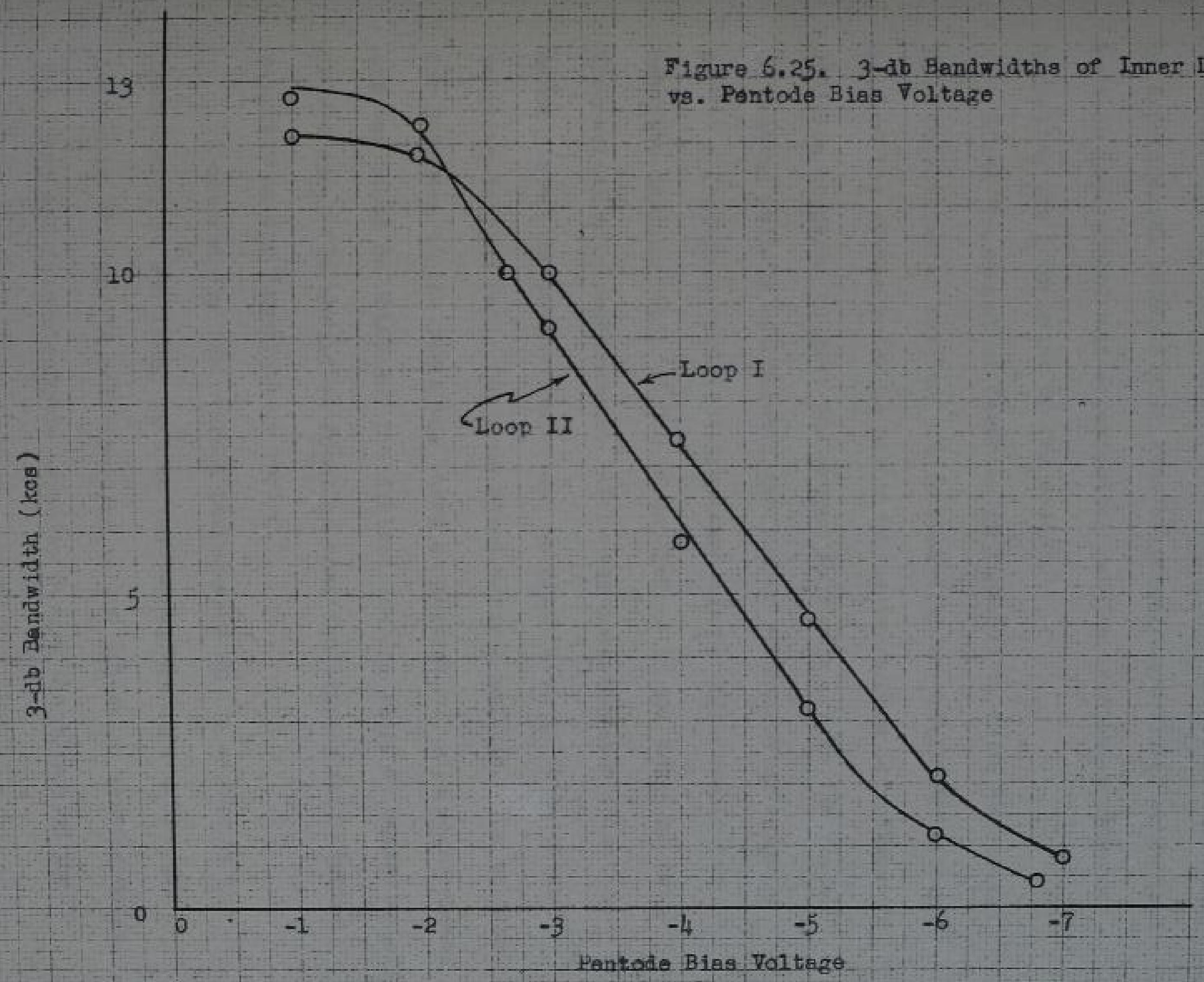

Figure 6.25. 3-db Bandwidths of Inner Loops vs. Pentode Bias Voltage

Before concluding the remarks on the tests of the inner loops it is noted for completeness that the tests were made with the larger (0.05 microfarad) capacity in the coupling network associated with tube V_1 of Figure 6.21. This enables accurate determination of experimental midband gain thereby permitting a closer comparison between theoretical and experimental results, for the purpose of adjusting the loop gains of the system.

Since the amount of feedback in loop III is to be determined by a potentiometer setting, it is essential to obtain the midband gain of the system prior to establishing the amount of voltage division required to give the proper loop gain. When the value of midband gain is established, the proper potentiometer range to permit a loop-gain variation from zero to the value needed for a relative-damping ratio of 0.3 is determined on the basis of the theoretical gain requirement. After the potentiometer selection takes place, the potentiometer may be calibrated in terms of the relative-damping ratio for some fixed control voltage. The calibration for purposes of the tests was based on theoretical gain requirements, and was based on obtaining the desired damping ratio at approximately 10,000 cps. (It should be recalled that the damping ratio varies slightly as the bandwidth is varied.)

In determining the overall performance of the system in the laboratory, it was found that at the relatively high values of loop gain of loop III corresponding to low values of damping ratio, the stray and interelectrode capacity across the output of the potentiometer caused undesirable deviation from the predicted amplitude characteristic (in the form of an excessive peak prior to high-frequency cutoff), while at lower values of loop gain corresponding to higher values of damping ratio the transmission was insensitive to the parasitic capacity and the anticipated results were obtained. Accordingly the voltage divider was compensated at the value of loop gain corresponding to a damping ratio of 0.3.

As discussed in Section 6.5.7, in order to obtain a suitable degree of relative stability under the condition that $\zeta = 0.3$ it is desirable to employ a pole at $s = (-1{,}000)$, corresponding to the capacitor of 0.001 microfarads in the coupling network following tube V_1 in Figure 6.21. The tests of overall performance were conducted with this value of capacity installed.

The amplitude-vs-frequency characteristic of the complete system was obtained at representative values of bias voltage for nominal values of damping ratio of 0.6 and 0.3. These results appear in Figures 6.26 and 6.27. The variation of bandwidth f_n with control voltage is shown in Figure 6.28. The variation in relative-damping ratio with bandwidth is indicated in Figure 6.29 for two nominal settings of the potentiometer in loop III. The variation in midband gain with control voltage is shown in Figure 6.30.

A number of figures showing the essential features of the performance have just been presented. It is now desirable to survey the results of the tests and determine what has been accomplished and established in the tests. It is also desirable to consider possible improvements, based on the knowledge of established performance. These considerations are taken up in the next section.

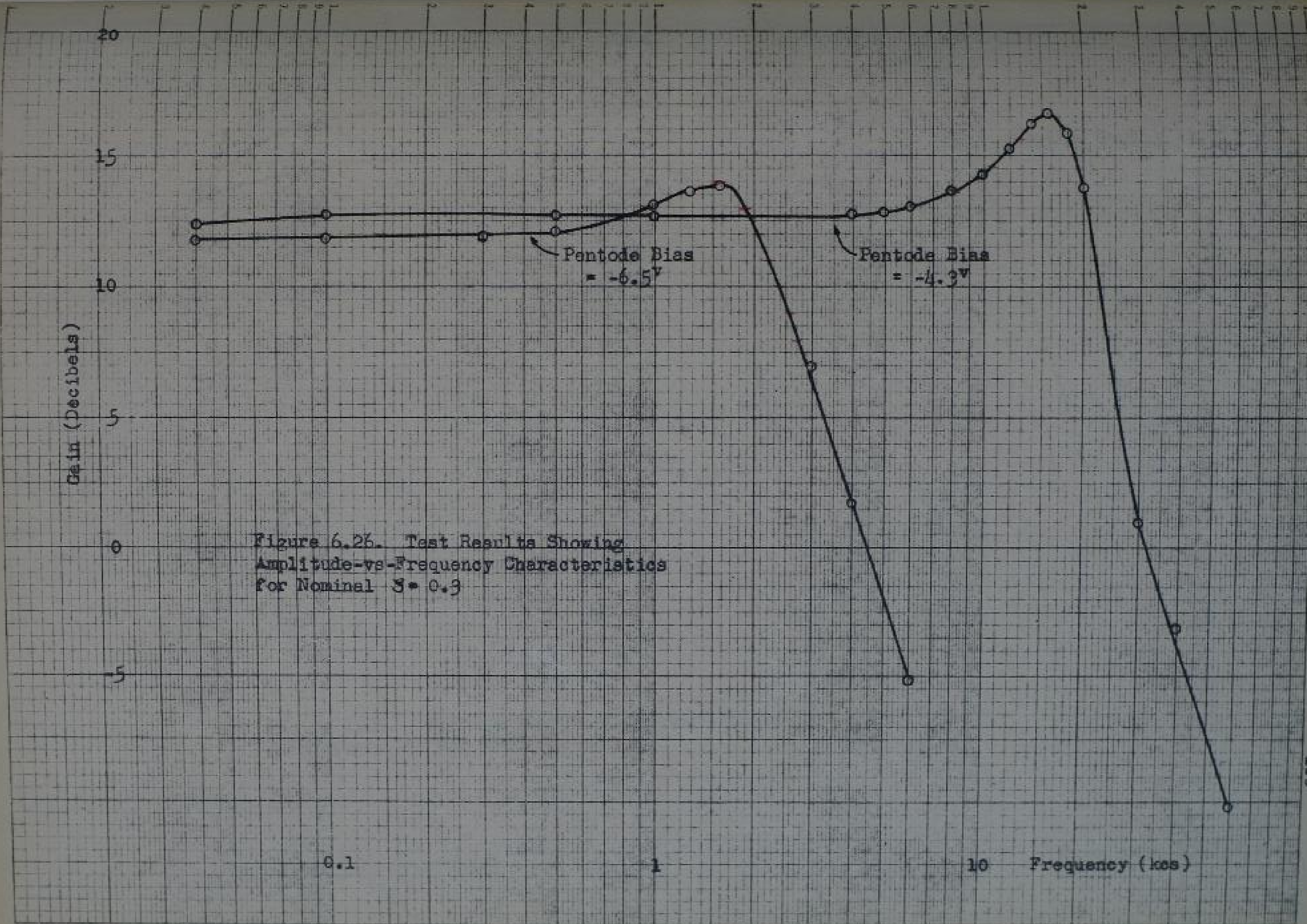

Figure 6.26. Test Results Showing
Amplitude-vs-Frequency Characteristics
for Nominal δ = 0.3

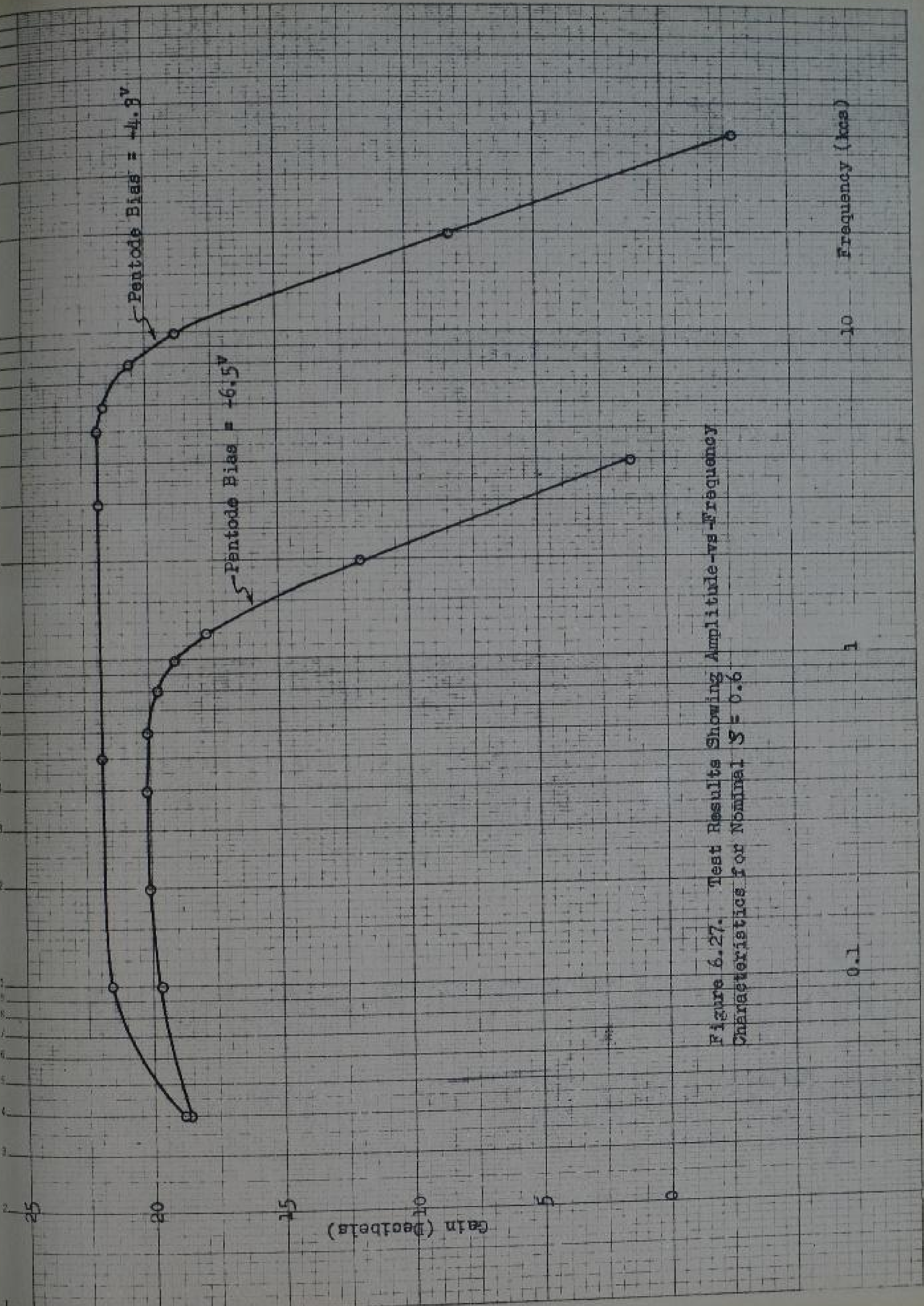

Figure 6.27. Test Results Showing Amplitude-vs-Frequency Characteristics for Nominal $\zeta = 0.6$

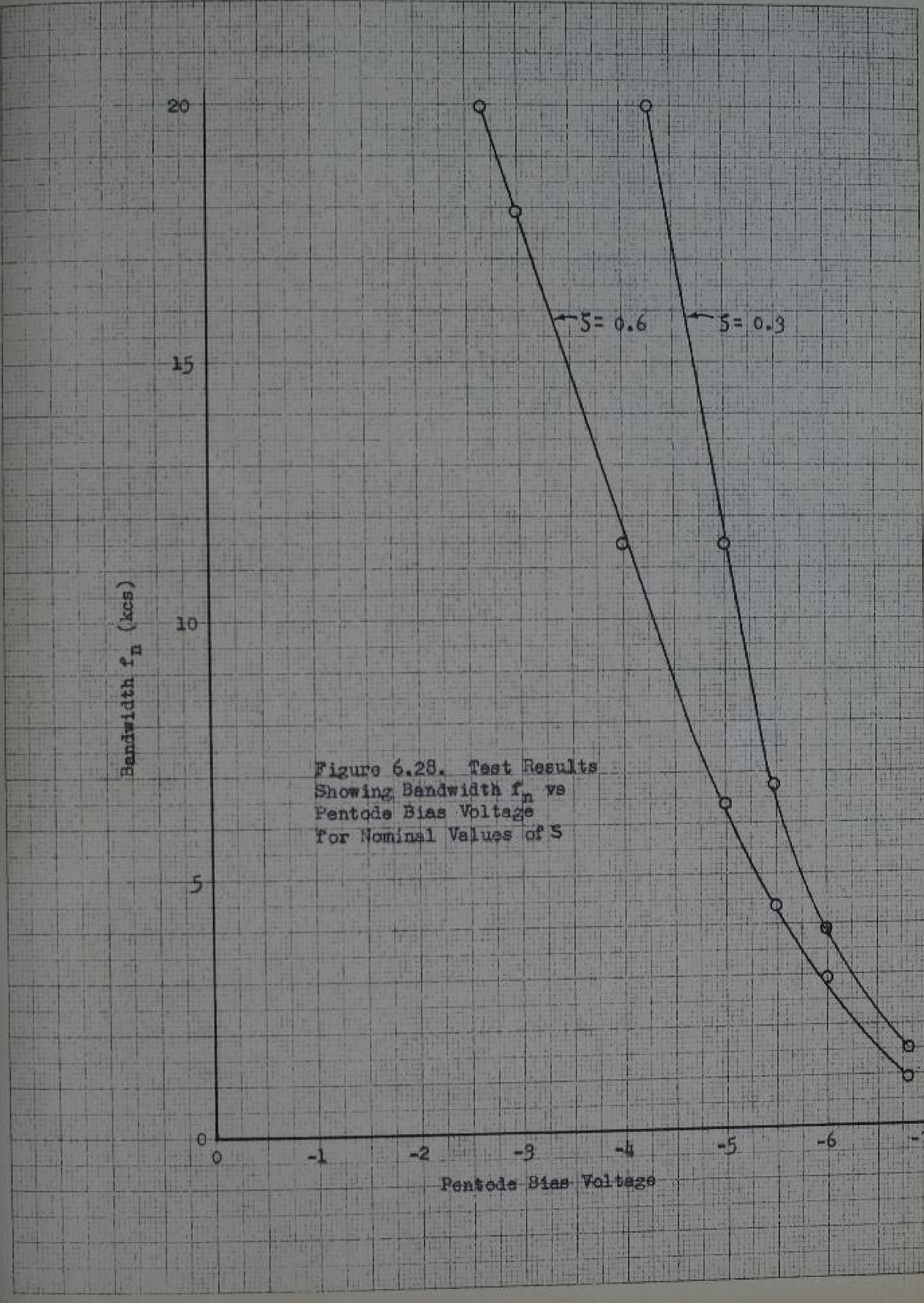

Figure 6.28. Test Results Showing Bandwidth f_n vs Pentode Bias Voltage for Nominal Values of ζ

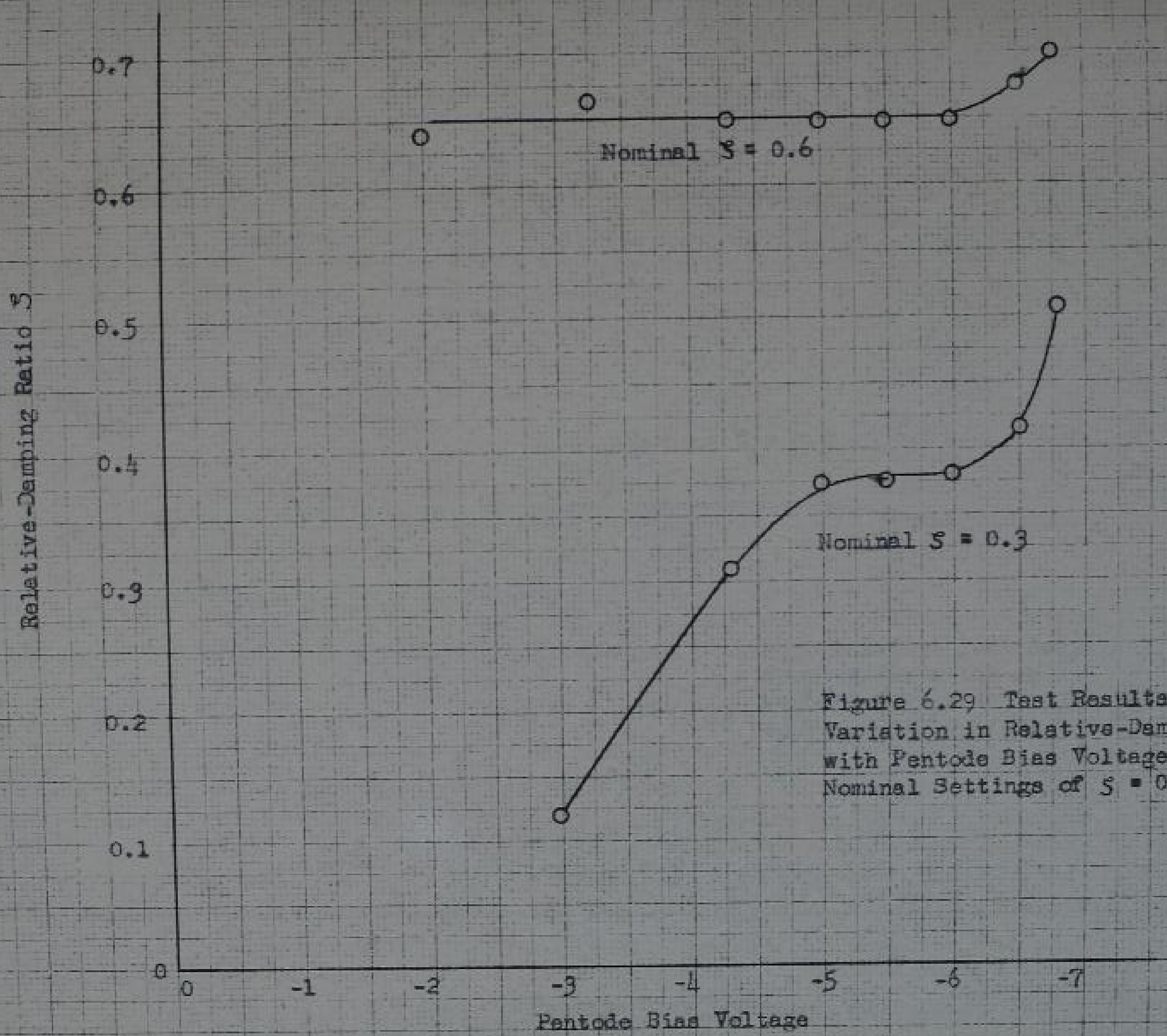

Figure 6.29 Test Results Showing Variation in Relative-Damping Ratio with Pentode Bias Voltage for Nominal Settings of ζ = 0.3 and 0.6

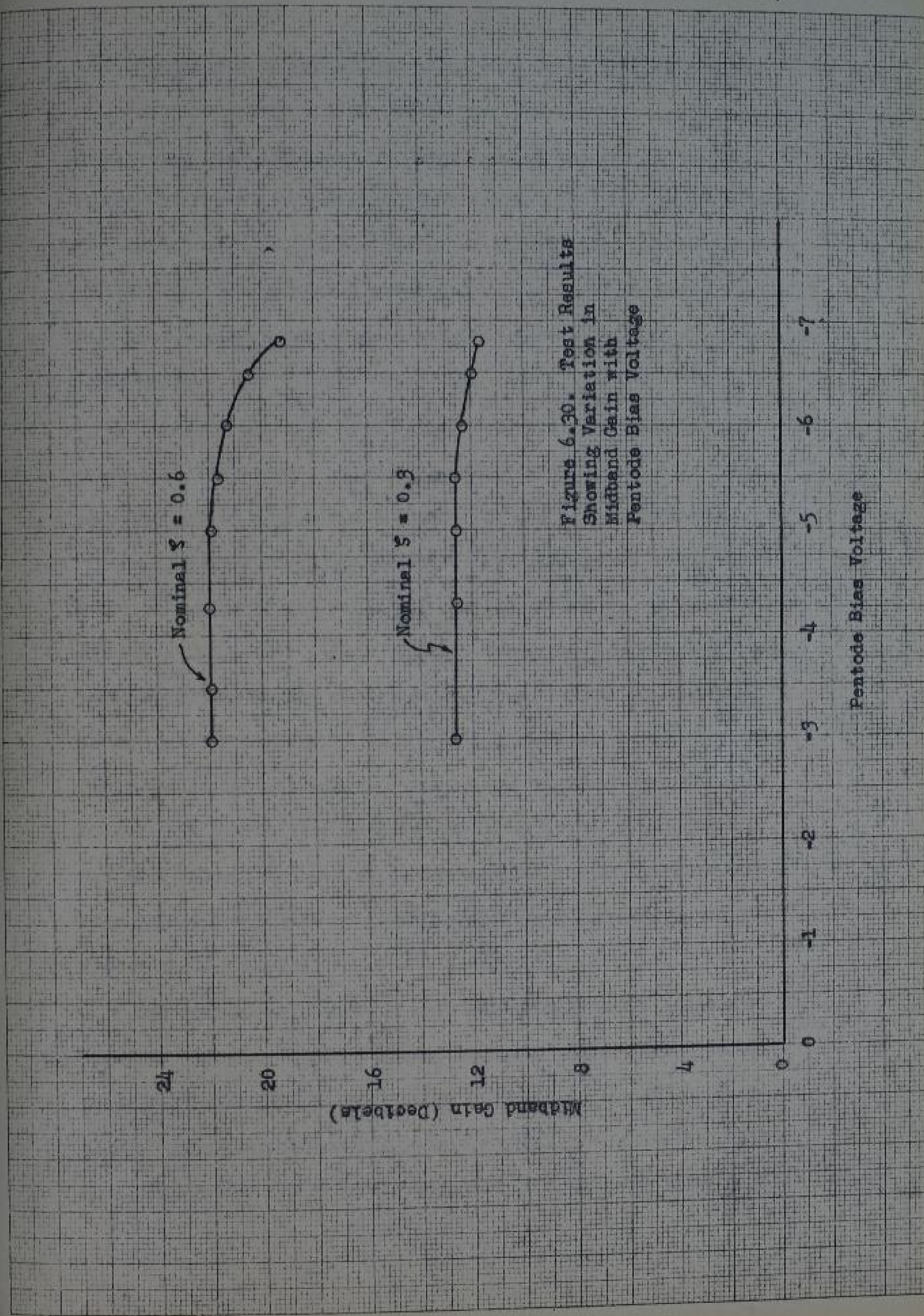

Figure 6.30. Test Results Showing Variation in Midband Gain with Pentode Bias Voltage

6.6. Discussion of System Performance in Terms of Test Results

From the standpoint of overall performance, the system behaves as expected from the theory. Comparison of the theoretical and experimental amplitude-vs-frequency characteristics justifies the statement just made. These characteristics, given in Figures 6.17 and 6.27 for $\zeta = 0.6$ and Figures 6.19, 6.20 and 6.26 for $\zeta = 0.3$, show that the test performance compares favorably with the predicted performance.

It may be observed from Figure 6.28 that the variation of bandwidth with control voltage deviates from linearity at large negative values of control voltage. It may also be observed from Figure 6.29 that the damping ratio corresponding to a setting at a nominal value of 0.3 falls off sharply at low values of bias voltage. The latter effect occurs outside the desired frequency band, while the former occurs at the lower values of bandwidth. Accordingly, both effects may be largely eliminated by employing resistors in the cathode circuits of the variable-gain pentodes to reduce the available gain in the inner loops. The effect of the gain reduction is to shift the desired frequency range more nearly into the region of linear variation of overall bandwidth with control voltage.

The variation in midband gain with bias voltage, indicated in Figure 6.30, is in general accord with the theoretical results. It is seen that at a nominal value of $\zeta = 0.6$, the decrease in midband gain over the useful range is about two decibels. With a nominal value of $\zeta = 0.3$, the decrease over the same range of bandwidth is less than one decibel.

As to operation under overload, it was found that the system would become unstable at low values of damping ratio if the overload were of the order of twenty-five hundred percent of the nominal test input of twenty millivolts. The oscillations under the overload condition were low-frequency oscillations. Upon removal of the overload the system returned rapidly to the stable condition.

To summarize the performance, it may be said that the behavior is as expected from the theory. The bandwidth can be varied dynamically over more than a decade in the nominal range of frequencies extending from 10,000 cps to 1,000 cps. The relative-damping ratio may be preset between limits of 0.3 and 1.0 by a single potentiometer adjustment to control the rate of cutoff of the system. Stable operation may be expected within the stated limits, provided extreme overloads do not occur. Reasonably linear variation of bandwidth with control voltage may be obtained. Slight variations in midband gain and rate of cutoff as the bandwidth changes are a result of the scheme used to provide integration in the inner loops. The lower cutoff frequency is in the vicinity of 10 cps, but can be lowered to about 3 cps if sharp cutoff is not required at the high-frequency end of the amplitude-vs-frequency characteristic.

6.7 Summary and Conclusions

In this chapter an illustrative design has been presented for the purpose of demonstrating the utility of the synthesis procedure in a practical setting. While the example is long and detailed, it is felt that a complete description is more likely to serve the purpose of many readers than a brief and incomplete discussion. Proceeding from the specifications, the transmission function was chosen. The transmission form was selected and a preliminary block diagram of the system was drawn. The physical realization of the system was carried out in detail and illustrated problems likely to be encountered in the realization of other slowly-variable systems, as well as procedures for dealing with the problems. In order to substantiate the theory, the system was built and tested. It was shown that the system performed substantially as expected on the basis of the theory. Although not emphasized in the previous discussion, it is felt that the system devised may be of practical utility. A more refined model of the system could, for example, be employed to study experimentally filtering problems of a statistical nature of the type described in Appendix I.

VII. RESULTS AND CONCLUSIONS

7.1. The Development and Application of the Synthesis Procedure

This thesis presents a discussion of the synthesis of slowly-variable systems. Early in the discussion it is concluded that if a slowly-variable system is to perform effectively in a filtering application, the distortion removed by virtue of the variation must exceed that which might be introduced by virtue of the variation; hence it is logical to proceed with the synthesis on a quasi-stationary basis, employing the familiar complex-frequency variable s with its full connotations carried over from the synthesis of fixed systems.

Methods for obtaining a rational function which incorporates the specifications are considered. It is concluded that while the Wiener o-m-s theory is, in general, unsatisfactory from the standpoint of providing (directly) a suitable function for representation of the specifications, the theory is useful in evaluating the performance of systems with transfer functions chosen from familiar sets of pole-zero configurations or obtained by approximation methods, since the performance of a given system can be compared with the o-m-s system on the basis of the mean-square-error criterion. It is further concluded that the Wiener o-m-s theory is useful in establishing the nature of the physical parameters which may be used to control the variation in a slowly-variable system.

The requirement that the parameters of a slowly-variable system be varied dynamically leads naturally to the choice of active systems as the basis for the synthesis of slowly-variable systems. Because no suitable procedure for the synthesis of active linear systems is available in a form readily adaptable to meet the needs of the discussion, a procedure for the synthesis of active systems is developed, stressing the use of signal-flow graphs.

The employment of active systems as the basis for the synthesis of slowly-variable systems suggests a consideration of practical factors associated with the realization of active systems. In particular, the question of sensitivity is considered in some detail. It is concluded that by appropriate choice of a system from the available systems, a system with comparatively low values of sensitivity can be obtained, although if it is desired that the system transmission vary with some particular element, it will be necessary, in general, to accept a moderately high value of sensitivity to changes in that element as a natural consequence of the specifications.

It is concluded that, while a pentode may be satisfactory for many applications as a variable-gain device, if the required variation in a parameter with a control voltage is other than linear it will be difficult to obtain the variation with any substantial degree of accuracy over a wide range of variation of the parameter.

An illustration of the application of the principles of the synthesis procedure to the realization of a variable-bandwidth filter is presented. It is shown that by employing feedback around integrators and making use of two variable elements in a multiloop arrangement, a system can be developed to provide a pass band extending from 10 cps to an upper cutoff frequency which is dynamically-variable over a decade of frequencies extending upward from 1,000 cps. It is shown that the rate of cutoff in the vicinity of cutoff for this system can be preset to include a wide range of values, as determined by the relative damping ratio of the system. Test results are given which show that the system performs substantially as expected from theoretical considerations.

Two units of the type realized may be placed in tandem to obtain a variable-bandwidth filter, providing a maximally-flat cutoff of the Butterworth type with n = 4. For this purpose one unit should be adjusted to give a relative-damping ratio of 0.924, while the other should give a damping ratio of 0.382.

7.2. Suggestions for Further Investigation

It is believed that the topic of rate of variation in slowly-variable systems should be investigated. One method of studying the rate of variation is through Zadeh's frequency analysis technique[1]. As a beginning to the investigation, the effect of the rate of variation in very simple networks may be considered. Closely allied with this investigation is the theory of modulators. There is a need for a means of distinguishing between a modulator and a slowly-variable filter. In practice reactance-tubes have been employed in both applications.

As the body of network theory grows the need for a unified approach to analysis and synthesis becomes greater. It is felt that signal-flow graphs are an effective means of providing a unified treatment of linear systems. It is therefore suggested that the synthesis of passive systems be considered on the basis of manipulation

1. Zadeh, L. A., "Frequency Analysis of Variable Networks", Proc. IRE 38, 3, March, 1950.

of signal-flow graphs. A detailed treatment of allowed transformations and node-identification procedures would do much to add to the practical application of the active-network synthesis procedures given in this thesis.

As another problem for study, it is suggested that the synthesis of fixed active networks for prescribed subsidiary partial sensitivities be considered. It is possible to synthesize active networks with a prescribed relation between the parameter(s) of the system and the elements of the system. The most familiar practical application of this procedure is the basic feedback amplifier design where a level factor is realized by a high-gain amplifier with a small amount of feedback. The extension of the fundamental notion suggested in this application to the realization of poles and zeros should be quite valuable in the synthesis of fixed active systems.

Further study should be directed to the problem of obtaining a prescribed relation between a parameter-control signal and the parameter which it controls. If such a study is fruitful it will add greatly to the practical value of the techniques given in this thesis for the synthesis of slowly-variable systems.

APPENDIX I

SOME CONSIDERATIONS OF OPTIMUM-MEAN-SQUARE FILTERING WHEN STATISTICAL PARAMETERS VARY SLOWLY

A. Preliminary Comments

In discussing the process of statistical filtering (i.e., filtering where the specifications on the filter have been set through knowledge of the statistics of the signal and noise) a number of newly expostulated ideas are involved. The essence of these ideas is a new approach to an old problem. The extent to which these ideas are applicable to specific physical problems is not clear but the philosophy of the approach is undoubtedly sound. The purpose in presenting this appendix is to consider how this approach might be used to specify a slowly-variable system and illustrate certain difficulties which arise in the realization of a system from the specifications.

B. The Optimum-Mean-Square (O-M-S) Linear System

The term "o-m-s linear system" implies that a system is doing something in the best way possible by linear means. What is the system doing? The mathematical answer, as formulated by Wiener[1] is that it is giving at the output the best possible reproduction on a linear least-square basis of a message m(t) when confronted with an input consisting of a linear combination of the message m(t) and a disturbance n(t) which may broadly be called noise. The system is specified in terms of certain statistical parameters which may be approximated from a sufficient number of samples of the quantities involved. The specification may take the form of the required

1. Wiener, N., The Extrapolation, Interpolation, and Smoothing of Stationary Time Series, The Technology Press and Wiley, 1949

transient response h(t) or the complex-frequency response H(s).

C. Functions Performed by an O-M-S System

The remarks of Section B need to be generalized by introducing the terms "lead" and "lag" as applied to the output of the system. If the input is considered to be a sequence of short pulses, it is evident from the superposition principle that in any realizable linear device the output will always lag the input. Still it may be possible, by performing just the right operation on the input, to cause the system to predict the input, although error must be expected. The o-m-s system is the system which gives the least mean-square error obtainable by linear means when the output is required to lead or lag the message portion of the input by a specified time. If the network has time to "digest" the information received it can do a better job of reproducing the desired function at the output. In other words, if a lag can be allowed the error in filtering is reduced. If an infinite amount of time is allowed for the "digestion" process, the system can do the best possible job of reproducing the desired function by a linear process. The unavoidable error will be the least possible error associated with the given input. Infinite lag, in practice, seems to be just a short time[1]. This is fortunate, since the amount of lag is substantially determined by the delay characteristic of the system, and large delays with linear electrical systems are not very practical. A great deal of attention is being directed to increasing the scope of time delay equipment at present.

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1. Stutt, C. A., Experimental Study of Optimum Filters, Tech. Report No. 182, M. I. T. Research Laboratory of Electronics, May 15, 1951, pp 102 and 113.

To summarize, the system is required to produce the least error on a mean-square basis when a given lead or lag is specified. The amount of lead or lag is determined by the amount of time (positive or negative) after the message appears at the input before the output is used. If the device which receives the output is patient, a lag may be allowed, whereas if the device must anticipate the received message, then a lead must be specified.

D. Some Basic Assumptions

In the material to follow, it is assumed that the cross-correlation between message and disturbance is zero. It is further assumed that the necessary statistical information is available. The entire treatment is on a quasi-stationary basis.

E. Mean-Square Error in O-M-S Filtering

The mean-square error in filtering may conveniently be broken down into four quantities by writing

$$E = E_1 + E_2 + E_3 - E_4 \tag{I.1}$$

where E is the total mean square error

E_1 is the input message power

E_2 is the portion of the output power contributed by the message

E_3 is the portion of the output power contributed by the noise

E_4 is the cross-power between message and system impulse response

and all powers are average powers based on a one-ohm load. Obviously E_1 is independent of the choice of a filter. It is also clear from the descriptions of the quantities that E_2 and E_3 are influenced only by the amplitude-vs-frequency characteristic of the filter.

The portion E_4 of the error depends on both the gain and phase characteristics of the system, as well as the lead or lag specified. If the o-m-s system is considered, it can be shown that E_4 is twice the sum of E_2 and E_3; hence the error for optimum filtering is

$$E_o = E_1 - (E_2 + E_3) \tag{I.2}$$

$$= \text{input message power - total output power.}$$

F. An Illustrative Example

The best way to discuss o-m-s filtering is probably through continuous association with an example. The example selected for illustrative purposes is not necessarily representative of any particular envisioned application. The example is a standard example in the literature[1]. It has not been investigated from the standpoint of variation in the stationary property of the input message and noise. It will now be investigated in that regard.

Consider the problem of separating a message with a power density spectrum given by

$$\phi_m(\omega) = \frac{M^2 b^2}{(\omega^2 + \beta^2)(\omega^2 + b^2\beta^2)} \tag{I.3}$$

from a disturbance consisting of white noise with

$$\phi_n(\omega) = a^2 , \tag{I.4}$$

the separation to be effected by a linear system.

The goal is to arrive at a system to satisfy the following requirements: (1) the mean-square error in filtering shall be

1. Stutt, C. A., Op. cit., p 95.

near the optimum value for any dc message-to-noise power ratio[1] at which effective filtering can be achieved, and for the particular value of lead or lag specified, and (2) the system shall be realizable without unreasonable complexity.

Since the variable parameter in the signal statistics is assumed to be the dc message-to-noise power ratio denoted by the symbol W^2, this ratio must vary slowly if the spectra are to retain the significance attached to them in the example.

The requirement that the error be near a minimum for a range of values of W demands that this quantity or some quantity closely related to this quantity be susceptible to measurement. In general the type of measurement made will be determined by the particular conditions of the physical situation. In this example if the message and noise can be isolated from each other the measurements can be made, but this separation is just what the system is trying to accomplish, so the possibility is a ridiculous one. On the other hand, it may well be true that a partial separation can be made by sharp filters operating in different portions of the frequency spectrum. The first major problem to be resolved is evidently the problem of measuring a suitable parameter of the input. If this parameter cannot be measured, it is evident that the filter cannot be dynamically controlled to maintain the optimum state.

The second requirement on the system will usually necessitate an approximation to the optimum-mean-square filter, since the transfer function of the o-m-s system may not be realizable (with lumped elements)

1. The dc message-to-noise power ratio is interpreted as the ratio of message power to noise power averaged over a long interval passed by a flat low-pass filter of infinitesimal bandwidth.

and is usually a complicated function of the dc message-to-noise power ratio W^2.

The first step in the attack on the problem is to establish the transfer function (or the impulse response) of the optimum system and the corresponding mean-square error. It may then be determined whether it is feasible to proceed by examining the magnitude of the error. The possibility of realizing the optimum transfer function directly seems remote in the case of slowly-variable systems since the transfer function is a complicated function of the variable parameter. Stutt[1] has given a time-domain synthesis technique for experimentally synthesizing the optimum impulse response on an oscilloscope screen through manipulation of a network. To employ this technique for realizing a slowly-variable system would be quite difficult or impossible, except in simple cases. The system realized according to this technique might be quite complex and tend to violate the requirement on simplicity.

The discussion has led to the next major problem. How can the optimum system function be approximated by a system of reasonable simplicity? This problem is a very difficult one to solve for fixed systems (except in special cases) and is consequently of enormous complexity for the general situation involving several variable parameters. However, if an approximating function is obtained by any means it can be compared with the optimum system through examination of the incremental error δE_f added in filtering by virtue of the approximation. This incremental error may be obtained by use of either of the formulas given in (I.5) in terms of the time domain

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1. Stutt, C. A., _Op. cit._, pp. 32-48.

or the frequency-domain quantities. The incremental error is

$$\delta E_f = \int_0^\infty \left[h_f(t) - h_o(t)\right] dt \int_0^\infty \left[h_f(\sigma) - h_o(\sigma)\right]\left[\phi_m(t-\sigma) + \phi_n(t-\sigma)\right] d\sigma$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \Phi_n \left[A_f^2 - A_o^2\right] d\omega + \int_{-\infty}^{\infty} \Phi_m \left[A_f^2 - A_o^2 + 2A_o \epsilon^{-jB_o'} - 2A_f \epsilon^{-jB_f'}\right] d\omega\right\} \qquad (I.5)$$

where the subscript "f" refers to the approximating filter

the subscript "o" refers to the optimum filter

the subscript "m" refers to the message

the subscript "n" refers to the noise

the symbol h refers to the impulse response

the symbol ϕ refers to the autocorrelation function

the symbol Φ refers to the power density spectrum

the symbol α refers to the specified lead (and is negative for a lag)

the symbol A refers to the amplitude-vs-frequency function

the symbol B refers to the phase-vs-frequency function

and the symbol $B' = B - \alpha\omega$.

If a suitable approximation to the transfer function is established, the approximating system may be analyzed and the required dynamic variation in the system established. Alternately, if the required dynamic variation can be incorporated in the approximation the analysis is circumvented to some extent. One method of attack on the problem of approximation is to plot the pole, zero and level variation in the o-m-s function and attempt to approximate the variation by a somewhat simpler scheme of pole, zero and level variation. If a delay of the form $\epsilon^{-j\omega T}$ is present in the transfer function of the o-m-s system, delay lines may be employed or rational approximations may be used.

To illustrate the problems discussed in the preceding paragraphs, a solution of the illustrative problem is presented.

Case I. Zero Lag Specified

Optimum Transfer Function and Optimum Error. The optimum transfer function can be found from the Wiener theory[1] to be:

$$H(j\omega) = \frac{\beta[X - (b + 1)][j\omega + \beta(X + b + 1)/2}{(j\omega + \beta X/2)^2 + (\beta/4)[X^2 - 2(1 + b^2)]} \qquad (I.6)$$

where
$$X = 2b^{1/2}(1 + W^2)^{1/4}\left[0.5 + \frac{1 + b^2}{4b(1 + W^2)^{1/2}}\right]^{1/2}$$

and
$$W = (\text{dc message to noise ratio})^{1/2} = M/a\beta^2 \quad .$$

The optimum error is given by an expression which is too unwieldy for inclusion, in the general case. For the particular cases considered the optimum error will be expressed either in the form of a graph, or in some cases by an equation.

The Approximating Transfer Function. The transfer function chosen to approximate that of (I.6) on the basis of simplicity and a knowledge of the shape of the message and noise power-density spectra is

$$H_f(s) = \frac{k}{s + k} \qquad (I.7)$$

where $s = \sigma + j\omega$, and the numeric k, as yet undetermined, may be allowed to vary as the dc message-to-noise power ratio changes slowly.

Error for the Approximating Filter

The mean-square error is found for the approximating filter in order that it may be compared with that of the optimum filter

1 Wiener, N., Op. cit., pp. 81-92.

and also considered in its own right. The error for the approximating filter is

$$E_f = \phi_m(0)\left[1 - \frac{\Omega(\Omega+b+1)}{(\Omega+1)(\Omega+b)} + \frac{\Omega(b+1)}{bW^2}\right]$$

or

$$\frac{E_f}{\phi_m(0)} = \frac{b}{(\Omega+1)(\Omega+b)} + \frac{\Omega(b+1)}{bW^2} \qquad (I.8)$$

where $\Omega = k/\beta$ and is the normalized bandwidth of the approximating filter. For a given set of spectral values b and W, the error of the approximating filter can be minimized by appropriate choice of the bandwidth Ω. In general the minimization process requires that a fourth order algebraic equation be solved. However, for the special case b = 1, the value of Ω which minimizes the expression for error is

$$\Omega_1 = W^{2/3} - 1\,. \qquad (I.9)$$

Equation (I.9) gives only limited information in regard to the minimization. It is desirable to know whether the minimum is sharply defined or relatively flat. This is of importance in the approximation of the expression for Ω_1 given in (I.9) by a simpler function of W. It is therefore desirable to plot the error for various values of W. This is done in Figure I.1 for the case b = 1. Figure I.1 shows that the minimum is relatively flat for reasonably large values of the message-to-noise level, but as the level approaches low values the bandwidth for minimum error becomes rather sharply defined. On the other hand, it is in the region of relatively low level that the error tends to be greatest, and there is some question whether satisfactory filtering can be achieved, even with the most suitable bandwidth.

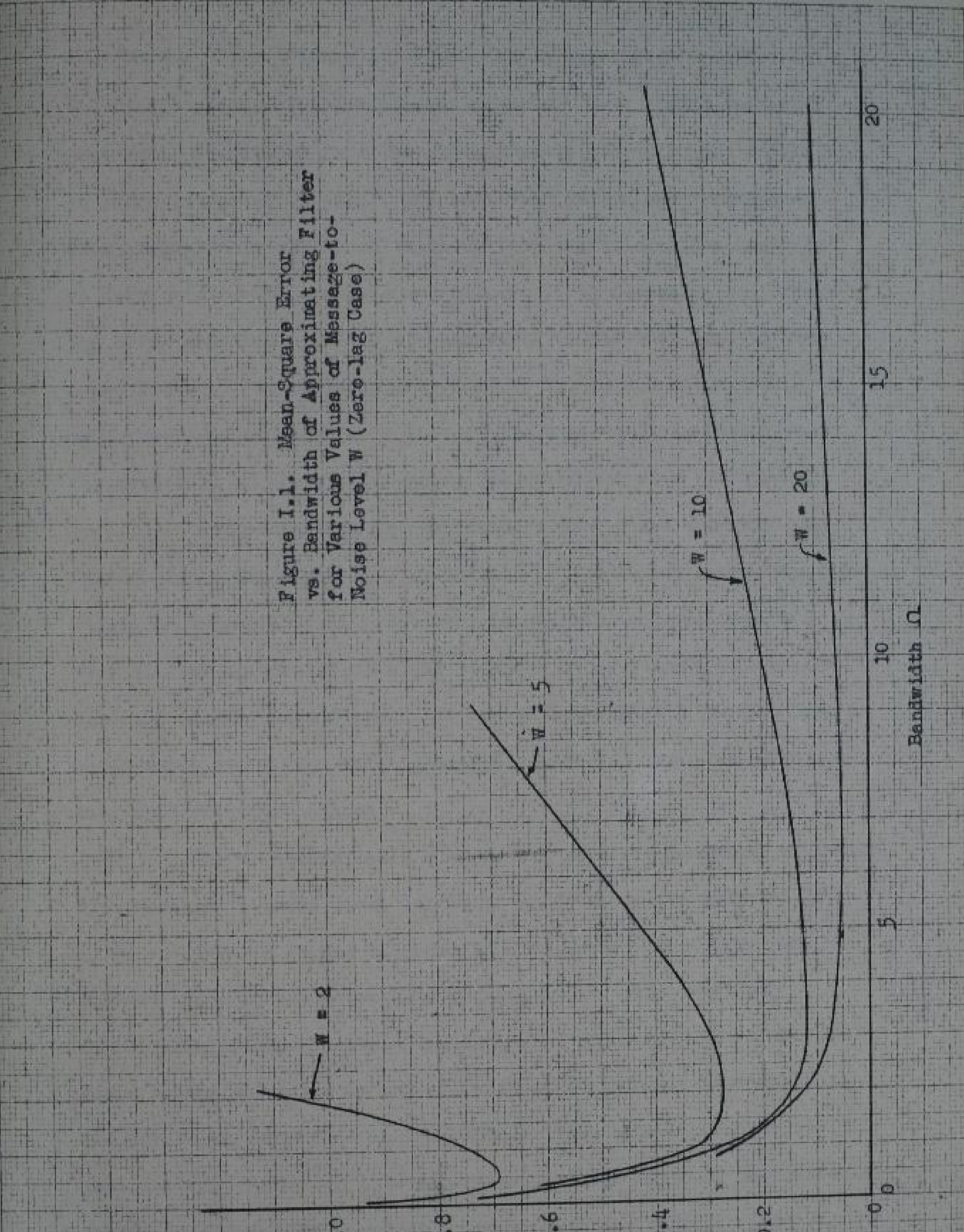

Figure I.1. Mean-Square Error vs. Bandwidth of Approximating Filter for Various Values of Message-to-Noise Level W (Zero-lag Case)

On the whole, Figure I.1 gives some promising information about the final outcome of the synthesis process. This figure indicates that small deviations from the optimum bandwidth would not result in large increases in error, and indicates that such phenomena as slow variation in supply voltages or tube characteristics need not be deletorious. To complete the picture it is desirable to compare the error obtained with the approximating filter with that of the o-m-s filter. The comparison is made in Figure I.2 where the o-m-s performance is compared with that of the approximating filter when the bandwidth of the approximating filter varies with W according to (I.9). For comparison the performance of a fixed approximating filter with a bandwidth of 2 is included. The figure shows that for values of W greater than 5 the variable filter achieves results very similar to results obtained with the o-m-s filter, and is clearly superior to the fixed filter. It may be shown that a similar result obtains for b = 2, for which the same trends are apparent, although they are not as pronounced.

In order to more closely examine the required bandwidth variation with message-to-noise level for least error the curve computed from (I.9) is given in Figure I.3, and if a tolerance of 5% from the minimum error at any given W is permitted, any single-valued characteristic drawn between the indicated boundaries would serve, assuming the variation is not so rapid that the quasi-stationery assumption is violated. This is a fortuitous situation since it obviates the problem of obtaining a precise variation of the variable parameter of the slowly-variable system with the control signal.

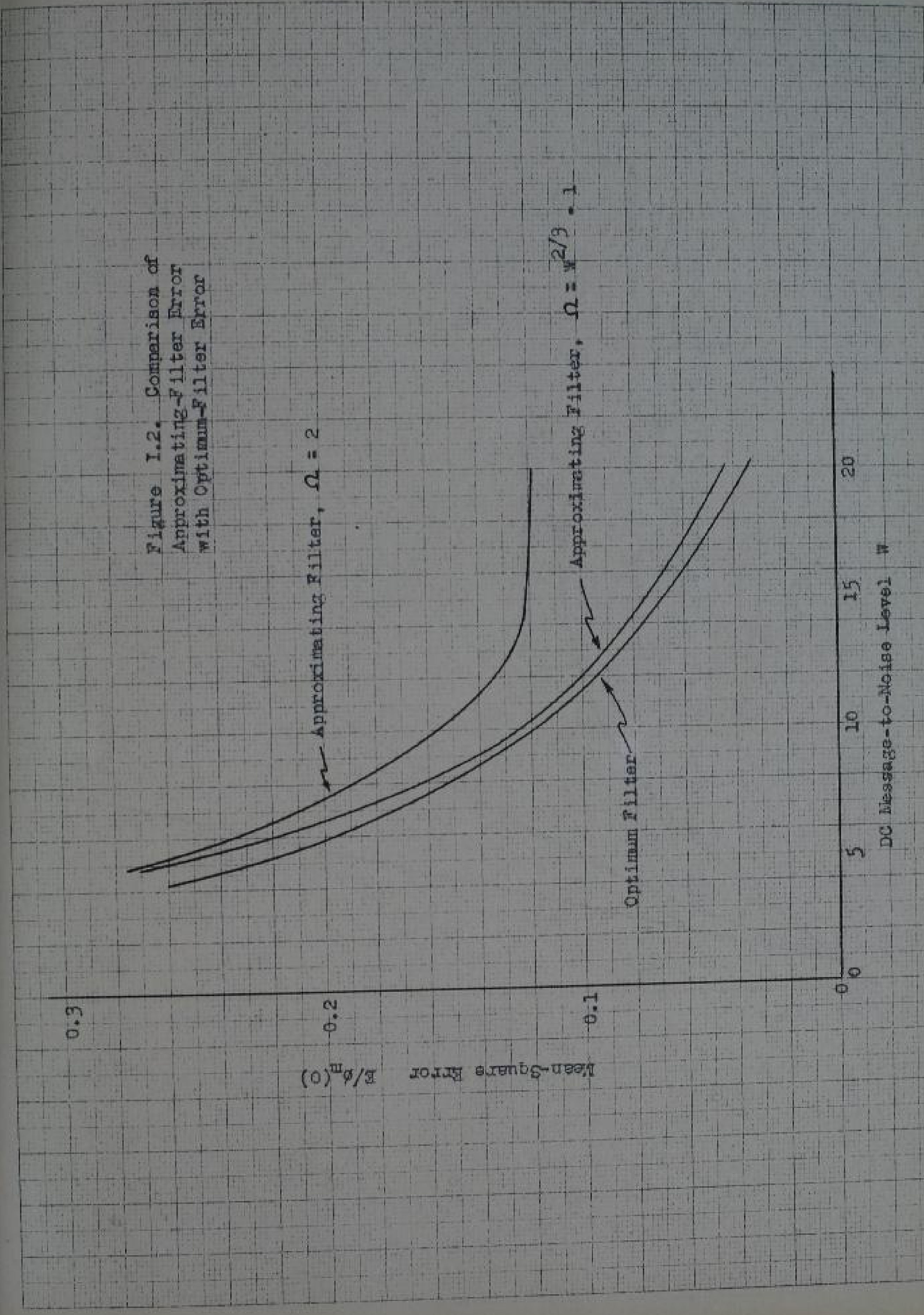

Figure I.2. Comparison of Approximating-Filter Error with Optimum-Filter Error

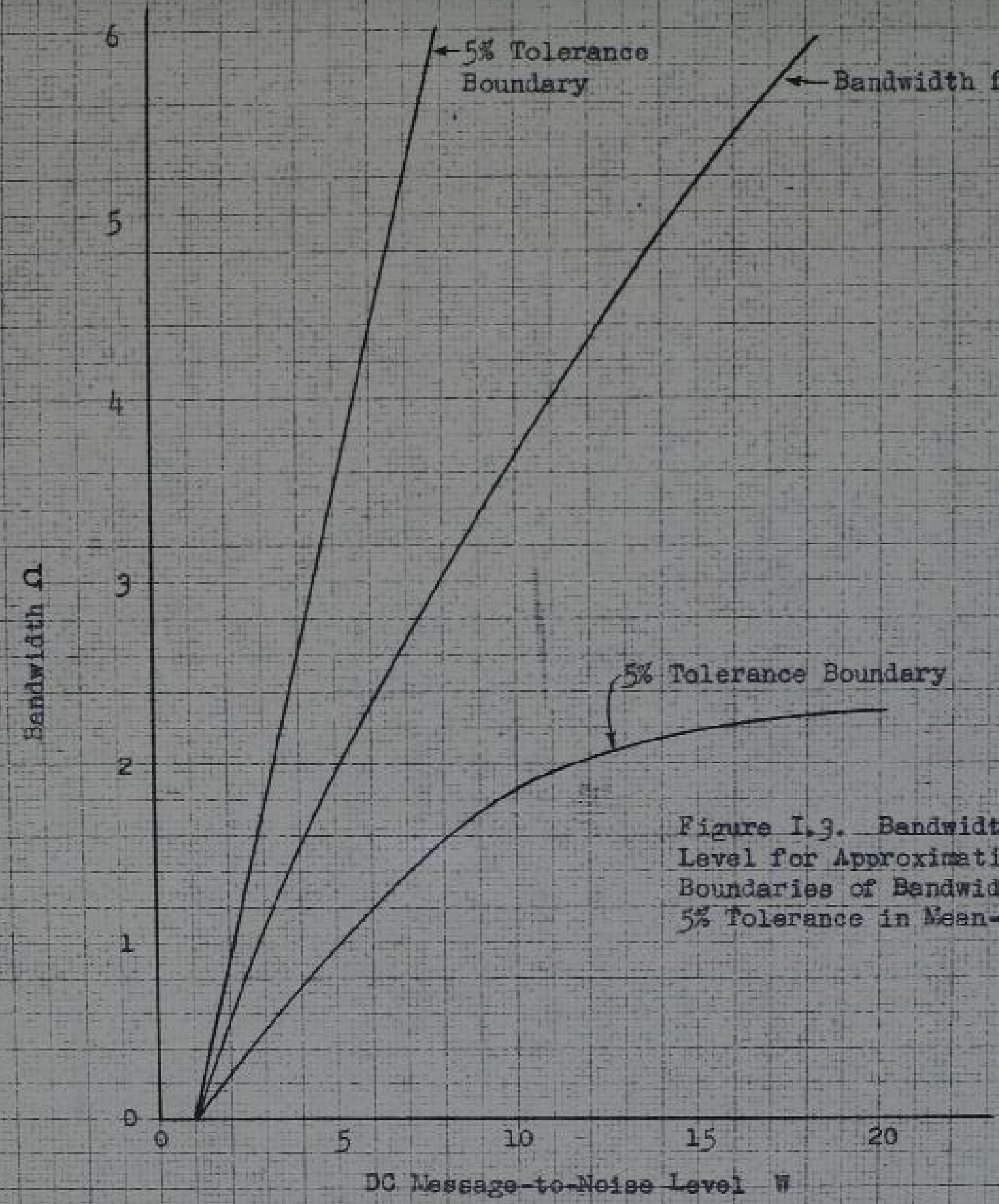

Figure I.3. Bandwidth vs. Message-to-Noise Level for Approximating Filter Showing Boundaries of Bandwidth Variation for 5% Tolerance in Mean-Square Error

Case II. Filtering with Lag

Optimum Transfer Function and Optimum Error. The optimum transfer function is found from Wiener's theory[1] to be (for infinite lag)

$$H(j\omega) = \epsilon^{-j\alpha\omega} \frac{M^2b^2/a^2}{\omega^4 + \beta^2(1 + b^2)\omega^2 + \beta^4 b^2 + M^2b^2/a^2} \quad (\alpha \to \infty) \qquad \text{(I.10)}$$

The irremovable error is that obtained with the infinite-delay filter and is

$$E_i = \phi_m(0) \frac{2b(b + 1)}{X[X^2 - (1 + b^2)]} \qquad \text{(I.11)}$$

where X has the same meaning as in (I.6).

Since the infinite-delay filter is the best filter which the o-m-s theory can provide it is used to provide the ultimate error characteristic as a goal for the approximating filter performance.

The Approximating Transfer Function. As with the zero-lag case, the transfer function chosen to approximate the optimum infinite-delay filter is the function

$$H_f(s) = \frac{k}{s + k} \qquad \text{(I.12)}$$

Decrease in Error Resulting from Allowed Lag. It is instructive to study the error when a message lag is permitted in conjunction with the error which is obtained in the zero-lag situation, when the same filter is used in both situations. The decrease in error δE_ℓ

1. Wiener, N., Op. cit., pp. 88-94.

obtained by holding all factors constant except the lag, which is allowed to vary, is given by

$$\delta E_{\ell} = 2\phi_m(0) \int_0^{\infty} h(\tau) \left[\phi_m(\alpha - \tau) - \phi_m(\tau)\right] d\tau \ . \tag{I.13}$$

Applying (I.13) to the case b = 1, it may be shown that

$$\delta E_{\ell}\Big|_{b=1} = \frac{2\Omega\epsilon^{-T}(\Omega - 2 + T\Omega - T) + 2\Omega(2 - \Omega)\epsilon^{-\Omega T}}{(\Omega - 1)^2}$$

$$+ \frac{2\Omega(\Omega + 2)\epsilon^{-\Omega T} - 2\Omega(\Omega + 2)}{(\Omega + 1)^2} \tag{I.14}$$

where $\Omega = k/\beta$, as before, and $T = \alpha\beta$ is a normalized time lag hereafter referred to as the lag associated with the message at the output of the filter. If the bandwidth of the approximating filter is taken as unity, (I.14) reduces to

$$\delta E_{\ell}\Big|_{b,\Omega = 1} = \phi_m(0)\left[\epsilon^{-T}(T^2 + 2T + 3/2) - 3/2\right] . \tag{I.15}$$

In order to find the best value of lag T associated with a given bandwidth of the approximating filter, it is necessary to minimize (I.14) with respect to T. The best lag T_o (in the sense that the mean-square error is least) is related to the bandwidth by a transcendental equation

$$\epsilon^{-(\Omega - 1)T_o} = \frac{(\Omega + 1)^2(T_o + 1 - \Omega T_o)}{4\Omega} . \tag{I.16}$$

It follows from (I.16) and (I.14) that if the optimum lag T_o is specified the error in filtering (with b = 1) is given by the error when no lag is allowed minus the decrease in error

$$\delta E\Big|_{b = 1} = 2\epsilon^{-T_o}(1 + T_o) - \frac{2\Omega(\Omega + 2)}{(\Omega + 1)^2} \qquad \text{(I.17)}$$

It should be noted that in (I.17) the lag T_o and the bandwidth are not independent because of the relation (I.16).

<u>Performance of Approximating Filter.</u> Equation (I.16) has been solved to give the optimum value of message lag inherent in the performance of the approximating filter as a function of the bandwidth of the filter. The solution is given in Figure I.4. Using this figure in conjunction with (I.17) and Figure I.2 it is possible to graph the total mean-square error for a filter of bandwidth equal to 2, and this is done in Figure I.5. Figure I.5 also gives the variation in error with message-to-noise level for the infinite-delay filter, which is obtained using (I.11). Figure I.5 shows that for a level W of 5 or more the fixed filter is virtually as good as the optimum filter. In this situation it is probably undesirable to vary the bandwidth of the filter.

It is of interest to examine the decrease in error (relative to the zero-lag performance) as a function of the message lag T. The graph of the decrease is plotted in Figure I.6 based on (I.15). The best lag in this instance is shown to be 0.707. Examination of the phase characteristic of the approximating filter with a unit bandwidth reveals that the geometric mean of the filter delay computed at zero frequency and at a frequency corresponding to the value of the bandwidth is precisely 0.707, illustrating the close relationship between the best lag inherent in the filter and the filter delay.

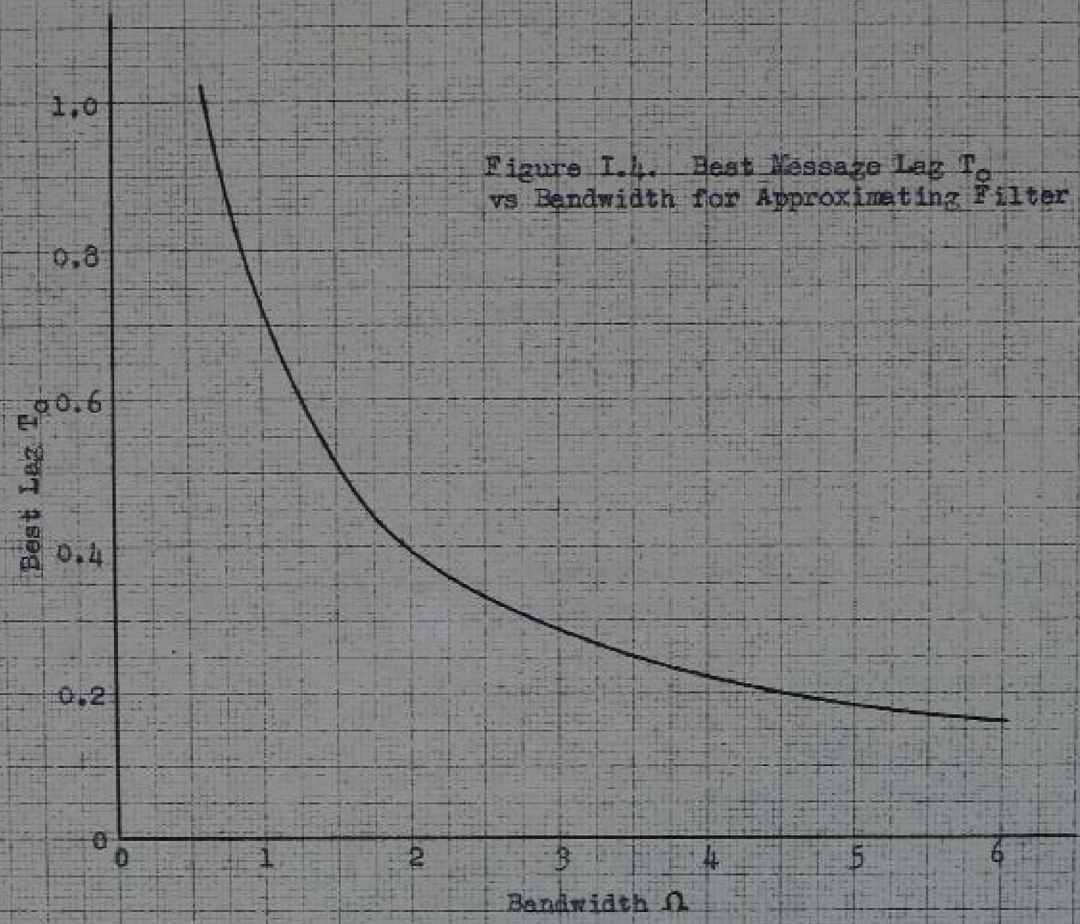

Figure I.4. Best Message Lag T_o vs Bandwidth for Approximating Filter

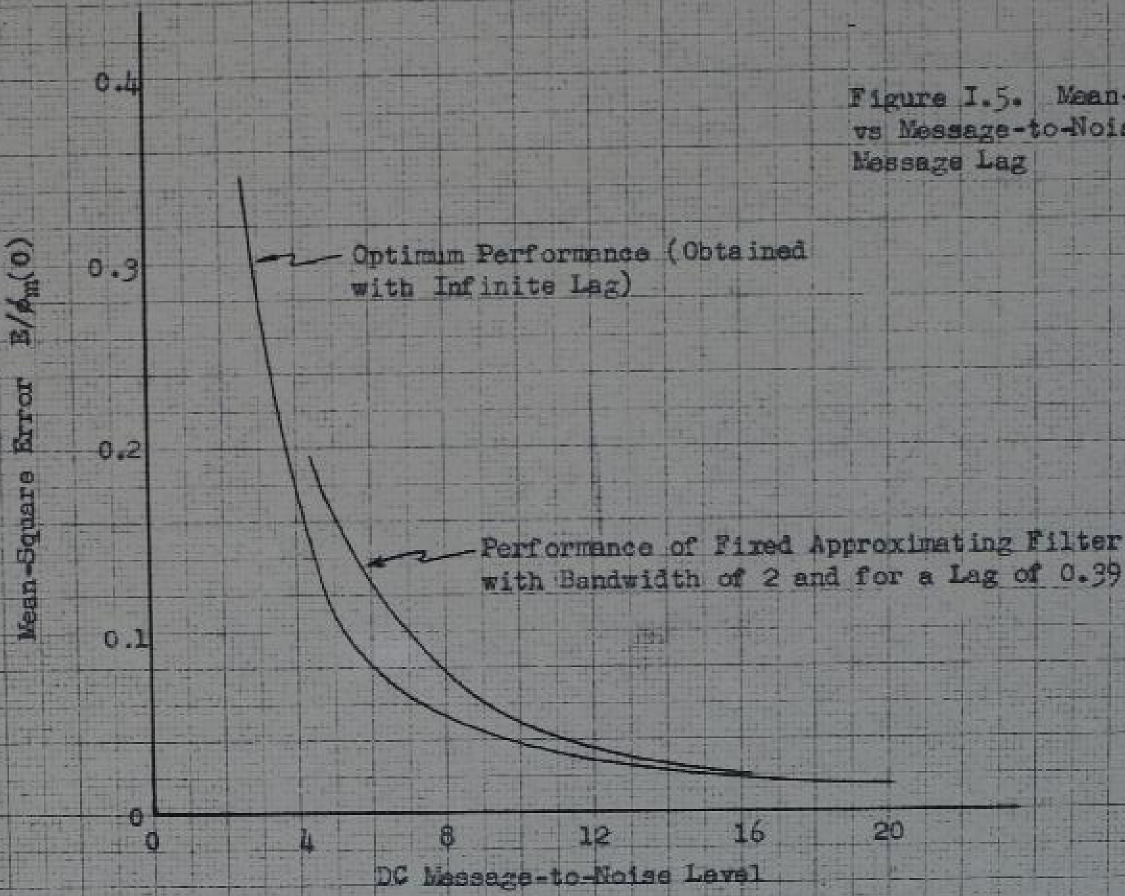

Figure I.5. Mean-Square Error vs Message-to-Noise Level with Message Lag

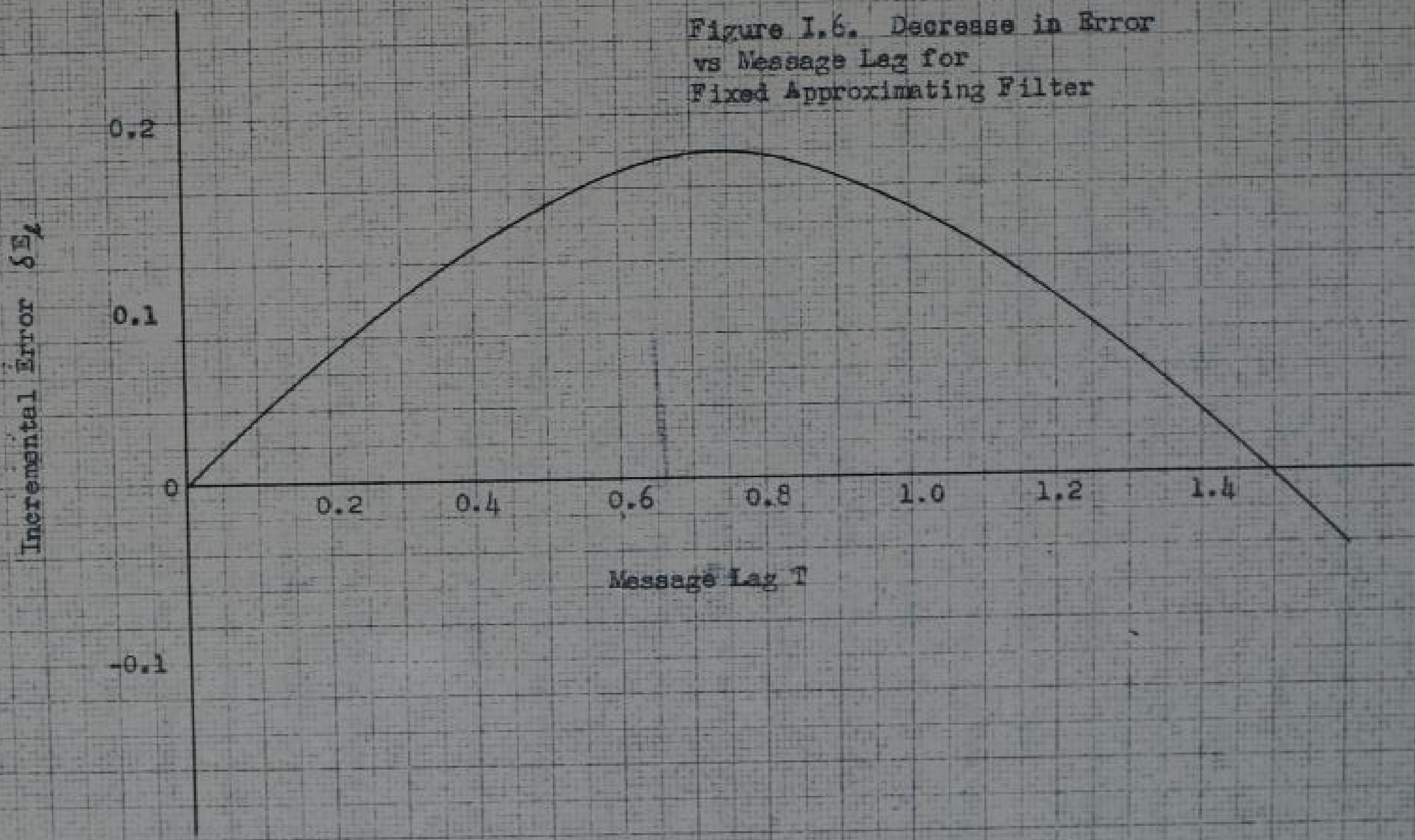

Figure I.6. Decrease in Error vs Message Lag for Fixed Approximating Filter

While the geometric mean relationship is not a general one, it illustrates the notion that as long as the phase characteristic does not deviate substantially from linearity in the pass band the best message lag and the filter delay will be of the same order of magnitude. This point is significant in the approximation of the optimum linear operator by a realizable transfer function.

Improvement in Performance of the Approximating Filter.

Thus far only a single variable has been allowed in the approximating filter characteristic, namely, the bandwidth. That this is sufficient for the cases discussed has been established for values of level W roughly of the order of five or above. However, for lower values of the level it may still be possible to achieve some measure of satisfactory filtering, although this will hinge largely on what is meant by "satisfactory". It is noticeable from Figure I.5 that the performance of the approximating filter in the vicinity of low values of level is not very satisfactory from the standpoint of comparative error.

Speaking with the zero-lag case in mind, it may be shown that as the level decreases the optimum-filter amplitude-characteristic is compressed horizontally, lowered on the vertical scale, and suffers a change in shape, i.e., the filter has two maxima in its amplitude characteristic for high values of W, but only one at low levels (when plotted along the entire axis of real frequency). It is precisely this complicated variation which led to the choice of a simple approximating function, since it is desired to avoid the complexity in variation, if possible.

The curve of dc gain as a function of level for the optimum filter is given in Figure I.7, where it is seen that as the level diminishes from 5, the gain falls off rather rapidly. This illustrates the fact that when the quality of filtering becomes poor because of the low message-to-noise level the o-m-s system tends toward a complete cutoff. The bandwidth variation with level is not shown, but it also diminishes as the level diminishes and as a consequence the delay increases thereby causing poorer zero-lag performance. Gain variation, although not investigated for the approximating filter, might lead to better performance in the region of low message-to-noise level, but continued reduction of bandwidth is not desirable, because of the increase in delay which results.

Delay Equalization. It is of interest to determine the effect of phase distortion in the lag filter. This effect may be studied through the contribution E_4 to the total error which, as previously stated, is the only contribution to the error affected by the phase characteristic of the filter. When the approximating filter is endowed with a linear phase characteristic over the entire frequency axis (admittedly an impractical situation) the contribution of the error E_4 to the total error is

$$E_{4e} = \frac{2\Omega}{\pi(\Omega^2-1)} \left[1 + \frac{(\Omega^2-2)\tan^{-1}(\Omega^2-1)^{1/2}}{(\Omega^2-1)^{1/2}}\right] \tag{I.18}$$

while, with no equalization, the contribution is

$$E_{4ne} = \frac{2\Omega}{(\Omega-1)^2}\left\{\epsilon^{-T}\left[\Omega - 2 + (\Omega-1)T\right] + (2-\Omega)\epsilon^{-\Omega T}\right\} + \frac{2\Omega\epsilon^{-\Omega T}(\Omega+2)}{(\Omega+1)^2} \quad . \tag{I.19}$$

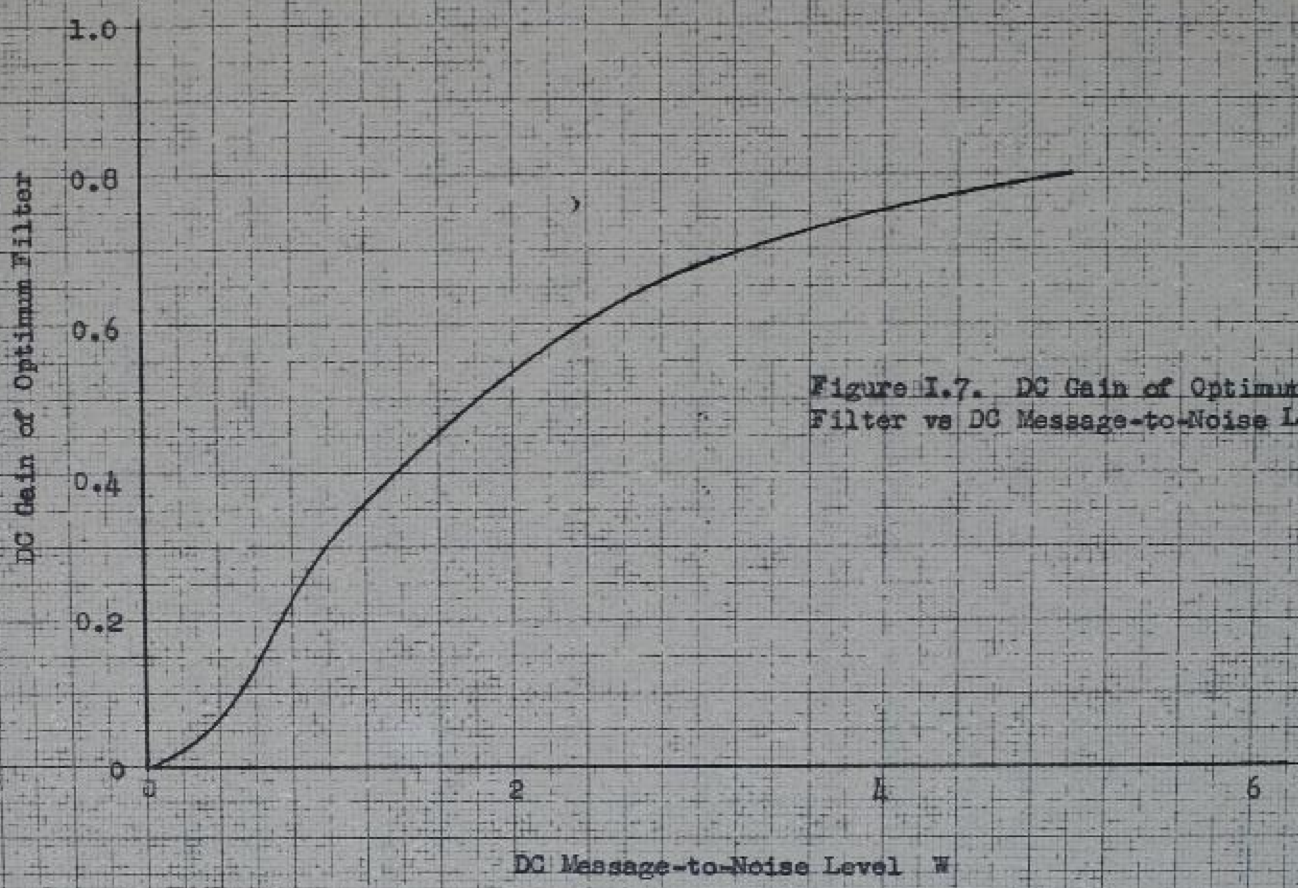

Figure I.7. DC Gain of Optimum Filter vs DC Message-to-Noise Level

In Figure I.8 the difference between the error with equalization and the error without equalization is given. This figure illustrates again that except at small bandwidths where the delay in the filter is large, and the phase characteristic deviates substantially from linearity, little can be gained by equalization in this example.

G. Conclusions

In approaching the problem of synthesis of a slowly-variable system on an o-m-s basis problems tend to arise in a logical sequence. While liberties may be taken at each step if necessary, favorable solutions to these problems must be found if a satisfactory solution to the overall problem is to be reached. The sequence of problems is indicated in question form.

1. Are the necessary correlation functions (or power density spectra) available?

2. Can the variable statistical parameter be measured?

3. How much lead or lag is specified?

4. What is the transfer function of the o-m-s system?

5. How does the optimum error vary, and what do the numerical values of error mean in terms of suitable performance?

6. How can the optimum transfer function be approximated to satisfy the requirements of physical realizibility and instrumentation of the system on a slowly-variable basis?

7. How can the approximating transfer function be realized?

8. Are the errors inherent in the approximating system of such magnitude that the purpose of the system is defeated?

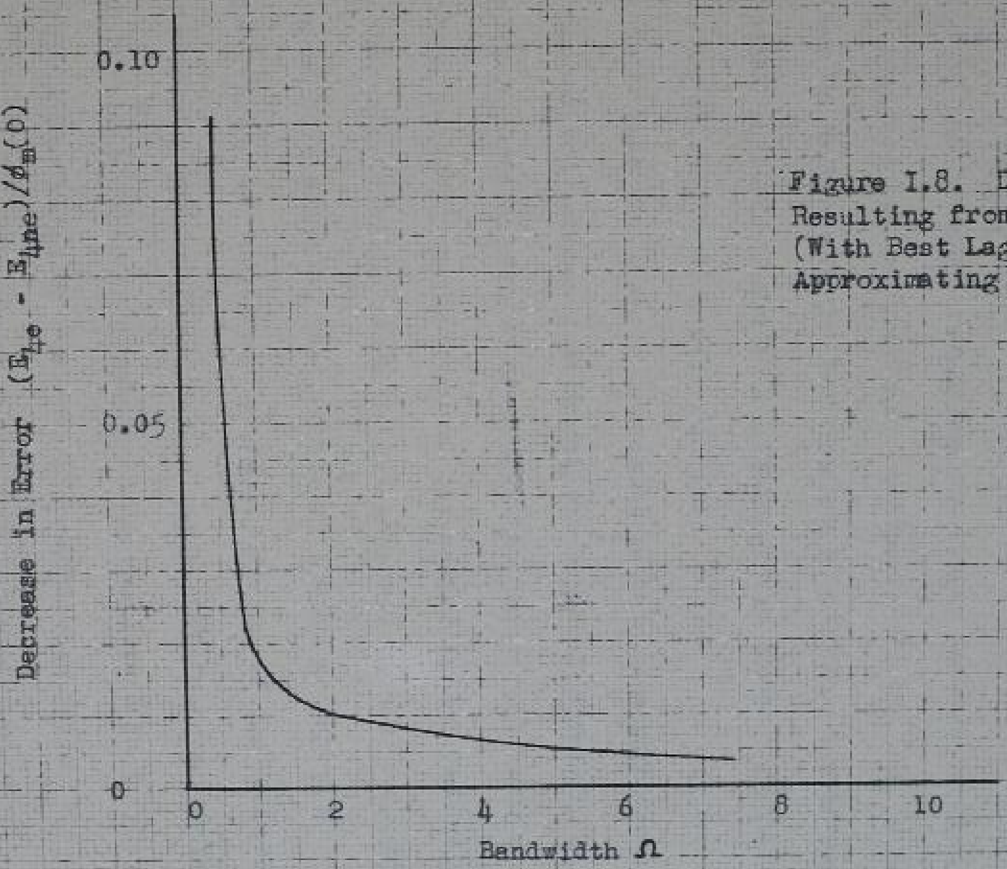

Figure I.8. Decrease in Error Resulting from Delay Equalization (With Best Lag Associated With Approximating Filter)

9. How does the rate of variation enter the picture? (An alternate way of phrasing this question is: how rapidly may the parameters vary before the system fails to perform adequately?)

10. Can the performance be improved by the addition of non-linear devices, or should a non-linear device be chosen initially?

It is hoped that some progress toward the answer to question 7 of this sequence has been made in the main body of this thesis. The location of the question regarding non-linear devices at the last of the sequence may be condemned in some instances. It is felt, however, that in many cases the performance of a linear system may be investigated initially, and if a linear system proves unsatisfactory resort to a non-linear system will be made.

It is felt that the example presented in this appendix reveals the close connection between specifications based on conventional techniques (i.e., specification of bandwidth, delay, and gain) and specifications based on performance of the system in the time domain. It is the author's opinion, based on the knowledge gained from an examination of this example, that (trite as it may sound) if a conventional filter is indicated by the conditions of the problem, design based on conventional methods is quite satisfactory; while if the nature of the problem is complicated to the extent that insight attained from a knowledge of conventional techniques fails, the Wiener theory assumes its rightful place as a set of tools for system design. Perhaps the easiest way for the reader to bring himself to agreement with this viewpoint is to consider the problem of prediction in the light of conventional filter specifications.

APPENDIX II. SOME PROPERTIES OF THE DIFFERENTIAL AMPLIFIER

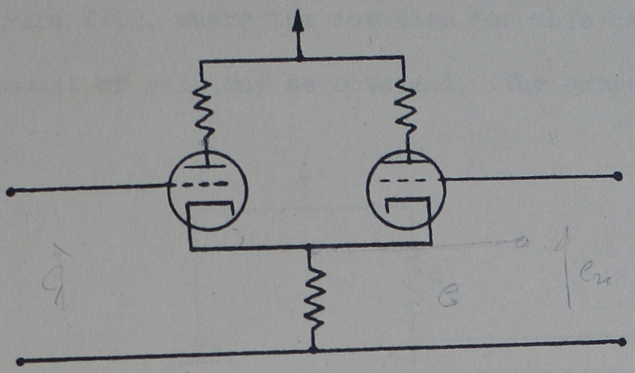

Figure II.1. The Differential Amplifier

The circuit shown in Figure II.1 is usually referred to as the differential amplifier. A partial analysis of the differential amplifier is found in the literature[1]. A brief discussion of this circuit is given here as a supplement to Chapter VI of this thesis.

A. Use as an Adder.

The circuit of Figure II.1 may be used as an adding circuit if the output is taken from the cathode resistor. The gain in this application is small. If the output voltage is evaluated with the plate load resistors set equal to zero, for simplicity, the gain is given by

$$\frac{\mu}{2(1+\mu) + r_p/R_k}$$

based on the usual linear approximation.

The output impedance is small in the application as an adder, and is of the order of $(1/2g_m)$.

1. Seely, S., Electron-Tube Circuits, McGRAW-HILL BOOK Co., New York, N.Y., 1950, pp. 113-117.

B. Use as a Subtractor.

When the differential amplifier is used as a subtracting device as shown in Figure II.2, where the notation for this section is given, a reasonable amount of gain may be obtained. The expression for

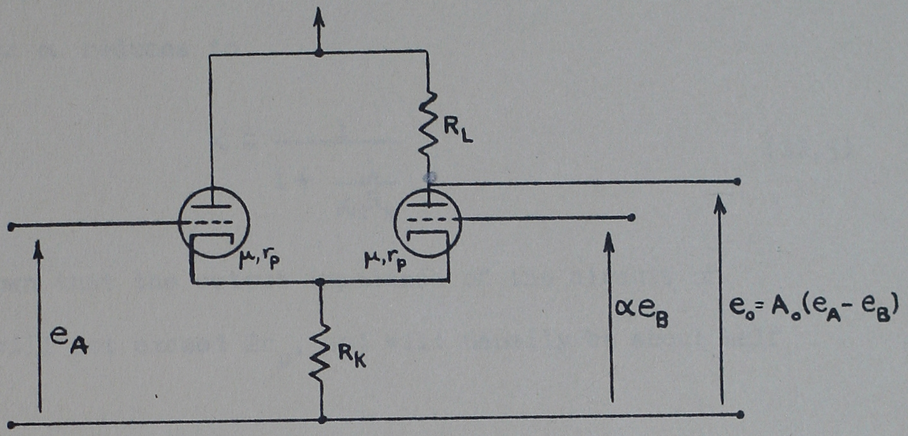

Figure II.2. The Differential Amplifier Used as a Subtracting Device

the gain A_o may be written in the form

$$A_o = \frac{1}{\frac{1}{\mu} + \frac{2r_p}{\mu R_L} + \frac{r_p}{\mu(\mu+1)R_k} + \frac{r_p^2}{\mu(\mu+1)R_k R_L}} . \qquad \text{(II.1)}$$

It may be seen from (II.1) that the factors which operate to produce high gain are high values of μ, R_L, and R_k, accompanied by a low value of r_p.

The voltage weighting factor α which must be considered in the application because of asymmetry in the action of the differential amplifier on its two inputs is given by

$$\alpha = \frac{1}{1 + \frac{r_p}{(\mu + 1)R_k}} \quad . \qquad (II.2)$$

For large values of amplification factor (compared to unity) the expression for α reduces to

$$\alpha \doteq \frac{1}{1 + \frac{1}{g_m R_k}} \quad . \qquad (II.3)$$

It can be shown that the output impedance of the circuit of Figure II.2 will not exceed $2r_p$, and will usually be about half that amount.

Table II.1 is presented for use in the application of the circuit of Figure II.2 to the simple feedback system indicated in the block diagram of Figure II.3, where the differential amplifier

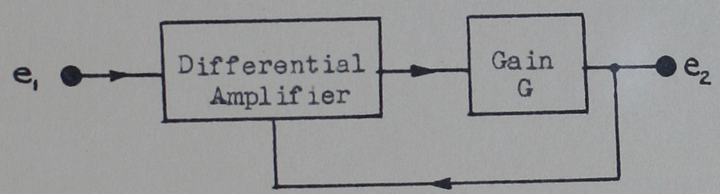

Figure II.3. A Feedback Configuration

is followed by a positive or negative gain G.

Table II.1. Application of Differential Amplifier in a Simple Feedback System

Sign of G	Type of Feedback	e_A	e_B	Sign of A_o
+	negative	e_1	e_2	+
+	positive	e_2	e_1	-
-	negative	e_1	e_2	-
-	positive	e_2	e_1	+

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