SURVEY OF FUZZY SET THEORY IN ACTUARIAL LIFE MODELING

by

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Bachelor of Science
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DEDICATION

This thesis is dedicated to the entire Boni family in Abidjan, Ivory Coast, my affectionate companion Ornella Pitah, my lovely friends and professors whose advices guided me to completion of this masterwork; and last but not least the all-mighty Lord Jesus Christ.

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TABLE OF CONTENTS

Page

List of Tables vii

List of Figures viii

Abstract ix

Part 1: Structural approach to Fuzzy logic and Actuarial modeling 1

1. Introduction 1
   - 1.1. Life Contingent Events 2
   - 1.2. Fuzzy Logic 2
   - 1.3. Literature Review 4
   - 1.4. The Current Research 5
2. Traditional actuarial Life 6
   - 2.1. Market Setup: Underwriting and Preferred Lives 6
   - 2.2. Survival Probability Models 11
   - 2.3. Benefits and Premium Calculations 18
3. Fuzzy Mathematics 24
   - 3.1. Fuzzy Set Methodology 24
   - 3.2. Fuzzy clustering algorithms 32

Part 2: Applications to Life Contingencies 38

1. Classification of Preferred Policyholders in Life Insurance 38
2. Fuzzy Survival Probability 44
3. Computation of Fuzzy Premiums 48
4. Fuzzy Insurance Benefits 54
5. Fuzzy Clustering of Policyholders 62

Conclusions and further research 70

Appendix 71

References 77

LIST OF TABLES

Table **Page**
Table 1. Frequency and Severity of Claims 64
Table 2. Clusters centers or centroids. 66
Table 3. 20%-level set of the membership 67
Table 4. Risk Measure per Age group 68
Table 5. Life Table for the total population of United States, 2011 71
Table 6. Life Table for the total population of United States, 2011 (continued) 72
Table 7. Expectation of Life by age, race, ethnicity and sex in United States, 2011 73
Table 8. Number of survivors out of 100,000 individuals by age, race, ethnicity and sex in the United States, 2011 74
Table 9. Standard results for t-norms and t-conorms for Fuzzy set operations 76

LIST OF FIGURES

Figure **Page**
Figure 1. Transition of Temperatures 3
Figure 2. Crisp set and Fuzzy set membership graphs... 4
Figure 3. Survey of Evolution (Zimmerman 2001) 5
Figure 4. Predictive Modeling for Life Insurance by Deloitte Consulting, LLP 10
Figure 5. Visualizing the mortality function on a timeline 15
Figure 6. Trapezoidal and Triangular Fuzzy Number 28
Figure 7. The midpoint bias of the Butterfly problem in data mining 34
Figure 8. The midpoint bias of the Butterfly problem in data mining 34
Figure 9. Butterfly Fuzzy Clustering for m=2 37
Figure 10. Short, Medium, Long future lifetime membership for 65 years old 45
Figure 11. A Fuzzy random variable representation.. 46
Figure 12. Combining for the Fuzzy survival probability 48
Figure 13. Membership function for a 10-year continuous life insurance for a 55 year-old 50
Figure 14. Membership function for the triangular fuzzy number p 51
Figure 15. Membership function for a triangular fuzzy interest rate 52
Figure 16. Membership function of the net single premium of a pure endowment.......... 53
Figure 17. Membership function of insurance benefit for a continuous life insurance 58
Figure 18. Comparative graph for the Crude Premium and the Group Premium............. 69

ABSTRACT

SURVEY OF FUZZY SET THEORY IN ACTUARIAL LIFE MODELING

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This thesis describes Actuarial science and Fuzzy Logic as relatively recent fields of mathematics introducing methods for containing uncertainty and vagueness in the line of business in which it is being used. Whereas actuaries work on the financial risk in (re)insurance of future events, 'Fuzzicists' aim at modeling the degree to which such events may occur. In the process of researching and writing, the author conducted a literature search and review of Fuzzy Set Theory with a structural approach to actuarial modeling. Following the recent development and discoveries of fuzzy logic, life insurance actuaries gained ultramodern modeling techniques, replacing the sole use of probabilities that had started to become insufficient. This thesis is slated to span the applications of Fuzzy Mathematics in the actuarial modeling of Life Contingencies.

PART 1: STRUCTURAL APPROACH TO FUZZY LOGIC AND ACTUARIAL MODELING

1. Introduction

The year 1965 marked the birth of fuzzy logic as forefather Lotfi A. Zadeh published his paper entitled "Fuzzy Sets" in the journal of Information and Control. He introduced an alternative logic to the well-known Boolean logic that an event is either True (=1) or False (=0) or, as formerly stated in Aristotle's *law of excluded middle*, an element is either contained in a set or not contained in a set. From this point of view, only a few elements of the real world can be properly represented, for everything has to be black and white and there are no shades of grey. On a note from Albert Einstein "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality." Fuzzy logic proposes that sets of objects had boundaries not sharply defined, awarding elements to be contained in a set to a grade of membership. Today on its 50th year anniversary, it has evolved into a whole field of mathematics with its very own analysis, operations and rules. Pioneer work in the theory of Fuzzy Sets extends to actuarial science and, specifically for our purposes, life contingencies models. We must introduce and define the basic engineering of these fields.

1.1. Life Contingent Events

Actuarial Science is the field of study that uses mathematical and statistical methods to assess risk in insurance, finance, and other industries/professions. The study is frequently associated with insurance and stock markets where its principles are commonly applied. In Life Insurance, actuaries aim to assess the risk of losing a life, to an insurer by modeling the policyholder's life expectancy. A person's life becomes a probabilistic set and its distribution is represented with various assumptions. One future lifetime becomes a random variable, and probabilities of death or survival are calculated. "Life contingencies" is a term used to describe survival models for human lives and resulting cash flows that start/stop contingent upon the state of a human life.

1.2. Fuzzy Logic

The traditional way of thinking in mathematics complies with Boolean logic (Boole 1847) and dates to 300 B.C. with Aristotle's law of the excluded middle. It states that for any two contradictory propositions (i.e. where one proposition is the negation of the other) one must be true (1), and the other false (0). Later adapted to algebra as X must be either in a set A or not in A.

Fuzzy logic is rather an extended logic dealing with linguistic ambiguity and handling the concept of partial truths. Truth values of a variable may be any real number between 0 and 1. By linguistic ambiguity arises matters of daily life having two or more aspects whose boundaries have not been unanimously agreed upon. One such example is the temperatures for Cold, Hot and Warm. The three functions are plotted in Figure 1, where the sense of transition between these aspects is more visible. Other examples may have more or less variables, such as Young-Adult-Old; Short-Tall, etc.

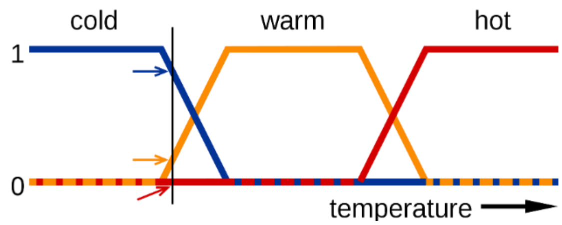


Figure 1. Transition of Temperatures

Fuzzy Set Theory (Zadeh 1965) is the mathematical field based on Fuzzy logic, dealing with the sets whose elements have degrees of membership. Define a Fuzzy set as a pair (U, m) for a set U and a membership function $m: U \rightarrow [0, 1]$. For each $x \in U$, the value *m(x)* is the grade of membership of *x* in the fuzzy set. In particular, *x* is not included if $m(x) = 0$, and *x* is fully included if $m(x) = 1$. That means for all $x \in U: m(x) \in (0,1)$, *x* is at the same time partly included and partly not included, hence the concept of sets with no sharp boundaries. Classical sets are special cases of fuzzy sets called crisp sets, with membership function $m: U \rightarrow \{0,1\}$.

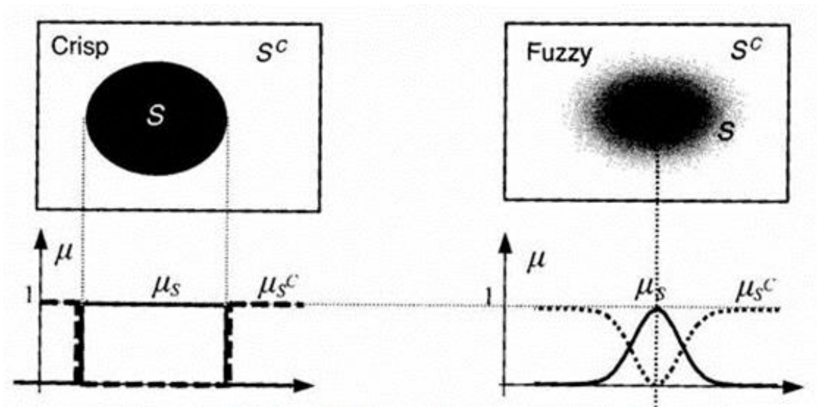


Figure 2. Crisp set and Fuzzy set membership graphs

1.3. Literature Review

Most insurance executives deal better with the crisp/traditional logic, and often transform imprecise statements into rigid rules. This is the case of Belgian insurers using fuzzy statistical evidence, such as "Young drivers provoke more automobile accidents" to set up the rating rule "Drivers under 23 years old will pay $150 deductible if they provoke an accident" (Lemaire 1990). Thus, the initial statement was distorted and "Young" was equated to "under 23," when 23 is only perhaps 80% young.

Since 1965, the count of publications on Fuzzy set theory has grown to exceed 50,000 today (Chen et al). We have experienced what is called a fuzzy boom since the 1990s thanks to pioneers in actuarial science such as Shapiro, Lemaire and Liu. Today, there are more researchers in Fuzzy Logic than in Actuarial Science, with important contributions from Japan, China & Russia. The evolution of the study in the literature started with linguistic variables and fuzzy sets, followed by fuzzy numbers arithmetic, fuzzy inference systems and fuzzy linear programming, and more recently fuzzy clustering with soft computing. It may be found in a variety of applications such as helicopter autopilot, home electronics, vehicle control, camera stabilization. The figure below from Zimmerman's "Fuzzy Sets Theory and its applications" 2001 provides a

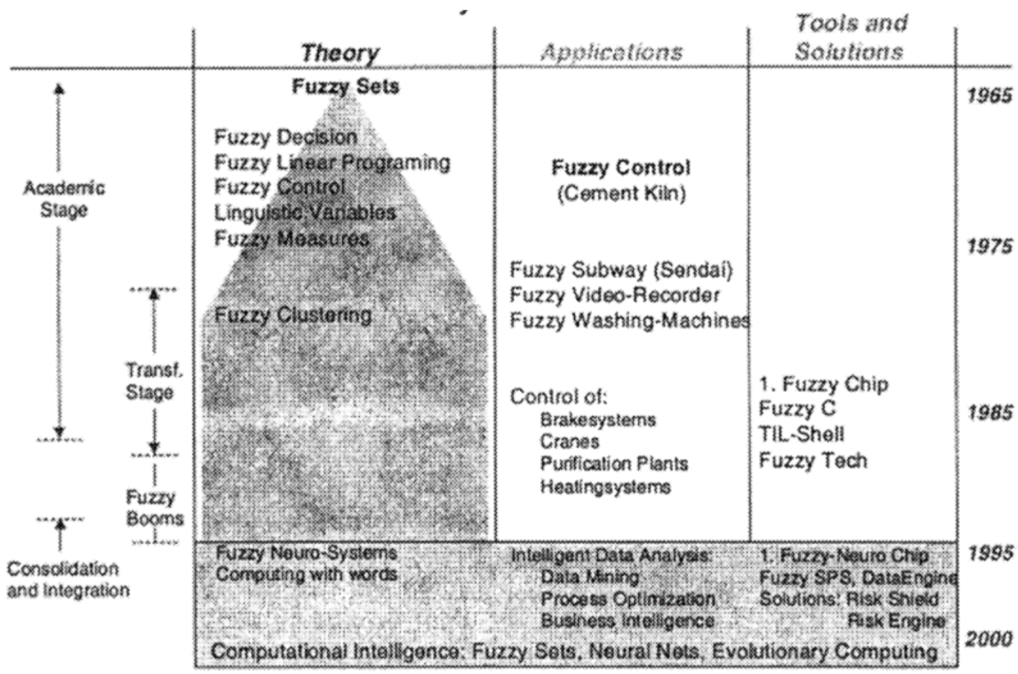


Figure 3. Survey of Evolution (Zimmerman 2001)

better grasp of the evolution.

1.4. The Current Research

This investigation aims at presenting the applications of Fuzzy Set Methodology in an actuarial science framework with focus on modeling life contingencies. The approach is meant to define where actuarial science and fuzzy logic intersect. First, the traditional mechanism of life insurance will be explained, from the underwriting process to the classification of policies in preference classes. This will require a review of probability theory in human life modeling, with the customary use of survival models and life tables for premium calculations. Second, the applications of fuzzy mathematics will be fully described through a series of theorems and definitions, including fuzzy rules, analysis, and clustering algorithms.

The second part of the research will survey the use of the aforementioned applications in life insurance. This will involve the translation of the medical records of applicants for life insurance using a fuzzy decision-making process, followed by the classification of the policyholders by risk levels using a fuzzy system of preference classes. Next, the author will remodel actuarial survival probabilities, insurance benefits and premium computations using fuzzy parameters, to eliminate the inaccuracy caused by the fluctuation of interest rates. Finally, each policyholder's risk derived from actuarial life tables using only the age factor will be rearranged in fuzzy clusters.

2. Traditional actuarial Life

2.1. Market Setup: Underwriting and Preferred Lives

Life insurance is a contract between a person and an insurance company. The insurance company promises a compensation to a designated person with a certain amount of money upon death of the insured, in return for periodic payments. Life insurance has its own jargon and the designated person is called the *beneficiary*; the money received upon death is the *sum insured* or *death benefit*; the insured is also called the *policyholder* and his/her payments are called *premiums*. *Underwriting* is the process of determining which risk class an insured belongs to, based on several factors such as age, gender, physical condition, medical history, financial background, personal habits, profession, hobbies, etc. Underwriting is an important aspect for an insurance company: it classifies the applicants according to their level of risk, protects the company from fraud or identity thefts, benefits consumers by keeping the insurance more affordable with low premiums, and helps the solvency of the company.

The consumer chooses the type of *insurance coverage* he/she wishes. The traditional insurance contracts are:

- *Whole life insurance* pays a lump sum benefit on the death of the policyholder whenever it occurs. Premiums are often payable up to a maximum age (80).
- *Term insurance* pays a lump sum benefit on the death of the policyholder, provided death occurs before the end of a specified term. Subtypes of these contracts include term insurance renewable every year, and term insurance convertible to whole life insurance.
- *Pure Endowment* pays the insured himself if he survives a specified period but pays nothing in case of a death prior the specified date.
- *Endowment insurance* offers a lump sum benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs first. (Term insurance + pure endowment).

Life insurance policies may involve a single premium at the beginning of the contract, or a series of premiums payable until death/end of term. Another type of insurance contract to consider is *Life Annuity*. An annuity in financial mathematics is a contract that offers a regular series of payments to the buyer. If the annuity depends on survival of an individual, it is called a 'life annuity' and the recipient is the *annuitant*. Note that the beneficiary can be the policyholder himself in this case. These contracts are often purchased by older lives to provide income in retirement. They include:

- *Whole life annuity* pays until the death of the annuitant.
- *Term life annuity* pays up to an agreed upon date, and may stop upon death of the annuitant, if sooner.

Other types of insurance contracts that are of more recent vintage and are more attuned to the current economy are:

- *With-profit insurance* shares profits earned on the invested premiums are shared with the policyholders in the form of cash dividend, reduced premiums, or increased sum insured
- *Universal life insurance* puts premiums into an investment fund and deducts insurance charges from the fund periodically.

Once the type of contract is selected, underwriters do a classification of the risk level of the applicant following the guidelines of the insurance company executives. Insurance Risk Classes are groups of people with similar characteristics and risk level.

The classes may be defined based on age, sex, income and physical condition. In life insurance, an underwriting decision is made whereby the applicant may be either denied coverage or put into one of the following generic classes:

- *Preferred Class*: healthy, middle-aged individuals with stable income and a better than average risk of insolvency or mortality. They are charged lower (preferred) rates.
- *Standard Class*: This is for a more or less healthy person, complying with the definition of normal or typical risk the carrier desired to insure. This class is charged the standard rate.
- *Substandard/Rated Class*: In this class are applicants for higher coverage with only tolerable physical or medical condition. They represent an above average risk and are charged higher premium rates than the Preferred and the Standard classes.
- *Postponed*: Occasionally cases are postponed until additional information is gathered, some time passes, or until negative factors change in favor of assigning a risk class. For instance, a person with a medical condition could be subject to additional tests, such as a stress test to check cardiovascular functions.
- *Declined*: if the client represents an uninsurable risk, coverage is declined.

This is the type of underwriting scheme used by most life insurance carriers, in particular by the American International Group (AIG) in its 'Field Underwriting Guide'. Other derivatives schemes can be found as in Figure 4, which shows the risk classes Deloitte Consulting uses for 'Predictive modeling for Life Insurance.' Note to the reader, declination of insurance coverage does not reflect the health or likelihood of dying, but solely a risky venture for the solvency of the company.

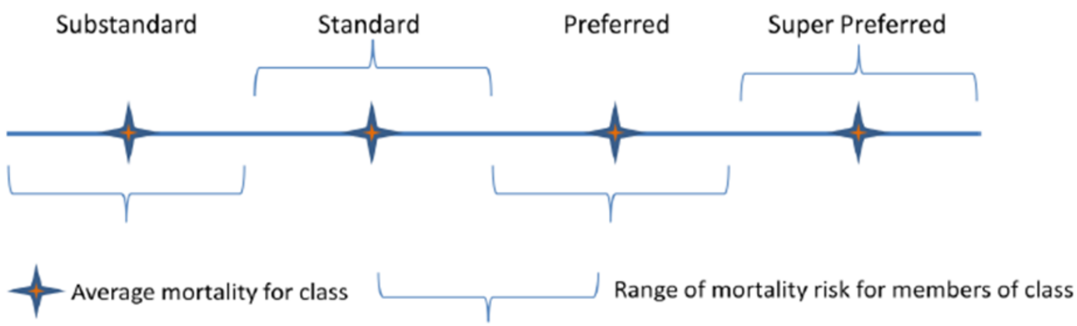


Figure 4. Predictive Modeling for Life Insurance by Deloitte Consulting, LLP

Life insurers agree that mortality is unevenly distributed in the population. For instance, with all factors being equal, females outlive males; the use of tobacco is detrimental to health and critically affects mortality. United States mortality tables are divided by sex, race and ethnicity and all give different statistics (See Tables 5, 6, 7 & 8).

Preferred programs expanded in the 1980s with the HIV/AIDS scare (Hughes 2012). Prior to this date, a single premium rate was charged for each age/sex cohort with occasionally a nonsmoker discount. All 35 year-old male customers would pay the same premium. This method had some negative outcomes by creating a **pooling of risks** – customers of higher mortalities offset costs of favorable mortalities, and vice-versa. To address this problem, carriers began demanding blood samples to determine if an applicant was HIV-positive. Initially, this additional information would only suggest denial of coverage. But eventually, blood panels revealed a wealth of data on a person's well-being, later used to assess mortality risk in more refined ways.

The main objectives for a preference program are to have rates in line with each risk profile, to reduce **premium cross-subsidization**, and to shape the industry into a competitive market. By design these programs operate more closely to individual risk setting. But the practice of grouping lives has been more prominent in life insurance sales historically, and requires less underwriting.

Preferred programs are complicated to develop and require a chief pricing actuary. A preferred premium must correlate to the expected risk profile of the best class. Selection decisions outside the agreed-upon bounds (i.e. exceptions) can affect the distribution of risks and profitability. If underwriting guidelines are applied liberally, more applicants will qualify for the best class. On the other hand, if they are too rigorous, premium rates will be too high, driving away applicants the preferred class was designed to attract (Hughes 2012). Most companies require that clients apply for a stated minimum death benefit of perhaps $250,000 in order to qualify for preferred status. Industry surveys say that 10-45% of all applicants qualify for preferred classification.

2.2. Survival Probability Models

2.2.1. Survival Models In Statistics

In Probability theory, the *sample space* Ω for a random phenomenon is the set of all possible outcomes (Ross 2013). The *event* E is any subset of the sample space and the *probability of event* E occurring is $P(E) \in [0,1]$. There exists a duality property between an event E and its complement $\mathrm{P}(E^C) + P(E) = 1$. Two important equalities in probability are:

$$\forall A, B \in \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

and the *conditional probability* of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A *random variable* X is a numerical outcome of a random experiment. The *distribution* of X is the collection of outcomes and their probabilities. X is said to be *discrete* if it has a countable number of outcomes; and *continuous* if it has an infinite continuum of possible values (e.g. blood pressure, weight). The *cumulative distribution function (or cdf)* is given by: $F_X(x) = P(X \leq x)$ or $F_X(x) = P(X = x)$.

The derivative of the cdf is called the *probability density function* (*or pdf*) in the continuous case, and the *probability mass function (or pmf)* in the discrete case. It is given by:

$$f_X(x) = \frac{dF_x(x)}{dx}$$

Each distribution has an *expected value* denoted

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx \quad and \quad E[X] = \sum_{-\infty}^{\infty} x \cdot P(X = x)$$

and a *variance* $V[X] = E(X - E[x])^2$.

With the above considerations in mind, the author will attempt to define an actuarial survival model.

The distribution of a future lifetime may be represented using probability theory. Actuaries and Statisticians agree that any survival function for a lifetime distribution must satisfy the following conditions:

i. All lives must die before a stated terminal age ω.

ii. Survival functions are non-increasing over time.

iii. The probability that a life aged x survives the next t minutes goes to 1 as $t \to 0$.

Let *(x)* denote a "*life aged x*", for x ≥ 0. Death may occur at any age greater than x, and the future lifetime of (x) is a continuous random variable denoted T_x. That means T_0 represents the *future lifetime at-birth* or for a life aged 0; and $x + T_x$ represents the *age-at-death* for (x). Denote the cdf of T_x as F_x, so $F_x(t)$ is the probability that (x) does not survive beyond age $x + t$:

$$F_x(t) = P[T_x \leq t]$$

The complement S_x is the survival function and $S_x(t)$ is the probability that (x) survives at least t more years defined as:

$$S_x(t) = 1 - F_x(t) = P[T_x > t]$$

The previous conditional probability formula gives the following relations:

$$F_x(t) = P[T_x \leq t] = P[T_0 \leq x + t | T_0 > x] = \frac{P[x < T_0 \leq x + t]}{P[T_0 > x]}$$

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \quad or \quad S_0(x+t) = S_0(x) \cdot S_x(t)$$

Condition (i) can be translated as $S_x(0) = 1 \; \forall x \leq \omega$, the terminal age assumed, and condition (ii) as $\lim_{t \to \infty} S_x(t) = 0$. In order for the mean and the variance of T_x to exist, other assumptions need to be made:

1. $S_x(t)$ is differentiable for all $t > 0$. Note this means that $\frac{d}{dt} S_x(t) \leq 0 \forall t > 0$.
2. $\lim_{t \to \infty} t \cdot S_x(t) = 0 \; and \; \lim_{t \to \infty} t^2 \cdot S_x(t) = 0$.

2.2.2. Actuarial Survival Models

Actuarial Mathematics uses another notation for life models: the International Actuarial Notation. Following the development of the survival models from Dickson et al (2009), the previously defined quantities are expressed using this notation as follows:

$$ {}_tp_x = S_x(t) \quad and \quad {}_tq_x = 1 - {}_tp_x = F_x(t). $$

Also, a *deferred mortality probability* ${}_{u|t}q_x$ is defined as the probability that *(x)* dies between ages $x + u$ and $x + u + t$:

$$ {}_{u|t}q_x = P[u < T_x \leq u + t] = S_x(u) - S_x(u + t) = {}_up_x - {}_{u+t}p_x $$

The last term is derived following the above conditional survival formula in simple steps:

$$ {}_{u+t}p_x = S_x(u + t) = S_x(t) \cdot S_{x+t}(u) = {}_tp_x \cdot {}_up_{x+t} $$

Deaths do not exactly occur at integer ages taken as points of time. To properly represent this, actuaries compound the possibilities of death on infinitesimal periods of life Δt for a life (x). We define the *force of mortality* at fixed age *x* by μ_x (Daykin, Macdonald, 2004):

$$ \mu_x = \lim_{dt \to 0+} \frac{1}{dt} P[T_0 \leq x + dt | T_0 > x] = \lim_{dt \to 0+} \frac{P[T_x \leq dt]}{dt} $$

$$ = F_x'(0) = -S_x'(0) = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x) $$

This quantity will help connect others. Note the equation for the pdf of the lifetime distribution can be calculated as:

$$ f_x(t) = {}_tp_x \mu_{x+t} = \frac{d}{dt} {}_tq_x = -\frac{d}{dt} {}_tp_x $$

This is because by definition the pdf $f_x(t)$ is the derivative of the $F_x(t)$, the complement of $S_x(t)$. Finally, the force of mortality and the survival functions can be newly expressed as the equations below, whose measurement can be clarified in Figure 5:

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}$$

$$_tp_x = S_x(t) = e^{-\int_0^t \mu_{x+s} ds} \quad and \quad _tq_x = \int_0^t {_sp_x} \mu_{x+s} ds$$

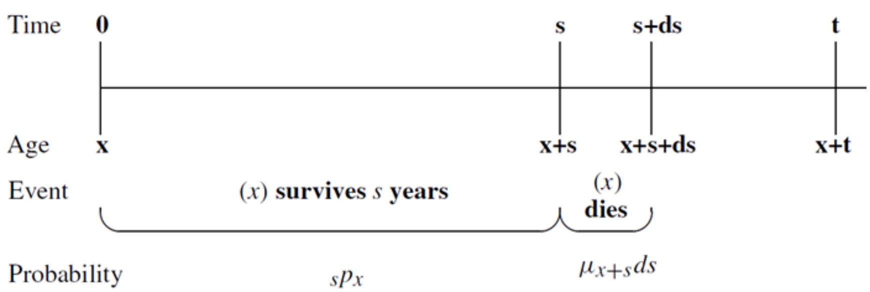


Figure 5. Visualizing the mortality function on a timeline

The mean of T_x is the expected future lifetime of (x). It is denoted $\dot{e}_x$, for the *complete expectation of life* and is the following:

$$\dot{e}_x = \mathrm{E}[T_x] = \int_0^\infty t \ {_tp_x} \mu_{x+t} \, dt.$$

The *second moment* of T_x is $\mathrm{E}[T_x^2]$ and is expressed as:

$$\mathrm{E}[T_x^2] = \int_0^\infty t^2 \ {_tp_x} \mu_{x+t} dt = \int_0^\infty t^2 \left(-\frac{d}{dt} \ {_tp_x}\right) dt$$

$$= -t^2 \ {_tp_x}\Big|_0^\infty + \int_0^\infty 2t \ {_tp_x} dt$$

$$= 2\int_0^\infty t\ {}_tp_x dt.$$

The *variance* of T_x is

$$V[T_x] = E[T_x^2] - (\dot{e}_x)^2$$

The *curtate future lifetime* is defined as the integer part of future lifetime K_x for a life (*x)* $K_x = \lfloor T_x \rfloor$, obtained by the floor function.

The actuarial science literature contains three pioneer efforts to find the exact force of mortality (Huang et al 2011). First is the law of *Abraham de Moivre* (1725) in "Annuities upon Lives," with the quantity ω denoting the **ultimate age**, i.e. the terminal age at which no lives remain. This is usually taken to be $\omega = 100$ or $\omega = 120$. *De Moivre's law of mortality* is expressed as:

$$\mu_{x+t} = \frac{1}{\omega - x - t} \quad and \quad {}_tp_x\mu_{x+t} = \frac{1}{\omega - x}$$

Second is the *Gompertz law of mortality*. In this case, Gompertz argues that life decays exponentially over time. For flexibility, constants B and c are given values such that $0 < B < 1$ and $c > 1$. For a life aged $x > 0$,

$$\mu_{x+t} = Bc^{x+t} \quad and \quad {}_tp_x\mu_{x+t} = Bc^{x+t}e^{\frac{B}{lnC}C^x(1-C^t)}$$

Another law is that of *Makeham* (1860), which extends the Gompertz force of mortality by adding a term $A \geq -B$ that is not age-related for accidental deaths and improves the fit of the model to mortality data at younger ages. For a life aged x>0,

$$\mu_{x+t} = A + Bc^{x+t} \quad and \quad {}_tp_x\mu_{x+t} = (A + Bc^{x+t})e^{-At+\frac{B}{lnC}C^x(1-C^t)}.$$

Unless stated otherwise, the assumptions in this research will be the Makeham's law of mortality with parameters $A = 0.00022, B = 2.7 \times 10^{-6}$ and $C = 1.124$ (Dickson et al. 2009).

Finally is the *Weibull assumption* (1951) that mortality is modelled as follows:

$$\mu_{x+t} = k(x+t)^n \ \ and \ \ {}_tp_x\mu_{x+t} = k(x+t)^n e^{\frac{k}{n+1}[x^{n+1}-(x+t)^{n+1}]}$$

2.2.3. Life Tables

Life tables are tables containing the mortality statistics that allow the calculation of life expectancy for a group of individuals. Given the survival models as above, we choose the parameters A, B and c to fit the data of the life table in use. Consider a life table starting from age x_0 to a maximum age ω. l_x for $x_0 \leq x \leq \omega$ is a function for the *number of persons alive of age x*. Define l_{x_0} to be any positive number and call it the *radix* of the table, and for $0 \leq t \leq \omega - x_0$,

$$l_{x_0+t} = l_{x_0}\ {}_tp_{x_0} \ \ and \ \ {}_tp_x = \frac{l_{x+t}}{l_x}$$

The *number of deaths* occurring during the interval between l_x and l_{x+1} is denoted d_x:

$$d_x = l_x - l_{x+1} = l_x q_x$$

where q_x is the probability that a person dies between x and $x+1$.

Individuals of the same age in life tables are assumed to have the same life expectancy. See Tables 5, 6, 7 and 8 and Figure 19 in Appendix, excerpted from 'United States Life Tables of 2011' by the Division of National Vital Statistics.

2.3. Benefits and Premium Calculations

2.3.1. Financial Mathematics: Interest Theory

The notation in actuarial science is consistent with that of interest theory. Define i to be the constant interest rate, and the *force of interest* or continuously compounded interest rate δ:

$$\delta = \log(1+i)\,, \quad 1+i = e^{\delta}$$

The discount rate, or the present value of $1 in one year, is expressed as:

$$v = \frac{1}{1+i} = e^{-\delta}.$$

The *nominal* rate of interest compounded *p* times per year is expressed as:

$$i^{(p)} = p\left((1+i)^{\frac{1}{p}} - 1\right) \iff 1+i = (1+\frac{i^{(p)}}{p})^{p}$$

The effective rate of discount per year is $d = 1 - v = iv = 1 - e^{-\delta}$.

The nominal rate of discount compounded p times per year is

$$d^{(p)} = p(1 - v^{\frac{1}{p}}) \Leftrightarrow \ v = (1 - \frac{d^{(p)}}{p})^{p}.$$

2.3.2. Insurance benefits & Annuities

The continuous future lifetime benefit random variable Z combines both the survival distribution and interest theory. It is the present value of a benefit of $1 payable immediately on death:

$$Z = v^{T_x} = e^{-\delta T_x}.$$

Given (x), the **International Actuarial Notation** defines the method for notation of the *expected present value* (EPV) of a life insurance contract in actuarial science.

$\boldsymbol{A_{x:\overline{n}|}}$ is the present value of insurance for (x) paying 1 on the insured event for n years;

$\boldsymbol{a_{x:\overline{n}|}}$ is the present value of an annuity for (x) paying 1 per annum for n years at the end of each year. To simplify the word editing, we will remove the bars above and to the right of n. Thus $A_{x:\overline{n}|} = A_{x:n}$ and $a_{x:\overline{n}|} = a_{x:n}$

The notation symbols and letter meanings are explained below using an example containing all of them. Denote an insurance contract of n years starting u years from now with interest compounded m times a year and benefit payable at the beginning of the year with

$${}_{u|}\ddot{\bar{A}}{}^{1(m)}_{x:\overline{n}|} \ \ or \ \ {}_{u|n}\ddot{\bar{A}}{}^{(m)}_{x}$$

- u is the *deferred period*,

- (m) represents the *interest rate i compounding frequency*, and

- the superscript "1" is the *endowment indicator* –

when placed above x, it indicates that benefit is payable only if (x) dies within n years, and when placed above n, it indicates that benefit is payable if (x) survives n years. The absence of a superscript means that the insurance pays on the earliest of death or n-years.

- The mark above "A" or annuity "a" represents the *time of payment* –

For $\bar{A}_x$ or $\bar{a}_x$, the *line* is for continuity i.e. payment is made continuously or immediately at the moment of death. For $\ddot{A}_x$ or $\ddot{a}_x$, the *double dot* indicates that payments are made at the beginning of the year. For A_x or a_x, the absence of mark indicates that payments are made at the end of the year.

For instance, the EPV of a life insurance of $1 benefit payable immediately on death is:

$$\bar{A}^{1}_{x:n} = E[Z] = E\left[e^{-\delta T_x}\right] = \int_0^n e^{-\delta t}{}_t p_x \mu_{x+t} dt$$

and the EPV of a life insurance of $1 benefit payable at the end of year of death is:

$$A^{1}_{x:n} = \sum_0^n v^t{}_t p_x \mu_{x+t}$$

As $n \to \infty$ or for a maximum age ω, $n \to \omega$, the quantities become A_x or a_x.

The second moment of the death benefit EPV is of the form ${}_{m|}^{2}\bar{A}_{x:\overline{n|}}$, ${}_{m|}^{2}\ddot{A}_{x:\overline{n|}}$ or ${}_{m|}^{2}A_{x:\overline{n|}}$

where the upper-left superscript "2" is for double force of interest. For example:

$${}^{2}\bar{A}_{x:\overline{n|}} = \int_0^n e^{-2\delta t}{}_t p_x \mu_{x+t} dt + e^{-2\delta n}{}_n p_x$$

The variance of the EPV and for a sum insured S is given by:

$$V[Z] = V\left[e^{-\delta T_x}\right] = {}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2$$

$$V[SZ] = V\left[Se^{-\delta T_x}\right] = S^2\left({}^2\bar{A}_{x:\overline{n|}} - \bar{A}^2_{x:\overline{n|}}\right)$$

The present value RV for the annuity payment series, Y are those of interest theory with the time now a (curtate) lifetime distribution.

$$a_{K_x+1} = \frac{1 - v^{K_x+1}}{d}$$

All other results can be derived, such as the following relationships between whole insurance and annuity contracts, and between annuities:

$$\ddot{a}_x = \frac{1 - A_x}{d} \quad ; \quad \bar{a}_x = \frac{1 - \bar{A}_x}{d}$$

$$a_{x:\overline{n|}} < a^{(m)}_{x:\overline{n|}} < \bar{a}_{x:\overline{n|}} < \ddot{a}^{(m)}_{x:\overline{n|}} < \ddot{a}_{x:\overline{n|}}$$

2.3.3. Premium Calculations

Consider the *net premium;* that is, the remaining part of the premium once the company expenses are removed. Other concepts are the *premium income* and the *insurance benefit outgo,* both life contingent. The *net future loss function* is L_0^n for a sum insured S and premium rate P for life insured (x). It is calculated at curtate/integer ages $K_x = \lfloor T_x \rfloor$.

$$L_0^n = PV\ of\ benefit - PV\ of\ premiums.$$

$$L_0^n = Sv^{\min(K_x+1,n)} - P\ddot{a}_{\min(K_x+1,n)}$$

The expected value $E[L_0^n]$ involve the actuarial accumulation functions.

$$E[L_0^n] = SA_{x:\overline{n|}} - P\ddot{a}_{x:\overline{n|}}$$

The Equivalence principle states that the net premium has to be set such that $E[L_0^n] = 0$. This is meant to give a fair price for coverage. High prices drive away the customers, and low prices put the company solvency in danger. The equation for the *Net premium* is:

$$P = S\frac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} = S(\frac{1}{\ddot{a}_{x:\overline{n|}}} - d)$$

Example 2-1. Consider a 20-year endowment insurance with sum insured $100,000 issued to a life aged 45 under which the death benefit is payable immediately at death. Using Makeham's law with an interest rate of 5% per year, find the net premium payable in a year if premiums are payable annually.

Solution. Use the equation above for P. From interest theory,

$$e^{\delta} = 1.05 \ \ and \ \ d = 1 - \frac{1}{1.05} = .0479619;$$

The SSSM gives the mortality rate for $x = 45$ and for $t \in [0,20]$,

$$\mu_x = .00022 + 27 \times 10^{-6} \cdot 1.124^x \ \ and \quad {}_tp_{45} = e^{-\int_0^t \mu_{45+s} ds}$$

If premium payments are made annually and are life contingent, then the present value is

$$\bar{a}_{45:\overline{20|}} = \int_0^{20} e^{-\delta t} {}_tp_{45} dt = \int_0^{20} 1.05^{-t} e^{-\int_0^t .00022+27\times 10^{-6}\cdot 1.124^{45+s} ds} dt = 12.9295$$

Finally, the net premium follows.

$$P = S\left(\frac{1}{\bar{a}_{45:\overline{20|}}} - d\right) = 100{,}000\left(\frac{1}{12.9295} - .047619\right) = \$\ 2972.35.$$

2.3.4. Modeling Issues

Classical probability theory has limited effectiveness when dealing with problems in which some of the principal sources of uncertainty are non-statistical in nature. Life tables group the risk of death by age and give the statistical mean as life expectancy. The mortality forces have parameters to help fit the data found in life table. Arguing the efficacy of the model leads to the pooling of risks by life insurance companies, whereby all individuals of the same risk class are insured at the same rate. Higher mortalities may offset costs of favorable mortalities (and vice-versa). Preferred programs are a refinement of the concept.

Age is inevitably correlated to mortality, as over time one's health decays. In reality, each individual has his own life expectancy depending not only on age but on his physical condition, gender, family medical history, financial background, and many other factors. Fuzzy mathematics, and precisely fuzzy statistical clustering, is the science able to combine these attributes.

Furthermore, premium calculations often use a constant rate i and a force of mortality δ_t over time. The actuary may use time-series regression to forecast future interest rates but this does not allow for extreme events ranked as a "Black Swan" events. The Black Swan theory of Nicholas Taleb (2001) describes extreme outliers in a theatrical way that shows a major impact, and yet causes can only be found after the fact through retrospective analysis. Fuzzicists use the theory of possibility to extend the grasp of probability. Hence, the next section presents Zadeh's engineering and its applications.

3. Fuzzy Mathematics

Let us adopt the formal framework for mathematicians to present new concepts: Definitions, Theorems and Examples.

3.1. Fuzzy Set Methodology

The theory for classical/crisp sets remains the same as it is normally defined by a collection of elements or objects. $x \in X$ that can be finite, countable, or infinite. Each element can either belong to or not belong to a set $A \subseteq X$. A good general reference for the theory of fuzzy sets is H.J. Zimmerman "Fuzzy Set Theory and its Applications" (2001). The notation given in that text is used for this thesis.

3.1.1. Fuzzy Set Theory

Definition 3-1. If X is a collection of objects *x*, then a ***fuzzy set*** *A* in X is a set of ordered pairs: $A = \{(x, m_A(x)) | x \in X\}$ and $m_A: X \to M$ is the membership function of A for all *x* in *X*, where M is a bounded subset of $\mathbb{R}^+$ or $[0, \infty)$ called the membership space.

The membership function is not limited to values between 0 and 1. If $sup_x m_A(x) = 1$, then the fuzzy set *A* is ***normal***. A fuzzy set can always be *normalized* by dividing $m_A(x)$ by its supremum as shown below.

$$\frac{m_A(x)}{sup_x m_A(x)}$$

One may omit elements with membership grade of 0 in writing the fuzzy sets. This is illustrated below.

Example 3-1. (Finite set.) Let $X = \{1,\ 2,\ 3, \dots, 10\}$ the set of houses with *x* the number of bedrooms. The fuzzy set "comfortable house for 4 individuals" is

$$A\ =\ \{(1,.2),(2,.5),(3,.8),(4,1),(5,.9),(6,.4)\}$$

(Infinite Set) Let $X = \mathbb{R}$, and A= “real numbers considerably larger than 25”

$$m_A(x) = \begin{cases} 0,\ if\ x \leq 25 \\ \dfrac{1}{1 + (x - 25)^{-2}},\ x > 25 \end{cases}$$

B= “real numbers almost equal to 10” = $\{(x, m_B(x)) | m_B(x) = [1 + (x - 10)^2]^{-1}\}$

Definition 3-2. The *Support* of a fuzzy set *A*, *S(A)* is the crisp set of all $x \in X$ such that $m_A(x) > 0$. The crisp set of elements in *A* with grade of membership greater or equal to $\alpha \in M$ is called the ***α-level set*** or ***α-cut****:* $A_\alpha = \{x \in X | m_A(x) \geq \alpha\}$.

Note $A'_\alpha = \{x \in X | m_A(x) > \alpha\}$ is called *"strong α-level set*" or *"strong α-cut”*.

Example 3-2. Recall the finite set of Example 3-1. The following are α-level sets:

$$A_0 = S(A) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}\ A_{.5} = \{2,\ 3,\ 4,\ 5\}\ \ A_{.8} = \{3,\ 4, 5\}\ \ A_1 = \{4\}$$

The strong α-level set for $\alpha = .8$ is $A'_{.8} = \{4, 5\}$

Definition 3-3. For finite *A*, the ***cardinality*** is defined as

$$|A| = \sum_{x \in X} m_A(x)\ \ or\ \ |A| = \int_x\ m_A(x)dx\,,$$

and the *relative cardinality* of *A* is $\|A\|$ when divided by $card(X)$, the number of elements.

$$\|A\| = \frac{|A|}{card(X)}$$

Example 3-3. Same A as in finite set of example 3-1, $|A| = .2 + .5 + .8 + 1 + .9 + .4 = 3.8$ and the relative cardinality $\|A\| = {}^{3.8}/_{10} = .38$

Definition 3-4. (Zadeh 1968) The *intersection* of fuzzy sets A and B is $C = A \cap B$ and the membership function is defined pointwise by

$$m_C(x) = \min\{m_A(x),\ m_B(x)\},\ x \in X.$$

The *union* is $D = A \cup B$ and the membership function is defined pointwise by

$$m_D(x) = max\{m_A(x),\ m_B(x)\},\ x \in X$$

Definition 3-5. The *complement of a normal fuzzy set* is the set A^C and the membership function is defined by $m_{A^C}(x) = 1 - m_A(x),\ \ x \in X$

Example 3-4. Recall the finite set of example 3-1, and let B be the fuzzy set "large house" with $B = \{(3,.4),(4,.7),(5,.9),(6,1),(7,1),(8,.5)\}$. Then the intersection is

$$C = A \cap B\ =\ \{(3,.4),(4,.7),(5,.9),(6,.4)\}$$

and the union $D = A \cup B =\ \{(1,.2),(2,.5),(3,.8),(4,1),(5,.9),(6,1),(7,1),(8,.5)\}$

The complement "not large house" may have small or extra-large houses:

$$B^C = \{(1,1),(2,1),(3,.6),(4,.3),(5,.1),(8,.5),(9,1),(10,1)\}$$

Now, consider the infinite set A in example 3-1, A="real numbers considerably larger than 25" and C="$x \cong 26$" with $m_C(x) = \frac{1}{1+(x-26)^4}$.

Their intersection and union are:

$$m_{A\cap C}(x) = \begin{cases} 0,\ if\ x \le 25 \\ \min\left\{\dfrac{1}{1+\dfrac{1}{(x-25)^2}}, \dfrac{1}{1+(x-26)^4}\right\}, if\ x > 25 \end{cases}$$

$$m_{A\cup C}(x) = \max\left\{\dfrac{1}{1+\dfrac{1}{(x-25)^2}}, \dfrac{1}{1+(x-26)^4}\right\}, \forall x \in \mathbb{R}$$

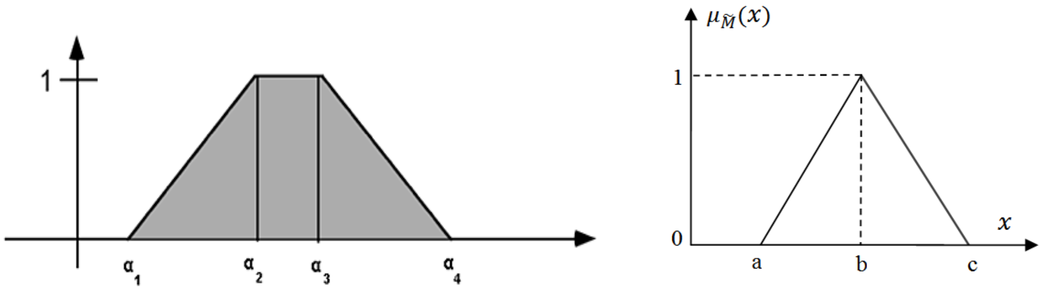
Definition 3-6. A ***type* 2 *fuzzy set*** is a fuzzy set whose membership function is also a fuzzy set on the space M. More generally, for some integer *m>1,* a *type m fuzzy set* in X is a fuzzy set whose membership values are type *m-1* fuzzy sets on M.

From a practical point of view, type m fuzzy sets for $m \ge 3$, are extremely difficult to measure or visualize. Examples are fuzzy sets with membership function as a probabilistic set (Hirota 1981), or an *intuitionistic* fuzzy set of ordered triples (Atanassov and Stoeva 1983).

Definition 3-7. A ***fuzzy number*** K is a fuzzy subset of the real line whose membership function is a continuous mapping defined by $m_K\colon \mathbb{R} \to M \cap [0,1]$ and is represented solely by (a_1, a_2, a_3, a_4) such that:

- $m_K(x) = 0$ for $x \in (-\infty, a_1] \cup [a_4, \infty)$,
- $m_K(x)$ increases linearly on $[a_1, a_2]$,
- $m_K(x) = 1$ for $x \in [a_2, a_3]$,
- $m_K(x)$ decreases linearly on $[a_3, a_4]$,

In the case $a_2 = a_3$, it is a ***triangular*** fuzzy number; otherwise it is a ***trapezoidal***.



3.1.2. Fuzzy Set-Theoretic Operations

Operations in this section apply to both normal and other fuzzy sets over the same underlying set.

Operation 3-1. (Zadeh 1965) The *algebraic sum* of two fuzzy sets (probabilistic sum) $A + B$ is defined by $m_{A+B}(x) = m_A(x) + m_B(x) - m_A(x) \cdot m_B(x)$

The *algebraic product* of two fuzzy sets $A \cdot B$ is defined by $m_{A \cdot B}(x) = m_A(x) \cdot m_B(x)$

Figure 6. Trapezoidal and Triangular Fuzzy Number

Operation 3-2. The *bounded sum* of fuzzy sets $A \oplus B$ is defined by

$$m_{A \oplus B}(x) = \min\{1, m_A(x) + m_B(x)\}.$$

The *bounded difference* $A \ominus B$ is defined by $m_{A \ominus B}(x) = \max\{0, m_A(x) + m_B(x) - 1\}$.

Operation 3-3. The *Cartesian product* of fuzzy sets $A_1, \ldots, A_n$ in $X_1, \ldots, X_n$ respectively is a subset of $X_1 \times \ldots \times X_n$ with membership function

$$m_{(A_1 \times \ldots \times A_n)}(x) = \min_i \{ m_{A_i}(x_i) | x = (x_1, \ldots, x_n),\ x_i \in X_i \}$$

Operation 3-4. The *n^th^ power* of a fuzzy set A is A^n with membership function

$$m_{A^n}(x) = [m_A(x)]^n,\ x \in X$$

This mapping is a *concentration* if $n > 1$, and a *dilation* if $n < 1$. When combined in order to increase the membership grade of certain elements and/or reduce the grade of others, it is an *intensification.*

Example 3-5. Define the fuzzy sets $J = \{(a, .5), (b, 1), (c, .2)\}\ and\ \ H = \{(a, .3), (b, .7)\}$ The above definitions are then illustrated by the following:

$$J + H\ = \{(a, .65), (b, 1), (c, .2)\}$$

$$J \cdot H = \{(a, .15), (b, .7)\}$$

$$J^2 = \{(a, .25), (b, 1), (c, .04)\}$$

$$J \oplus H = \{(a, .8), (b, 1), (c, .2)\}$$

$$J \ominus H = \{(b, .7)\}$$

$$J \times H = \{[(a; a), .3], [(b; a), .3], [(c; a), 0], [(a; b), .5], [(b; b), .7], [(c; b), .2]\}$$

Min and *max* for intersections and unions are not generally smooth functions. Their explicit formula is a sequence of interval brackets or piecewise-continuous functions. Fuzzy mathematics, though named “fuzzy” as if to emphasize imprecision, focuses on improving precisions. Fuzzicists eventually decided to consider softer definitions of "intersection". Many were suggested, with all satisfying the following properties (Dubois & Prade 1980):

i. Cumulative effects: $m_{A\cap B}(x) \leq \min\{m_A(x), m_B(x)\}$, $if\, m_A(x) < 1,\ m_B(x) < 1$

ii. Interactions between criteria: Assume $m_A(x) < m_B(x) < 1$. Then the effect of a decrease of $m_A(x),\ on\ m_{A\cap B}(x)$ may depend on $m_B(x)$.

iii. Compensations between criteria: $if\ m_A(x), < 1,\ \ m_B(x) < 1$ the effect of a decrease of $m_A(x)\ on\ m_{A\cap B}(x)$ can be erased by an increase of $m_B(x)$ (unless $m_B(x) = 1$)

Earlier mathematics applications of triangular norm satisfy these criteria (Menger 1942). Hence, intersections can be defined by t-norms and unions by a t-conorms.

Definition 3-8. A *t-norm* is any bivariate function *t* from $M \times M \rightarrow [0,1]$ such that for all fuzzy sets *A*, *B*, *C* and *D in X* each with its own membership function, $t(a, b)$ is commutative, associative and monotonic, and $\forall x \in X$:

$$t(0,0) = 0,$$

$$t(m_A(x), 1) = t\big(1, m_A(x)\big) = m_A(x)$$

$$t(m_A, m_B) = t(m_B, m_A)\ commutativity.$$

$$t\big(m_A, t(m_B, m_C)\big) = t(t(m_A, m_B), m_C)\ associativity$$

$$if\ m_A \leq m_C\ and\ m_B \leq m_D, then\ t(m_A, m_B) \leq t(m_C, m_D)\ \text{ monotonicity}$$

Definition 3-9. A *t-conorm* or *s-norm* is a commutative, associative and monotonic bivariate function s from membership space $M \times M \rightarrow [0,1]$ such that

$$\forall x \in X, \qquad s(1,1) = 1;\ \ s(m_A(x), 0) = s\big(0, m_A(x)\big) = m_A(x).$$

For mathematical derivations, proofs and other t-norms the reader is referred to Klement et al. (1994)

Theorem 3-5. (Alsina 1985). *t-norms* and *t-conorms* are dual such that the function *t* is defined as $t(m_A, m_B) = 1 - s((1 - m_A), (1 - m_B))$.

Lemma 3-1. (Hamacher 1978). The intersection of two fuzzy sets A and B may be defined as a *t-norm* with

$$t(m_A, m_B) = \frac{m_A \cdot m_B}{p + (1-p)[m_A + m_B - m_A \cdot m_B]}, \qquad for\ p \geq 0$$

and the union of two fuzzy sets A and B is defined as a *t-conorm* with

$$s(m_A, m_B) = \frac{(p' - 1)m_A \cdot m_B + m_A + m_B}{1 + p' \cdot m_A \cdot m_B}, \qquad for\ p' \leq -1$$

Lemma 3-2. (Yager 1980). The intersection of two fuzzy sets A and B may be defined as a *t-norm* with

$$t(m_A, m_B) = 1 - \min\{1, [(1 - m_A)^p + (1 - m_B)^p]^{1/p}\}, \qquad for\ p \geq 1$$

and the union of two fuzzy sets A and B is defined as a *t-conorm* with

$$s(m_A, m_B) = \min\left\{1, ({m_A}^p + {m_B}^p)^{1/p}\right\}, \qquad for\ p \geq 1.$$

For intersections or unions between more than two fuzzy sets, the method recommended is to merge them two-by-two or progressively. This means, first combine $A \cap B,$ then $(A \cap B) \cap C,$ then $(A \cap B \cap C) \cap D, etc.$, as t-norms and t-conorms are associative and commutative by Definition 3-8.

Note that the Hamacher norm for $p = 1$ corresponds to the t-norm "Algebraic product". Taking the infinity norm for the Yager norm (as $p \to \infty$), both the intersection and the union give the minimum operator and the maximum operator, respectively. Many other norms are used in Fuzzy Set Theory, see Table 9 for a list of common t-norms and s-norms (Bonissone and Decker 1986).

An important arithmetic result in Dubois & Prade (1978, 1980), for the sum and product of fuzzy numbers, is associativity and commutativity of inverses.

Theorem 3-2. (Dubois, Prade 1980) Let A, B be *trapezoidal fuzzy numbers* with membership notations $m_A(x) = U_A(x)$ and $m_B(x) = U_B(x)$. U_{A1} is the increasing part of $U_A(x)$ on $[a_1, a_2]$, and U_{A2} is the decreasing part on $[a_3, a_4]$. Their inverses are

$$V_{A1} = U_{A1}^{-1} \; and \; V_{A2} = U_{A2}^{-1}.$$

Then the sum $C = A \oplus B$ has membership functions on $[a_1, a_2]$ and $[a_3, a_4]$

$$U_{C1} = [U_{A1}^{-1} + U_{B1}^{-1}]^{-1} \;\; or \;\; V_{C1} = V_{A1} + V_{B1}$$

$$U_{C2} = [U_{A2}^{-1} + U_{B2}^{-1}]^{-1} \;\; or \;\; V_{C2} = V_{A2} + V_{B2}$$

and the product $D = A \odot B$ has membership functions on $[a_1, a_2]$ and $[a_3, a_4]$

$$U_{D1} = [U_{A1}^{-1} \cdot U_{B1}^{-1}]^{-1} \;\; or \;\; V_{D1} = V_{A1} \cdot V_{B1}$$

$$U_{D2} = [U_{A2}^{-1} \cdot U_{B2}^{-1}]^{-1} \;\; or \;\; V_{D2} = V_{A2} \cdot V_{B2}$$

3.2. Fuzzy clustering algorithms

In Data mining, clustering techniques are used to put together objects showing similar characteristics within the same group, and to separate objects with different characteristics. To do so, one must write algorithms that permit iterations before stabilizing. These clustering techniques are made for detection and handling of noisy data or outliers. There are two approaches: **Hard clustering** and **Soft clustering**. Hard data clustering divides data elements into clusters in such a way that one data item can belong to one cluster only. This is the crisp version for data mining. Soft clustering, also known as *fuzzy clustering*, allocates data elements to one or more clusters based on their membership levels in the different clusters.

Fuzzy C-Means (Dunn 1973) is the most popular and efficient technique of soft computing. Note this is the fuzzy logic version of the most popular hard clustering method: *K-means algorithm*. Other names for it are Soft computing and Fuzzy K-means. The fuzzy c-means (FCM) algorithm requires steps such as the calculation of cluster centers, assignment of points to centers by taking their Euclidian distances, and continuous iteration until the cluster centers stabilizes (Thomas 2012).

Definition 3-10. Consider the set of data $X = \{x_1, \ldots x_n\}$, and $V_{c,n}$ the set of real $c \times n$ matrices ($2 \leq c \leq n$). The matrix $\widetilde{U} = \left[m_{jk}\right] \in V_{c\times n}$, with $m_{jk} \in [0,1],\ 1 \leq j \leq c,\ 1 \leq k \leq n$ is called *a* ***fuzzy-c partition*** if it satisfies the following conditions [Bezdek 1981]:

$$\sum_{j=1}^{c} m_{jk} = 1 \quad and \quad 0 < \sum_{j=1}^{n} m_{jk} < n$$

Example 3-6. Let $X = \{x_1, x_2, x_3\}$. A fuzzy *2-partitions can be*

$$\widetilde{U}_1 = \begin{bmatrix} 1 & .5 & 0 \\ 0 & .5 & 1 \end{bmatrix} \quad or \quad \widetilde{U}_2 = \begin{bmatrix} .8 & 1 & .9 \\ .2 & 0 & .1 \end{bmatrix}.$$

In $\widetilde{U}_1$, x_1 *and* x_3 are fully included in clusters c_1 and c_2 respectively, and x_2 is equally contained in both. In $\widetilde{U}_2$, x_2 is fully included in clusters c_1, when x_1 *and* x_3 are still 20% and 10% in cluster 2, respectively. Note the fuzzy c-partition conditions are met: the sum of each column is 1, and the row sums are always less than 3, the number of data.

Fuzzy c-means became more popular for symmetric data such as $\widetilde{U}_1$, where the K-means algorithm would fail. This is due to the presence of midpoints ($m_{ik} = 0.5$). Eventually the algorithm would insert it in a random cluster when it may as well belong to another.

Example 3-7. This is the case for the popular butterfly example (Zimmerman 1994).

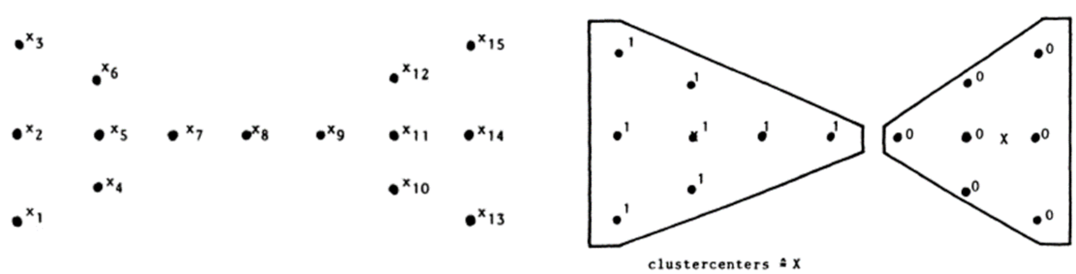


Figure 7. The midpoint bias of the Butterfly problem in data mining

Let us define an algorithm to find these fuzzy c-partitions. For an FCM algorithm, it is necessary to choose a few parameters. These are the *desired number of clusters c*

($2 \le c \le n$*);* an *exponential weight r* $(1 < r < \infty)$ often called the *fuzzy parameter;* the *type of norm* $\|\cdot\|$ (here the Euclidean distance will serve as norm); and a *termination criterion* $\varepsilon > 0$. A method to initialize the membership matrix $\widetilde{U}^{(l)} \in V_{c,n}$, for $l \ge 0$ for each iteration is also necessary. Here membership values and cluster centers are given by: $\forall\ 1 \le i \le c,\ 1 \le j \le c$ and $1 \le k \le n$,

$$m_{jk} = [\sum_{i=1}^{c} \frac{\|x_k - c_j\|}{\|x_k - c_i\|}]^{-2/_{r-1}} \quad with \quad c_j = \frac{\sum_{k=1}^{N} m_{jk}^{r} \cdot x_k}{\sum_{k=1}^{N} m_{jk}^{r}}$$

To summarize the steps:

Step 1. Choose c, r, ε.

Step 2. Initialize $\widetilde{U}^{(l)} = \left[m_{jk}\right]^{(l)} \in V_{c,n}$, for $l \ge 0$, set $l = 0$.

Step 3. Calculate the c fuzzy cluster centers $\{c_j^{(l)}\}$ by using $\widetilde{U}^{(l)}$.

Step 4. Calculate the new membership matrix $\widetilde{U}^{(l+1)}$by using $\{c_j^{(l)}\}$**.** If

$x_k \neq c_j^{(l)}$, Else set $m_{jk} = \begin{cases} 1\ for\ j = k \\ 0\ for\ j \neq k \end{cases}$

Step 5. Calculate $\Delta = \left\|\widetilde{U}^{(l+1)} - \widetilde{U}^{(l)}\right\|$.

Step 6. If $\Delta > \varepsilon,\ set\ l = l + 1$ and go to *Step 2.* If $\Delta \le \varepsilon \rightarrow stop$

Example 3-7.(continued) The data of the butterfly were processed with a fuzzy 2-means algorithm. Choose $c = 2, \varepsilon = .01, m = 1.25$ with the Euclidean norm. In 6 iterations the clustering results in the memberships and cluster centers as shown in Figure 8. The butterfly fuzzy c-partition gives the membership level of each point in its cluster. The partition is:

$$\widetilde{U} = \begin{bmatrix} .99 & 1 & .99 & 1 & 1 & 1 & .99 & .47 & .01 & 0 & 0 & 0 & .01 & 0 & .01 \\ .01 & 0 & .01 & 0 & 0 & 0 & .01 & .53 & .99 & 1 & 1 & 1 & .99 & 1 & .99 \end{bmatrix}$$

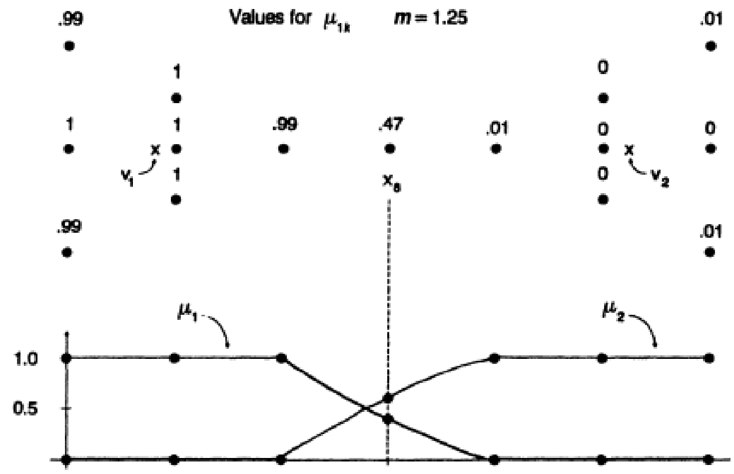


Figure 8. Butterfly Fuzzy Clustering for m=1.25

To improve the fuzzy c-partitions, one only needs to increase the fuzzy parameter *m* and choose c suitably. The same process is done in Figure 9 for *m* = 2 and all other default parameters. The new butterfly fuzzy c-partition is

$$\widetilde{U} = \begin{bmatrix} .86 & .97 & .86 & .94 & .99 & .94 & .86 & .5 & .14 & .06 & .01 & .06 & .14 & .03 & .14 \\ .14 & .03 & .14 & .06 & .01 & .06 & .14 & .5 & .86 & .94 & .99 & .94 & .86 & .97 & .86 \end{bmatrix}.$$

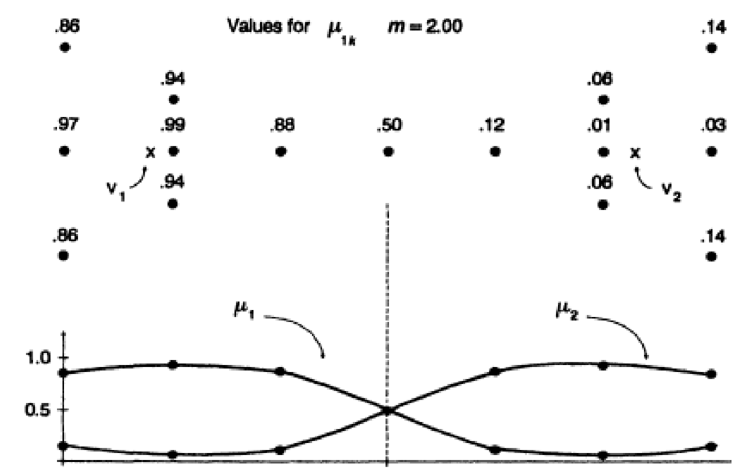


Figure 9. Butterfly Fuzzy Clustering for m=2

PART 2: APPLICATIONS TO LIFE CONTINGENCIES

This part illustrates the use of the above applications in a life insurance setting. It involves the translation of medical records for applicants through fuzzy decision-making processes. Through a series of case studies, we will observe the classification of policyholder risks using a fuzzy system for preference; the definition of survival probability formula using fuzzy parameters to counter interest rate fluctuations; and lastly, the clustering of policyholder sociodemographic data by the fuzzy c-means algorithm.

1. Classification of Preferred Policyholders in Life Insurance

Let X be the set of applicants for a life insurance. The carrier may have a set of pricing policies or a preference program. For instance, one carrier may offer a bonus of 15% more coverage, if the applicant is not a smoker or has not smoked for a minimum of 12 months prior to application. Another may be even more generous and give a bonus of 50% more coverage with no increases in premium if the applicant achieves the highest degree of health defined by the company. This corresponds to an applicant who has *not smoked* for a year, a *resting pulse* of 72 or below, a *blood pressure* below 134/80, a *cholesterol reading* below 200, and does not participate in *hazardous sports*. To reach the perfection standard laid out by the CDC/WHO (Centers for Disease Control and

Prevention/World Health Organization), the applicant must follow weekly exercise programs, be within *Body Mass Index (BMI)* specified height/weight restrictions, and have no family history of deaths prior to 50 years old due to kidney/heart disease, stroke or diabetes. However, this collection of individuals is extremely uncommon. Thus, for marketing purposes, the company may want to accept the preferred status for a person lacking only a few of these criteria. Fuzzy Set Theory will be very resourceful in modeling these preference classes for underwriting purposes.

Case Study 1. (Lemaire 1990) For simplicity, limit the study to 4 variables $t_i,\ for\ i = 1,2,3,4$ and the fuzzy sets: Cholesterol in Blood (A), Systolic Blood Pressure (B), BMI (C) and Cigarette Consumption (D). Each applicant $x \in X$ is represented with its information by $x(t_1, t_2, t_3, t_4)$. Lemaire uses the following membership functions for the fuzzy variables.

$\boldsymbol{t_1}$ **Blood Cholesterol** (mg/dl)

$$m_A(x, t_1) = \begin{cases} 1,\ if\, t_1 \leq 200 \\ 1 - 2\left(\dfrac{t_1 - 200}{40}\right)^2,\ if\ 200 \leq t_1 \leq 220 \\ 2\left(\dfrac{240 - t_1}{40}\right)^2,\ \ if\ 220 \leq t_1 \leq 240 \\ 0,\ \ if\ 240 < t_1 \end{cases}$$

$\boldsymbol{t_2}$ **Blood Pressure** (mmHg)

$$m_B(x,t_2) = \begin{cases} 1, \; if \; t_2 \le 130 \\ 1-2\left(\frac{t_2-130}{40}\right)^2, \; if \; 130 \le t_2 \le 150 \\ 2\left(\frac{170-t_1}{40}\right)^2, \; if \; 150 \le t_2 \le 170 \\ 0, \; if \; 170 < t_2 \end{cases}$$

t_3 Body Mass Index (%)

$$m_C(x,t_3) = \begin{cases} 0, & if \; t_3 \le 60 \\ 2\left(\frac{t_3-60}{25}\right)^2, & if \; 60 \le t_3 \le 72.5 \\ 1-2\left(\frac{85-t_3}{25}\right)^2, & if \; 72.5 \le t_3 \le 85 \\ 1, & if \; 85 < t_3 \le 110 \\ 1-2\left(\frac{t_3-110}{20}\right)^2, & if \; 110 \le t_3 \le 120 \\ 2\left(\frac{130-t_3}{20}\right)^2, & if \; 120 \le t_2 \le 130 \\ 0, & if \; 130 < t_3 \end{cases}$$

t_4 Cigarette Consumption

$$m_D(x,t_4) = \begin{cases} 1, & if \; t_4 = 0 \\ 0, & if \; t_4 > 0 \end{cases}$$

Choose at random an applicant $x = x(210mg/dl, 145mmHg, 112\%, 0)$

The fuzzy set $E = A \cap B \cap C \cap D$ determines how fit a customer is for the preferred program. Recall the t-norms for intersection of fuzzy sets in Theorem 3-5 and Table 9. The pricing actuary of the life insurance carrier will choose which operator works best among the following.

- **Minimum operator**,

$$m_E(x; 210,\ 145,\ 112,\ 0) = \min(.875,\ .71875,\ .98, 1) = .71875$$

- **Algebraic product**, $m_E(x) = (.875)(.71875)(.98)(1) = .6163$

- **Bounded difference**,

$$m_E(x) = \max[0,\ .875 + .71875 + .98 + 1 - 3] = .57375$$

- **Hamacher operator** for *p=1/2*,

$$m_E(x; 210,\ 145) = \frac{(.875)(.71875)}{.5 + (1 - .5)[.875 + .71875 - (.875)(.71875)]} = .6402$$

$$m_E(x; 210,\ 145,\ 112{,}0) = \frac{(.6402)(.98)}{.5 + (1 - .5)[.6402 + .98 - (.6402)(.98)]} = .629$$

- **Yager operator** for *p=2*,

$$m_E(x) = 1 - \min\{1, [(1 - .875)^2 + (1 - .71875)^2 + (1 - .98)^2]^{\frac{1}{2}}\} = .69157$$

Thus, no computation of $x(210, 145, 112, 0)$ health will give a preference status if the requirement is 100% grade of membership. Note that a smoker is never preferred; every operator gives 0% membership. The pricing actuary may allow a few infringements to perfection with a statement such as: "*An applicant is considered preferred if he meets at least 75% of the requirements of the CDC/WHO health index*." This step may require, in crisp set theory, the creation of new membership functions. If the actuary uses only the minimum operator which is the strictest operator, underwriters will obtain rules defined by

$$t_1 \leq 214.2; \qquad t_2 \leq 144.2; \qquad 76.2 \leq t_3 \leq 117.1; \quad t_4 = 0$$

This means, any applicant with information not in one of these intervals will be below 75% and cannot be a preferred policyholder.

Fuzzicists may, as in Definition 3-2, take the alpha-level set to refine the results and create classes such as the one from Section 2.1. Take, for any α, E_α to be the crisp set of policyholders with grade of membership greater than α. Choose $\alpha = 75\%$, so that after evaluating the membership of an applicant in the fuzzy set E above, he becomes preferred if $m_E(x) \geq 0.75$. Clearly, the policyholder $x(210, 145, 112, 0)$ is still not part of the preferred program under any operator/t-norm.

Then, we build another preference class, as for Deloitte Consulting (Figure 4). Say "*An applicant is considered **Superpreferred** if he meets at least 75% of the requirements of the CDC/WHO health index, and he is considered **preferred** if he qualifies from 65% to 75%.*" The same process works and the actuary does not need to build membership functions. By taking the alpha cut $E_{.65}$, the applicant $x(210, 145, 112, 0)$ falls in the range for the preferred program benefits only if the actuary decides to use *minimum operator* or *Yager t-norm with p=2.* Otherwise $x(210, 145, 112, 0)$ may fall in the standard class or some residual classes.

In reality, each criterion has its own importance. To show this difference, Fuzzicists use the operations of *concentration, dilation,* and *intensification*. Suppose blood pressure better predicts future health complications, while cholesterol level does less well. The actuary may then concentrate the fuzzy number t_1 for the cholesterol by taking the square; while dilating t_2 blood pressure by taking the square root. Then we have the following:

- **Min operator**,

$$m_E(x; 210,\ 145,\ 112,\ 0) = \min(.875^2,\ \sqrt{.71875},\ .98,\ 1) = .7656$$

- **Algebraic product**, $m_E(x) = (.875^2)\sqrt{.71875}(.98)(1) = .6361$

- **Bounded difference**,

$$m_E(x) = \max[0,\ .875^2 +\ \sqrt{.71875} + .98 + 1 - 3] = .59267$$

- **Hamacher operator** for *p=1/2*,

$$m_E(x; 210,\ 145) = \frac{(.875^2)(.71875)^{.5}}{.5 + (1-.5)[.875^2 +\ .71875^{.5} - (.875^2)(.71875)^{.5}]} = .6608$$

$$m_E(x; 210,\ 145,\ 112{,}0) = \frac{(.6608)(.98)}{.5 + (1-.5)[.6608 + .98 - (.6608)(.98)]} = .6497$$

- **Yager operator** for *p=2*,

$$m_E(x) = 1 - \min\{1, [(1 -.875^2)^2 + (1 -.71875^{.5})^2 + (1 - .98)^2]^{\frac{1}{2}} = .7198$$

Then $x(210, 145, 112, 0) \in E_{.75}$ when using the minimum operator, i.e. it is a Superpreferred policy, and $x(210, 145, 112, 0) \in E_{.65}$ when using the Hamacher operator for $p = .5$ and the Yager operator for $p = 2$, making him a Preferred policy.

This shows how fuzzy decision-making processes can be used to translate medical records and facilitate the classification of policyholder risks. It is in fact a faster and simpler process for underwriters. Preferred classes offer bonuses on coverage but the author has yet to show how to calculate these benefits and premiums, using fuzzy set theory. For that it is necessary to have fuzzy survival functions.

2. Fuzzy Survival Probability

The future lifetime may be represented as a fuzzy random variable (FRV) when one adds to it some linguistic variables; a basis for fuzzy logic. Assume for a moment the awful event where a medical doctor tells someone they only have a short time left to live. *Short future lifetime (S)*, *Medium future lifetime (M)*, and *Long future lifetime (L)* can be considered FRVs over the lifetime probability space Ω mentioned above. In this case, fuzzy sets and survival probabilities are combined. This scenario is best illustrated in Puri and Ralescu (1986) and Shapiro (2013). The following case study puts the problem in context.

Case Study 2. (Shapiro 2013) Consider the task of giving post-retirement planning advice to new retirees. At this juncture, it may be necessary to know how far their future lifetime will extend. The linguistic lifetime scale S, M, L cited above can be retained. Puri and Ralescu describe a function T that assigns a membership value to each retiree death probability time event $\omega_i \in \Omega$. This is done so that $T(\omega_i)$ is equated to the highest of the membership functions $m_S(\omega_i), m_M(\omega_i), m_L(\omega_i)$. Retirement is assumed to be 65 years of age, so future lifetime starts from $(x) = 65$. Using the Gompertz law of mortality, we build simplistic fuzzy survival probabilities for S, M, L.

Sivanandam et al. (2007) offers a catalog of methods for the development of membership functions (MF) but a simplistic model for a fuzzy set A is as follows:

$$m_A(x) = \begin{cases} \dfrac{x - x_L}{x_M - x_L}, & x_L \leq x \leq x_M \\ \dfrac{x_U - x}{x_U - x_M}, & x_M \leq x \leq x_U \\ 0, & otherwise \end{cases}$$

where x_L is the lower bound, x_M is the midpoint and x_U is the upper bound of the fuzzy number. In the same fashion, the lifetime scale MFs and their graphs are defined below (Figure 10).

$$m_S(t) = \begin{cases} 1, & 0 \le t \le 5 \\ \dfrac{15 - t}{10}, & 5 \le t \le 15 \\ 0, & otherwise \end{cases}$$

$$m_M(t) = \begin{cases} \dfrac{t - 10}{5}, & 10 \le t \le 15 \\ \dfrac{20 - t}{5}, & 15 \le t \le 20 \\ 0, & otherwise \end{cases}$$

$$m_L(t) = \begin{cases} 1, & t \le 25 \\ \dfrac{x - 15}{10}, & 15 \le t \le 25 \\ 0, & otherwise \end{cases}$$

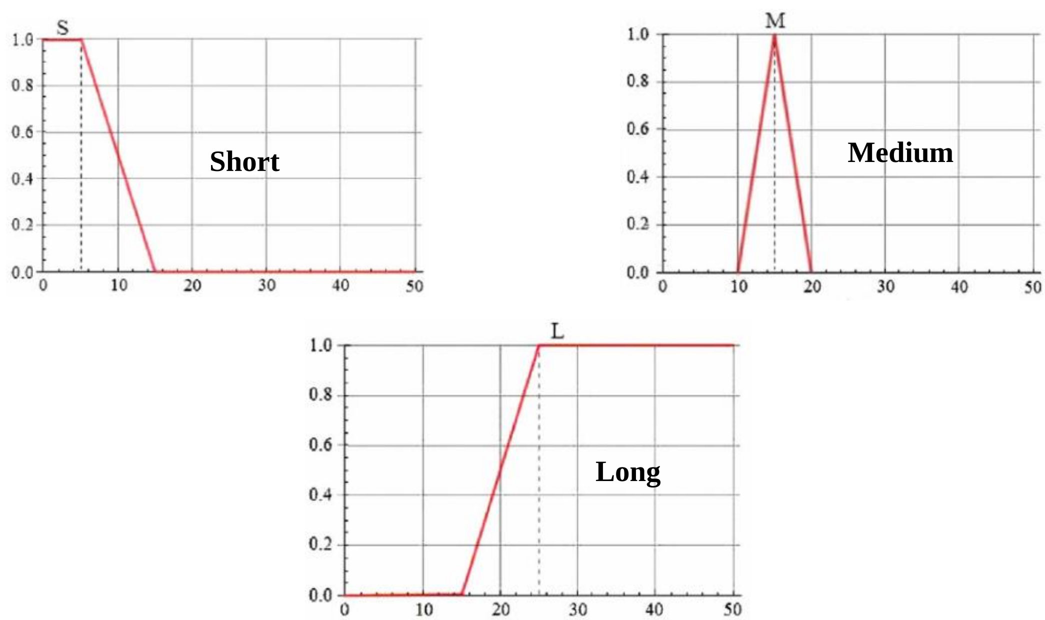


Figure 10. Short, Medium, Long future lifetime membership for 65 years old

However, this only gives the lifetime as a fuzzy variable. The purpose is to make it a *fuzzy random variable*; that is a RV for which each value has a membership grade in the linguistic scale considered. Figure 11 offers a simple representation of this:

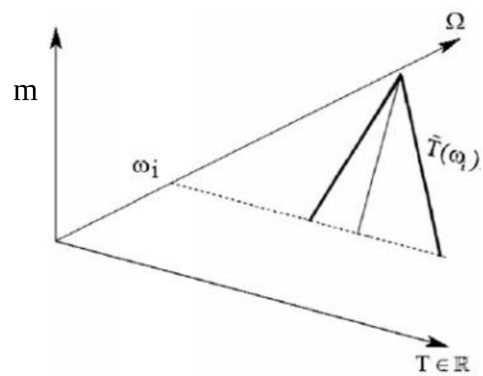


Figure 11. A Fuzzy random variable representation

Each event of death $\omega_i \in \Omega$ has probability density $P(\omega_i) \in \mathbb{R}$ (contained in $[0,1]$ if normalized), and this event has a degree of membership in the three groups Short, Medium or Long future lifetime. Finally, $T(\omega_i)$ outputs the highest degree of membership of ω_i, and is also a fuzzy random variable:

$$T(\omega_i) = \max\{m_S(\omega_i), m_M(\omega_i), m_L(\omega_i)\}$$

Now, let us consider the *fuzzy death risk* in a future lifetime. It weighs the risk of death in the whole linguistic groups instead of each individual event $\omega_i \in \Omega$. This means, for instance, if $m_S(\omega_i) > 0$ for all $i \in [1, n]$, such that every $\omega_1 \dots \omega_n \in S$, then the risk of the short lifetime is $\mathrm{P}\big(T(S)\big)$, the probability of the FRV short future lifetime S. It is equal to the expectation of the membership function in general (Zadeh 1968), such that

$$P\big(T(S)\big) = \int_0^n m_S(t) f_x(t)\mathrm{dt} = \int_0^n m_S(t)\ {}_t p_x \mu_{x+t} \mathrm{dt} = \mathrm{E}[m_S].$$

In our case:

$$P\big(T(S)\big) = \begin{cases} \int_0^n 1 \cdot\ {}_t p_x \mu_{x+t} \mathrm{dt}, & 0 \le t \le 5 \\ \int_0^n \frac{15-t}{10} \cdot\ {}_t p_x \mu_{x+t} \mathrm{dt}, & 5 \le t \le 15 \\ 0, & otherwise \end{cases}$$

Or equivalently,

$$P\big(T(S)\big) = \begin{cases} {}_n q_x, & 0 \le t \le 5 \\ \frac{1}{10}\big(15\ {}_n q_x - \dot{e}_x\big), & 5 \le t \le 15 \\ 0, & otherwise \end{cases}$$

Note that $\dot{e}_x$ is the *complete expectation of life* and $f_x(t)$ uses the Makeham's law of mortality for $x \ge 65$ as the assumed age for retirement. This gives the mortality probability. Using Definition 3-5 for the complement of a fuzzy variable, the finding of the survival probabilities becomes a simple process.

$$P(T(S)^C) = \begin{cases} {}_n p_x, & 0 \le t \le 5 \\ \frac{10 + \dot{e}_x - 15\ {}_n q_x}{10}, & 5 \le t \le 15 \\ 0, & otherwise \end{cases}$$

Hence, this combines the membership function and the mortality probabilities as in Figure 12. The same argument derives the Medium and Long fuzzy survival probabilities.

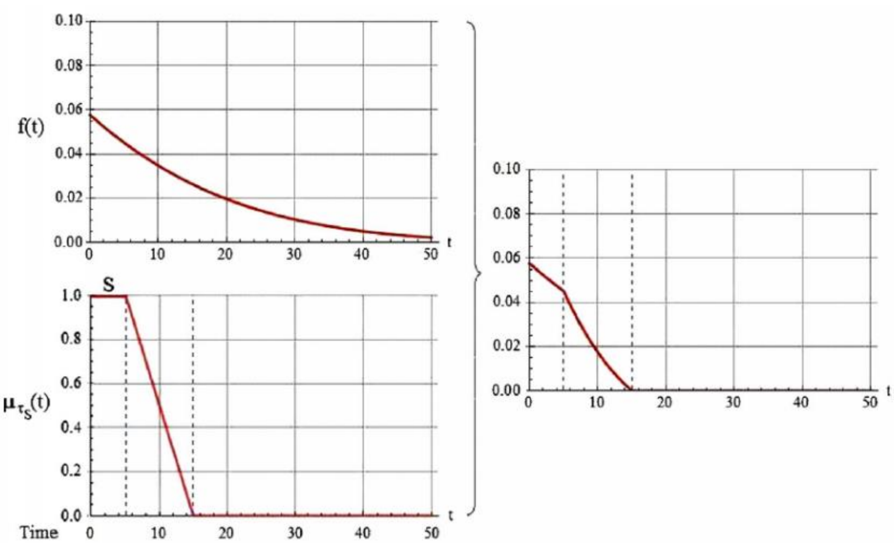


Figure 12. Combining for the Fuzzy survival probability

$f_x(t)$

3. Computation of Fuzzy Premiums

This se $m_S(t)$ es more numerical results for better understanding of the theory.

Case Study 3. (Buckley 1987, Lemaire 1990) *Fuzzy interest* $P(T(S))$

Compute the net single premium of an insurance benefit $S = \$1000$ on a 10-year pure endowment policy, issued to a life x aged (55), with ${}_{10}p_{55} = .87$ using a fuzzy interest rate *i*. The interest rate *i* (approximately 6%) is defined as a fuzzy probabilistic set (Hirota 1981). This is the trapezoidal fuzzy number below.

$$m_i(z) = \begin{cases} 0, & if\ z \le .03 \\ m_{i1} = 50z - 1.5, & if\ .03 < z \le .05 \\ 1, & if\ .05 < z \le .07 \\ m_{i2} = 4.5 - 50z, & if\ .07 < z \le .09 \\ 0, & if\ .09 < z \end{cases}$$

The net single premium is expressed as the actuarial present value of a pure endowment:

$$S \cdot {}_nE_x = S \cdot {}_np_x v^n = S \cdot {}_np_x (1 + \tilde{\imath})^{-n}$$

The tilde (~) above the "i" is meant to differentiate the fuzzy variables from the non-fuzzy (or crisp) ones. By plugging in the quantities from our assumption, we obtain the fuzzy present value below.

$$S \cdot {}_{10}E_{55} = S \cdot {}_{10}p_{55}(1 + \tilde{\imath})^{-10} = 1000 * .87(1 + \tilde{\imath})^{-10}$$

Following Theorem 3.6, take the inverse of the membership function of the interest rate $m_i(z)$. Precisely, ${m_{i1}}^{-1}(z)$ and ${m_{i2}}^{-1}(z)$, i.e.

$${m_{i1}}^{-1}(z) = .03 + .02z \quad and \quad {m_{i2}}^{-1}(z) = .09 - .02z$$

It is the inverses that go through all the computations, and as for normal piecewise functions, we have:

$$1000 * .87\left(1 + {m_{i1}}^{-1}(z)\right)^{-10} = 870(1.09 - .02z)^{-10}, \qquad and$$

$$1000 * .87\left(1 + {m_{i2}}^{-1}(z)\right)^{-10} = 870(1.03 + .02z)^{-10}$$

Again, take the inverses of the two new results in order to have the membership functions for the fuzzy set of $S \cdot {}_{10}E_{55}$. Notice the change in the intervals for z since the exponent "-10" is negative. We obtain the following membership function and the corresponding graph:

$$m_{S\cdot{}_{10}E_{55}}(z) = \begin{cases} 0, & if\ z \le 367.5 \\ 54.5 - 50\left(\frac{870}{z}\right)^{1/10}, & if\ 367.5 < z \le 442.26 \\ 1, & if\ 442.26 < z \le 534.1 \\ 50\left(\frac{870}{z}\right)^{1/10} - 51.5, & if\ 534.1 < z \le 647.36 \\ 0, & if\ 647.36 < z \end{cases}$$

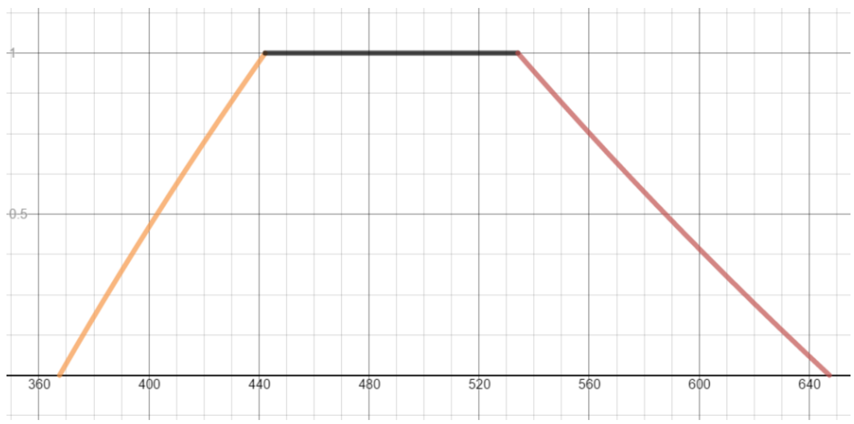


Figure 13. Membership function for a 10-year continuous life insurance for a 55 year-old

Case Study 4. (Lemaire 1990) *Fuzzy interest Rates and fuzzy survival probabilities*

Using the same assumptions as Case study 3 above, we compute the net single premium of a pure endowment with sum insured $S = \$1000$ for a life aged $x = (55)$ over $n = 10\ years$. In this case however, not only the interest rate is fuzzy but so is the survival probability. For notation purposes, write $\tilde{p} = \widetilde{{}_{10}p_{55}}$, for the fuzzy short-term survival probability for a 55 year-old. One may relate to the Short future lifetime in Case

Study 2. Below is the membership function for the triangular fuzzy number $\tilde{p}$ along with its graph:

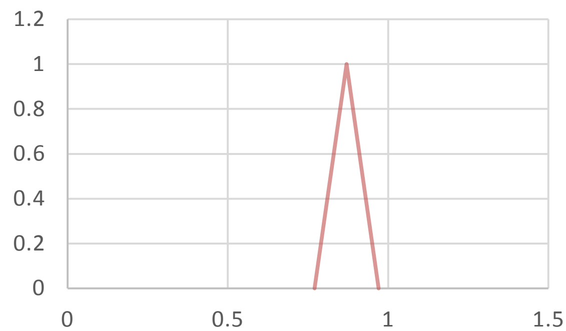


Figure 14. Membership function for the triangular fuzzy number $\tilde{p}$

$$m_{\tilde{p}}(z) = \begin{cases} 0, \ if \ (z \leq .77) \cup (z \leq .97) \\ m_{\tilde{p}1}(z) = 10z - 7.7, \ if \ .77 < z \leq .87 \\ m_{\tilde{p}2}(z) = 9.7 - 10z, \ if \ .87 < z \leq .97 \end{cases}$$

For convenience, we use a simpler membership function for the interest rate *i*, also approximately 6%. It is defined as the triangular fuzzy number below.

$$m_{\tilde{\imath}}(x) = \begin{cases} m_{\tilde{\imath}1}(z) = 50z - 2, \quad if \ .04 < z \leq .06 \\ m_{\tilde{\imath}2}(z) = 4 - 50z, \quad if \ .06 < z \leq .08 \\ 0, \quad otherwise \end{cases}$$

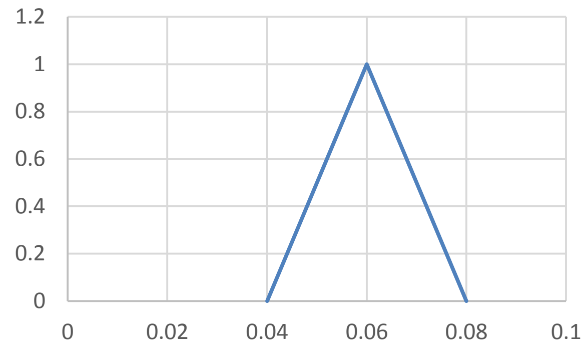


Figure 15 Membership function for a triangular fuzzy interest rate $\tilde{\imath}$

The new actuarial present value of the pure endowment can be expressed taking into account the new fuzzy variable by

$$S \cdot {}_{n}E_x = S \cdot \tilde{p}(1+\tilde{\imath})^{-10}$$

To find the membership function, we must use Theorem 3-6. First, we determine the inverses of $m_{\tilde{p}1}(z)$ and $m_{\tilde{p}2}(z)$, precisely $m_{\tilde{p}1}{}^{-1}(z)$ and $m_{\tilde{p}2}{}^{-1}(z)$, i.e.

$$m_{\tilde{p}1}{}^{-1}(z) = .77 + .1z \quad and \quad m_{\tilde{p}2}{}^{-1}(z) = .97 - .1z$$

Next, the inverse of the membership function of the interest rate $m_i(x)$:

$$m_{i1}{}^{-1}(z) = .04 + .02z \quad and \quad m_{i2}{}^{-1}(z) = .08 - .02z$$

To facilitate the calculation, we use the algorithm for multiplication of two trapezoidal fuzzy numbers from Dutta et al (2011) along with Theorem 3.6:

$m_{X \cdot Y}(z)$

$$= \begin{cases} \dfrac{-\big((b-a)p + (q-p)a\big) + \sqrt{\big((b-a)p + (q-p)a\big)^2 - 4(b-a)(q-p)(ap-z)}}{2(b-a)(q-p)}, & ap \le z \le bq \\ \dfrac{-\big((r-q)c + (c-b)r\big) + \sqrt{\big((r-q)c + (c-b)r\big)^2 - 4(c-b)(r-q)(cr-z)}}{2(b-a)(q-p)}, & bq \le z \le cr \end{cases}$$

Hence, by the multiplication of two fuzzy numbers and multiplication of a fuzzy number by a scalar, we choose $X = S \cdot (1+\tilde{\imath})^{-10}$ and $Y = \tilde{p}$ to do the multiplication and obtain these new membership functions:

$$m_X(z) = m_{S\cdot(1+\tilde{\imath})^{-10}}(z) = \begin{cases} m_{X1}(z) = 54 - 50\left(\frac{z}{1000}\right)^{-1/10}, & if\ 463.2 < z \leq 558.4 \\ m_{X2}(z) = 50\left(\frac{z}{1000}\right)^{-1/10} - 52, & if\ 558.4 < z \leq 675.6 \\ 0, & otherwise \end{cases}$$

$$and \qquad m_Y(z) = m_{\tilde{p}}(z) = \begin{cases} 0, \ if\ (z \leq .77) \cup (z \leq .97) \\ m_{\tilde{p}1}(z) = 10z - 7.7, & if\ .77 < z \leq .87 \\ m_{\tilde{p}2}(z) = 9.7 - 10z, & if\ .87 < z \leq .97 \end{cases}$$

The corresponding values are therefore:

$$a = 463.2, b = 558.4, c = 675.6, \qquad and \qquad p = .77, q = .87, r = .97$$

So that the combination of both membership functions by the multiplication algorithm gives:

$$m_{S\cdot nE_x}(z) = \begin{cases} -6.2828 + .0525\sqrt{38.08z + 728.136}, & if\ 356.66 < z \leq 485.81 \\ 7.7323 - .0427\sqrt{46.88z + 2122.9}, & if\ 485.81 < z \leq 655.33 \\ 0, & otherwise \end{cases}$$

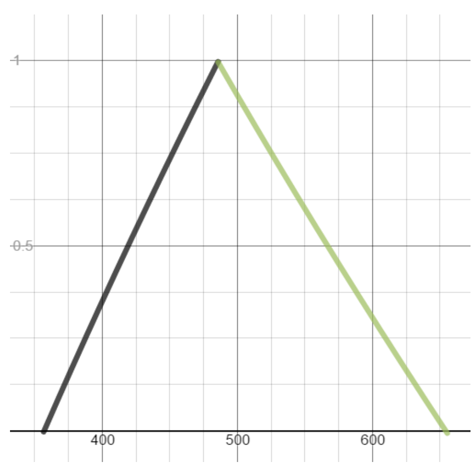


Figure 16. Membership function of the net single premium of a pure endowment

The premiums computed with the fuzzy logic approach reveal more information than our usual crisp premiums. First, it gives a bargaining advantage to the underwriters/company to discuss premium rates. Second, since the premium is calculated for each interest rate $i \in [0.3, 0.9]$, the fuzzy approach provides a range for the premium that entails with the probability that a change in interest rates will happen. That means, not only are you aware of the rate change according to the fluctuation in interest rates, but also the chance of that event occurring. Third, with the fuzzy survival probability, the insurance company is equipped with a short term lifetime probability distribution, allowing for accidental deaths or black-swan events. Another important application of fuzzy set theory is the segmentation of the policyholders into clusters by sociodemographic traits.

4. Fuzzy Insurance Benefits

Case Study 5. Consider the crisp models of distribution of future lifetime given by the laws of DeMoivre, Gompertz, Makeham and Weibull. One wants to find expressions for the insurance benefit of an *n*-year continuous life insurance (Huang et al 2011). In this case, the only fluctuating variable is the interest rate *i* (approximately 6%) from Case Study 3. This is the trapezoidal fuzzy number below.

$$m_i(z) = \begin{cases} 0, & if\ z \leq .03 \\ m_{i1} = 50z - 1.5, & if\ .03 < z \leq .05 \\ 1, & if\ .05 < z \leq .07 \\ m_{i2} = 4.5 - 50z, & if\ .07 < z \leq .09 \\ 0, & if\ .09 < z \end{cases}$$

Note that the general equation for the insurance benefit is:

$$\bar{A}^1_{x:\overline{n}|} = \int_0^n Se^{-\tilde{\delta}t}\ {}_tp_x\mu_{x+t}dt$$

Again, the tilde (~) is meant to differentiate the fuzzy variables from the non-fuzzy (or traditional) ones. The force of interest $\tilde{\delta} = \log(1 + \tilde{\imath})$ is inherently fuzzy. Its membership function is:

$$m_{\tilde{\delta}}(z) = \begin{cases} 0, & if\ z \leq \log(1.03)\ and\ \log(1.09) < z \\ m_{\tilde{\delta}1} = 50e^z - 51.5, & if\ \log(1.03) < z \leq \log(1.05) \\ 1, & if\ \log(1.05) < z \leq \log(1.07) \\ m_{\tilde{\delta}2} = 54.5 - 50e^z, & if\ \log(1.07) < z \leq \log(1.09) \end{cases}$$

This is found by taking the inverse of the pieces of the membership function of the interest rate $m_i(z)$. Precisely, ${m_{i1}}^{-1}(z) = .03 + .02z \ \ and\ \ {m_{i2}}^{-1}(z) = .09 - .02z$.

To go through the computations:

$$\log\left(1 + {m_{i1}}^{-1}(z)\right) = \log(1.03 + .02z)\,, \qquad and$$

$$\log\left(1 + {m_{i2}}^{-1}(z)\right) = \log(1.09 - .02z)$$

And taking the inverse one more time:

$$m_{\tilde{\delta}1} = 50e^z - 51.5\ \ and\ \ m_{\tilde{\delta}2} = 54.5 - 50e^z$$

Using the *DeMoivre assumption*, the insurance benefit for *n*-year continuous life insurance is newly expressed as:

$$\bar{A}^{1}_{x:\overline{n|}} = \frac{S}{\omega - x}\int_0^n e^{-\tilde{\delta}t}dt = \frac{S}{\omega - x}\cdot\frac{1}{\tilde{\delta}}\cdot 1 - e^{-\tilde{\delta}n}$$

So the membership value $\forall n < \omega \in \Omega, S > 0$ is:

$$m_{\bar{A}^{1}_{x:\overline{n|}}}(z) = \frac{S}{\omega - x}\cdot m_{\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}}(z) = \frac{S}{\omega - x}\cdot m_{\frac{1}{\tilde{\delta}}}\cdot m_{1-e^{-\tilde{\delta}n}}(z)$$

$$= \frac{S}{\omega - x}\cdot\frac{1-(1+m_{\tilde{\imath}}(z))^{-n}}{\log(1+m_{\tilde{\imath}}(z))}$$

Since $\frac{S}{\omega - x}$ is a real number or a constant, we may focus on the membership function. To simplify the calculation, we split the membership function into two pieces and we combine them at the end. As in Case Study 4, we can rename the quantities to facilitate multiplication.

Since $m_X = m_{\frac{1}{\tilde{\delta}}} = \frac{1}{\log(1+m_{\tilde{\imath}})}$, we only need to take the inverse of the above results for the force of interest:

$$\frac{1}{\log\left(1+m_{i1}{}^{-1}(z)\right)} = \frac{1}{\log(1.03+.02z)}, \qquad and$$

$$\frac{1}{\log\left(1+m_{i2}{}^{-1}(z)\right)} = \frac{1}{\log(1.09-.02z)}$$

Again, taking the inverses of the two new results in order to have the membership functions for the fuzzy set of $m_{\frac{1}{\tilde{\delta}}}$:

$$m_X(z) = m_{\frac{1}{\tilde{\delta}}}(z) = \begin{cases} 0, & if\ z \le \log(1.09)^{-1}\ and\ \log(1.03)^{-1} < z \\ 54.5 - 50e^{1/z}, & if\ \log(1.09)^{-1} < z \le \log(1.07)^{-1} \\ 1, & if\ \log(1.07)^{-1} < z \le \log(1.05)^{-1} \\ 50e^{1/z} - 51.5, & if\ \log(1.05)^{-1} < z \le \log(1.03)^{-1} \end{cases}$$

Next, the membership functions for the fuzzy set of $m_Y = m_{1-e^{-\tilde{\delta}n}} = 1 - (1 + m_{\tilde{\imath}})^{-n}$.

Using the same argument as above, the inverses are:

$$1 - \left(1 + m_{i1}{}^{-1}(z)\right)^{-n} = 1 - (.02z + 1.03)^{-n}, \quad and$$

$$1 - \left(1 + m_{i2}{}^{-1}(z)\right)^{-n} = 1 - (1.09 - .02z)^{-n}$$

Taking the inverses of the two new results in order to have the membership functions for the fuzzy set of $m_{1-e^{-\tilde{\delta}n}}$:

$$m_Y(z) = m_{1-e^{-\tilde{\delta}n}}(z) = \begin{cases} 50(1-z)^{-1/n} - 51.5, & if 1 - 1.03^{-n} < z \le 1 - 1.05^{-n} \\ 1, & if\ 1 - 1.05^{-n} < z \le 1 - 1.07^{-n} \\ 54.5 - 50(1-z)^{-\frac{1}{n}}, & if\ 1 - 1.07^{-n} < z \le 1 - 1.09^{-n} \\ 0, & otherwise \end{cases}$$

The next step is to find $m_{X \cdot Y} = m_{\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}}$ by following the technique of Dutta et al. (2011) and Taleshian and Rezvani (2011) to multiply the two trapezoidal fuzzy numbers:

$$m_{X \cdot Y}(z)$$

$$= \begin{cases} m_{XY_1}(z) = \dfrac{-(aq + bp - 2ap) + \sqrt{(aq - bp)^2 - 4(b-a)(q-p)z}}{2(b-a)(q-p)}, & ap \le z \le bq \\ 1, & bq \le z \le cr \\ \dfrac{\left((s-r)d - (d-c)s\right) - \sqrt{\left((s-r)d - (d-c)s\right)^2 - 4(d-c)(s-r)(ds-z)}}{2(d-c)(s-r)}, & cr \le z \le ds \end{cases}$$

Here, the corresponding letters are constants and we have:

$$a = \log(1.09)^{-1}, b = \log(1.07)^{-1}, c = \log(1.05)^{-1}, d = \log(1.03)^{-1} \ and$$

$$p = p_n = 1 - 1.03^{-n}, q = q_n = 1 - 1.05^{-n}, r = r_n = 1 - 1.07^{-n}, s = s_n = 1 - 1.09^{-n}$$

So that the combination of both their membership functions by the multiplication algorithm gives:

$$m_{X\cdot Y_1}(z) = \frac{-3.2p_n - 11.6(q_n - p_n) + \sqrt{\left(3.2p_n + 11.6(q_n - p_n)\right)^2 - 12.8(q_n - p_n)(11.6p_n - z)}}{6.4(q_n - p_n)}$$

$$m_{X\cdot Y_2}(z) = \frac{13.3s_n - 33.8(s_n - r_n) + \sqrt{(13.3s_n + 33.8(s_n - r_n))^2 - 53.2(s_n - r_n)(33.8s_n - z)}}{26.6(s_n - r_n)}$$

As $n \in \mathbb{Z}$ for years in integer form, we can approximate the membership function of the insurance benefit for term life insurance of n years or for whole life insurance. Whole life insurance can be treated by taking the limit of n to infinity. A simple but lengthy result for $n = 1$ gives the membership functions below, along with the graph of the functions:

$$m_{\frac{1}{\tilde{\delta}}\cdot 1-e^{-\tilde{\delta}}} = \begin{cases} 0, & if\ z \le \frac{1-1.03^{-1}}{\log(1.09)}\ and\ \frac{1-1.09^{-1}}{\log(1.03)} < z \\ \frac{-.308+\sqrt{.015+.237z}}{.118}, & if\ \frac{1-1.03^{-1}}{\log(1.09)} < z \le \frac{1-1.05^{-1}}{\log(1.07)} \\ 1, & if\ \frac{1-1.05^{-1}}{\log(1.07)} < z \le \frac{1-1.07^{-1}}{\log(1.05)} \\ \frac{1.68-\sqrt{.278-.912z}}{.456}, & if\ \frac{1-1.07^{-1}}{\log(1.05)} < z \le \frac{1-1.09^{-1}}{\log(1.03)} \end{cases}$$

Figure 17. Membership function of insurance benefit for a continuous life insurance

In the following page, we have the full expression of the membership function with the sequences for all n expressed.

$$m_{\bar{A}^{1}_{x:\overline{n}|}}(z) = \frac{S}{\omega - x} \cdot m_{\frac{1}{\bar{\delta}} \cdot 1 - e^{-\bar{\delta} n}}$$

$$= \begin{cases} 0, & if\ z \leq \dfrac{1 - 1.03^{-n}}{\log(1.09)}\ and\ \dfrac{1 - 1.09^{-n}}{\log(1.03)} < z \\ \dfrac{-3.2(1 - 1.03^{-n}) - 11.6(1.03^{-n} - 1.05^{-n}) + \sqrt{\left(3.2(1 - 1.03^{-n}) + 11.6(1.03^{-n} - 1.05^{-n})\right)^2 - 12.8(1.03^{-n} - 1.05^{-n})(11.6(1 - 1.03^{-n}) - z}}{6.4(1.03^{-n} - 1.05^{-n})} & if\ \dfrac{1 - 1.03^{-n}}{\log(1.09)} < z \leq \dfrac{1 - 1.05^{-n}}{\log(1.07)} \\ 1, & if\ \dfrac{1 - 1.05^{-n}}{\log(1.07)} < z \leq \dfrac{1 - 1.07^{-n}}{\log(1.05)} \\ \dfrac{13.3(1 - 1.09^{-n}) + 33.8(1.07^{-n} - 1.09^{-n}) - \sqrt{\left(13.3(1 - 1.09^{-n}) + 33.8(1.07^{-n} - 1.09^{-n})\right)^2 - 53.2(1.07^{-n} - 1.09^{-n})(33.8(1 - 1.09^{-n}) - z}}{26.6(1.07^{-n} - 1.09^{-n})} & if\ \dfrac{1 - 1.07^{-n}}{\log(1.05)} < z \leq \dfrac{1 - 1.09^{-n}}{\log(1.03)} \end{cases}$$

The approach is similar for the other life distribution laws (Gompertz, Makeham, Weibull). The same function $\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}$ is used in every assumption, only to be multiplied by their very specific survival probability of ${}_tp_x\mu_{x+t}$. Recall section 2.2.2 for these actuarial survival probabilities. Thus:

- The insurance benefit of the n-year continuous life insurance *using Gompertz assumption*

$$m_{\bar{A}^1_{x:\overline{n|}}}(z) = S \cdot \int_0^t BC^{x+t} e^{\frac{B}{\ln C}C^x(1-C^t)} dt \cdot m_{\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}}(z)$$

- The insurance benefit of the n-year continuous life insurance *using Makeham assumption*

$$m_{\bar{A}^1_{x:\overline{n|}}}(z) = S \cdot \int_0^t (A + BC^{x+t}) e^{-At+\frac{B}{\ln C}C^x(1-C^t)} dt \cdot m_{\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}}(z)$$

- The insurance benefit of the n-year continuous life insurance *using Weibull assumption*:

$$m_{\bar{A}^1_{x:\overline{n|}}}(z) = S \cdot \int_0^t k(x+t)^n e^{k(n+1)[x^{n+1}-(x+t)^{n+1}]} dt \cdot m_{\frac{1-e^{-\tilde{\delta}n}}{\tilde{\delta}}}(z)$$

5. Fuzzy Clustering of Policyholders

A Non-Life Approach

Age is always treated as a factor in mortality, because inevitably the older you get the more susceptible to die you are. The goal for preference classes is to have premium rates according to each policyholder's risk profile. It is common practice to group policyholders for reasons of efficiency; even if the rating operates by design more effectively on an individual basis. The premium rate must be estimated for the group. Each underwriting structure regroups policyholders by blocks of ages, and most premium estimations are based on these classes. The crisp concept uses curtate age (section 2.2.2); that is the integer age at the last birthday. An example of grouping is Brockham and Wright (1991):

$$17-18, 19-21, 22-24, 25-29, 31-34, 35-44, 45-54, 55+$$

Limitations to the model occur when we wish to assign a 30-year-old to a group. Given the random nature of insurance, the evidence favoring the 4th group over the 5th group is not likely to be conclusive. The theory of Fuzzy Set aims to resolve this ambiguity. In the next Case Study, Verrall & Yakoubov provide a fuzzy approach to grouping policyholder ages using past claims for non-life insurance.

Case Study 6. (Verrall & Yakoubov 1999) *Policyholder risk grouping by age.*

This is a case study of general insurance that can easily be applied to life insurance. The data for the study consist of approximately 50,000 motor policies of all ages. The youngest ages are grouped under the label "<25" and the oldest under "83+" to remove the absent data or discrepancies. We only focus on two types of claims: Body Injuries (BI) and the Material Damage (MD). For each age, we retrieve the number of claims, or *Frequency*, along with the total cost of the claim, or *Severity*. For each claim many factors enter in the cause of an accident rather than age only. The Department of Motor Vehicles "*Unit for Accidents: Causes and Prevention*" gives an exhaustive list among which the most often cited are age, gender, driver's years of exposure, car group, mechanical failure, location and road conditions and weather conditions. Most insurance companies use only age as the key to grouping policyholders. Hence, to remove distortion due to the uneven mix of policyholder age, we adjust the data as follows:

$$Adjusted\ Frequency = frequency \times severity$$

Table 1 displays the Frequency and Severity of the MD and BI claims in the data set used in Verrall et al (1999). The exposure tab gives a numerical value for the number of earned driver years. The abbreviation AdjFreq means Adjusted Frequency and CruPrem mean Crude Premium. It is the a priori estimation of what a policyholder of age (x) will need to pay in premium, before any extra fees such as taxes, state surcharges, transaction fees, etc. It equals the expenses the insurance carrier is at risk of incurring for the policyholder (x); hence:

$$Crude\ Premium = MD\ AdjFreq + BI\ AdjFreq$$

Table 1. Frequency and Severity of Claims

	Frequency		Severity		AdjFreq			
Age	MD	BI	MD	BI	MD	BI	CruPrem	Exposure
< 25	0.30691	0.04384	515.90	5381.68	121.10	248.14	369.25	43.54
25	0.27846	0.05967	439.60	2242.76	109.88	337.70	447.58	31.99
26	0.13580	0.01598	302.74	9742.90	53.59	9.42	144.01	79.66
27	0.18732	0.01767	364.29	4286.98	73.91	10.01	173.93	36.11
28	0.20380	0.01002	395.48	5254.42	8.42	56.73	137.15	571.41
29	0.18907	0.01126	362.36	4598.03	74.61	63.75	138.35	79.95
30	0.18175	0.01486	434.33	6041.05	71.72	84.10	155.82	1113.44
31	0.14277	0.01142	406.46	7614.30	56.34	64.64	12.98	1671.45
32	0.15469	0.00729	331.00	5928.38	61.04	41.26	102.30	2007.57
33	0.12644	0.00651	30.71	5423.97	49.89	36.85	86.74	1857.12
34	0.12914	0.00861	416.71	6037.58	5.96	48.73	99.69	1921.73
35	0.14105	0.00641	369.25	6043.16	55.66	36.29	91.94	1885.85
36	0.12895	0.00794	414.31	4742.69	5.88	44.96	95.85	2082.56
37	0.14444	0.00698	365.89	5473.63	57.00	39.53	96.52	2004.59
38	0.12641	0.00967	423.00	5445.77	49.88	54.74	104.62	1907.93
39	0.12772	0.00872	458.40	4757.70	5.40	49.37	99.77	1823.65
40	0.12218	0.00732	356.89	3381.56	48.21	41.40	89.61	1739.68
41	0.11796	0.01222	422.91	3424.66	46.55	69.14	115.69	1666.95
42	0.11471	0.00828	401.68	6085.11	45.26	46.85	92.11	1614.38
43	0.11017	0.00646	416.18	4804.83	43.47	36.55	8.02	1675.11
44	0.11005	0.00758	385.92	5724.75	43.42	42.88	86.30	1596.01
45	0.11247	0.00287	345.49	4551.51	44.38	16.26	6.64	155.33
46	0.11479	0.00776	36.96	3789.87	45.29	43.89	89.19	164.99
47	0.11970	0.00874	385.04	7947.82	47.23	49.45	96.68	1456.71
48	0.11975	0.01023	411.26	3776.65	47.25	57.88	105.13	1493.30
49	0.12098	0.01052	343.41	4671.31	47.74	59.54	107.27	1209.86
50	0.11047	0.00440	337.79	8008.71	43.59	24.90	68.49	1301.91
51	0.14009	0.01302	386.38	415.16	55.28	73.69	128.96	1221.93
52	0.12067	0.00546	396.67	7867.67	47.61	3.90	78.52	1165.49
53	0.12340	0.00507	391.90	2685.65	48.69	28.70	77.40	1129.33
54	0.12615	0.00788	455.80	6436.11	49.78	44.62	94.40	887.81
55	0.10273	0.00654	535.29	6999.40	4.54	37.03	77.57	972.51
56	0.09071	0.00542	366.74	5385.20	35.79	3.65	66.44	94.04
57	0.07957	0.00918	387.84	3859.75	31.40	51.96	83.36	415.89
58	0.11671	0.00824	382.74	2726.71	46.05	46.63	92.68	463.46
59	0.10629	0.00518	496.70	11538.43	41.94	29.34	71.28	49.95
60	0.07552	0.00252	37.90	1599.23	29.80	14.25	44.05	505.56
61	0.08699	0.00829	619.72	9716.70	34.33	46.89	81.22	46.85
62	0.08507	0.00597	503.03	3677.61	33.57	33.79	67.36	426.37
63	0.06762	0.00432	353.15	3397.81	26.68	24.43	51.11	442.31

Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.

	Frequency		Severity		AdjFreq			
Age	MD	BI	MD	BI	MD	BI	CruPrem	Exposure
64	0.06307	0.00293	696.58	27154.86	24.89	16.60	41.49	433.88
65	0.07488	0.00326	346.98	2545.45	29.55	18.42	47.97	39.95
66	0.08823	0.01307	514.26	6541.60	34.81	73.97	108.79	389.50
67	0.08404	0.00615	343.60	4642.38	33.16	34.80	67.96	31.47
68	0.08037	0.00423	304.66	1415.02	31.71	23.94	55.65	30.88
69	0.08252	0.00236	35.23	23768.18	32.56	13.34	45.91	269.91
70	0.06287	0.00286	31.02	7089.09	24.81	16.17	4.98	222.70
71	0.07759	0.00517	749.87	9937.03	3.62	29.27	59.89	246.06
72	0.07509	0.00901	401.62	3782.42	29.63	51.00	8.63	211.86
73	0.08597	0.00000	351.00	0.00	33.92	0.00	33.92	17.25
74	0.04838	0.00000	43.71	0.00	19.09	0.00	19.09	157.85
75	0.04505	0.00000	379.38	0.00	17.77	0.00	17.77	155.39
76	0.09957	0.01494	378.72	3991.06	39.29	84.53	123.82	127.82
77	0.06820	0.00000	662.06	0.00	26.91	0.00	26.91	74.65
78	0.11220	0.00863	276.11	636.36	44.27	48.85	93.12	73.73
79	0.04088	0.00000	172.09	0.00	16.13	0.00	16.13	46.70
80	0.05759	0.01920	857.47	6532.27	22.73	108.65	131.38	33.15
81	0.06099	0.02033	1782.07	636.36	24.07	115.06	139.13	31.30
82	0.04193	0.00000	193.53	0.00	16.55	0.00	16.55	3.35
83+	0.00854	0.00000	396.89	0.00	3.37	0.00	3.37	74.50

Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.

A fuzzy c-means algorithm (recall Section 3.2) is applied to the adjusted data for the optimal number of clusters $c = 6$ (Bezdek 1981). The *exponential weight* or *fuzzy parameter* $r = 2$; $\|\cdot\|$ is the Euclidean norm; the *termination criterion* $\varepsilon = 0.05$. For each cluster, we are to determine the center of the adjusted MD, the adjusted BI, and the Crude Premiums. The cluster centers (centroids) derived from Table 1 are allotted in Table 2.

Table 2. Clusters centers or centroids.

Centers of the Six Clusters

Clusters	1	2	3	4	5	6
MDADJ	114.48	48.39	52.16	48.53	36.80	21.94
BIADJ	292.07	90.79	64.61	43.44	29.85	4.47
Crude premium	406.55	139.18	116.77	91.97	66.65	26.41

Notes: MDADJ = MD Adjusted Frequency; and BIADJ = BI Adjusted Frequency.

The resulting table of centroids is then used to calculate the membership of each age to a cluster. We use the adjusted frequencies and determine the membership values for the corresponding age. As in Brockham and Wright (1991), we wish to create age groups for underwriters. With less computing skills or tools, underwriting procedure would require crisp age groups. Hence, to separate the element that belongs to more than one cluster, we proceed with an alpha-cut of 20%. The level set of the fuzzy membership is shown in Table 3.

Table 3. 20%-level set of the membership

	Cluster					
Age	1	2	3	4	5	6
< 25	1.00	0.00	0.00	0.00	0.00	0.00
25	1.00	0.00	0.00	0.00	0.00	0.00
26	0.00	1.00	0.00	0.00	0.00	0.00
27	0.00	0.69	0.31	0.00	0.00	0.00
28	0.00	0.00	0.56	0.44	0.00	0.00
29	0.00	0.00	0.66	0.34	0.00	0.00
30	0.00	0.51	0.49	0.00	0.00	0.00
31	0.00	0.00	1.00	0.00	0.00	0.00
32	0.00	0.00	0.00	1.00	0.00	0.00
33	0.00	0.00	0.00	0.76	0.24	0.00
34	0.00	0.00	0.00	1.00	0.00	0.00
35	0.00	0.00	0.00	0.76	0.24	0.00
36	0.00	0.00	0.00	1.00	0.00	0.00
37	0.00	0.00	0.00	1.00	0.00	0.00
38	0.00	0.00	0.38	0.62	0.00	0.00
39	0.00	0.00	0.00	1.00	0.00	0.00
40	0.00	0.00	0.00	1.00	0.00	0.00
41	0.00	0.00	1.00	0.00	0.00	0.00
42	0.00	0.00	0.00	1.00	0.00	0.00
43	0.00	0.00	0.00	0.44	0.56	0.00
44	0.00	0.00	0.00	1.00	0.00	0.00
45	0.00	0.00	0.00	0.00	0.71	0.29
46	0.00	0.00	0.00	1.00	0.00	0.00
47	0.00	0.00	0.00	1.00	0.00	0.00
48	0.00	0.00	0.61	0.39	0.00	0.00
49	0.00	0.00	0.73	0.27	0.00	0.00
50	0.00	0.00	0.00	0.00	1.00	0.00
51	0.00	0.00	1.00	0.00	0.00	0.00
52	0.00	0.00	0.00	0.39	0.61	0.00
53	0.00	0.00	0.00	0.37	0.63	0.00
54	0.00	0.00	0.00	1.00	0.00	0.00
55	0.00	0.00	0.00	0.28	0.72	0.00
56	0.00	0.00	0.00	0.00	1.00	0.00
57	0.00	0.00	0.23	0.44	0.33	0.00
58	0.00	0.00	0.00	1.00	0.00	0.00
59	0.00	0.00	0.00	0.00	1.00	0.00
60	0.00	0.00	0.00	0.00	0.32	0.68
61	0.00	0.00	0.00	0.54	0.46	0.00
62	0.00	0.00	0.00	0.00	1.00	0.00
63	0.00	0.00	0.00	0.00	0.73	0.27
64	0.00	0.00	0.00	0.00	0.30	0.70
65	0.00	0.00	0.00	0.00	0.54	0.46
66	0.00	0.38	0.62	0.00	0.00	0.00
67	0.00	0.00	0.00	0.00	1.00	0.00
68	0.00	0.00	0.00	0.00	1.00	0.00
69	0.00	0.00	0.00	0.00	0.37	0.63
70	0.00	0.00	0.00	0.00	0.28	0.72
71	0.00	0.00	0.00	0.00	1.00	0.00
72	0.00	0.00	0.00	0.52	0.48	0.00
73	0.00	0.00	0.00	0.00	0.00	1.00
74	0.00	0.00	0.00	0.00	0.00	1.00
75	0.00	0.00	0.00	0.00	0.00	1.00
76	0.00	0.73	0.27	0.00	0.00	0.00
77	0.00	0.00	0.00	0.00	0.00	1.00
78	0.00	0.00	0.00	1.00	0.00	0.00
79	0.00	0.00	0.00	0.00	0.00	1.00
80	0.00	1.00	0.00	0.00	0.00	0.00
81	0.00	1.00	0.00	0.00	0.00	0.00
82	0.00	0.00	0.00	0.00	0.00	1.00
83+	0.00	0.00	0.00	0.00	0.00	1.00

Underwriting usually anticipates that risk progress smoothly with age, and so age groupings must be adjacent. Verrall et al. uses a **risk measure** to determine which adjacent ages are in the same group. This is the following equality:

$$R_i = \frac{1}{\|i\|} \sum_{ages\ in\ i} \sum_{k}^{c} \mu_{jk} \|v_k\|$$

From the beginning of the case study, it is assumed that <25 and 83+ were whole groups. Applying the risk measure to the 20 percent-cut data gives the results as in the Table 4 below. Hence, for the six clusters, we can count 7 age groups:

$$\leq 25, 26-27, 28-31, 32-47, 48-51, 52-68, 69+$$

Table 4. Risk Measure per Age group

Group, i	1	2	3	
R_i	406.29	135.65	114.79	
Group, i	4	5	6	7
R_i	90.15	100.15	71.78	60.92

In figure 18, we have a comparative graph for the crude risk premium and premium based on risk groups. The group premium gives a very good fit of the model for crude premium. The fuzzy clustering allows the creation of risk groups with very smooth transition between ages. The accuracy may slightly be off at groups 1 and 2, for the little information available for these drivers; hence such high premiums.

Age is only one factor among many others, yet the insurance industry made it as the primary indicator of risk in any type of coverage.

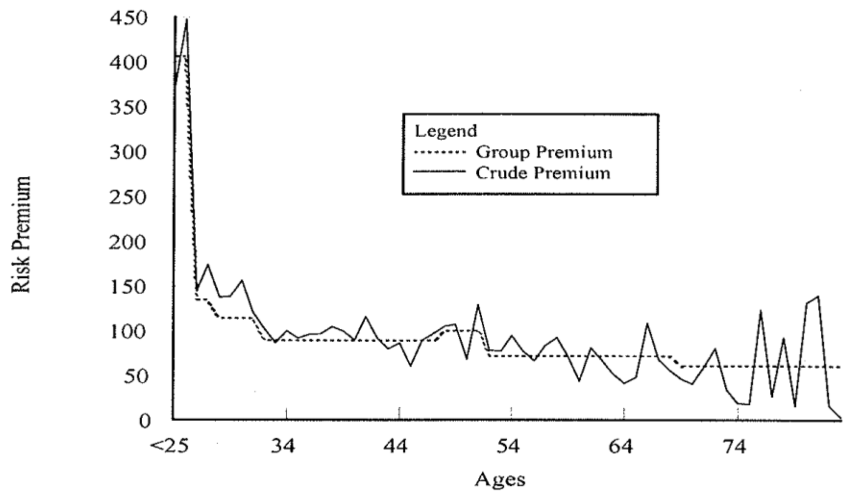


Figure 18. Comparative graph for the Crude Premium and the Group Premium.

This experiment can be applied to a life contingency study of multiple state models. Simply replace the Material Damage (MD) claims by Disability Claims and the Body Injuries (BI) by Death Claims.

CONCLUSIONS AND FURTHER RESEARCH

This paper presented successful concepts and techniques of fuzzy set theory as used in the actuarial science environment. The applications focused on life contingencies and life insurance from the underwriting to claims. Fuzzy actuarial mathematics offers a new and promising way of treating uncertainty, with a useful addition to the modeling tools. The Actuarial Research Clearinghouse is correct to forecast a drastic increase of interest for fuzzy methods in the future. New research may shift toward multiple state models including joint-survivorship, disability, sickness, retirement or withdrawal. Also involved are the hybrid models and the company-sponsored insurance with multiple options. Perhaps these are building blocks that will suggest other fields to develop fuzzy set applications.

Table 5. Life Table for the total population of United States, 2011

Spreadsheet version available from: ftp://ftp.cdc.gov/pub/Health_Statistics/NCHS/Publications/NVSR/64_11/Table01.xlsx.

Age (years)	Probability of dying between ages x and $x+1$ q_x	Number surviving to age x l_x	Number dying between ages x and $x+1$ d_x	Person-years lived between ages x and $x+1$ L_x	Total number of person-years lived above age x T_x	Expectation of life at age x e_x
0–1	0.006058	100,000	606	99,470	7,870,915	78.7
1–2	0.000415	99,394	41	99,374	7,771,445	78.2
2–3	0.000264	99,353	26	99,340	7,672,071	77.2
3–4	0.000208	99,327	21	99,316	7,572,731	76.2
4–5	0.000167	99,306	17	99,298	7,473,415	75.3
5–6	0.000151	99,289	15	99,282	7,374,117	74.3
6–7	0.000134	99,274	13	99,268	7,274,835	73.3
7–8	0.000120	99,261	12	99,255	7,175,567	72.3
8–9	0.000106	99,249	11	99,244	7,076,312	71.3
9–10	0.000092	99,239	9	99,234	6,977,068	70.3
10–11	0.000084	99,230	8	99,225	6,877,834	69.3
11–12	0.000090	99,221	9	99,217	6,778,609	68.3
12–13	0.000117	99,212	12	99,207	6,679,392	67.3
13–14	0.000172	99,201	17	99,192	6,580,185	66.3
14–15	0.000246	99,184	24	99,171	6,480,993	65.3
15–16	0.000326	99,159	32	99,143	6,381,822	64.4
16–17	0.000404	99,127	40	99,107	6,282,679	63.4
17–18	0.000486	99,087	48	99,063	6,183,572	62.4
18–19	0.000570	99,039	56	99,011	6,084,509	61.4
19–20	0.000655	98,982	65	98,950	5,985,499	60.5
20–21	0.000743	98,917	73	98,881	5,886,549	59.5
21–22	0.000826	98,844	82	98,803	5,787,668	58.6
22–23	0.000886	98,762	88	98,719	5,688,865	57.6
23–24	0.000919	98,675	91	98,630	5,590,146	56.7
24–25	0.000930	98,584	92	98,538	5,491,517	55.7
25–26	0.000934	98,493	92	98,447	5,392,978	54.8
26–27	0.000943	98,400	93	98,354	5,294,532	53.8
27–28	0.000958	98,308	94	98,261	5,196,178	52.9
28–29	0.000983	98,214	96	98,165	5,097,917	51.9
29–30	0.001016	98,117	100	98,067	4,999,752	51.0
30–31	0.001055	98,017	103	97,966	4,901,685	50.0
31–32	0.001094	97,914	107	97,860	4,803,719	49.1
32–33	0.001132	97,807	111	97,751	4,705,859	48.1
33–34	0.001167	97,696	114	97,639	4,608,107	47.2
34–35	0.001203	97,582	117	97,523	4,510,468	46.2
35–36	0.001250	97,465	122	97,404	4,412,945	45.3
36–37	0.001313	97,343	128	97,279	4,315,541	44.3
37–38	0.001389	97,215	135	97,148	4,218,262	43.4
38–39	0.001476	97,080	143	97,008	4,121,114	42.5
39–40	0.001576	96,937	153	96,860	4,024,106	41.5
40–41	0.001685	96,784	163	96,702	3,927,245	40.6
41–42	0.001813	96,621	175	96,533	3,830,543	39.6
42–43	0.001972	96,446	190	96,351	3,734,009	38.7
43–44	0.002171	96,256	209	96,151	3,637,659	37.8
44–45	0.002405	96,047	231	95,931	3,541,508	36.9
45–46	0.002652	95,816	254	95,688	3,445,577	36.0
46–47	0.002910	95,561	278	95,422	3,349,888	35.1
47–48	0.003196	95,283	305	95,131	3,254,466	34.2
48–49	0.003513	94,979	334	94,812	3,159,335	33.3
49–50	0.003851	94,645	364	94,463	3,064,523	32.4
50–51	0.004204	94,281	396	94,083	2,970,060	31.5
51–52	0.004563	93,884	428	93,670	2,875,977	30.6
52–53	0.004928	93,456	461	93,226	2,782,307	29.8
53–54	0.005304	92,995	493	92,749	2,689,081	28.9
54–55	0.005702	92,502	527	92,238	2,596,332	28.1
55–56	0.006131	91,975	564	91,693	2,504,094	27.2
56–57	0.006596	91,411	603	91,109	2,412,401	26.4
57–58	0.007096	90,808	644	90,486	2,321,292	25.6
58–59	0.007621	90,164	687	89,820	2,230,806	24.7
59–60	0.008164	89,476	730	89,111	2,140,986	23.9
60–61	0.008732	88,746	775	88,359	2,051,875	23.1

See footnote at end of table.

APPENDIX

Table 6. Life Table for the total population of United States, 2011 (continued)

Age (years)	Probability of dying between ages x and $x+1$ q_x	Number surviving to age x l_x	Number dying between ages x and $x+1$ d_x	Person-years lived between ages x and $x+1$ L_x	Total number of person-years lived above age x T_x	Expectation of life at age x e_x
61–62	0.009335	87,971	821	87,560	1,963,516	22.3
62–63	0.009983	87,150	870	86,715	1,875,956	21.5
63–64	0.010715	86,280	924	85,818	1,789,241	20.7
64–65	0.011568	85,355	987	84,862	1,703,423	20.0
65–66	0.012586	84,368	1,062	83,837	1,618,562	19.2
66–67	0.013763	83,306	1,147	82,733	1,534,725	18.4
67–68	0.015057	82,160	1,237	81,541	1,451,992	17.7
68–69	0.016380	80,923	1,326	80,260	1,370,451	16.9
69–70	0.017756	79,597	1,413	78,890	1,290,191	16.2
70–71	0.019299	78,184	1,509	77,429	1,211,301	15.5
71–72	0.021039	76,675	1,613	75,868	1,133,871	14.8
72–73	0.022997	75,062	1,726	74,199	1,058,003	14.1
73–74	0.025182	73,335	1,847	72,412	983,805	13.4
74–75	0.027634	71,489	1,975	70,501	911,392	12.7
75–76	0.030322	69,513	2,108	68,459	840,892	12.1
76–77	0.033309	67,405	2,245	66,283	772,432	11.5
77–78	0.036740	65,160	2,394	63,963	706,149	10.8
78–79	0.040688	62,766	2,554	61,489	642,186	10.2
79–80	0.045172	60,212	2,720	58,852	580,697	9.6
80–81	0.050072	57,493	2,879	56,053	521,844	9.1
81–82	0.055306	54,614	3,020	53,103	465,791	8.5
82–83	0.061241	51,593	3,160	50,013	412,688	8.0
83–84	0.067893	48,434	3,288	46,789	362,674	7.5
84–85	0.075594	45,145	3,413	43,439	315,885	7.0
85–86	0.084649	41,733	3,533	39,966	272,446	6.5
86–87	0.094437	38,200	3,607	36,396	232,480	6.1
87–88	0.105152	34,593	3,637	32,774	196,083	5.7
88–89	0.116835	30,955	3,617	29,147	163,309	5.3
89–90	0.129516	27,338	3,541	25,568	134,163	4.9
90–91	0.143215	23,798	3,408	22,094	108,595	4.6
91–92	0.157937	20,389	3,220	18,779	86,501	4.2
92–93	0.173671	17,169	2,982	15,678	67,722	3.9
93–94	0.190385	14,187	2,701	12,837	52,043	3.7
94–95	0.208029	11,486	2,389	10,292	39,207	3.4
95–96	0.226531	9,097	2,061	8,066	28,915	3.2
96–97	0.245796	7,036	1,729	6,171	20,849	3.0
97–98	0.265711	5,307	1,410	4,602	14,677	2.8
98–99	0.286142	3,897	1,115	3,339	10,075	2.6
99–100	0.306941	2,782	854	2,355	6,736	2.4
100 and over	1.000000	1,928	1,928	4,382	4,382	2.3

SOURCE: CDC/NCHS, National Vital Statistics System.

Table 7. Expectation of Life by age, race, ethnicity and sex in United States, 2011

Age	All races and origins			White			Black			Hispanic[1]			Non-Hispanic white[1]			Non-Hispanic black[1]		
	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female
0	78.7	76.3	81.1	79.0	76.6	81.3	75.3	72.2	78.2	81.6	79.0	83.8	78.8	76.4	81.1	74.9	71.7	77.9
1	78.2	75.8	80.5	78.4	76.0	80.6	75.2	72.1	78.0	81.0	78.5	83.2	78.2	75.9	80.4	74.8	71.6	77.7
5	74.3	71.9	76.6	74.4	72.1	76.7	71.3	68.2	74.1	77.1	74.5	79.3	74.2	71.9	76.5	70.9	67.8	73.8
10	69.3	66.9	71.6	69.5	67.1	71.8	66.4	63.3	69.1	72.1	69.6	74.3	69.3	67.0	71.5	66.0	62.8	68.8
15	64.4	62.0	66.7	64.5	62.2	66.8	61.4	58.4	64.2	67.1	64.6	69.3	64.3	62.0	66.6	61.1	57.9	63.9
20	59.5	57.2	61.7	59.7	57.4	61.9	56.6	53.6	59.3	62.3	59.8	64.4	59.5	57.2	61.7	56.3	53.2	59.0
25	54.8	52.5	56.9	54.9	52.7	57.0	52.0	49.1	54.4	57.5	55.1	59.5	54.7	52.6	56.8	51.6	48.7	54.2
30	50.0	47.9	52.0	50.1	48.0	52.2	47.3	44.6	49.6	52.6	50.3	54.6	50.0	47.9	52.0	47.0	44.2	49.4
35	45.3	43.2	47.2	45.4	43.4	47.3	42.7	40.1	44.9	47.8	45.6	49.7	45.3	43.3	47.2	42.4	39.7	44.7
40	40.6	38.6	42.4	40.7	38.7	42.6	38.1	35.5	40.3	43.0	40.9	44.8	40.6	38.6	42.4	37.8	35.2	40.0
45	36.0	34.0	37.8	36.1	34.2	37.9	33.6	31.1	35.7	38.3	36.2	40.0	36.0	34.1	37.7	33.3	30.8	35.5
50	31.5	29.7	33.2	31.6	29.8	33.3	29.3	26.9	31.3	33.7	31.7	35.3	31.5	29.7	33.2	29.1	26.6	31.1
55	27.2	25.5	28.8	27.3	25.6	28.8	25.3	23.0	27.1	29.3	27.4	30.8	27.2	25.6	28.8	25.1	22.8	26.9
60	23.1	21.6	24.5	23.2	21.6	24.5	21.5	19.4	23.2	25.0	23.3	26.3	23.1	21.6	24.5	21.3	19.3	23.0
65	19.2	17.8	20.3	19.2	17.8	20.3	18.0	16.2	19.4	20.9	19.3	22.0	19.1	17.8	20.3	17.9	16.1	19.2
70	15.5	14.3	16.5	15.5	14.3	16.4	14.7	13.2	15.8	17.0	15.7	17.9	15.4	14.3	16.4	14.6	13.1	15.7
75	12.1	11.1	12.9	12.1	11.0	12.8	11.7	10.4	12.5	13.4	12.3	14.1	12.0	11.0	12.8	11.7	10.4	12.5
80	9.1	8.2	9.6	9.0	8.2	9.6	9.1	8.0	9.6	10.2	9.2	10.6	9.0	8.2	9.6	9.0	8.0	9.6
85	6.5	5.9	6.9	6.5	5.8	6.8	6.8	6.0	7.2	7.4	6.6	7.6	6.5	5.8	6.8	6.8	6.0	7.2
90	4.6	4.1	4.8	4.5	4.0	4.7	5.1	4.5	5.3	5.2	4.6	5.2	4.5	4.0	4.7	5.1	4.5	5.3
95	3.2	2.9	3.3	3.1	2.8	3.2	3.8	3.4	3.9	3.6	3.2	3.5	3.1	2.8	3.2	3.8	3.4	3.9
100	2.3	2.1	2.3	2.2	2.0	2.2	2.8	2.6	2.9	2.5	2.3	2.4	2.2	2.0	2.2	2.9	2.6	2.9

[1]Life tables by Hispanic origin are based on death rates that have been adjusted for race and ethnicity misclassification on death certificates.
SOURCE: CDC/NCHS, National Vital Statistics System.

Table 8. Number of survivors out of 100,000 individuals by age, race, ethnicity and sex in the United States, 2011

Age	All races and origins			White			Black			Hispanic[1]			Non-Hispanic white[1]			Non-Hispanic black[1]		
	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female	Total	Male	Female
0	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000	100,000
1	99,394	99,342	99,448	99,489	99,447	99,534	98,850	98,740	98,964	99,490	99,451	99,523	99,493	99,448	99,542	98,857	98,750	98,967
5	99,289	99,226	99,356	99,391	99,338	99,447	98,698	98,570	98,829	99,397	99,350	99,438	99,398	99,341	99,457	98,691	98,568	98,824
10	99,230	99,160	99,302	99,334	99,275	99,397	98,616	98,481	98,756	99,347	99,296	99,393	99,340	99,277	99,406	98,603	98,473	98,746
15	99,159	99,077	99,246	99,267	99,195	99,343	98,525	98,370	98,686	99,287	99,225	99,343	99,272	99,197	99,352	98,504	98,355	98,672
20	98,917	98,739	99,106	99,034	98,875	99,203	98,206	97,894	98,531	99,095	98,942	99,249	99,034	98,875	99,202	98,169	97,854	98,510
25	98,493	98,116	98,888	98,626	98,285	98,987	97,616	96,987	98,257	98,776	98,468	99,106	98,610	98,263	98,973	97,543	96,891	98,220
30	98,017	97,449	98,612	98,165	97,644	98,717	96,934	95,986	97,878	98,447	97,989	98,946	98,118	97,583	98,673	96,811	95,818	97,812
35	97,465	96,722	98,237	97,627	96,936	98,358	96,148	94,948	97,324	98,068	97,449	98,746	97,540	96,836	98,269	95,969	94,709	97,216
40	96,784	95,866	97,733	96,967	96,100	97,879	95,159	93,723	96,552	97,590	96,812	98,435	96,832	95,950	97,743	94,919	93,403	96,399
45	95,816	94,691	96,973	96,024	94,950	97,152	93,796	92,108	95,419	96,906	95,936	97,957	95,839	94,750	96,963	93,488	91,706	95,213
50	94,281	92,831	95,764	94,528	93,120	95,996	91,712	89,673	93,658	95,809	94,567	97,145	94,287	92,865	95,750	91,326	89,156	93,406
55	91,975	90,009	93,972	92,292	90,375	94,278	88,487	85,786	91,045	94,097	92,404	95,887	92,005	90,081	93,978	88,009	85,126	90,744
60	88,746	85,999	91,515	89,174	86,495	91,929	83,864	80,083	87,402	91,661	89,248	94,155	88,854	86,183	91,583	83,221	79,213	86,978
65	84,368	80,723	88,027	84,915	81,380	88,535	77,760	72,557	82,581	88,218	84,911	91,554	84,574	81,072	88,148	76,934	71,466	82,013
70	78,184	73,558	82,807	78,803	74,318	83,371	69,915	63,341	75,956	83,045	78,616	87,384	78,425	73,991	82,938	68,979	62,216	75,226
75	69,513	63,804	75,184	70,134	64,578	75,747	60,041	52,185	67,206	75,688	70,035	81,063	69,725	64,234	75,280	59,052	51,097	66,358
80	57,493	50,846	64,010	58,030	51,513	64,517	47,885	39,372	55,549	65,221	58,368	71,531	57,615	51,184	64,030	46,938	38,383	54,695
85	41,733	34,665	48,470	42,075	35,102	48,829	33,579	25,356	40,709	50,440	42,690	57,134	41,722	34,804	48,410	32,748	24,587	39,964
90	23,798	17,846	29,226	23,895	17,970	29,346	19,251	12,924	24,558	31,891	24,599	37,422	23,667	17,750	29,066	18,675	12,467	24,035
95	9,097	5,812	11,914	8,973	5,723	11,777	8,241	4,767	11,057	14,233	9,550	17,098	8,880	5,662	11,656	8,003	4,584	10,798
100	1,928	987	2,672	1,826	925	2,544	2,355	1,144	3,278	3,745	2,090	4,408	1,807	926	2,518	2,316	1,101	3,201

[1]Life tables by Hispanic origin are based on death rates that have been adjusted for race and ethnicity misclassification on death certificates.
SOURCE: CDC/NCHS, National Vital Statistics System.

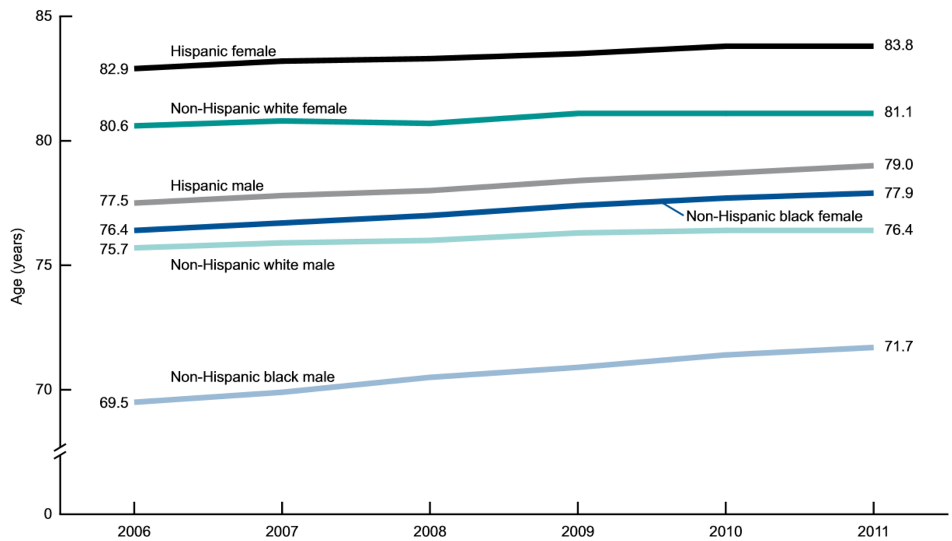


Figure 19. Life expectancy at birth by origin, race and sex from 2006 to 2011

Table 9. Standard results for t-norms and t-conorms for Fuzzy set operations

	t-norm $t(x,y)$	t-conorm $c(x,y)$
Algebraic product-sum	xy	$x+y-xy$
Hamacher product-sum	$\frac{xy}{x+y-xy}$	$\frac{x+y-2xy}{1-xy}$
Einstein product-sum	$\frac{xy}{1+(1-x)(1-y)}$	$\frac{x+y}{1+xy}$
Bounded difference-sum	$\max(0,x+y-1)$	$\min(1,x+y)$
Dubois and Prade (1980b) operators ($0 \leq p \leq 1$)	$\frac{xy}{\max(x,y,p)}$	$1-\frac{(1-x)(1-y)}{\max[(1-x),(1-y),p]}$

REFERENCES

Alsina, C. (1985). *On a family of connectives for fuzzy sets.* Fuzzy Sets and Systems.

Andres, D.J. Belvis, C. Terceno, A. (1996). Fuzzy Methods Incorporated in The Study of Personal Insurance.

Arias, E. Ph.D. (2015). *United States Life Tables of 2011*. Division of National Vital Statistics Reports. Volume 64, Number 11, September 22, 2015

Baruah, H.K., Mahanta, S. Chutia, R. (2010). *Fuzzy arithmetic without using the method of α- cuts*. International Journal of Latest Trends in Computing (E-ISSN: 2045-5364) Volume 1, Issue 2, December 2010

Bezdek, J.C. (1981). *Pattern Recognition with Fuzzy Objective function Algoritms*. New York, N.Y.: Plenum Press.

Brockman, M.J. Wright, T.S. (1992). *Statistical Motor Rating: Making Effective Use of Your Data*. Journal of the Institute of Actuaries 119:457-543

Buckley, J. (1987). *The Fuzzy mathematics of Finance*. North Holland: Fuzzy Sets and Systems 21. 257-273

de Andrés-Sánchez, J. Puchades, L.G.V. (2012). *Using Fuzzy Random Variables in Life Annuities Pricing*.

Dickson, D.C.M. Hardy, M.R. Waters, H.R. (2009). *Actuarial Mathematics for Life Contingent Risks*. New York: Cambridge University Press.

Dubois, D. Prade, H. (1980). *Systems of Linear Fuzzy Constraints*. North Holland: Fuzzy Sets and Systems 3. 37-48

Dutta, P. Boruah, H. Ali, T. (2011) *Fuzzy Arithmetic with and without using α-cut method: A Comparative Study*. Dept. of Mathematics, Dibrugarh University, Dibrugarh,-786004, India

Grover, N. (2014). *A Study of Various Fuzzy Clustering Algorithms*. International Journal of Engineering, Hamacher, H. Leberling, H. Zimmermann, H.J. (1978). *Sensitivity analysis in fuzzy linear programming*. Fuzzy Sets and Systems.

Klement, E.P. Mesiar R. Pap, E. (2000). *Generated Triangular Norms*. Kybernetica

Horgby, P. (1998). *Fuzzy Classification by Fuzzy Inferences*. Fuzzy Sets and Systems.

Hosler V.R. (1992). *The Application of Fuzzy Sets to Group Health Underwriting*. Actuarial Research Clearinghouse.

Huang Q., Lin L., Sun T. (2011). *Some Actuarial Formula of Life Insurance for Fuzzy Markets*. Journal of Applied Mathematics.

Hughes, M. (2012). *Preferred Risks in Life Insurance*. France: SCOR Global Life Americas.

Jandaghi G.; Moazzez H.; Moradpour Z. (2015) *Life Insurance Customers Segmentation Using Fuzzy Clustering*. World Scientific News.

Kosko, B. (1990). *Fuzziness vs. Probability*. International Journal General Systems. California: Gordon and Breach Science Publishers S.A.

Larsen, R.; Marx, M. L. (1981) *An Introduction to Mathematical Statistics and its Applications*, Pearson Prentice Hal

Lemaire, J. (1990). *Fuzzy Insurance*. Pennsylvania: Wharton School, Insurance Departement.

Liu B. (2006). *A survey of credibility theory*. Fuzzy optimization and decision making.

Puri, M.L.; Ralescu, D.A. (1986) *Fuzzy Random Variables*. Journal of mathematical analysis and applications.

Ross, S. (2012) *A First Course in Probability (9th Edition)*, Pearson

Shapiro, A.F. (2005) *Fuzzy Logic in Insurance: The First 20 Years*. Actuarial Research Clearinghouse

Shapiro, A.F. (2013). *Modeling Future Lifetime as a Fuzzy Random Variable.* Insurance: Mathematics and Economics.

Shapiro, A.F. (2013). *Implementing Fuzzy Random Variables – Some Preliminary Observations.* Actuarial Research Clearinghouse.

Taleshian, A. Rezvani, S. (2011). *Multiplication Operation on Trapezoidal Fuzzy Numbers*. Journal of Physical Sciences, Vol. 15, 2011, 17-26 ISSN: 0972-8791, www.vidyasagar.ac.in/journal

Verrall, R. Yakoubov, Y. (1999). A Fuzzy Approach to Grouping by Policyholder Age in General Insurance. Journal of Actuarial Practice vol. 7.

Wit, W.G. (1982). *Underwriting and Uncertainty*. Insurance: Mathematics and Economics.

Yager, R.R. (1980). On a general class of fuzzy connectives. Fuzzy Sets and Systems.

Young, V. (1996). *Insurance Rates Changing: A Fuzzy Logic Approach*. The Journal of Risk and Insurance.

Zimmerman, H.-J. (2001). *Fuzzy Sets theory and its applications*. New York: Spring Science+Business Media, LLC. Kluwer Academic Publishers

Zimmerman, H.-J. (2010). *Fuzzy Set Theory.* Wiley Interdisciplinary Reviews: Computational Statistics.

BIOGRAPHY

Jean Guy Daniel Boni graduated from his hometown Saint Viateur High School, Abidjan, Cote d'Ivoire, in 2008. He moved to the United States for secondary education where he followed two-year undergraduate coursework at Marymount University in Arlington, VA; before graduating from George Mason University in 2012 with a Bachelor of Science in Mathematics and a concentration in Actuarial Science. As of 2015, he completed his graduate work in Mathematical Science along with a Graduate Certificate in Actuarial Science. Conducting several researches in the field of actuarial science, he worked at the National Social Security Fund in Cote d'Ivoire, where efforts are being made to reform the pension and retirement programs.