ELEMENTARY MATHEMATICS SPECIALIST COACHES' CONSTRUCTION OF A HYPOTHETICAL LEARNING TRAJECTORY FOR RATIONAL NUMBER EQUIPARTITIONING

by

Kimberly Morrow-Leong
A Dissertation
Submitted to the
Graduate Faculty
of
George Mason University
in Partial Fulfillment of
The Requirements for the Degree
of
Doctor of Philosophy
Education

Committee:

Jennifer M Suh ______ Chair

______

Anastasia P. Samaras ______

______ Program Director

______ Dean, College of Education and Human Development

Date: July 30, 2019 ______

Summer Semester 2019
George Mason University
Fairfax, VA

Elementary Mathematics Specialist Coaches' Construction of a Hypothetical Learning Trajectory for Rational Number Equipartitioning

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at George Mason University

by

Kimberly Morrow-Leong
Master of Education
George Mason University, 2010
Master of Arts
San José State University, 1993
Bachelor of Arts
University of California Santa Barbara, 1988

Director: Margret Hjalmarson, Professor
College of Education and Human Development

Summer Semester 2019
George Mason University
Fairfax, VA

Dedication

This is dedicated to my husband Greg, my daughter Cassandra, and all the people I've met along the way who have been a part of my village.

Acknowledgements

The list of people to thank for this part of my life journey is as long as the journey itself.

To Dr. Jennifer Suh – You will never know how much you have done to change my life. You always believe in what I can do, but more importantly, I marvel at the number of people you have influenced with your work.

To Dr. Margret Hjalmarson – Because you have always asked me what I wanted to be when I grow up, I have kept wondering what the answer might eventually be. Now I know. Your calm and your patience have taught me how to do the job I will soon be doing.

To Dr. Anastasia Samaras – You've challenged me, mentored me, and in the end you trusted me. I appreciate the work you have done and the patience you have shown.

To Dr. Sara Delano Moore – Not all friendships start over fractions, but I think the best ones do. Thank you for your consistent support. Remember, the best dissertation is . . .

To the rest of the Mason MEL faculty and doctoral students – Dr. Courtney Baker, Dr. Theresa Wills, Dr. Toya Frank, Dr. Kat Meints, Dr. Terrie Galanti, Kim Fair, Dasha Gerasimova, Lesley King, Dr. Spencer Jamieson, Dr. Dori Hargrove, Patti Freeman, Danielle Steelman, Sarah DeLeeuw, Andrea Weiss, I have enjoyed and learned from our time together. You are amazing! Thank you.

A sincere and gracious thank you goes to all of the mathematics specialist candidates I have had the great fortune to teach at Mason. You are the exciting future!

To Dr. Bill Bintz, Suyi Chuang, Patty Clark, Pat Dell, Judy Dietrich, Dr. Michael Fish, Jason Gauthier, Vicky Gibson, Linda Gojak, Dr. Victoria Jacobs, Carol Knight, Steve Leinwand, Annette Lewis and all of the Dream Team, Becky Molitor and the GBW staff, Dr. Julie McNamara, Terrie Pressman, Beth Skipper, Genni Steele, Diane Taylor, Patrick Vennebush, and to many, many others who knowingly and unknowingly provided inspiration – a sincere and heartfelt thank you!

To my mother, who inspired me to this set this life goal. I know you do not remember saying to me, "I don't care what kind, you just need to grow up and be a doctor," but I remember. Done. Thank you.

Table of Contents

List of Tables

List of Figures

List of Abbreviations

Association of Mathematics Teacher Educators .... AMTE
Association of Supervision and Curriculum & Development .... ASCD
Cognitively Guided Instruction........ CGI
Common Core State Standards, Mathematics .... CCSSM
Council for the Accreditation of Educator Preparation .... CAEP
Developing Mathematical Ideas .... DMI
Elementary Mathematics Specialist .... EMS
Elementary Mathematics Specialist Coach........ EMC
Elementary Mathematics Specialist & Teacher Leaders .... ems & tl
Fairfax County Public Schools .... FCPS
Hypothetical Learning Trajectory........ HLT
Learning Trajectory .... LT
Learning Trajectory/Progression .... LT/P
Looking at Student Work........ LASW
Mathematical Knowledge for Teaching........ MKT
Mathematically significant pedagogical Opportunities to build on Student Thinking........ MOST
Mathematics Specialist .... MS
National Council of Supervisors of Mathematics........ NCSM
National Council of Teachers of Mathematics .... NCTM
National Governors Association and Council of Chief State School Officers .... NGA & CSSO
National Mathematics Advisory Panel .... NMAP
Pedagogical Content Knowledge........ PCK
Pre-Service Teacher........ PST
Professional Noticing........ PN
Virginia Mathematics and Science Coalition .... VMSC

Abstract

ELEMENTARY MATHEMATICS SPECIALIST COACHES' CONSTRUCTION OF A HYPOTHETICAL LEARNING TRAJECTORY FOR RATIONAL NUMBER EQUIPARTITIONING

Kimberly Morrow-Leong, Ph.D.

George Mason University, 2019

Dissertation Director: Dr. Margret Hjalmarson

This qualitative study investigates how mathematics coaches use and refer to different sources of learning trajectories when examining written artifacts of student thinking. As an aspect of horizon mathematical knowledge for teaching, knowledge of learning trajectories reflects the mathematics coach's personal amalgamation of different learning trajectories, which is used as a road map for student learning. Using the framework of professional noticing and Confrey's (2012) framework for unpacking learning trajectories to guide the study design, mathematics coaches discuss and sort artifacts of student thinking and are then asked about their interpretation of student thinking. A qualitative design explores what features of the work mathematics coaches attend to, interpret, and plan to address in their coaching practice. A thematic network analysis method was used to map the coaches' actions against references they make to existing learning trajectories. This study adds to the body of knowledge of how learning trajectories guide educators' planning and assessment decisions and with implications for

understanding the identity language coaches use when discussing student work. The variety and richness of justifications coaches cite for pairing and grouping students based on their work is an unexplored area of research that warrants further study.
Keywords: Professional noticing, learning trajectories, hypothetical learning trajectory, grouping students, elementary mathematics specialist coach, equipartitioning, student work, mathematical knowledge for teaching, identity, evaluative language, thematic network analysis, commognitive theory, horizon content knowledge

Chapter One

We cannot observe that which we do not see nor wish to see.

Introduction

Regina reached over and picked up Jessie's assessment paper, glanced at it and placed it confidently in the advanced stack of student work saying, "Jessie knows this. She just forgot to write the answer. She had a bad day, but I know that she knows this."

Kelly picked up an assessment, scanned it in one glance, and dropped it in the progressing pile. "Can you tell me how you made that decision, Kelly?" I asked. "Well," she said, "there are no numbers here. Leenah just doesn't get it. She never does. She's in my low group."

Sarah sat hunched over a single paper with her lips pursed in concentration. Her pencil moved back and forth across the paper as if ticking off a list and counting, but not making any marks. Then she looked up and said, "Look at this one, guys. I am not exactly sure what her thinking is yet, but it's pretty cool. Take a look!"

From the back of the room, Ken, distractedly shuffles through a stack of assessments, sighs loudly and directs his question to no one in particular. "Why are we doing this? All of my kids know how to solve this problem. Look, they just need to memorize their facts, and they'll be fine!"

Lindsay, the school math coach, spoke next. She spoke softly, but confidently, "Let's try something different. Regina, give your class' papers to Ken. Ken, give yours to Sarah, and Sarah, you give yours to Regina." As the teachers passed over their stacks, Lindsay distributed a set of post-its to each of them, and then began to speak again.

As you look at each paper, write down observations on what you see. You don't have to evaluate or grade the work, but make a note of one thing you are pretty sure each student understands. Make a note of it on a sticky note and then find evidence of *why* you think that. Make a note of the evidence that convinced you. (Lindsay)

While this is a fictional portrayal of actual teachers scoring problem-based mathematics assessments, the teachers' words ring familiar for anyone who has engaged in such rubric-scored cognitively demanding tasks (Stein, Grover, & Henningsen, 1996;). It is traditionally a large part of a teacher's job responsibilities to read and evaluate student work. What is different now is the expectation that teachers apply a rubric-scored approach to evaluate student work on open-ended and rigorous tasks, a type of evaluation process previously associated with instruction in writing (Carter, 2009). Rightness and wrongness of student responses becomes less important but the depth and connectedness of the students' responses emerge as a more salient indicator of students' growth and progress.

Another thing that is different is the presence of a mathematics coach in this meeting. This math coach is part of the teaching faculty of the school, but he or she does not have any students assigned to them. Instead she spends her days in other teachers' classrooms or in planning or professional development sessions like the one represented here. Typically a mathematics coach is an experienced teacher who has taken additional courses to specialize in mathematics education in order to become a mathematics specialist (Virginia Mathematics and Science Consortium [VMSC], 2016). While some mathematics specialists may take other positions working directly with students, the mathematics coach has taken on the new goal of working with adult learners.

The caricatures above demonstrate that teachers' assessment of student thinking in mathematics can be non-specific, based on scant evidence, and worse yet, have little impact on instruction. On the other hand, their assessments can also be thoughtful and deep, reflecting on the gains students have made, and using this information to plan for future instruction. However, common grading practices often do not encourage teachers to record the profound understandings of the sometimes subtle growth students experience (MacMillan, Myran, & Workman, 2002).

In these cases the teachers may rely on intuition rather than on evidence presented in the student work. One goal of assessment might be to focus on evidence of student thinking, target the needs of particular students, and finally act as a tool for instructional planning.

One common reason that subtle student understandings are invisible is due to the nature of assessments. Another reason may be the lack of information or differing levels of fidelity in the hands of practicing teachers with differing priorities (Wilson, Mojica, & Confrey; 2013). Studies of teachers' interactions with textbooks show that there is wide variability of fidelity and level of adaptation of the materials that accompany a standard mathematics textbook. This variability included not just adherence to the intended use of certain tasks and activities but also to conceptions of the consistencies of standards-based l or problem-solving lessons (Remillard, 2005). Since such inconsistency manifested in a carefully designed textbook package, it is not a stretch to consider that they would also exist in learning trajectories that are not yet clearly defined by research (Lobato & Walters, 2017).

The general idea of a learning trajectory (LT) is not a new one: it is a sequence or progression of learning that unfolds out of learning tasks. While this is oversimplified, it is consistent with any definition of the LT that exists. Lobato and Walters (2017), in their chapter in the decennary volume on the state of mathematics education, declined to offer a single definition of the LT, instead broadening the definition to include any sequence that describes consistent growth in mathematics learning. van Hieles described a hierarchical model of geometric thought (van Hiele, 2004) long before it might have been called a learning trajectory, although the current definition now includes levels of thinking. The developmental studies of children conducted by Piaget and Inhelder (1974) might also qualify as LTs under the current conception.

However, the current phrase *learning trajectory* dates back to 1995 and Simon's introduction of the *hypothetical learning trajectory* (HLT). With this term he was endeavoring to describe a stage in the instructional planning process whereby a teacher gathers information at her disposal, plans a lesson, teaches it, and then uses what she learns to plan the next lesson. The entire process was entitled the Mathematics Teaching

Cycle, but specifically, the HLT is the teacher's most reasonable guess about the best lesson goal, plan of activities, and the learning process for a given group of students. Interestingly, this began as a plan for teaching, but as the idea gained momentum in the professional field, the term HLT became more reflective of the learning of students that takes place rather than of the teacher's planning process. In a later article, Simon along with Tzur (2004) further drew emphasis away from the role of the teacher and directed the readers' attention to a mechanism for outlining the connection between the learning task and the students' conceptual learning, thereby reinforcing the turn to student learning, even while referencing the teacher as the supervisor and planner of these activities. Others adopted the HLT language as well (Clements & Sarama, 2004; Gravemeijer, Bowers, & Stephan, 2003; Hadjidemetriou & Williams, 2002; Jones et al., 2001; Steffe, 2003), and with each reference, the term referred more and more to the prediction of a sequence of student learning in a given domain, accompanied by given tasks. As a result of this change in direction, from the teachers' acts to the students' learning, the community changed the focus. From the vantage point of many years later, Empson (2011) pointed out the contrast, which had previously been virtually overlooked, when she wrote: "a learning trajectory did not exist for Simon in the absence of an agent and a purpose and . . . it was introduced in the context of a theory of teaching" (Empson, 2011, p. 573).

Empson's intention was to critique the value of the learning progression as a practical concept, noting how dependent an LT is on the teaching environment and the tasks employed, as well as a concern about how tightly LTs are tied to specific domain, limiting their value in broader problem solving contexts. Empson (2011) was not the only researcher to revisit the learning trajectory phenomenon. The science education community was investigating the same pedagogical problem and adopted the phrase *learning progression* to refer to a very similar idea (Corcoran, Mosher, & Rogat, 2003). But unlike the field of mathematics education, the science community was clear from the beginning that they were interested in the thought tendencies students exhibited as they learned concepts in science. As a researcher in mathematics education, Battista (2011) took a more positive view of learning trajectories than Empson, stating flatly that teachers need LPs (Battista opts for the term *learning progression*) Most recently, the summative chapter by Lobato and Walters (2017) collected and organized a broad swath of research related to all manner of student learning in mathematics and science that was presented as sequentially ordered. The taxonomy they created accommodates the broad collection of conceptions and potentially resolves the conflicting terms, collapsing learning trajectory and learning progression into an LT/P.

One of the benefits of Lobato and Walters' (2017) broad sweeping inclusionary stance in the learning trajectories taxonomy is a reconceptualization of standards and standards documents as a form of learning trajectory. A widely known and implemented set of standards is the Common Core State Standards (National Governors' Association and Council of Chief State School Officers [CSSO], 2010). Since nearly all states and territories, and even the Department of Defense schools have adopted the Common Core Standards, Mathematics (CCSSM) standards, the impact of this particular learning trajectory is monumental in the United States (ASCD, n.d.). Even those states that did not adopt (Virginia and Minnesota) and those who adopted and adapted the CCSSM (Indiana and North Carolina are just two examples) are all impacted by the prevalence of CCSSM-based curricular materials and the de facto learning trajectory presented in them.

From this point of view, the potential impact of this particular learning trajectory is even greater than that of other learning trajectories. The authors of the CCSSM document their interactions with authors of some of the most well-studied research trajectories in publication (Clements & Samara, 2004; Confrey, et al., 2014; Petit, Laird, Marsden, & Ebby, 2010; Steffe & Olive, 2010). They noted the impact these interactions had on the development of the CCSSM (Daro, Mosher, & Corcoran, 2011). In the end the authors stated that the standards are *inspired by* research on learning trajectories, but note that this not the same as the standards being *based on* research. As a matter of fact, Daro et al. clearly stated that the standards balance three often competing demands of standards for learning:

> the pull of three important dimensions of progression: cognitive development, mathematical coherence, and the pragmatics of instructional systems. The situation differs for elementary, middle, and high school grades. In brief: elementary standards can be more determined by research in cognitive development, and high school more by the logical development of mathematics. Middle grades must bridge the two, by no means a trivial span. (Daro et al., 2011, p. 41)

In sum, there is evidence that the CCSSM references current conceptions of developmental learning trajectories, but the discipline logic that intervenes in the upper grades references a different type of learning trajectory altogether.

Despite the connection between the CCSSM and current learning trajectory research, it is important to note there is growing evidence that some standards may not be placed at the appropriate grade level, according to the cognitive development of most children in that age group (Steffe, Norton, Hackenberg, & Thompson, 2012). This is only problematic because the high stakes testing that is based on current standards demands that students meet grade level standards. This emphasis on “success” may incentivize short term and surface level procedural instruction in order for students to post better scores. Continued research into learning trajectories can continue to inform the content and leveling of standards, adaptations can be made, and more reasonable rigorous standards can be implemented in schools nationwide. It is important to note that the research into learning trajectories can continue to inform teacher and mathematics coach practice in other ways as well.

A more focused issue is how teachers use their understandings of learning trajectories in instruction. Knowledge of mathematics on the horizon is one way to characterize how teachers’ awareness of students’ progress is seen throughout a mathematical domain (Ball, Thames, & Phelps, 2008). One of the components of subject matter knowledge, content horizon knowledge ties what students learn in earlier grades to what they learn in later grades. Teachers who do not engage horizon content knowledge may teach students a grade-specific “rule” that later becomes obsolete, or “expires” (Karp, Bush, & Dougherty, 2014). For example, a second-grade teacher may say “You can’t subtract a higher number from a lower number.” The intent is to help students make sense of the practical use of the subtraction symbol: subtraction in second grade is only done by subtracting the lesser value from the greater value, as in 14 - 8 = 6. However, in seventh-grade students need to call upon a different understanding. They need to understand the subtraction *operation* and indeed will learn to subtract a greater value from a lesser value, such as 8 - 14 = -6. A teacher with horizon content knowledge makes pedagogical decisions with a mind toward communicating to students a more precise understanding of the subtraction operation rather than an expedient, but less than accurate, rule.

Not only does a leading theory about teachers’ mathematical knowledge for teaching include a broad understanding of student learning (Ball, 1993), results from practice-based professional development also demonstrate the importance of teachers examining student learning across grade levels. One lesson study project showed that there are immense benefits for teachers working in multi-grade level teams (Suh & Seshaiyer, 2015). By selecting a single lesson or task and then adapting it for third-grade, sixth-grade, or even eighth-grade, each teacher on the team was challenged to consider the content vertically and recognize a wide range of mathematical thinking in more than one content domain. In this context, upper grade teachers can recognize learning milestones for their students whose mathematical thinking operates at a lower grade level. Lower grade teachers have the opportunity to connect the content they teach with the content that is coming in later years, which informs vertical lesson planning for all teachers. In both of these projects, teachers' recognition of and work with content from another grade level not only informed them about the mathematical content of other grade levels, it helped them understand the progression of their own students.

In the two studies described, the growth in teachers' horizon content knowledge (Ball, 1993) was incidental to the goals of the research projects. In the lesson study example, the primary goal was to provide teachers with content-rich problems and ask them to engage students with a rich task that included algebraic thinking in an elementary school setting (Suh & Seshaiyer, 2015). That teachers gained knowledge of a broad swath of mathematical content knowledge was an additional positive outcome. The second study started as a vertical collaboration between multiple grade levels, but the result was essentially to standardize instruction in addition strategies in grades K-5 (Cameron, Loesing, Rorvig, & Chval, 2009). When the project began, teachers were using different approaches to teaching addition, had differing expectations for student success, and all were unclear when students should move to more sophisticated strategies. In essence, through the process of examining carefully culled samples of student work from each grade level, the team of teachers was able to construct a sequence of expected strategy use for addition on which each of them could agree. Additionally, the teachers came away with a deeper understanding of addition strategies and common student errors that occurred across all grade levels, and more importantly they came away with a sense of empowerment from identifying a school-wide problem and a solution of their own devising.

A more recent study had a different approach. The researchers purposefully engaged teachers in a study of the equipartitioning learning trajectory and then observed the changes in their instruction that resulted (Wilson et al., 2013). In this case the learning trajectory had already been derived through research and it was used as a teaching tool in a professional development setting. Nevertheless, the team of researchers found that the teachers not only understood their own students' work more profoundly, they also began to reformulate their own understanding of the mathematics in light of the learning trajectory information. Deliberately augmenting teacher content knowledge for teaching was effective in improving the kind of specialized content knowledge that directly connects to student learning. The success of these three research projects in increasing teachers' mathematical content knowledge for teaching supports the notion that investigating what teachers know about learning trajectories is an important line of inquiry. It may be even more important for their instructional coaches.

It is exciting to consider the magnitude of influence these three studies had on the school learning teams where the projects unfolded (Cameron et al., 2009; Suh & Seshaiyer, 2015; Wilson et al.,2013). All three were led by university research teams, clearly a resource not available to every school. Additionally most, but not all, of the teachers involved in these innovative programs were teachers in grades K-5, which means that they were licensed to be generalist teachers, with many responsibilities in addition to knowing and teaching mathematics well. The National Mathematics Advisory Panel recognized this reality and made a recommendation that the nation's elementary schools move toward a model of the *expert* teacher of mathematics, taking up the responsibility of teaching all K-5 mathematics sections (National Mathematics Advisory Panel [NMAP], 2008). The change to the expert teacher model permits a school division to invest in the content knowledge and the pedagogical content knowledge more heavily, albeit for fewer teachers. While the Advisory Panel did not specify the path for these expert individuals, it did raise attention for the need for additional expertise in mathematics education in our nation's schools.

The state of Maryland had already started addressing the concern by writing and introducing legislation in support of the mathematics coach. The *mathematics coach*, unlike the expert teacher described above, would not be a teacher. Instead they would be a support for existing teachers, building the teachers' capacity to be experts in mathematics teaching and learning. While the Maryland initiative was not able to get legislation for an Elementary Mathematics Instructional Leader (EMIL) passed until 2010, the writers were able to introduce a version of the program in Virginia, and with funding from the National Science Foundation, the Commonwealth of Virginia introduced an Elementary Mathematics Specialist Master's degree and license endorsement for practicing teachers in 2007 (Campbell, 2011). The coursework and preparation for the Mathematics Specialist in Virginia is arguably the strongest in the nation (VMSC, 2016), including five courses in elementary mathematics content and five courses in leadership, followed by an independent internship experience.

By virtue of the additional coursework, Virginia mathematics specialists may be more likely to have the knowledge and expertise to work in schools and to implement versions of the programs described above or to support the professional growth of teachers. But they may also have other roles in education, including district office positions or work to support Title I programs (Salkind, 2010). A mathematics coach is a mathematics specialist working in a school to support teachers (McGatha & Rigelman, 2017). In the opening vignette, we met Lindsay, a mathematics coach who is leading a group of teachers reviewing a problem-based assessment that they had recently administered to their students. Lindsay's goal in this interaction is to step into the teachers' habitual practice of "grading" student work and refocus their efforts on identifying evidence of students' progress along a LT within a mathematical domain. She recognizes the teachers' familiar language and deftly steps in to focus the lens on the evidence of student thinking and away from the evaluative statements that characterized most of the opening comments (Davis, 1997). Another one of her strategies is to limit the teachers' familiarity with the creators of the work samples, thereby limiting their references to prior performance. This is not because Lindsay does not have confidence in the teachers' abilities to evaluate students' progress, but rather she does this exercise so that they can learn to make even stronger assessments of their students' work in the future, with more confident references to evidence and to the relevant learning trajectory.

The use of learning trajectories in teachers' professional development has a strong and growing body of research to support the practice. Whether the learning trajectories teachers and coaches reference have been determined by research and presented in a professional development setting, or determined by local curriculum and standards, or through teachers' exploration of student thinking along with coaches, the results have shown productive gains in aspects of teacher practice (Franke, 2018; Krupa & Confrey, 2010; Sarama & Clements, 2009). Despite these positive outcomes, we do not know much about *how* teachers and coaches use the learning trajectory to assess and/or learn from student work. Moreover, because of the muddying influence of so many sources of knowledge, identifying the indicators of the sources which are referenced is challenging and even elusive. But given that learning trajectories influence the teachers' and the coaches' interpretations of student thinking and progress, this is valuable information.

Statement of the Problem

Learning trajectories are essentially a map of student learning; some are descriptive of student development and others are prescriptive sequences. Depending on a teacher's pedagogical content knowledge (Shulman, 1986), the different trajectories that educators have access to may impact the decisions and judgments they make. The National Mathematics Advisory Panel (2008) emphasized the importance of teacher content knowledge in general in increasing student achievement, including knowledge of learning trajectories. At the time the panel advocated for more dedicated mathematics specialist elementary teachers with additional specialized training.

Seven years after the NMAP report was released, a study of instructional quality found evidence that greater mathematical content knowledge, as measured by MQI metrics (Hill et al., 2008), is associated with greater teacher quality (Hill, Blazar, & Lynch, 2015). Virginia mathematics specialists, who have between 24-33 additional credits of mathematics content and leadership coursework, likely have higher mathematical content knowledge, despite no known study to explore whether this is true or not. Specifically, we do not know much about how the additional coursework and knowledge in mathematics that Virginia mathematics specialists have acquired impacts their understandings of student learning trajectories. Curiously, the same large longitudinal study showed early on that increases in coaches' content knowledge on an earlier version of the Mathematical Knowledge for Teaching assessment (Hill & Ball, 2009) shows no correlation with any measures of teacher improvement in a variety of categories (Burroughs, Yopp, Sutton, & Greenwood, 2017). Although to be fair, Burroughs et al. admitted that it is not clear whether the lack of impact is because there actually is no effect or if it is the result of the content courses provided within the study. In either case, little is also known about how coaches*,* as mathematics specialists with additional training in content and leadership, use the knowledge they have in their professional practice when they are in the role of mathematics coach, in particular in the act of examining and assessing artifacts of student thinking.

Research Questions

The purpose of this study is to explore elementary mathematics specialist coaches' references to learning trajectories as they examine artifacts of student thinking in order to understand what elementary mathematics coaches notice in student work, the resources they reference in order to make sense of the work and how they reference them, and how this information is used in practice. Using the professional noticing framework (van Es & Sherin, 2008) as a guide, engagement includes what coaches attend to in student work and the sources of the learning trajectories and learning progressions, broadly defined, that inform their interpretations, and the instructional or coaching decisions they propose for the students or teachers.

1. What evidence of students' mathematical thinking do elementary mathematics coaches attend to while examining students' written artifacts?
2. What learning trajectories or other similar sequencing sources do elementary mathematics coaches reference in order to interpret students' prior, current, and future understandings, based on an examination of student work?
3. How do elementary mathematics coaches use knowledge of learning trajectories or other similar sequencing sources, along with evidence gathered from artifacts of student thinking, to make instructional and coaching decisions?

Definitions

Elementary mathematics coach (EMC). The elementary mathematics specialist (EMS) has had more education in elementary mathematics and mathematics pedagogical content knowledge than the average teacher. The amount of additional education varies. There are three varieties of specialization for an EMS: elementary mathematics coach, mathematics specialist teacher, and interventionist teacher (McGatha & Rigelman, 2017). This study focuses only on the elementary mathematics coach.

Equipartitioning. Equipartitioning is the construct of ‘‘cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing ''fair shares'' for each of a set of individuals'' (Confrey, Maloney, & Corley, 2014, p. 724).

Learning trajectory/progression (LT/P). Narrowly defined, a learning trajectory consists of a mathematical goal, a developmental sequence of learning, and the tasks that moves students toward that goal (Clements & Sarama, 2004). Broadly defined the LT/P is any sequence that describes the order in which students learn a topic. This may include a variety of resources commonly available in schools, such as a state curriculum, a district pacing guide, or a commercial textbook. It may also include third party materials that reference research-based learning trajectories or similar online resources.

Professional noticing (PN). Professional noticing is a construct that describes the unique character of the work that experts do in their field of expertise (Mason, 2011). Professional noticing consists of three separate phases. The first is what the individual attends to in the professional environment. The second phase can occur nearly simultaneously and it includes a judgment or interpretation of what has been observed, and finally, the third phase is responding and acting (Jacobs, Lamb, & Philipp, 2010).

Virginia mathematics specialist. The Virginia Mathematics Specialist has completed enough coursework (at least 24 credits in mathematics content and leadership) in addition to their teaching license. Also required for the VA Mathematics Specialist endorsement is a master’s degree and at least 3 years of teaching experience (VMSC, 2016).

Horizon content knowledge (HCK). Horizon content knowledge is a variety of mathematical knowledge for teaching (MKT). MKT includes knowledge that is unique to the art of teaching. It includes general content knowledge, knowledge of tools and strategies that is only used for teaching, and HCK. HCK is the awareness of the mathematics that is beyond a student’s immediate grade level. For example, intentionally selecting a model for multiplication that works for a third grader but which is also a reliable tool for a sixth grader demonstrates HCK (Ball, 1993).

Chapter Two

Introduction

This study was designed to examine how Virginia elementary mathematics specialist coaches who currently practice in an elementary school engage with the products of student thinking. Specifically, the study is designed to explore a representative sample of written work in order to delve into what the coaches attend to, note their understandings and use of any form of student learning trajectory or learning progression to interpret student thinking, and devise a plan for future instruction. It also explores the coaching actions that are inspired by the examination of student work samples.

The role of the embedded mathematics coach puts them in a unique position in the school building. They are solely focused on mathematics instruction across all of the grades and even beyond (de Araujo, Webel, & Wray, 2017). Because they serve teachers across all grades, they are more likely to develop a working knowledge of all of the grade level standards, but a highly trained mathematics specialist is also knowledgeable of much more detail about student learning. For example, understanding more about how students learn mathematics, and in what sequence, may focus more attention on student thinking and learning, a generative change that positively feeds the cycle of learning about students' understanding of mathematical ideas (Franke, Carpenter, Levi, & Fennema, 2001). As a matter of fact, in an analysis of the existing literature on mathematics coaching, the authors identified two coaching practices directly related to mathematics as the most productive coaching activities for the mathematics coach: engaging in mathematics and examining student work (Baker, Bailey, Larsen, & Galanti, 2017). Since "productive," in the case of mathematics coaching, is having a positive impact on teacher practice and student achievement, it is reasonable to conclude that coaches zeroing in on students' mathematical work and using the mathematical thinking to guide coaching practice is more likely to achieve the often elusive goal of professional development: to improve student achievement.

The goals of this study lie at the intersection of at least four significant areas of educational research: mathematics coaching practice, the theoretical discussion of learning trajectories, the professional noticing construct, and protocols for examining student work. The first two areas address the choice of participants and the subject of the research. Focusing on the work of coaches, who practice a less explored form of teacher professional development, the framework begins by identifying the qualifications and work of mathematics coaching as form of professional development. Teachers' professional development experiences are varied (Borko, 2004), but the most successful experiences follow a certain pattern: they are focused on content, engage active learning, are consistent with the teachers' current beliefs and practices, endure over time, and are experienced collaboratively (Desimone, 2009). The practice of mathematics coaching is consistent with all of these features of successful professional development. A second area of exploration is that of learning trajectories, an idea broadly defined as a sequence of learning (Lobato & Walters, 2017). The variations in this definition will be explored in this chapter and within this framework they will define the ways in which mathematics coaches decide how they know student learning is moving forward.

The two additional areas that form this framework are professional noticing and protocols for looking at student work (LASW). If the first two areas reflect the *who* and *what* of the study, the final two areas frame *what* mathematics coaches say and do as they describe student work. Professional noticing separates the act of noticing into three separate but related steps: attending to details, interpreting details, and acting on the information (Mason, 2011). The framework makes the argument that examining student work not only occurs in-the-moment, but also in the less time-sensitive situation of reading student work samples. Finally, an analysis of protocols for examining student work, more commonly, and seemingly casually, referred to as *looking at student work* (LASW) provide a frame for the mindsets and dispositions of coaches as they examine samples of student work.

To bound the task of reviewing the research literature relevant to this study, I chose to discuss studies first that broadly inform mathematics education. Based on searches done during the pilot study (Morrow-Leong, 2013), the literature search started with foundational books, articles, or studies and used these to search for broader uses in other studies as well as the most current information. For example, to begin a search on the concept of learning trajectories, a Google scholar search for what was already known to be the foundational article was entered (Simon, 1995), and all articles that cited that piece within the area of mathematics were examined. The combined collection of articles was sorted into groups, with six or seven research teams that were working on different trajectory projects and domains identified. The literature citations in these studies then led to a review of learning trajectory research (Battista, 2004) and also within science education (Corcoran, Mosher, & Rognat, 2003). Finally, to obtain the most current developments, a recently published chapter (Lobato & Walters, 2017) was used as a reference to explore possible missed learning trajectory research. Similarly, two recently published edited volumes provided a current lens on the topics of professional noticing and elementary mathematics coaching (McGatha & Rigelman, 2017; Schack, Wilhelm, & Fisher, 2017). This supplemented the search that was first developed within the pilot study.

Mathematics Coaches and Specialists

A mathematics specialist is a relatively new idea – an educator whose professional time is devoted to improving mathematics instruction in schools. What is a mathematics specialist? Many states have adopted some form of teacher licensure that includes specialized training in mathematics pedagogy and which is more likely to address the area of elementary education. As of this writing, 19 states have adopted a licensing option for a specialty endorsement in mathematics, and eight more are in the process of doing so (Elementary Mathematics Specialist & Teacher Leaders [ems & tl], 2019). In part, this is in response to the 2009 report from the NMAP report which identified a need for more teachers with expertise and additional training in elementary mathematics education. Clearly there is recognition of the need for focused attention on the improvement of mathematics education.

History of the mathematics specialist in Virginia. For many years there was disagreement about the name and role of a specialist who works in mathematics education. In a 2009 position statement for the National Council of Teachers of Mathematics, McGatha (2009) referred to the mathematics specialist as an educator who works directly with students and reserved the “coach” title for the educator who worked directly with teachers (McGatha, 2009). In Virginia, where the first full scale degree and endorsement program originated (Campbell, 2011), the mathematics specialist role was, and is, not that strictly defined. By 2017, McGatha and Rigelman (2017) had established a nomenclature that seems to have been adopted by many in the community. The mathematics specialist is an umbrella term for individuals who have the requisite extra training, but who could serve as a coach, an interventionist, or in the specialist teacher role identified by the National Math Advisory Panel. In this study we will specifically address the elementary mathematics specialist serving in the role of a mathematics coach in an elementary school.

The commonwealth of Virginia was one of the first states in the nation to offer a mathematics specialist endorsement to the professional teaching license (Campbell, 2011). Originally part of a National Science Foundation grant through the Virginia Mathematics and Science Initiative , the mathematics specialist degree program is now offered through 12 universities in Virginia (Virginia Mathematics and Science Coalition [VMSC], 2016). The course requirements for the program include five content area courses in K-8 mathematics, including number and operations, rational numbers and proportional reasoning, algebra and functions, geometry and measurement, and probability and statistics. The course sequence also includes a series of leadership courses specific to mathematics education that include topics such as learning theory, diagnosis of student understanding, formative assessment, access for diverse learners, adult learning, instructional decision making, data analysis and discussion, lesson studies, and the development of effective task-based mathematics (VMSC, 2016). While there are still few job positions posted that require this specific coursework, in some areas it is becoming more of an expectation that candidates for instructional support positions in mathematics will have earned the degree or endorsement.

Despite the robust support from the Virginia Mathematics and Science Coalition, the role of the mathematics specialist, once hired into a school still remains unclear. In an early survey of coaches and their principals, Salkind (2010) found that principals and coaches did not share the same view of what the coach's role and responsibilities in the school should be. This often resulted in conflicts, which impacted efforts to impact instruction. Data collection and identification is also complicated by the titles and funding sources for mathematics specialist coaches. For example, "mathematics resource teacher" is a common title, however, the job description varies across counties, and more importantly, as Salkind (2010) showed, the expectations of school administrators varied, even when job descriptions remained consistent.

Elementary mathematics specialists in schools. Once a licensed mathematics specialist is installed in an elementary school context and is given responsibility for supporting the instruction of teachers of mathematics, the impact can be significant but slippery to measure because current research on the effectiveness of coaches in the school environment is still sparse. In part, this may be because the "coach" in the research literature often holds another title and is a supporting resource for a larger-scale project with separate goals, such as described by Cobb and Jackson (2011). Another reason for the sparse literature is the relative newness of the role, but results of studies on the impact on teacher practice and student achievement are starting to emerge more frequently. For example, one study showed that the mathematics coach can impact student achievement but that this effect can take as long as 3 years to emerge (Campbell & Malkus, 2009). Other studies can more confidently point to changes in teacher instructional practices that have been shown elsewhere to result in greater student achievement. McGatha, Davis, and Stokes-Levine (2017) highlight studies that show the coach's impact on teacher practice, including teachers engaging in less directive instruction (Polly, 2012), coaching acts that evoke teachers' *pedagogical curiosity* about student learning (Olson & Barrett, 2004), and teachers' engagement in more frequent best teaching practices (Race, Ho, & Bower 2002). As research continues into what impact coaches have on the success of students and their teachers, the question remains: What do mathematics specialists need to know in order to be successful?

What do elementary mathematics specialists need to know in order to be successful? Clearly they need a knowledge of K-8 mathematics content that exceeds that which is required for general teacher licensure, but they also need knowledge that is specific to working with adult learners (teachers), and for working with leaders in a school context. In 2013, the Association of Mathematics Teacher Educators (AMTE, 2013) published guidelines for the preparation of elementary mathematics specialists, setting the bar for preparation. One standard in the AMTE guidelines is a fuller understanding of the specific knowledge needed for understanding children's mathematics and for teaching children, often referred to as pedagogical content knowledge (Shulman, 1986). This knowledge is different from, and distinguished from, the mathematics required to do everyday mathematics or secondary and college mathematics (Ball et al., 2008). The National Council of Teachers of Mathematics' (NCTM) accreditation standards from the Council for the Accreditation of Educator Preparation (CAEP, 2012) reflect a similar vision of the teacher knowledge required to perform the duties of the mathematics specialist, including the work of the specialist serving in a departmentalized teaching context. These standards show that elementary mathematics specialist coaches should not only possess the knowledge expected of the classroom teacher, but should also have additional training that supports mathematics instructional practice with adult learners across a school building or district.

Coaches in practice. The daily work of coaches varies greatly. One impact that coaches can have on the school environment is focusing attention on longer term goals, including long-term planning or on the long arc of student learning in a particular domain, both across and within a grade level. For example, one coach in an early study of mathematics coaches encouraged teachers to focus more on planning content and units into the future (Becker, 2001). This long-term planning may have been partly because of the coach's schedule of visits. There were breaks of several weeks between the coach's visits, during which time the teacher continued the work the coach began during model lessons. The situation illustrated the impact a coach can have on teachers' longer term planning. In another more recent study, Krupa and Confrey (2010) conducted a detailed analysis of how coaches spend their time working within a school environment. Digging deep into the categories of coaching work, they identified at least two categories that show that long term planning or a focus on students' learning sequences was exclusively part of their practice for 7.5% of their time. Another 6.6% of the time focused on assessment rubrics or on discussing content, which may also address some of the bigger ideas that span whole school years or across grades. As a matter of fact, given their work with teachers at different grade levels, McGatha (2008) found that coaches focus more on the reasoning behind the sequence of student activities, tasks, and standards than teachers do. Coaches appear to focus more on broader swaths of mathematical content knowledge than the average teacher might.

The far-reaching visions shown by coaches may also have an impact on teachers' practices, causing them to be more attuned themselves to long term goals for student learning. In the early Becker (2001) study, one finding was that no matter the style of coaching, teachers with coaches had a more coherent view of their curriculum and were more prone to focus on the big ideas important to a grade level than previously. As a matter of fact, Kazemi & Franke (2004) directly attributed teachers' greater capacity to formulate their own hypothetical learning trajectories (HLT) for their students to facilitate sessions devoted to looking carefully at student work and reflecting on student thinking. While these sessions were not with school-based coaches, the facilitators led the session in the way a coach might. Not only did the teachers develop their own hypothetical learning trajectories based on close examinations of student thinking, they also formulated instructional trajectories that extended beyond immediately upcoming lessons.

Coaches' mathematical knowledge for teaching. The coaches' broader view of content and curriculum may come from the fact that their practice focuses on a range of grade levels, but it also may originate in a greater mathematical knowledge specific to the domain of teaching in general. The idea of pedagogical content knowledge (PCK) existing as separate from general content knowledge was a radical, yet obvious, idea when it was first introduced (Shulman, 1986). Teachers need to know a different variety of content than the general public needs because they have the unique role of supporting student learning. Ball and Cohen (1999) referred to this knowledge specifically in the domain of mathematics as mathematical knowledge for teaching (MKT). MKT includes two dimensions, one related to the subject itself and the other related to instruction within the discipline.

Because coaches' responsibilities span so many grades there are two sub-domains of mathematical content knowledge that may figure more prominently in a coaching practice. HCK addresses MCK as it spans the grades. As the word "horizon" implies, HCK includes a broader view, recognizing that choices made in kindergarten about models and representations may have implications far beyond that grade level. Similarly, knowledge of content and curriculum may also be of critical importance for coaches not only because they work with all grades, but as the Becker (2001) study, previously described, teachers who work with coaches may be more attuned to their own curriculum.

It is interesting that coaching programs and initiatives do not always assume that the coach has greater knowledge than the teacher. In fact, one comprehensive study of coaching programs noted that only half of the programs studied assumed or provided extensive extra training for the coaches. The other half assumed no extra training in mathematics or leadership than that which the classroom teacher commonly has (Yopp, Burroughs, Sutton, & Greenwood, 2017). On the other hand, Campbell and Malkus (2011) described another program that includes extensive extra training for coaches, including content and leadership courses that lead to a post-graduate degree. This program includes PCK as well as the specialized mathematics knowledge for teaching unique to teaching mathematics. While the knowledge that teachers need in order to do their work shares properties with the knowledge coaches need to do their work: coaches may even require more, although the list of what is necessary for teachers is already lengthy! It is logical to assume that the knowledge required to engage in a productive coaching practice includes at least the same level of knowledge of mathematics and pedagogy as teaching does.

Professional Noticing

The subject of this study is the knowledge and practices of elementary mathematics specialist coaches. In order to explore aspects of the coaching process, it is important to identify an approach that can focus attention on important coaching moves and actions. The professional noticing framework recognizes expertise and provides a tool for unraveling the choices made as the professional makes split second decisions in the context of working in their profession. Recognizing the mathematics coach as a professional with the skills of a teacher, but also with the additional skills required to coach adults, the work of coaches warrants another level of study.

Artifacts of students thinking are likely the most plentiful and arguably the most valuable resource in a school building, so it is important for teachers and elementary mathematics specialists to recognize the value of this plentiful classroom resource, as there are always opportunities to listen to students do mathematics, and to look at the work they do. However, if teachers do not systematically use the student work artifacts that are generated in the classroom to explore the thinking that students are doing, this value is lost. The construct of professional noticing offers a framework for describing what coaches and teachers pay attention to when they engage with student thinking in the course of their teaching or coaching practice.

"Noticing" is a common English word, but it also describes specific acts of a professional working within their field (Sherin, Jacobs, & Philipps, 2011). While noticing is not generally done consciously or mindfully, the professional noticing construct is more complex. Logically, what one notices in normal circumstances is only marginally within the realm of their control (Mason, 2011), but with attention and focus, professionals can learn to engage more mindfully and focus their attention on targeted aspects of practice (Gawande, 2017). Professional noticing in education is a fertile field for researchers exploring the practice of teaching precisely because what one notices is intimately tied to many other constructs related to the practice of teaching and coaching (Mason, 2011). Specifically, professional noticing focused on assessment of student thinking and their progress toward learning goals is one aspect of practice that promises to be fruitful.

Professional noticing (PN) consists of three separate but related acts (Sherin et al.,2011). While the acts are separate in the PN construct, in reality the three often occur nearly simultaneously in the most accomplished practitioners. The first is what we commonly think of as noticing: (1) seeing and *attending to* certain features within the classroom setting. The first act is instigated by something outside of the observer.(2) The second act is an *interpretive* process that takes place entirely within the mind of the observer, and therefore, the interpretation is subject to the varied experiences of each observer. (3) In the third phase, teachers make informed decisions about the next course of action (van Es & Sherin, 2008). In terms of noticing student thinking, van Es (2011) outlined a framework that was conceived to apply to the live classroom environment. Indeed much of the professional noticing literature in mathematics education targets episodes that occur while a teacher is engaged in the classroom (Luna & Sherin, 2017; McDuffie et al., 2014; Walkoe, 2015). While the live classroom is a vital grain size and locus for study, professional noticing can also reasonably apply to teacher and coach activities that take place outside of the highly reactionary and fast-paced classroom environment.

Levels of professional noticing. The video club format of professional development frames a study that applies the same professional noticing framework to teacher practice (van Es & Sherin, 2008). The video club engaged teachers in monthly meetings where they watched videos of each other teaching within the context of the professional development session. The whole program's focus was on children's mathematical thinking. While the discussions of teacher and student interactions took place once a month with a live gathering of participants, the groups nevertheless responded to the events that had been filmed live. One step removed from a live response, the video club study (van Es & Sherin, 2008) recorded and then classified the teachers' talk about student actions into three categories. These categories reveal a window into *what* teachers noticed as they listened to the classroom interactions as well as *how* they responded. Teacher statements in the *grounded narrative* category offered a sequential description of the events in the classroom setting. These comments generally did not hold an evaluative overtone, but read more like the plot sequence or other kind of story line. The narrative; however, did not typically focus on content or pedagogical content-related details.

Another category of teacher statement does reflect a dive into discussions of teaching and learning, but these statements take on an evaluative note. *Evaluative discourse* describes the judging behaviors of teacher-observers as they comment on both students and teachers, describing what the teacher should have done, what did not go well, how students performed, etc. (van Es & Sherin, 2008) The third discourse category is *interpretive discourse* and it is most closely tied to the act of professional noticing. The teacher-observer cites evidence of student thinking and then makes inferences about the student's thought process or current understanding. This sequence of actions precisely corresponds to components that describe professional noticing. The important outcome is that not all teacher interaction with student work or with students in the classroom engages the mindful and informed decision-making that characterizes the mature stage of professional noticing. The study concludes that teachers can develop these skills and learn to attend more to evidence of student thinking, interpret the evidence rather than evaluate it, and respond mindfully. There is no known research on what the highly trained mathematics specialist notices and how they respond in similar contexts.

While one might think that years of teaching experience would have an impact on teacher reactions, this may not necessarily be the case. A study of preservice teachers used a similar video club format to determine if the acts inherent to professional noticing were something that could be cultivated in teachers with little to no experience in the classroom. Students in methods classes examined four or five video clips of instruction through a variety of lenses. Three of the lenses focused on teaching, learning, and tasks, and the fourth included a social justice lens (McDuffie et al., 2014). What is notable is that this study identified reactions from preservice teachers similar to those identified in the van Es (2011) video study, despite the different experience levels of the teachers. Responses primarily included evaluative statements or were simply narrative statements of what happened. Interestingly, Jacobs, Lamb and Philip (2010) noted in their methodology that they consciously decided not to sample research participants based on years of experience, noting that more years of experience had not previously been a significant predictor of teachers' capacity to engage in productive professional noticing of children's mathematical thinking. It would be reasonable to assume that more years of experience would cause teachers to be more responsive to the evidence contained in student thinking and displayed in student artifacts. This is either not the case, or there is another explanation for why teachers did not show evidence of responsiveness.

In the end, the McDuffie et al. (2014) study identified four levels of noticing. The *baseline* level generally matches the grounded narrative from Sherin et al., (2011) in that it includes general descriptions of the teaching vignette that are vague and lack details. At the other end of the spectrum, the *making connections* level roughly corresponds to the behaviors identified in the interpretive discourse type of engagement. Both interpretive discourse and the making connections level describe teacher statements that highlight aspects of student thinking or that make explicit connections between teaching moves and student thinking. The differences fall in the middle. While the van Es et al. (2008) study focused exclusively on teacher discourse after viewing the video clips, the McDuffie et al. study looked beyond discourse to attentional behaviors as well. The *attention* level represents emergent pedagogical practices: in this case the preservice teachers (PST) began to notice teacher moves or attend to students' mathematical thinking in ways they had not done previously. The *awareness* level represents a more targeted level of noticing. The PSTs operating at this level cite evidence for their observations and discuss why certain outcomes occurred. This is similar to the results Crespo (2000) found when she asked preservice teachers to engage with students via a letter writing exchange. Initially they focused on errors, but as the project progressed they began paying attention to features of student work and then started to show awareness of the children's mathematical thinking.

Table 1

Responses to Observations of Teaching Videos

van Es et al. (2011)	McDuffie et al. (2014)
Grounded Narrative	Baseline
Evaluative Discourse	Attention Awareness
Interpretive Discourse	Making Connections

Curiously, McDuffie et al. (2014) observed that most PSTs began their study already comfortably operating at the attention level. An interesting, and contrasting, finding in the van Es (2008) study was observed when teachers in the video club watched the video clip of their own classroom. The teachers almost exclusively commented on their own videos at the most basic and descriptive level. They described what was present in the clip, but they did not engage in a higher level analysis of either their teaching or their students' mathematical thinking. We are left to wonder about the impact of watching one's own teaching and whether it prevents the individual from engaging in productive discussions about instruction, at least initially. It is also a reason to reconsider whether teacher learning can sometimes be more fruitful when teachers engage in observing the work of other teachers instead of their own.

The progression of teacher engagement with student thinking and student work from superficial observations to reflective discussions of children's mathematical thinking is not unique. Generally, both of the progressions described a similar pattern, beginning with fact-based accounts of what took place, to evaluative reports on the quality of student work or teacher action, and finally leading to an interpretive response that centers on evidence and is grounded in content. But as we have seen, experience alone does not lead teachers to engage productively with student thinking.

A professional development experience was required for some teachers to focus their attention on the details related to student thinking. In other words, connected and interpretive professional noticing is not a natural byproduct of years of teaching experience. The second phase of the professional noticing framework described by Sherin et al. (2011) seems to be the pivotal moment. When teachers engage in interpretive, rather than narrative or evaluative behaviors or statements, the act of professional noticing becomes a productive and generative process. It is useful to unpack and look separately at each of the three components of professional noticing in order to identify aspects of practice that may be relevant.

Attending to children's mathematical thinking. When teachers begin to attend to the mathematical thinking of students, they are turning attention to the lived experiences in the classroom. They may attend only to narrations of the general action in the classroom (van Es, 2011) or remain centered on the teacher's actions. Chamberlin (2003) named "de-centering" as a the primary challenge for teachers learning to more deeply engage with student thinking. De-centering means that the teacher subsumes their ever-present thoughts about the practice of teaching and focuses instead on the process of learning that is happening in the students, particularly when the students' thinking seems illogical to the observer. Not surprisingly, Empson and Jacobs (2008) specified a complementary teacher behavior that is present before teachers engage in de-centering. As a teacher responds to a student's explanation of their thinking, the *directive* teacher listening stance is characterized by the teacher who directs the student's thinking to the expected outcome rather than taking the student explanation at face value. They cite a particularly vivid example from another study: "I immediately doubted his accuracy because it did not concur with what I had in my mind" (Nicol, 1998, as cited in Empson & Jacobs, 2008, p. 268). Baldinger (2015) noted a similar phenomenon in secondary teachers who were engaged with analyzing student work. She noted that the majority of the preservice teacher participants in her study considered the work of students through a lens of what she termed *mathematical analysis*. The teachers focused entirely on the mathematical accuracy, even painting broad generalizations about accuracy based on scant evidence. The piece of work was treated only as mathematics rather than as the product of a student. In some cases, it was mathematics that the PST was still struggling to understand. The student work could have been a textbook entry, and the response might have been the same. Baldinger also noted PSTs directly comparing the student work to their own work on the same problem, and interestingly, she noted this action as further toward sophisticated teaching practice than the mathematical analysis. The practices described here are of teachers (or PSTs) who are not yet attending to children's mathematical thinking: they may be looking at student work, but they are attending to mathematical accuracy and not attending to the origin of the thinking that produced the work.

In order for teachers to attend to students' thinking, they must first de-center from their own practice and focus on students. They may start to identify features and details of the mathematics with which students struggle (Chamberlin, 2003; Talanquer, Bolger, & Tomanek, 2015). Or they may engage in *observational listening* (Empson & Jacobs, 2008) where teachers listen to students' explanations of their thinking, but do not pursue that line of thought. Instead the teacher begins to mold and form the students' thinking into an expected response. Only one teacher in Baldinger's (2015) study engaged with the student work from a pedagogical point of view. None of these studies makes any suggestion for why an examination of student thinking would not center on the students themselves. Proximity, or the fact that student performance may be perceived to be closely tied to teacher behavior may play a role. But it raises interesting questions about whether a mathematics coach would have the same perspective while examining student work.

Interpreting children's mathematical thinking. The second part of the professional noticing construct describes teachers' actions as they begin to interpret students' thinking. In a fascinating study, a researcher conducted a case study of an early career teacher as she began a journey of self-reflection and professional change (Davis, 1997). In some respects the study was also a form of self-study in that through collaboration the researcher also delved into his own beliefs about teaching and learning. Davis zeroed in on the listening practices of the teacher, using classroom incidents to characterize the purposes of her listening to student responses. He identified the evaluative listening stance which is characterized as "limited and limiting" (Davis, 1997, p. 359) to both teachers and students. The teacher is not listening openly but rather is listening for an expected response of some sort. Alternative responses are jarring and unexpected and are therefore distractions. Talanquer, Bolger, and Tomanek (2015) invoked the same evaluative frame when they described teachers' responses to written work that showed very few inferences about student thinking. In contrast, the movement of teacher comments from evaluative to interpretive is noted as progress toward a more thoughtful analysis of student work (van Es & Sherin, 2008). Evaluative comments rush to judge, often with little or no evidence. In schools, this may be heard in comments such as these: "This is good work," "Joey's last homework was just great," or the less complimentary, "Frances' last paper was terrible." The comments are not accompanied by evidence to justify the opinion and curiously in some communities, none may be expected.

By contrast, Davis' interpretive stance demands evidence. He described the teacher's questioning patterns at this stage as information-seeking rather than as response-seeking. This may seem insignificant, but the change from the teacher's point of view is profound. She was no longer seeking pre-determined answers, but rather was looking to understand what her students were thinking, and the questions were tinged with genuine curiosity. Olson referred to this strategy as "evoking pedagogical curiosity" (2005). Empson and Jacobs (2008) further stated that this curiosity about the possible strategies, approaches, and thoughts of students can be a powerful hook to motivate teachers to learn even more about children's mathematical thinking. In their work, they referred to this productive variety of listening as responsive, indicating that the teacher responds to students by eliciting even more information. The interpretive listening stance is an integral part of the professional noticing framework because it establishes the necessary prerequisite awareness of and curiosity about students' individual thinking that maintains the educator's focus on the data they will need for the next stage. They are now ready for what Kazemi and Franke (2004) call a *transformation of participation*, a shift from an insular view of what they know and of what students can do to one that is open to new ideas. Is there evidence of *transformation of practice* in the views of coaches as well as they engage with student work?

Deciding how to respond on the basis of children's understandings. The third phase of the professional noticing process reflects a new engagement with the work that students do, either live in the classroom (Liu, 2014), or with student work samples. These samples may have been examined either in collaborative teams (Brodie, 2014; Kazemi & Franke, 2004; McDuffie, 2014; van Es & Sherin, 2008), in an interview setting (van den Kieboom, Magiera, & Moyer, 2017), or as an individual activity (Baldinger, 2015; Son, 2013).

One of the most fruitful avenues for examining student thinking is through the mistakes they make. There isn't widespread agreement in the literature on the definitions of a mistake, with "errors," "mistakes," "misconceptions," or "slips" all appearing throughout. To simplify matters for our purpose, I will decide on a distinction and use it throughout. There are four basic types of errors: one sort is a conceptual error (Son, 2013). The *conceptual* error is based on a student's misconception of a primary mathematical concept. It appears pervasively and repeatedly, in many different representations. The second is a procedural error. Educators often confuse the two varieties of errors. In Son's (2013) study, 56% of the preservice teachers examined a sample of student work that showed the student's lack of a conceptual understanding of similarity and suggested procedural assistance, particularly related to how the student might solve the proportion, ignoring the fact that the student did not show evidence of understanding the concept of similarity! A *procedural* error is a miscue in the individual's algorithm. In the similarity example given above, the student may correctly describe what sides are similar in a given figure yet incorrectly use the proportion structure to represent those relationships. Brodie (2014) adopted the term *slip* to refer to the kind of mistake that is a cognitive gaffe. It is characterized by its sudden appearance, and the individual's quick response to step in and fix it. Another source of errors are *omissions*. For example, in a problem describing two groups of students sharing a certain number of sandwiches and comparing which group got more, the student may give two accurate representations of sharing and then fail to answer the final comparison by saying which group got more. Mistakes, errors, or "forgettings" are commonplace in a classroom, but refining the identification of these incorrect outcomes can focus educator attention on the source of the outcome, and facilitate an effective plan to address it.

In my practice, different sources of incorrect responses often fall under the umbrella of student forgetfulness. The student "forgot" some detail and for that reason they made a mistake. I always wonder if the student "forgot" or if they simply did not understand something well enough in the first place to use it meaningfully. For this reason, errors due to forgetting might also be classified as a slip, as a conceptual error, as a procedural error, or as an omission. Either way, forgetting something is often a teacher's characterization of students' errors, and it may not be based on how student themselves might be thinking.

Table 2

Description of Error Types

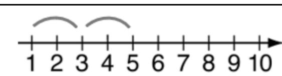
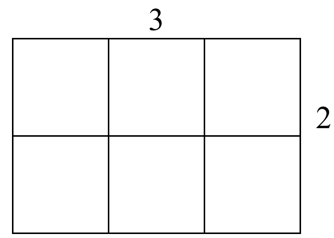
Error Type	Description	Examples, using 2 x 3 = 5 as the error.
Slip (Brodie, 2014)	The mistake does not reflect the student's understanding, but instead can be characterized as an unexpected drop in awareness. The individual quickly resolves the mistake when it is pointed out or they notice it themselves.	Oops! I wrote a five. I meant to write six! I saw a six in my head and then somehow I still wrote a five!
Procedural error (Son, 2013)	A procedural error may be the result of overgeneralizing a pattern from one operation to another, or any variety of mistakes that come from misapplying or executing a procedure in the wrong context.	 "I counted 1,2,3, then 3,4,5. That's two threes." This error is a procedural misunderstanding of counting three on a number line. The student counted tick marks using one-to-one correspondence rather than counting intervals between them.
Conceptual error (Son, 2013)	A conceptual error is a misunderstanding of a fundamental idea underlying the mathematics. For example, the misunderstanding may be in not understanding the meaning of an operator.	"Show me 2 x 3"  This is a conceptual error because the student shows each factor as a quantity, rather than showing one as a *group* and the other as *how many groups*.

Table 2 (continued).

Error Type	Description	Examples, using 2 x 3 = 5 as the error.
Omission	An omission occurs when a student fails to include a key piece of an answer. They may put a group of base 10 blocks together to show two addends, but not give the sum. It would be inappropriate to assume that they can find the sum, and it would also be inappropriate to assume they can’t.	 This is an error of omission. The student has drawn an accurate representation of 2 x 3, but has not shown that they understand that the product is 6.

In summary, the terms “error,” “mistake,” and even “misconception” alone have no meaning because they do not indicate the origin of the incorrect response. Without carefully examining student work and asking for more details it may be impossible to classify student errors accurately. However, addressing student errors offers teachers the opportunity to target their interpretation of student work and helps them move from a focus on correctness to a focus on the meaning of student thinking (Crespo, 2000) and to the potential for professional growth.

A teacher acting on their interpretation of student thinking is the least well-developed phase of the literature on professional noticing. Leatham, Peterson, Stockero and van Zoest (2014) admitted that the work of interpreting accurately and then acting on what one has heard is complicated work, and of course highly variable in situ. Their study's focus was to identify moments during teaching that were the richest opportunities for what one might term a "teachable moment," but which they called a MOST (mathematically significant pedagogical opportunities to build on student thinking). They began the investigation because they noted that many of these MOST moments, or moments to further student thinking, went unnoticed by teachers. These lost moments were also lost opportunities for learning, so their goal was to understand the kinds of student thinking that are the most necessary for teachers to understand and therefore, respond to. Moreover, they also wanted to discover how teachers can generate MOST moments with students and how teachers can best capitalize on these moments. In other words, they wanted to increase the effectiveness of the teachers' professional noticing.

Capturing and responding to students in the live classroom environment is not just challenging for decision-making on a class-level, it is also challenging in the one-on-one interactions that teachers have with students. In the course of their investigation, Schneider and Gowan (2013) noted one critically important observation: it was more difficult for teachers to give feedback that helps students learn than it was to analyze a student response and plan the next instructional step. They hypothesized that formative assessment targeted specifically at one student's needs was more challenging because the teachers in this study were perhaps not yet able to identify the actual reasons for students' errors. This failure to understand students' actual misconception shortcut their efforts to provide feedback. It also may explain globally why the last phase of the professional noticing framework is less developed than the other two; it is necessary that teachers have the capacity to accurately interpret student thinking before one can study what they do next.

Professional growth while engaged with student work. Nevertheless, there is some evidence that significant professional growth can take place in teachers who have learned to focus deeply on children's mathematical thinking. Teachers have been observed productively adding to their repertoire of understandings of student thinking during a particular lesson (Doerr, 2006). The change in teacher behavior based on the process of closely examining student work and responding productively can also be persistent, lasting even years after the professional development that effectuated the change in the first place. For example, Franke et al. (2001) followed up with a group of teacher-participants and found that they had not only maintained their focus on student thinking, they had also experienced generative change, or change that compounds into even greater change. These changes reflected continued growth in understanding the development of student thinking and a focus on evidence of that thinking, which continued at least 4 years into the future. Furthermore, they continued to seek out colleagues who wished to do the same!

The previously described project, Cognitively Guided Instruction (Empson & Levi, 2011), has experienced great success over the years, both in impacting teachers, but also in contributing to the body of knowledge on student thinking. In the descriptions of their methodologies, they state that they provide professional development to teachers that includes doing mathematics, reading research, and analyzing videos, in addition to the time spent discussing what they have learned along with other teachers as we also saw in Jacobs, Lamb, and Philip (2010). What stands out as atypical is the task of reading research, specifically research that is targeted to understanding children's mathematics. Interestingly, the research area of professional noticing has recently taken a turn toward a similar goal. One study of preschool teachers' professional noticing narrowed the focus of the study to the specific mathematical practice of problem solving (Fernández, Llinares, & Valls, 2013). Another study that limited and specified a targeted domain of mathematical thinking was the study of similarity discussed earlier (Son, 2013). Another study featured the topic of relational thinking, specifically the role of the equal sign, and gave PSTs the opportunity to analyze videos of master teachers engaged with students (van den Kieboom et al., 2017). One of the findings of this study is that the participants reported a stronger knowledge of relational thinking and felt better prepared to teach it. This was a direct result of several exercises in professional noticing targeted to this area of mathematics. As more studies explore the direct impact of professional noticing within particular domains of mathematics, specific details of the necessary pedagogical content knowledge required to teach or coach also promises to grow.

Finally, Spitzer, and Phelps-Gregory (2017) addressed the issue of mathematical domains more broadly. They described a methodology for analyzing teachers' professional noticing of any mathematical domain, a process that begins with identifying a mathematical goal. By deconstructing the goal into sub goals, the teacher can use the sub goals to identify students' understanding of mathematical ideas at a very precise level, and focus their professional noticing onto these smaller grain size ideas. With enough detail and enough time deconstructing mathematical goals relevant to their teaching, teachers can essentially create a form of learning progression that will inform the connections they can make to their observations of student thinking. This approach shows promise, not just for teachers, but also for researchers who can identify problematic mathematical topics and explore both teacher noticing of student thinking as well as student thinking itself. Ironically, this reflects a return to a model of a learning trajectory that is more about the act of teaching just as Simon (1995) originally described..

LTs and Learning Progressions

As the discipline of professional noticing begins the process of drilling down farther into the details of what teachers notice and to what they respond, the study of what students do, and should know becomes more critical, as these understandings become important data sources. Learning trajectories (and learning progressions) formally present what students know and therefore, become important tools.

Students' growth and development can take surprising twists and turns, sometimes following the curriculum set for them and at other times meandering around important ideas without achieving the objective of the day, week, or even of the year. Nevertheless, teachers still hold the key to guiding students toward a more robust understanding of mathematics. With the introduction of the CCSSM (NGA, 2010) a close examination of learning paths may be more feasible, as much of the U.S. population is following the same curricular sequence. However, a sequence of content presented by grade is no guarantee that each level of content is appropriate for the students in that grade to learn. Much research is needed in order to conclude definitively that, for example, modeling the division of fractions in sixth-grade is appropriately accessible for every student in the United States. In the meantime, learning trajectories and progressions may become key tools for assisting teachers as they learn more details about the means by which students learn mathematics.

The information shared in learning trajectories research varies widely, each having a different focus and emphasis. These might include what aspect of mathematics is studied, the age of the students, the data collection strategies, the goals of the research, and much more. Each choice profoundly determines the outcome of the research. Some projects have a narrow but detailed focus on the early understandings of fractions (Steffe & Olive, 2010), while others take on an entire mathematical domain (Battista, 2011).

Sequences provided for teaching within content area domains is not a uniquely modern phenomenon: any teaching or training activity must include some form of a plan that leads students from the beginning to a mature understanding of the domain content and skills. This is not only true for school learning. Lave (2011) described the process as apprentice tailors enter the community of practice (Wenger, 1998) and begin to engage. The strict sequence of mastery skills and assessment routines followed by the male tailors within the Vai and Gola community of Liberia are notable. The sequence of skills that are learned on the periphery include sewing on a button, but eventually move to full participation when the apprentice earns the opportunity to tailor a suit. Students learning mathematics are also on an apprenticeship path. The Common Core specifically identifies that path by stating that the goals of the standards are “college and career readiness” (NGA & CCSSO, 2010). It is beyond the scope of this paper to argue whether this is an appropriate set of standards and goals for moving students toward the stated goals; however, an understood set of goals, tasks, and a predictable sequence of expected competencies guide the master's efforts to mold the apprentice.

The unfolding of topics and content that students learn in a classroom is not an obvious sequence, and opinions about this progression vary greatly. One of the key considerations for describing the sequence students follow is the focus and intent of the sequence. One of the outcomes of the research done by Confrey and colleagues (2014) as part of the development of the equipartitioning learning trajectory is a clear distinction between a sequence of content based on mathematical ideas that originate from an idealized version of mathematics as an end goal, and a view of mathematics as something that is created within the individual (Confrey, 2012; Wilson et al.,2013). In other words, a mathematics learning sequence can be derived as a top-down model, where the content that students must learn is derived from a vision of what mastery of that content looks like. Because this vision of mathematical learning is conceived by those who have already achieved mastery, the content looks much like the mature version. On the other hand, a developmental view of mathematics learning builds from the earliest conceptions of number and space and assumes that learning will unfold in fairly predictable ways, given that students have exposure to productive mathematical tasks. In this view, student thinking and learning unfolds in "stages" or "levels" that have been shown empirically to be predictable (Sarama & Clements, 2009).

Origins of learning trajectories. When Simon (1995) published this article he casually mentioned the hypothetical learning trajectory. A careful reading shows that he may well have preferred that the mathematics education community instead elect to study the model of the iterative teaching decision-making process that he called the *Mathematics Teaching Cycle*. Beginning with a reasonably educated guess about the interactions between a lesson's goal, the activities of the lesson, and the thinking and learning in which the student might engage, the teacher lays out how a lesson might unfold: it's this part of the iterative design cycle that was called the hypothetical learning trajectory (HLT). The rest of the cycle is concerned with the changes in lesson design both live in the class and also for future instruction. But this is not what caught fire.

It is clear that Simon's focus in 1995 was on the individual teacher's work within a single classroom, even within a short time frame. He specifically referred to "the teacher's prediction as to the path by which learning might proceed" (Simon, 1995, p. 135). In the ensuing years, the "H" has largely been dropped, and more definitive statements about *the* learning trajectory for certain content domains have emerged. Few writers retain what I believe is Simon's original meaning within the context of this paper, including perhaps Simon himself (Simon and Tzur, 2004). In this piece, the authors present the HLT as a "vehicle" for instructional planning. The importance of tasks and a focus on students' understanding remain. For example, the lesson shared in Simon's original piece begins as an exploration of the "multiplicative relationships involved, *not* to teach about area" (Simon, 1995, p. 123), but eventually he admitted "Although my primary focus was on multiplicative relationships, not on area, it seemed clear that an understanding of area was necessary in order for students to think about constituting the quantity (area) and evaluating that quantity" (Simon, 1995, p. 127). In this responsive shift in focus, the hypothesized learning trajectory for the group of students changes significantly, even in some sense shifting domain, and the hypothesis about the students' learning path changes. But in 2004, Simon and Tzur used the more utilitarian word "vehicle" to describe the elaborated hypothetical learning trajectory, conveying the idea that the HLT "carries" the lesson instead of driving it. The agency appears less in the hands of the teacher and is more dependent on the sequence of activities designed to elicit activity-effect in students. Despite this subtle but noticeable shift in emphasis from the teacher as the author of the HLT to a research-determined HLT, the authors acknowledged that as knowledge of student learning processes grows, learning trajectories in general can become more precise and predictive than they have been in the past. However the HLT, as originally presented, has evolved significantly in the intervening years.

Currently, Confrey and her colleagues (Confrey et al., 2014) have outlined learning trajectories for all of the mathematical domains presented in the Common Core standards, building on current empirical research on student learning, particularly in the area of rational number understandings. They use the term "learning trajectory" in a definitive way. The LT is a "researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, and forms of interaction), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, Nguyen, Mojica, & Meyers, 2009, p. 347). Where the Common Core omits intermediary standards they deem necessary, they have added “bridging standards” to complete the connections (Maloney, 2013). In some ways the bridging standards are reminiscent of the sub goals proposed by Spitzer and Phelps-Gregory (2017) in that they identify points between the major mile markers and benchmarks toward a learning goal or between standards. However, in all cases, the LT from Confrey et al. (n.d.) is not an individual teacher construction as the sub goals are but rather a compilation of extant research in the domain.

A research taxonomy of learning trajectories. Lobato and Walters (2017) compiled a long list of separate and distinct learning trajectories, sorted them, and created a taxonomy of existing learning trajectories for mathematics and science learning. Some of the features used to sort the learning trajectories they examined included the *object* of learning, the *target phenomenon* studied, the *theoretical perspective* that informed the work, and the *scale* (scope) of the content addressed. The *objects* of learning may best be described as the source data for the conjectures. For example they name cognitive conceptions, textbook tasks, and observable strategies (Lobato & Walters, 2017) as sources of information used to craft a learning trajectory. Each of these objects is explored in detail in order to craft the sequence of the learning trajectory. The *target* phenomenon of the study is not specifically named by the authors, but they refer to the subjects of study. For example they name individuals’ learning, the practices of a mathematics classroom, or the intertwining of teaching and learning as phenomena studied in relation to learning trajectories. In other words, the target is the place to look for data to craft a learning trajectory. The *theoretical perspective* is typically well-defined in a research environment, as is the last topic, *scale*, which takes in anything from a single fraction-related topic to a broad, sweeping survey that includes many domains and grade levels.

Perhaps more importantly, the authors pointed out that the audience of a learning trajectory is important information for determining how the learning trajectory is presented. Their audience is clearly the field of researchers engaged in learning more about learning trajectories because the suggested approaches to learning trajectories or learning progressions (LT/Ps) that they present are broad categories that are useful for this audience. There are seven approaches to LT/Ps. Approach one describes any sequence that identifies cognitive levels through which students pass. The hierarchy of each level may be strong or weak, which refers to how rigid the progression through the milestones is purported to be. The van Hiele (2004) levels are an example of a strong hierarchy in that a student at level 3 is assumed to have passed through levels 0, 1, and 2. In contrast, Battista's levels of sophistication in spatial reasoning are far more flexible and students are expected to move around levels: the assigned level is more of a tendency (Battista, 2004).

Approach 3 reflects a Piagetian perspective as LT/Ps in this category explore students' construction and reconstruction of schemes through the vehicle of the teaching experiment. This work contributes important understandings to our collective theoretical knowledge of student learning in mathematics. However the controlled experimental conditions do not contribute immediately to our understanding of learning in the classroom environment. Conversely, Approach 4, the Hypothetical Learning Trajectory (HLT), is dependent on the classroom teacher's predictions about student learning, their interpretations of student thinking during the lesson, and their subsequent modifications to the unfolding of the mathematical content. Learning trajectories in the HLT approach are highly dependent on the choices teachers make, although critics point out that teachers often are not given adequate information and resources to do this work productively.

Two of the approaches identified by Lobato and Walters (2017) relate to the process-oriented development of mathematics. Approach 2 focuses attention on the level of student discourse in the classroom environment. The LT/Ps classified under the second approach may refer to the quality of a student's narrative in describing a concept or it may reflect the budding sophistication of students' argumentation skills. On the other hand, Approach 5 describes the progression of a community, specifically that of the interdependent classroom environment. As a matter of fact, the unit of analysis is a group of students and their teacher in the class, but not the individual students or teachers involved. LT/Ps in this category identify the nature of group interactions and consider how the group transforms collectively. Both of these approaches provide frameworks for making sense of effective processes that support mathematics learning, which is integrated with, yet distinct from, mathematical content.

Approach 6 is arguably the most impactful approach in that it is identifying a category of learning sequences that directly impacts state and local policies and therefore, classroom practice: *disciplinary logic and curricular coherence*. The term *disciplinary logic* refers to a top-down approach, where the discipline under study determines the most "logical" sequence for learning tasks. Lobato and Walters(2017) make an interesting distinction that is at the root of this approach: "These LT/Ps are typically *informed by research* versus being the *product of research"* (emphasis in the original), unlike the others (Lobato & Walters, 2017, p. 87). While the research about students' anticipated progression through a topic can have some influence on the creation of the sequences formed through curricular coherence approach, the logic of the discipline takes a lead role. Experts determine the order in which content will be taught. The learning progressions associated with the Common Core are an example of an LT/P from this approach, to which the CCSS authors readily admit (Daro et al., 2011). Most other standards of learning align with this approach as well. The inclusion of standards in a taxonomy that describes learning trajectories or learning progressions is a new connection, linking standards to an idea that traditionally reflected only a research-informed area of study. Interestingly, including standards under the LT/P umbrella may put pressure on the standards writers to be more responsive to the research agenda that informs them.

With a lens toward the approaches to LT/Ps that are most impactful in the classroom environment, Approach 7 stands out. Focused on *observable strategies and learning performances*, LT/Ps in this approach are recognizable as similar to rubric-scored tasks or other similarly scored assessments. Typically these tasks are accompanied by student work samples that serve as exemplars. Because communication is through student work, exemplars are familiar to practicing teachers and comfortable places to engage with a learning trajectory. A typical drawback is also familiar to teachers. Assessment of student thinking can result in a false positive assessment: the student can produce a correct answer but does not necessarily possess the expected underlying conceptual understanding of the mathematical relationships. The reverse may be true as well.

Learning trajectories in practice. Lobato and Walters (2017) have nudged the research field to view the entirety of learning trajectory research as connected. They identified common themes worthy of reframing and exploring. With its focus on a theoretical frame, the taxonomy reflects a research lens, in particular a lens on cataloguing existing LT/P research, which is appropriate for its intended audience. However, what lens is useful to investigate the sway learning trajectories have, or might have, at the school or local level? If we were to study the use of learning trajectories "in the wild," what would be important constructs to help understand the phenomenon?

Despite the fact that teachers make daily decisions about what their students should learn, it is not entirely clear how those decisions are made (Regis, 2008; Shavelson & Borko, 1979). Some teachers use textbooks, but others are following a plan for mastery that is independent of what is in the book (Fuson, Carroll, & Drueck, 2000). Even in a study that details an examination of science teachers' processes for planning inquiry-based lessons (Mangiante, 2012), not much detail emerges to address the anticipated path of student learning during the inquiry lesson. On the other hand, the Anticipation practice, the first of the five practices indicated for productive classroom discussions (Stein & Smith, 2011) challenges teachers during their lesson planning to anticipate the responses students might produce during the lesson. Teachers who participate in specific curricular programs built to represent targeted learning trajectories may be more likely to anchor their planning and instruction based on anticipated stages of student thinking (Franke et al., 2001; Clements & Sarama, 2007), but most teachers do not fall into that category. For most teachers, their references to any classification of learning trajectory is unidentified. Not enough is known about what knowledge of learning trajectories coaches possess, nor how they use this information in their coaching practice.

The lack of clarity around the role of LTs in the field of practice with coaches, and teachers, contrasts sharply with the seven categories in the taxonomy proposed by Lobato & Walters (2017). Many of the categories in the taxonomy are not relevant to the practitioner, which then poses the question – what is an effective strategy to understand the learning trajectories/progressions importance and relevance to practitioners?

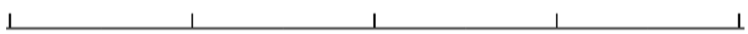
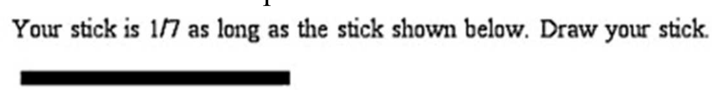
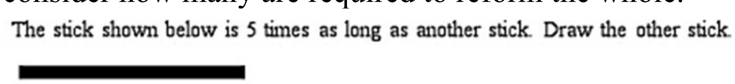
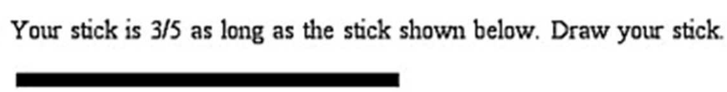
Rational number learning trajectories. The topic or content of a learning trajectory is the mathematics to be learned. Another name for this characteristic of a learning trajectory can be a *domain* or even a *strand*. Two good examples that identify the content of a learning trajectory come from very different research projects. First, the van Hieles (2004) outlined a series of levels of geometric thinking that count as one of the earliest conceptualizations of understanding children's unique mathematical thinking (Steffe & Olive, 2010). While van Hieles had one of the first descriptions of mathematical thinking from the child's point of view, Empson and Levi (2011) offered one of the most surprising results. One of the first fraction tasks they found that very young students could do was partition an area model into fractional parts, as long as each sharer got at least a one unit whole. For example, the classic brownie problem states that four students are sharing seven brownies. Despite the fact that most curricula would classify this task as a division of fractions task and place it in fifth- or sixth-grade, Empson and Levi found it to be conceptually (if not notationally) accessible to first and second graders. The domain of fractions/rational numbers is broad and complex (Vergnaud, 2004) and is united in that it is uniquely distinguished from the field of counting or additive activity (Confrey & Smith, 1995). Beyond that, the characteristics of the projected or hypothetical learning path for fractions varies considerably. Highlighting features of some of the most influential conceptions, recognizing that some have been omitted because they do not hold as much sway in schools in the geographic region where this study takes place. However, non-traditional sources of learning trajectories such as state standards are included, following the lead of Lobato & Walter (2017) who classified standards of learning as such.

Equal sharing: Cognitively Guided Instruction. Empson and Levi (2011) approached the study of rational number learning from the perspective of student thinking as students work on carefully selected tasks. Beginning with a task that innocuously begins with two whole numbers, commonly people sharing something that resembles an area model. The classification of the child's progression along the equal sharing learning path is determined by the actions of the child rather than an outcome of their work. Since equal sharing is a common activity, there is an entry point for every student. In one study, the team identified an early stage called "No-Coordination." The No-Coordination strategy involved a sharing of a quantity with no regard for recognizing the relationship between the number of objects and the number of sharers, often disregarding any remaining whole pieces rather than partitioning them. The child operating at the next stage (Non-Anticipatory Sharing) will partition the leftover pieces, but may not be able to name them as they do so, using terms like “some more” rather than an actual fractional quantity. Once the student starts relating the partitioned pieces to the number of sharers, they begin to move closer to Additive Coordination, beginning to reference the number of sharers as they name new pieces (Hunt & Empson, 2015). The student using the Ratio equal sharing strategy is able to abstract co-varied quantities, using a subset of the entire set to represent the quantities. A simple example of this strategy is recognizing that if 200 students share 500 cupcakes at an event, the problem can be simplified to 2 students sharing 5 cupcakes, with each child getting 2 1/2 cupcakes. Finally, the Multiplicative Coordination strategy does not require a drawn representation because the student recognizes the relationship can be solved with division.

Partitive fraction scheme: Steffe and Olive (2010). The grain size of a learning trajectory describes how broad or how narrow the learning goals are. For example, Norton and Wilkins (2010, 2012) studied a very fine grained learning trajectory that described students’ progression through a short but important list of fraction partitioning and iteration schemes. The experiments were originally designed by Steffe (2010). Based on the theories of Piaget and Inhelder (1973), Steffe and Olive (2010) structured the children’s learning itinerary based on observations of students as they encounter new tasks and progressively reorganize their current mental schemes to accommodate new challenges. The work is intricate as the authors hypothesize and then validate the existence of certain patterns of actions. The subset of schemes relevant to the task used in this study are described and demonstrated in the table below.

Table 3

Equipartitioning Schemes (Steffe, 2010)

Scheme	Task Example
Equipartitioning *Partitioning and iterating occur sequentially.*	Jason "partitioned the stick into four parts by using his concept of four as a template for partitioning" (Steffe, 2010, p. 316). 
PUFS *Partitioning and iterating are done only with the whole and a unit fraction.*	PUFS is the first genuine fraction scheme because it includes both partitioning the whole and iterating pieces to check the relationship to the whole. 
Splitting *Partitioning and iterating occur simultaneously.*	Splitting requires the student to partition a piece and also consider how many are required to reform the whole. 
PFS *Partitioning and iterating are done for unit fractions and proper fractions.*	Students make a proper fraction by recognizing that it is composed of iterations of a unit fraction. In this case they partition the whole into fifths and then iterate it three times to create 3/5. 

While these studies outline subtle details of how students learn certain ideas, the drawback to these findings is that it is challenging to apply the new information to the broader case of classroom use.

Equipartitioning: Confrey (2012). Confrey's interest and investment in learning trajectories related to rational numbers spans the grades, from preschool years into high school. The unifying conceptual ideas is that of the splitting conjecture, or equipartitioning. Equipartitioning is a non-additive process for subdividing a unit whole into equal sized pieces. With area models, this may be achieved by folding or by cutting and arranging to form equal portions. With a discrete model, even the youngest child can using distribution strategies to fair share a collection of objects without even knowing how many objects are present in the collection. Equipartitioning is a prerequisite to the task of naming the fractions that are formed by the action.

One of the most important contributions Confrey (n.d) has made is the addition of bridging standards. A bridging standard unpacks a standard of learning at one grade and matches it to a standard in the next grade (Confrey, et al., 2014). When the leap in learning is too wide, she and her team have identified skills and understandings that logically must be built in the intervening year and amend the standards document to reflect the missing stepping stones. Her team has done this specifically for the Common Core (Confrey, et al., 2014), but the principle is broader than its application in one standards document. As students move towards any mathematic goal, what are the anticipated (and not anticipated) student responses to tasks that outline a pathway to the final goal. At any given step along the way, what skills and understandings are in place, and which are likely to be acquired next? Bridging standards can help teachers identify a progressional view of learning that focuses more on what is going well and identifying a path forward.

Common Core State Standards and other standards documents. When Lobato and Walters (2017) expanded the conception of a learning trajectory it broadened the concept wide enough to include trajectories that are referred to as guided by *discipline logic* (Stzajin, Confrey, Wilson, & Edington; 2012). Discipline logic is a top-down learning trajectory, indicating that the learning milestones are set by the goals outlined for students. Curriculum standards are typically prescriptive and oriented from the top down because they reflect what students should achieve. In contrast, a developmental logic view describes the learning that is taking places and uses that information to predict what might come next in a learning sequence. The learning trajectory that describes young students' acquisition of shape composition skills is an example of a learning trajectory based on the domain development of children (Clements, Swaminathan, Hannibal, & Sarama, 1999). The trajectory describes how and through what tasks students learn to compose geometric shapes to form new ones. This process is distinctly different from the goals of curriculum standards, which may not take individual differences into account.

Most standards are separated into domains, such as those presented in Principles and Standards (2000) or by the Common Core (CCSSO-NGA, 2010). As with other broad topics or domains, domains span multiple grades and may be represented in a variety of tasks from other domains. For example, the Common Core does not technically introduce Numbers and Operations in Fractions until Grade 3, but a careful look at a subset of standards in the primary grades reveal that standards related to partitioning shapes in the geometry domain foreshadow the equal partitioning that is necessary for working with fractions.

Looking at Student Work

Despite the fact that mathematics specialists may not have training above and beyond that of the teachers they serve (Campbell & Malkus, 2009), this may be more a reflection of reality rather than an aspiration. Their distance from the classroom environment gives coaches the opportunity to step back and consider artifacts of student thinking as both an insider and as an outsider. The professional noticing construct establishes the importance of the teachers being aware of student thinking and strategies, relying on evidence rather than on making assumptions about performance. As a matter of fact, Jacobs, Lamb, and Phillip (2010) pointed out that developing the expertise to recognize and attend to students' different strategies and make sense of them is the foundation of learning how to respond appropriately to those strategies.

Recognizing and responding to students' thinking seems an obvious requirement for educators, but it is not always what happens in classrooms. In classrooms that are not student-centered, discussions can unfold in the manner Davis (1997) described: the teacher listens to *respond* to the student, rather than listening to *understand* the student's thinking. Hufferd-Ackles, Fuson, and Sherin (2004) described a similar condition: at the lowest level of productive classroom discourse, the teacher's questions serve to maintain behavioral control and to go only in the direction of teacher to student. Similarly, certain kinds of questions and questioning patterns *funnel* student thinking to mimic the teacher's own (Wood, 1994). These patterns of action in a classroom are anathema to the student-centered classroom. Learning to attend productively and astutely to student strategies is part of the professional noticing construct. The same is expected of the mathematics coach.

Best practices in mathematics coaching. Frameworks for identifying and clarifying best practices for mathematics teachers are more available than they were even 10 years ago. *Principles to Actions*, released in 2014, describes eight practices that exemplify the productive teaching actions that are most supported by research on mathematics teaching. The last of the list calls for teachers to "elicit and use evidence of student thinking" (Leinwand et al., 2014, p. 53). This practice puts student thinking squarely as a focus area for teaching and therefore coaching practice.

Coaching is a different role than teaching, however, calling for an additional list of best practices. The existing research is not broad enough to state definitively what these practices might be; however, an examination of the features of success professional development along with a review of the existing coaching practice literature shows a list of seven coaching practices that show the greatest promise for positively impacting learning in the classroom (Gibbons & Cobb, 2017). Most notable is coaches engaging teachers in the act of looking at student work, with the intent of focusing on a range of student ideas and for devising community-generated terms to name student strategies. Often focused sessions examining student work are facilitated in order to direct attention to student work and the mathematical goals of the student work sessions (Bella, 2004; Blythe, Allen, & Powell, 2008; Daehler & Folsom, 2014; Goldsmith & Seago, 2011, 2013). Coaches can press teachers to be more specific about their comments, cite evidence, and encourage teachers to consider how to respond to student thinking (Gibbons & Cobb, 2017). Interestingly, Jacobs and colleagues (2010) observed that while over half of the teachers observed in one of their studies did attend to the students' strategies, less than a fifth of the group provided any evidence to support their assessment of student thinking. More importantly, about one quarter of the participants referred to the students' understanding of the mathematics in order to formulate a response (Jacobs, Lamb, & Phillip, 2010). This is a curious outcome, as one might assume that it is the work of teachers to listen and respond to student thinking! Clearly there is more to understand about teachers' interactions with artifacts of student thinking. The role of the coach is to facilitate focused efforts examining student thinking as an assessment tool, but also as a tool for planning instruction. The anecdote that opened Chapter 1 describes a coach beginning to lead teachers in a reflective analysis of student work.

Skillful use of artifacts. The "skillful use of artifacts" can be presented in two different lines of thinking (Goldsmith & Seago, 2013). The focus on evidence of student thinking is a key part of the skillful use of artifacts, more specifically it is part of the educators' attention to student thinking. Also included in the attention to thinking category is the distinction between describing student work and interpreting their thinking based on the evidence they find (Blythe et al.,2008). Goldsmith and Seago (2013) also suggested that during a productive study of student work teachers examine student work samples and use evidence to devise multiple plausible interpretations of the inscriptions on the artifact of student work. Engaging in hypothetical work like this not only focuses attention on different solution strategies, it also engages teachers in the mathematics of the task students completed in a different way.

Attention to mathematical content. Attention to the mathematical content of the task and of the student work is another promising addition to the list of best coaching practices (Gibbons & Cobb, 2017). Engaging teachers with the mathematical content of a task not only activates their thinking as teachers, it also places them in the position of learner, for some offering them the opportunity to *construct* a mathematical concept for the first time, rather than taking it as a given fact or procedure. Doing the work of the task in a setting with the coach and a team of teachers also gives teachers the opportunity to anticipate student responses and consider how to respond to student errors and misconceptions with the team of colleagues. Additionally, a focus on anticipating student responses to a task gives teachers and coaches collectively an opportunity to map the task and possible responses back to a learning trajectory, identifying critically important ("big") mathematical ideas, as well as the separate skills and understandings that undergird broad standards or learning goals. It may also be true that student work is a neutral place to engage in challenging coaching conversations, particularly if the samples are mixed or completely anonymous, as it allows the difficult conversations about improving teaching practice to be de-personalized (Chamberlin, 2003).

Coaches as content leaders. The concept of the mathematics instructional coach as a leader (Bitto, 2015) as well as a mathematics specialist (McGatha & Rigelman, 2017) communicates the critically important role this professional can play in effectuating the scope of change that is needed in mathematics education in order for students to become productive, numerate citizens of the 21st century (NMAP, 2008). Under the guidance of the mathematics coach using best teaching and best coaching practices, the careful and structured examination of student work can not only impact teachers' pedagogical knowledge, it can also offer opportunities for teachers to build their mathematical knowledge for teaching, particularly as it refers to their understanding of students' learning trajectories in the different domains of mathematics. This begs the question – what do the coaches know about students' learning trajectories themselves? In the digitally connected world of social media and teacher marketplaces offering "take-away" quick lessons, what information do coaches use to vet and test materials against their understood progression of students' mathematical ideas? Finally, coaches may choose to conduct structured examinations of student work, a coaching practice that is well-documented, but they may also reference the same knowledge of student learning trajectories as it informs other aspects of their practice.

Conceptual Framework

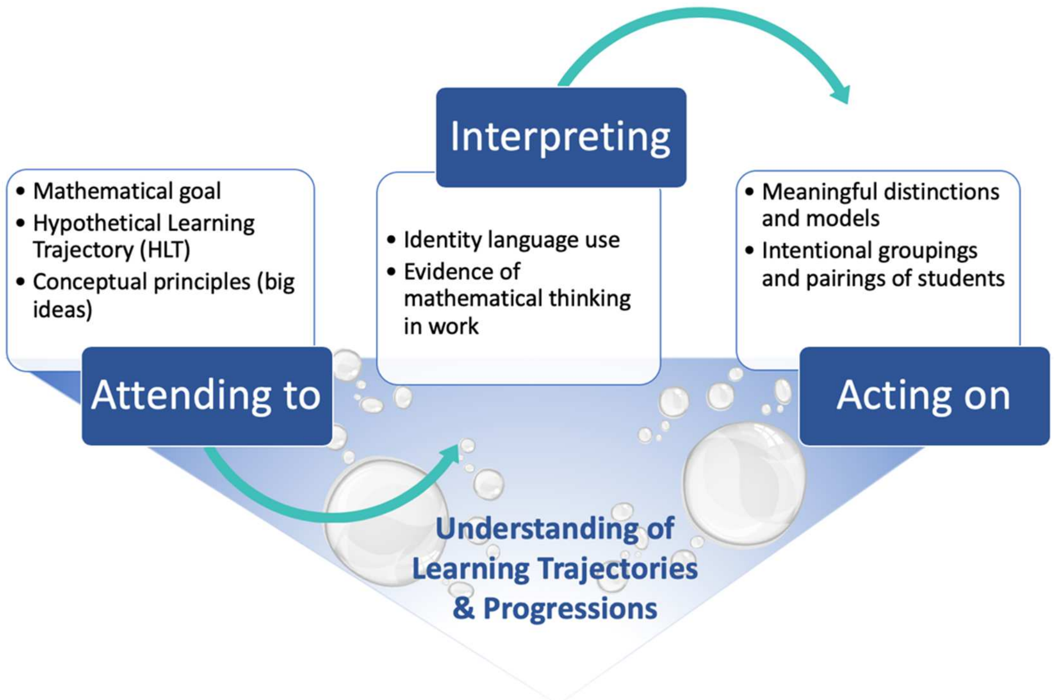


Figure 1. Projection of knowledge of learning trajectories/progressions onto the process of professional noticing and the unpacking of learning trajectories

The conceptual framework that guides this study is an application of the professional noticing framework. Professional noticing is the result of professional knowledge gained only with experience and knowledge of the domain in question (Mason, 2011). Because of this distinction, identifying what the professional attends to is of great significance, as it is more likely to indicate its value in practice. By focusing the professional, in this case the mathematics coach, on student thinking, on the student misconceptions and the representations that draw their attention, we can tentatively assume that these areas of focus have merit. Although there are a number of areas of focus that this framework could have centered on, the application and impact of learning trajectories is currently a relevant topic. Since the introduction of common standards across most of the United States (CCSSM, 2010), the content of the grade level mathematics can therefore be explored in more depth. The learning trajectories that are emerging from research promise to guide this process. Using the professional noticing framework to identify what coaches notice and respond to in student work promises to offer information on how these trajectories are currently used in schools, in this case in the hands of likely the most knowledgeable of mathematics educators. In order to give coaches time to reflect and respond, examining student work in the interview setting gives a context for examining the professional noticing of the elementary mathematics coach

Research Purpose

The purpose of this study is to explore elementary mathematics specialist coaches' references to learning trajectories as they examine artifacts of student thinking in order to understand what elementary mathematics coaches notice in student work, the resources they reference in order to make sense of the work and how they reference them, and how this information is used in practice. For the purposes of this study, learning trajectories are defined in a manner consistent with Lobato and Walters' (2017) conception, sometimes called LT/P, which includes resources that describe in what order mathematical content should be sequenced. A distinction is made between an implicit and an explicit learning trajectory. An explicit learning trajectory refers to a source to which the coach gives the authority to sequence content. In order to be explicit, the source will be specifically mentioned by the coach. The implicit learning trajectory is one that shapes the coach's thinking, indicating some order of topics that is not specifically referenced. In order to elicit more information, implicit learning trajectory references will need to be uncovered. Finally, this study seeks to explore how mathematics coaches might use their knowledge of students' learning trajectories in their coaching practice.

Summary

Chapter 3 shares the design and procedures of the study, including the plan for exploring coaches' engagement with student work samples and the references to and use of learning trajectories that frame their thinking. Using the conceptual framework as a guide, the analysis of the data in Chapter 4 will revisit the phases of the professional noticing framework and report on the data within that frame. Chapter 5 will discuss the results of the data and share the implications for practice and make recommendations for further research.

Chapter Three

Introduction

This chapter describes the methodology used to investigate elementary mathematics coaches' engagement with student work. It begins with identifying the purpose of the study and the research questions. The chapter will include methodology; the rationale for using this methodology; and will present study methods, including descriptions of the participants, the setting of the study, the different data sources that were collected and analyzed, and data analysis. This chapter will also address threats to validity.

Purpose of the Study

The purpose of this study is to explore elementary mathematics specialist coaches' references to learning trajectories as they examine artifacts of student thinking in order to understand what elementary mathematics coaches notice in student work, the resources they reference in order to make sense of the work and how they reference them, and how this information is used in practice. This is a critical awareness for coaches in the regular course of their work with teachers and students; however, how coaches do this and the knowledge base they access as they examine student work is not well understood. Broadly classified by Lobato and Walters (2017), the term learning trajectory (LT; progression) reflects a whole taxonomy of approaches to learning sequences in mathematics and science. The taxonomy includes many of the resources commonly available to teachers, and certainly to coaches, but also some sources that may only be available to some. One way to unearth how these reference sources are used by mathematics coaches, particularly those who are highly educated in this specialty, is to engage coaches in the act of examining student work and study their responses. The framework of professional noticing provides a means for understanding the possible mechanisms, including identifying what coaches notice, how they interpret it, and what they do in response. To that end, forming a picture of a coach's response is an essential understanding.

Research Questions

The purpose of this study is to explore mathematics coaches' references to learning trajectories as they examine artifacts of student thinking in order to understand what elementary mathematics coaches notice in student work, the resources they reference in order to make sense of the work and how they reference them, and how this information is used in practice.

1. What evidence of students' mathematical thinking do elementary mathematics coaches attend to while examining students' written artifacts?

2. What learning trajectories or other similar sequencing sources do elementary mathematics coaches reference in order to interpret students' prior, current, and future understandings, based on an examination of artifacts of student thinking?

3. How do elementary mathematics coaches use knowledge of learning trajectories or other similar sequencing sources, along with evidence gathered from artifacts of student thinking, to make instructional and coaching decisions?

Methodology

In deciding on the methodology for this investigation, the first decision was to select a general approach. A quantitative approach was immediately dismissed because not enough is known about the process of coaches' engagement with student work exemplars to assign any quantifiable units of measure. Moreover, the investigation focused on examining an under-studied sense-making process, which also made it unsuitable for a quantitative design. A mixed methods approach does not meet the study's objectives for the same reasons.

Rationale for methodology. A qualitative research design not only allows a clearer avenue to explore what is currently an open question, a qualitative research design is also consistent with the researcher's constructivist philosophy, as it also a form of constructing a reality (Saldaña, 2015), this time the reality of mathematics coaches in a particular context. The qualitative research design also meets the five features of qualitative research set out by Yin (2011). First, the topic of the study is meaningful to the professional lives of mathematics coaches, as well as to those who are debating whether to expend valuable resources on coaching positions in K-8 schools. Understanding the impact of a coaching practice in a school is of interest to both parties. EMCs have the potential to facilitate improved instruction, specifically as they facilitate teacher understanding of the progression of student learning. Secondly, this study is focused on the views and perspectives coaches have on student learning, which span a broader age range than those of a single classroom teacher. School coaches uniquely have a mathematics-centered view on student learning and sources for understanding it. Third, the study will focus on the work coaches do in schools so it is wholly based in the context in which the coaches work, including acknowledgment of the resources provided by their local situation and within the execution of their daily duties. Fourth, the understanding that children's learning progresses in somewhat predictable ways is not new. While there is some evidence of what coaches should know (AMTE, 2013; Burroughs et al.,2017; de Araujo, Webel, & Wray, 2017), it is unclear what knowledge and understandings they have about students' learning trajectories globally and how this information might be used to support teachers. What is new in this study is the exploration of what coaches know about the progression of student learning, how they reference that information, and how they use the understandings they have of student learning to guide their coaching practice. In short, how do they form working hypothetical learning trajectories (HLTs) for students and use this to inform their practice with teachers? And lastly, this study will include multiple sources of evidence, including written work, an assessment activity, and an interview.

A qualitative study design allows a "customized, inductive, emergent process that permits more of a researcher's personal signature in study design, implementation, and write-up" (Saldaña, 2015). Having served as a mathematics coach in three different variations of the job, the researcher's perspective on what a coach does is broad and encompasses many tasks, so it uniquely informed the interview design, anticipated categories of responses, and allowed more probative questioning based on background knowledge of the systems and expectations in local contexts. In this respect an emic approach to the study was unavoidable. Of course this is also a potential threat to validity, and the issue is addressed throughout. Because the purpose and context of the study closely align with Yin's (2011) features of qualitative research studies, this served as confirmation that the research purpose is well-suited to a qualitative study.

Researcher identity statement and approach to research. The researcher's epistemological stance and identity also play critical roles in the selection of a methodological approach. I am a White, cis-gender, married, mother, daughter, sister, female, early researcher and mathematics education leader, as well as a former mathematics coach and teacher. I prefer the pronouns she and her. I am mostly trilingual and have co-built a family unit that blends three cultures, and as such, I do not fully fit into any one of these cultures. The experience of being a language learner in many different contexts makes me more sensitive to the acknowledgment of the challenges of students who are new to American culture and English. As a White female I am aware that this category overpopulates the field in which I work and the field in which this study will be conducted, education.

As an educator I subscribe to a constructivist philosophy (Bruner, 1996), that each individual constructs knowledge through their actions and interactions to varying degrees with the physical world and within the social world. In this regard, I do not hold a fully sociocultural view of learning (Cobb, 1994; Vygotsky, 1980). While we all certainly learn from others, I hold that each individual can learn ideas formulated apart from her interaction with others. In my philosophical view, each individual creates an ontological version of reality, which shares properties with the reality composed by others. This differs from the radical constructivist view (von Glasersfeld, 1995), who posited as many different constructs of the world as there are individuals. Although I find his work persuasive, there is too much that is observable, tangible, and agreed upon by human beings for each person to be generating a fully different version of reality. This view is consistent with Sfard's (1991) view of the simultaneous existence of the personal "conception" and the "reified concept" that are both associated with the same mathematical idea. This vision of some common ontological reality between individuals implies that the analysis of the data collected within this study is the result of the construction of both the mathematics coach being interviewed and myself as the researcher. It was my responsibility as the researcher to work to align my construction of their experiences as closely to their recounting as possible, and not over-impose my version of reality into the analysis. Mindful of my identity as someone who has performed the job of mathematics coach in the past and has more recently participated in the education of other elementary mathematics specialists (EMSs), I actively took care not to impose my assumptions about the data onto participants' statements. On the other hand, the same experience did inform the follow up questions that I asked during an interview. The impact of various relevant aspects of my identity were scrutinized within the context of bias and validity checks at regular intervals during the data collection and analysis (Maxwell, 2005).

Methods

The phenomenon addressed is not teachers or coaches, nor is it coaches doing assessment. The phenomenon is more exactly the *process* that coaches follow when they examine student work, interpret that work by placing it along some form of continuum of learning, make instructional recommendations, and use that information to engage their teachers in learning more about students. Specifically, the phenomenon of interpreting student work against the backdrop of some form of professional knowledge of students' learning paths is part of the lived experience of all educators, including coaches. However, the unit of analysis in this study is not the coach and the coach's lived experience. The unit of analysis, rather, is the link between student thinking and a learning trajectory through the eyes of an elementary mathematics coach, and not necessarily on the lived experience as recounted by the coaches. The point of view that this study seeks to illuminate comes from outside the coach's experience.

Study design. One of the principal tools in the mathematics education literature for close examinations of teachers' professional noticing is the video club format (Gamoran & van Es, 2008; Luna & Sherin, 2017; van Es & Sherin, 2008; Walkoe, 2015). Teachers take turns sharing lessons they teach on video, and the group discusses what they see in the classroom. It is a revelatory tool that leads to teachers engaging more productively with student thinking. However, there are some drawbacks to this approach. The primary drawback is securing the time teachers need to devote over several months, but also problematic is the response of the teacher whose video is being examined. They tend to offer cursory, narrative descriptions of the actions in the classroom, not focused on work of students themselves (van Es & Sherin, 2008). For this study, the problematic factor in deciding on a study design for examining student work is the fact that coaches do not have students of their own. For these reasons the video club format was considered and rejected.

Another setting considered for examining the coaches' professional noticing was the live coaching session. This was rejected for two reasons. The first reason is the focus of the coaching session – the work of coaches is teacher-centric. The coach's role is to bring out the thinking and beliefs of teachers and hold them up for the teacher's own professional growth (McGatha et al., 2017). This puts the coach in the secondary focus position, which will not help achieve the purpose of this research study, which is to study a coaches' engagement with student thinking. The second reason for rejecting the live meeting is the hectic and rushed aspect of in-the-moment examinations. In a similar case with teachers, Liu (2014) found that in-the-moment noticing was profoundly impacted by teachers' own beliefs, knowledge, and goals, rather than by the student thinking taking place in the classroom. Of course this was a study of teachers, but it is possible that a coach's in-the-moment reactions during a coaching event with teachers may also be more oriented to the teachers' experience. For these reasons, observation of coaches working with teachers in situ and in practice was determined to be of less value than a session focused entirely on the coaches' beliefs, noticings, and understandings. The coaches' reactions to written artifacts of student thinking are the unit of analysis.

Phases of the study. There were four phases of data collection. First, the coach participants completed demographic and practice-related survey items through an online

Google survey at their convenience. As part of this phase, participants also engaged with the mathematical task at the center of the project (see Appendix A), sharing multiple strategies to solve it, followed by a written account of a mathematical goal of the task, and their predictions of what students might do to solve the same problem. The second phase of the data collection was a 30-min voice recorded one-on-one interviews with the researcher. In two cases the interviews took place at the participants' school and in one case at a local library. The third phase took place during the same location visit and included a video recorded student work sorting session. The final location was in an online room through Blackboard Collaborate (2019), where the participants member checked the transcripts of their interviews and answered follow up questions that were clarifications following their initial interviews.

Participants

The focus of the study is on practicing elementary mathematics coaches because coaches have a view on a school context that differs from that of a typical teacher, although it is important to note that this is not always the case with teachers (Suh & Seshaiyer, 2015). Coaches have awareness of the vertical progression of mathematics topics by virtue of their training and because they work with multiple grades on a daily basis (Gibbons, 2017). The job responsibilities for some coaches may include substantial work with students, in a role more akin to that of an interventionist. For this reason, participants indicated in the initial survey what percentage of time in a typical week is spent working with teachers and working with students. Coaches should indicate that they spend 85% of their time working directly with teachers to improve practice, even if that includes being in a teacher's classroom to co-teach or model.

Recruitment. Recruitment was done via email. Mathematics leaders sent an email to local coaches and invited them to participate in an intake survey. To ensure that participants met the eligibility criteria, the recruitment survey verified that participants met the eligibility as a Virginia Mathematics Specialist coach, as well as serving as a full time mathematics coach. This is another opportunity for a validity check. As a former mathematics coach, it would be easy to select individuals whose coaching practice looks like the one I had. The initial recruitment survey gathered some basic demographic data so that selection could be conscious and mindful of creating diverse representation in the sample of participants. Recruitment was conducted in four school divisions, and ten individuals expressed an interest through the recruitment survey. Only three met the recruitment criteria and they were all coaches in the same county.

The three coaches selected for this study represent a specifically focused and theoretically selected sample of professional educators (Bloomberg & Volpe, 2012). They are all elementary mathematics specialist coaches (EMC), and were recruited based on the level of their experience and formal education in the area of elementary mathematics. Each of the coaches has worked as an elementary mathematics coach for at least 6 years and each has completed the coursework in their state that qualifies them for the elementary mathematics specialist endorsement to their elementary teaching license. They practice at different K-6 elementary schools that are part of the same large Mid-Atlantic suburban county. Due to the relatively small population of elementary mathematics coaches, this report will not reveal non-essential demographic information about the coaches or their schools so that their privacy is not jeopardized. Some information, such as years of experience or special licensure, has been generalized also in order to conceal participants' identities. All names of participants and schools are pseudonyms.

Rationale for the selection of participants.-Each mathematics coach participant holds a K-8 Virginia Mathematics Specialist endorsement to an existing teaching license. Finally, each participant has completed their preparation as a mathematics specialist at a Virginia college or university that follows the program outlined by the Virginia Mathematics and Science Coalition (VMSC, 2016). These programs require at least 24 graduate credit hours in coursework focused on mathematics content and leadership. (See Appendix B for the full endorsement requirements). This program also meets the standards laid out by the Association of Mathematics Teacher Educators standards for elementary mathematics specialists (AMTE, 2013). A detailed description of the mathematics coach is taken from the EMS framework developed by McGatha & Rigelman (2017), and is described in detail in Chapter 2.

Currently the list of universities offering this program includes the following institutions: George Mason University, Virginia Commonwealth University, Virginia Polytechnic Institute and State University, Norfolk State University, James Madison University, The College of William & Mary, Old Dominion University and Longwood University (VMSC, n.d.). For more information on programs outside of Virginia, see Rigelman & Wray (2017).

Data Sources

The data sources for this research project were determined by the purpose of the project and the conceptual framework that defines the purpose. The purpose was to explore the mathematics coaches' recognition, understanding, and use of learning trajectories in their coaching practice. The conceptual framework proposes that the professional noticing construct offers recognizable behaviors that can be observed, so the three components of professional noticing create three categories that govern the collection of data (van Es & Sherin, 2008). These categories include: what individuals attend to, how they interpret it, and how they respond. The learning trajectory component of the conceptual framework offers a lens on the participants' references to the mathematical sequence of student learning. Learning trajectories have three parts (Sarama & Clements, 2009): the mathematical goal, the developmental path to that goal, and the tasks that can help students move toward the goal. Because these are very different goals, the sources of data are also varied.

Appendix C includes a detailed table that maps the research questions to the data sources and to the specific questions on the survey, in the interview protocol, during the student work sorting, and at the conclusion of the student work sort. In summary, the demographic survey is designed to verify that the coach's work situation matches the eligibility requirements and informs the researcher's understanding of the coach's context. The completion of the task and reflection on student understanding of the task is important for orienting the coach to the task before sorting the work and for accessing information about the coach's existing hypothetical learning trajectory around this task.

The face-to-face interview questions replicated some of the questions asked during the demographic survey in order to solicit additional information and to triangulate data already provided. It also asked about the coach's practice, the coach's impressions of the task, and any follow up questions from the demographic survey. The student work sorting task had opening requests for the coach to examine the work and to think out loud as they look at the samples. After all samples had been addressed, they are asked to sort the samples using "whatever criteria" they would like. Finally, the questions after the sort asked coaches to reflect on the task and about their thoughts on using work samples like this during professional development with their teacher-clients. A copy of the Institutional Review Board Exempt status statement is provided in Appendix D. Additional survey details are also shared below.

Questionnaires. The first questionnaire participants completed was for recruitment. The primary role of the initial contact was to determine the initial eligibility of the participant. The second questionnaire collected more detailed information in four categories:

1. Personal Demographic – This is data that relates to the coaches' identity, as a mathematics learner, as a former teacher, and as a coach. This also includes information about licensure, as one of the requirements for eligibility is holding a VA Mathematics Specialist endorsement.
2. School Demographic Information – This includes the coach's current school, the school population, the number of teachers in the coaching practice, and how the coach's position is funded. This information is

collected in order to understand whether a county, state, or federal agency may be having an impact on the coach's mathematical learning goals.

3. The Coach as Learner – This section gathered information about the coach's identity as a *learner* of mathematics. This includes questions about their past experiences in mathematics classes, the mathematics courses they have taken (other than the VA MS courses), and their current disposition toward mathematics. This data is gathered in order to determine the quantity and nature of general mathematics knowledge (Ball, 1993) they may have.
4. Job experiences as a coach – Gibbons (2017) outlined a list of potential best practices for coaches. A portion of those potentially related to focusing on student learning or on long-range academic goals for students. These categories are directly related to the examination of the coach's references and uses of learning trajectories in their practice. The questions in this part of the questionnaire asked for a percentage of time dedicated to a variety of coaching tasks (the list should add up to 100%). Questions also asked about the coach's favorite professional resources.

Questionnaire responses were collected using a Google Form managed only by the researcher and stored in an Excel spreadsheet protected by a password.

Mathematics task. The choice of mathematical task was pivotal in this study. The task is a high cognitive demand task – there are many ways students can approach it, and it is accessible to many grade levels (see Appendix A). As presented on paper, the task does not rely solely on procedures (Stein et al., 1996). The content of the task relates to rational number understandings, and in terms of the Common Core progressions, it is accessible to elementary students as part of Numbers and Operations and Fractions, but it is also meaningful for middle school students under the Ratio and Proportional Relationships domain. The problem is also very well-known and was proven to be a fruitful source of data in the pilot study (Morrow-Leong, 2014), as well as in another study (Suh, Birkhead, Galanti, & Seshaiyer, 2019; Suh, Birkhead, Farmer, Galanti, Nietert, Bauer & Seshiayer, 2019). The rational number domain was selected as the source for the mathematical task because it is an area with a varied and sometimes contradictory body of research (Behr, Cramer, Harel, Lesh, & Post, 2010; Confrey & Smith, 1995; Empson & Levi, 2011; Lamon, 2012; Petit et al.,2010; Steffe & Olive, 2009). Despite the depth and breadth of knowledge, it remains unclear what an optimal learning trajectory for this domain might be.

The mathematics task has two roles in this study. The coach completed the task and anticipated what students might do with the task. It is important for the coach to engage with the mathematics first and then anticipate what students might do when they solve the task (Stein & Smith, 2011). Moreover, the coach solved the problem first in their preferred way and then showed two additional representations. This step took place before the interview. During the interview, the coach examined examples of student work on the exact same task. The coach's solutions to the task were collected, as well as their predictions of what students would do with the task. Paper copies are stored in a locked file cabinet in a home office and in the advisor's office.

Voice recording of the interview. The 30-min interview was conducted face-to-face using VoiceMemo on an iPad mounted on an overhead stand. The interview took place at a location convenient to the participant. The questions asked during the interview in some cases duplicated the questions asked during the demographic survey. Other questions asked about the coaches' practice with teachers, including school-wide initiatives and coaching activities. The detailed interview protocol can be found in Appendix E. The initial interview was 30 min long and led immediately into the work sorting session. This interviewed was transcribed.

Video recording of the student work sorting activity. The student work sorting activity was 45 min as the coach actively interacted with 12 student work samples. At the table were colored markers, white paper, sticky notes, and the video equipment described earlier. This interview segment was recording using the standard video recording software installed on the iPad. Using a fly-on-the-wall strategy the interviewer encouraged the coach to think out loud as they examined the student work and made notes. The list of probing questions is noted in the interview protocol in Appendix E. Because the first step of professional noticing is noting what individuals *attend to* when they look at artifacts of student thinking. What do they see? What do they choose to talk about? What do they consider actionable? The purpose of the interview is to listen closely to how the coach attended to student thinking and interpreted it. Most importantly, what learning trajectories did coaches use to anchor their descriptions of student thinking? The video recorded interviews featured only the participants' hands and the papers to which they were referring. The discussion was transcribed. Informed consent was gathered for each step of the study, and more specifically for the video and audio recordings. See Appendix F for IRB audio and video approval documentation.

Student work samples. The core of the sorting portion of the interview is a set of 12 student work samples. Ten samples are student work samples from another study (Suh, Birkhead, Galanti, et al., 2019) taken in their entirety or in part. Samples were abbreviated, traced, recolored, and cleaned for legibility but remained true to the students' depictions. Two additional samples were inspired by student work from the pilot study (Morrow-Leong, 2013). These solutions were pared down from a four-field trip location problem to a two-field trip location problem, while still maintaining the integrity of the students' solution strategies. The samples were chosen from a much larger set in order to target common student solution strategies and errors on the task. There is very little redundancy but the collection also represents a set smaller than most classrooms would generate. The 12 samples were randomly lettered for easy reference and were printed one sample per page. Each participant was provided their own unique set of samples so that they could write on them if desired. Participants were advised that the students were assumed to all be in one class, that the thinking on the work samples was original, and that although a light box had been used to retrace the work for greater clarity, it was still genuine student thinking. See Appendix G for details about the samples, including a mathematical summary of the salient mathematical features.

Results of the sort. After the participant examined the student work samples, the interviewer asked them to sort the set using any reasoning they wished, talking through their reasoning as they went. Once the sorting was completed, the participants' sorting order was preserved and recorded, including their written comments on the student work, when it was applicable. Finally, the interviewer asked what was an important next task for the entire class of sample students, based on these work samples. The coach's suggestion and their reasoning for it were noted. The work samples were collected, the teachers' notes on the samples coded, and the discussion pairings that the teacher shared were recorded. If any other notes or solutions were presented during the interview and sort, they were gathered as well.

Procedures

The data collection took place in four phases following recruitment. The recruitment letter can be found in Appendix H. The first phase was a survey and exploration of a mathematics task. The second and third phase took place in person and consisted of a semi-structured individual interview and a sorting activity with various student work samples selected for their variety of strategies, placements along a rational number learning trajectory, and degree of accuracy. The final phase was conducted online primarily for member checking transcripts and other information. Below is additional detail on the different phases.

Phase 1. The demographic survey gathered information about the individuals' experiences with mathematics and the school context. Appendices F and G show a complete listing of the questions to be asked, but they can be grouped into four categories:

1) Demographic information about licensure, preparation, years teaching, years coaching, and ethnicity (U.S. Census Bureau, n.d.), experiences in

other mathematics courses and disposition towards mathematics. Contact information.

2) Identity as a mathematics teacher and learner
3) School context, location, and time spent weekly in activities as mathematics coach
4) Directions for the mathematics task to be completed

Demographic information is collected in order to assure that the mathematics coaches meet the study parameters and to gather information about their experience teaching and coaching mathematics, including the number of years, grade levels, as well as any additional courses they have taken in mathematics. The second category gathered information about the coach's school. This information included the number of teachers and the range of grades for which they coach, the number of students, and the name of the school. The name of school and principal was collected so that I was able to later look up the school data made available by the district and describe the school population. Finally, there were three questions that addressed the coach's experience with mathematics and as a coach. These questions gave insight into the coach's relationship to mathematics.

The third set of questions asked about the coach's average allocation of time across a normal week. The purpose was to anticipate the kinds of activities the coach engages teachers in so that I could adjust the interview questions to target those activities. It also reassured me that the coach spends at least 85% of their time working with teachers. The full set of interview questions is in Appendix E.

Finally, participants were given the task from which the student work samples have been drawn. There were directions to complete the task using at least three different solutions (see Appendix I). It also asked the coach to anticipate the responses students might give and identify their reasons for each prediction, including incorrect solutions (Stein & Smith, 2011). Additional questions asked the coach to give information about possible student responses to it and a reasonable mathematical goal for using the task in class. Coaches completed this task so that they were familiar with the mathematics behind the student work they would be examining during the interview. It was also another way to determine what resources coaches referenced when predicting and describing anticipated student responses.

During the first phase of the research procedure, participants received a link to additional survey questions, a consent form, instructions for the mathematics task, and finally an invitation to make an interview appointment. Respondents who were not selected received a gracious thank you note, appreciative of their willingness to participate, yet declining it at that time. Directions on the mathematics task asked participants to return their solutions by scanning their work and sending it to me via email. This was the first opportunity to analyze data from participants. Research memos were written following the first reading of the participants' presentation of the solution to the mathematics task and prediction of student responses.

Phase 2. The second phase of data collection consisted of conducting individual interviews. Consistent with qualitative research practice, interviews of the selected teachers took place at a convenient, comfortable, and quiet place for the coach in order to capture a natural setting for the phenomenon (Kvale & Brinkmann, 2009). The live interview had the primary purpose of asking questions about the coach's practice with teachers and the resources that inform their practice. An iPad mounted on a 4-ft stand sat on the table with the arm extended over the work space in front of the participant. A microphone hung from the extension arm. The first part of the interview began with questions about the coach's practice and was recorded using the VoiceMemo application and lasted from 20-30 min.

Phase 3. The second part of the interview used the standard video recording application installed on the iPad and lasted 45 min. The goal in this phase was to conduct a student work sort, probe the coaches to understand the details that inform their sorting of the tasks in more depth, and discover his or her suggestion of what this group of students might do in the lesson that immediately follows the task (see Appendix E for the questioning protocol).

The visual frame for video was measured ahead of time and included a space that captured three standard 8 ½ x 11 papers wide and two papers tall and only included the participants' hands, forearms, and the papers they were sorting. Their voices were also recorded so that participants' references to work samples could later be identified: "*this* student," " . . . put in a group with *that* one," etc. Figure 2 is an example of the overhead screen view.

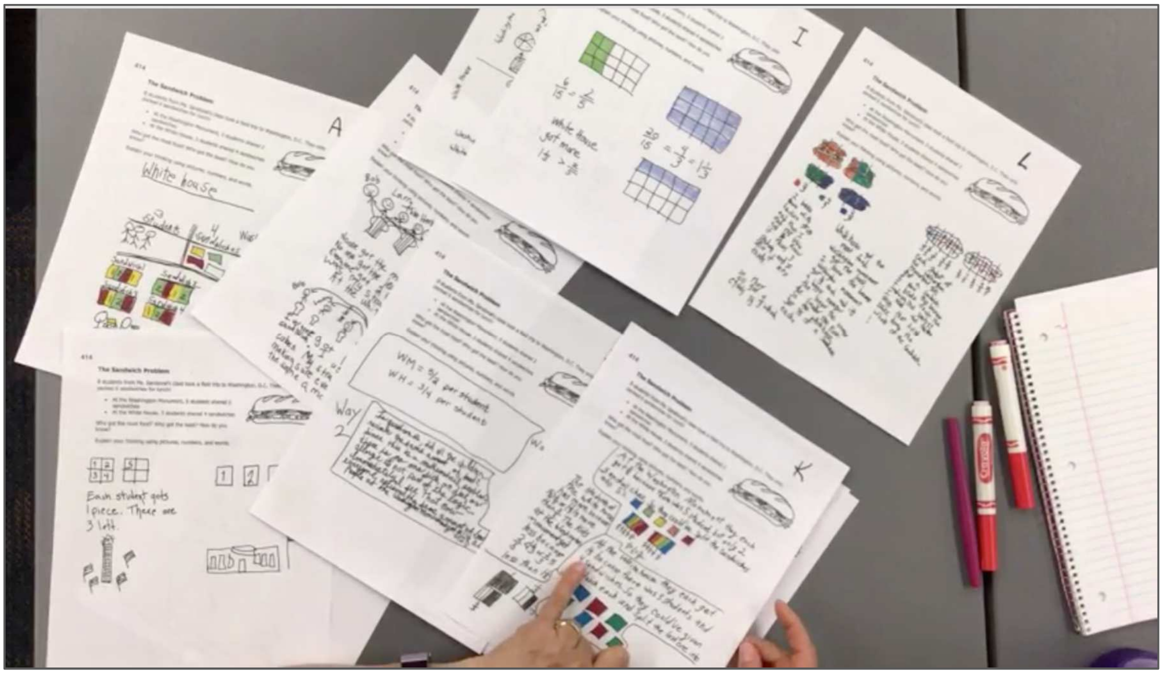


Figure 2. Overhead view of the process of sorting student work.

During the student work sort, the researcher handed the teacher a set of 12 work samples from students the teacher did not know and asked the teacher to take a quick glance at the work samples and share their initial observations about them. Consistent with the "deliberate naïveté" interviewer stance (Kvale & Brinkmann, 2009), the initial directions were deliberately left open so that the researcher's comments did not interfere with the coach's initial thoughts. As the coaches' statements and observations about the student work began to narrow, the interviewer prompted the teacher to describe details and supply evidence of what they were seeing (Appendix E). The questions provided in the semi-structured interview were prepared in anticipation of needing probing questions, but not all of them were asked.

After the initial examination, the participant was asked to sort the student work according to any rules that they thought were important.

- Could you **sort** these papers? You can use whatever criteria you would like, but please share your thinking with me as you are sorting. I'd like to know your reasons.
- Could you **explain** how you are sorting the papers? What are you looking for? How are you making your decisions?

For all participants, their choice was to form groups for instruction the next day. As they described their groupings, they were asked to use a sticky note to record their reasoning for each of their groupings. At times, the researcher asked questions to further understand the reasoning behind their pairings and groupings.

At the conclusion of each interview, the video and audio files were sent to a third-party transcribing company and were completed within 24 hr. In the meantime, the researcher completed a researcher memo following each interview, reflecting on the presence of all of the predicted or current code categories and noting any in vivo codes to use during the analysis. Once the transcripts were returned, they were reviewed for accuracy. The video transcripts were also reviewed in order to make references to student work samples clear. For example, if the coach said, "This student should be paired with this student." They were referring to a paper they held in their hand. All such references were clarified, using parentheses to note which student work sample was indicated. The format of the transcription company product streamlined this process, showing video along with a moving cursor indicating that place in the transcript. Once the references in the transcript were resolved, the transcripts were downloaded and saved to a word processing program.

Phase 4. The final phase of the data collection process included member checking the gathered data with participants for accuracy and fidelity, as well as asking some clarification questions unique to each participant. Triangulation of data (Creswell, 2008) was achieved by comparing the coach's anticipated student responses, their statements during the sorting portion of the interview, and in the final phase: member checking (Creswell, 2008). The meeting was scheduled with participants and took place using Blackboard Collaborate Ultra (2019) and Google docs (Google, n.d.) for sharing information. This too was recorded for later reference but was not transcribed.

Analysis

Data collection in this study was inherently sequential. The mathematics coach completed a survey and did the student mathematics task prior to the interview and sorting activity with the researcher. Because of this dual stage process, there was a space of time to read the initial data and establish in vivo codes for analysis before collecting the second phase of data (Creswell, 2008). Because there were multiple participants, the schedule of interviews also contributed to data coming in at irregular intervals, allowing time and space for the researcher to code transcripts, write analytical memos, and consider choices for probative questions in subsequent interviews (Campbell, 2013). However, the conceptual framework offered a way to frame the investigation and offered a set of priori codes to begin the first round of coding (Miller, 1992 as cited in Bloomberg & Volpe, 2012). Saldaña (2014) suggests *in vivo* coding on the first pass through the data, but it is also reasonable that some structure be provided through a small collection of *a priori* codes that are derived from the conceptual framework and the task itself (Glaser, 1992). The conceptual framework of coaches referencing learning trajectories to inform practice includes two functional frames: professional noticing (van Es & Sherin, 2008) and the working definition of a learning trajectory (Sarama & Clements, 2009).

Professional noticing includes three sometimes simultaneous actions. First, the observer *attends* to some aspect of practice. Second the observer *interprets* what they see, and finally they *act in response* to what they have seen. This yielded three initial a priori codes for analyzing data: attending to, interpreting, and acting in response to. The use of gerunds as a morphological structure for the a priori code names reminds us that noticing and responding to student work is a dynamic process and that the codes should also be as dynamic (Nathaniel & Andrews, 2007). These codes were eventually rejected as not useful in shedding light on the process (Bloomberg & Volpe, 2012), however; they were intimately tied to the conceptual framework, so the initial choice was logical. In the end, these a priori codes instead became organizing principles for reporting the results, which will become evident in Chapter 4.

The learning trajectory construct is built from three separate and distinct components. A learning trajectory (Sarama & Clements, 2009) starts with a *mathematical goal* to set the learning path, a *developmental sequence* that describes the stages of learning along that path, and finally it includes the *tasks* that students work through in order to move along the learning trajectory. This construct yields three a priori codes that focus on the specifics of the mathematics and student work: *mathematical goal*, *developmental* *sequence*, and *tasks*. If the professional noticing codes address the *how* of coaches' interactions, learning trajectory codes reflect with *what* the coaches are interacting. These a priori codes also proved to be too broad and all-encompassing to be codes or even themes. Instead they too were incorporated into the organization of the analysis.

Starting to code. These two sets of broad themes became two nodes in the thematic network analysis of the data (Attride-Stirling, 2001). The thematic network analysis tool is a six step process for analyzing the data. The first step in the process is deciding on the level of text in which to divide the material. There were many different types of data collected for the study, so each needed to be treated differently. For example, coding the interviews was facilitated by the transcripts. The transcripts provided by the transcription company presented the text in coherent paragraphs. Other data sources included examples of the participants' work done by hand which were captured with screen grabs and saved directly into the coding structure of Dedoose (2019). On the other hand, some of the data was much more formulaic and could be coded by matching the codes directly to the questions. Creating workable units of analysis was the first step in coding a thematic network analysis. The second step was organizing the data in order to code the text segments.

Since the demographic data came in early, the next set of codes created allowed that data to be coded by question. This included basic demographic data as well as the school information. The next set of data to be entered was the participants' work on the task, including their solutions, the mathematical goals of the task, the prerequisite skills, future skills, and the set of anticipated student strategies. During these initial coding sessions in vivo codes began to emerge and were created as needed and in response to the data. For example codes for mathematical strategies and representations were added as they were encountered in the participants' work. The audio taped interviews were coded by question, but as the interviews began, new coding strategies were needed, and in vivo codes were allowed to emerge from the responses. For example, one of the questions asked participants which resources they always bring to their planning meetings. The entire response to this question was coded as Resources, but it also led to a new list of codes identifying individual resources, added as participants mentioned them.

Coding the live interviews. Based on data from the pilot study (Morrow-Leong, 2013). I anticipated possible open codes that focused on the tools and resources commonly used in schools, the mention of standards or learning progressions, as well as language used to describe student thinking. Although I hesitated to color the field by imagining any other codes before data collection even began, some of the mathematical codes were predictable based on the task in question

The sorting portion of the interview offered the richest view of the mathematical thinking of the coach participants and was the source of much of the primary data, leaving the additional data sources as points of triangulation. The first pass on this data included cleaning the transcript, identifying acronyms and clarifying participant statements. Transcripts of the sorting interviews were filled with many grammatical pointers that referred to the work samples participants were holding or were moving around. The second pass on transcripts needed resolving these grammatical pointers such as "this one," "that one," and other such visual cues. This was simply done with parenthetical references to the appropriate sample(s), and it became a critically important source of data. Without these resolved pointers, the participants' statements were nearly meaningless. For example, this statement was nearly meaningless until it was modified with the appropriate pointers resolved:

> I don't think I'd do either of those (Samples I, H, F). This one's not really right (Sample F). Probably this one (Sample L, K). Not necessarily that it's least sophisticated but I feel like . . . Or this one (Sample B). This one actually shows it being . . . It's dividing. (Isabella)

The primary source of codes was the collection of 12 work samples and as such were established as *a priori* codes. Each utterance was coded for the work samples to which it referred. This coding was done at the larger grain size of an utterance rather than at the sentence or phrase grain size. This decision later allowed for code co-occurrences of work samples to emerge from the data.

The sorting portion of the interview offered many additional opportunities for codes taken directly from participants' words. Coding for the mathematical strategies students in the work samples used added to the codes established while reviewing participants work. Another area that emerged was the coaches' references to problem solving or communication or other mathematical behaviors and codes for mathematical practices were added and later collapsed into a theme for practices and processes.

While coding the coaches' engagement with student work samples, it became apparent that participants were at times making instructional decisions, pairing or grouping students for future discussion or activity. Because this seemed to be a universal practice across coaches, the sorting transcripts were revisited and coded again with references to grouping or pairing of students. Another possible area of interest were the coaches' references to the act of looking at students' work, either as an individual activity or as team-based activities with teachers at their schools. These were coded for future exploration.

The third pass at coding revisited previous interviews and documentation sources in order to apply the newly devised codes referring to groupings and pairings and looking at student work. Additionally the participant work was revisited in order to apply any of the new references to mathematical strategies or to the process or practice standards that may have been miss on the first passes. After coding the interviews, I printed a codebook list and began grouping similar codes into themes. From these themes came several clusters.

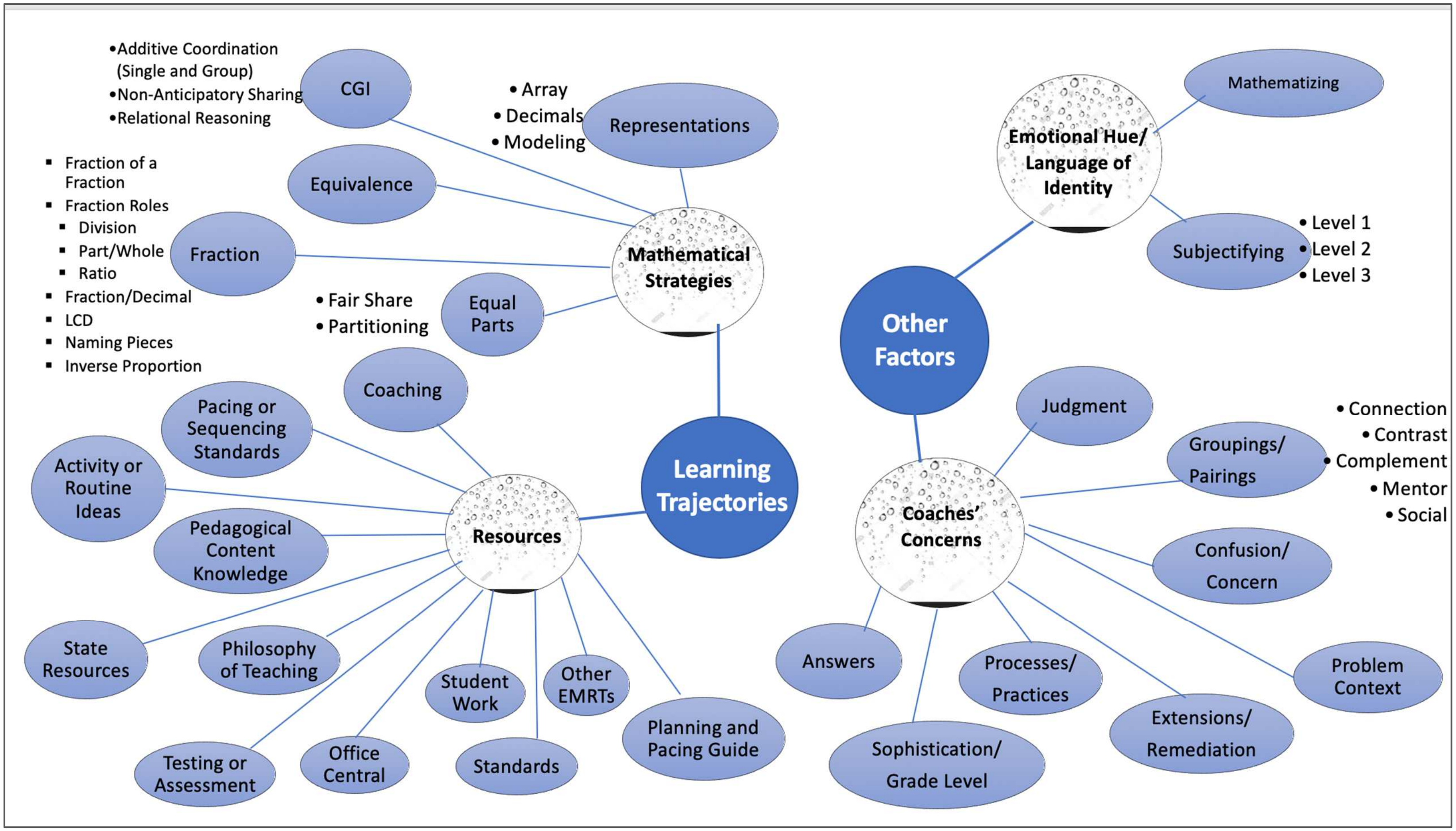


Figure 3. Thematic network analysis.

Clustering and constructing networks. In the third step of the thematic network analysis approach themes were organized into a hierarchy that generates broader themes. Figure 3 shows the basic themes in the bubbles of the network, which then are clustered into the basic or global themes of mathematical strategies, sorting task, emotional hue/language of identity, and resources for planning. Learning trajectories is the broader theme that linked together the Resources and Mathematical Strategies nodes, indicating that these two categories contribute to our broader understanding of the impact of learning trajectories in analyzing a task like this one.

The remaining two broader nodes connect only in that they are not strictly based on mathematics. Recognizing that Processes/Practices, Problem Context, and Mathematizing are indeed considered mathematics, this seems contradictory. To clarify, the codes in this branch of the network did not arise organically out of the student work samples. Instead they were researcher-created codes that reflect ideas the surfaced in the data, not expressed by the coaches or the students through their work samples. For example, “Answers” is a code created by the researcher/analyst to highlight the coachers’ reactions to incorrect or incomplete answers offered by the students. In the end the finding seemed incomplete and was abandoned, but the code remains as a curious possible path for future exploration.

Analyzing the Work Samples

Although the work samples were coded during the analysis, they were not included in the thematic network because the work samples appeared to be more of a resource than an actual theme in the data. That does not mean that they did not offer interesting and valuable insight. The code co-occurrence analysis tool in the qualitative software tool yielded even more interesting patterns by providing a visual image of the relationships between the work sample codes. This image shed light on how the coaches made sense of the sorting task and in the end, what they determined to be a grouping and pairing for instruction task. The chart that was generated provided information that indicated what samples were often grouped together.

This data was input into Kumu, software that generates a graphical representation of relationships between pieces of data (Mohr & Mohr, 2019). The software generated a weighted network analysis map of the relationships between all 12 work samples. The global map is presented in Figure 4, but individual work sample maps are later displayed as referenced in chapter four in order to highlight important connections. Both of these tools combined offered a graphical conceptualization of the relationships between work samples that emerged during the interviews.

In the weighted network analysis map in Figure 4, each of the circles is a node that represents one of the labeled work samples (see Appendix G to see details of the work samples and corresponding letters). The arcs between each node are connections between each of the work samples. A connection was made when a participant deliberately made an instructional grouping or pairing of the work samples from each of those student work samples. The weight of each line and the depth of the color indicated the frequency of the connection between each work sample, across all three coach participants.

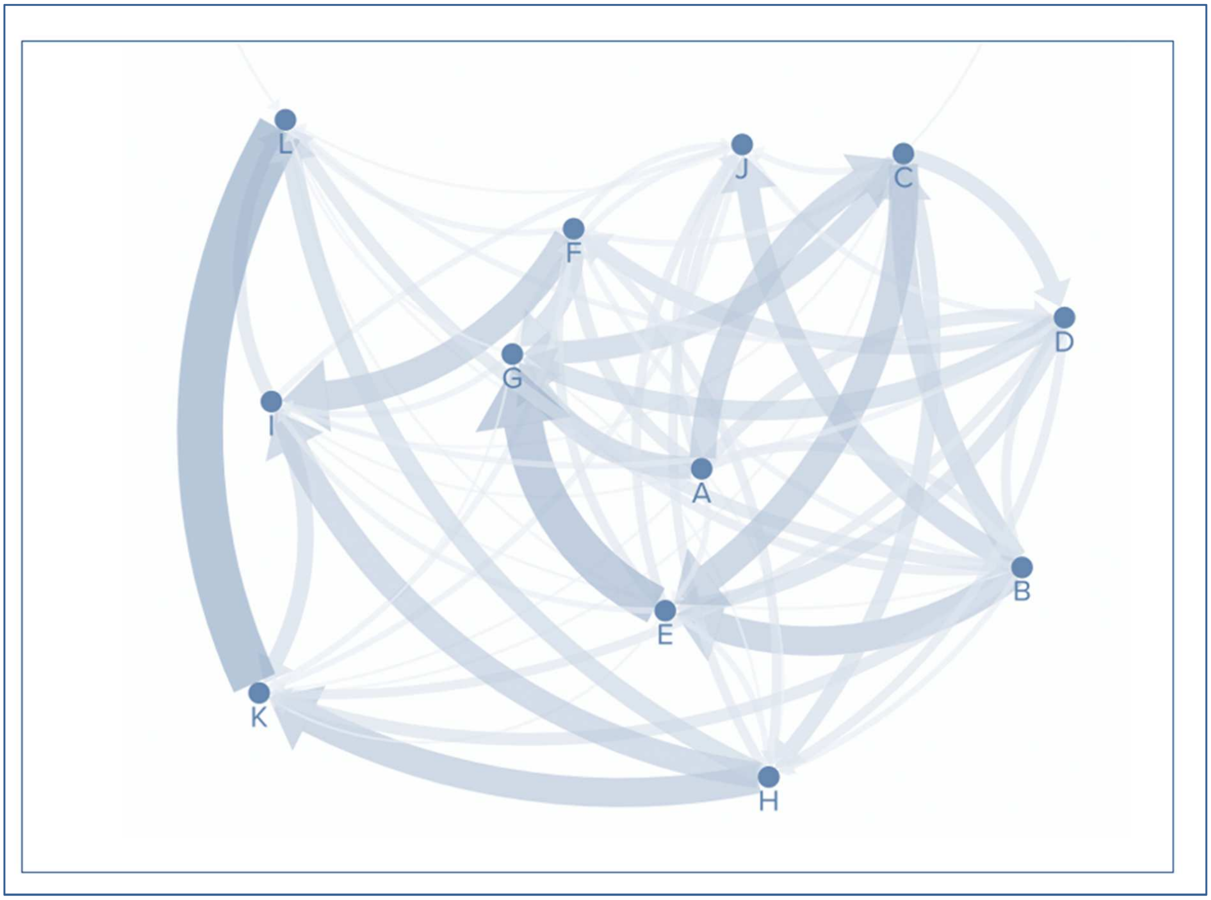


Figure 4. Weighted network analysis of co-occurrence of work sample codes.

Table 4 below shows an example of how each utterance mentioning a work sample was coded for sample co-occurrences.

Table 4

Network Connections Made between Nodes

Utterance	Connections and nodes
Who do I want to pair these guys (Samples D, B) up with?	Sample D and Sample B
. . .I had a hunch as to what this person (Student I) was thinking. I might group this person (Sample F). Because he's got great modeling but just isn't sure what to call this piece, and I feel like this (Sample I) is a representation of that (Sample F).	Sample F and Sample I
So I might just bring him (Sample F) in here with this person (Sample K) to try and get more explanation and have him listen and then say, "Well, I'm wondering that you had mentioned this is one-fifth.	Sample F to Sample K
What do you think about what this person (Sample I) was explaining, or something. I might actually put it in here (Samples F, G, E, D).	Sample I to Sample F Sample I to Sample G Sample I to Sample E Sample I to Sample D Sample F and Sample G Sample F and Sample E Sample F and Sample D Sample G and Sample E Sample G and Sample D Sample E and Sample D

Note: In some cases the connection is directional, which is indicated by the preposition "to." In other cases, no direction is implied. These are shown using the conjunction "and."

The fourth pass. At a certain point, after the initial data analysis had been completed, an interesting phenomenon kept appearing in the participants' words. The words that the participants were saying appeared to be qualitatively different. A review of a short segment of one transcript sparked an early labeling of utterances as distinct from each other. In short, some utterances referred to the students' mathematical work, and other utterances referred to the student as the doer of mathematics. Determined to understand this phenomena, more reading about engagement with student thinking and student work helped frame the issue more clearly. The details about the new framework helped to distinguish between utterances that subjectify and those that mathematize are included in Chapter 4;however it is important to mention that this discovery lead to a fourth pass to look at all of the data related to student work. The four additional codes provided data in the texts on what the authors call emotional hue (Heyd-Metzuyanim & Sfard, 2012).

Checking validity.

As previously mentioned, the fact that I had previously been an elementary mathematics coach increased the risk of validity threats. After listening to the first interview, I realized that I had perhaps contributed too much to the conversation with the coach in order to show solidarity as a fellow coach. This realization allowed me to check that instinct and in subsequent interviews, my voice as the interviewer was far more neutral.

Summary

Gathering data for this research study occurred in four phases. The first phase was a demographic survey exploring the work of the elementary mathematics specialist coach, including the resources they employ in their practice. The second and third phases took place in a live interview format. Much of the second phase was used to triangulate data gathered during the initial demographic survey and data that was later gleaned from the sorting interview. It also provided a picture of the coaching practice and of the coaches' interactions with artifacts of student thinking. The final phase of the study was primarily designed as a member check to confirm that the researcher had appropriately recorded the participants' thoughts and actions, but it also served as a venue for clarifying some questions and probing a little deeper in others. Since data collection was completed over a span of a few weeks, analysis occurred in conjunction with the data collection in most cases. Chapter 4 will explore more details of the data analysis process.

Chapter Four

The findings of this study will be organized into four separate sections. The first section (meet the coaches) describes the three coach participants, their backgrounds, and descriptions of their current coaching practice, including professional resources on which they rely. The remaining three sections (coaches' resources, coaches attending to student thinking, coaches interpreting student thinking, and coaches acting on student thinking) will be organized according to the professional noticing framework, identifying the nature of coaches' interactions with written artifacts of student thinking and relating their actions to the components of the professional noticing framework. Broadly, the findings show that coaches engage in practices of observation that reflect the outcomes described in the professional noticing literature (Mason, 2011; Sherin et al., 2011; Sherin & van Es, 2002, 2008).

Another focus of the study is the coaches' awareness and use of learning trajectories in practice, therefore the report will explore the coaches' references to any assumed sequence of learning. The discussion will draw upon a structure for systematizing the understanding and unpacking of learning trajectories in order to inform instruction, both for coaches and for teachers (Confrey, 2012; Confrey et al., 2014). Additionally, Confrey's structure for unpacking and understanding learning trajectories will also frame the data as the discussion addresses evidence of the coaches' constructed and hypothetical learning trajectories as evidenced in their engagement with the mathematical task and with the students' work samples. But first, a detailed look at the coaches' description of their practice offers a basic understanding of their work as context to explore their learning trajectories.

Meet the Coaches

Participants in this study were all coaches in a K-6 school, and have worked as coaches for at least 6 years. In addition to an elementary state teaching license, each participant also has an additional endorsement as a mathematics specialist. Additional information on the coaches below offers a view on their experiences with mathematics as well as resources that they use in their practice. The data will help inform our understanding of the coaches' engagement with the mathematics in the task.

Rachel. Rachel taught in an upper elementary grade level for 6 years before she accepted a coaching position at Rhodes Elementary School. She has been coaching teachers in grades K-6 in elementary schools for 6 years now. Rachel completed her mathematics specialist degree program entirely within the state that offers the program. Currently she holds an elementary teaching license, with a K-8 mathematics specialist endorsement added on to that license. In high school, Rachel took mathematics courses up through Precalculus, as an undergraduate took two mathematics-for-educators courses, but otherwise described no other university mathematics training. Rachel, however, has had extensive experience with local professional development opportunities, both with her district and at a local university. Her disposition towards mathematics as a child was positive, and remains so into adulthood.

Like most mathematics coaches, Rachel's job includes many different tasks, but more typically she co-teaches with the classroom teachers to model and develop best practices. Through co-teaching she is able to support first-year teachers as well as other teachers given the "gift" of a coach by the principal. Her role as a coach in this school seems to be both as a "fixer," and as a change agent. For example she works weekly with students, using a mathematics "recovery "program, as well as pulling groups of kindergarten students into the hallway to work on specific skills. She has the advantage of working with instructional assistants whom she guides in a kindergarten "pullout" program two or three times a week. On the other hand, her school is in the process of thinking through its philosophy regarding fact fluency. Before her arrival at Rhodes, teachers believed that they were not allowed to emphasize fluency with their students, an idea that Rachel believed originated in a movement to eliminate timed tests. She was able to act as a change agent by sharing the article *Fluency without Fear* from Boaler (2015), a resource that she learned about through the regularly scheduled meetings of mathematics coaches held by the county mathematics office. This resource allowed her to guide teachers in the school to understand and promote fact fluency with more depth. Apart from the district pacing guide, three other resources important to Rachel were the extended curriculum framework and lessons provided by the state department of education, a planning and resource book for coaches, and a popular website that offers ready-made ideas for daily number routines.

Rachel shared that her biggest goal of the current year was working collaboratively with teachers to create quarterly assessments, which is the result of the recent adoption of revised state standards. She indicated that the best part of her coaching work was watching teachers make significant shifts in their thinking. The most challenging aspect of her coaching practice was working with teachers, or teams of teachers, who were not yet willing to question their practice. One way she was able to work around that constraint was to work with one teacher in one classroom in order to show success. When other teachers see the teachers experience success, they seek out the assistance of the coach themselves.

Faith. During her 12 year teaching career before becoming a mathematics coach, Faith taught in the primary grades. She entered the work of coaching when she was invited by administration to try a new position the district had created to find and share the mathematics successes happening throughout the district. This early experience coaching in another county has given her a longer career as a mathematics coach than many others. Currently Faith coaches at Cushing Elementary School, a pre-kindergarten to Grade 6 school. She holds a license in elementary education from her state, and currently holds an additional endorsement in another specialty area, as well as the endorsement as a mathematics specialist, which she has held for a number of years, She completed the coursework for this endorsement through a university degree program within her state.

Faith had positive experiences with mathematics up through Algebra 1, but does not remember much about her high school courses, with the exception of an unfortunate experience in a high school Geometry course that she still associates with unpleasant memories. As an undergraduate in college, Faith recalled taking a mathematics methods course for teaching but does not recall any others. Like the other participants in the study, Faith has also had extensive experience taking courses through her district, as well as through a third party consulting company that once offered professional development through the district. She often teaches summer workshops as well.

In a typical week, Faith spends over 50% of her time either coaching teachers or co-teaching. For her personally, the goal of coaching was to ensure that teachers recognize that all students can learn, and she focused her coaching activities on that goal. In the past she has had the freedom to build her own coaching practice, but at other schools this has not necessarily been the case: principals often decide who needs her assistance. At other times she has taught one grade or a class of mathematics when there was a need.

Faith frequently used the state's curriculum framework to unpack and understand the standards with teachers and the district's pacing guide to assure that teachers recognize the progression and flow of the lessons. She also referenced *Elementary and Middle School Mathematics* (Van de Walle, Karp, & Bay-Williams, 2010) as a source for both content and pedagogy, *Content-Focused Coaching* (West & Staub, 2003) for her own reference as a coach, and the trio of Empson and Levi's (2011) *Cognitively Guided Instruction* books (Carpenter, Franke, Levi, & Ferguson, 1999; Carpenter, Franke, Johnson, Turrou, & Wager, 2016; Empson & Levi, 2011), primarily to view the clips of student thinking and to understand the continuum of computation strategies explicated in these volumes. In team meetings, Faith's emphasis was on using student thinking to plan instruction or to design problems for future lessons. Specifically, she worked with teachers to select numbers for different problems so that they can have experiences with the problems before their students do them. Even after naming this extensive list, Faith still asserted that "less is more" at a grade level team meeting.

For Faith, the best part about being a mathematics coach was doing mathematics with teachers. She loved leading other teachers in professional development, especially when they have "lightbulb" moments. She said the most challenging aspect of her practice was helping school administration understand their role in supporting mathematics coaches and their teachers. From these experiences with principals, Faith has learned how to address teachers' concerns when principals similarly place similar strict controls on the teachers' actions: it allows her to better understand and support teachers in that situation.

Isabella. During her 10 year career as a teacher, Isabella taught at grade levels that span the entire elementary school are range. She has been coaching for over 10 years in grades K-6 schools, all within the same district. Isabella holds an elementary license from her state with a specialty area endorsement beyond the standard license. This also in addition to the mathematics specialist endorsement, which she has held for 5 years. All of the coursework for her mathematics specialist endorsement was completed within the recommended state program. As a child, Isabella found mathematics to be easy and quite pleasant. Her high school courses included mathematics up through Calculus, which was followed by additional Calculus courses as an undergraduate. As with other participants, Isabella has taken extensive coursework through her district, and has even facilitated courses for other teachers and coaches. She, too, has participated in district professional development, which in the past included extensive training with Developing Mathematical Ideas (n.d.) and another third party consulting organization. She has even taught courses for teachers within the district.

Isabella's coaching practice at Waverly Elementary School consists primarily of working with teachers. She noted that she spends 20% of her time on average planning with collaborative grade level teams and another 35% of her time either conducting professional development with the teachers or co-teaching in their classrooms. Because it is a large school she has to make careful choices on how she spends her time, but she appears to make these decisions herself. In the past she has taught a mathematics class for an over-enrolled grade level, but primarily spends her time with teachers. She and the teachers have systematized their planning process, incorporating an online system document system to do their lesson planning, including a process for reflecting on a unit and making changes for the following year. Because of a recent standards update, much of the most recent year has been focused on understanding and unpacking the new standards and adjusting the planning materials. She says that the best part of her job is working with teachers to help them better understand the mathematics they teach so that they have the tools they need to help students learn. To Isabella, balancing her time poses the most challenge, but she also admitted that working with teachers who are not ready to change their practice can be challenging.

Coaching Resources

Data about coaches' use of instructional resources was triangulated through several different data sources. In the initial survey, the following two questions were asked, with the purpose of discovering what constituted coaches' daily resources as well as the resources they seek out when teachers are looking for new ideas.

> What are your three most important go-to professional resources? Why did you choose those resources?
>
> What resources do you bring with you when you do planning with teachers, either long term or short term planning? Why?

The face-to-face interview questions about the coaches' planning meetings with teachers again sought information about common resources and new resources. A final question allowed follow up questions to the information already provided. Additionally, more information about common resources often arose during the student work sorting potion of the interview, as coaches mentioned lessons or strategies that they work with in their practice with teachers.

Two of the coaches have had extensive additional training or courses in addition to the degree program. For example, the *Developing Mathematical Ideas* (n.d.) series of courses was referenced, as was *Cognitively Guided Instruction* (Empson & Levi, 2011). These two resources offered more than simply instructional activities. The instruction and content was shared as meaningful formative experiences. For example, one coach said, "Anyway, then we came across CGI and we started learning about kids' thinking. We had a person come in and train us and we just started like that, just started to take off." CGI also serves as a source for coaches and teachers to sequence students' calculation strategies during collaborative team meetings with teachers.

Alternatively, the coaches mentioned resources in both their surveys and in the face-to-face interview that figured less prominently as inspiration but rather as sources of tasks or activities to serve immediate needs. Each of the coaches listed quick reference sources of activities that they provided for teachers, sometimes looking them up online, such as various number routines and tasks, or specific lessons taken from activity books referenced in the county's Planning and Pacing Guide (2016), all in addition to coach-made activities. Notably, no coach mentioned the use of teacher open source/inexpensive materials as a source of information, nor did they mention the official adopted textbook as a resource.

Another level of resource that coaches use with teachers is not simply a collection of activities, but also resources that teaches mathematics along with pedagogical information. *Elementary and Middle School Mathematics* (Van de Walle et al., 2010) is a good example. This resource contains hundreds of useful activities, but it also shares information teachers need in order to understand the mathematics they teach, otherwise known as specialized content knowledge and pedagogical content knowledge (Ball, Thames, & Phelps, 2008). The district planning and pacing guide (PPG) was also highlighted as a source of valuable information about student learning and student misconceptions. Both of these resources were mentioned in the survey, in the interview portion, as well as during the sorting portion of the interview. Additionally, the *Investigations in Number, Data, and Space* textbook series (Russell, et al. 2012) and the fact fluency resources from Boaler (2015) were similarly cited as informative and useful beyond providing useful classroom activities. In one case, the Boaler materials were used to transform habits and expectations in the school community. These resources were shared as part of the face-to-face interview, primarily while explaining aspects of the coaches' practice.

Finally, coaches called upon district and state resources for information on the learning standards that are assessed at the end of the school year. In addition to standards documents, coaches also used the state's lesson resources and the state's expanded information on standards. Additionally, the district planning and pacing guide was mentioned numerous times by each participant as frequent resource. Coaches report that the PPG is used not only to help sequence and pace instruction in schools, but it also contains information about student learning sequences, common student misconceptions, and strategies for using a variety of representations and manipulates. Since the district's planning and pacing guide was mentioned frequently as a valuable resource, it may warrant further investigation.

Coaches Engaged in Professional Noticing

Professional noticing is a phenomenon identified as part of the practice of an expert in a professional arena (Mason, 2011). In this study, the professionals were former teachers who transitioned to practicing elementary mathematics coaches. Not only were they licensed and professional teachers, their education, experience, and expertise qualifies them as experts in mathematics education, therefore the observations made by these individuals can be considered under the category of "expert." As experts, coaches draw upon a variety of resources to make sense of aspects of teaching and learning. According to Mason (2011), the expert has enough experience and awareness of a situation to filter out the irrelevant details and focus only on the ones that matter. Of course in the filtering process the choices made gradually reveal the priorities and character of the expert, perhaps even revealing a unique professional "voice" (Beijaard, Meijer, & Verloop, 2004). A professional voice unique to the elementary mathematics specialist coach, that differs from the teacher's and the administrator's voice, may also be recognizable.

Professional noticing is really a three-part phenomenon that can sometimes unfold nearly simultaneously. In this reporting, each element will be deliberately shared as distinct from the other two in order to frame results from the data. The first engagement of PN is *attending to* a situation and deciding what is notable and important. Even in the controlled environment of looking at student work in a quiet room, attending to what is important can be rife with complications. Multiple representations, multiple misconceptions, and simply the variety of thought that emerges from an open task can thwart a focused reading.

Once the expert makes clear where their attention is directed, they focus on making sense of what they see or hear (Berliner, 2001). In the professional noticing framework this is referred to as *interpreting* what is seen or noticed. Interpreting is not just making sense of student thinking, it is also making connections to broader ideas that may be applicable. In other words, it is considering, "What is this a case of?" (Shulman, as cited in Sherin & Van Es, 2002). The elementary mathematics coach can look at this from two points of view. They may interpret student work from the perspective of understanding student thinking. On the other hand, the coach may interpret the work samples as a reflection of the instruction and classroom environment from which the work samples came. Often they may be thinking of both tasks. In the final step of PN, the expert takes action, in this case making connections between details in the student work and the broader principles that govern teaching and learning, all within the constraints of the local school environment (van Es & Sherin, 2008). Again in the case of mathematics coaches, this action may be related to student instruction, or to the mentorship of teachers.

In this study, EMCs were asked to carefully examine student work and to share their thoughts out loud as they worked. What they noticed, how they chose to reflect on what they saw, and how then to respond depicts an expert view of these student work samples.

Coaches attending to student thinking. Attending to the details of student work on a task like the sandwich task is surprisingly challenging. Interpreting a variety of strategies and making sense of different representations calls on extensive specialized content knowledge that is used strictly in teaching, but not in other fields that frequently use mathematics (Ball & Cohen, 1999). Appendix G provides a list of all of the student work exemplars accompanied by a description of the mathematical features that were planned and intentionally inserted, as well as others that were anticipated for the sandwich task. Samples that exemplify learning trajectories and other learning sequences were included in order to determine which ones coaches noticed or identified.

Predicting or anticipating student strategies. In anticipation of the sort and the interview, participants solved the sandwich task and then predicted the strategies that students might use. Their predictions were remarkably uniform. All three participants anticipated an equal sharing strategy that Confrey et al. (2014) would call “Split All” and Empson and Levi (2011) would call “Additive Coordination.” The first two are examples of the “one item at a time” variety and the last shows the “sharing groups of items” variety. The anticipated solutions look remarkably similar even if none used a formal nomenclature (see Figure 5.

In this case the coaches each described an implied sequence of topics for content that they expected students to progress through in relation to this task. In effect, the coaches formed a *hypothetical learning trajectory* (Simon, 1995) for the students around this task. Additionally, the students’ answers to this sandwich task figured prominently in all three coaches’ discussions, but for different reasons.

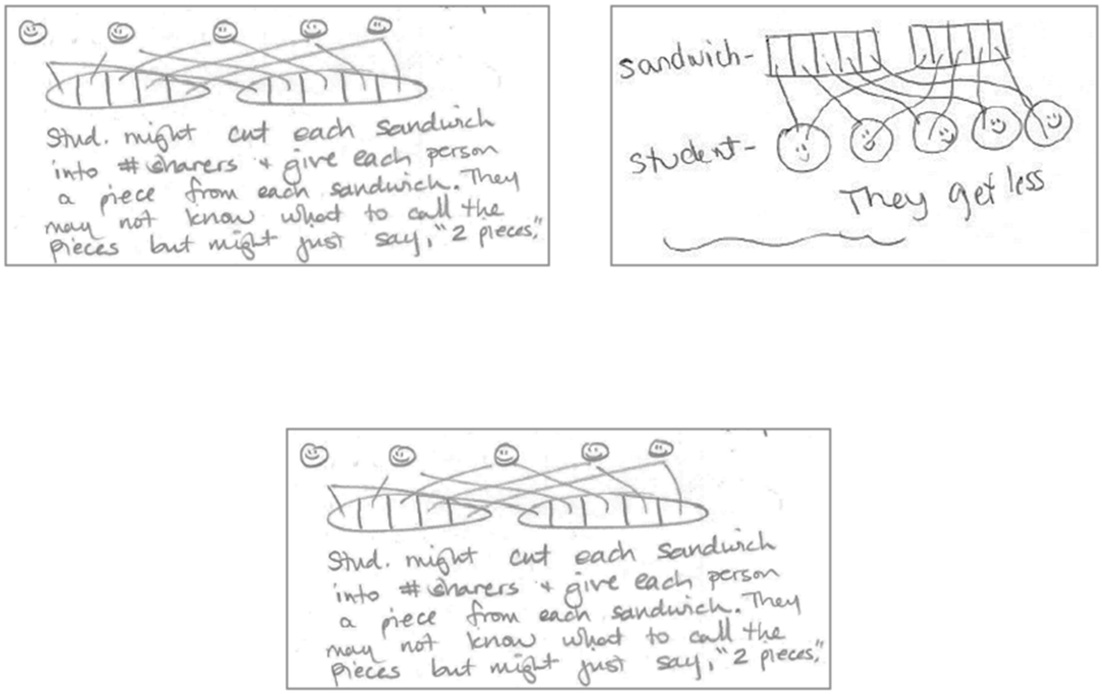


Figure 5. Additive coordination by Isabella, Faith, and Rachel, respectively.

In this case the coaches each described an implied sequence of topics for content that they expected students to progress through in relation to this task. In effect, the coaches formed a *hypothetical learning trajectory* (Simon, 1995) for the students around this task. Additionally, the students' answers to this sandwich task figured prominently in all three coaches' discussions, but for different reasons.

Coaches' hypothetical learning trajectories (HLTs). A hypothetical learning trajectory describes the detailed plan teachers create as they examine their students' current understandings, gather data and evidence, and make decisions for future lessons. It is a *trajectory* because students are moving forward in their learning and teachers are drawing from evidence in the past to predict future needs. It is *hypothetical* because, despite multiple sources and resources, teachers still can only make educated guesses of what skills and information students need to experience next in order to continue their learning. Some of the resources that teachers and coaches used were quite detailed. For example, the learning trajectory from Confrey et al. (2014) contains a detailed analysis and sequencing of what students can be expected to do as they learn to partition shapes and lines into equivalent fractional parts, otherwise known as equipartitioning. On the other hand, resources like a standards document, are more broad. For example, a first-grade standard might indicate that students should count to 100, while a second-grade standard might indicate that students add double digit numbers within 100. There are many sub skills and understandings that must be in place for a first grader to move from the first-grade standard to the second-grade standard. Filling in the spaces between standards requires a deeper mathematical knowledge for teaching (Ball et al., 2008).

The evidence of the coaches' hypothetical learning trajectories emerged not just in their overt mapping and sequencing of the instructional goals of this task. Their HLTs also became evident in the nature of what they noticed and attended to in the students' work and the references they made to other guiding resources. Two of the coaches made additional references to the strategy sequencing embedded in the CGI framework, another to the state standards, and yet another relied on her experience to suggest an extension of the task. Rachel suggested the following: "This group of kids (Samples I, F, K, L, H), I mean, could possibly even be ready to think about what if I have two-and-a-half sandwiches shared by five students? And just think of it like an extension." While Rachel herself did not mention the connection, her instinct to change a dividend from a whole number to a non-whole number is also indicative of a higher level of sophisticated thinking within the CGI equal groups framework. Rachel is not alone in thinking that this is an extension to the current task (Empson & Levi, 2011).

All three coaches identified the sandwich task as primarily a comparison task, specifically the comparison of fractions, which anchors the task in either third- or fourth-grade, depending on local standards. Within these coaches' curriculum, a fourth-grade standard is applicable. Using "Compare Fractions" as an anchor, the coaches hypothesized the following trajectories as seen in Chapter 5.

Table 5

Hypothetical Learning Trajectories (HLTs) Presented by Each Coach

	Prerequisite Skill	Mathematical Goal	Future Lesson Goal
Isabella	• Equipartitioning • "halving" (non-anticipatory sharing)	• Compare Fractions	• Fractions as Division
Rachel	• Modeling Fractions • Equipartitioning • Equal pieces	• Compare Fractions (VA 4.2, 5.2) • Equal Sharing (VA 2.4, 3.2) • Fractions as Division (VA 4.2)	• Ordering Fractions • Mixed Numbers • Problem Solving

Table 5 (continued).

	Prerequisite Skill	Mathematical Goal	Future Lesson Goal
Faith	• Quantities to 5 • Equal share problems • Understand "most" "least" • Part/Whole interpretation of fractions	• Compare Fractions • Part/Whole interpretation of fractions • Equivalence • Relational Thinking	• Relational Thinking

Coaches' next steps. At the conclusion of each interview participants were asked, "What do you think the lesson goal should be for the next day?" Isabella had hypothesized that students working on this task would be moving to lessons on fractions as a division operation, based on the hypothetical learning trajectory she had envisioned for this task. After examining the work samples, her HLT was adjusted. She did not at all mention fraction division as a future goal, but instead focused her instructional recommendations on further practice equipartitioning and the skill of presenting a complete answer to a problem.

Rachel envisioned the students ordering fractions, working with mixed numbers, and an unspecified goal of problem solving. At the conclusion of the student work sorting interview, Rachel had refined the HLT of these students to include skills that she had noted previously as either a lesson goal or as prerequisite knowledge: equipartitioning with concrete objects for some and for others, answering the question, which she reports this way: ". . . They've got the stuff split up, but they might be struggling with answering the questions (Samples B, C, E)?" She was prepared to extend the task, but not by moving to a fraction division or an ordering of fractions task as she had hypothesized. Instead she suggested that students extend their knowledge of equipartitioning by doing a task with a fractional quantity in the dividend instead of a whole number.

Faith predicted that students would move to learning relational (or algebraic) thinking for comparing quantities. At the end of the task she adjusted her HLT for this group of students to include topics of equivalence, writing answers, and explaining their reasoning. This excerpt sums up her position well.

> Now that I've talked this through a little bit, I'm feeling like, let's go back to the question. Let's look at the goal. Let's see if we can just take this one idea and see if we can get the kids to understand, okay when we answer the question, get that comprehension piece and ensure that they understand what's happening in this story. Did we get all the parts? Let's go back and check. That could be one avenue we could go. (Faith)

Despite the fact that coaches do not generally teach students, previous studies have noted that coaches are school leaders in focusing attention on the long arcs of mathematical content development across the grades and beyond (Becker, 2001). This knowledge may help coaches, for example, work to negotiate school-wide initiatives in mathematics. Rachel described her efforts to provide alternatives to timed tests while still promoting fluency throughout the school:

> And we've done a huge shift away from timed tests, . . . And we had a big conversation about, the way the standards are worded now is that fluency is important, but what does that mean? What does fluency mean? (Rachel)

This is similar to the school-wide examination of addition strategies that helped one school agree on what to expect as students moved through the grades (Cameron et al., 2009).

A hypothetical learning trajectory is a teacher's prediction of what students will learn from a given lesson. It informs instruction and planning, and it is drawn from evidence from instruction and planning. The HLT set in the morning before a lesson is amended with each passing moment in the lesson, and at the end it might be an utterly unrelated topic the next day. In this case, each coach abandoned their predicted lesson trajectory (or rather they modified it) and set their sights on understandings previously thought to be in place. In this case, the coaches uniformly mentioned equipartitioning as a skill students needed to revisit. **Conceptual principles.** "Compare Fractions" is a stated mathematical goal of this task for all three coaches, and all three coaches also noted the conceptual idea of equipartitioning as a prerequisite skill or experience required to do the task, even if they called it by slightly different names (equal pieces, equal share problems). Making comparisons is an important content goal and standard, but it may not rise to the level of conceptual principle. A conceptual principle is "an underlying cognitive principle, identified by research, that supports the development of ideas" (Confrey, 2012, p. 724). It is also "a big idea" and a generalization that has multiple meanings for experts, but which may be collapsed into a single category for the general public (Confrey, et al., 2014). Equipartitioning meets the criteria for a big idea, an assertion which is also backed by the research of Empson and Levi (2011). For example, Sample B demonstrated that the student could equipartition into three equal shares, even distributing a whole sandwich to each individual rather than dividing up each sandwich before distributing. Hunt and Empson (2015) distinguished the equal sharing work on Sample B from the equal sharing work on Sample G; Sample G partitions equally, but uses knowledge of simple fractions (or trial and error) to create fourths, a fraction that is easily done by repeated halving. They call this Non-Anticipatory Sharing because it does not take into account the number of sharers as Sample B does. Moreover, Confrey et al. (2014) is more likely to highlight the fact that Sample B is partitioned, but is *not* equipartitioned, given that the root of the equipartitioning conceptual principle is equality. The shape of the bread would make each partition different from the others. Interestingly, none of the coaches noted the irregularity of the shape of the object being subdivided in Sample B, nor did they recognize the incorrect partitioning in the last sandwich shown in Sample E. On the other hand, only Isabella noticed that Sample G showed evidence of the Non-Anticipatory strategy of making familiar cuts in the square (in half and then in half again).

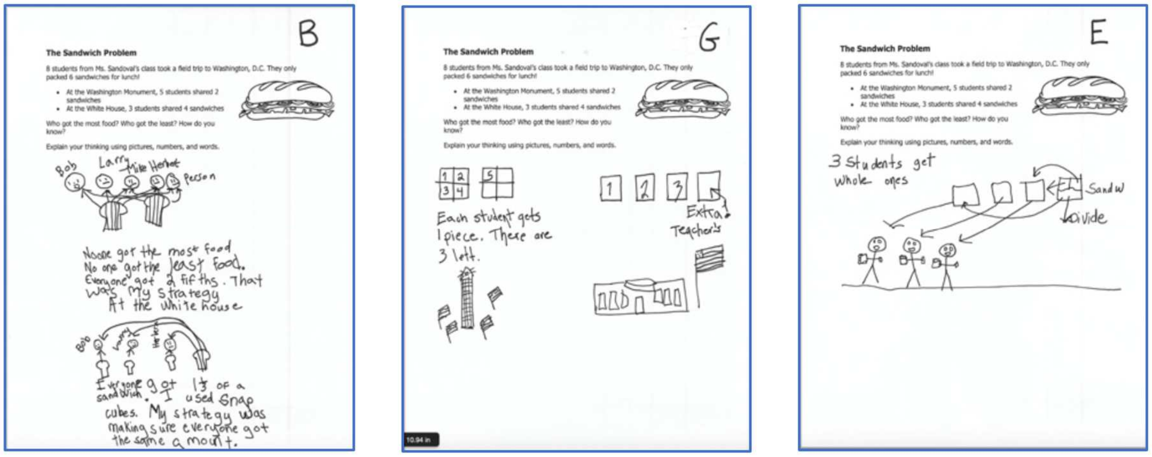


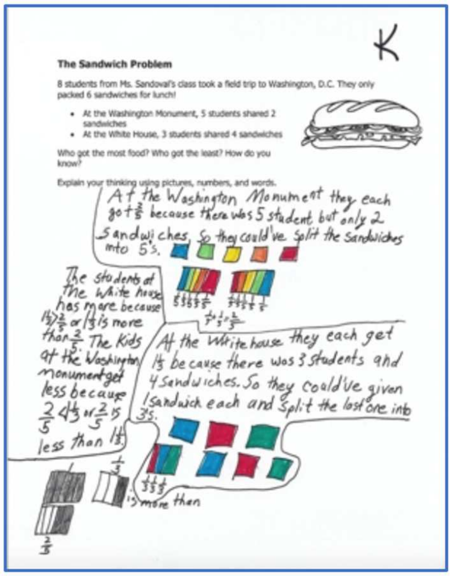
Figure 6. Samples B, G, and E show subtle variations of equipartitioning.

Task goals. There is not universal agreement on the goal of this task. Rachels' goals are highly dependent on the state standards, which is evidenced by the fact that she listed all relevant standards when setting the mathematical goal of the task. Because of Rachel's focus on the state standards, she also noted that recognizing fractions in the role of a division statement (Kieren, 1980) is an appropriate goal in the same grade level standards as the comparison of fractions, while Isabella saw that as a future goal, and Faith did not note it at all. Interestingly, Faith did cite "relational thinking" as a current and future goal for this task and gave this example of evidence of relational thinking in her own work in advance of the sorting interview. From Faith's anticipated student strategies: "They (Washington Monument) get less because they won't have a whole sandwich." And from Faith's own solution we saw, "The students sharing sandwiches at the White House get more food because there are more sandwiches than students."

Relational thinking is not named as part of the state standards, but it is an end goal in the CGI developmental sequence for tasks, which is evidence that the source of Faith's hypothetical learning trajectory for a partitioning task may include principles from the CGI framework. It is important to note that the term *relational thinking* applies to a certain kind of algebraic-like reasoning (Empson, Levi, & Carpenter, 2011). Rachel noted that the task could be solved without calculation, which she referred to as "Words/Reasoning," but the two coaches' meanings were the same. Given her use of the word "reason" it seems likely that Rachel might refer to this strategy as more evidence of the process standard Reasoning & Proof (NCTM, 2000), than as a specific goal within an equal groups or equipartitioning learning trajectory itself.

There are other notable differences that emerged between the three coaches' HLTs. Isabella and Faith recalled ideas that were not present in their state curriculum standards, and which echo language found in the Cognitively Guided Instruction literature. For example, Isabella mentioned that students will often partition shapes into halves first, expanding the meaning of "half" to refer to all fractional pieces. "Halving, so yeah, he ran out and divided that into three equal parts. Okay, so what I've seen is that this is uncommon because a lot of times they just start with halves is what I've seen." (Isabella). While the state standards do mention splitting into two pieces, they do not identify it as a common overgeneralization when students are tasked with partitioning into another quantity. Isabella attributed her awareness of this common student misconception to CGI materials, where it is identified as indicative of the "Non-Anticipatory" equal sharing strategy (Empson & Levi, 2011).

Despite the presence of CGI-influenced elements in both Isabella and Faith's hypothetical trajectories, neither of the coaches highlighted the difference between sample K and sample L.



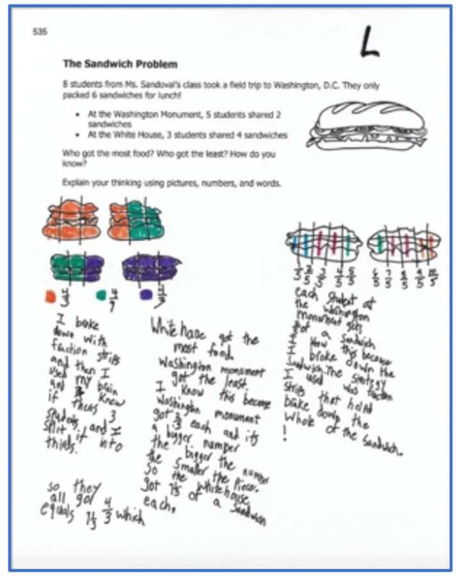


Figure 7. Samples K and L show two varieties of Additive Coordination.

Sample solutions K and L were intentionally selected in order to contrast Additive Coordination (while sharing one item at a time) and Additive Coordination (while sharing a group). In Sample K the student shares 1/5 from each of the two sandwiches to arrive at a portion of 2/5. In Sample L, 2/5 is taken as a group from one sandwich, continuing on linearly. Interestingly, Samples K and L had the highest rate of co-occurrence of any of other pair of samples in the sorting tasks because they tended to be grouped together as equal exemplars. This may be an example of coaches not recognizing or lending value to the distinction between the two different strategies, despite their familiarity with the CGI sequences of strategies for equal groups partitioning.

Faith and Isabella's approach to stating learning goals differed in another important way from Rachel's. Rachel set the goal of problem-solving as a future lesson topic to follow the sandwich problem. However Faith and Isabella place problem-solving at the center of the task goal. For example, Faith frequently referred to the context of the problem as she was examining the student work samples, as we can see in the excerpt below. As she speaks, Faith is grouping students according to their partitioning strategy. She noted that not one of them had arrived at an answer to the task as it was asked. "I think that this group too, they've got the stuff split up, but they might be struggling with answering the questions (Samples B, C, E). So who got more in those two scenarios? Which group of kids got more?" Her questions are directed at the students, who are of course not present.

That which coaches attend to can be found in their hypothetical learning trajectory, but what coaches do with this information follows.

Coaches interpreting student thinking. The second phase in the process of professional noticing is interpreting or making sense of events or details in the professional setting. The expert draws upon their stores of knowledge and experience in order to apply meaning to whatever is happening in the situation (van Es & Sherin, 2008). In a sense, this is a form of interpretation, especially when the pertinent details are presented in static written form. As coaches began the sorting phase of the interview, they were given the set of 12 work samples, two pens, and were asked to look at the work samples, note what they saw, and then speak out loud about what they noticed. Attending to and interpreting student thinking are often intimately linked, so coaches may do both acts in quick succession in this interview context. The area of interest here remains the interpretations that coaches make and the sense they make of the student work samples.

Coaches grouping work samples mathematically. In all three interviews, the coaches started with Sample A, tried to make sense of it, and then moved on to Sample B because some samples were easier to understand than others. As they progressed through the stack of work samples, they began to make connections between students' representations and strategies that were mathematical in nature.

Tens, tenths, 0.1, and $\frac{1}{10}$. The student whose work is shown in Sample H (figure 8) shared an answer that is a study in contrasts: the answer itself was expressed as a decimal, even a repeating decimal, which was a response that does not appear in any grade associated with the sandwich task. On the other hand, the student's response did not actually answer the question posed by the task: Which students got more? He simply said that they didn't get the same, with the actual comparison left unsaid.

The coaches' responses varied in significant ways. Rachel interpreted this work as an example of a student who "gets it," "got the right answer," and she then grouped it with samples K and L as the group who "understands division and the equal groups and kind of naming the pieces." She did state that she would like to see another strategy but accepted the response in Sample H as progressive enough to warrant an extension task. Isabella was far less accepting of the response as given. Acknowledging that Sample H has a correct answer, she twice said that it "needs more explanation," and therefore she could not hold it up as a “model for discussion,” implying that samples with drawn models would be more appropriate for classroom discussion.

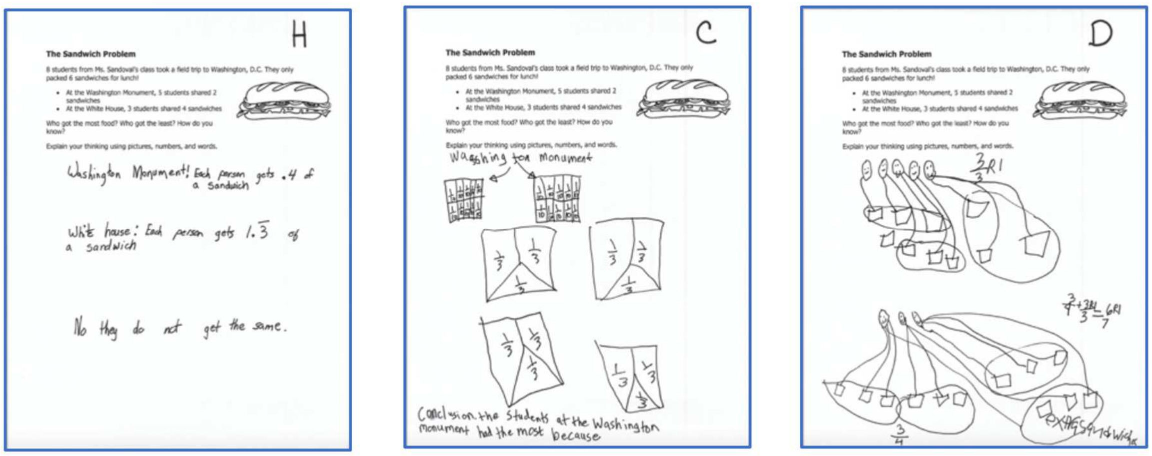


Figure 8. Samples H, C and D as examples of equipartitioning by tenths.

Faith went a step further and focused on two features of the work in Sample H. First, she noted that the student did not offer a complete answer to the task and might possibly be incorrectly interpreting the task. More importantly she made a connection between Sample H and Samples C and D for a mathematical reason. Reading the numbers in Sample H as “four tenths” and “one and three tenths, repeating” reminded her that the implied partitioning was into tenths (although the student showed no evidence of making that connection), and she immediately made a connection to Sample C, who had explicitly made partitions of one tenth.

> This guy (Sample H) needs to be up with this guy (Sample C) because this could, they could see the tenths, the four-tenths. It could also extend their thinking into

> some decimal work possibly, if he could explain (Sample H) where did that come from? How did he think about it? (Faith)

Not only would the focus on tenths encourage student C to consider why she chose to name the pieces tenths, it would also encourage students C and H to “do some comparison work . . . because we can see the four-tenths and the one and a third kind of sort of.” (Faith). Similarly, Faith made a connection between Sample C and Sample D using the same rationale as she used for matching Sample H with Sample C: a focus on tenths. However, this connection was less sure, but she recognized that there may be commonalities in the discussion of ten pieces.

> (Sample D) One, two, three, four, five, six, seven, eight, nine, ten. Which is what this little friend is wanting to do into pieces (Sample C), but. They break it up into five pieces (Sample D). If they have two sandwiches, then they’d have 10 pieces. They’ve broken each sandwich into fifths, possibly if these are all fifths.”

This is in contrast with Rachel, who observed that Sample D was broken into “lots of little pieces,” showing no reference to the fact that there are 10-15 pieces. Is this an important detail? It depends on the situation of course, but there is far more potential for targeted teaching and learning when the teacher or educator recognizes specific details in student work and links it to student thinking.

Seeing fifteenths. In the previous discussion, coaches recognized and highlighted tenths in the student work samples and used this common feature to tie together some of the most and least sophisticated student samples. An exploration of representations of fifteenths is equally interesting. All three coaches looked at Sample F and tried to make sense of how the student named the fraction using the denominator 15. Isabella observed that the student's model was drawn accurately in that there was a third for each student in the group and the final third was shared by all five students at the Washington Monument. Faith recognized that the drawing was accurate as well, but shared that she would ask the student the next day to try drawing another model, with the thought that they would recognize the value of each sector in the circle they sketched.

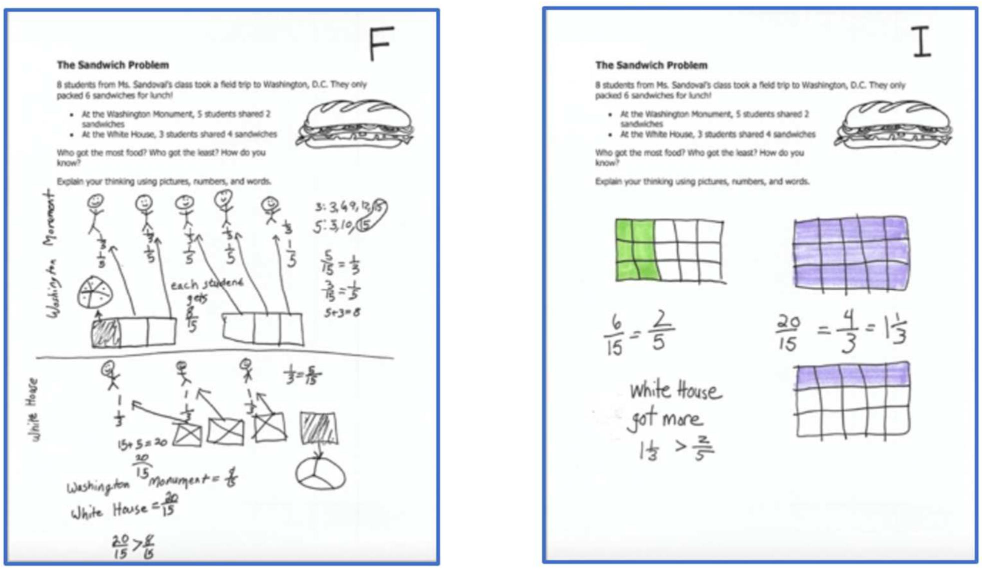


Figure 9. Samples F and I as examples of equipartitioning by fifteenths.

Naming and recognizing the value of each of the five students' share of two sandwiches highlights a key difference in the responses of the three coaches. The table below outlines their verbal responses, and the discussion that follows will show that the manner in which the reader describes an error like this one is a reflection of their interpretation of the student's understanding of the mathematics. In chapter two we clarified the difference between types of errors. A *conceptual* error indicates a misunderstanding of a concept while a *procedural* error is a missed or incorrect step in a process. A *slip* is a true accident and is typically noticed quickly, and an *omission* occurs when the learner fails to include something that is critically important in the context.

Table 6

Classification of Error Types by Coach

	Utterance	Presumed error type
Rachel	They're understanding that they gave them each one-third, and then they split this into fifths, but they called it one-fifth instead of one-fifteenth because they split this third into fifths. So this would be fifths, fifths, fifths.	*Possibly* Procedural *Possibly* Conceptual
Faith	. . . We're getting thirds and fifths and fifteenths. Where's that coming from? . . . Just looking to see why he went back to the fifteenths (Sample F). Unless he was just trying to get them all into fifteenths . . . he (Sample F)was trying to get a fifth of the third (I think). . .	Procedural (35:37) *moving later to* Conceptual (41:54)
Isabella	. . . then they're really giving each person a fifth of a third and they somehow are trying to . . . I don't know where this is coming from. How are they seeing fifteenths, or is it just because one-third plus one-fifth? Are they adding? So, is this one-third plus one-fifth equals eight-fifteenths is what they're thinking that's how much that is. So then it really should be one-fifth of one-third is what each of them is getting, not one-fifth.	Procedural (28:15) *moving quickly to* Conceptual (28:51)

Rachel immediately recognized that student F had assigned the incorrect value to sectors of the circle, assigning them the value of one-fifth, when the actual value of a sector was 1/5 of 1/3. Rachel's response to the error makes it appear that she might believe that this error is in name only. In other words, ". . . but they *called it* one-fifth instead of one-fifteenth . . ." (emphasis added) reflects a procedural error implying that the student incorrectly named the portion of the sandwich. Yet, it is also possible that she interpreted this as a conceptual error because she said, "They split this third into fifths." It is clear that the coach understands the student's error, but it is less clear how she interpreted the nature of the student's error. The evidence appears to show that the student does not have an understanding of unitizing and naming fractional parts that allows them to accurately identify a fifth of a third as a fifteenth. In isolation, identifying a portion of the circle as a fifth is correct. In this context, it is not, so this would be classified as a conceptual error. The distinction is important because a mini-lesson the next day would include either a lesson on two levels of unitizing (Steffe & Olive, 2010) or a procedural lesson on naming fractions.

Faith grappled with the student work in Sample F and came back to it several times in her attempt to make sense of the student's thinking. Initially she appeared to believe that the student began with trying to find a common denominator of 15 in order to compare the fractional values. She even noted that this is a skill that fourth graders commonly learn so it was not surprising to see evidence of this on the student's paper. In this case, the error was procedural because the student was not successfully finding the right numerator needed to compare. In the final utterance, she hesitantly observed that the student was trying to find a fifth of a third, her change of thought marked by a parenthetical “I think” at the end. This change of interpretation changed not just her window into the problem but also her perception of the student’s conceptions of equipartitioning: the student’s error was no longer to be remedied by a tutorial in common denominators but instead called for a mini-lesson in accurately naming a fraction of a fraction for which an accurate model already exists. In terms of addressing misunderstandings, this is a profound shift in approach for the teacher.

Isabella worked through the same process Faith did, making sense of the student work finding a common denominator before recognizing that the error was a conceptual error. For Isabella, she made sense of that after about 30 s examining the student work compared to the 6 min it took Faith. This is not meant to imply that speed in reading student work is required: it is not. But it does show that sometimes reading a student’s work in depth takes longer than is typically allotted to do so. Faith spent all of those intervening 6 min exploring the work in Sample F, considering the pedagogical opportunities it afforded for a whole class discussion.

Strategies, representations, and misconceptions. The framework for unpacking learning trajectories (Confrey, et al., 2014) starts with understanding the importance of identifying the core conceptual principles that thread throughout the K–8 curriculum. This was explored earlier, within the context of exploring hypothetical learning trajectories. The framework also highlights the importance of students having a flexible understanding of different strategies and representations. It also calls on teachers and coaches to identify and respond appropriately to the misconceptions that may arise while students are exploring these different strategies and representations. The three coaches in this study were tasked with making sense of twelve different solutions. In the previous section, their interpretation of student misconceptions/errors shed light on the nature of the students' grasp of the conceptual principle of equipartitioning. In this section we will explore how the coaches were able to recognize a "kernel of right thinking" (Confrey, 2012) and leverage this information to inform a plan for instruction.

In the final phase of the sorting, all three coaches assigned Samples F and I to the same group because both of the representations presented the shares of sandwich in terms of fifteenths. A quick look at these two samples might not necessarily show a similarity. The representation in Sample F is an example of an equipartitioned figure, demonstrating a life-like distribution of equal portions of two sandwiches. In this respect it is an equal groups model (cite). Sample I, however, is an array model of a single serving of sandwich – it does not show the unit whole of two full sandwiches (Lamon, 2012). In essence the arrays are a re-presentation of the answer to the task rather than a representation of a process that led to that solution (Moore, Morrow-Leong, & Gojak, 2020). The array is arranged as 3 x 5 and each cell represents a fifteenth. The appropriate number of cells is shaded to show the correct answer.

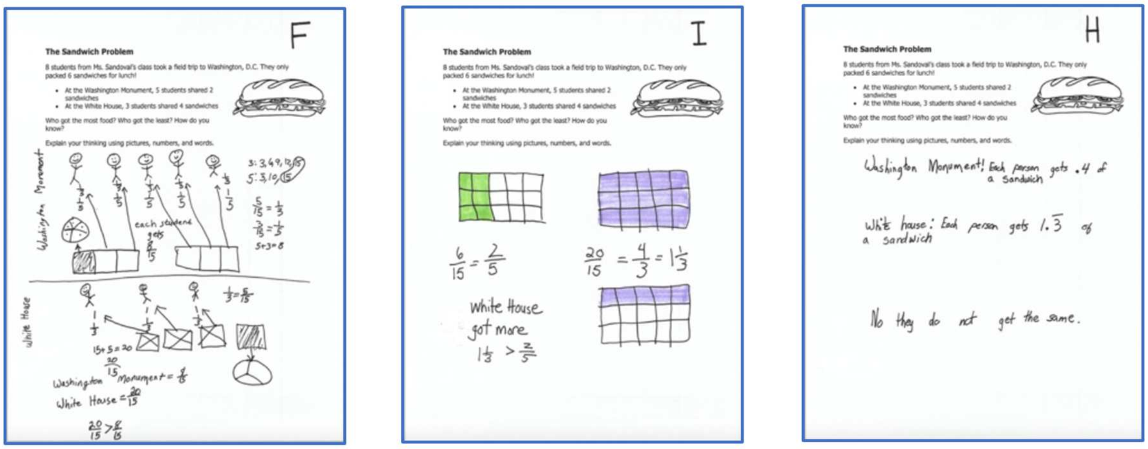


Figure 10. Sample I is paired with Sample F for one reason and with Sample H for another reason.

In contrast, Isabella's first sort paired Sample I with Sample H because she believed that each sample represented the work of students in higher grades than the others. Faith said the same plainly but clearly:

> So they found equivalent denominators or a common denominator. Interesting. I don't know. I feel like these guys are older grades who have done this before and have probably been told in order to compare you have to have common denominators so, figure out a way to make them common. (Faith)

As the interview progressed, all three coaches moved Sample F and Sample I away from other groups in order to pair them together because they shared a common denominator-dependent strategy. Isabella created categories using sticky notes and the change in placement she made was physically obvious during the interview. Originally, Sample I was in the category of most successful solutions she titled, "Got it! Has an answer" because she had associated the array and common denominator with the work of a successful older child. However, after further consideration, Sample I was moved to join Samples F, H, and J in the category entitled, "Got it! Needs more explanation:"

> I feel like this (the array in Sample I) is a representation of that (the drawn model in Sample F) . . . So, I might just bring him (Sample F) in here with this person (Sample I) to try and get more explanation and have him (Sample F) listen. (Isabella)

Isabella believed the partitioning representation in sample F would complement the abstract array representation in sample I, and that the 3 x 5 array in Sample I could help student F make sense of the fractional value of 1/5 of 1/3 that was accurately modeled but incorrectly named. Reconciling the similarities between an array and a partitioned pictorial model in the sandwich task requires a certain degree of flexibility and a deep specialized content knowledge (Ball et al., 2008). There is evidence that the coaches in this study demonstrated this depth of knowledge and flexibility and are well-prepared to facilitate the act of unpacking the learning trajectories that inform instruction.

Mathematizing or subjectifying. Focusing on the mathematics that underlies student work is more than just an aspect of assessment. Within the professional noticing framework, the interpretation stage is a locus for understanding the depth of teachers' knowledge, but more importantly, their responsiveness to evidence of student thinking and resistance to snap judgments about student performance (Mason, 2002). Furthermore, it is important to acknowledge the importance of the mathematical identity of the students behind the work. While the study of identity is less important for coaches examining the work of students they do not know, it is still true that habits of behavior and language patterns might reveal distinctions between the work and the individual. Heyd-Metzuyanim and Sfard (2011) acknowledged the interplay between the mathematizing as subjectifying. Mathematizing is referring to the mathematical content and understanding in the student work. Subjectifying addresses issues of identity as well as other affective characteristics of student behavior. In the absence of live students, subjectifying is grounded in the language of the coach (or teacher) who is reading the work. Heyd-Metzuyanim and Sfard identified three levels of generality. In one sense, subjectifying is transferring from the centrality of the mathematics to the centrality of students. There are three levels of subjectifying. The first level refers to a specific act or action of the student. In contrast, the second level refers to the act of ascribing a word like "always" or "never" to a student behavior. A student may do this themselves, and say something like, "I can't do this!" or, "She is gifted," or "He's a level 2." These statements are not flexible; they are assigned to the student and there is no indication that the student can move out of that identity.

Table 7

Understanding Utterances as Mathematizing or Subjectifying

Classification	Utterance
Mathematizing	And hopefully here (Sample K) you can see there's one whole and one-third, here's one whole and one-third. Oh look, if I put this here, that's another whole and one-third which they can see here as well.

Table 7 (continued)

Classification	Utterance
Level 1 Subjectifying	This one was able to partition three or deal three wholes and then only have to partition the one leftover whole (Sample K). (Isabella)
Level 2 Subjectifying	I put these guys together because they understand equal groups, but they . . . I don't really know what's going on with this one because it seems like they only answered half the question (Sample A).(Rachel)
Level 3 Subjectifying	This one (Sample C) might be a little younger. (Faith)

All of the coaches' utterances were coded as mathematizing or subjectifying. Utterances categorized as subjectifying were further categorized as either level 1, level 2, or level 3. In some cases a sentence was coded in both categories if the focus of the utterance changed mid-sentence. The raw data was converted to a percentage because interview lengths varied significantly. The results are displayed in Figure 11.

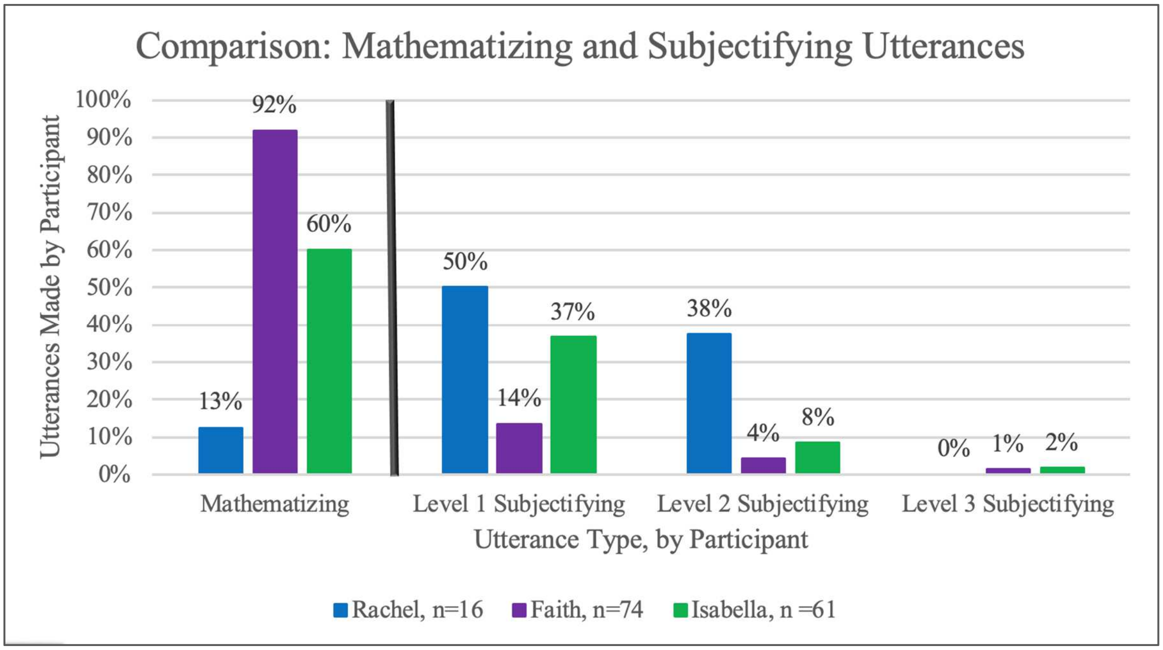


Figure 11. Comparison of mathematizing and subjectifying in utterances, displayed by coach

Notably absent were level 3 subjectifying utterances, with the exception of two estimations of the students' ages. Another interesting result is that Faith's statements about the work samples were almost exclusively mathematizing, but the manner in which she speaks might be interpreted differently by another researcher. Faith often spoke about the students using third person pronouns (they, them), but her statements remained focused on the mathematics. Grammatically, the subject of her sentences were often the creator of the sample, but the subject of the utterance was nearly always the mathematics. It seems as if she was reenacting the scene with the student doing the work. The following excerpt is indicative of this phenomenon.

> (Sample D) Extra sandwiches. All right so what did they do there? One, two, three, four, five. One, two, three. There's my five students. Three students. They've taken the sandwich and cut it up. Extra sandwiches. Okay. I'm guessing he's saying, "remainder one." (Faith)

For Faith and Isabella, the proportion of mathematizing utterances exceeded the total of all subjectifying utterances. This was not the case for Rachel. It is important to note that Rachel spoke far less than the other two coaches (n = 16), partly because of the technical issues with video recording her session but also because she spent long stretches of time thinking and considering the student work. But, on the other hand, Rachel's language often spoke more to the student and their apparent level of understanding rather than to the mathematics itself. In this excerpt note the confidence Rachel has in her global assessments of the student's understanding of the task.

> I think that they (Sample A left) don't understand what's going on with . . . Well, they split the sandwich into fifths and they gave each person two of the fifths. So they get two fifths. They understand that. And then they understand that this is the white that they get. It seems like they just have a misconception of who gets . . . Of what the question is asking. But they understand the two fifths and the one-and-one-third. (Rachel)

In this excerpt, note the frequent use of the word "understand," a contention that is impossible to make with surety based on a single work sample. Furthermore, the word "misconception" dangles without a specific reference point.

In the professional noticing framework, attending to the conditions in the professional environment is markedly different for professionals and experts that it is for novices. In their study of teachers, Jacobs et al. (2010) reported instances of over-attribution and generalization of student understanding, particularly in preservice teachers and teachers just beginning in the professional development program. With more experience in the professional development program, the teachers' expertise increased, and they were less likely to generalize about student thinking. The coaches in the present study focused on evidence of student thinking in a manner and frequency that was more consistent with the emerging teacher leaders in the Jacobs et al. study.

Coaches acting on student thinking. An act of professional noticing begins when the expert makes note of a detail in a complex situation, makes sense of what they see, and acts. Using years of experience and education, coaches in this study looked at examples of student work and reflected on what came to their attention. What these coaches noted about this task and the hypothetical learning trajectories around this task, adjusted following their engagement with 12 work samples. The second phase of noticing requires the expert to interpret, reflect on, or otherwise see their observations in context. In this study, we turned the lens on the coaches' engagement and interpretation of the mathematics in the students' work. Considering the second element critical for unpacking learning trajectories, we noted how coaches connected surprisingly different strategies, representations, and misconceptions across the set of exemplars. Focusing on the details of the mathematics is also pivotal to recognizing the importance of establishing dialogue about learners that recognizes the difference between the learner and the mathematics they learn.

Meaningful distinctions and multiple models. The cognitive principles of the mathematics curriculum are the big ideas that thread through a learning trajectory (Confrey, et al., 2014). As students move toward a goal they pass through intermediary steps, and at multiple points along the way formulate working models and strategies they may not be viable in later years. Some standards represent giant leaps of conceptual development from one grade level to the next, and the end result is a lack of a road map for teachers to follow as students move from one standard to the next grade level, particularly if they are new teachers. For example, CCSS 2.G.3 (NGA &CCSSO, 2010) indicates that students will partition circles and rectangles into two, three, or four equal partitions. In third grade (3.G.2), the standard not only does not specify a specific number of partitions, it also asks that students name the fraction. Bridging standards serve as a roadmap between the big leaps like this between standards (Maloney, 2013). Confrey et al. (n.d.) have written bridging standards designed to slip seamlessly between the Common Core standards, but other bodies of research have stages of development of mathematical reasoning that could serve the same role. Confrey et al.'s bridging standards and the strategy progression from CGI (Empson & Levi, 2011) will be used to explore the intervening learning goals required by the sandwich task. This choice was made because Confrey et al.'s bridging standards are comprehensive for the equipartitioning learning trajectory and because some of the CGI strategies were explicitly referenced by participants during their interviews.

Meaningful distinctions tap into mathematical as well as cognitive conceptions and make distinctions between some mathematical behaviors that are recognizable by mathematics educators even if they are not meaningful to non-educators. For example, Hunt et al. (2015) distinguished between Non-Anticipatory and Emergent Anticipatory equal sharing. The functional difference between the two is to tell the difference between the student who cannot yet consistently name the fractional amount a sharer gets in an equal groups problem, and one who recognize that the fraction name is directly related to the number of sharers. Outside the classroom, this may be irrelevant information, but it is important information for those invested in assessment and instructional planning.

Groupings and pairings. The examination of student work began as a "talk aloud" activity as the participants began to describe what they saw in the student work. As part of adhering to the noticing framework, the interviewer asked the participants to make decisions about grouping or sorting the students as a form of acting on their interpretations of student thinking. Some samples of student work were frequently paired as potential discussion "partners" and will be explored in the context of the coaches' comments about that work. Further, some samples had greater, or more connections, to all of the others, which will lead to an exploration of the stated reasons for either frequent or infrequent instances of pairings with other work samples. Keeping in mind that the student work sample is a proxy for a real student in a classroom, the pairings of work samples are done with a live pairing for discussion in mind.

Samples L and K. Work sample K is an example of additive coordination, sharing one item at a time. Work sample L is also additive coordination, but sharing groups of items at a time. In sample K one fifth is taken from each of the two sandwiches while in sample L a group of two fifths is taken from only one sandwich until that one is exhausted. Recognizing that both fifths form a group is an important bridging standard, and a step toward a mature equipartitioning.

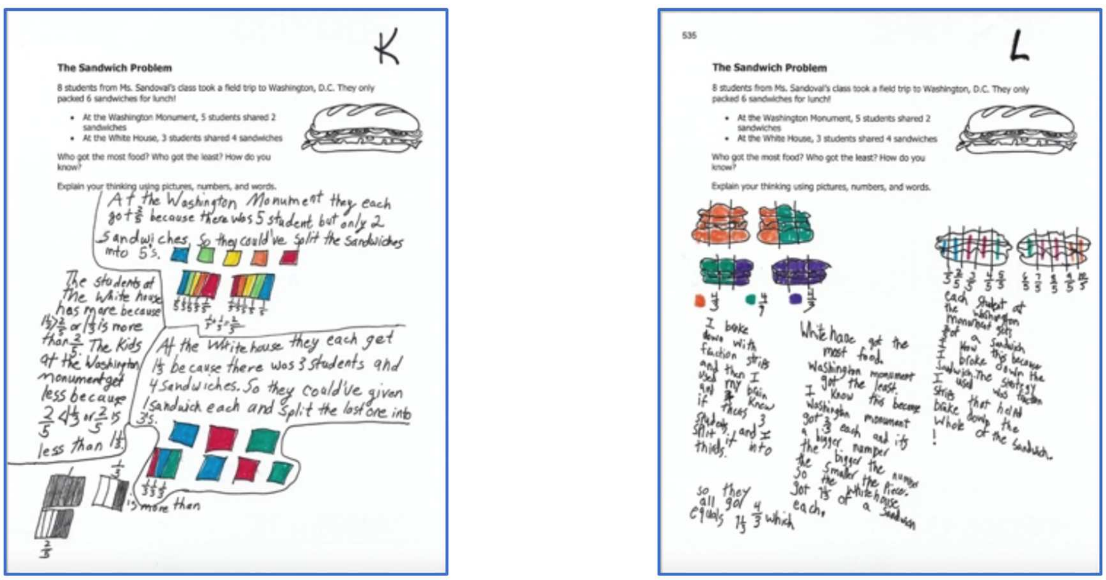


Figure 12. Samples K and L demonstrate two versions of Additive Coordination.

The participants' pairing of student L and student K is in part uninteresting, and fascinating. Faith hesitated to put both students in the same discussion group because their strategies were "too much alike" or "the same," referring to their value to the other. At one point in the discussion Faith mentioned pairing sample K with sample H, and sample L with sample D, but when pressed for her reasoning, she replied, "I'm not 100% clear on that. I'm still trying to figure will it go here?" More importantly, Faith did not engage with the differences between the student strategies, despite the fact that the two strategies represent a meaningful distinction in the CGI strategy sequence.

Isabella had a similar reaction, but spoke more of the students' competency on the task, saying that they "explained well," "had the same level of education," and had the "most sophisticated" strategies. Like Faith, Isabella did not distinguish between the students. Of greater note is the degree of subjectification of these two students. One participant even said, "These guys are easy." In a similar vein, Rachel made subjectifying statements about these students as well, saying, "So this group (Samples H, K, L), they understand the division and the equal groups and kind of naming the pieces..." On the other hand, Rachel did mention that she would change the values in the problem and encourage the students to come up with another strategy to solve this task.

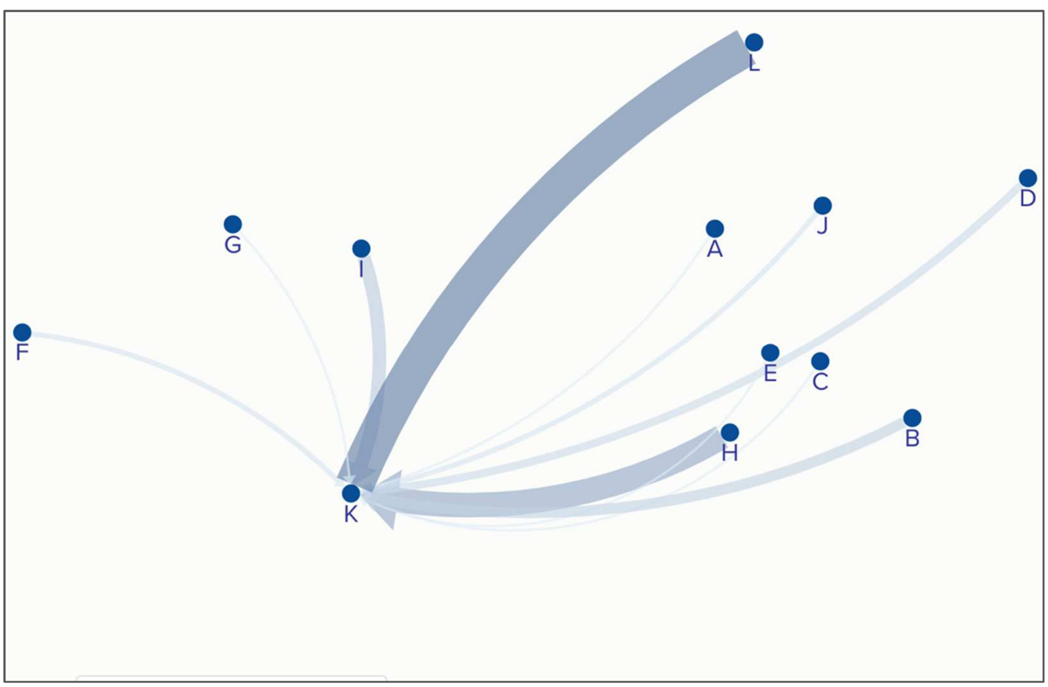


Figure 13. Samples K and L were paired more frequently than any other pair.

The network density map pictured in Figure 13 demonstrates the relative weight of the connection using both depth of color and line weight as an indicator of the strength of their connection. After coding for any instance of grouping or pairing of two work samples, the instances were counted and graphed to show the strength of the connection. In this case the connection between sample K and L is strong while the connection between K and A is weak. Samples K and L are notable because the students who produced the work were paired to be in the same discussion group more frequently than any other two samples in the set. However, using the participant statements as a guide, the instructional "action" in this case may not represent thoughtful and considered decisions. Other pairings may be less frequently mentioned, but they are possibly more meaningful.

Samples E and G. Samples E and G are noteworthy subjects for examination because the coaches frequently treated them as a pair of related exemplars. As a matter of fact these samples were paired together almost as frequently as samples K and L. The students who created samples E and G demonstrate similar skills and challenges with equipartitioning.

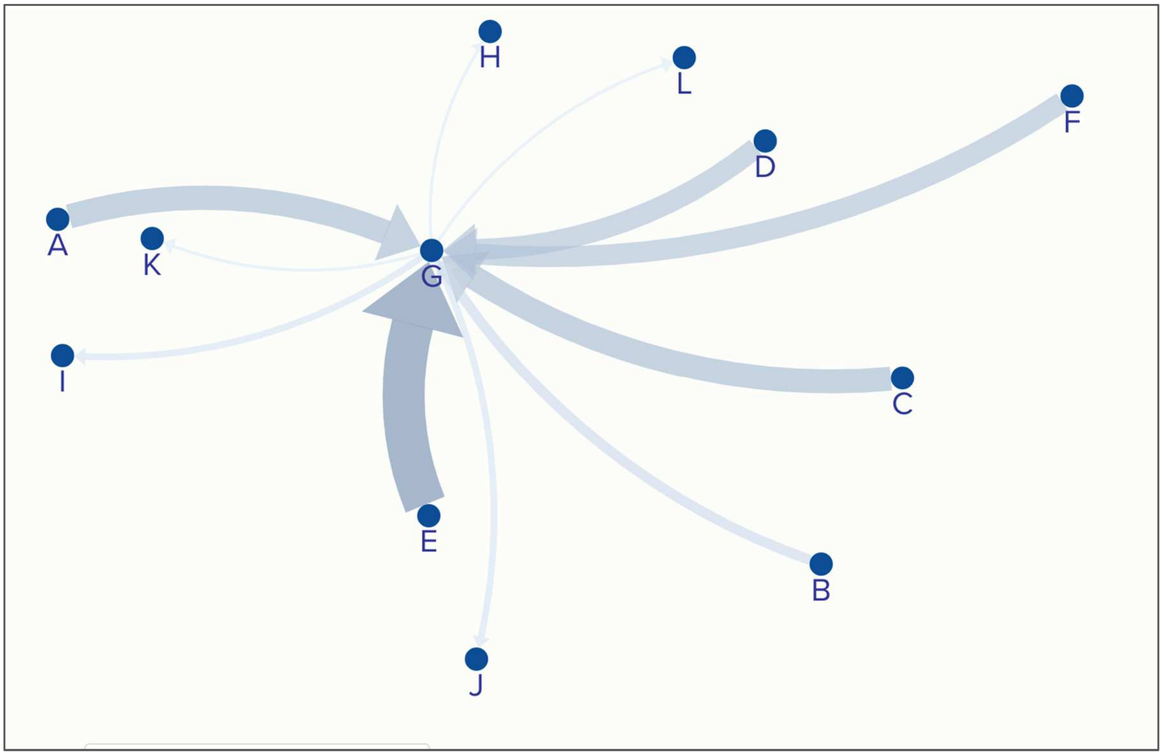


Figure 14. Samples G and E were paired frequently for a similar partitioning error.

Figure 14 shows the network density map for Sample G. The arrows indicate that students A, D, F, C, and particularly E were purposefully placed in groups with student G. The weight and intensity of the shading indicate how frequently the coaches signaled their intention to pair those students in a discussion group. Digging a little deeper into the coaches' statements can illuminate more detail about their intentions in placing student E in a group with student G, as well as the role of the other students' thinking.

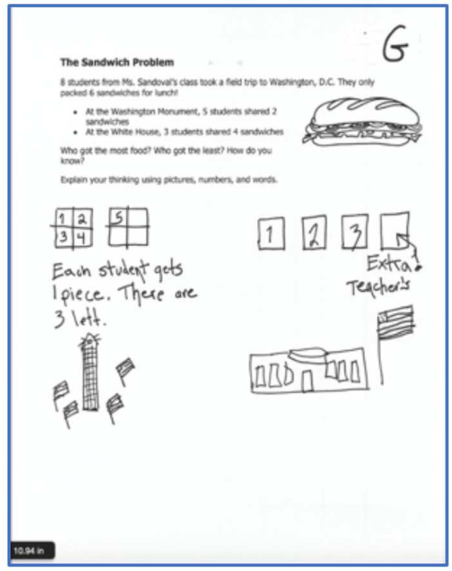


Figure 15. Samples E and G demonstrate three instructional cues.

Samples E and G demonstrate three areas of interest. First, sample E only shows half of the answer, modeling just one of the equal sharing situations of the comparison. Second, sample G shows a Non-Anticipatory sharing strategy, which is evidenced first by the fact that he partitions the sandwiches at the Washington Monument into fourths when the task indicates that there are five sharers, and also by the fact that the fourth sandwich at the White House does not get distributed to the students. The groupings and pairings suggested by the coaches varied in their level of productivity with respect to addressing specific student learning needs. The least productive comments come from observations but lack detail, do not connect to conceptual principles, and do not offer a path forward. This excerpt is one of few examples observed in the three coach interviews: "So I think these guys (Samples E & G) maybe are struggling with equal groups or what to do with leftovers” (Rachel). In contrast, this excerpt from Isabella’s interview offers evidence, but still does not reference a standard or a bridging standard that could help place this piece of work in a learning trajectory:

> Okay. These guys (Samples F, E, D, G) modeled it but had a problem naming it and also with what to do with the extra (Samples G, E ... This one
>
> also had an extra sandwich. I don't know if I would put him with them. He just didn't do anything with the extras (Sample G). (Isabella)

Because E and G have similar early concepts of the “big idea” there are many opportunities to pair them for learning. The first observation is that both student E and student G may not understand the comparison context, particularly student E. Faith targets this concern and prioritizes a plan to ensure their understanding of the context, saying:

> So, let's think about the comprehension piece to ensure that everybody understands what the question is asking. Then, we'll come and start to talk about a couple of the strategies. We'd never share all of these, but then we have to think about how we are going to pull in this child (Sample G) and this child (Sample E). Seems like they had a way to get started, then they weren't real sure what's next. (Faith)

Samples E and G show evidence of similar approaches and Isabella intentionally pairs them together so that four students could share strategies that are complementary.

She groups them and gives their group a directive: “clarify thinking through questions.” Here is the full statement, that better shows her purposeful planning.

> So then it's one whole and then the last one in thirds which is the same as this person (Sample E) and then this person (Sample G) could hear what they did with that extra one. So I kind of like this group together. (Samples A C E G) I don't know what to call them. Clarify thinking through questions. (Faith)

A bit later Isabella elaborated on her thinking, specifying the exact connections she hopes students will make. “I would probably put this person (Sample D) in here (Samples A, C, E, G) just because this one (Sample G), he has a remainder and doesn't want to do anything with it . . .” After matching sample G to sample D, who discussed the idea of remainders, she considered the group as a whole and decided that student G would benefit from working with students A, C, and E.

> just because this one (Sample G), he has a remainder and doesn't want to do anything with it, but then I also feel like he's a little close to this one (Sample A) in that he counted wrong and here he has good thinking, but he counted wrong or divided it up wrong. So I feel like he (Sample G) would fit there. (Isabella)

The differences between the most responsive and least responsive instructional planning action may lie in the attention to details, including an interpretation of the work that focuses on the kernel of students’ “right thinking” (Confrey et al., 2014). It is also grounded in a deep knowledge of meaningful distinctions of mathematical content along a learning trajectory that includes bridging standards, all of which inform the next steps that are taken. ***The case of sample B.*** After many passes through the data it became apparent that sample B captured a great deal of attention from the coaches and deserved a closer look. Sample B was connected with every student work sample. More importantly, student B was not always paired with another student because the work was wanting, nor because it was exemplary.

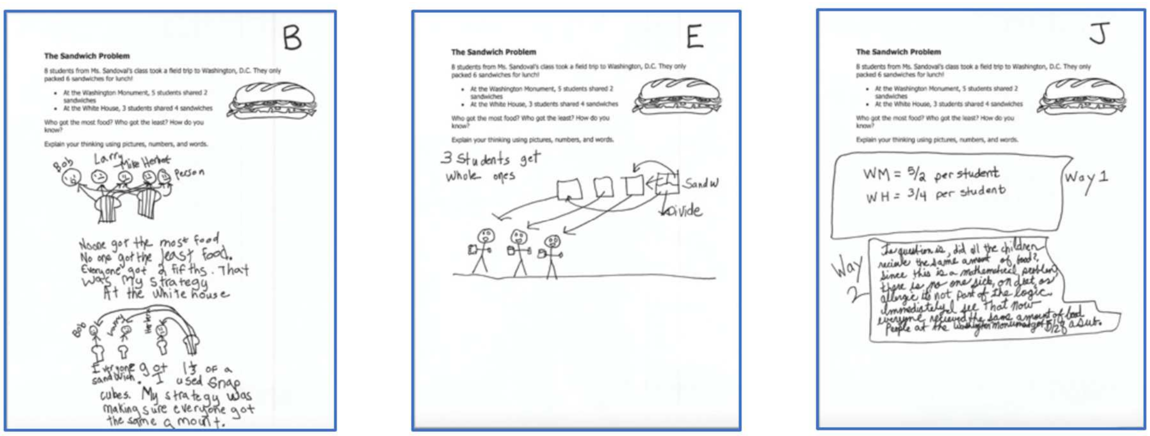


Figure 16. Sample B as mentor and Sample B as partner.

The focus of the pairing was sometimes because of the answer in Sample B, or because of the representation of the partitioning in it, or because of the written explanation given. It’s important to note that at times the coaches positioned the work in Sample B as that of a mentor and in other pairings as that of the mentee. For example, Faith paired Sample B with Sample E the strategies in Sample E and Sample B were similar, although Sample E had not shown a completed representation, as Sample B showed:

> (Samples) B and E, if these guys work together they kind of have this sense of this idea, and they are similar strategies where they might work well together and this would pull this little guy (Sample E) up to continue. (Isabella)

While this partnership was based on similar strategies and designed to mentor student E, Isabella suggested pairing student B with student J so that student B could see an alternative approach. She said, "I think this person (Sample J) needs to talk to this person (Sample B) and see which one is the right answer." Faith made the same observation, saying, ". . . in this case, I might say, 'Can you prove to me that five halves is the same as two-fifths or is it?'" This was a strategic pairing so that student J could see a visual representation of how student B arrived at an answer of 2/5. She was firm in this decision, reiterating it by saying "He (Sample J) needs to see the two-fifths. He (Sample J) needs to be able to see that one."

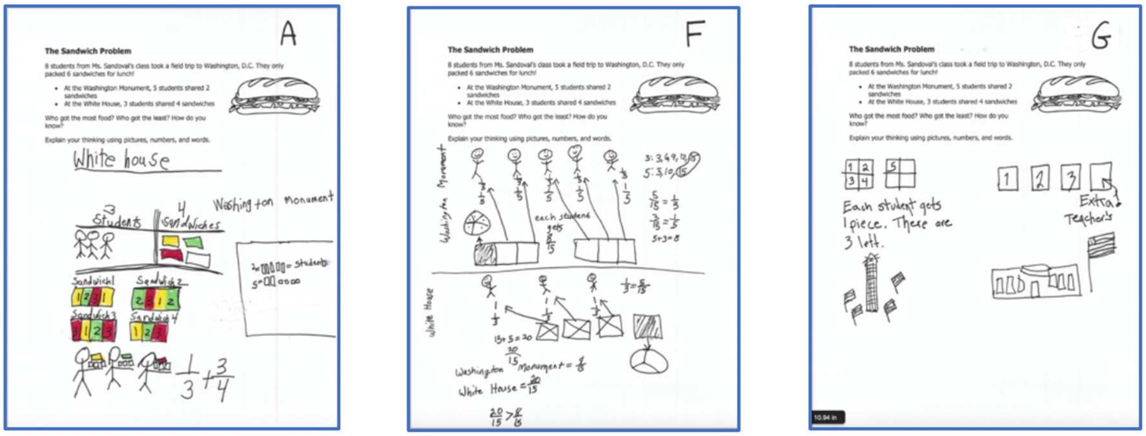


Figure 17. Sample B Links Three Other Strategies.

The work in Sample B also contributed to the variety of approaches or strategies placed in a discussion group. For example, Faith added Sample B to a group with Samples A, F, and G because her strategy was noticeably different from theirs, yet still similar enough to have productive conversations:

> I'm feeling like we need another person. And the reason, oh wait, I did say I would pull this guy. I would pull another person like these (Samples C or B) in because I want to have a mix of strategy levels . . . so that they can build on each other. (Faith)

Despite the fact that Sample B was often held as a potential productive contributor to discussion groups, Sample B also had an answer that challenged the coaches. Rachel was particularly concerned that Sample B did not share an expected answer to the task, despite the mostly appropriate drawn representation of the situation. In Sample B's surprising interpretation of the problem, the students at both field trip destinations had the same amount of food for what was generally considered a comparison task across the two destinations. For example, if there are two sandwiches for five people to share at the Washington Monument, everyone will get the same amount. The other two coaches noted the unexpected answer and also recognized the same answer in Sample J. So while Faith put Sample B with Sample J so that Sample J could have a model for creating a visual representation, Isabella chose to place Sample B in a group with Samples K and L so that they could share their more expected interpretation of the problem situation.

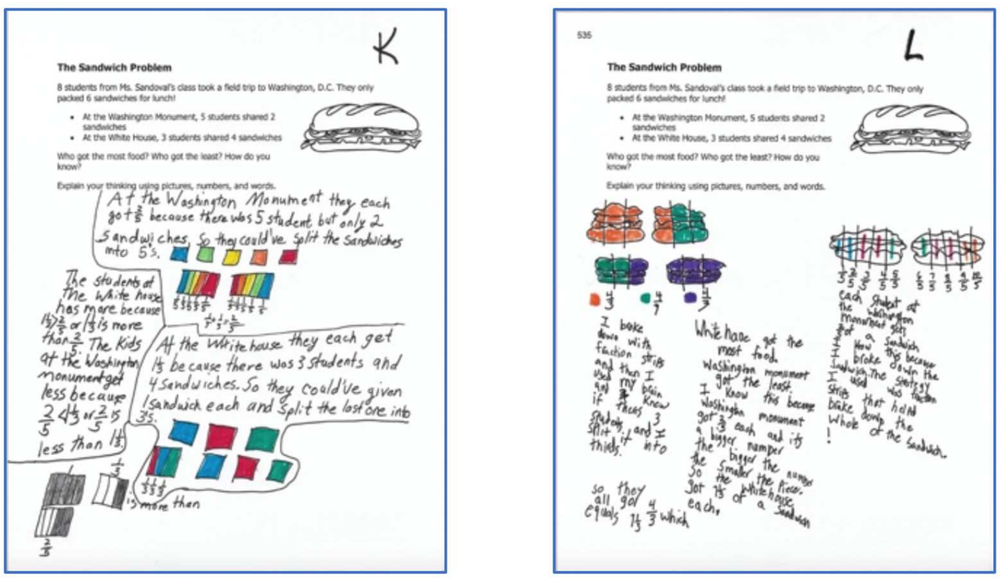


Figure 18. Samples K and L serve as exemplars.

It is important to reiterate that the directions for the sorting did not specify that the coaches sort the samples for instructional purposes, but they all gradually drifted in that direction as they became more familiar with the details of the student work. Sample B stood out during the interviews as a pivotal exemplar because it could be paired so many different ways. The flexibility of its connections to the other samples is visible in the network density map. Only Sample C was connected objectively more times with other samples than Sample B was. Sample C was intentionally paired with Sample H so that they could share their symbolic and pictorial representations of tenths of a sandwich, but Sample C was mostly put in groups because every sample had to be assigned a discussion group. That is what educators do; they make sure every student participates. Sample C was often put just anywhere another representation was needed.

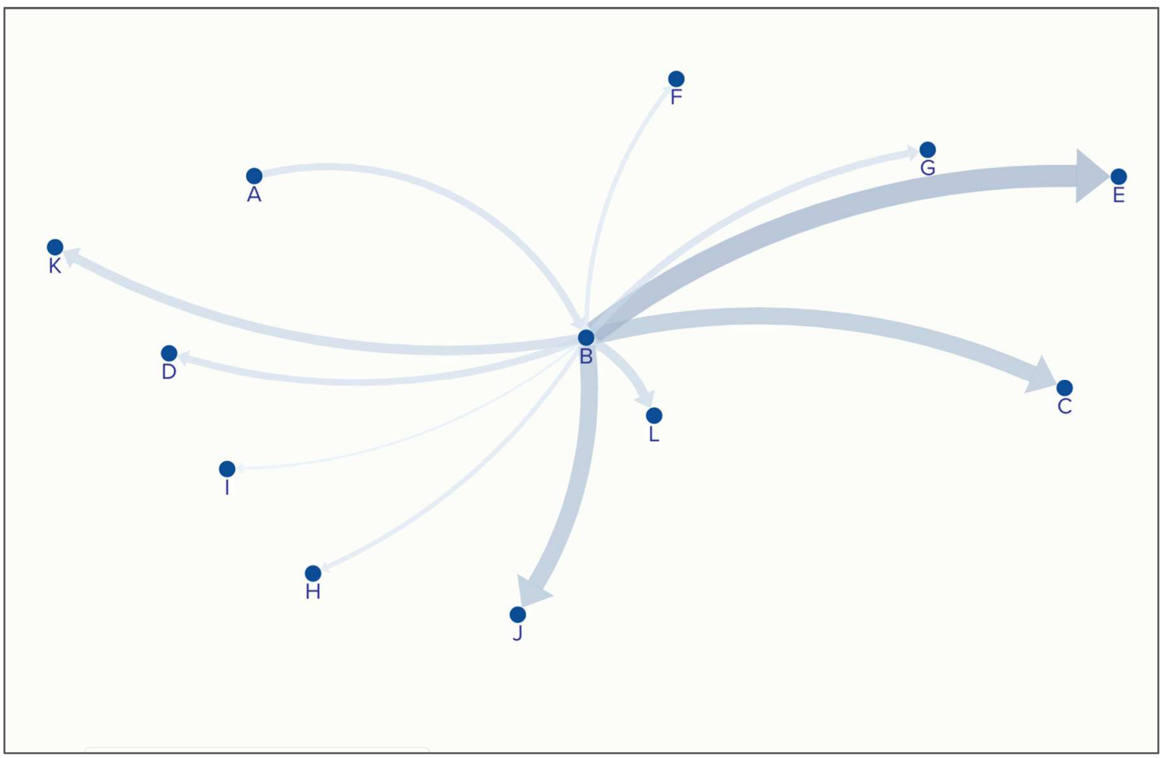


Figure 19. Sample B weighted network analysis.

The network density map for Sample B shows the strength of its connection to Sample E to whom student B served as a mentor. It also shows the strength of its connection to Sample J, as both offered the same unusual interpretation of the problem situation and were paired by the coaches to address that concern. It also shows Sample B's "convenience" connection to Sample C, often tied together because their strategies looked similar but over all were not compatible. The rest of Sample B's connections were weaker, but it is more important to note that Sample B had a connection to every other work sample by at least one of the coaches.

As the coaches made groupings and pairings of student work samples, their reasoning shed light on how they used their current understandings of the mathematical topics that underlie this task. When coaches made note of subtle differences between two work samples, they showed evidence that these differences had instructional import. Similarly, when they did not place two samples in the grouping or a pairing, they were indicating that the students were less likely to offer contributions to a productive conversation. While there is evidence for every pairing noted in this study, it is possible that the order in which the samples were presented, for example A, B, C, may also have had undue influence on the pairings. A clear example is Sample B and Sample C. In the previous network density map, Samples B and C were frequently matched, but there were no statements from the coaches that make a strong argument for why this was true. In future studies it would be advisable to re-letter Samples B and C, and likely samples H and I, and K and L. Or, alternatively, one might offer the stack of papers to teachers *out* of alphabetical, or in random order, to see if the same pairings were prevalent.

Chapter Summary

The purpose of this study was to explore elementary mathematics coaches' references to learning trajectories as they were examining artifacts of student thinking. The first results reported were the resources coaches used in their practice. The sources varied from entire programs that encourage different ways of thinking about mathematics, but they also included resources used to locate daily use activities for classroom teachers. What was not mentioned may be even more important: the coaches did not mention the use of the adopted textbook nor did they mention the use of open source outlets for lesson materials. For these participants the county pacing guide as well as the states curriculum resources were also noted as critically important.

The professional noticing framework provided a lens for reporting additional results from the data. Using information gathered from the participants reflections on the task at the center of the study, including the mathematical goals they assigned to it, the prerequisite skills, and the next steps for that lesson combined to create a hypothetical learning trajectory for each coach. Although there were differences between the three coaches HLTs, the more important finding is that student performance on the task caused each coach to modify their lesson goals significantly, in response to details in the student work. Because most students did not demonstrate skills in equipartitioning that are needed to solve this task, the coaches quickly revised their learning goals, modifying their original proposed learning trajectory, generating a new learning goal. Interestingly, despite the fact that two of the coaches had extensive experience with CGI, they did not make a note of every distinction that is meaningful in that sequence of strategies, as represented in the student work. They both did, however, reference the idea of relational thinking, and end goal for CGI. The coach without those experiences relied on her extensive knowledge of the state content standards to inform her hypothetical learning trajectory.

The second phase of the professional noticing framework encouraged a look at how these coaches interpreted the thinking of the students whose work samples they explored. One interesting finding is that coaches often noticed similarities between students' work samples that were unexpected. The connections they maintained were based on mathematical features in the written work that might have seemed unrelated to the untrained eye, but which nevertheless shared important mathematical details that later proved to be productive connections for grouping students for instruction. Because coaches focused so singularly on mathematics in the student work samples, an additional level of coding explored this language feature in depth. Making the distinction between mathematizing, focusing on the mathematics, and focusing on the subject of the student, quantified and verified the researchers' initial observations about the coaches' focus on mathematics. In general, coaches did not use the students' mathematical work to categorize the students themselves as "deficient" or "successful." Instead altogether these coaches focused a large majority of their time either on the mathematics or on the work presented in that single sample. They did not generalize the students overall competency based on this single sample.

After coaches were asked to interpret the student work, they were asked to sort the papers according to a set of criteria that they established. The similarities and differences that they highlighted for fronted mathematical features that were interesting to them, and meaningful distinctions between samples of student work emerged. Some of the meaningful differences that coaches noted were described and patterns emerged. Some groupings were done for convenience and without thought, but these were the exception and not the rule. Other groupings addressed particular student needs, and these groupings seemed to fall into predictable categories. The pairing and grouping approaches used by the coaches in this study included students whose thinking is similar, while other groupings matched students whose thinking was in stark contrast to each other's'. Even more interesting are the pairings specifically designed to complement each student's demonstrated need, sometimes making pairings that provided each student a partner whose strength complemented another's area for growth. Finally, some partnerships were established so that one student could mentor another, but these partnerships varied widely. In summary, the coaches' pairings and groupings were thoughtful and primarily established in order to meet the needs of each student.

Chapter Five

The purpose of this study was to explore elementary mathematics specialist coaches' references to learning trajectories as they examined artifacts of student thinking in order to understand what elementary mathematics coaches noticed in student work, the resources they referenced in order to make sense of the work and how they referenced them, and how this information is used in practice. Using the professional noticing framework (van Es & Sherin, 2008) as a guide, engagement includes what coaches attend to in student work and the sources of the learning trajectories and learning progressions, broadly defined, that inform their interpretations, and the instructional or coaching decisions they propose for the students or teachers.

1. What evidence of students' mathematical thinking do elementary mathematics coaches attend to while examining students' written artifacts?
2. What learning trajectories or other similar sequencing sources do elementary mathematics coaches reference in order to interpret students' prior, current, and future understandings, based on an examination of student work?
3. How do elementary mathematics coaches use knowledge of learning trajectories or other similar sequencing sources, along with evidence

gathered from artifacts of student thinking, to make instructional and coaching decisions?

Discussion

This study was unique in that it engaged in an exploration of the mathematical content knowledge of elementary mathematics specialist coaches. Using the categorization of the Elementary Mathematics Specialist laid out by McGatha and Rigelman (2017), the three participants were all elementary mathematics specialist coaches. Although many coaches have some additional training in mathematics and leadership, the participants in this study not only had a teaching license, they also have earned a graduate level degree in mathematics education (VMSC, 2016). Although these individuals serve in the role of EMS coaches, the manner in which the coaches approached this task was more akin to the approach teachers might take. This is important because the responses the coaches gave in most cases reflected a teacher perspective rather than that of a teacher leader, therefore, the results can reasonably offer a lens on the expert teacher's approach to the tasks rather than that of the coach.

The data in this study was collected in four different phases. Phase 1 included a demographic survey that gathered information about the elementary mathematics specialist coaches' coaching practice, the instructional resources on which they rely, and additional experiences that may influence their understandings of mathematics. At the conclusion of the survey coaches completed a mathematics task and then recorded how they anticipated students would respond to the task. Coaches were also asked to describe a mathematical goal for the task, name the prerequisite skills, and a possible follow up lesson. The task ended with solutions the coaches anticipated students would share. Phases 2 and 3 took place in a live interview setting. Phase 2 continued the survey during the interview to probe deeper into the coaches' practice with teachers and the resources they use in that practice. Phase 3 asked coaches to examine and sort student work and think aloud about what they were seeing and thinking. Phase 4 was a follow up meeting for clarifying any remaining concerns.

Coaches and their resources. The data showed that coaches relied heavily on state-produced materials, third party materials like Cognitively Guided Instruction (Empson & Levi, 2011) resources, and the local district pacing guide. The instructional resources which coaches count on in their practice and referred to by name during the interview fell into four identifiable categories.

The most common resources cited came from either state or local district sources. The state materials included standards for learning but also included a curriculum framework and lessons that address specific standards. The pacing guide from this county was also cited frequently as an important resource. One coach even stated the planning and pacing guide was "like our bible," while another indicated that the document contained much more useful information than just a pacing guide and standards. It was apparent that these coaches relied heavily on resources from both the county and the state.

The second category of resource was broad: it included the kinds of resources that provide teachers with daily tasks and activities that serve a particular need. Daily number routines, sources for cognitively demanding tasks, and other quickly available resources fill out this category. It should be noted, however, that many of the resources cited in this category are also referenced in the county planning and pacing guide described previously.

The third and fourth categories are uniquely distinguished by the roles they play in guiding practice rather than by the information they provided. It might be more accurate to call the third group “thought guides” than “resources.” For example, the *Developing Mathematical Ideas* (DMI) texts (DMI, 2019) are a resource for professional development, not specifically for classroom use, and it was mentioned as part of one coach’s training yet was also shared as a resource. In the fourth category of resource, there are sources of tasks and activities for students, but the resource is used primarily for information, and guidance and inspiration. The most typical example in the fourth category is the book often called the “Van de Walle book” (Van de Walle, 2010). It not only offers productive activities, it also offers guidance on teaching and learning. In this respect, resources like this one encompass categories two and three, with the notable exception that it is contained within a single resource.

Elementary mathematics specialist coaches formulate hypothetical learning trajectories based on the resources they have available. The resources to which coaches refer directly impact what details they attend to in students’ work, whether those are resources that represent practical daily activities or resources that guide and inspire.

What these data do not tell us are what philosophical or pedagogical sources have impacted the coaches’ practices in the past and influence their current behaviors. With the exception of the CGI materials and the mindset materials from Boaler (2016), the majority of the resources named or referenced were practical. They filled a need in the classroom environment, and defined coaches as resource providers. Yet, despite many similarities in their practice and in their education, the coaches still described school-based practices that differed widely.

Coaches' hypothetical learning trajectories. The hypothetical learning trajectories of coaches are drawn from their general experience teaching and coaching, from specific resources, and of course from local standards, forming a hypothetical learning trajectory that guides both instruction and coaching. What coaches notice and attend to in student work is also a reflection of their understandings and beliefs, which are in turn impacted by the education, training, and other professional activities that built their career.

Despite the fact that coaches cited the same mathematical goal for the sandwich task at the center of this study, their individual hypothetical learning trajectories differed significantly. One coach devised an HLT that drew heavily from the state standards to list prerequisite content, as well for possible current task goals the task addressed. The lessons that should come next were also heavily influenced by the sequence laid out in the state standards. On the other hand, the other coaches drew from the state standards to create an HLT but also drew from strategy sequencing information influenced by Cognitively Guided Instruction (Empson & Levi, 2011). Perhaps more importantly, all three coaches referenced the district pacing guide as an important tool for planning with teachers, specifically citing the additional content that had been added to elucidate the mathematical content.

More generally, elementary mathematics specialist coaches formulate hypothetical learning trajectories based on the resources they have available, and the resources to which they refer directly impact the details they attend to in students' work. Additionally, what coaches understand about learning trajectories impacts how they interpret the work they see. In other words, that which is named is acknowledged.

The importance of the mathematics. The second component of the professional noticing framework moves experts from attending to details in student work to interpreting what they see or hear. The coaches identified significant mathematical details in the student work samples, and in the end, grouped them according to fine-grained mathematical details. In some cases, it is likely that the students would not immediately recognize why they had been grouped together. For example an array representation of fifteenths was paired with a representation of an equipartitioned model. Because of the difference between the two models, it was not immediately apparent why those two students had been grouped, yet the connection was deep and was enough for students to engage in a productive discussion about the mathematics at the core of each representation.

Confrey's (2012) framework for unpacking learner trajectories also underscores the importance of mathematics. The framework acknowledges that when students learn new ideas they approach the learning with existing knowledge and must make sense of and integrate the new information. This inevitably ends in students making developmentally appropriate mistakes and demonstrating common misconceptions.

Educators must be able to anticipate and should be able to find the "kernel of right-thinking” (Confrey, 2012, p. 7) and respond appropriately.

Focusing on the mathematics in student work may be a more productive instructional strategy than simply evaluating students, as evaluating students is fraught with challenges related to equity and identity in the mathematics classroom (Davis, 1997). The coaches in this study demonstrated a deep and connected understanding of the mathematical content in the student work samples. Since these participants were purposely selected as experts, this may imply that experts are more finally attuned to the details in student work. It may also speak to the complexity of the coaches’ work with students of all ages. Not only do they have the obligation to understand fine details of student work, they also have the obligation to use that information in their work with teachers and with other stakeholders in the school community.

Mathematizing and subjectifying. The analysis of mathematizing or subjectifying utterances was presented as part of the interpreting phase of the professional noticing framework. An exploration of the coaches’ interpretations of students’ mathematical thinking, based on the written work samples, shed light on the importance of understanding students’ strategies, both the common and the unexpected. Directing attention to the mathematics in student work is accomplished through mathematizing utterances, ones that focus on the details of the mathematics. In contrast, at times the focus was not directed at the mathematizing work but rather at the student who generated the mathematics. An utterance that refers to the student and his or her actions is a subjectifying utterance. An utterance that holds the mathematics in focus is a mathematizing utterance (Heyd-Metzuyanim & Sfard, 2012).

Heyd-Metzuyanim and Sfard's study (2012) focused on the critical role that identity plays in the formation of students' mathematical experiences. Language that assigns judgment or intrinsic personal value to the students' work can hinder a less successful student from achieving eventual success. They may begin to believe that their success is an innate part of their identity, rather than simply being the outcome of a single event or assignment. Perhaps more importantly, teachers may begin to assign traits and values to the individual that they have generalized from the category that labeled them. Subjectifying language is indicative of something Gee (2018) would call a "categorical error" in his interpretation of Ryle's writings (Gee, 2018). A categorical error assigns a student to a category based on a single metric or score and then erroneously ascribes all properties of the category to the student. This is problematic because one single result from an assessment does not imply that the student embodies all of the qualities one might ascribe to that category. Categorical errors are the inherent risks of the use of subjectifying language. In this study, coaches with some variation, were far more likely to use mathematizing language, thus setting students in the position of being learners and doers, rather than embodying performance on one task.

Overall, when coaches are able to name and describe the details in student work with precision they may be more likely to focus on the evidence of mathematical learning and not subjectify the student as the doer of mathematics.

Responding to student thinking. The final phase of the professional noticing framework is the response. Once the educator has attended to details in student work and interpreted the work, the next natural step is to respond. Because this study involved coaches working in a very controlled environment, they could respond to student work more deliberately and thoughtfully than in a hectic planning or professional development session. The coaches in this study demonstrated a deep and connected analysis of the student work samples that focused heavily on the mathematical details.

Recognizing that students' progression along a learning trajectory occurs in smaller steps than state standards often indicate, it is important for educators to recognize the smaller steps and intricacies of students learning "between the standards" (Confrey, 2012). In other words, there are distinctions that must be made between mathematical ideas in order to distinguish the different responses that students offer. The coaches in this study recognized connections between strategies, connections between models, and found strengths in student thinking that might surprise others and used these connections to create productive parings and groupings of students for future discussion or instruction.

Grouping and pairing strategies. The coaches demonstrated five different strategies for creating these pairings and groups of students based on the details in their mathematical thinking (see table 8).

Table 8

Classifying Utterances as Mathematizing or Subjectifying

Grouping Strategy	Excerpt
Mentor	*He (Sample J) needs to be able to see that one (Sample B).*
Connector	*Well, maybe these two (Samples G, E) could be together just because they're both doing the same thing …*
Contrastor	*I'm feeling like we need another person. And the reason, oh wait, I did say I would pull this guy. I would pull another person like these (Samples C or B) in because I want to have a mix of strategy levels … they can build on each other.*
Complementor	*Yeah. I'm just coming back to see the different things here that I wanted to make sure that Student J is able to understand where the five halves. They've got five halves and we've got two-fifths (Samples J B). So, I want to be able to compare.* *…* *I might say, "Can you prove to me that five halves is the same as two-fifths or is it?" (Samples J B) You know? That's why I put compare strategies.*
Social	*So this one (a group) has three students and this one (another group) has three students. In which case I would look, personality wise, which group would fit best. There you have it. I don't know.*

One of the categories is familiar to most readers. Some students were paired together because one student appeared to have greater knowledge and as such would be able to share their understanding with other students. This is an unbalanced grouping because some students are intended to bring more of a contribution than others. This is a ***Mentor*** pairing.

The remaining three pairing strategies are balanced pairings, meaning that each student is placed in a group because the teacher intends for them to make a specific contribution to the discussion and not to be tutored or to teach.

The first strategy in this group is the *Connector* strategy. The Connector strategy places students in groups because their strategies (or other features) are similar. We often saw work Samples A, B, and C grouped together because each of the students partitioned their sandwiches similarly. Interestingly, their mathematical thinking may still be meaningfully different, but the intention of the teacher is for the students to come together to share their similar approaches.

The second strategy in this balanced group is the *Contrastor* strategy. Students are placed in Contrastor groups so that they can be exposed to a very different idea than the one they themselves produced. For example, Sample B was often grouped with Sample C because each had equipartitioned the area models of the sandwiches in significantly different ways. Note that Samples A and C were paired using both the Connector and the Contrastor strategies by the coaches. This is not to be taken as a contradiction, but rather as an illustration of the intentionality of the coaches in their pairings. The strategy applies to the coaches' *intention* in making the pairing decision.

The third strategy in the balanced group was originally part of the Contrastor group, but it became apparent that the coaches had different intentions when pairing the students together. Drawing on the connection to the definition of a set of angles that make a sum of 90°, the *Complementor* groupings intentionally group students together because their strategies are complementary: one student's thinking fills a need for the other with the reverse true as well. While a normal Contrastor grouping could be more or less balanced, the Complementor grouping is intentionally balanced equally. A good example is pairing together Sample C with Sample H, with the goal of ensuring that Sample C offer strategies for modeling the actual sandwich situation, but also with the intention that Sample H would share their understanding of generating a decimal response. This is a balanced Complementor strategy for grouping rather than a Mentor grouping because Sample H, despite the beyond-grade-level response, shows no evidence of understanding how to generate a visual model of the situation. The model in sample C has modeling strengths to draw from, but it also has vagaries in the drawn models that could stimulate conversation. The coach's intention in this pairing was to spur student H to think about a model, while it pushed Sample C to defend their less than accurate equipartitioning and to move toward recognizing these fractions in decimal form.

The final strategy for grouping is based on the reality of the classroom situation. At one point in the sort Isabella is holding Sample J in her hand and is unsure where to place that student. She stated that she would make note of who was friends with student J, how they worked together in class and then place the student accordingly. This is the *Social* strategy. The strategy is mentioned here not because it is mathematically relevant, but rather because it acknowledges that there often are many reasons for grouping students when all else is equal. But the intention for including Social groupings is to foreground the mathematical strategies in contrast, and therefore highlight a mathematical focus as a favored strategy for grouping learners. It is important to note that while these strategies for grouping students were observed in coaches in the interview environment, they are also applicable for classroom teachers as well.

Implications

This study showed that elementary mathematics specialist coaches drew upon a variety of resources in order to support their planning with teachers, but showed a marked preference for state and local resources. If elementary mathematics coaches frequently draw upon state and local resources, these divisions should recognize their obligation to provide access to rich and exemplary resources.

Learning trajectories can be derived from a variety of sources, but often the sources are presented in ways that are elusive to the school-level practitioner. They may be hidden behind journal paywalls, concealed by technical language, or overlooked because they conflict with current standards. That does not mean that coaches (and teachers) choose their content freely - standards therefore become the sole determiner of the sequence of student concept development. But standards were not designed to be a fine-tuned tool. What coaches understand and describe about learning trajectories impacts how they interpret the work they see. After all, that which is named is acknowledged. An understanding of learning trajectories can help coaches and teachers move from general statements about students' mathematical work to targeting students' specific strengths and needs.

If attention to details of student thinking can result in more thoughtful instructional planning, in this case for pairings for discussion groups, then details about student thinking should be a greater focus for professional development. Even standards documents can incorporate exemplars of student work in order to illustrate common strategies, representations, and misconceptions, building this awareness into coach and teacher planning.

Grouping students for discussion using a variety of pairing techniques has the potential to give students opportunities to serve as both mentor and mentee. By naming strategies for grouping students in pairs or groups for discussion, there is an additional advantage of identifying categories that can be used with intentionality. These grouping strategies for student discussions can offer teachers and coaches, who may still be grouping solely based on the Mentor strategy, a workable set of alternatives. The categories may also be used and further research may determine if and when each strategy is effective.

Limitations

One clear limitation of this study is the limited sample size of three coaches, which is much too small to generate broad conclusions. Similarly, this study was not able to draw conclusions about the elementary mathematics specialist coach in general. Because participants in this study had a wide variety of experiences outside of the mathematics specialist degree program, there were many more sources from which they drew their understandings of learning trajectories to interpret student work, which impacted the data.

This study only addresses one conceptual principle: equipartitioning. Additional study would be needed to verify the findings in other domains. The equipartitioning principle is unique in that it was mapped out by a scholar who has spent a career studying related learning trajectories (Confrey et al., 2014). It is unclear whether there is sufficient detail in other domains to make this possible. Nor is it clear if there is widespread agreement on the nature of bridging standards between grade level standards.

This study positioned the elementary mathematics specialist coach, educated in a specific master of education curriculum, and currently serving as a full time mathematics as an expert. There are certainly many other definitions of "expert" that would yield different results and outcomes. Even the CAEP (2019) and AMTE (2013) standards are not sufficient for encompassing the variety of means by which educators become specialists in elementary mathematics education.

An additional limitation is that this study only addressed elementary mathematics specialist coaches in a single county, which limits its broader applicability. The resources available in every county vary, even in states that share the same common set of standards, and this factor may have been critically important to the results of this study, given the frequency with which the coaches mentioned the county planning and pacing guide.

Next Steps

This study could be replicated with participants from other school divisions which use different planning and pacing guides. While this study was focused on elementary mathematics specialist coaches, positioned as experts, the next phase might explore the engagements of individuals who either are not currently coaches, or have not completed the degree program that these individuals have. Similarly, extending this exercise to practicing teachers may offer a lens on teachers' hypothetical learning trajectories. It may even demonstrate whether coaches and teachers, because of their job descriptions, have different views on what constitutes a hypothetical learning trajectories.

The depths of mathematical understanding that were revealed by this study show that there is great promise in exploring student work as a tool for professional development, either with coaches or not. Realistically, student work is the most plentiful resource in a school building, and coaches can easily design and create discussion protocols or use existing resources for protocols for looking at student work (Baldinger, 2015; Bella, 2004; Blythe et al., 2008; Cameron et al., 2009; Daehler & Folsom, 2014; Kazemi & Loef Franke, 2004). These discussions can take advantage of the plentiful resource and build teachers' understanding of bridging standards, and their skills in recognizing details in student work.

Finally, the student work analysis strategy may also serve as a potential assessment tool for coaches' mathematical knowledge for teaching, as it may go into more depth than the other assessment tools currently available.

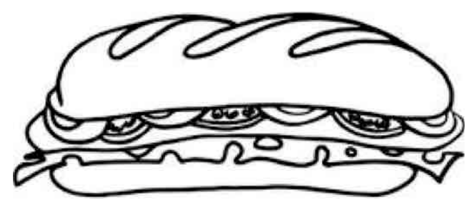
Appendix A

The Task

The Sandwich Problem

8 students from Ms. Sandoval's class took a field trip to Washington, D.C. They only packed 6 sandwiches for lunch!

- At the Washington Monument, 5 students shared 2 sandwiches
- At the White House, 3 students shared 4 sandwiches



Who got the most food? Who got the least? How do you know?

Explain your thinking using pictures, numbers, and words.

Appendix B

Virginia Mathematics Specialist Program Sample Syllabi

VMSI PROGRAM OVERVIEW AND ANNOTATED SYLLABI 9

Overview of the Virginia Mathematics Specialist Preparation Program

Twelve Virginia universities currently offer a master's degree program to prepare mathematics specialists, core mathematics, and leadership courses make up the programs in addition to some unique set of program expectations defined by the university. What follows is a description of the core courses as developed and piloted with support through the series of four five-year NSF projects, listed previously, during the collaborative efforts of Virginia Commonwealth University (VCU), University of Virginia (UVA), Norfolk State University (NSU), and Longwood University (LU) under the VMSC Statewide Master Degree Programs initiative (VMSC, n. d.).

Mathematics Content and Leadership Courses

From the beginning, designers of the VMSPP realized that teaching courses to prepare teachers for a mathematics specialist role would be unique; instructors would be teaching the coaches of the mathematics teachers. The 2002 Task Force Report highlighted, modifying existing college mathematics courses such as number theory, geometry, or algebra would not meet the needs of mathematics specialists (VMSC, 2005, p. 16). New 3-hour graduate level mathematics and mathematics education leadership courses needed to be created. Courses were needed that connected mathematics content knowledge to content pedagogical knowledge and that allowed teachers to understand the developmental progression of mathematical ideas necessary for planning instruction and assessment. In addition, these future leaders would need to recognize how the information from their coursework bridged to their own teaching practices and how information from their coursework would be reflected in the coaching practice. The Middle School Mathematics Specialist Task Force (2008) supported the recommendations made by the 2002 Task Force and reiterated that, "helping participants recognize how assignments from their coursework translated into their practice both as teachers and as coaches is a critical obligation of the curriculum and the course instructors" (VMSC, n. d., p. 16-18).

A brief description of six mathematics content and three mathematics education leadership courses follow. An annotated syllabus for each of these courses is located in the following section of this report.

Numbers and operations. This introductory course addresses fundamental mathematical ideas concerning the operations of arithmetic and the base-ten number system. Connections between the operations are explored in various contexts including whole numbers, problem solving, decimals, and fractions. The structure of the number system is used to develop understandings of our base-ten system. The course also uses cases about students' thinking and the computational methods they use and episodes in the history of the number system that illuminate the developmental progression of the mathematics and the learning trajectories of children.

Rational numbers and proportional reasoning. In this course, students explore the conceptual and procedural basis of rational numbers; fractions, decimals, and percents as well as the essential role that proportional reasoning plays in mathematics. The logic, estimations, interpretations, and procedures used when ordering and computing with fractions and decimals are explored using multiple interpretations and representations including visual and physical

Appendix B (Continued)

Virginia Mathematics Specialist Program Sample Syllabi

representations. Episodes from the history of the number systems are explored and compared with the developmental sequence and learning trajectories of children learning this material.

Algebra and functions. Students develop skills in representation, generalization, and development of mathematical arguments through the exploration of the properties of arithmetic operations, the relationship between operations and operating on particular numbers. Additional topics from algebra that are explored are variables, patterns, and functions; modeling and interpretations of graphs; linear functions and non-linear functions, including quadratics and exponentials.

Algebra for middle school specialists. This course extends the understanding of topics introduced in the Functions and Algebra I course, introduces new topics from secondary mathematics, and integrates graphing technology into the study of the algebra topics. Class activities focus on extending students' skills in representation, generalization, and developing mathematical arguments. Topics include but are not limited to linear equations and inequalities; modeling and interpreting graphs; linear and non-linear functions; logarithms; factoring, zeros, and intercepts; domain and range; exponents and radicals; and some number theory related to the real number system.

Geometry and measurement. This course explores the foundations of informal geometry and measurement in 1, 2, and 3 dimensions. The van Hiele model for geometric learning is used as a framework to explore how children build their understandings of length, area, volume, angles, and geometric relationships. Visualization, spatial reasoning, and geometric modeling are stressed along with transformational geometry, congruence, and similarity.

Probability and statistics. Various elementary statistical measures and graphical representations are used to describe, compare, and interpret data sets. The basic laws and concepts of probability are explored including sample spaces, probability distributions, and random variables. A statistical project is required that uses hypothesizing, experimental design, the collection of data, and comparisons of different populations.

Leadership I. This introductory course is designed to build an understanding of the content and process standards identified by the National Council of Teachers of Mathematics (NCTM) *Principals and Standards for School Mathematics* (2000) and the K-8 *Virginia Mathematics Standards of Learning and Curriculum Framework*. In addition, connections are made within the mathematics content as participants develop their knowledge about mathematics, mathematics content pedagogy and diagnosing student understanding. A focus is given to students as mathematics learners with attention to learning theory, formative assessment, and diverse learners; teachers as learners through study groups and observation of another teacher's classroom; and the instructional program through the design, teaching, and evaluation of student-centered lessons.

Leadership II. This course is designed to build skills, understandings, and dispositions required for optimal mathematics education leadership roles in K-8 schools; in particular the different roles of the school-based mathematics specialist. The course develops skills to coach and work

Appendix B (Continued)

Virginia Mathematics Specialist Program Sample Syllabi

with adult learners, understanding mathematics content pedagogy necessary to support teachers, using research in selected topics for instructional decision making, and building deeper understandings of the mathematics that underpins the K-8 mathematics curriculum.

Leadership III. This course builds skills, understandings, and dispositions required for optimal mathematics education leadership roles in K-8 schools; attention is given to data analysis and collaborative data-driven discussions for instructional planning and for mathematics program decision making. In addition, students engage in learning to participate in and to facilitate the Lesson Study process; to develop and use formal and informal formative assessments to guide instruction; to develop or modify tasks for effective task-based mathematics instruction, and to support other teachers effective mathematics lesson planning.

Considerations when Planning a Course

Courses in the VMSPP have been taught in various formats, and each format presents different advantages and challenges to the students and the instructors. As part of the VMSI, courses have been offered in residential summer institutes with about 55 hours of class time and significant daily in class and homework assignments including readings, doing mathematics, and writing reflection papers. As traditional semester classes, taught in 15 three-hour weekly sessions each with homework assignments including readings, doing mathematics, writing reflection papers, and writing cases. A third option, more often used with the leadership courses was to split the time between summer sessions and Saturday classes.

Summer residential institute format. Students who participated in a program offered entirely as residential institutes took courses each of three consecutive summers. Content courses were taught simultaneously during the first summer institute so that on a given day participants experienced one course in the morning and another in the afternoon for the five weeks. Feedback on this schedule was not as positive, so the schedule was adjusted to have one course follow the next. In the following summer institutes, two content courses were offered in succession over a five week period. Specifically, the two content courses were offered in intense 2½-week sessions designed for two 3 ½ hour blocks per day.

In addition, to the two content courses each summer, the first half of a leadership course was also scheduled, held four times spread out over the five-week institute. The second half of the leadership course was held on four Saturdays spread throughout the fall semester for 6 hours each time. This allowed participants to work with students and teachers in their schools when completing class projects.

Semester format. Students enrolled in a semester content course met one night a week for 3 hours. During the summer, the course was taught either for two full weeks or for two or three days spread over several weeks. Leadership courses were generally offered in the fall semester sometimes overlapping into the second semester which gave participants more time to work with students and teachers in completing class projects.

Appendix B (Continued)

Virginia Mathematics Specialist Program Sample Syllabi

Blended learning format. The core courses in the Mathematics Specialist Program were originally designed to be face- to-face courses. However, in reaching out to rural school districts across Virginia in one of the NSF grants, *Research the Expansion of K-5 Mathematics Specialist Program into Rural School Systems*, it was evident that travel to class would be a major obstacle to teachers who wanted to participate. In an effort to address the travel challenge or teachers, the program was offered in a combination of the hybrid formats; blended and residential summer institutes as well as blended semester courses.

Technology allowed the face-to-face courses to be repurposed to fit a blended format. To maintain the cohesiveness of a cohort, the blended courses met twice for a 2-day, Friday and Saturday, face-to-face meeting, at the beginning and then midway through the course, with the remainder of the classes meeting synchronously online. The sequence of courses remained similar to the original design with a few exceptions. Each cohort began with the *Number and Operations* course. The use of Digimemo L2, while not without challenges, in conjunction with a universities online collaboration platform allowed students to share work and to participate in small group chat rooms. This allowed online classes to be dynamic and interactive. The online classes used whole-group and small-group real-time discussions in class. Group projects were assigned with the expectation that students would use the online infrastructure to meet with their groups online. Formative evaluation reports from Horizon Inc. made to the course development teams about the change in students' knowledge as measured on pre- and post-course assessments in the blended courses revealed no significant difference from students who participated in the face-to-face course.

The cohort program for the K-5 Mathematics Specialist Program into Rural School Systems grant included three face-to-face mathematics content classes, and three other core content classes were offered as blended classes. The face-to-face class offered the first summer was *Geometry and Measurement*; the second summer was *Algebra and Functions,* and the third summer was *Mathematics for Diverse Populations*. An additional course was taught in the blended format each of the first two summers; *Rational Numbers and Proportional Reasoning* and *Probability and Statistics*. All three leadership courses were taught in a blended format during the school year. *Leadership I* and *II* were taught the fall and spring semester following the first and second summers respectively and *Leadership III* followed the third summer institute during the fall semester.

The format in which a course is offered impacts participant experiences in different ways. In the summer residential institute format, students are immersed in the work and have the opportunity for additional collaboration with their peers after class hours. Participants do not, however, have the opportunity, as they do in the school year semester format, to do the mathematics with their own students, to interview students about their understanding of the mathematics, or write their own case studies. During the school-year format, there may be more time between classes for reflection and making connections than is readily available during the residential institutes when classes meet all day on consecutive days. Some participants reported, however, that the residential institutes allowed them to focus on the coursework, and they were not interrupted by the daily demands of home or work. Time can become an issue in any format, so careful planning and pacing are essential.

Appendix C

Alignment of Research Questions to Data Collection

Research Questions Data to be collected	Questions and Tasks
Demographic information: Verifies that the coach's work situation matches the eligibility requirements and informs the researcher's understanding of the coach's context. **Data Source:** • Demographic Survey	Last Name, First Name How long were you a teacher? What grades have you taught? How long have you been a math coach? What grade levels have you coached? What licensure do you currently hold in Virginia? What endorsements do you hold? How long have you been eligible for the Virginia Mathematics Specialist endorsement? At what university did you do your MS coursework? Did you take any course outside of a Virginia university that applied to your endorsement? If so, which one(s)? What math courses did you take in high school? What math courses did you take as an undergraduate? Are there any other math courses that you have taken besides those included above, including long term professional development courses offered by your district? As a coach, I think mathematics is _________ As a learner, I think mathematics is ________ As a child, my relationship with mathematics was _______.

Appendix C (Continued)

Alignment of Research Questions to Data Collection

Research Questions Data to be collected	Questions and Tasks
Demographic information: Verifies that the coach's work situation matches the eligibility requirements and informs the researcher's understanding of the coach's context. **Data Source:** Demographic Survey	Name of your school ______ Principal _______ Do you know what funds support the position you hold (e.g. resource teacher, Title I, etc.)? Number of students in the school (approximate) Number of teachers you coach Grades of teachers you coach Can you tell me something I need to know about the students in your school? What percentage of your typical week is spent doing the following tasks? ☐ Coaching individual teachers outside of classroom ☐ Co-teaching, modeling, observation, or other classroom-based coaching activity (including planned pre- and post-classroom discussions) ☐ Leading a CLT/PLC? ☐ Conducting professional development ☐ Working with a small group of students ☐ Gathering resources for teachers ☐ Providing support for instruction (test administration, copying, preparing materials for instruction, etc.) ☐ Any task directly supporting parents ☐ Working with administration on projects related to mathematics ☐ Substituting for a teacher in a classroom ☐ Other? _________________

Appendix C (Continued)

Alignment of Research Questions to Data Collection

Research Questions Data to be collected	Questions and Tasks
Demographic information: Verifies that the coach's work situation matches the eligibility requirements and informs the researcher's understanding of the coach's context. **Data Source:** Demographic Survey	The best part about being a math coach is ___________ . The most challenging part of being a math coach is ________ What is your most important go-to professional resource? Why did you choose that resource? What resources do you bring with you when you do planning with teachers, either long term or short term planning? Why?
How do elementary mathematics coaches use knowledge of LTs or other similar sequencing sources, along with evidence gathered from artifacts of student thinking, to make instructional and coaching decisions? **Data Sources** • Coach's solution to the task • Coach's list of anticipated student responses and reasoning for them • Coach's list of prerequisite skills for students and reasoning for them.	**The Task** 1) Please complete the task as it is designed for students. Please show your preferred solution strategy and at least two additional solutions or representations. 2) On the second page please state the mathematical goal for which this task is appropriate, and for what grade(s). You may have more than one answer to this question. 3) Record the strategies and approaches that you expect students would use to solve the task. Please include reasons for your thoughts. For example, what tells you that students would solve the task that way?

Appendix C (Continued)

Alignment of Research Questions to Data Collection

Research Questions Data to be collected	Questions and Tasks
How do elementary mathematics coaches use knowledge of LTs or other similar sequencing sources, along with evidence gathered from artifacts of student thinking, to make instructional and coaching decisions? **Data Sources** • Audio of part 1 of the interview	**Interview questions to ask first (before looking at samples):** o When you plan with teachers, what resources to you keep at the ready? Why? o If you finish working with a teacher or a CLT and promise to send a resource to them, where is it likely to come from, and why? o In the survey you mention ______, can you tell me more? o What are the big mathematical concepts that are addressed by this task? Tell me more. Why do you think that? o Tell me about what students need to understand or know how to do in order to be successful at this task. How do you know that these are necessary skills? o Tell me some of the ways students might solve this problem. Why do you think that? (probing…) Are there any other incorrect (correct) ways to approach the problem? o Tell me what you were thinking about as you were working on the task on your own? You can talk about either the task or the goals or the anticipated student approaches. o What would surprise you to see on a student's paper? Why?

Appendix C (Continued)

Alignment of Research Questions to Data Collection

<table>
<tr><th>Research Questions
Data to be collected</th><th>Questions and Tasks</th></tr>
<tr><td>

What evidence of students' mathematical thinking do elementary mathematics coaches attend to while examining students' written artifacts?

Data Sources

- Video of coach sorting the student work samples (part 2 of the interview)
- The written comments made by the coach about the student work

</td><td>

Introduce Work Samples

- I have 12 examples of student work on the sandwich
- I am asking you to look at these samples just as if these were students in your own school, and you need to assess this task along with your teachers.
- You can write comments on the work.
- Please share your thinking with me as you are looking at the work. I am interested in what you are thinking
- Are you ready to begin?

During the Examination of the Examples

- Please take a **look** at these work samples.
- Could you **sort** these papers? You can use whatever criteria you would like, but please share your thinking with me as you are sorting. I'd like to know your reasons.
- Could you **explain** how you are sorting the papers? What are you looking for? How are you making your decisions?
- As individual student work samples are considered, ask these questions:
  - What would you say if you approached this student's desk and saw this paper?
  - Could you tell me more about why you think that?

</td></tr>
</table>

Appendix C (Continued)

Alignment of Research Questions to Data Collection

<table>
<tr><th>Research Questions
Data to be collected</th><th>Questions and Tasks</th></tr>
<tr><td></td><td>o What might this student be thinking?
o What do you think they understand?
o What understandings are they still working on? How do you know?
o What should the class learn next?</td></tr>
<tr><td>What LTs or other similar sequencing sources do elementary mathematics coaches reference in order to interpret students' prior, current, and future understandings, based on an examination of artifacts of student thinking?

Data Sources

• Video of coach sorting the student work samples
• The written comments made by the coach about the student work
• Stacks of student work sorted according to coach's perception</td><td>During the Examination of the Student Work
o Please take a look at these work samples.
o Could you sort these papers? Please sort them from the student who appears to struggle with this mathematical idea to the student who is the most successful. You can use whatever criteria you would like, but please share your thinking with me as you are sorting. I'd like to know your reasons.
o Could you explain how you sorted the papers? What were you looking for? How did you make your decisions?
o What would you say if you approached this student's desk and saw this paper?
o What might this student be thinking? What do you think they understand? What understandings are they still working on? How do you know?
o What should the class learn next?</td></tr>
</table>

Appendix C (Continued)

Alignment of Research Questions to Data Collection

<table>
<tr><th>Research Questions
Data to be collected</th><th>Questions and Tasks</th></tr>
<tr><td>Data Sources

• Interview questions regarding coaches’ recommendations for future work
• Video recording of coach</td><td>After the Sorting
Reflecting on the Task
o Was it important for you to solve the problem first before looking at this student work? What did you learn from it?
o Have you reconsidered the big mathematical concepts after looking at these samples? For which grade levels?
o What concepts do you now think the teacher was trying to assess with this task? Why?
o As a coach, what would you like to hear the teacher talk say about these samples?
Using the student work in the coaching practice
o How would you use this set of student work while working with a teacher or team of teachers as their coach? What goal would you want to achieve? How would you use this set of samples help you to achieve that goal?
o Have you used student work during CLT’s or team meetings before?
o Is there anything else that you want to share?</td></tr>
</table>

Appendix D

Institutional Review Board EXEMPT Status

GEORGE MASON UNIVERSITY

Office of Research Development, Integrity, and Assurance

Research Hall, 4400 University Drive, MS 6D5, Fairfax, Virginia 22030
Phone: 703-993-5445; Fax: 703-993-9590

DATE:	June 10, 2019
TO:	Jennifer Suh, PHD
FROM:	George Mason University IRB
Project Title:	[1440807-2] Mathematics Specialist Coaches' Engagment with Artifacts of Student Thinking
SUBMISSION TYPE:	Amendment/Modification
ACTION:	DETERMINATION OF EXEMPT STATUS
DECISION DATE:	June 10, 2019
REVIEW CATEGORY:	Exemption categories #2 & 3

Thank you for your submission of Amendment/Modification materials for this project. The Institutional Review Board (IRB) Office has determined this project is EXEMPT FROM IRB REVIEW according to federal regulations.

Please remember that all research must be conducted as described in the submitted materials.

Please note that any revision to previously approved materials must be submitted to the IRB office prior to initiation. Please use the appropriate revision forms for this procedure.

If you have any questions, please contact Kim Paul at (703) 993-4208 or kpaul4@gmu.edu. Please include your project title and reference number in all correspondence with this committee.

Please note that all research records must be retained for a minimum of five years, or as described in your submission, after the completion of the project.

Please note that department or other approvals may also be required to conduct your research.

GMU IRB Standard Operating Procedures can be found here: https://rdia.gmu.edu/topics-of-interest/human-or-animal-subjects/human-subjects/human-subjects-sops/

This letter has been electronically signed in accordance with all applicable regulations, and a copy is retained within George Mason University IRB's records.

- 1 -

Generated on IRBNet

Appendix D (Continued)

Institutional Review Board EXEMPT Status

An Exploration of the Elementary Mathematics Specialist Coaches' Construction of Student Learning Trajectories

This research is being conducted by Dr. Jennifer Suh and Kimberly M Leong at George Mason University. The purpose of this study is to explore how mathematics coaches examine artifacts of student thinking and use this information in their coaching practice. As a whole we do not know much about how the additional coursework and knowledge in mathematics that Virginia mathematics specialists have acquired impacts their understandings of student learning. Little is also known about how they use this knowledge in their professional practice when they are in the role of mathematics coach. This study explores the coaches' knowledge of and engagement with student thinking.
If you agree to participate in the study, the research is a 3 phase process.

- Phase 1: (following recruitment) includes a demographic survey, a request for a solution to a mathematics problem, and a prediction of students' responses to the problem. This phase should take about 60 minutes.
- Phase 2: The second phase is a one-on-one interview which includes questions about coaching practice, a task sorting student work, and the use of the information to make recommendations for future instruction and coaching. The interview session will be videotaped and audio recorded. Only your hands and voice will be captured in the video. The interview will likely last 75 minutes in a private room.
- Phase 3: The final phase is scheduled online and is designed for member checking transcripts and other data gathered. This phase will take about 30 minutes.

All questionnaires, interview questions, and the mathematical task will be shared in a Google Form or with a link to a Google doc. The estimated time for each phase includes: Recruiting survey - 15 minutes, Demographic survey - 30 minutes, Complete math task - 30 minutes, Interview with doctoral candidate - 75 minutes, Follow up member checking interview - 30 minutes. The interview will take place face-to-face at an appropriate location convenient for participants. Individual work on surveys and the math task will take place independently at the participants' preferred location. The follow up meeting can take place via a Blackboard Collaborate Ultra meeting room.

There are no foreseeable risks for participating in this research. Additionally, there are no benefits to you as a participant other than to further research in the teaching and learning of mathematics.

The data in this study will be confidential. If you agreed to participate, your name will be removed from all submitted work and in its place will be a code to identify you. Through this code the researchers will be able to link your work to your identity. Only the researchers will have this code. No identifying information about you, your job or your school will be included in any presentation or paper published about this study.

Your participation is voluntary, and you may withdraw from the study at any time and for any reason. If you decide not to participate or if you withdraw from the study, there is no penalty or loss of benefits to which you are otherwise entitled. For the interview, only participants that agree to be interviewed will sign, "I have read this form and agree to be a part of this study. I agree that the interviews will be video and audio-taped." Dr. Suh's contact information is included at the bottom of this form. If you have any further questions with regards to this research you may contact them using this information.

Dr. Suh 703 973 4642

jsuh4@gmu.edu

IRBNet number: 1440807-1

IRB: For Official Use Only

GEORGE MASON UNIVERSITY
Institutional Review Board

Project Number: 1440807-1

Page **1** of **1**

Appendix E

Semi-Structured Interview Protocol

Thank you for agreeing to take part in this study. I appreciate your willingness to share your thinking with me, and to contribute to our understanding of the work of mathematics coaches.

I am not evaluating any student's, teacher's, or coach's performance, but rather I am trying to understand the process of assessing student learning through their written work and using it for professional development.

Do you mind if I record our session?

Questions to ask first (before looking at samples):

Planning

- o When you plan with teachers, what resources to you keep at the ready? Why?
- o If you finish working with a teacher or a CLT and promise to send a resource to them, where is it likely to come from, and why?
- o In your survey you mentioned __________ , can you tell me more about it?

The Task

- o Tell me what you were thinking about as you were working on the task on your own? You can talk about either the task or the goals or the anticipated student approaches.
- o What would surprise you to see on a student's paper? Why?

Introduce Work Samples

- o I have 12 examples of student work on the sandwich. I am asking you to look at these samples just as if these were students in your own school, and you need to assess this task along with your teachers.
- o You can write comments on the work.
- o Please share your thinking with me as you are looking at the work. I am interested in what you are thinking
- o Could you **explain** how you are sorting the papers? What are you looking for? How are you making your decisions?

Appendix E (Continued)

Semi-Structured Interview Protocol

- As individual student work samples are considered, ask these questions:
- What would you say if you approached this student's desk and saw this paper?
- Could you tell me more about why you think that?
- What might this student be thinking? What do you think they understand? What understandings are they still working on? How do you know?

After the Sorting

Reflecting on the Task

- Was it important for you to solve the problem first before looking at this student work? What did you learn from it?
- Have you reconsidered the big mathematical concepts after looking at these samples? For which grade levels?
- What concepts do you now think the teacher was trying to assess with this task? Why?
- As a coach, what would you like to hear the teacher talk say about these samples?

Using the student work in the coaching practice

- How would you use this set of student work while working with a teacher or team of teachers as their coach? What goal would you want to achieve? How would you use this set of samples help you to achieve that goal?
- Have you used student work during CLT's or team meetings before?
- Is there anything else that you want to share?

Are you ready to begin?

Appendix F

Informed Consent

INFORMED CONSENT FORM

An Exploration of the Elementary Mathematics Specialist Coaches Construction of Student Learning Trajectories

RESEARCH PROCEDURES
This research is being conducted to study how the knowledge and application of student learning trajectories influences the mathematics coaches' practice of working with teachers.

The study consists of three parts:

1. The first part is online and includes a recruitment survey (15 minutes). It also includes a demographic survey, the completion of a mathematics task and a list of anticipated student strategies. Your work will be collected and used for the study (60 minutes).

2. The second part is a face-to-face interview arranged at your convenience. During the interview you will answer questions about student thinking, resources you use in your practice, and your work as a mathematics coach. You will also look at a collection of student work samples and talk about your professional assessment of the student thinking. All work samples will be collected. The interview will include video (and audio) taping of the interview, including only your hands and possibly the clothes you are wearing. An image of your face will not be collected (75 minutes).

3. The third phase is an online follow up meeting to verify that your interview transcript's accuracy (30 minutes).

If you agree to participate, you will be asked to allow the researchers to analyze your work on a mathematics task, your predictions of student thinking, your words and reflections on the student work samples and video-recordings of your interview and analysis of student work in this project for researching purposes. No additional assignments will be required of you.

RISKS:
There are no foreseeable risks for participating in this research.

BENEFITS
There are no direct benefits for participating in this study.

CONFIDENTIALITY:
Only the researchers will be able to view and have access to the video recordings. Your face will not be included. The data in this study will be confidential. All forms used in this study will be coded with an ID number. (1) Your name will not be included on the collected data; (2) a code will be placed on the collected data; (3) through the use of an identification key, the researchers will be able to link the assignments to your identity; and (4) only the researchers will have access to the identification key. All personal information collected will be stored securely in a locked folder on a computer, only accessible to researchers.
All audio/video taped material will be collected on a dedicated and password protected iPad during

IRB: For Official Use Only

George Mason University
Institutional Review Board

Project Number: 1440807-2

Page 1 of 3

Appendix F (Continued)

Informed Consent

collection and uploaded to a password protected cloud folder. A digital media copy will be stored and protected in a locked file under the supervision of the PI. The audio/video taped material will be stored for at least 5 years after the study ends and will be destroyed after that.
While it is understood that no computer transmission can be perfectly secure, reasonable efforts will be made to protect the confidentiality of your transmission.
The de-identified data could be used for future research without additional consent from participants.

PARTICIPATION

1. Participants should have completed the majority of the coursework required to apply for an endorsement on their license as a Virginia Mathematics.
2. Participants should be a full-time mathematics leader or coach who works directly with teachers most of the workweek (>85% of normally scheduled work hours).
3. Completion of the majority of the Mathematics Specialist coursework at a Virginia university
4. Participants must be willing to participate in this research study.

Your participation is voluntary, and you may withdraw from the study at any time and for any reason. If you decide not to participate or if you withdraw from the study, there is no penalty or loss of benefits to which you are otherwise entitled. There are no costs to you or any other party. Participants will receive a gift card and one professional book as a thank you for participating.

CONTACT

This research is being conducted by doctoral candidate Kimberly Morrow-Leong, under the supervision of the principal investigator, Dr. Jennifer Suh, Professor in the College of Education and Human Development. You may contact them for questions or to report a research-related problem at 703-675-9697, 703-993-9119, respectively. You may contact the George Mason Institutional Review Board (IRB) Office at 703-993-4121 if you have questions or comments regarding your rights as a participant in the research. This research has been reviewed according to George Mason University procedures governing your participation in this research. IRBNet number: 1440807-1.

Appendix F (Continued)

Informed Consent

Addendum G – Audio or videotape

1. Describe the use of audio or videotape (including purpose): **aVideo will be used to analyze the participants' physical interaction with student work samples and to generate written transcripts of the interview and work sample sorting activity. The audio allows the researcher to capture the participants' observations and analyze their comments. Participants' faces will not be recorded. The focus of the lens will be on their hands and the desk surface where they are working.**

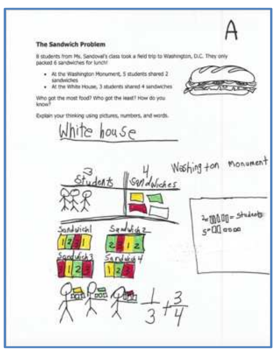
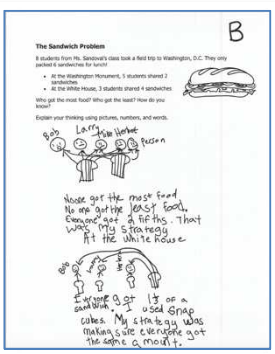
2. If the audio/video tape consent is separate from the informed consent discuss method of audio/video consent and attach consent form: **Audio/Video consent is included in the consent form.**

3. What are your plans for storage of the audio/video taped material during the course of the data collection? **All audio/video taped material will be collected on a dedicated and password protected iPad during collection and uploaded to a password protected cloud folder. A digital media copy will be stored and protected in a locked file under the supervision of the PI. The audio/video taped material will be stored for at least 5 years after the study ends and will be destroyed after that.**

4. What are the plans for ultimate disposition or storage of the audio/video taped material (*ensure that this information is included in the consent form*)? **Video will be used to analyze the participants' interaction with student work samples. Video will be destroyed after five years.**

Appendix G

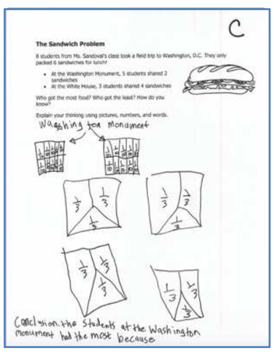
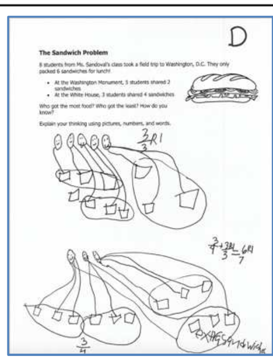
Student Work Samples

Work Sample	Description
	Sample A represents the work of a student who demonstrates proficiency partitioning an area model. While the student accurately partitions the first three sandwiches into four equal parts, she still reduces the size of the final sandwich to show three equal parts of the same size, rather than maintaining the unit whole. The quantity given to each of the three sharers is not accurately named, and it appears that is due to a miscount, although there are confusing aspects to the child's coloring of the partitions as well. Finally, this work sample only addresses one of the two sandwich situations. The second one is treated superficially and without clarity.
	The drawings in sample B are correctly partitioned. However, the shape of the bread in the sandwich is irregular, which makes the partitions technically inaccurate. The student's answer reflects a possible misinterpretation of the comparison situation. The student observes that everyone at each location gets the same amount of sandwich, but does not compare the amount between students at different locations

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

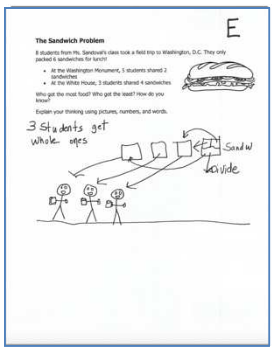
Student Work Samples

Work Sample	Description
	The student work in Sample C reflects an unusual unitizing of two sandwiches, rather than of one sandwich. At least this is a possible explanation for why the students chose to partition the sandwiches from one location into tenths. While it is appropriate to partition a sandwich in order to share with three people, the drawing this student creates incorrectly partitions the square into three parts. The parts are not equal. Lastly, the student does not finish answering the question. The reader is left with a dangling "because."
	The work in sample G represents the two problem situations. However, the use of remainders is inappropriate with fractions, and it is not clear how the students named the fractions. Additionally, this work does not show a representation of a whole sandwich, a necessary understanding for naming the correct fraction. Finally, the student does not give an answer to the question in the task.

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

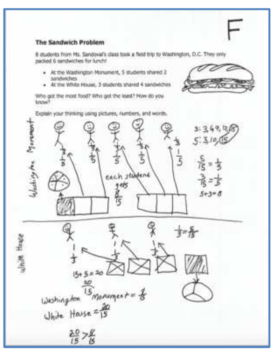
Student Work Samples

<table>
<tr><th>Work Sample</th><th>Description</th></tr>
<tr><td>
</td><td>The student work in Sample E shows that each sharer gets an equal portion of a sandwich. The student appears to be aware that each student on the field trip we'll get at least one whole sandwich.
However, for the fourth sandwich the student indicates that it is necessary to "divide," but then creates an equal share for the remaining students. Finally, the student does not solve the task for the second field trip destination.</td></tr>
</table>

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

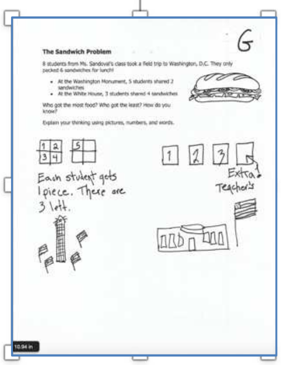
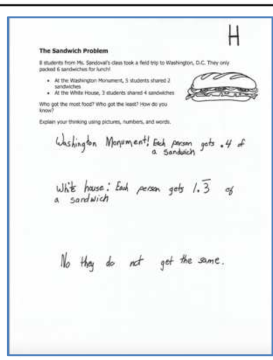
Student Work Samples

Work Sample	Description
	The student work in Sample F shows two sandwiches partitioned into three equal pieces. Three is neither the number of students at that destination, nor the number of sandwiches at that destination. It is possible that the student partitioned the sandwiches in such a way that they would have five pieces to give to the five students. In this case the student was able to split the number of sandwich pieces in half. Yet, there was still one piece left. The student recognized that they needed to share with five students, so they drew a circle and partitioned it into five equal pieces. This portion of the sandwich is named $\frac{1}{5}$, but it is actually $\frac{1}{5}$ of one third, which means it is a fifteenth. Incorrectly naming this fraction gives a portion that is incorrect, yet it is still less than the amount students at the other destination got, so the student's answer remains correct. At the second destination, the student partitions the first three sandwiches into four equal parts, which is not incorrect, but is still inappropriate for three sharers.

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

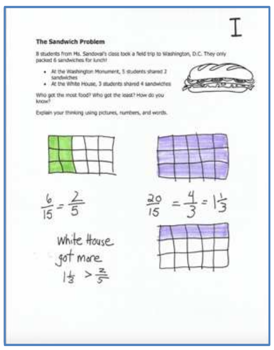
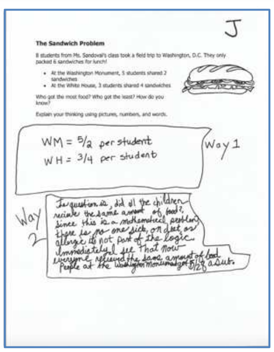
Student Work Samples

Work Sample	Description
	The student work in sample G shows two sandwiches partitioned into four equal pieces each, although it is not clear why the students chose to partition into fourths when there are five students at that destination. It is possible that fourths are a fraction that the student is able to easily create so that they can create five equal pieces. This student proposes different solutions for working with the remaining pieces. In one case, the pieces are leftover and in the other case the remaining pieces are given to the teacher.
	The student work in sample H reflects a use of decimal notation that is appropriate for a student in a higher grade than this task is designed for. On the other hand, this student does not show any evidence that he understands what 0.4 or $1.\overline{3}$ represents in a physical or drawn representation. Finally, the student's written answer states that the quantities are not the same but does not indicate which destination got more sandwich.

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

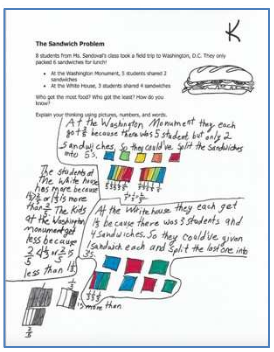
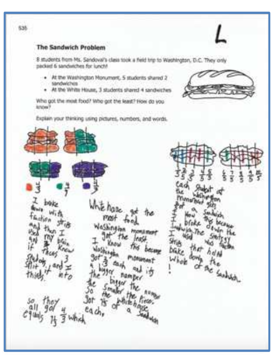
Student Work Samples

Work Sample	Description
	The student work in Sample I uses an array to represent the quantity of sandwich that each student received at each destination. The student does not show what one sandwich looks like, let alone how all of the sandwiches at each destination are shared. While the answer is correct, the student does not also explain *why* the sandwiches are partitioned into fifteenths.
	The student work in sample J reflects a procedure that has been used incorrectly. The appropriate ratios for these problem situations are $\frac{2}{5}$ and $\frac{4}{3}$, but this student shows the reciprocal of those answers. In "Way 2," the student's explanation shows that they may not understand the comparison situation because they reflect that each student got the same amount which is true only at a single destination

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix G (Continued)

Student Work Samples

Work Sample	Description
	The student work in sample K shows that the student has color-coded each sharer and has given each sharer $\frac{1}{5}$ of each sandwich. Similarly, when five sandwiches are shared with three individuals, each of the whole sandwiches is represented in a single color. The final sandwich is represented in fourths and is appropriately partitioned into three equal pieces. This work sample is designed to contrast with work sample L, while still reflecting a correct response. This sample might be referred to according to the Empson and Levi (cite) categorization of additive coordination (one at a time) equal groups partitioning
	The student work in Sample L is accurately partitioned, and each student gets the correct amount of sandwich. The representation does leave some open questions. For example, how did the student know that each sharer would get 1 $\frac{1}{3}$ sandwiches before they started drawing this representation? This sample might be referred to according to the Empson and Levi (cite) categorization of additive coordination (group) equal groups partitioning, which contrasts with the similar strategy shown in Sample K.

Student work from Morrow-Leong, 2013; Suh, Birkhead, Galanti, et al. 2019; Suh, Birkhead, Farmer, et al., 2019

Appendix H

Recruitment Survey

(Any answer of "No" to questions 1 -4 will redirect the individual to a slide thanking them for considering participation.) A yes to all four leads to the basic contact information collection.

1. Do you currently hold an endorsement on your license as a Virginia Mathematics Specialist or are you eligible for one?
2. Do you work in an elementary school as a mathematics coach, Math Resource Teacher, Title I coach, or do the work of a mathematics coach for teachers under any other title?
3. Is your primary job responsibility supporting teachers?
4. Are you willing to participate in this research study? The commitment will include doing a math task, thinking about and recording how students might approach the task, and answering some informational and demographic questions about your experience as a mathematics specialist/coach (~20 minutes). It will also include a 75-minute interview that includes examining samples of student work. This will be arranged in a place and time convenient for you.

Please enter your contact information.

- Last Name
- First Name
- Phone number where you can be reached (please specify if you prefer text or voice)
- Personal email address
- Gender: M F Other _____ OR Decline to say

Appendix H (Continued)

Recruitment Survey

- Ethnicity: (Check as many as apply)
  - White
  - Black or African American
  - American Indian or Alaska Native
  - Asian
  - Native Hawaiian and Other Pacific Islander
  - Other _______
  - Hispanic or Latino
  - Decline to say

  **These are US Census categories. This survey allows for multiple ethnicities to be checked.
- In what county do you work?
- What school(s) do you support?
- What is your job title?
- What grade level teachers do you serve?
- What percentage of your time is dedicated to supporting teachers (not students) in a normal week?

Thank you! I appreciate your time. I will be in touch with you soon.
Your participation is entirely voluntary and you may decline to participate at any time.

Appendix I

Task Directions

Here is a link to a task designed for students in grades 3- 6.

1) Please complete the task as it is designed for students. Please show your preferred solution strategy and at least two additional solutions or representations.
2) On the second page please state the mathematical goal for which this task is appropriate, and for what grade(s). You may have more than one answer to this question.
3) Record the strategies and approaches that you expect students would use to solve the task. Please include reasons for your thoughts. For example, what tells you that students would solve the task that way?

*Final thank you screen.

References

AMTE (2013). Standards for elementary mathematics specialists: A reference for teacher credentialing and degree programs. Association for Mathematics Teacher Educators. Retrieved from https://amte.net/sites/all/themes/amte/resources/EMS_Standards_AMTE2013.pdf

ASCD. (n.d.). Status of State Adoption of Common Core Standards. Retrieved from http://www.ascd.org/common-core-state-standards/common-core-state-standards-adoption-map.aspx

Attride-Stirling, J. (2001). Thematic networks: an analytic tool for qualitative research. *Qualitative Research*, *1*(3), 385-405. https://doi.org/10.1177/146879410100100307

Baker, C. K., Bailey, P., Larson, S. M., & Galanti, T. M. (2017). A critical analysis of emerging high-leverage practices for mathematics coaches. In C. A. Browning, (Ed.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning.* (pp. 183-192). Charlotte, NC: Information Age Publishing.

Baldinger, E. E. (2015). Interpreting student work: What secondary mathematics teacher candidates bring to preparation. Presented at the NCTM Research Conference, Boston, MA: National Council of Teachers of Mathematics.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*, 373-397. https://doi:10.1086/505446

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond, & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). Jossey-Bass. Retrieved from http://eric.ed.gov.mutex.gmu.edu/?id=ED431368

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education*, *59*, 389-407. https://doi.org/10.1177/0022487108324554

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, *6*, 185-204. https://doi.org/10.1207/s15327833mtl0602_6

Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *Montana Mathematics Enthusiast*, *8*(3), 507-570.

Becker, J. R. (2001). Classroom coaching: An emerging method of professional development. In *Proceedings of the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 751-760). Snowbird, UT: CSMEE Publications.

Behr, M. J., Cramer, K., Harel, G., Lesh, R., & Post, T. (2010). The Rational Number Project. Retrieved from http://www.cehd.umn.edu/ci/rationalnumberproject/

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher Education*, *20*, 107-128. https://doi.org/doi:10.1016/j.tate.2003.07.001

Bella, N. J. (2004). *Reflective analysis of student work: Improving teaching through collaboration*. Thousand Oaks, CA: Corwin Press.

Berliner, D. C. (2001). Learning about and learning from expert teachers. *International Journal of Educational Research*, *35*: 463–482. https://10.1016/S0883-0355(02)00004-6

Bitto, L. E. (2015). *Roles, responsibilities, and background experiences of elementary mathematics specialists* (Doctoral dissertation). Retrieved from Proquest Dissertations & Theses Global. (UMI No 3663010)

Blackboard (2019). Blackboard collaborate. Retrieved from https://www.blackboard.com/online-collaborative-learning/blackboard-collaborate.html

Bloomberg, L. D., & Volpe, M. (2012). *Completing your qualitative dissertation: A road map from beginning to end*. Los Angeles: SAGE Publications, Inc.

Blythe, T., Allen, D., & Powell, B. S. (2008). *Looking together at student work* (2nd ed.). New York, NY: Teachers College Press.

Boaler, J. (2015). *Fluency without Fear*. Retrieved July 14, 2019, from https://www.youcubed.org/evidence/fluency-without-fear/

Boaler, J. (2016). *Mathematical mindsets: unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey-Bass.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, *33*(8), 3-15. https://doi.org/10.3102/0013189X033008003

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics*, *85*(2), 221-239. https://doi.org/10.1007/s10649-013-9507-1

Bruner, J. S. (1996). *The Culture of Education*. Washington, DC: Harvard University Press.

Burroughs, E. A., Yopp, D. A., Sutton, J. T., & Greenwood, M. (2017). Knowledge for coaching. In C. A. Browning (Ed.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (pp. 175-182). Charlotte, NC: Information Age Publishing.

CAEP Standards - National Council of Teachers of Mathematics. (2012). Retrieved from https://www.nctm.org/Standards-and-Positions/CAEP-Standards/

Cameron, M., Loesing, J., Rorvig, V., & Chval, K. (2009). Using student work to learn about teaching. *Teaching Children Mathematics*, *15*(8), 488-493. Retrieved from http://www.jstor.org.mutex.gmu.edu/stable/41199338

Campbell, P. F. (2011). Elementary mathematics specialists: A merger of policy, practice, and research. In W. F. Tate, K. D. King, & C. R. Anderson (Eds.), *Research and practice pathways in mathematics education: Disrupting tradition* (pp. 93-104). Reston, VA: National Council of Teachers of Mathematics.

Campbell, P. F., & Malkus, N. N. (2009). School improvement through elementary mathematics coaches: Impact on teacher beliefs and student achievement. Paper presented at *the Annual Meeting of the American Educational Research Association*, San Diego, CA.

Campbell, T. (2013). *A constructivist grounded theory study of refugee parent experiences with education and the influence on their young children's education* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3562272)

Carpenter, T. P., Franke, M. L., Johnson, N. C., Turrou, A. C., & Wager, A. A. (2016). *Young children's mathematics: Cognitively guided instruction in early childhood education*. Chicago, IL: Heinemann.

Carpenter, T., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Carter, S. (2009). Connecting mathematics and writing workshop: It's kinda like ice skating. *The Reading Teacher*, *62*(7), 606-610. https://doi.org/10.1598/RT.62.7.7

Chamberlin, M. T. (2003). *Teacher Investigations of Students' Work: Meeting the Challenge of Attending to Students' Thinking* (Vol. 2). International Group for the Psychology of Mathematics Education. Retrieved from https://eric.ed.gov/?id=ED500937

Clements, D. H., & Sarama, J. (2004). Learning Trajectories in Mathematics Education. *Mathematical Thinking and Learning*, *6*(2), 81-89. https://doi.org/10.1207/s15327833mtl0602_1

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the building blocks project. *Journal for Research in Mathematics Education*, *38*(2), 136-163. https://doi.org/10.2307/30034954. https://www.jstor.org/stable/30034954

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, *30*, 192-212. https://doi.org/10.2307/749610

Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A Learning Trajectory. *Mathematical Thinking and Learning*, *6*(2), 163-184. https://doi.org/10.1207/s15327833mtl0602_5

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematic development. *Educational Researcher*, 23 (7), 13-20. https://doi.org/10.2307/1176934

Cobb, P. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, *13*(1), 6-33. Retrieved from https://eric.ed.gov/?id=EJ960944

Confrey, J. (2012). Articulating a learning sciences foundation for learning trajectories in the CCSS-M. In *Proceedings of the 34th conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2-22). Kalamazoo, MI.

Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM*, *46*(5), 719-733. https://doi.org/10.1007/s11858-014-0598-7

Confrey, J., Maloney, A., Nguyen, K., Lee, K. S., Panorkou, N., Corley, A., … Gibson, T. (2014, May 26). Learning trajectories for the K-8 Common Core Math Standards. Retrieved from http://www.turnonccmath.net/

Confrey, J., Maloney, A., Nguyen, K., Lee, K. S., Panorkou, N., Corley, A., … Gibson, T. (2014, May 26). Learning Trajectories for the K-8 Common Core Math Standards. Retrieved from http://www.turnonccmath.net/

Confrey, J., Maloney, A., Nguyen, K., Mojica, M., & Meyers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using LTs. In *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (pp. 345–352). Thessaloniki, Greece. Retrieved from https://trove.nia.gov.au/work/36940283

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66. https://doi.org/10.2307/749228

Corbin, J., & Strauss, A. (2008). *Basics of Qualitative Research (3rd ed.): Techniques and Procedures for Developing Grounded Theory*. Thousand Oaks: SAGE Publications, Inc. https://doi.org/10.4135/9781452230153

Corcoran, T., Mosher, F. A., & Rogat, A. (2003). Learning progressions in science: An evidence-based approach to reform. *Consortium for Policy Research in Education*, 337-676. Retrieved from http://www.sciencedirect.com/science/article/pii/S0375947403018098 https://doi.org/10.12698/cpre.2009.rr63

Crespo, S. (2000). Seeing more than right and wrong answers: Prospective teachers' interpretations of students' mathematical work. *Journal of Mathematics Teacher Education*, *3*(2), 155-181. Retrieved from https://doi.org/10.1023/A:1009999016764

Creswell, J. W. (2008). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (3rd ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Daehler, K., & Folsom, J. (2014). *Making sense of student work: A protocol for teacher collaboration*. San Francisco, CA: WestEd.

Daro, P., Mosher, F., & Corcoran, T. (2011). Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction. *CPRE Research Reports*. https://doi.org/10.12698/cpre.2011.rr68

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, *28*(3), 355-376. https://doi.org/10.2307/749785

de Araujo, Z., Webel, C., & Wray, J. A. (2017). Preparing elementary mathematics specialists: Essential knowledge, skills, and experiences. In C. A. Browning (Ed.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (pp. 19-32). Charlotte, NC: Information Age Publishing.

Dedoose. (2019). Retrieved from https://www.dedoose.com/home

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181-199. https://doi.org/10.3102/0013189X08331140

Developing mathematical ideas. (n.d.). Retrieved July 15, 2019, from http://mathleadership.org/

Doerr, H. M. (2006). Examining the tasks of teaching when using students' mathematical thinking. *Educational Studies in Mathematics*, *62*(1), 3-24. Retrieved from http://link.springer.com/article/10.1007/s10649-006-4437-9. https://doi.org/10.1007/s10649-006-4437-9

Empson, S. B. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast*, *8*(3), 571-596. Retrieved from https://scholarworks.umt.edu/tme/vol8/iss3/6

Empson, S., & Jacobs, V. R. (2008). Learning to listen to children's mathematics. *ResearchGate*, 257-281. Retrieved from https://www.researchgate.net/publication/260001751_Learning_to_Listen_to_Children's_Mathematics

Empson, S., & Levi, L. (2011). *Extending children's mathematics: Fractions & decimals: Innovations in Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Elementary Mathematics Specialist & Teacher Leaders (n.d.). Math specialist certification (by state). Retrieved, from http://www.mathspecialists.org/state-certifications.html

Fernández, C., Llinares, S., & Valls, J. (2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast*, *10*(1), 441-468. Retrieved from https://scholarworks.umt.edu/tme/vol10/iss1/19

Franke, M. L. (2018, April). *How and why attention to student thinking supports teacher and student learning: The case of Cognitively Guided Instruction (CGI)*. Presented at the 2018 NCSM Annual Conference,Washington, DC.. Retrieved from https://www.mathedleadership.org/events/conferences/DC2/index.html

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A Follow-Up study of professional development in mathematics. *American Educational Research Journal*, *38*(3), 653-689. https://doi.org/10.3102/00028312038003653

Fuson, K. C., Carroll, W. M., & Drueck, J. V. (2000). Achievement results for second and third graders using the standards-based curriculum everyday mathematics. *Journal for Research in Mathematics Education*, *31*(3), 277-295. https://doi.org/10.2307/749808

Gamoran, S. M., & van Es, E. A. (2008). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20-37. https://doi.org/10.1177/0022487108328155

Gawande, A. (2017). Want to get great at something? Get a coach. Retrieved from https://www.ted.com/talks/atul_gawande_want_to_get_great_at_something_get_a_coach

Gee, J. P. (2018, October). From grammar to society. Arizona State University, Temple campus. Retrieved from https://www.youtube.com/watch?v=LKiaYBVAEUk

Gibbons, L. (2017). Examining mathematics coaching practices that help develop schoolwide professional learning. In M. B. McGatha & N. R. Rigelman (Eds.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (Vol. 2, pp. 167-174). Charlotte, NC: Information Age Publishing, Inc.

Gibbons, L. K., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. *Journal of Teacher Education*, *68*(4), 411-425. https://doi.org/10.1177/0022487117702579

Glaser, B. (1992). Basics of grounded theory analysis. Mill Valley, CA: Sociology Press.

Glesne, C. (2011). *Becoming qualitative researchers: An introduction*. Boston, MA: Pearson Education, Inc.

Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teachers' noticing. In M. Sherin, V. Jacobs, & R. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 169–187). New York, NY: Routledge.

Goldsmith, L. T., & Seago, N. M. (2013). *Examining mathematics practice through classroom artifacts*. Boston, MA: Pearson.

Google (n.d.). Google docs. Retrieved from https://www.google.com/intl/en-GB/docs/about/

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). Chapter 4: A hypothetical learning trajectory on measurement and flexible arithmetic. *Journal for Research in Mathematics Education. Monograph*, *12*, 51-66. Retrieved from http://www.jstor.org/stable/30037721

Hadjidemetriou, C., & Williams, J. (2002). Children's graphical conceptions. *Research in Mathematics Education*, *4*(1), 6-87. https://doi.org/10.1080/14794800008520103

Heyd-Metzuyanim, E., & Sfard, A. (2012). Identity struggles in the mathematics classroom: On learning mathematics as an interplay of mathematizing and identifying. *International Journal of Educational Research*, *51-52*, 128-145. https://doi.org/10.1016/j.ijer.2011.12.015

Hill, H., & Ball, D. L. (2009). The curious — and crucial — case of mathematical knowledge for teaching. *Phi Delta Kappan*, *91*(2), 68-71. https://doi.org/10.1177/003172170909100215.

Hill, H. C., Blazar, D., & Lynch, K. (2015). Resources for teaching: Examining personal and institutional predictors of high-quality instruction. *AERA Open*, *1*(4). https://doi.org/10.1177/2332858415617703.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*, 430-511. https://doi.org/10.1080/07370000802177235

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2), 81-116. https://doi.org/10.2307/30034933.

Hunt, J. H., & Empson, S. B. (2015). Exploratory study of informal strategies for equal sharing problems of students with learning disabilities. *Learning Disability Quarterly*, 38(4), 208-220. https://doi.org/10.1177/0731948714551418

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202. Retrieved from http://www.jstor.org/stable/20720130

Jones, G. A., Langrall, C. W., Thornton, C. A., Mooney, E. S., Wares, A., Jones, M. R., … Nisbet, S. (2001). Using students' statistical thinking to inform instruction. *The Journal of Mathematical Behavior*, *20*(1), 109-144. https://doi.org/10.1016/S0732-3123(01)00064-5

Karp, K. S., Bush, S. B., & Dougherty, B. J. (2014). 13 rules that expire. *Teaching Children Mathematics*, *21*(1), 18-25. https://doi.org/10.5951/teacchilmath.21.1.0018

Kazemi, E., & Loef Franke, M. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, *7*, 203-235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Kieren, T. (1980). The rational number construct — its elements and mechanisms. In T. Kieren (Ed.), *Recent research on number learning* (pp. 125–149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education. Retrieved from https://files.eric.ed.gov/fulltext/ED212463.pdf

Krupa, E., & Confrey, J. (2010). Teacher change facilitated by instructional coaches: A customized approach to professional development. In *32nd Annual Conference of North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH.

Kvale, S., & Brinkmann, S. (2009). *Interviews: Learning the craft of qualitative research interviewing* (2nd. ed.). Thousand Oaks, CA: SAGE Publications, Inc.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding : Essential content knowledge and instructional strategies for teachers*. Routledge, NY. https://doi.org/10.4324/9780203803165

Lave, J. (2011). *Apprenticeship in critical ethnographic practice*. Chicago, IL: The University of Chicago Press.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Zoest, L. R. V. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088

Leinwand, S., Brahier, D. J., Huinker, D., Berry, R. Q., Dillon, F. L., Larson, M. R., … Smith, M. S. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics* *problem solving, learning and teaching* (pp. 1-33). Mahwah, NJ: Lawrence Erlbaum Associates.

Liu, Y. (2014). *Teachers' in-the-moment noticing of students' mathematical thinking: A case study of two teachers* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3622401)

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 74-101). Reston, VA: National Council of Teachers of Mathematics.

Luna, M. J., & Sherin, M. G. (2017). Using a video club design to promote teacher attention to students' ideas in science. *Teaching and Teacher Education*, *66*, 282-294. https://doi.org/10.1016/j.tate.2017.04.019

McMillan, J. H., Myran, S., & Workman, D. (2002). Elementary Teachers' Classroom Assessment and Grading Practices. *The Journal of Educational Research*, *95*(4), 203–213. https://doi.org/10.1080/00220670209596593

Maloney, A. (2013, October 31). personal conversation.

Mangiante, E. S. (2012). Effective urban elementary teachers of inquiry science: Beliefs, knowledge, and resources shaping teacher planning. (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3503342)

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London, United Kingdom: Routledge Falmer.

Mason, J. (2011). Noticing: Roots and branches. In M. Sherin, V. Jacobs, & R. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35-50). New York, NY: Routledge.

Maxwell, J. (2005). *Qualitative research design: An interactive approach*. Thousand Oaks, CA: SAGE Publications, Inc.

McDuffie, A. R., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T. G., … Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education*, *17*(3), 245-270. https://doi.org/10.1007/s10857-013-9257-0

McGatha, M. (2008). Levels of engagement in establishing coaching relationships. *Teacher Development*, *12*(2), 139-150. https://doi.org/10.1080/13664530802038147

McGatha, M. (2009). Research brief: Mathematics specialists and mathematics coaches: What does the research say? Reston, VA: National Council of Teachers of Mathematics.

McGatha, M. B., Davis, R., & Stokes-Levine, A. (2017). Mathematics specialists: What does the research say? In C. A. Browning, (Ed.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning.* (pp. 153-166). Charlotte, NC: Information Age Publishing.

McGatha, M. B., & Rigelman, N. R. (2017). In C. A. Browning, (Ed.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (p. xiv). Charlotte, NC: Information Age Publishing.

McMillan, J. H., Myran, S., & Workman, D. (2002). Elementary teachers' classroom assessment and grading practices. *The Journal of Educational Research*, *95*(4), 203-213. https://doi.org/10.1080/00220670209596593

Mohr, J., & Mohr, R. (2019). *Kumu, Inc.* Retrieved from https://kumu.io/

Moore, S. D., Morrow-Leong, K., & Gojak, L. (2020). *Mathematize it!: Going beyond key words to make sense of word problems, grades 3–5*. Thousand Oaks, CA: Corwin.

Morrow-Leong, K. (2013). *Uncovering evidence of models and structure: Teachers examining students' task-based assessments*. Unpublished manuscript, College of Education and Human Development, George Mason University, Fairfax, Virginia.

Nathaniel, A. K., & Andrews, T. (2007). How grounded theory can improve nursing care quality. *Journal of Nursing Care Quality*, *22*(4), 350-357. Retrieved from EBSCOhost database. https://doi.org/10.1097/01.ncq.0000290417.27393.91

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards. Washington DC: NGA Center and CCSSO. Retrieved from https://www.corestandards.org/Math

National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. U.S. Department of Education. Retrieved from https://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

Nicol, C. (1998). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics*, *37*(1), 45-66. Retrieved from . https://doi.org/10.1023/A:1003451423219

Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *Journal of Mathematical Behavior*, *29*(4), 181-194. https://doi.org/10.1016/j.jmathb.2010.10.001

Norton, A., & Wilkins, J. L. (2012). The splitting group. *Journal for Research in Mathematics Education*, *43*(5), 557–583. https://doi.org/10.5951/jresematheduc.43.5.0557

Olson, J. C. (2005). Evoking pedagogical curiosity: A coaching approach to support teacher's professional growth. *The Mathematics Educator*, *8*(2), 84-94.

Olson, J., & Barrett, J. (2004). Coaching teachers to implement mathematics reform recommendations. *Mathematics Teacher Education and Development*, 63-78. https://eric.ed.gov/?id=EJ852387

Petit, M. M., Laird, R. E., Marsden, E. L., & Ebby, C. B. (2010). *A focus on fractions: Bringing research to the classroom*. Routledge, NY. https://doi.org/10.4324/9780203855515

Piaget, J., & Inhelder, B. (1973). *The psychology of the child*. New York, NY: Basic Books.

Polly, D. (2012). Supporting mathematics instruction with an expert coaching model. *Mathematics Teacher Education & Development*, *14*(1), 78-93. Retrieved from https://mted.merga.net.au/index.php/mted/article/view/20

Race, K., Ho, E. and Bower, L. (2002, April) Documenting in□classroom support and coaching activities of a professional development program directed towards school□wide change: An integral part of an organization's evaluation efforts. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Regis, T. P. (2008). *Analyzing how grade four teachers plan to teach inquiry-based curriculum materials and the influences on their preparation of mathematics lessons* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3374916)

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211-246. https://doi.org/10.3102/00346543075002211

Rigelman, N. R., & Wray, J. A. (2017). Current state of mathematics specialist state certification and standards. In M. B. McGatha & N. R. Rigelman (Eds.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (Vol. 2, pp. 33-38). Charlotte, NC: Information Age Publishing, Inc.

Russell, S. J., Economopoulos, K., Hollister, A., Bastable, V., Bloomfield, K., Horowitz, N., … Wittenberg, L. (2012). Investigations in number, data, and space. Retrieved July 15, 2019, from https://www.pearsonschool.com/index.cfm?locator=PSZpOz

Saldaña, J. (2014). *Thinking qualitatively: Methods of mind*. Los Angeles, CA: SAGE Publications, Inc.

Salkind, G. M. (2010). *Coaches' and principals' conceptualizations of the roles of elementary mathematics coaches* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3416120)

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Schack, E. O., Wilhelm, J. A., & Fisher, M. H. (Eds.). (2017). *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*. Cham, Switzerland: Springer.

Schneider, M. C., & Gowan, P. (2013). Investigating teachers' skills in interpreting evidence of student learning. *Applied Measurement in Education*, *26*(3), 191-204. https://doi.org/10.1080/08957347.2013.793185

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4-13. https://doi.org/10.3102/0013189x027002004

Shavelson, R. J., & Borko, H. (1979). Research on teachers' decisions in planning instruction. *Educational Horizons*, *57*(4), 183-189. Retrieved from http://www.jstor.org/stable/42924342

Sherin, M., & van Es, E. A. (2002). Using video to support teachers' ability to interpret classroom interactions. In D. Willis, J. Price, & N. Davis (Eds.), *Proceedings of SITE 2002--Society for Information Technology & Teacher Education International Conference* (pp. 2532–2536). Nashville, TN: Association for the Advancement of Computing in Education (AACE). Retrieved from https://www.learntechlib.org/primary/p/17741/

Sherin, M. G., Jacobs, V. B., & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. B. Jacobs, & R. A. Philipp (Eds.), *Mathematics*

Teacher Noticing: Seeing Through Teachers' Eyes (pp. 3-13). New York, NY: Routledge.

Shulman, L. (1986). Those who understand: Knowledge growth for teaching. *Educational Researcher*, *15*(2), 4-14. http://www.jstor.org/stable/1175860

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114. https://doi.org/10.2307/749205

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91-104. https://doi.org/10.1207/s15327833mtl0602_2

Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, *84*(1), 49-70. https://doi.org/10.1007/s10649-013-9475-5

Spitzer, S. M., & Phelps-Gregory, C. M. (2017). Using mathematical learning goals to analyze teacher noticing. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 303–319). Springer, Cham. https://doi.org/10.1007/978-3-319-46753-5_18

Stake, R. E. (2013). *Multiple case study analysis*. New York, NY: Guilford Press.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior*, *22*(3), 237-295. https://doi.org/10.1016/S0732-3123(03)00022-1

Steffe, L. P. (2010). The partitioning and fraction schemes. In L. P. Steffe & J. Olive (Eds.), *Children's Fractional Knowledge* (pp. 315-340). Boston, MA: Springer. https://doi.org/10.1007/978-1-4419-0591-8_10

Steffe, L. P., Norton, A., Hackenberg, A., & Thompson, P. (2012, April). *Students' fraction knowledge and the Common Core State Standards (CCSS)*. Presented at the NCTM Research Presession, Philadelphia, PA.

Steffe, L. P., & Olive, J. (2010). *A new hypothesis concerning children's fractional knowledge*. New York, NY: Springer.

Stein, M. K., & Smith, M. (2011). *5 practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488. https://doi.org/10.3102/00028312033002455

Sztajin, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher*, *41*, 147-156. https://doi.org/10.3102/0013189X12442801

Suh, J.M., Birkhead, S., Galanti, T., Farmer, R., & Seshaiyer, P. (2019). The use of lesson study to unpack learning trajectories and deepen teachers' horizon knowledge (756-781). In R. Huang & Takahashi, A. (Eds.), *Theory and Practices of Lesson Study in Mathematics: An International Perspective.* Springer. Retrieved from https://drive.google.com/file/d/1E1B1h1GibZ-gSoZQnSHI66AE3v3tVzhU/view?usp=sharing

Suh, J., Birkhead, S., Farmer, R. R., Galanti, T., Nietert, A., Bauer, T., & Seshaiyer, P. (2019). Split it! Unpacking the equipartitioning learning trajectory. *Teaching Children Mathematics*, *25*(6), 362-369. https://doi.org/10.5951/teacchilmath.25.6.0362

Suh, J., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education*, *18*, 207-229. https://doi.org/10.1007/s10857-014-9282-7

Talanquer, V., Bolger, M., & Tomanek, D. (2015). Exploring prospective teachers' assessment practices: Noticing and interpreting student understanding in the assessment of written work. *Journal of Research in Science Teaching*, *52*(5), 585-609. https://doi.org/10.1002/tea.21209

U.S. Census Bureau. (n.d.). Race & ethnicity. Retrieved from https://www.census.gov/mso/www/training/pdf/race-ethnicity-onepager.pdf

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: teaching developmentally* (7th ed.). Boston, MA: Pearson/Allyn and Bacon.

van den Kieboom, L. A., Magiera, M. T., & Moyer, J. C. (2017). Learning to notice student thinking about the equal sign: K-8 preservice teachers' experiences in a teacher preparation program. In E. O. Schack, J. A. Wilhelm, & M. H. Fisher (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 141-159). Cham, Switzerland: Springer.

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. B. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3-13). New York: Routledge.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244-276. https://doi.org/10.1016/j.tate.2006.11.005

van Hiele, P. M. (2004). The child's thought and geometry. In *Classics in Mathematics Education Research* (pp. 60-67). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (2004). Multiplicative structures. In *Classics in Mathematics Education Research* (pp. 84-97). Reston, VA: National Council of Teachers of Mathematics.

Virginia Mathematics and Science Coalition. (2016). *Virginia's mathematics specialist initiative: Overview of program and course annotated syllabi for preparing mathematics specialists* (p. 205). Retrieved from http://www.vamsc.org/wp-content/uploads/VMSI_AnnotatedSyllabi_Final.doc.pdf

Virginia Mathematics and Science Coalition. (n.d.) Master's Degree Programs – Virginia Mathematics and Science Coalition. Retrieved from http://www.vamsc.org/index.php/masters-degree-programs/

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London, United Kingdom: Routledge.

Vygotsky, L. S. (1980). *Mind in society: The development of higher psychological processes* (14th ed.). Boston, MA: Harvard University Press.

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*(6), 523-550. https://doi.org/10.1007/s10857-014-9289-0

Wenger, E. (1998). *Communities of practice: learning, meaning, and identity*. Cambridge, United Kingdom: Cambridge University Press.

West, Lucy, & Staub, Fritz. (2003). *Content-focused coaching: Transforming mathematics lessons*. Chicago, IL: Heinemann.

Wiley, K. E. (2014). *Middle school mathematics teachers' use of advanced mathematical knowledge in practice: An interpretative phenomenological analysis* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global (UMI No. 3665237)

Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior*, *32*(2), 103-121. https://doi.org/10.1016/j.jmathb.2012.12.003.

Wood, T. (1994). Patterns of interaction and the culture of mathematics classrooms. In *Cultural Perspectives on the Mathematics Classroom* (pp. 149-168). Dordrecht, the Netherlands: Springer.

Yin, R. K. (2011). *Qualitative research from start to finish*. New York, NY: Guilford Press.

Yopp, D. A., Burroughs, E. A., Sutton, J. T., & Greenwood, M. C. (2017). Variations in coaching knowledge and practice that explain elementary and middle school mathematics teacher change. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-017-9373-3

Biography

Kimberly Morrow-Leong graduated from Independence High School, San José, California, in 1984. She received her Bachelor of Arts in French Language from the University of California, Santa Barbara in 1988. Following that she earned a Master of Arts degree in Linguistics, with an emphasis in teaching English to speakers of other languages (TESOL) and earned her initial credential as an elementary school teacher. She taught middle school mathematics in Milpitas Unified School District, Prince William County Public Schools, and at All Saints Catholic School for a total of 10 years, after which she served as a mathematics specialist coach in both Loudoun and Fairfax County Public Schools. She is the 2009 recipient of the Presidential Award for Excellence in Mathematics and Science Teaching (PAEMST) for Virginia. In 2010 she earned a Masters of Education in Mathematics Education leadership from George Mason University, earning the Exxon Mobil Mathematics Specialist Excellence scholarship in 2009. She has since worked for the National Council of Teachers of Mathematics, Math Solutions, American Institutes for Research, and served as Program Chair for the NCSM 50th Anniversary conference. She is more recently a co-author of *Mathematize It! Going Beyond Key Words to Make Sense of Word Problems, Grades 3- 5* and is a frequent speaker. She is happiest when working with teachers and students, putting pencils down and getting messy with manipulatives!