## FORMATION OF THE POWER DENSITY SPECTRUM IN THE ACCRETING COMPACT X-RAY OBJECTS

by

Andre Makeev<br>A Dissertation<br>Submitted to the<br>Graduate Faculty<br>of<br>George Mason University<br>in Partial Fulfillment of<br>The Requirements for the Degree<br>of<br>Doctor of Philosophy<br>Computational Astrophysics

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at George Mason University

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## Dedication

I dedicate this dissertation to my parents Raya and Valeri

## Acknowledgments

I would like to thank many people who were around me during this time, and who helped me to get it completed. Mostly these are folks who I had pleasure to work with during my graduate research at the Naval Research Laboratory (NRL). Lev Titarchuk has been a great advisor, patiently answering my numerous (and often repeated) questions, and teaching me the proper ways of doing physics. His positive attitude, sense of humor, and encouragement were indispensable at times when some things seemed to be undoable. Astrophysicists from NRL's High-Energy Space Environment branch made me always feel welcome, whenever I was avidly seeking for the answers: Michael Wolff, Paul Ray, Kent Wood provided me with solid advises, readily shared their expertise, valuable experience, and vast archives of great literature. I am very grateful to Chul Gwon, who helped me to resolve any programming/computer-related problems I had. Communications with my friend Nick Shaposhnikov, from NASA, were of great aid in the early stages of work, when the scope of the thesis project was outlined. I would like to thank my parents Valeri and Raya, and my girlfriend Nataliya for their loving support, understanding, and wise advises during these years. I thank my friends Aous Abdo, Justin Finke, and Mark Christophersen for their involvement and instilling the hope that this will soon be over. I want to express my special thank you to my colleague at NRL Alexander (Sasha) Chehtman, who, during Lev's absence, basically took supervision over my work in his hands. Our discussions were the ones most enlightening physics lectures, and taught me some basic concepts, which I thought I learnt years ago.. A rigorous, self-consistent, analytical approach in solving physics problems is one of the many great lessons I learnt from Sasha. I really appreciate Sasha's readiness to help me each time I asked, no matter how busy he was. Undoubtly, completion of this work in time would not have been possible without his participation.

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## Abstract

FORMATION OF THE POWER DENSITY SPECTRUM IN THE ACCRETING COMPACT X-RAY OBJECTS<br>Andrey Makeev, PhD<br>George Mason University, 2010<br>Dissertation Director: Dr. Peter A. Becker

One of the remarkable phenomena, characterizing both Galactic and extra-Galactic Xray binary systems, is the substantial variability of a photon flux, detectable in a very broad range of timescales. For instance, the accretion flow near a black hole event horizon can produce X-ray variability on a millisecond timescale. At the same time aperiodic changes from the extended accretion disk formed around the same black hole can occur on timescales of order of several months to years. A complex structure, involving high and low frequency nearly periodic oscillations and aperiodic features, observed in X-ray lightcurves, is the subject of intensive studies. The characteristic quantities, extracted from temporal analysis, carry specific physical meaning and contain direct observational information about dynamics of the accreting X-ray source. It is the established fact that X-ray spectral and timing properties are tightly correlated. Combined together, the photon energy spectrum and the power density spectrum analyses, form a powerful framework that brings up the complete (in the energy/space domain) picture of the physical processes at work in the accreting system. Simultaneous study of spectral and timing characteristics allows for comprehensive probing of the geometry of accretion flows, reliable identification of the type of an X-ray source (black hole vs neutron star), constraining mass, size, and
spin of accreting stellar-mass compact objects. Up until now there is no self-consistent physical model of the formation and evolution of the X-ray variability. This leaves a relative freedom in interpretation of the characteristic quantities obtained from the timing analysis. The current work aims at development of the physical alternative to the commonplace ad hoc description of the Fourier power density spectrum of X-ray timing signal. In the following study we employ the diffusion theory to directly solve for the X-ray luminosity fluctuations. The basic underlying physical assumption is that the observed variability of Xray luminosity originates as the result of local fluctuations of the accretion rate, at all radii in the disk, that diffusively propagate outward. Energy dissipation (and X-ray emission) occurs in a narrow, shock-like region, called the transition layer, where the Keplerian flow becomes non-Keplerian in order to adjust itself to the slowly-rotating surface of a neutron star or the innermost stable orbit around a black hole. The X-ray time signal from the transition region, as seen by a remote observer, is obtained by integrating over the emission zone. The signal's power spectrum is then calculated and analyzed. Our diffusion model of the power spectrum formation operates with parameters that are physical characteristics of the accretion flow: the diffusion time scale, the Reynolds number (which is connected to the viscosity $\alpha$-parameter), Keplerian and magnetosonic quasi-periodic oscillation frequencies, radial size of the transition layer, and viscosity index, related to the viscosity distribution law in the system. These quantities constitute the core of temporal data used along with the spectral information to study physics of accretion. The proposed propagating fluctuation model can reproduce fundamental properties of the variability observed in X-ray light curves of accreting black hole and neutron star systems, as well as explain the power spectrum evolution during the spectral state transitions of the source.

## Chapter 1: Introduction

### 1.1 Overview

This thesis focuses on the timing properties of neutron stars and black holes in binary systems, emitting electromagnetic radiation in the X-ray band. One of the primary tasks of X-ray astrophysics is determination of physical characteristics of compact objects (cataclysmic variables, black holes and neutron stars) such as mass, size, distance, temperature, luminosity, type of object, etc. Of the same importance is understanding of the physical processes that cause these objects to radiate massive amounts of energy in different bands of electromagnetic spectrum. High-energy photons emitted by X-ray sources are detected individually by space-based observatories and analyzed to extract the principal quantities of interest: incoming photon's energy, its direction, time of arrival, photon flux, and, in some cases, polarization. The following spectral and temporal studies are mainly based on the analysis and interpretation of the compact object's energy spectrum and its light curve (photon count rate as a function of time). The spectral analysis provides the most important piece of information, the photon energy spectrum. Variability (timing) studies constitute a solid complement to spectral analysis and often carried out alongside.

An important task of X-ray temporal analysis is to explain the mechanism of the observed broad-band variability. X-ray lightcurves from Galactic binary systems and Active Galactic Nuclei (AGN) show significant fluctuations on a very broad range of timescales, which is difficult to explain based on the compact nature of the X-ray emission region. Study of timing flicker noise by Lyubarskii (1997) [33] proposed an elegant solution, where fluctuations in the accretion rate produced at different radii in the accretion flow propagate inward, to finally modulate the emission at the central, X-ray emitting region. Lyubarskii's propagating fluctuation model produces a broad range of variability timescales. In this
research we attempt to extend Lyubarskii model of distributed fluctuations in the accretion disk, and explain the cause of luminosity fluctuations and observed features in the X-ray power spectra.

### 1.2 Scope of work

The first Chapter of the thesis makes an introduction to the process of accretion onto compact stellar objects like neutron stars and black holes. The basic mass transfer mechanisms in close-binary systems are listed. The concept of the Eddington limit, which is used throughout the thesis for approximate calculations, is explained. X-ray spectral states, X-ray timing analysis and existing models of power spectrum formation are briefly discussed. Chapter 2 contains derivation of the basic conservation equations in the disk geometry, which we use later to write down the equation for diffusive propagation of the fluctuations of mass accretion rate. We also provide an introduction to the model of the viscous transition (adjustment) layer (also known as the Centrifugal Barrier Model, CBM) of Titarchuk, Lapidus, Muslimov, 1998 [57] in accreting systems with slowly-rotating black holes or weakly-magnetized neutron stars. In Chapter 3 a simplified initial-value problem for the accretion rate variations is formulated and solved analytically for the two parts of transition region. A brief discussion is presented regarding the discovered instability of the transition layer. Important physical conclusions are drawn based on the analysis of the power spectrum of the obtained solution. Chapter 4 gives a detailed description of the numerical method for solving the diffusion equation with the time and space dependent perturbation source term. Numerical solution obtained for both the compact non-Keplerian transition layer, and the extended Keplerian configurations. The question of the X-ray lightcurve variability dependence on the accretion rate is considered, based on the form of the diffusion operator and obtained results. The extra broadband power spectrum of the two configurations is presented and analyzed. An interesting, and potentially helpful in data analysis, fact of steepening of the power spectrum slope for the Keplerian disk, with the increasing viscosity index is discussed there as well. Chapter 5 briefly summarizes the
results of this research.

### 1.3 Black holes and neutron stars

Existence of neutron stars was proposed by theorists not long after discovery of neutron. The first self-consistent neutron star model was developed by Oppenheimer and Volkov in 1939 [37]. The current understanding is that neutron stars are created when normal stars with masses of $4 \lesssim M \lesssim 8 M_{\odot}$ burn out nuclear fuel, and the deficit of the outward thermal pressure is followed by the gravitational collapse. As the star collapses it loses most of its outer material. The free electrons are forced by strong gravitation to combine together with protons to form neutrons. Modern nuclear models predict the maximum stable mass of the neutron star to be $\sim 3.0-3.2 M_{\odot}$. If mass of a collapsing star exceeds this value, the central core continues to contract into an infinitesimal gravitational singularity, to form a black hole. Black hole space-time is distorted to the extent that even photons cannot escape past a certain radius, called the event horizon. The event horizon (the Schwarzschild radius) of a black hole is determined only by its mass

$$
\begin{equation*}
R_{\mathrm{S}}=2 G M / c^{2} \tag{1.1}
\end{equation*}
$$

The effective potential of a test particle, in a gravitational field of a non-rotating (Schwarzschild) black hole, is determined by a black hole mass and test particle's angular momentum. Stable orbits occur in the local minima of the effective potential. The innermost stable circular orbit around a black hole is given by [48]

$$
\begin{equation*}
R_{0}=3 R_{\mathrm{S}}=\frac{6 G M}{c^{2}} \tag{1.2}
\end{equation*}
$$

The primary mechanism by which compact objects like black holes and neutron stars become detectable from Earth is called the accretion. Accretion occurs when the gaseous matter
from the neighboring star or surrounding environment gets drawn onto a compact object, often in form of a disk, heats up due to magnetic friction to temperatures of order $\sim 10^{6} \mathrm{~K}$ (which corresponds to X-ray band in the spectrum) and emits radiation.

### 1.4 Accreting X-ray close-binary systems

A binary system consists of the two stars orbiting around their center of gravity. A significant fraction (up to $50 \%$ out of $\sim 10^{11}$ in the Galaxy, according to Shakura \& Sunyaev, 1973, [47]) of known stars form binary systems. A binary system is called "close", if the two components can undergo mass exchange. In a close-binary system separation distances are of order of the diameter of the components. In this work we consider close-binary systems where one object is a compact accreter (a black hole or a neutron star), the other, its donor companion, a normal star which loses mass. The flow of mass between the two stars is determined by the binary parameters of the system, as well as by the process of mass removal from the donor star. The three main mechanisms of mass transfer between the two stars are

- Roche-lobe overflow of the companion star. Typically, gas leaving the companion star via Roche lobe overflow, has considerable angular momentum relative to the compact object and therefore cannot fall on it directly. This leads to formation of an (accretion) disk around the compact star.
- Stellar wind from the companion. Massive young stars may produce a significant outflow of plasma, accelerated by radiation pressure.
- "Capture of circumstellar material from a Be star primary. A Be star is a B star which rotates so rapidly that an instability results via which material streams out from the equatorial plane and an expanding atmosphere is formed", (taken from http://imagine.gsfc.nasa.gov/docs/science/know_l2/mass_exchange.html).


### 1.5 The Eddington Limit

If the luminosity of a star is greater than the Eddington luminosity, the radiative pressure will eject out the accreting matter. This condition sets the limitation for the highest possible accretion rate onto a compact object, at which the accreter can still keep electrons from being blown away. One can derive the Eddington limit from the condition of the hydrostatic equilibrium, at which the radiation pressure gradient is balanced by the gravitational force, under assumption of spherical accretion. Consider the radiative energy flux at radius $R$

$$
\begin{equation*}
F=\frac{d E}{d t d A}=\frac{L}{4 \pi R^{2}} . \tag{1.3}
\end{equation*}
$$

Photon's momentum is $p=E / c$, therefore the momentum flux is

$$
\begin{equation*}
\frac{d p}{d t d A}=\frac{L}{4 \pi r^{2} c} . \tag{1.4}
\end{equation*}
$$

The radiation force exerted on a free electron (protons are connected with electrons via Coulomb interaction) is

$$
\begin{equation*}
\frac{d p}{d t}=\sigma_{T} \frac{d p}{d t d A}=\sigma_{T} \frac{L}{4 \pi R^{2} c}, \tag{1.5}
\end{equation*}
$$

where $\sigma_{T}=8 \pi / 3 r_{e}^{2} \approx 6.65 \times 10^{-25} \mathrm{~cm}^{2}$ is the Thomson scattering cross-section, $r_{e}=$ $e^{2} / m_{e} c^{2} \approx 2.8 \times 10^{-13} \mathrm{~cm}$ is the classical electron radius. Equalizing the pressure gradient (1.5) to the gravitational force yields the balance equation

$$
\begin{equation*}
\sigma_{T} \frac{L}{4 \pi R^{2} c}=\frac{G M m_{p}}{R^{2}}, \tag{1.6}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{\mathrm{Edd}}=\frac{4 \pi G M m_{p} c}{\sigma_{T}} . \tag{1.7}
\end{equation*}
$$

Substituting the physical constants, one obtains

$$
\begin{equation*}
L_{\mathrm{Edd}} \approx 1.2 \times 10^{38}\left(\frac{M}{M_{\odot}}\right) \mathrm{erg} / \mathrm{s} \tag{1.8}
\end{equation*}
$$

It is convenient to introduce the Eddington accretion rate as a normalization

$$
\begin{equation*}
\dot{M}_{\mathrm{Edd}}=\frac{L_{\mathrm{Edd}}}{c^{2}}=\frac{4 \pi G M m_{p}}{\sigma_{T} c}, \tag{1.9}
\end{equation*}
$$

such that the dimensionless accretion rate is expressed as a fraction of $\dot{M}_{\text {Edd }}$

$$
\begin{equation*}
\dot{m}=\frac{\dot{M}}{\dot{M}_{\mathrm{Edd}}} \tag{1.10}
\end{equation*}
$$

### 1.6 X-ray spectral states

All known X-ray binary systems are usually observed in one of the five canonical "spectral states". The following classification is phenomological and is based on the relative strength of the "soft" black-body-like and "hard" power-law components in the photon energy spectrum, the magnitude of X-ray flux in $\sim 1-20 \mathrm{keV}$ range, and the time variability.

- The low-hard state is characterized by low luminosity and hard nonthermal powerlaw like energy spectrum $d N / d E \propto E^{-\alpha}$, with the photon index $\alpha=0.5-2$ and an exponential cutoff at around $100-200 \mathrm{keV}$. Sources observed in this state often exhibit rapid and strong broad-band variability with a fractional root mean square amplitude reaching $40 \%$.
- The high-soft state has higher photon flux and is dominated by the soft thermal component from an accretion disk, with a typical temperature of order of 1 keV . The power-law tail $(\alpha \sim 2.5)$ is present, but is much weaker than in the low state. No highenergy cutoff has been observed for the high-soft state. This state shows suppressed
temporal variability, which is of order of a few percent.
- The intermediate state is, naturally, occupied by the source between transitions from/to low/high states. Photon energy spectrum in the intermediate state shows both soft blackbody-like component and hard power-law.
- The very high state is characterized by several times higher luminosity than the high state. In this state the thermal and power-law components have similar fluxes. The power-law tail with a photon index $\sim 2.5-3$ does not seem to have a cutoff. The broad-band noise has fractional root mean square variability of order $1-10 \%$.
- The quiescent state is dominated by nonthermal power-law emission with photon index steeper than in the low-hard state and is characterized by very faint X-ray flux.

An intuitive graphical sketch of the five X-ray spectral states, Figure 1.1, showing how the geometry of the accretion flow changes as the mass accretion rate varies, was borrowed from [10].


Figure 1.1: Accretion flow in different X-ray spectral states as a function of the Eddingtonscaled (1.10) mass accretion rate. The Advection-Dominated Accretion Disk (corona) is indicated by dots, the thin disk by the horizontal bars. Image taken from A. A. Esin, J. E. McClintock, R. Narayan, "Advection-dominated Accretion and the Spectral States of Black Hole X-Ray Binaries: Application to Nova MUSCAE 1991" [10].

### 1.6.1 Power spectrum of an X-ray lightcurve

The Fourier power spectral analysis is a prime mathematical tool used in X-ray astronomy to perform temporal/variability studies. Typical X-ray timing analysis routine involves extracting (from data files) a photon lightcurve, where $N$ selected photons from an energy range of interest are "binned" into evenly-spaced time bins with $x_{i}$ counts per bin. The binned lightcurve is then used to calculate the power density spectrum, which is essentially the properly normalized squared Fourier amplitude, and describes how variability power is distributed as a function of frequency. As the result a lightcurve with $N$ bins, comprised of
counts, $x_{i}$, produces a power spectrum, with $N / 2+1$ independent amplitudes. A common convention is to use either Leahy normalization

$$
\begin{equation*}
P_{j}=2\left|X_{j}\right|^{2} / N_{\mathrm{ph}}, \tag{1.11}
\end{equation*}
$$

or (RMS/mean) $)^{2} / \mathrm{Hz}$ normalization

$$
\begin{equation*}
P_{j}=2\left|X_{j}\right|^{2} /\left(N_{\mathrm{ph}} \times\langle\text { Rate }\rangle\right), \tag{1.12}
\end{equation*}
$$

where the discrete Fourier transform of a time sequence $x_{k}$ is defined as

$$
\begin{equation*}
X_{j}=\sum_{k=0}^{N-1} x_{k} \exp (2 \pi i j k / N), \quad j=[-N / 2, \ldots, 0, \ldots, N / 2] \tag{1.13}
\end{equation*}
$$

Provided below brief description of the key power spectrum constituents, terminology and definitions are taken from from the book "Compact X-ray stellar sources" (W. Lewin and M. van der Klis, Cambridge University Press 2006) [30]. "A number of variability components or power-spectral components together make up the power spectrum. An aperiodic component by definition covers several, usually many, frequency resolution elements. Broad structures are called noise and narrow features quasi-periodic oscillations (QPOs); "broad-band noise" and "QPO peaks" are common terms. Least-square fitting techniques are used to measure these components." Some of the typical terms used include: "power law noise is noise that (in the frequency range considered) follows a power law $P_{\nu} \propto \nu^{-\alpha}$." "Band-limited noise (BLN) is defined here as noise that steepens towards higher frequency (i.e., its local power-law slope $-d \log P_{\nu} / d \log \nu$ increases with $\nu$ ) either abruptly (showing a "break" at break frequency $\nu_{\text {break }}$ ) or gradually." "The term peaked noise is used for noise whose $P_{\nu}$ has a local maximum at $\nu>0$. Various modified power laws (broken, cutoff) as well as broad Lorentzians are used to describe band-limited BLN." "A quasi-periodic oscillation (QPO) is a finite-width peak in the power spectrum. It can usually be described with a

Lorentzian $P_{\nu} \propto \lambda /\left[\left(\nu-\nu_{0}\right)^{2}+(\lambda / 2)^{2}\right]$ with centroid frequency $\nu_{0}$ and full width at half maximum (FWHM) $\lambda$. This is the power spectrum of an exponentially damped sinusoid $x(t) \propto e^{-t / \tau} \cos \left(2 \pi \nu_{0} t\right)$, but the underlying signal may well be different from this. $\lambda$ is related to the coherence time $\tau=1 / \pi \lambda$ of the signal, and is often reported in terms of the quality factor $Q \equiv \nu_{0} / \lambda$, a measure for the coherence of the QPO. Conventionally signals with $Q>2$ are called QPOs and those with $Q<2$ peaked noise. A sharp QPO peak is one with high Q. The strength (variance) of a signal is proportional to the integrated power $P=\int P_{\nu} d \nu$ of its contribution to the power spectrum, and is usually reported in terms of its fractional root-mean-squared (rms) amplitude $\mathrm{r} \propto P^{1 / 2}$, which is a measure for signal amplitude as a fraction of the source flux."

### 1.7 Existing models of power spectrum formation

Lyubarskii (1997) [33] was the first to suggest a self-consistent model for time variability production in accretion powered X-ray sources. His model assumes that the small-amplitude local (at each radius) fluctuations in the accretion rate are initiated by small variations in the viscosity. Such fluctuations propagate inward and modulate the accretion rate in the inner region of the disk, where the most of the observable X-ray flux is generated. His model predicts that for the radially-independent amplitude of variations the power spectrum of Xray luminosity decays with frequency as the power law $P_{\nu} \propto \nu^{-1}$, while for the amplitude of the variations which increases with radius, the power law of the power spectrum continuum becomes steeper than -1 . Lyubarskii's treatment, however, provides no physical model for the sources of the initial perturbations. Titarchuk, Shaposhnikov and Arefiev (2007) [61] (hereafter TSA2007) continued development of this model by formulating and solving the problem of local driving perturbation diffusion in a disk-like configuration. "The problem of the diffusive propagation of the space distributed high-frequency perturbations is formulated as a problem in terms of the diffusion equation for the surface density perturbations. The
parameters of the resulting PDS, diffusion timescale of the diffusion propagation of the local perturbations $t_{0}$ and the power law index of the viscosity distribution over radius are essential parameters of diffusion in a given bounded configuration. The PDS of the Green's function is a WRN power spectrum. Specifically, the low-frequency (LF) asymptotic form of the WRN PDS, when the frequency is less than the inverse of diffusion timescale in the disklike configuration $t_{0}^{-1}$, is characterized by a flat shoulder (white noise). In other words, the LF white noise shoulder is insensitive to the source and viscosity distributions over radius. The high-frequency (HF) asymptotic form of WRN is a power law $\nu^{-\alpha}$ with index $\alpha$, which is determined by the viscosity and perturbation source distributions over the accretion configuration." (L. Titarchuk and N. Shaposhnikov, "On the nature of the variability power decay toward soft spectral states in X-ray binaries: case study in Cygnus X-1", 2008) [60]. TSA2007 came to the conclusion that the White-Red Noise shape of the power spectrum continuum is an outcome of the solution of the initial-value problem for the spatially-distributed initial condition, defined over a bounded medium, which is given by a weighted sum of the related exponential shots. This thesis research represents the next stage in the development of the diffusive perturbation propagation model. We employ the TSA2007 formalism and combine it with the framework of the Centrifugal Barrier Model of Titarchuk, Lapidus and Muslimov 2008 [57] to consider the problem of the power spectrum formation in the non-Keplerian adjustment layer around accreting compact objects, as well as in the standard Keplerian disk. We solve the complete diffusion equation with the non-separable time and space dependent perturbation source and study the power spectrum properties as a function of accretion rate (via the Reynolds number of the flow) and investigate how the two power-spectral components (from compact transition zone and from extended disk) exhibit themselves, depending on a spectral state. Both analytical and numerical solutions are presented.

## Chapter 2: Radial structure of the accretion disk

In a close-binary system the gas material flowing out from the companion star, as a result of the Roche lobe overflow, typically has considerable angular momentum relative to the compact object, such that the material cannot fall on the accreter directly, but instead begins moving in circular orbits, forming an accretion disk. It is a good approximation to assume that the gas particles move in the disk with the Keplerian angular velocity

$$
\begin{equation*}
\omega_{\mathrm{K}}=(G M)^{1 / 2} R^{-3 / 2} \tag{2.1}
\end{equation*}
$$

In this chapter we derive the basic one-dimensional conservation equations that govern dynamics of accretion disks which obey (2.1) and consider the viscous boundary (transition) layer formed at the innermost part of the disk, in a case when it extends down to the surface of the accreting object (or the last stable circular orbit around a black hole), and when (2.1) is no longer valid. Adjustment of the Keplerian flow to the inner boundary is presented within the formalism of Titarchuk, Lapidus and Muslimov 2008 [57]. Solving the equation for the radial distribution of the angular velocity in the transition zone, we demonstrate that the transition layer size is a strong function of Reynolds number of the flow, and thus a function of mass accretion rate. We use the argument by Titarchuk and Fiorito 2004 [55] that at the point of adjustment formation of weak or strong shocks is likely, to treat the transition layer and the Keplerian disk as the two independent configurations. In this case the observed power spectrum is given by the sum of the power spectra from the individual components. Study of the emission mechanism in the non-Keplerian transition region is as important as in the Keplerian disk, since one half of all gravitational energy is released in this narrow region, and most of the detectable X-ray radiation produced there as well.

### 2.1 Conservation equations and viscous energy dissipation in a differentially rotating disk

Let the axially-symmetric accretion disk be characterized by the surface density (mass per unit surface area) which is the gas density $\rho$ integrated over the disk's height $\Sigma(R, t)=$ $\int_{0}^{2 H} \rho(z) d z$. Consider a thin cylindrical shell of disk material of radius $R$, thickness $d R \ll$ $R$, height $2 H$. In absence of sources/sinks within the shell the mass-continuity equation $\partial \rho / \partial t+\operatorname{div}(\rho \mathbf{v})=0$ written in cylindrical coordinates $(R, \varphi, z)$ reads

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\frac{1}{R} \frac{\partial}{\partial R}\left(R \rho v_{R}\right)+\frac{1}{R} \frac{\partial}{\partial \varphi}\left(\rho v_{\varphi}\right)+\frac{\partial}{\partial z}\left(\rho v_{z}\right)=0 . \tag{2.2}
\end{equation*}
$$

Azimuthal symmetry implies that the $\varphi$-derivative in (2.2) is zero. Integrating the entire equation vertically from 0 to $2 H$, we obtain

$$
\frac{\partial}{\partial t} \int_{0}^{2 H} \rho(z) d z+\frac{1}{R} \frac{\partial}{\partial R}\left(R v_{R} \int_{0}^{2 H} \rho(z) d z\right)+\int_{0}^{2 H} \frac{\partial}{\partial z}\left(\rho v_{z}\right) d z=0
$$

The third integral (a perfect differential), contributes zero, since $\rho(z=0)=\rho(z=2 H)=0$, yielding the mass conservation equation

$$
\begin{equation*}
\frac{\partial \Sigma}{\partial t}+\frac{1}{R} \frac{\partial}{\partial R}\left(R v_{R} \Sigma\right)=0 \tag{2.3}
\end{equation*}
$$

The radial velocity component in the continuity equation (2.3) is negative ( $v_{R}<0$ ), reflecting the fact that the accreting material is moving towards the compact object. In a typical disk the azimuthal velocity component of accreting gas is much higher than the radial one $\left|v_{\varphi}\right| \gg\left|v_{R}\right|$. The total flux of material passing through radius $R$ in the disk towards the
center (the mass accretion rate) can be determined as follows: consider a disk segment confined by $R, d \varphi, d z$. The area element through which mass is moving in is $d A=R d \varphi d z$. The amount of mass passing through $d A$ during time interval $d t$ is $d M=-\rho v_{R} d t d A$ (minus sign appears because $\left.v_{R}<0\right)$. Correspondingly $\dot{M}=-\rho v_{R} d A$, and after performing integration over $\varphi$ and $z$, we obtain the accretion rate $[\mathrm{g} / \mathrm{s}]$

$$
\begin{equation*}
\dot{M}=-2 \pi R \int_{0}^{2 H} \rho v_{R} d z=-2 \pi R \Sigma v_{R} \tag{2.4}
\end{equation*}
$$

In a case when a mass accretion source is present in the shell the corresponding term appears on the right-hand side of the mass conservation law

$$
\begin{equation*}
\frac{\partial \Sigma}{\partial t}+\frac{1}{R} \frac{\partial}{\partial R}\left(R v_{R} \Sigma\right)=S(R, t) \tag{2.5}
\end{equation*}
$$

The angular momentum balance in the disk can be derived in a similar manner. The ring of gas has mass $M=4 \pi R d R H \rho$ and angular momentum $L=M v_{\varphi} R=M \omega R^{2}=$ $4 \pi R d R H \rho \omega R^{2}$. The rate of change of $L$ is given by the net flow of angular momentum from neighboring annuli at $R$ and $R+d R$ plus the net torque $N_{\text {visc }}$ of viscous forces acting on the annulus of gas between $R$ and $R+d R$

$$
\frac{\partial L}{\partial t}+\operatorname{div}(L \mathbf{v})=N_{\mathrm{visc}}
$$

or writing in cylindrical coordinates

$$
\begin{equation*}
\frac{\partial L}{\partial t}+\frac{1}{R} \frac{\partial}{\partial R}\left(R L v_{R}\right)+\frac{1}{R} \frac{\partial}{\partial \varphi}\left(L v_{\varphi}\right)+\frac{\partial}{\partial z}\left(L v_{z}\right)=N_{\text {visc }} \tag{2.6}
\end{equation*}
$$

Substitution of $L$ into (2.6) gives, after dropping the $\varphi$-derivative term and dividing through by $4 \pi R d R H$,

$$
\frac{\partial}{\partial t}\left(\rho \omega R^{2}\right)+\frac{1}{R} \frac{\partial}{\partial R}\left(R \rho \omega R^{2} v_{R}\right)+\frac{\partial}{\partial z}\left(\rho \omega R^{2} v_{z}\right)=\frac{1}{4 \pi R d R H} N_{\mathrm{visc}} .
$$

After performing vertical integration from $z=0$ to $z=2 H$, with the assumption that the angular velocity $\omega$ does not vary along $z$-axis for given radius $R$, we have

$$
\frac{\partial}{\partial t}\left(\Sigma \omega R^{2}\right)+\frac{1}{R} \frac{\partial}{\partial R}\left(R \Sigma \omega R^{2} v_{R}\right)=\frac{1}{2 \pi R d R} N_{\mathrm{visc}} .
$$

"The net torque on a ring of gas between $R$ and $R+d R$ is subject to competing torques" (J. Frank, A. King and D. Raine "Accretion power in astrophysics" 2002) [12] on the two edges of the ring

$$
\begin{equation*}
N_{\mathrm{visc}}=G(R+d R)-G(R)=\frac{\partial G}{\partial R} d R \tag{2.7}
\end{equation*}
$$

where the torque between the two neighboring rings is (Appendix C)

$$
\begin{equation*}
G(R)=2 \pi R^{2} \nu \Sigma R \frac{d \omega}{d R} \tag{2.8}
\end{equation*}
$$

Thus the angular momentum balance is expressed by equation

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\Sigma \omega R^{2}\right)+\frac{1}{R} \frac{\partial}{\partial R}\left(R \Sigma \omega R^{2} v_{R}\right)=\frac{1}{2 \pi R} \frac{\partial G}{\partial R} . \tag{2.9}
\end{equation*}
$$

Using the fact that for a constant gravitational potential $\partial \omega / \partial t=0$ [12], and expressing $\partial \Sigma / \partial t$ from the mass conservation law (2.3), equation (2.9) can be simplified to

$$
\begin{equation*}
-\dot{M} \frac{d}{d R}\left(\omega R^{2}\right)=\frac{d G}{d R}, \tag{2.10}
\end{equation*}
$$

where $\dot{M}$ is given by (2.4). It is convenient to write the viscous torque $G$ in terms of the $r \varphi$-component of the viscous stress tensor $\tau$ (Appendix C). Integrating $\tau_{r \varphi}$ vertically we obtain

$$
\begin{equation*}
W_{r \varphi}=\int_{0}^{2 H} \tau_{r \varphi} d z=2 H \cdot \eta R \frac{d \omega}{d R} \approx \nu \Sigma R \frac{d \omega}{d R}, \tag{2.11}
\end{equation*}
$$

where $\eta=\rho \nu$ is dynamic viscosity and we used $\Sigma \approx 2 H \rho$. Substitution of (2.11) into (2.8) gives

$$
\begin{equation*}
G(R)=2 \pi R^{2} W_{r \varphi}, \tag{2.12}
\end{equation*}
$$

producing the angular momentum equation expressed in terms of $W_{r \varphi}$

$$
\begin{equation*}
\dot{M} \frac{d}{d R}\left(\omega R^{2}\right)=-2 \pi \frac{d}{d R}\left(W_{r \varphi} R^{2}\right) . \tag{2.13}
\end{equation*}
$$

The energy flux radiated from the disk surface can be obtained from the power $P_{\text {visc }}$ [erg $\mathrm{s}^{-1}$ ] dissipated by the viscous torques [12]. For the annulus of matter confined between $R$ and $R+d R$

$$
\begin{equation*}
P_{\mathrm{visc}}=\omega \cdot N_{\mathrm{visc}} . \tag{2.14}
\end{equation*}
$$

Substituting formula (2.7) for the torque and using the identity

$$
\omega \frac{\partial G}{\partial R}=\frac{\partial}{\partial R}(\omega G)-G \frac{\partial \omega}{\partial R},
$$

we obtain

$$
\begin{equation*}
P_{\mathrm{visc}}=\frac{\partial}{\partial R}(\omega G) d R-G \frac{\partial \omega}{\partial R} d R . \tag{2.15}
\end{equation*}
$$

The energy flux $F\left[\mathrm{erg} \mathrm{s}^{-1} \mathrm{~cm}^{-2}\right]$ is then given by the ratio of the power of viscous dissipation [second term in equation (2.15)] to emission area

$$
\begin{equation*}
F(R)=\frac{P_{\mathrm{visc}}^{\mathrm{diss}}}{d A_{\mathrm{ring}}}=\frac{G \omega^{\prime} d R}{2 \times 2 \pi R d R}=\frac{G \omega^{\prime}}{4 \pi R}, \tag{2.16}
\end{equation*}
$$

where each ring has two ( $z+$ and $z-$ ) plane faces. Using expression (2.12) for $G(R)$ we finally obtain

$$
\begin{equation*}
F(R)=\frac{1}{2} W_{r \varphi} R \frac{d \omega}{d R} . \tag{2.17}
\end{equation*}
$$

For a steady-state accretion $\dot{M}=$ const, and integration of (2.13) gives

$$
\begin{equation*}
\dot{M} \omega R^{2}=-2 \pi W_{r \varphi} R^{2}+C . \tag{2.18}
\end{equation*}
$$

The integration constant, according to Shakura \& Sunyaev 1973 [47], is determined by the condition that $W_{r \varphi}$ vanishes at the innermost stable circular orbit $R_{0}=3 R_{\mathrm{S}}$ in the Schwarzschild gravitational field of a black hole or at the neutron star's surface

$$
\begin{equation*}
C=\dot{M} \omega\left(R_{0}\right) R_{0}^{2} \tag{2.19}
\end{equation*}
$$

This allows us to express $W_{r \varphi}$ from (2.18):

$$
\begin{equation*}
W_{r \varphi}=-\frac{1}{2 \pi} \dot{M} \omega(R)\left[1-\frac{\omega\left(R_{0}\right)}{\omega(R)}\left(\frac{R_{0}}{R}\right)^{2}\right] . \tag{2.20}
\end{equation*}
$$

Thus, the energy flux radiated from the disk surface is

$$
\begin{equation*}
F(R)=-\frac{1}{4 \pi} \dot{M} \omega(R) R \frac{d \omega}{d R}\left[1-\frac{\omega\left(R_{0}\right)}{\omega(R)}\left(\frac{R_{0}}{R}\right)^{2}\right] \tag{2.21}
\end{equation*}
$$

For the Keplerian motion $\omega(R)$ is given by (2.1), and consequently

$$
\left\{\begin{array}{l}
\frac{d \omega}{d R}=-\frac{3}{2} \sqrt{G M} R^{-3 / 2} R^{-1}  \tag{2.22}\\
\frac{\omega\left(R_{0}\right)}{\omega(R)}=\left(\frac{R}{R_{0}}\right)^{3 / 2}
\end{array}\right.
$$

Using these relations, we obtain

$$
\begin{equation*}
F(R)=\frac{3}{8 \pi} \frac{G M \dot{M}}{R^{3}}\left[1-\left(\frac{R_{0}}{R}\right)^{1 / 2}\right], \tag{2.24}
\end{equation*}
$$

The luminosity [ $\mathrm{erg} \mathrm{s}^{-1}$ ] of the annulus of the disk confined between radii $R_{1}$ and $R_{2}$ is given by the energy flux integrated over the area of the annulus

$$
\begin{equation*}
L=4 \pi \int_{R_{1}}^{R_{2}} F(R) R d R=\frac{3}{2} G M \dot{M}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right), \tag{2.25}
\end{equation*}
$$

in case when $\dot{M}$ is independent of $R$. These results are well-known from the standard theory of accretion disks of Shakura \& Sunyaev 1973 [47]. We will use formulas (2.24), (2.25) to derive the equation for accretion rate fluctuations $\triangle \dot{M}$ as a function of radius and time.

### 2.2 Viscous transition layer

### 2.2.1 Sub-Keplerian motion in the vicinity of the compact object

In cases when the accretion disk extends down to the surface of the neutron star or the innermost stable circular orbit around a slowly-rotating black hole, relation (2.1) is not entirely correct. A neutron star cannot rotate faster than the break-up speed (Appendix A) at its equator $\omega_{*}<\omega_{\mathrm{K}}$ [12]. In fact, the majority of neutron stars are observed to rotate slower than a few revolutions per second [18]. Neutron stars in close-binary systems
can get spun up to a few hundreds Hz by the transfer of angular momentum from the companion star. The most rapidly rotating neutron star currently known, eclipsing binary millisecond pulsar PSR J1748-2446ad, has rotational frequency $f_{*}=716 \mathrm{~Hz}$, mass $<2 M_{\odot}$, and estimated upper limit on radius $R_{*} \sim 16 \mathrm{~km}$ [18]. A simple calculation (2.1) for a test particle moving in the Keplerian orbit at $R=R_{*}$ gives $f_{\mathrm{K}} \approx 1282 \mathrm{~Hz}$, i.e. the Keplerian rotation at the surface of a star is $\sim 1.8$ times faster than the fastest-spinning neutron star. The difference, of course, is more dramatic for the slower spinning stars. With decreasing radius the angular velocity of the flow will remain Keplerian (and therefore increasing) before some critical point, after which it begins to rapidly decrease (adjust) to the value $\omega_{*}$ in a thin boundary layer of hot plasma [12], [47]. The motion in the boundary layer is not Keplerian and is not described by (2.1). It is a rather remarkable fact, that a substantial fraction (approximately one half) of the total accretion energy is deposited in such nonKeplerian adjustment boundary layer (the other half is radiated from extended Keplerian disk) [12], [20], [41]. In fact, a dominant portion of all detectable X-ray radiation (photons with $h \nu \gtrsim 1 \mathrm{keV}$ ) is generated in the compact area of the size of a few $R_{\mathrm{S}}$ around the accreter, within which the boundary layer is contained. The following argument is reconstructed from derivations by J. Frank, A. King and D. Raine "Accretion power in astrophysics" 2002 [12] and S. Kato, J. Fukue and S. Mineshige "Black hole accretion disks" 1998 [20]. For a massive compact central object $M$ of "radius $R_{*}$ the gravitational potential energy released by the accretion of a mass $m$ on to its surface is" [12]

$$
\begin{equation*}
\triangle E_{\mathrm{acc}}=G \frac{M m}{R_{*}} \tag{2.26}
\end{equation*}
$$

"If all the kinetic energy of infalling matter is given up to radiation at the stellar surface $R_{*}$, then the accretion luminosity is" [12]

$$
\begin{equation*}
L_{\mathrm{acc}}=G \frac{M \dot{M}}{R_{*}} \tag{2.27}
\end{equation*}
$$

where $\dot{M}=d m / d t$. On the other hand [20], the luminosity of the Keplerian disk can be estimated as follows: a test particle $m_{\mathrm{p}}$ rotating in a circular orbit of radius $R$ has total energy

$$
\begin{equation*}
E=T+U=\frac{m_{\mathrm{p}} v_{\phi}^{2}}{2}-\frac{G M m_{\mathrm{p}}}{R}=-\frac{G M m_{\mathrm{p}}}{2 R}, \tag{2.28}
\end{equation*}
$$

where for Keplerian orbit

$$
\begin{equation*}
v_{\phi}=\omega_{\mathrm{K}} R=\sqrt{G M} R^{-\frac{1}{2}} . \tag{2.29}
\end{equation*}
$$

If the particle falls from infinity to radius $R$ with zero total energy, the amount of excess energy released during the accretion is

$$
\begin{equation*}
\triangle E_{\mathrm{rad}}=0-(T+U)=\frac{G M m_{\mathrm{p}}}{2 R} . \tag{2.30}
\end{equation*}
$$

In other words, in the case of the standard non-relativistic accretion disk its luminosity is given by

$$
\begin{equation*}
L_{\mathrm{disk}}=\frac{G M \dot{M}}{2 R_{\mathrm{in}}} \tag{2.31}
\end{equation*}
$$

where $R_{\mathrm{in}}$ is the inner radius of the Keplerian flow. We notice that the luminosity (2.31) of the Keplerian disk is only a half of the total available accretion luminosity (2.27). "This discrepancy arises because the matter just outside the boundary layer still retains one half of the potential energy it has lost as kinetic energy" (J. Pringle "Accretion discs in astrophysics" 1981) [41]. Thus, the rest of the accretion luminosity must be emitted in the boundary layer very close to the central star.

### 2.2.2 X-ray emission zone

It is helpful to have an order of magnitude estimates of the size of the emission region responsible for X-ray production for characteristic black hole/neutron star objects. Suppose that the accretion disk extends all the way down to the surface $R_{*}$ of a neutron star or the
innermost stable circular orbit (1.2) $R_{0}$ around a black hole. In fact, $R_{0}$ is often used as an approximate neutron star radius, such that $R_{*} \approx R_{0}$. Consider an annulus of the disk confined between $R_{0}$ and some radius $R^{\prime}>R_{0}$, such that ring's radial size is $\delta=R^{\prime}-R_{0}$. For optically thick accretion flow, the radiation from the ring will reach thermal equilibrium with the matter before being detected by a remote observer. Assuming that thermal photons are radiated away from the disk's two surfaces, the luminosity [51] of matter, drifting towards the accreter, is given by

$$
\begin{equation*}
L=\frac{1}{2} \frac{G M \dot{M}}{R_{0}}=\text { Flux }_{\mathrm{BB}} \times \text { Area }=\sigma T^{4} \cdot 2 \cdot 2 \pi R_{0} \delta \tag{2.32}
\end{equation*}
$$

where $\sigma=5.67 \times 10^{-5} \mathrm{erg} \mathrm{cm}^{-2} \mathrm{~K}^{-4} \mathrm{~s}^{-1}$ is the Stefan-Boltzmann constant. Therefore temperature of an optically thick ring related to its size by

$$
\begin{equation*}
T=T_{\mathrm{opt} \mathrm{thk}}=\left(\frac{G M \dot{M}}{8 \pi R_{0}^{2} \sigma}\right)^{1 / 4} \delta^{-1 / 4} \tag{2.33}
\end{equation*}
$$

or expressing $\delta$ as a function of $T$

$$
\begin{equation*}
\delta=\left(\frac{G M \dot{M}}{8 \pi R_{0}^{2} \sigma}\right) T^{-4} \tag{2.34}
\end{equation*}
$$

Consider two characteristic compact objects - a $1.4 M_{\odot}$ neutron star, and a massive $10 M_{\odot}$ black hole. For a neutron star $R_{0}^{\mathrm{NS}}=3 R_{\mathrm{S}}=6 G M / c^{2}=1.24 \times 10^{6} \mathrm{~cm}$, while for a black hole $R_{0}^{\mathrm{BH}}=8.85 \times 10^{6} \mathrm{~cm}$. It is more intuitive to use units of km to get the feel of the scale. So, let us put approximately $R_{0}^{\mathrm{NS}} \approx 12 \mathrm{~km}, R_{0}^{\mathrm{BH}} \approx 90 \mathrm{~km}$. Taking the mass transfer rate to be the Eddington accretion rate (1.9), we have $\dot{M}_{\mathrm{NS}} \approx 1.96 \times 10^{17} \mathrm{~g} / \mathrm{s}$ for a neutron star, and $\dot{M}_{\mathrm{BH}} \approx 1.4 \times 10^{18} \mathrm{~g} / \mathrm{s}$ for a black hole. Many orbital X-ray observatories have effective detection threshold $h \nu_{\min } \sim 1 \mathrm{keV}$, which, using $E=k_{\mathrm{B}} T, k_{\mathrm{B}}=8.617 \times 10^{-5}$
$\mathrm{eV} / \mathrm{K}$, is equivalent to temperature $T \approx 1.16 \times 10^{7} \mathrm{~K}$. Substituting these numbers into (2.34) gives the radial size of X-ray emission zone: $\delta_{\mathrm{NS}}=\delta_{\mathrm{BH}}=0.92 \times 10^{6} \mathrm{~cm}(9 \mathrm{~km})$. Notice that $\delta$ is independent of mass of the object, since $M^{2}$ present in both numerator and denominator in (2.34), when formula for $\dot{M}_{\text {Edd }}$ is substituted. Therefore we find that X-ray (photons with $h \nu \geq 1 \mathrm{keV}$ ) emission zone extends from $R_{0}^{\mathrm{NS}}$ to $R_{0}^{\mathrm{NS}}+\delta \approx 21 \mathrm{~km}$ $\sim 5 R_{\mathrm{S}}$ for a neutron star, and from $R_{0}^{\mathrm{BH}}$ to $R_{0}^{\mathrm{BH}}+\delta \approx 99 \mathrm{~km} \approx 3.3 R_{\mathrm{S}}$ in case of a black hole. These elementary calculations demonstrate that most of the detectable X-ray radiation is produced in a compact region around the central object of the size of only a few Schwarzschild radii.

### 2.2.3 TLM98 model of the adjustment layer

In 1998 Titarchuk, Lapidus, Muslimov [57], hereafter TLM98, proposed the comprehensive physical model (Centrifugal Barrier model) of a bounded compact coronal region around an accreting black hole or neutron star, "that is a consequence of dynamical adjustment of the Keplerian disk flow to the innermost sub-Keplerian boundary condition near the central object", (TLM98). In the following section we provide a brief overview of their model, and present a simple yet convincing argument, which will allow us to treat a problem of diffusive propagation of perturbations in the boundary layer analytically.

According to TLM98 "the disk structure begins deflecting from a Keplerian one at a certain point to adjust itself to the boundary condition at the surface" of a neutron star, or at the innermost stable circular orbit around a black hole. We will adopt TLM98 notation and call the ring of the disk confined between $R_{0}$ and the last Keplerian orbit $R_{\text {adj }}$ the transition layer. Recall that the angular momentum balance is governed by equation (2.13)

$$
\begin{equation*}
\dot{M} \frac{d}{d R}\left(\omega R^{2}\right)=-2 \pi \frac{d}{d R}\left(W_{r \varphi} R^{2}\right), \tag{2.35}
\end{equation*}
$$

where $W_{r \varphi}$ is the shear stress (2.11) exerted on the layer surface integrated over the disk height, $\eta$ is the turbulent viscosity. Substituting $W_{r \varphi}$, one obtains a second-order differential equation for angular velocity

$$
\begin{equation*}
\omega^{\prime \prime}+\frac{\gamma+3}{R} \omega^{\prime}+\frac{2 \gamma}{R^{2}} \omega=0 . \tag{2.36}
\end{equation*}
$$

One can notice that the single parameter that enters equation (2.36), and that regulates the adjustment of $\omega$ in the transition layer to $\omega_{\mathrm{K}}$, is the Reynolds number of the flow

$$
\begin{equation*}
\gamma=\frac{\dot{M}}{4 \pi \eta H}=\frac{\rho}{\eta} v_{R} R \equiv \operatorname{Re} \tag{2.37}
\end{equation*}
$$

where characteristic velocity and length scales are $v_{R}$ and $R$. To get an idea of what values of $\gamma$ one may expect from observing X-ray radiation from disk accretion onto compact objects, recall that the Reynolds number can be expressed as the inverse of the $\alpha$-parameter of Shakura \& Sunyaev, which is defined so that $\alpha<1$. Thus, $\gamma$ is always expected to be greater than unity. In fact, analysis of power spectra, done by [60] and [61], for a number of X-ray black hole and neutron star binary systems in different spectral states, suggests that the effective Reynolds number varies from $\gamma \sim 3$ in a low-hard state to approximately $\gamma \sim 80$ in a high-soft state. Utilizing the physical interpretation of the Reynolds number (relative strength of inertial forces compared to viscous forces), we infer that in the case of disk accretion inertial forces of the flow always dominate the viscous ones.

The inner boundary condition is dictated by the continuity of the flow and requires that the angular frequency of the flow matches the angular frequency $\omega_{0}$ of a neutron star, or the last stable orbit around a Schwarzschild black hole

$$
\begin{equation*}
\omega=\omega_{0} \quad \text { at } R=R_{0} . \tag{2.38}
\end{equation*}
$$

The outer boundary condition requires the smooth matching of the Keplerian flow at some (not a priori known) adjustment radius $R_{\text {adj }}$

$$
\begin{cases}\omega=\omega_{\mathrm{K}} & \text { at } R=R_{\mathrm{adj}}  \tag{2.39}\\ \frac{d \omega}{d R}=\frac{d \omega_{\mathrm{K}}}{d R} & \text { at } R=R_{\mathrm{adj}}\end{cases}
$$

The boundary problem formulated as the second-order differential equation (2.36) combined with the three boundary conditions (2.38)-(2.40) and unspecified value of $r_{\text {adj }}$ is perfectly determined: three constants matched up with three boundary conditions. Such formulation ([23], Ch. 9.3) uniquely determines the distribution of the angular velocity $\omega(R)$ in the boundary region, as well as the position of the adjustment point $R_{\text {adj. }}$ TLM98 solved (2.36) with the boundary conditions (2.38)-(2.40) for the case where parameter $\gamma$ is constant. Introducing dimensionless variables

$$
\left\{\begin{array}{l}
m=M / M_{\odot}  \tag{2.41}\\
\theta=\omega / \omega_{0} \\
r=R / R_{0}
\end{array}\right.
$$

where $R_{0}=3 R_{\mathrm{S}}, R_{\mathrm{S}}=2 G M / c^{2}$ is the Schwarzschild radius, the resulting angular velocity profile has the form

$$
\begin{equation*}
\theta(r)=D_{1} r^{-\gamma}+\left(1-D_{1}\right) r^{-2} \tag{2.44}
\end{equation*}
$$

Factor $D_{1}$ is given by the ratio

$$
\begin{equation*}
D_{1}=\frac{\theta_{\mathrm{adj}}-r_{\mathrm{adj}}^{-2}}{r_{\mathrm{adj}}^{-\gamma}-r_{\mathrm{adj}}^{-2}} \tag{2.45}
\end{equation*}
$$

where $\theta_{\text {adj }}=\theta_{\mathrm{K}}\left(r_{\text {adj }}\right)$. The Keplerian angular velocity, expressed in terms of new variables, can be obtained from

$$
\begin{equation*}
\theta_{\mathrm{K}}=\frac{\omega_{\mathrm{K}}}{\omega_{0}}=\frac{1}{\omega_{0}} \sqrt{\frac{G m M_{\odot}}{\left(R_{0} r\right)^{3}}} . \tag{2.46}
\end{equation*}
$$

TLM98 normalized the angular velocity by the value $\omega_{1728}=2 \pi \times 363 \mathrm{rad} / \mathrm{s}$, because of the remarkable 363 Hz quasi-periodic oscillation (QPO) feature, which is thought to be related to the rotation period of a neutron star 4 U 1728-34 [52]. Expressing rotational velocity of a compact object as $\omega_{0}=\omega_{1728}\left(\omega_{0} / \omega_{1728}\right)$, we have, after substituting numbers into (2.46)

$$
\begin{equation*}
\theta_{\mathrm{K}} \approx \frac{6}{m\left(\frac{f_{0}}{363}\right)} r^{-3 / 2} \tag{2.47}
\end{equation*}
$$

where $f_{0}=\omega_{0} /(2 \pi)$ is the rotational frequency at the inner boundary. The outer boundary condition (2.40), written in terms of new variables, now reads

$$
\begin{equation*}
\frac{3}{2} \theta_{\mathrm{adj}}=D_{1} \gamma r_{\mathrm{adj}}^{-\gamma}+2\left(1-D_{1}\right) r_{\mathrm{adj}}^{-2} . \tag{2.48}
\end{equation*}
$$

The principal difference of the radial profile (2.44) from Keplerian (2.1) is that it is not a monotonically-decreasing function of radius, but has a maximum at

$$
\begin{equation*}
r_{\max }=\left(\frac{\gamma}{2}\right)^{\frac{1}{\gamma-2}}\left[\frac{\theta_{\text {adj }}-r_{\text {adj }}^{-2}}{\theta_{\text {adj }}-r_{\text {adj }}^{-\gamma}}\right]^{\frac{1}{\gamma-2}}, \tag{2.49}
\end{equation*}
$$

after which $\theta(r)$ rapidly falls off in region $1<r<r_{\text {max }}$, to match $\theta=1$ at the inner boundary $r=1$ (Figure 2.1). It is easy to see from the boundary condition at the adjustment point (2.48) that the radial size $r_{\text {adj }}$ of the transition layer is solely determined by the parameter $\gamma$. Combination of equations (2.45), (2.48) along with (2.46) results in a non-linear equation
which can be solved for $r_{\text {adj }}$ as a function of $\gamma$

$$
\begin{equation*}
f\left(r_{\text {adj }}\right)=(4-2 \gamma) r_{\text {adj }}^{-(\gamma+2)}+(2 \gamma-3) \frac{6}{m}\left(\frac{f_{0}}{363}\right)^{-1} r_{\text {adj }}^{-(\gamma+3 / 2)}-\frac{6}{m}\left(\frac{f_{0}}{363}\right)^{-1} r_{\text {adj }}^{-7 / 2}=0 . \tag{2.50}
\end{equation*}
$$

A visual inspection of this equation suggests that it can yield an approximate form of the dependence $r_{\text {adj }}=f(\gamma)$ without invoking a numerical method. If set $f_{0}=363 \mathrm{~Hz}$, denote $a=6 / m$, and replace (approximate) the term $a(2 \gamma-3) r_{\text {adj }}^{-(\gamma+3 / 2)}$ by $a(2 \gamma-4) r_{\text {adj }}^{-(\gamma+2)}$, after a little algebra we obtain

$$
r_{\mathrm{adj}} \approx \exp \left[\frac{2}{2 \gamma-3} \cdot \ln \frac{(a-1)(2 \gamma-4)}{a}\right] .
$$

The logarithmic factor determines the scale of the function, whose principal behavior is


Figure 2.1: Angular velocity $\theta$ as a function of radius $r$ for a $1.4 M_{\odot}$ neutron star.


Figure 2.2: Radius of maximum $\theta$ and adjustment radius as functions of Reynolds number.
controlled by the exponent

$$
\begin{equation*}
r_{\text {adj }} \propto e^{2 /(2 \gamma-3)} . \tag{2.51}
\end{equation*}
$$

Hence, we conclude that the adjustment radius is a rapidly-decaying function of $\gamma$, where in the limit of infinitely large Reynolds number radial extent $r_{\mathrm{adj}}-1$ of the transition layer asymptotically approaches zero. The exact numerical solution $r_{\text {adj }}=f(\gamma)$ for a sample 1.4 $M_{\odot}$ neutron star, rotating with $f_{0}=300 \mathrm{~Hz}$, is shown on Figure 2.2. The functional dependence $r_{\text {adj }}=f(\gamma)$ is very insensitive to the parameters of the accreter, $m$ and $f_{0}$, so the Figure 2.2 represents the characteristic behavior of the adjustment radius of the boundary layer as a function of Reynolds number. Estimates for $1.4 M_{\odot}$ neutron star and $10 M_{\odot}$ black hole show that the region $\delta r=r_{\text {adj }}-r_{\max }$ shrinks by a factor $\sim 75$ (!), for both types of objects, as Reynolds number changes from $\gamma=3$ (typically associated with low-hard spectral state) and $\gamma=80$ (corresponds to high-soft spectral state). It is worth noting that calculations by [35] for inner radius of the Keplerian disk in the low-hard state
$\left(\leq 6-25 G M / c^{2}\right)$ are in a good agreement with the estimate for $R_{\text {adj }} \approx 18 G M / c^{2}$ for $\gamma=3$. For sufficiently large values of Reynolds number, "thickness" of the transition zone is many times less than the characteristic size of the system. For example, for parameter $\gamma=50$ the characteristic radius of the $1.4 M_{\odot}$ system $\left(R_{*}=R_{\max } \approx 13.2 \mathrm{~km}\right)$ by more than an order of magnitude exceeds the radial size ( $d_{\mathrm{TL}} \approx 1.2 \mathrm{~km}$ ) of the transition layer. Therefore, the transition layer can be approximated by a "thin shell", such that

$$
\begin{equation*}
d_{\mathrm{TL}}=R_{\mathrm{adj}}-R_{0} \ll R_{*}, \tag{2.52}
\end{equation*}
$$

for $\gamma \gg 1$. We will use this approximation as the main argument in simplifying the diffusion equation, when solving the problem analytically.

### 2.2.4 Two X-ray emitting configurations: compact adjustment region and extended disk

One should realize that ideal "adjustment of the Keplerian rotation to the sub-Keplerian inner boundary condition" (TLM98) (2.38) is only possible in a highly-unlikely case, when both $\omega$ and $d \omega / d R$ match with $\omega_{\mathrm{K}}$ and $d \omega_{\mathrm{K}} / d R$ at the adjustment point. This condition allows for unique determination of position of the adjustment radius. When the outer boundary condition (2.39), (2.40) is not satisfied, the solution of equation (2.36), satisfying the inner boundary condition will inevitably pass through the zone of super-Keplerian rotation [57]. Such a scenario is demonstrated on Figure 2.3, where the green line represents smooth adjustment of the angular velocity $\omega$ to the inner boundary condition; navy-blue line represents the adjustment where $d \omega / d r \neq d \omega_{\mathrm{K}} / d r$ at $r_{\text {adj }}$. In a real accretion flow presence of kinks [34], [26] is accompanied by discontinuities in the the velocity components derivatives, and thus implies discontinuities in $\omega^{\prime}$. The presence of a super-Keplerian region can significantly affect the flow dynamics in the transition layer, in particular, a centrifugal barrier can develop as accreting material enters the zone with super-Keplerian motion. Titarchuk and Fiorito, 2004, [55] argue that when adjustment to the sub-Keplerian flow is
not smooth strong or weak shocks are likely to be formed at $r_{\text {adj }}$. Extensive observational


Figure 2.3: Ideal adjustment (green line) of Keplerian rotation to sub-Keplerian BC and realistic adjustment (navy line) through a super-Keplerian motion for $1.4 M_{\odot}$ neutron star and parameter $\gamma=15$.
timing analysis conducted by [13] and [61] shows that there can be several components in the accretion flow, each having a different viscous timescale, exhibiting themselves as independent continuum components in the the power density spectrum. The characteristic timescales of the two most prominent components suggest that they may be related to compact (of order of a few $R_{\mathrm{S}}$ ) configuration, with timescales ranging from milliseconds to a few seconds, and independent extended configuration (of order of thousands of $R_{\mathrm{S}}$ ), with timescales ranging from several days to years. Notably, the high-frequency power spectrum component seems to evolve completely autonomously with respect to the low-frequency part. In the framework of the TLM98 transition layer model, presence of discontinuities and abrupt transitions in the solution at the adjustment radius can result in that the X-ray signal detected by a distant observer is comprised of the two weakly correlated components, one emerging from the compact non-Keplerian transition region, the other from the
extended Keplerian disk. Thus, their model with non-ideal adjustment can, potentially explain the observed two-component power spectra. We test this assumption by solving the diffusion propagation problem for the two independent bounded configurations and calculating their power spectra. Titarchuk, Shaposhnikov and Arefiev 2007 [61] demonstrate that the resulting power spectrum of a sum of two weakly correlated signals is given by the sum of power spectra of the two components. Namely,

$$
\begin{equation*}
\left\|\mathcal{F}_{X_{1}+X_{2}}(\omega)\right\|^{2}=\left\|\mathcal{F}_{X_{1}}(\omega)\right\|^{2}+\left\|\mathcal{F}_{X_{2}}(\omega)\right\|^{2} \tag{2.53}
\end{equation*}
$$



Figure 2.4: Schematic representation of the transition layer.

In the following Chapters we calculate the power spectra for the two configurations and demonstrate that observed two-component spectrum continuum can be reproduced by the TLM98 model. Keeping the contribution of the Keplerian disk to the power spectrum unchanged, we show that, depending on the accretion rate, the variability of the transition layer component can produce various observed patterns in the power spectrum, where the high-frequency component completely dominates the low-frequency one, opposite case where
the low-frequency component is dominating, and the case where both components have comparable power. The graphical illustration of the transition zone is presented on Figure 2.4.

# Chapter 3: Analytical solution of the simplified diffusion propagation problem for compact non-Keplerian configuration 

In this chapter we demonstrate that the generic shape of the Fourier power spectrum of X-ray luminosity fluctuations emerging from the transition region near the compact object can be obtained in analytical form. Using equations of mass conservation and angular momentum transport in a disk-like configuration, derived in Chapter 2, we write the evolution equation for the mass transfer rate fluctuations $\triangle \dot{M}$, specify the boundary conditions, and show that for the source term with separable space and time parts the solution of this problem is reduced to the convolution of the solution of the initial-value problem with the time part of the source term. Employing the convolution theorem (Appendix B), it follows that the resulting power spectrum is given by the product of the power spectrum of the solution of the initial-value problem and the power spectrum of the source function. We demonstrate that due to the nature of the problem with the dynamic adjustment of the Keplerian flow to the sub-Keplerian boundary condition at the surface of a neutron star (or the innermost stable circular orbit around a black hole) the diffusion operator inevitably changes its sign. Therefore, we consider the two problems in the two space domains. We then solve the diffusion problem for the trivial initial conditions to find accretion rate fluctuations $\triangle \dot{M}(R, t)$ at each radius in the region, integrate this solution over the radial range, and finally calculate the power spectrum of the integrated X-ray signal. Analysis of the power spectrum continuum is done by comparison to the power spectrum of a single exponential shot. We discover a strong dependence of the power spectrum on the Reynolds number, connected with the accretion rate, and therefore with the spectral state, and introduce characteristic physical quantities that have immediate interpretation in observational analysis.

### 3.1 Formulation of the problem of power spectrum formation

### 3.1.1 X-ray luminosity fluctuations in the vicinity of a compact object

In the case when the accretion rate is the function of both radius and time, the total luminosity (2.25) of the annulus of the disk confined between radii $R_{1}$ and $R_{2}$ is given by

$$
\begin{equation*}
L(t) \approx \frac{3}{2} G M \int_{R_{1}}^{R_{2}} \dot{M}(R, t) R^{-2} d R . \tag{3.1}
\end{equation*}
$$

Therefore, any perturbations $\triangle \dot{M}$ of accretion rate in the energy release zone will cause variations $\triangle L$ in observed luminosity. Dropping a constant factor in front of the integral, we have

$$
\begin{equation*}
\triangle L(t) \propto \int_{R_{1}}^{R_{2}} \triangle \dot{M}(R, t) R^{-2} d R \tag{3.2}
\end{equation*}
$$

We will adopt the notation $\left\|\mathcal{F}_{f}(\omega)\right\|^{2}=\mathcal{F}_{f}^{*}(\omega) \mathcal{F}_{f}(\omega)$ for the power density spectrum (Appendix B) of a function $f(t)$, where $\mathcal{F}_{f}(\omega)$ is the Fourier transform of $f(t), \mathcal{F}_{f}^{*}(\omega)$ its complex conjugate. Hence, the power spectrum of the luminosity fluctuations is

$$
\begin{equation*}
\left\|\mathcal{F}_{\Delta L}(\omega)\right\|^{2} \propto \mathcal{F}_{\triangle L}^{*}(\omega) \cdot \mathcal{F}_{\Delta L}(\omega) \tag{3.3}
\end{equation*}
$$

where $\triangle L(R, t)$ is given by (3.2).

### 3.1.2 Diffusion equation for accretion rate fluctuations

In order to calculate the power spectrum (3.3) one needs to know $\triangle \dot{M}(R, t)$. Combination of the mass conservation law and the angular momentum balance allows us to construct the evolution equation expressed entirely in terms of $\dot{M}$ [and $\Delta \dot{M}$ as we will demonstrate later]. This equation combined with the appropriate boundary and initial conditions can
then be solved for $\triangle \dot{M}(R, t)$. Expressing $\dot{M}$ from equations (2.5) and (2.9) gives

$$
\begin{equation*}
\dot{M}=\frac{G^{\prime}-2 \pi R \omega R^{2} S}{v_{R}^{-1}\left(\omega \dot{R}^{2}\right)+\left(\omega R^{2}\right)^{\prime}}, \tag{3.4}
\end{equation*}
$$

where dot stands for $\partial / \partial t$, prime stands for $\partial / \partial R$. For particle orbits in a fixed gravitational potential [12] $\partial \omega / \partial t=0$, what simplifies (3.4) to

$$
\begin{equation*}
\dot{M}=\frac{G^{\prime}}{\left(\omega R^{2}\right)^{\prime}}-\frac{\omega R^{2}}{\left(\omega R^{2}\right)^{\prime}} 2 \pi R S . \tag{3.5}
\end{equation*}
$$

Multiplying the continuity equation (2.5) by $2 \pi R \nu \omega^{\prime} R^{2}$ and acting on it with $\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}$, we obtain

$$
\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(2 \pi R \nu \omega^{\prime} R^{2} \frac{\partial \Sigma}{\partial t}\right)+\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left[\nu \omega^{\prime} R^{2} \frac{\partial}{\partial R}\left(2 \pi R v_{R} \Sigma\right)\right]=\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(2 \pi R \nu \omega^{\prime} R^{2} S\right) .
$$

Substitution of formulas (2.4) and (2.8) for $\dot{M}$ and $G$, under assumption that kinematic viscosity $\nu$ is independent of time, reduces the previous equation to

$$
\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\frac{\partial G}{\partial t}\right)+\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial \dot{M}}{\partial R}\right)=\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(2 \pi R \nu \omega^{\prime} R^{2} S\right) .
$$

We are free to exchange the space and time derivatives in the first term, and, with the help of relation (3.5), finally obtain

$$
\begin{equation*}
\frac{\partial \dot{M}}{\partial t}+\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial \dot{M}}{\partial R}\right)=\frac{2 \pi}{\left(\omega R^{2}\right)^{\prime}}\left[\frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{3} S\right)-\omega R^{3} \dot{S}\right] . \tag{3.6}
\end{equation*}
$$

Constructed is the diffusion (heat) equation where the left-hand side is written for variable $\dot{M}$ which is the function of time and radius, while the source-related terms are isolated on the right-hand side. Equation (3.6) for the accretion rate can also be used for the accretion rate perturbations $\triangle \dot{M}(R, t)$ (as demonstrated in Appendix D ) in the framework of the linear perturbation theory

$$
\begin{equation*}
\frac{\partial \Delta \dot{M}}{\partial t}=-\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial \triangle \dot{M}}{\partial R}\right)+f_{\mathrm{dr}}(R, t) . \tag{3.7}
\end{equation*}
$$

We used the collective "driving force" notation $f_{\mathrm{dr}}(R, t)$ for the source terms located on the right-hand side of equation (3.6).

### 3.2 Transition layer instability

The space operator

$$
\begin{equation*}
\Lambda_{R}=-\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial}{\partial R}\right) \tag{3.8}
\end{equation*}
$$

in the diffusion equation (3.7) involves the angular velocity radial derivative $\omega^{\prime}(R)$. Notice that $\omega^{\prime}$ changes its sign at $R=R_{\max }$. Let us inspect the behavior of the diffusion coefficient in (3.8) in the vicinity of this special "turnover" point. First, consider the multiplicative term

$$
\begin{equation*}
\left(\omega R^{2}\right)^{\prime}=(2-\gamma) \omega_{0} D_{1} R_{0}^{\gamma} R^{1-\gamma} \tag{3.9}
\end{equation*}
$$

Its sign is determined by the product $(2-\gamma) D_{1}$. Coefficient $D_{1}$, defined by (2.45), can be written as

$$
D_{1}=\frac{\theta_{\text {adj }} r_{\text {adj }}^{2}-1}{r_{\text {adj }}^{2-\gamma}-1} .
$$

It is easy to see that $D_{1}$, as well as critical radius $r_{\max }$ (2.49), become undefined for $\gamma=2$. We shall, therefore, limit ourselves to studying the case of Reynolds number $\operatorname{Re}=\gamma>2$, Under this condition, since $\theta_{\text {adj }}>1$ and $r_{\text {adj }}>1$, the numerator of $D_{1}$ always remains positive, while the denominator is always negative. Thus, $D_{1}<0$ and the derivative $\left(\omega R^{2}\right)^{\prime}>0$. Now, since $\omega^{\prime}<0$ for $R>R_{\max }$ and $\omega^{\prime}>0$ for $R<R_{\max }$, the diffusion coefficient in $\Lambda_{R}$ is positive for $R>R_{\max }$, but turns negative for $R<R_{\max }$. Figure 3.1


Figure 3.1: Factor in the diffusion coefficient, determining its sign, as a function of radius in transition region.
shows the radial dependence of the diffusion coefficient in the transition layer for a $10 M_{\odot}$ black hole, rotating with $f_{0}=30 \mathrm{~Hz}$, for parameter $\gamma=20$.

Negative diffusion coefficient in the region $R<R_{\text {max }}$ indicates the onset of instability in the disk as accreting matter proceeds inwards past the critical point $R_{\text {max }}$. The diffusion taking place in the disk for $R>R_{\max }$ reverses into the process of clumping of matter (into separate
rings, which continue to move towards the accreter) at radius $R=R_{\text {max }}$, causing the disk break up. The outward angular momentum transfer, occurring in the Keplerian accretion disk, changes its direction at this point, such that the infalling matter starts to transfer its angular momentum to the central object, causing it to spin up. The diffusion coefficient will alternate sign from positive (in the outer region) to negative (in the inner region) for any accretion disk configuration, involving dynamical adjustment of the Keplerian flow to the sub-Keplerian inner boundary. In this sense the instability emerges as an intrinsic property of the any disk diffusion problem with the adjustment.

In order to isolate the critical point we formulate and solve the two problems defined in two regions of the transition zone:


### 3.2.1 Formulation of the diffusion problem and plan for the analytical solution

Let us assume that the perturbation sources are distributed over the disk, and that they are separable, i.e. can be presented in form of a product of space and time parts

$$
\begin{equation*}
f_{\mathrm{dr}}=\mathcal{R}(R) \varphi(t) \tag{3.10}
\end{equation*}
$$

The problem of diffusive propagation of driving perturbations in the transition layer can now be formulated as the nonhomogeneous equation (with $\triangle \dot{M}$ replaced by $u$ for compactness)

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\Lambda_{R} u+\mathcal{R}(R) \varphi(t) \tag{3.11}
\end{equation*}
$$

combined with the appropriate boundary conditions (B.C.) at $R_{0}, R_{\max }, R_{\text {adj }}$ subject to the initial condition $u(R, 0)=\mathcal{I}(R)$.

The general solution of the nonhomogeneous equation is given by the sum:

$$
\begin{equation*}
u(R, t)=u_{\mathrm{g}}^{\mathrm{h}}(R, t)+u_{\mathrm{p}}^{\mathrm{n}}(R, t), \tag{3.12}
\end{equation*}
$$

where the first term on the right-hand side is the general solution of the homogeneous problem

$$
\left\{\begin{array}{l}
\frac{\partial u_{\mathrm{g}}^{\mathrm{h}}}{\partial t}=\Lambda_{R} u_{\mathrm{g}}^{\mathrm{h}}  \tag{3.13}\\
u_{\mathrm{g}}^{\mathrm{h}}=\mathcal{I}(R), \\
+ \text { B.C. }
\end{array}\right.
$$

The second term in (3.12) is the particular solution of the problem for the nonhomogeneous equation (3.11). It is not hard to show (Appendix E) that for the specific (homogeneous) initial condition $u(R, 0)=0$, the particular solution of (3.11) can be presented as a convolution

$$
\begin{equation*}
u_{\mathrm{p}}^{\mathrm{n}}=\int_{0}^{t} \varphi\left(t^{\prime}\right) Y\left(R, t-t^{\prime}\right) d t^{\prime} \tag{3.16}
\end{equation*}
$$

where the kernel $Y(R, t)$ is the solution of the initial-value problem for distributed perturbations at the initial moment

$$
\left\{\begin{array}{l}
\frac{\partial Y}{\partial t}=\Lambda_{R} Y  \tag{3.17}\\
Y\left(R, t-t^{\prime}\right)_{t=t^{\prime}}=\mathcal{R}(R) \\
+ \text { B.C. }
\end{array}\right.
$$

Thus, the resulting perturbation $\triangle \dot{M}(R, t)$ is a sum of the two components - one is given by the solution of the problem (3.13)-(3.15), the other is given by the convolution (3.16). It should be pointed out that when "the observational time intervals are much longer than the characteristic diffusion timescale of the perturbation in the disk $t_{0}$, the contribution of the second (first in our case) component to the resulting signal (the amplitude of perturbations determined by the solution of the homogeneous problem) decays exponentially with time." (Titarchuk, Shaposhnikov and Arefiev 2007) [61]. Recent observational analysis by [60] suggests that the typical diffusion timescales for the compact non-Keplerian configuration vary roughly from $\sim 0.1 \mathrm{~s}$ to $\sim 1 \mathrm{~s}$. The duration of observation can be taken to be of order $\sim 1 \mathrm{hr}$, i.e. much longer than diffusion time scale. We, therefore, shall keep only the observationally significant part of the solution

$$
\begin{equation*}
u(R, t)=\int_{0}^{t} \varphi\left(t^{\prime}\right) Y\left(R, t-t^{\prime}\right) d t^{\prime} \tag{3.20}
\end{equation*}
$$

Lets now specify the boundary conditions for the diffusion problem formulated over the transition region $R_{0} \leq R \leq R_{\text {adj }}$. We set $\triangle \dot{M}\left(R_{0}\right)=0$ at the inner radius where infalling gas meets the surface of the neutron star or passes through the last stable circular orbit around a black hole. Accretion rate perturbations also vanish at the outer boundary $R_{\text {adj }}$ due to presence of the centrifugal barrier shock: $\triangle \dot{M}\left(R_{\mathrm{adj}}\right)=0$. In addition we require $\triangle \dot{M}$ to be continuous and bounded at the "turnover" point $R_{\max }$.

### 3.2.2 Solution of the initial-value problem for region 1

We will use method of separation of variables to seek the solution of the initial-value problem

$$
\left\{\begin{array}{l}
\frac{\partial Y}{\partial t}+\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial Y}{\partial R}\right)=0, \quad R_{\max } \leq R \leq R_{\mathrm{adj}}  \tag{3.21}\\
Y(R, 0)=\mathcal{R}(R) \\
Y\left(R_{\max }\right)< \pm \infty ; \quad Y\left(R_{\mathrm{adj}}\right)=0
\end{array}\right.
$$

in form $Y=X(R) T(t)$. Angular velocity entering the diffusion equation (3.21) is given by $\omega=\omega_{0}\left[D_{1} r^{-\gamma}+\left(1-D_{1}\right) r^{-2}\right]$, where $r=R / R_{0}$, coefficient $D_{1}$ is defined by formula (2.45). After substitution $Y=X T$ in (3.21), choosing the separation constant $-\lambda^{2}$, we obtain

$$
\left\{\begin{array}{l}
\frac{d}{d R}\left(\nu \omega^{\prime} R^{2} \frac{d X}{d R}\right)-\lambda^{2}\left(\omega R^{2}\right)^{\prime} X=0  \tag{3.24}\\
T(t)=A e^{-\lambda^{2} t}
\end{array}\right.
$$

Spatial equation (3.24) combined with the boundary conditions $X\left(R_{\max }\right)<\infty, X\left(R_{\text {adj }}\right)=$ 0 , forms the eigenvalue problem of finding such (eigenvalues) $\lambda_{k}$, that there exist non-trivial solutions (eigenfunctions) $X_{k}$ of (3.24). In fact, it is not hard to see that (3.24) has the form of the Sturm-Liouville equation (which we will, using some simplifications, show later). Let us assume that its eigenvalues $\lambda_{k}$ are known and its eigenfunctions $\left\{X_{k}\right\}$ form a complete orthogonal basis. Then the particular solution of the problem (3.21)-(3.23) is

$$
Y_{k}(R, t)=X_{k}(R) A_{k} e^{-\lambda_{k}^{2} t} .
$$

The general solution is written as a linear combination of particular solutions

$$
Y(R, t)=\sum_{k=1}^{\infty} X_{k}(R) A_{k} e^{-\lambda_{k}^{2} t}
$$

In order to determine coefficients $A_{k}$, we note that at the initial moment $t=0$

$$
\begin{equation*}
Y(R, 0)=\sum_{k=1}^{\infty} X_{k}(R) A_{k} \tag{3.26}
\end{equation*}
$$

But since $\left\{X_{k}(R)\right\}$ form the complete eigenfunction basis, equation (3.26) represents the eigenfunction expansion of $Y(R, 0)$ in $\left\{X_{k}\right\}$, with the Fourier coefficients

$$
A_{k}=\frac{1}{\left\|X_{k}\right\|^{2}} \int_{R_{\max }}^{R_{\mathrm{adj}}} Y(R, 0) X_{k}(R) \rho(R) d R
$$

where the weight function $\rho(R)$ is the coefficient at $\lambda^{2} X$ term in the spatial equation (3.24), with the eigenfunction's norm given by

$$
\begin{equation*}
\left\|X_{k}\right\|^{2}=\int_{R_{\max }}^{R_{\mathrm{adj}}} X_{k}^{2}(R) \rho(R) d R \tag{3.27}
\end{equation*}
$$

Therefore the solution of the initial-value problem can be written as a series

$$
\begin{equation*}
Y(R, t)=\sum_{k=1}^{\infty} e^{-\lambda_{k}^{2} t} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} X_{k}(R) \tag{3.28}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{k}=\int_{R_{\max }}^{R_{\mathrm{adj}}} \mathcal{R}(R) X_{k}(R) \rho(R) d R, \tag{3.29}
\end{equation*}
$$

where we used the initial condition $Y(R, 0)=\mathcal{R}(R)$.
Let us find $\lambda_{k}$ and $X_{k}$. The diffusion operator in equation (3.24) can be simplified, if we recall that the transition layer can approximated by a "thin shell" $(2.52)$ in the limit $\gamma \gg 1$

$$
\begin{equation*}
d_{\mathrm{TL}}=R_{\mathrm{adj}}-R_{0} \ll R_{\max } \tag{3.30}
\end{equation*}
$$

Expanding $\omega(R)$ in Taylor series around $R=R_{\max }$, we have

$$
\begin{equation*}
\omega(R)=\omega\left(R_{\max }\right)+\omega^{\prime}\left(R_{\max }\right)\left(R-R_{\max }\right)+\frac{1}{2} \omega^{\prime \prime}\left(R_{\max }\right)\left(R-R_{\max }\right)^{2}+\cdots \tag{3.31}
\end{equation*}
$$

The second term in the expansion is zero, so (3.31) reduces to

$$
\omega(R)=\omega\left(R_{\max }\right)-\frac{1}{2} a_{0}\left(R-R_{\max }\right)^{2}+\cdots
$$

where $a_{0}=-\omega^{\prime \prime}\left(R_{\max }\right)$ is a positive constant. If we introduce $x=R-R_{\max }$, such that $0 \leq x \leq x_{\text {adj }}$ in region 1, then

$$
\begin{align*}
& \omega(x)=\omega_{\max }-\frac{1}{2} a_{0} x^{2}+\cdots  \tag{3.32}\\
& \omega^{\prime}(x)=-a_{0} x+\cdots \approx \omega^{\prime \prime}\left(R_{\max }\right) x \tag{3.33}
\end{align*}
$$

After substituting linear approximation (3.33) for $\omega^{\prime}$, equation (3.24), with derivatives expressed in terms of new variable $x$, becomes

$$
\begin{equation*}
\frac{d}{d x}\left[\nu \omega^{\prime \prime}\left(R_{\max }\right) x R^{2} \frac{d X}{d x}\right]-\lambda^{2}\left(\omega R^{2}\right)^{\prime} X=0 \tag{3.34}
\end{equation*}
$$

The "thin shell" condition (3.30) implies $x \ll R_{\max }$. Therefore, we can approximately set $R=R_{\max }+x \approx R_{\max }$. in (3.34). The tenuity of the transition region also allows us to
assume that viscosity does not vary with radius $\nu \approx \nu_{0}$. Exploiting these approximations the quantity $\nu_{0} \omega^{\prime \prime}\left(R_{\max }\right) R_{\max }^{2}$ can be pulled outside the derivative sign to yield the SturmLiouville (self-adjoint) form of the equation for $X$

$$
\begin{equation*}
\frac{d}{d x}\left(x \frac{d X}{d x}\right)+\lambda^{2} \rho(x) X=0 \tag{3.35}
\end{equation*}
$$

where the weight factor is $x$-independent, and is given by

$$
\begin{equation*}
\rho=-\frac{\left(\omega R^{2}\right)^{\prime}}{\nu_{0} \omega^{\prime \prime}\left(R_{\max }\right) R_{\max }^{2}}=\frac{(\gamma-2) \omega_{0} D_{1}}{\nu_{0} \omega^{\prime \prime}\left(R_{\max }\right)} R_{0}^{\gamma} R_{\max }^{-\gamma-1}=\text { const } . \tag{3.36}
\end{equation*}
$$

The self-adjointness of the spatial equation asserts our assumption about orthogonality of its eigenfunctions. Equation (3.35) can be transformed to

$$
\begin{equation*}
x^{2} X^{\prime \prime}+x X^{\prime}+\lambda^{2} \rho x X=0, \tag{3.37}
\end{equation*}
$$

which is the Bessel's equation

$$
\begin{equation*}
x^{2} y^{\prime \prime}+a x y^{\prime}+\left(b x^{m}+c\right) y=0, \tag{3.38}
\end{equation*}
$$

with $a=1, b=\lambda^{2} \rho, c=0, m=1$ : The solution of (3.38) is represented by a linear combination of Bessel functions $J_{\xi}$ and $Y_{\xi}$

$$
y(x)=x^{\frac{1-a}{2}} \cdot\left[C_{1} J_{\xi}\left(\frac{2}{m} \sqrt{b} x^{\frac{m}{2}}\right)+C_{2} Y_{\xi}\left(\frac{2}{m} \sqrt{b} x^{\frac{m}{2}}\right)\right], \quad \xi=\frac{1}{m} \sqrt{(1-a)^{2}-4 c} .
$$

In our case index $\xi=0$. The boundary conditions (3.23) in terms of variable $x$ read

$$
\begin{aligned}
& X_{k}(x=0)< \pm \infty, \\
& X_{k}\left(x=x_{\text {adj }}\right)=0 .
\end{aligned}
$$

Since the function $Y_{0}(x)$ diverges to negative infinity at $x=0$, we set $C_{2}=0$ to obtain the (unnormalized) spatial eigenfunctions

$$
\begin{equation*}
X_{k}(x)=C_{1} J_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right), \quad \rho>0 . \tag{3.39}
\end{equation*}
$$

The normalized eigenfunctions are given by

$$
\begin{equation*}
\hat{X}_{k}=\frac{1}{\left\|X_{k}\right\|} X_{k}, \text { where }\left\|X_{k}\right\|^{2}=\int_{0}^{x_{a d j}} X_{k}^{2}(x) \rho(x) d x \tag{3.40}
\end{equation*}
$$

The eigenvalues $\lambda_{k}$ can be determined from the boundary condition at $x_{\text {adj }}$

$$
\begin{equation*}
J_{0}\left(2 \lambda_{k} \sqrt{\rho x_{\mathrm{adj}}}\right)=0 . \tag{3.41}
\end{equation*}
$$

Use of the Bessel function's principal asymptotic form [1]

$$
\begin{equation*}
J_{\xi}(z) \approx \sqrt{\frac{2}{\pi z}} \cos \left(z-\frac{\xi \pi}{2}-\frac{\pi}{4}\right), \quad \text { for } z \gg 1 \tag{3.42}
\end{equation*}
$$

makes (3.41) easily solvable for $\lambda_{k}$ (Appendix F). Thus we find

$$
\begin{equation*}
\lambda_{k}=\frac{\pi}{8}(4 k-1)\left(\rho x_{\mathrm{adj}}\right)^{-1 / 2}, \quad k=1,2, \cdots \tag{3.43}
\end{equation*}
$$

The first five spatial eigenfunctions (3.39) are displayed on Figure 3.2. One can spot increasingly better matching of $X_{k}$ with the outer boundary condition $X_{k}\left(r_{\text {adj }}\right)=0$, as with
increasing $k$ (and $\lambda_{k}$ ) asymptotic form (3.42) becomes more accurate. We have now all the


Figure 3.2: First several spatial eigenfunction $X_{k}(r)$ (3.39), where dimensionless radius $r=R / R_{0}$.
necessary ingredients to calculate $Y(R, t)$. According to formula (3.20), the accretion rate fluctuations $\triangle \dot{M}(R, t)$ in the region are presented as a convolution of the solution $Y(R, t)$ of the initial-value problem with distributed initial condition and the time part $\varphi(t)$ of the source term

$$
\begin{equation*}
\triangle \dot{M}(R, t)=\int_{0}^{t} \varphi\left(t^{\prime}\right) Y\left(R, t-t^{\prime}\right) d t^{\prime} \tag{3.44}
\end{equation*}
$$

In the simplest possible case, when $\varphi\left(t^{\prime}\right)$ is a point source at time instant $t^{\prime}=0, \varphi\left(t^{\prime}\right)=\delta\left(t^{\prime}\right)$, we obtain

$$
\begin{equation*}
\triangle \dot{M}(R, t)=Y(R, t) \tag{3.45}
\end{equation*}
$$

The power density spectrum of the luminosity fluctuations $\triangle L_{X}(t)$ from region $R_{\max } \leq$ $R \leq R_{\text {adj }}$ (equivalent to $0 \leq x \leq x_{\text {adj }}$ ) can be calculated, using equations (3.2), (3.3), and
the solution (3.28) of the initial-value problem for $Y(R, t)$

$$
\begin{align*}
& \left\|\mathcal{F}_{\triangle L_{X}}(\omega)\right\|^{2} \propto  \tag{3.46}\\
& \int_{0}^{\infty}\left[\int_{R_{\max }}^{R_{\mathrm{adj}}}\left(\sum_{m=1}^{\infty} e^{-\lambda_{m}^{2} t^{\prime}} \frac{c_{m}}{\left\|X_{m}\right\|^{2}} X_{m}\right) R^{\prime-2} d R^{\prime}\right] e^{i \omega t^{\prime}} d t^{\prime} \times \\
& \int_{0}^{\infty}\left[\int_{R_{\max }}^{R_{\mathrm{adj}}}\left(\sum_{k=1}^{\infty} e^{-\lambda_{k}^{2} t} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} X_{k}\right) R^{-2} d R\right] e^{-i \omega t} d t
\end{align*}
$$

Converting to variable $x=R-R_{\text {max }}$ in the spatial integrals, we can exploit the same "thin shell" argument as was used, when we linearized the diffusion coefficient in the eigenvalue problem (3.34), namely that $R=R_{\max }+x \approx R_{\max }$. Hence, we can pull the constant factor $R_{\max }^{-2}$ outside the integral, and actually drop it, since normalization constant is not important for the power spectrum continuum shape. Moving space and time integrals under summation symbols, we have, after performing time integration

$$
\begin{equation*}
\left\|\mathcal{F}_{\triangle L_{X}}(\omega)\right\|^{2} \propto \sum_{m=1}^{\infty} \frac{c_{m}}{\left\|X_{m}\right\|^{2}} \frac{1}{\lambda_{m}^{2}-i \omega} \int_{0}^{x_{\mathrm{adj}}} X_{m} d x^{\prime} \sum_{k=1}^{\infty} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} \frac{1}{\lambda_{k}^{2}+i \omega} \int_{0}^{x_{\mathrm{adj}}} X_{k} d x \tag{3.47}
\end{equation*}
$$

It is convenient to combine factors determining the "weight" of each term in the sum

$$
\begin{equation*}
B_{m}=\frac{c_{m}}{\left\|X_{m}\right\|^{2}} \int_{0}^{x_{\mathrm{adj}}} X_{m}\left(x^{\prime}\right) d x^{\prime} \quad B_{k}=\frac{c_{k}}{\left\|X_{k}\right\|^{2}} \int_{0}^{x_{\mathrm{adj}}} X_{k}(x) d x \tag{3.48}
\end{equation*}
$$

such that the power spectrum expression becomes more compact

$$
\begin{equation*}
\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|^{2} \propto \sum_{m=1}^{\infty} \frac{B_{m}}{\lambda_{m}^{2}-i \omega} \cdot \sum_{k=1}^{\infty} \frac{B_{k}}{\lambda_{k}^{2}+i \omega} . \tag{3.49}
\end{equation*}
$$

The upper limit on the product of series (3.49) can be evaluated, using the following reasoning: for any two complex quantities $x$ and $y$

$$
\begin{equation*}
\bar{x} x+\bar{y} y=|x|^{2}+|y|^{2} \geq(\bar{x} y+x \bar{y}), \tag{3.50}
\end{equation*}
$$

[which follows immediately from the fact that $(x-y)^{2} \geq 0$ ]. Thus, if we assume that there are only two terms $a_{1}, a_{2}$ in the power spectrum sum, we would have

$$
\left(a_{1}+a_{2}\right)\left(\bar{a}_{1}+\bar{a}_{2}\right)=\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}+\bar{a}_{1} a_{2}+a_{1} \bar{a}_{2} \leq 2\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right) .
$$

Identity (3.49) can be generalized to infinitely large number of terms ([61] Appendix C). Hence, we obtain the upper limit estimate for the spectrum

$$
\begin{equation*}
\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|^{2} \leq 2 \sum_{k=1}^{\infty} \frac{B_{k}^{2}}{\lambda_{k}^{4}+\omega^{2}}, \tag{3.51}
\end{equation*}
$$

where the weight factors $B_{k}$ are given by (3.48). In order to proceed we need to calculate the expansion coefficients $c_{k}^{2} /\left\|X_{k}\right\|^{2}$ and the radial integral of the eigenfunction $X_{k}(x)$ in (3.48). Let us elaborate more on the (unspecified yet) initial condition (3.22). In order to satisfy the boundary conditions (3.23) and, at the same time, have a relatively simple integral in the equation for the expansion coefficients (3.29), one can implement the following arrangement: introduce a small quantity $\epsilon$ and define the initial condition as

$$
\mathcal{R}(R)= \begin{cases}\mathcal{R}_{0}=\text { const }, & \text { for } R_{\max } \leq R \leq R_{\mathrm{adj}}-\epsilon,  \tag{3.52}\\ -A\left(R-R_{\mathrm{adj}}+\epsilon\right)^{2}+B, & \text { for } R_{\mathrm{adj}}-\epsilon \leq R \leq R_{\mathrm{adj}}\end{cases}
$$

where coefficients $A$ and $B$ are determined from

$$
\begin{cases}-A\left(R-R_{\mathrm{adj}}+\epsilon\right)^{2}+B=\mathcal{R}_{0}, & \text { at } R_{\mathrm{adj}}-\epsilon  \tag{3.54}\\ -A\left(R-R_{\mathrm{adj}}+\epsilon\right)^{2}+B=0, & \text { at } R=R_{\mathrm{adj}}\end{cases}
$$

Example of such initial condition is demonstrated on the Figure 3.3 for $R_{\text {adj }}=2, \epsilon=0.1$, and $\mathcal{R}_{0}=1$. We constructed a function which is a constant on the interval $R_{\max } \leq R \leq R_{\text {adj }}-\epsilon$,


Figure 3.3: Example of the initial condition (3.52), (3.53) with $R_{\text {adj }}=2, \epsilon=0.1$, and $\mathcal{R}_{0}=1$.
which then smoothly turns into a parabola at radius $R_{\text {adj }}-\epsilon$, to match the zero boundary condition at $R_{\text {adj }}$. For the purpose of calculation of $c_{k}$ one can always set $\epsilon \rightarrow 0$ (with both boundary conditions fulfilled), thus approximating function $\mathcal{R}(R)$ in the integral (3.29) with $\mathcal{R}_{0}$. This is completely valid approximation, because in the limit of infinitely small $\epsilon$ contribution of the region $R_{\text {adj }}-\epsilon \leq R \leq R_{\text {adj }}$ to the integral (3.29) will be infinitely
small. It is not hard to find $c_{k}$ and $\left\|X_{k}\right\|^{2}$ for the uniform initial perturbation distribution (Appendix G):

$$
\begin{equation*}
\frac{c_{k}}{\left\|X_{k}\right\|^{2}} \propto \frac{1}{\lambda_{k} J_{1}\left(2 \lambda_{k} \rho^{1 / 2} x_{\mathrm{adj}}^{1 / 2}\right)} . \tag{3.56}
\end{equation*}
$$

As for the radial integral, substitution of (3.39) gives

$$
\begin{equation*}
\int_{0}^{x_{\mathrm{adj}}} X_{k}(x) d x=\int_{0}^{x_{\mathrm{adj}}} J_{0}\left(2 \lambda_{k} \rho^{1 / 2} x^{1 / 2}\right) d x . \tag{3.57}
\end{equation*}
$$

The integral of the Bessel's function of zero-th order with an argument $a x^{1 / 2}$ is expressed via regularized hypergeometric function

$$
\begin{equation*}
\int J_{0}\left(a x^{1 / 2}\right) d x=x_{0} \tilde{F}_{1}\left(; 2 ;-\frac{a^{2} x}{4}\right) \tag{3.58}
\end{equation*}
$$

where ${ }_{0} \tilde{F}_{1}(; \alpha ; \beta)={ }_{0} F_{1}(\alpha ; \beta) / \Gamma(\alpha)[\operatorname{and} \Gamma(2)=1]$, thus giving

$$
\begin{equation*}
\int_{0}^{x_{\mathrm{adj}}} X_{k}(x) d x=x_{\mathrm{adj} 0} F_{1}\left(2 ;-\lambda_{k}^{2} \rho x_{\mathrm{adj}}\right) . \tag{3.59}
\end{equation*}
$$

Notice that (3.56), (3.59) involve $\rho$ [equation (3.36)], which depends on a number of parameters, all of which can be determined for a given compact object (with mass $m$ and rotational velocity $f_{0}$ ). Angular velocity second derivative $\omega^{\prime \prime}\left(R_{\max }\right)$ is given by

$$
\begin{equation*}
\omega^{\prime \prime}\left(R_{\max }\right)=\omega_{0}\left[\gamma(\gamma+1) D_{1} R_{0}^{\gamma} R_{\max }^{-\gamma-2}+6\left(1-D_{1}\right) R_{0}^{2} R_{\max }^{-4}\right] . \tag{3.60}
\end{equation*}
$$

Now, we have everything to calculate the final power spectrum. Disregarding a constant factor in front of the sum, we obtain the power spectrum profile emergent from region
$R_{\max } \leq R \leq R_{\text {adj }}:$

$$
\begin{equation*}
\left\|\mathcal{F}_{\Delta L_{X} 1}(\omega)\right\|^{2} \propto \sum_{k=1}^{\infty} \frac{\chi_{k}^{2}}{\lambda_{k}^{2}\left(\lambda_{k}^{4}+\omega^{2}\right)}, \tag{3.61}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi_{k}=\frac{x_{\text {adj } 0} F_{1}\left(2 ;-\lambda_{k}^{2} \rho x_{\text {adj }}\right)}{J_{1}\left(2 \lambda_{k} \rho^{1 / 2} x_{\mathrm{adj}}^{1 / 2}\right)} . \tag{3.62}
\end{equation*}
$$

The solution (3.28) of the initial-value problem [and which power spectrum is given by (3.61)] can be considered as the weighted sum of related exponential shots. The exponential time dependence of $Y(R, t)$ is the immediate consequence of solving the diffusion equation for the bounded configuration. In this respect it is instructive to analyze the power spectrum of the simple exponential shot

$$
\begin{equation*}
f(t)=e^{-t / t_{0}} \tag{3.63}
\end{equation*}
$$

defined for time $t>0$. Taking the Fourier transform of (3.63), we obtain the power density spectrum as the zero-centered Lorentzian

$$
\begin{equation*}
\left\|\mathcal{F}_{f}(\nu)\right\|^{2}=\frac{1}{2 \pi} \frac{1}{\left[t_{0}^{-2}+(2 \pi \nu)^{2}\right]} . \tag{3.64}
\end{equation*}
$$

It is easy to see that the power spectrum (3.64) becomes frequency-independent in the low-frequency limit, specifically, when $\nu \ll\left(2 \pi t_{0}\right)^{-1}$, while for high frequencies, i.e., when $\nu \gg\left(2 \pi t_{0}\right)^{-1}$, it behaves as the decaying power law $\sim \nu^{-2}$. Continuous transition from one extreme case to another occurs when $\nu \sim\left(2 \pi t_{0}\right)^{-1}$. One can calculate location of the anticipated "break" $\nu_{0}$ in the spectrum, setting $\nu_{0}=\left(2 \pi t_{0}\right)^{-1}$. Thus, for instance, for $t_{0}=10 \mathrm{~s}, \nu_{0} \approx 0.016 \mathrm{~Hz}$. Figure 3.4 shows the power spectrum of the exponential shot (3.63), plotted in log-log scale, for $t_{0}=10 \mathrm{~s}$. We readily identify the break in the spectrum located at $\sim 0.016 \mathrm{~Hz}$, as expected. Based on this simple analysis one can anticipate a somewhat similarly-shaped power spectrum continuum of the solution (3.28). Indeed,


Figure 3.4: Power spectrum of the single exponential shot $\exp \left(-t / t_{0}\right)$ for $t_{0}=10 \mathrm{~s}$.


Figure 3.5: Power spectrum continuum (3.61) of the solution $Y(R, t)$ of the initial-value problem (3.21)-(3.23) integrated over region $R_{\max } \leq R \leq R_{\text {adj }}$ for a sample black hole with $m=10, f_{0}=30 \mathrm{~Hz}, \gamma=10$.
expression (3.61) is nothing, but a weighted sum of the zero-centered Lorentzians. It can be shown, using (3.43) and the identity $\lambda_{k}=\left(\lambda_{k} / \lambda_{1}\right) \lambda_{1}$, where $\lambda_{1}=3 \pi / 8\left(\rho x_{\mathrm{adj}}\right)^{-1 / 2}$, that $\lambda_{k}$ related to $\lambda_{1}$ by

$$
\begin{equation*}
\lambda_{k}=\frac{4 k-1}{3} \lambda_{1}, \quad k=1,2, \cdots \tag{3.65}
\end{equation*}
$$

Hence, the eigenvalues are ordered such that $\lambda_{1}<\lambda_{2}<\cdots<\lambda_{k}$, which is, in fact, the property of the regular Sturm-Liouville problem (3.35). Keeping this in mind it is not hard to see that the power spectrum series (3.61) is dominated by the first term $k=1$ (which is also the slowest-decaying component in the solution (3.28)). Let us consider the behavior of the leading term of (3.61)

$$
\begin{equation*}
\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|_{k=1}^{2} \propto \frac{\chi_{1}^{2}}{\lambda_{1}^{6}+\lambda_{1}^{2} \omega^{2}} \tag{3.66}
\end{equation*}
$$

for the following cases

- $\omega \ll \lambda_{1}^{2}$ implies that $\lambda_{1}^{6} \gg \lambda_{1}^{2} \omega^{2}$. Neglecting the small term, we find that the power spectrum $\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|_{k=1, \mathrm{LF}}^{2} \propto \lambda_{1}^{-6}$ is frequency-independent (white noise) in the low frequency limit,
- $\omega \gg \lambda_{1}^{2}$ implies that $\lambda_{1}^{6} \ll \lambda_{1}^{2} \omega^{2}$. Neglecting the small term, gives that $\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|_{k=1, \mathrm{HF}}^{2} \propto$ $\lambda_{1}^{-2} \omega^{-2}$, i.e. the power spectrum is a power-law function of frequency (red noise) with index -2 in the high frequency limit,
- the characteristic angular velocity $\omega^{*}=\lambda_{1}^{2}$ determines the location of a "break" in the power spectrum continuum, where the flat low-frequency plateau smoothly turns into a power law for high frequency.

The limiting behavior of the spectrum continuum (3.66) is completely identical to the one of the single exponential shot (3.64). The power spectrum continuum (3.61) (normalized
to a maximum value) for a $10 M_{\odot}$ black hole, rotating with $f_{0}=30 \mathrm{~Hz}$, and parameter $\gamma=10$ is shown on Figure 3.5. A thick dashed line represents the power spectrum series with 100 terms in it. The solid red line represents the leading term $\left\|\mathcal{F}_{\Delta L_{X}}(\omega)\right\|_{k=1}^{2}$ of the same series. It is easy to see that the leading term completely dominates the series (the two lines are indistinguishable on the plot). As anticipated from the analysis of the limiting behavior of (3.66), the spectrum shape is the white noise for frequencies $\omega \ll \omega^{*}$; the red noise with the power law index -2 , for frequencies $\omega \gg \omega^{*}$, with the break (a "knee") in the continuum occurring at $\omega^{*}$. The successive $(k=2,3)$ terms have progressively weaker normalization, due to the high power of (increasing with $k$ ) $\lambda_{k}$ in the denominator of (3.61), with break locations $\omega_{k}^{*}=\lambda_{k}^{2}$ shifted at higher frequencies. We associate $t_{\mathrm{d}}=\lambda_{1}^{-2}$ as a characteristic diffusion timescale, based on our solution (3.28) of the diffusion problem (3.21)-(3.23). Physical meaning of $t_{\mathrm{d}}$ can be understood in a sense that any perturbation at any radius in the region will diffusively propagate outward over this timescale [59]

$$
\begin{equation*}
t_{\mathrm{d}} \sim\left(\frac{L}{l_{\mathrm{fp}}}\right)^{2} \frac{l_{\mathrm{fp}}}{v_{R}}, \tag{3.67}
\end{equation*}
$$

where $L$ is the characteristic thickness of the transition layer,

$$
\begin{equation*}
l_{\mathrm{fp}} \sim \frac{\eta}{\rho v_{R}}=\frac{1}{\sigma_{\mathrm{pert}} n} \tag{3.68}
\end{equation*}
$$

is the mean free path of the particle, $\eta$ is the turbulent viscosity, $\rho$ is the mass density, $v_{R}$ is the inward radial drift velocity, $n$ is the number density, $\sigma_{\text {pert }}$ is the perturbation interaction cross-section.

### 3.2.3 Power spectrum as a function of Reynolds number

As we showed, the power spectrum continuum is determined by the first eigenvalue of the spatial problem (3.24). Substituting formula (3.36) for $\rho$ into (3.43), $\lambda_{1}$ expressed in terms
of dimensionless radius $r$ reads

$$
\begin{equation*}
\lambda_{1}=\frac{3 \pi}{8}\left(\nu_{0} R_{0}^{-2}\right)^{1 / 2}\left[\frac{\omega_{r}^{\prime \prime}\left(r_{\max }\right)}{(\gamma-2) \omega_{0} D_{1}}\right]^{1 / 2}\left[\frac{r_{\max }^{\gamma+1}}{r_{\mathrm{adj}}-r_{\max }}\right]^{1 / 2} \tag{3.69}
\end{equation*}
$$

It is obvious from (3.69) that $\lambda_{1}$ can change substantially with varying $\gamma$, however it is not clear how exactly, because $r_{\max }, r_{\text {adj }}, \omega_{r}^{\prime \prime}\left(r_{\max }\right)$ and $D_{1}$ are all functions of $\gamma$. Before providing exact numerical answer let us try to simplify this equation and determine the principal behavior of $\lambda_{1}=f(\gamma)$. We can use approximate formulas $r_{\max } \approx(\gamma / 2)^{1 /(\gamma-2)}$ and $r_{\text {adj }} \approx(2 \gamma)^{1 /(\gamma-2)}$ (Appendix H ) along with (2.45) and (3.60) to obtain

$$
\begin{equation*}
\lambda_{1} \approx \frac{3 \pi}{8}\left(\nu_{0} R_{0}^{-2}\right)^{1 / 2}\left(\frac{\gamma^{2}+6\left(D_{1}^{-1}-1\right) \gamma / 2}{(\gamma-2)\left[(2 \gamma)^{1 /(\gamma-2)}-(\gamma / 2)^{1 /(\gamma-2)}\right]}\right)^{1 / 2} \tag{3.70}
\end{equation*}
$$

where $\gamma(\gamma+1) \approx \gamma^{2}$ and factor $(\gamma / 2)^{1 /(2-\gamma)} \approx 1$. Term $6\left(D_{1}^{-1}-1\right) \gamma / 2$ in the numerator gives little contribution and can be safely neglected. Finally, using Taylor series expansion $(1 / 4)^{1 /(\gamma-2)} \approx 1+1 /(\gamma-2) \ln (1 / 4)$ and the above approximations, yields

$$
\begin{equation*}
\lambda_{1} \approx \frac{3 \pi}{8}[-\ln (1 / 4)]^{-1 / 2}\left(\nu_{0} R_{0}^{-2}\right)^{1 / 2} \gamma \tag{3.71}
\end{equation*}
$$

such that $\lambda^{-2}$ has dimension of time. Our estimate (3.71) implies that $\lambda_{1}$ is a linear growing function of $\gamma$. Figure 3.6 shows $\lambda_{1}$ plotted as a function of $\gamma$ for $R_{0}^{2} / \nu_{0}=10 \mathrm{~s}$, using exact (3.69) (solid circles) and approximate (3.71) formulas. The difference between the two representations does not exceed $2.5 \%$ for the line's slope. Linear dependence of $\lambda_{1}$ on Reynolds number $\gamma$ translates into rapid power spectrum decay with increasing $\gamma$, according to equation (3.66), accompanied by shift of its characteristic features towards high frequency, as $\omega^{*} \sim \gamma^{2}$. We anticipate the attenuation effect to be especially pronounced at
lower frequencies (on the left side relative to the break in continuum), where $\lambda_{1}^{6}$ dominates the denominator of (3.66). Figure 3.7 shows the power spectra (3.61) calculated for $\gamma=5$, 15 , and 45 (corresponding to $\lambda_{1}=0.54,1.76$, and 5.52 ) superimposed on one plot. We identify the corresponding break locations at $\omega^{*} \approx 0.3,3.1$, and $30.4 \mathrm{rad} \mathrm{s}^{-1}$, as well as observe power spectrum variability decay with increasing $\gamma$. It is easy to see that a factor 3 change in Reynolds number results in a dramatic (more than 4 orders of magnitude) suppression of power spectrum normalization in the low frequency limit, and $\sim 2$ orders of magnitude suppression of variability in the high frequency limit. This result can potentially explain the observed variability decay of X-ray lightcurves during transition of a source from low-hard to high-soft state. Recall that the transition layer size shrinks very rapidly as $\gamma$ increases. Therefore, power spectrum decay is a direct consequence of the compression of the emission zone with increasing accretion rate. In Chapter 4 we return to this discussion and give a more detailed explanation, as we analyze the complete diffusion problem.


Figure 3.6: Eigenfunction $\lambda_{1}$ as a function of Reynolds number $\gamma$. Solid circles represent exact formula (3.69).


Figure 3.7: Power spectrum high-frequency shift/decay with increasing Reynolds number.

### 3.2.4 Solution of the initial-value problem for region 2

Let us now consider the diffusion problem in the inner region $R_{0} \leq R \leq R_{\max }$ of the boundary layer. We will use method of separation of variables to solve the initial-value problem

$$
\left\{\begin{array}{l}
\frac{\partial Y}{\partial t}+\frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left(\nu \omega^{\prime} R^{2} \frac{\partial Y}{\partial R}\right)=0, \quad R_{0} \leq R \leq R_{\max }  \tag{3.72}\\
Y(R, 0)=\mathcal{R}(R) \\
Y\left(R_{0}\right)=0 ; \quad Y\left(R_{\max }\right)< \pm \infty
\end{array}\right.
$$

Substitution of $Y(R, t)=X(R) T(t)$ into (3.72) gives

$$
\frac{1}{T} \frac{d T}{d t}=-\frac{1}{X} \frac{1}{\left(\omega R^{2}\right)^{\prime}} \frac{d}{d R}\left(\nu \omega^{\prime} R^{2} \frac{d X}{d R}\right)= \pm \lambda^{2}
$$

It is not difficult to show (Appendix I) that the choice $-\lambda^{2}$ of the separation constant is unacceptable, since it only allows for the trivial solution of the spatial equation. Thus, a positive constant must be chosen. In this case the spatial equation becomes

$$
\begin{equation*}
\frac{d}{d R}\left(R^{2} \nu \omega^{\prime} \frac{d X}{d R}\right)+\lambda^{2}\left(\omega R^{2}\right)^{\prime} X=0 \tag{3.75}
\end{equation*}
$$

Applying the same assumptions we used when solved the problem for region 1, the previous equation can be reduced to

$$
\begin{equation*}
\frac{d}{d x}\left(x \frac{d X}{d x}\right)-\lambda^{2} \rho X=0 \tag{3.76}
\end{equation*}
$$

where $x=R-R_{\max }$, such that $-x_{0} \leq x \leq 0, \rho=$ is determined by (3.36). Now, since $x$ is non-positive in the region, equation (3.76) can be expressed in terms of $|x|$

$$
\begin{equation*}
\frac{d}{d x}\left(|x| \frac{d X}{d x}\right)+\lambda^{2} \rho X=0, \quad 0 \leq|x| \leq x_{0} \tag{3.77}
\end{equation*}
$$

Solution of (3.77) is represented by the sum of Bessel's functions

$$
\begin{equation*}
X_{k}(y)=C_{1} J_{0}\left(2 \lambda_{k} \sqrt{\rho|x|}\right)+C_{2} Y_{0}\left(2 \lambda_{k} \sqrt{\rho|x|}\right) . \tag{3.78}
\end{equation*}
$$

The boundary condition at $x=0$ requires us to set $C_{2}=0$, because of the divergence of $Y_{0}(0)$. Therefore the eigenfunctions of the equation (3.75) are

$$
\begin{equation*}
X_{k}(x)=C_{1} J_{0}\left(2 \lambda_{k} \sqrt{\rho|x|}\right), \quad \rho>0 . \tag{3.79}
\end{equation*}
$$

The remaining boundary condition (at $x=-x_{0}$ )

$$
J_{0}\left(2 \lambda_{k} \sqrt{\rho x_{0}}\right)=0,
$$

can be used to determine the eigenvalues

$$
\begin{equation*}
\lambda_{k}=\frac{\pi}{8}(4 k-1)\left(\rho x_{0}\right)^{-1 / 2}, \quad k=1,2, \cdots \tag{3.80}
\end{equation*}
$$

Hence, we found that the solution for the spatial eigenvalue problem in region $R_{0} \leq R \leq$ $R_{\max }$ is identical to the solution in region $R_{\max } \leq R \leq R_{\mathrm{adj}}$, with $R_{\text {adj }}$ replaced by $R_{0}$ in the expression for $\lambda_{k}$. However, due to positive sign of the separation constant, solution for the time part became an exponentially growing function of time

$$
T(t)=A e^{\lambda^{2} t}
$$

Formally, the general solution of the diffusion equation can be written as a series

$$
\begin{equation*}
Y(R, t)=\sum_{k=1}^{\infty} e^{\lambda_{k}^{2} t} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} X_{k}(R) \tag{3.81}
\end{equation*}
$$

with

$$
\begin{gather*}
c_{k}=\int_{R_{0}}^{R_{\max }} \mathcal{R}(R) X_{k}(R) \rho(R) d R,  \tag{3.82}\\
\left\|X_{k}\right\|^{2}=\int_{R_{0}}^{R_{\max }} X_{k}^{2}(R) \rho(R) d R, \tag{3.83}
\end{gather*}
$$

where $X_{k}$ and $\lambda_{k}$ are given by (3.79), (3.80).
Demanding the spatial problem to have a non-trivial solution in $R_{0} \leq R \leq R_{\text {max }}$, resulted in an unstable solution for the time part, which makes impossible to calculate the power spectrum directly, because of divergence of the integral $\int_{0}^{\infty} e^{\lambda^{2} t} e^{ \pm i \omega t} d t$. This difficulty can be overcome if we treat the problem using the following physical reasoning. Let us introduce
a characteristic time $T_{\mathrm{pl}}$ which takes plasma to travel the distance from $R_{\max }$ to $R_{0}$

$$
\begin{equation*}
T_{\mathrm{pl}}=\left(R_{\max }-R_{0}\right) / v_{\mathrm{pl}} \tag{3.84}
\end{equation*}
$$

To keep calculations simple let us assume that the perturbation sources are distributed over the region according to the first spatial eigenfunction $\mathcal{R}(R) \propto X_{1}(R)$ (see Figure 3.8), such


Figure 3.8: First several spatial eigenfunctions $X_{k}(3.79)$.
that expansion coefficients are

$$
c_{k} \propto \int_{R_{0}}^{R_{\max }} X_{1}(R) X_{k}(R) \rho(R) d R
$$

Orthogonality of $\left\{X_{k}\right\}$ leaves only one non-vanishing coefficient

$$
\begin{equation*}
c_{1} \propto \int_{R_{0}}^{R_{\max }} X_{1}(R) X_{1}(R) \rho(R) d R=\left\|X_{1}\right\|^{2} \tag{3.85}
\end{equation*}
$$

reducing the sum (3.81) to a single term

$$
\begin{equation*}
Y_{1}(R, t)=e^{\lambda_{1}^{2} t} X_{1}(R) . \tag{3.86}
\end{equation*}
$$

Let us consider the time signal $\bar{Y}_{1}(t)=e^{\lambda_{1}^{2} t}$, on interval $0<t<T_{\mathrm{pl}}$. A non-periodic function defined over $0<t<T_{\mathrm{pl}}$ can be expanded into a Fourier series which is defined only in $0<t<T_{\mathrm{pl}}$ :

$$
\begin{equation*}
\bar{Y}_{1}(t)=\frac{a_{0}}{\sqrt{T_{\mathrm{pl}}}}+\sum_{n=1}^{\infty}\left[a_{n} g_{n}\left(\omega_{n} t\right)+b_{n} h_{n}\left(\omega_{n} t\right)\right] \tag{3.87}
\end{equation*}
$$

where the angular velocity of $\mathrm{n}-t h$ harmonic is

$$
\begin{equation*}
\omega_{n}=\frac{2 \pi}{T_{\mathrm{pl}}} n, \quad n=1,2, \cdots, \tag{3.88}
\end{equation*}
$$

functions

$$
\begin{aligned}
& g_{n}\left(\omega_{n} t\right)=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \cos \left(\omega_{n} t\right), \\
& h_{n}\left(\omega_{n} t\right)=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \sin \left(\omega_{n} t\right),
\end{aligned}
$$

form the orthonormal basis, $a_{n}, b_{n}$ are the Fourier coefficients. Parseval's theorem (Appendix B) states that the area under the power spectrum curve is equal to the area under the square of the magnitude of the time signal

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left|\bar{Y}_{1}(t)\right|^{2} d t=\int_{-\infty}^{\infty}\left\|\mathcal{F}_{\bar{Y}_{1}}(\omega)\right\|^{2} d \omega \tag{3.89}
\end{equation*}
$$

Substituting the Fourier series representation (3.87) of $\bar{Y}_{1}(t)$ into the left-hand side of the Parseval's identity we obtain after integration

$$
\int_{0}^{T_{\mathrm{pl}}}\left|\bar{Y}_{1}(t)\right|^{2} d t=a_{0}^{2}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right) .
$$

The right-hand side of this equation can also be written as an integral over $\omega$, using Dirac $\delta$-function

$$
a_{0}^{2}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)=\int_{-\infty}^{\infty}\left[a_{0}^{2} \delta(\omega)+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right) \delta\left(\omega-\omega_{n}\right)\right] d \omega
$$

Thus

$$
\begin{equation*}
\int_{0}^{T_{\mathrm{pl}}}\left|\bar{Y}_{1}(t)\right|^{2} d t=\int_{-\infty}^{\infty}\left[a_{0}^{2} \delta(\omega)+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right) \delta\left(\omega-\omega_{n}\right)\right] d \omega \tag{3.90}
\end{equation*}
$$

where the quantity under the integral on the right-hand side is the power spectrum of $\bar{Y}_{1}$

$$
\left|\left|\mathcal{F}_{\bar{Y}_{1}}(\omega) \|^{2}=a_{0}^{2}\right|_{\omega=0}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)\right|_{\omega=\omega_{n}}
$$

Thus, in order to find $\left\|\mathcal{F}_{\bar{Y}_{1}}(\omega)\right\|^{2}$ we need to calculate the Fourier coefficients $a_{0}, a_{n}$, and $b_{n}$ :

$$
\begin{aligned}
& a_{0}=\frac{\int_{0}^{T_{\mathrm{pl}}} \frac{1}{\sqrt{T_{\mathrm{pl}}}} \bar{Y}_{1}(t) d t}{\int_{0}^{T_{\mathrm{pl}}}\left(\frac{1}{\sqrt{T_{\mathrm{pl}}}}\right)^{2} d t}=\frac{1}{\sqrt{T_{\mathrm{pl}}}} \int_{0}^{T_{\mathrm{pl}}} e^{\lambda_{1}^{2} t} d t=\left.\frac{1}{\lambda_{1}^{2} \sqrt{T_{\mathrm{pl}}}} e^{\lambda_{1}^{2} t}\right|_{0} ^{T_{\mathrm{pl}}}=\frac{1}{\lambda_{1}^{2} \sqrt{T_{\mathrm{pl}}}}\left(e^{\lambda_{1}^{2} T_{\mathrm{pl}}}-1\right), \\
& a_{n}=\frac{\sqrt{\frac{2}{T_{\mathrm{pl}}}} \int_{0}^{T_{\mathrm{pl}}} \bar{Y}_{1}(t) \cos \left(\omega_{n} t\right) d t}{\frac{2}{T_{\mathrm{pl}}} \int_{0}^{T_{\mathrm{pl}}} \cos ^{2}\left(\omega_{n} t\right) d t}=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \int_{0}^{T_{\mathrm{pl}}} e^{\lambda_{1}^{2} t} \cos \left(\omega_{n} t\right) d t=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \cdot \lambda_{1}^{2} \frac{e^{\lambda_{1}^{2} T_{\mathrm{pl}}}-1}{\lambda_{1}^{4}+\omega_{n}^{2}},
\end{aligned}
$$

$$
b_{n}=\frac{\sqrt{\frac{2}{T_{\mathrm{pl}}}} \int_{0}^{T_{\mathrm{pl}}} \bar{Y}_{1}(t) \sin \left(\omega_{n} t\right) d t}{\frac{2}{T_{\mathrm{pl}}} \int_{0}^{T_{\mathrm{pl}}} \sin ^{2}\left(\omega_{n} t\right) d t}=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \int_{0}^{T_{\mathrm{pl}}} e^{\lambda_{1}^{2} t} \sin \left(\omega_{n} t\right) d t=\sqrt{\frac{2}{T_{\mathrm{pl}}}} \cdot \omega_{n} \frac{1-e^{\lambda_{1}^{2} T_{\mathrm{pl}}}}{\lambda_{1}^{4}+\omega_{n}^{2}} .
$$

Finally, dropping the DC term, we obtain the power spectrum from the inner region $R_{0} \leq$ $R \leq R_{\text {max }}$

$$
\left\|\mathcal{F}_{\Delta L_{X} 2}(\omega)\right\|^{2}= \begin{cases}\frac{2}{T_{\mathrm{pl}}} \frac{\left(e^{\lambda_{1}^{2} T_{\mathrm{pl}}}-1\right)^{2}}{\lambda_{1}^{4}+\omega_{n}^{2}}, & \omega=\omega_{n}  \tag{3.91}\\ 0, & \omega \neq \omega_{n}\end{cases}
$$

where $\omega_{n}$ are given by (3.88). We notice that due to the finite time interval of the Fourier expansion the spectrum is discrete, i.e. it is non-zero only at certain frequencies (harmonics). The behavior of the power spectrum is governed by relation between the diffusion timescale $t_{\mathrm{d}}=\lambda_{1}^{-2}$ of the flow, and the characteristic time $T_{\mathrm{pl}}$ of plasma inward motion in the exponent $e^{T_{\mathrm{pl}} / t_{\mathrm{d}}}$. Consider the following cases

- a quasi-stable regime $T_{\mathrm{pl}} \sim t_{\mathrm{d}}$ yields the power spectrum which behaves as the powerlaw with index -2 for high frequency harmonics and flattens out into the white noise in the low frequency limit.
- regime $T_{\mathrm{pl}} \ll t_{\mathrm{d}}$ leads to power spectrum extinction, which can be realized in the case of very fast, free-fall-like movement of gas, when the perturbations in the flow just do not have sufficient time to propagate and cause luminosity variability.
- regime $T_{\mathrm{pl}} \gg t_{\mathrm{d}}$ corresponds to unbounded exponential growth (destruction) of the power spectrum, in the case of slowly-moving gas, causing even minimal perturbations grow exponentially. In this regime the diffusion problem becomes highly non-linear.

It should be noted that it is likely that matter entering the unstable zone $R<R_{\max }$, with negative diffusion coefficient, will rapidly attain the velocity comparable to the free-fall speed

$$
\begin{equation*}
v_{f f}=c \sqrt{\frac{R_{\mathrm{S}}}{R}} \tag{3.92}
\end{equation*}
$$

Even for the lowest observable [61] value of Reynolds number $\gamma \sim 3$, which corresponds to the largest value $R_{\max } \lesssim 4.5 R_{\mathrm{S}}$, the estimated value for $v_{f f} \gtrsim 0.5 \mathrm{c}$. On the other hand, the velocity of the viscous diffusion propagation is comparable or less than the speed of sound [47]

$$
\begin{equation*}
v_{S}=\sqrt{\frac{k_{\mathrm{B}} T}{m_{\mathrm{P}}}} . \tag{3.93}
\end{equation*}
$$

Assuming the highest temperature of the photons, typically observed from the sources in the low-hard state, to be $k_{\mathrm{B}} T \sim 50 \mathrm{keV}$, we obtain that $v_{S} \lesssim 0.007 c$, i.e. at least a factor of 70 less than $v_{f f}$. Based on these estimates we anticipate that the most probable scenario, among the three listed above, is the second, which results in the power spectrum extinction. One can assume that such an accretion mode, past the critical point $R_{\max }$, is not observationally-relevant for timing analysis. In the next Chapter we provide a numerical solution of the diffusion problem for the transition region and the Keplerian disk. Due to intrinsic instability of the solution in the region $2, R_{0} \leq R \leq R_{\max }$, we do not attempt to solve the problem in that region. In fact, the stability criterion for the numerical method, used to solve the tridiagonal system of equations, is violated for the diffusion equation with negative diffusion coefficient (Appendix J). Special approach is required to treat such a highly-nonlinear problem, which is not a part of this research. Taking into account the fact that the signal from the unstable zone is unlikely to contribute much to the observed signal anyway, we will concentrate only on studying the diffusion equation defined for stable part of the transition layer, $R_{\max } \leq R \leq R_{\text {adj }}$.

## Chapter 4: Numerical solution of the diffusion problem

The evolution equation (3.7) for $\triangle \dot{M}$ is a diffusion (heat) partial differential equation in one dimension, that can be approximated in finite differences and solved numerically. Numerical approach removes limitations of the analytical solution, in particular, allowing us to

- drop all approximations made to render the problem solvable in elementary functions,
- use the source term of arbitrary complexity,
- test various forms of the radial dependence law of viscosity in the disk,
- solve a diffusion problem for the extended Keplerian disk configuration, using logarithmic scale in both space and time
- conduct "quick" computer experiments in order to study peculiarities of the system under consideration, etc.,
while providing a far superior computational speed (maintaining very good accuracy), as compared to implemented-on-a-computer analytical formulas. In this Chapter we will solve the diffusion propagation problem with time and space variable source of perturbations, for the two independent "configurations" - a compact (an order of a few $R_{\mathrm{S}}$ ) sub-Keplerian transition layer and an extended ( $\sim 10^{4} R_{\mathrm{S}}$ ) Keplerian accretion disk. We will calculate the power spectrum of the two solutions (integrated over the relevant spatial interval) for a broad range of frequencies, and demonstrate how the two power spectra can be combined together to yield a spectrum continuum that has direct connection to the observed power spectra from X-ray accreting binary systems. For this analysis we will consider a massive $10 M_{\odot}$ black hole, rotating with $f_{0}=30 \mathrm{~Hz}$.


### 4.1 Diffusion equation for compact non-Keplerian and extended Keplerian configurations and its approximation in finite differences

First, let us write down the evolution equation for accretion rate fluctuations $\triangle \dot{M}(r, t)$ expressed in terms of dimensionless variables (2.41) - (2.43) of TLM98, for the two disk configurations. It is common to assume that kinematic viscosity in accretion disk is a power law function of radius, normalized to the value on the outer edge of the disk

$$
\begin{equation*}
\nu=\nu_{0}\left(r / r_{\text {out }}\right)^{\psi} . \tag{4.1}
\end{equation*}
$$

For our simulations we will adopt that the non-Keplerian transition layer is confined between radius $r_{\text {max }}(2.49)$ of maximum angular velocity and the adjustment point $r_{\text {adj }}$, given by relation (2.50); while the Keplerian disk extends from the outer edge of the transition layer $r_{\text {adj }}$ to $r_{\text {out }}=10^{4}\left[r\right.$ is measured in units $R / R_{0}$, where $R_{0}=3 R_{\mathrm{S}}, R_{\mathrm{S}}=2 G M / c^{2}$ is Schwarzschild radius]. Employing viscosity distribution law (4.1) and substituting formula (2.44), for the non-Keplerian angular velocity $\theta(r)$ in the adjustment layer, into equation (3.7), one obtains

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\nu_{0} R_{0}^{-2} r_{\mathrm{adj}}^{-\psi} r^{\gamma-1} \frac{\partial}{\partial r}\left[\left(\mu r^{\psi+1-\gamma}+\xi r^{\psi-1}\right) \frac{\partial u}{\partial r}\right]+f_{\mathrm{dr}}(r, t) . \tag{4.2}
\end{equation*}
$$

where $u(r, t)=\triangle \dot{M}(r, t) ; \mu=\gamma /(2-\gamma), \xi=2 /(2-\gamma)\left(D_{1}^{-1}-1\right)$ are constant coefficients, $D_{1}$ is given by equation (2.45). Substituting Keplerian angular velocity profile $\theta(r)=\theta_{\mathrm{K}}(r)$, equation (2.47), into (3.7) we obtain the diffusion propagation equation for the extended disk

$$
\begin{equation*}
\frac{\partial u}{\partial t}=3 \nu_{0} R_{0}^{-2} r_{\text {out }}^{-\psi} r^{1 / 2} \frac{\partial}{\partial r}\left[r^{\psi-1 / 2} \frac{\partial u}{\partial r}\right]+f_{\mathrm{dr}}(r, t) . \tag{4.3}
\end{equation*}
$$

Diffusion equations (4.2), (4.3) have the form

$$
\begin{equation*}
\frac{\partial u}{\partial t}=f(r) \frac{\partial}{\partial r}\left[D(r) \frac{\partial u}{\partial r}\right]+S(r, t) \tag{4.4}
\end{equation*}
$$

This is a linear equation with coordinate-dependent coefficient of diffusion. Using (2nd order accurate) centered difference formula we can approximate the right-hand side of (4.4) at $i-$ th spatial node at $k$-th time instant as

$$
\begin{equation*}
f(r) \frac{\partial}{\partial r}\left[D(r) \frac{\partial u}{\partial r}\right]+S(r, t) \approx f_{i} \frac{D_{i+1 / 2} u_{i+1 / 2}^{\prime}-D_{i-1 / 2} u_{i-1 / 2}^{\prime}}{2(\triangle r / 2)}+S_{i}^{k} \tag{4.5}
\end{equation*}
$$

where the spatial step is $\triangle r$, and index $i$ runs through the spatial grid $1 \leq i \leq M$. Applying the same 2nd order formula for derivatives $u_{i-1 / 2}^{\prime}, u_{i+1 / 2}^{\prime}$

$$
\begin{equation*}
u_{i-1 / 2}^{\prime} \approx \frac{u_{i}-u_{i-1}}{\Delta r}, \quad u_{i+1 / 2}^{\prime} \approx \frac{u_{i+1}-u_{i}}{\Delta r} \tag{4.6}
\end{equation*}
$$

gives

$$
\begin{equation*}
f(r) \frac{\partial}{\partial r}\left[D(r) \frac{\partial u}{\partial r}\right]+S(r, t) \approx f_{i} \frac{D_{i+1 / 2}\left(u_{i+1}-u_{i}\right)-D_{i-1 / 2}\left(u_{i}-u_{i-1}\right)}{(\Delta r)^{2}}+S_{i}^{k} . \tag{4.7}
\end{equation*}
$$

Using $\partial u / \partial t \approx\left(u_{i}^{k+1}-u_{i}^{k}\right) / \Delta t$ for the time derivative, where the time step is $\Delta t$, and index $k$ runs through the time grid $1 \leq k \leq N$ we can write the explicit centered difference for the heat equation (4.4)

$$
\begin{equation*}
\frac{u_{i}^{k+1}-u_{i}^{k}}{\triangle t} \approx \frac{f_{i}}{(\triangle r)^{2}}\left[D_{i+1 / 2}\left(u_{i+1}^{k}-u_{i}^{k}\right)-D_{i-1 / 2}\left(u_{i}^{k}-u_{i-1}^{k}\right)\right]+S_{i}^{k} \tag{4.8}
\end{equation*}
$$

The implicit centered difference is based on the solution at time step $k+1$

$$
\begin{equation*}
\frac{u_{i}^{k+1}-u_{i}^{k}}{\Delta t} \approx \frac{f_{i}}{(\triangle r)^{2}}\left[D_{i+1 / 2}\left(u_{i+1}^{k+1}-u_{i}^{k+1}\right)-D_{i-1 / 2}\left(u_{i}^{k+1}-u_{i-1}^{k+1}\right)\right]+S_{i}^{k+1} \tag{4.9}
\end{equation*}
$$

Averaging the two approximations yields the desired scheme

$$
\begin{align*}
& \frac{u_{i}^{k+1}-u_{i}^{k}}{\triangle t} \approx \frac{f_{i}}{2(\triangle r)^{2}}\left\{D_{i+1 / 2}\left(u_{i+1}^{k}-u_{i}^{k}\right)-D_{i-1 / 2}\left(u_{i}^{k}-u_{i-1}^{k}\right)+\right.  \tag{4.10}\\
& \left.D_{i+1 / 2}\left(u_{i+1}^{k+1}-u_{i}^{k+1}\right)-D_{i-1 / 2}\left(u_{i}^{k+1}-u_{i-1}^{k+1}\right)\right\}+\frac{1}{2}\left(S_{i}^{k}+S_{i}^{k+1}\right)
\end{align*}
$$

The implicit scheme (4.10) is unconditionally stable and is second-order accurate in both space and time [17]. It is known as Crank-Nicolson method, and for the heat equation in one space dimension leads to the tridiagonal linear system to be solved at each time step, thus providing fast, memory-efficient, and easy-to-implement algorithm. Sorting out $k$ and $k+1$ terms on different sides of equation and calling $\alpha=f_{i} \Delta t / 2(\Delta r)^{2}, \bar{S}=\left(S_{i}^{k}+S_{i}^{k+1}\right) / 2$ gives

$$
\begin{align*}
& a_{i} u_{i-1}^{k+1}+b_{i} u_{i}^{k+1}+c_{i} u_{i+1}^{k+1}=  \tag{4.11}\\
& \alpha D_{i-1 / 2} u_{i-1}^{k}+\left[1-\alpha\left(D_{i+1 / 2}+D_{i-1 / 2}\right)\right] u_{i}^{k}+\alpha D_{i+1 / 2} U_{i+1}^{k}+\triangle t \bar{S},
\end{align*}
$$

with matrix elements

$$
\left\{\begin{array}{l}
a_{i}=-\frac{f_{i}}{2} \frac{\Delta t}{(\Delta r)^{2}} D_{i-1 / 2},  \tag{4.12}\\
b_{i}=1+\frac{f_{i}}{2} \frac{\Delta t}{(\Delta r)^{2}}\left[D_{i-1 / 2}+D_{i+1 / 2}\right], \\
c_{i}=-\frac{f_{i}}{2} \frac{\Delta t}{(\triangle r)^{2}} D_{i+1 / 2} .
\end{array}\right.
$$

Equation (4.11) can be conveniently presented in a matrix form $\mathbf{A} \mathbf{u}_{\mathbf{i}}^{\mathbf{k}+\mathbf{1}}=\mathbf{d}_{\mathbf{i}}^{\mathbf{k}}$ :

$$
\left(\begin{array}{ccccccccc}
b_{1} & c_{1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
a_{2} & b_{2} & c_{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{3} & b_{3} & c_{3} & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & . & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{M} & b_{M}
\end{array}\right)\left(\begin{array}{c}
u_{1}^{k+1} \\
u_{2}^{k+1} \\
u_{3}^{k+1} \\
\cdots \\
u_{M-1}^{k+1} \\
u_{M}^{k+1}
\end{array}\right)=\left(\begin{array}{c}
d_{1}^{k} \\
d_{2}^{k} \\
d_{3}^{k} \\
\cdots \\
d_{M-1}^{k} \\
d_{M}^{k}
\end{array}\right)
$$

where symbol $d_{i}^{k}$ represents the entire right-hand side of equation (4.11) and matrix elements $a_{i}, b_{i}, c_{i}$ as defined above. In this form the diffusion equation can be solved on a computer, using any tridiagonal matrix solver. Possible choices include Thomas method (the sweep method), variations of Gaussian elimination or Cholesky factorization methods, etc. I implemented the sweep method in C and compared it with the tridiagonal solver gsl_linalg_solve_tridiag (a variant of Cholesky decomposition provided in the GNU Scientific Library), to find that both produce exactly the same solution.

Complete formulation of the initial-boundary value problem requires specification of the boundary conditions, which in general can be mixed, i.e. given as a linear combination of Dirichlet (function's value is specified on the boundary) and Neumann (function's derivative is specified on the boundary) boundary conditions. Thus for one space dimension we would generally have

$$
\begin{cases}\alpha_{\mathrm{in}} u(r)+\beta_{\mathrm{in}} \frac{\partial u}{\partial r}=\gamma_{\mathrm{in}} & \text { at } r=r_{\mathrm{in}}  \tag{4.15}\\ \alpha_{\mathrm{out}} u(r)+\beta_{\mathrm{out}} \frac{\partial u}{\partial r}=\gamma_{\mathrm{out}} & \text { at } r=r_{\mathrm{out}}\end{cases}
$$

Recalling that our spatial grid is defined over range $1 \leq i \leq M$, we reserve indexes $i=0$ and $i=M+1$ for "inner" and "outer" boundaries, such that $u\left(r_{\text {in }}\right)=u_{0}, u\left(r_{\text {out }}\right)=u_{M+1}$.

It follows from (4.15) and (4.16) that

$$
\left\{\begin{array}{l}
u_{0}=\frac{\beta_{\mathrm{in}} u_{1}-\Delta r \gamma_{\mathrm{in}}}{\beta_{\mathrm{in}}-\triangle r \alpha_{\mathrm{in}}}  \tag{4.17}\\
u_{M+1}=\frac{\beta_{\mathrm{out}} u_{M}+\triangle r \gamma_{\mathrm{out}}}{\beta_{\mathrm{out}}+\triangle r \alpha_{\mathrm{out}}}
\end{array}\right.
$$

where we approximated $\frac{\partial u}{\partial r}\left(r_{\text {in }}\right) \approx \frac{u_{1}-u_{0}}{\Delta r}, \frac{\partial u}{\partial r}\left(r_{\text {out }}\right) \approx \frac{u_{M+1}-u_{M}}{\Delta r}$. Equation (4.11) written for $i=1$ and $i=M$ reads

$$
\left\{\begin{array}{l}
a_{1} \sqrt[u_{0}^{k+1}]{k+} \mathbf{b}_{\mathbf{1}} \mathbf{u}_{\mathbf{1}}^{\mathbf{k}+\mathbf{1}}+\mathbf{c}_{\mathbf{1}} \mathbf{u}_{\mathbf{2}}^{\mathbf{k}+\mathbf{1}}=\mathbf{d}_{\mathbf{1}}^{\mathbf{k}}  \tag{4.19}\\
\mathbf{a}_{\mathbf{M}} \mathbf{u}_{\mathbf{M}-\mathbf{1}}^{\mathbf{k}+\mathbf{1}}+\mathbf{b}_{\mathbf{M}} \mathbf{u}_{\mathbf{M}}^{\mathbf{k}+\mathbf{1}}+c_{M} \sqrt[u_{M+1}^{k+1}]{k+1}=\mathbf{d}_{\mathbf{M}}^{\mathbf{k}}
\end{array}\right.
$$

with bold-lettered parts being the first and the last equations of the linear system $\mathbf{A u} \mathbf{u}_{\mathbf{i}}^{\mathbf{k + 1}}=\mathbf{d}_{\mathbf{i}}^{\mathbf{k}}$. Elimination of $u_{0}^{k+1}, u_{M+1}^{k+1}$ (boxed terms) in (4.19), (4.20) by means of substitution of (4.17) (4.18) gives

$$
\left\{\begin{array}{l}
b_{1}^{*} u_{1}^{k+1}+c_{1} u_{2}^{k+1}=\left(d_{1}^{k}\right)^{*}  \tag{4.21}\\
a_{M} u_{M-1}^{k+1}+b_{M}^{*} u_{M}^{k+1}=\left(d_{M}^{k}\right)^{*}
\end{array}\right.
$$

where

$$
\begin{cases}b_{1}^{*}=b_{1}+a_{1} \frac{\beta_{\mathrm{in}}}{\beta_{\mathrm{in}}-\triangle r \alpha_{\mathrm{in}}} & \left(d_{1}^{k}\right)^{*}=d_{1}^{k}+a_{1} \frac{\Delta r \gamma_{\mathrm{in}}}{\beta_{\mathrm{in}}-\triangle r \alpha_{\mathrm{in}}}  \tag{4.23}\\ b_{M}^{*}=b_{M}+c_{M} \frac{\beta_{\mathrm{out}}}{\beta_{\mathrm{out}}+\triangle r \alpha_{\mathrm{out}}} & \left(d_{M}^{k}\right)^{*}=d_{M}^{k}-c_{M} \frac{\Delta r \gamma_{\mathrm{out}}}{\beta_{\mathrm{out}}+\triangle r \alpha_{\mathrm{out}}}\end{cases}
$$

Therefore, in order to account for the boundary conditions of type (4.15), (4.16), diagonal matrix elements $b_{1}, b_{M}$ have to be altered to $b_{1}^{*}, b_{M}^{*}$, as well as the right-hand side vector elements $d_{1}^{k}, d_{M}^{k}$ have to be altered to $\left(d_{1}^{k}\right)^{*},\left(d_{M}^{k}\right)^{*}$ according to (4.23), (4.24). This would also require calculation of coefficients $a_{1}, c_{M}$, which are not part of the tridiagonal matrix A. Consider, for instance, the case of Dirichlet or Neumann boundary conditions

$$
\left\{\begin{array}{l}
u\left(r_{\text {in }}\right)=x,  \tag{4.25}\\
\frac{\partial u}{\partial r}\left(r_{\text {out }}\right)=y .
\end{array}\right.
$$

Using (4.15), (4.16) and (4.23), (4.24) we obtain

$$
\left\{\begin{array}{l}
\left(d_{1}^{k}\right)^{*}=d_{1}^{k}-a_{1} x  \tag{4.27}\\
b_{M}^{*}=b_{M}+c_{M} \\
\left(d_{M}^{k}\right)^{*}=d_{M}^{k}-\Delta r c_{M} y
\end{array}\right.
$$

Likewise, for the boundary conditions

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial r}\left(r_{\text {in }}\right)=x  \tag{4.30}\\
u\left(r_{\text {out }}\right)=y
\end{array}\right.
$$

modified matrix/vector entries become

$$
\left\{\begin{array}{l}
b_{1}^{*}=b_{1}+a_{1},  \tag{4.32}\\
\left(d_{1}^{k}\right)^{*}=d_{1}^{k}+\triangle r a_{1} x \\
\left(d_{M}^{k}\right)^{*}=d_{M}^{k}-c_{M} y
\end{array}\right.
$$

### 4.2 Perturbation term in the diffusion equation

The origins of broadband variability observed in X-ray binary systems are attributed to an ensemble of oscillations and waves propagating in the innermost region of an accretion disk. As in other systems, one can expect various types of oscillatory motions in accretion disk. These motions, in general, occur as the outcome of restoring forces acting on perturbations [20]. As a result, an accretion disk represents a complex dynamical system of many variables with its defining physical quantities (radius, height, density, temperature, viscosity, pressure, speed of sound, accretion rate, etc.) experiencing quasi-periodic fluctuations around their equilibrium values in a steady state, with characteristic timescales. Some of these timescales, ordered from fastest to slowest, include [41], [30]: the dynamical timescale $t_{\varphi} \sim R / v_{\mathrm{K}}=\left(R^{3} / G M\right)^{1 / 2}$, the $z$-timescale $t_{z}=2 H / v_{\mathrm{S}}=R / v_{\mathrm{K}}=t_{\varphi}$, the thermal timescale $t_{\mathrm{th}}=\Sigma v_{\mathrm{S}}^{2} / F(R) \sim R^{3} v_{\mathrm{S}}^{2} / G M \nu$, and the viscous timescale $t_{\nu}=R / v_{R} \sim R^{2} / \nu \sim \alpha^{-1}(H / R)^{-2} t_{\varphi}$. It is reasonable to assume that local mechanical oscillations, which can translate into fluctuations of the mass accretion rate $\triangle \dot{M}$, occur at all radii in the disk, can have radial, azimuthal, and vertical modes, and can be of different frequency and amplitude, possibly conforming with some radial distribution law. Let us demonstrate that such perturbations happen on the local dynamical timescale $t_{\varphi}$. Local disk perturbations may be caused by various oscillatory and instability-related processes in the rotating gas, including particle orbits eccentricities initiated by irregularities in mass supply or disturbances from the companion star, relaxation fluctuations due to Rayleigh-Taylor instabilities in the disks with inhomogeneous vertical structure [54], inertial and coupled oscillations, $p$ - and $g$-mode trapped disk oscillations [57], [20], oscillations due to magnetic force (MRI), etc. At basics, however, all such oscillation mechanisms essentially come down to perturbing Keplerian orbits of gas particles. In a non-relativistic disk, under an assumption of "smallness" of perturbations it can be shown that their dominant oscillation frequency in either $\hat{r}$, or $\hat{\varphi}$, or $\hat{z}$ directions scales with radial position as the Keplerian angular velocity $\nu_{\mathrm{dr}} \propto \nu_{\mathrm{K}}(\mathrm{R})$. Due to presence of viscosity between adjacent layers
of rotating matter these oscillations diffuse out into neighboring disk annuli, and therefore decay with time.

### 4.2.1 Small perturbations of a Keplerian orbit

To prove our statement let us now analyze the problem of stability of the modified Keplerian orbit. We will consider the simplest case of the circular orbit and prove that the small perturbations of such orbits are stable and have the same period as the original orbit. Assume that a particle of mass $m$ is orbiting another particle of mass $M \gg m$ at distance $r=r_{0}$, such that its orbit lies in the $\hat{r} \hat{\varphi}$-plane of the cylindrical coordinates $(r, \varphi, z)$. The effective gravitational potential of the two interacting masses is given by

$$
\begin{equation*}
U_{\mathrm{eff}}(r)=-G \frac{m M}{r_{0}}+\frac{l^{2}}{2 \mu r_{0}^{2}}, \tag{4.35}
\end{equation*}
$$

where the reduced mass $\mu=m M /(m+M)$, and the angular momentum $l=\mu r_{0}^{2} \dot{\varphi}$ is conserved. Since $m \ll M, \mu \approx m$. It is easier to proceed if we move to the non-inertial (in our case rotating) frame of reference. Let us choose the frame of reference placed at the center of the unperturbed Kepler orbit, and uniformly-rotating with the angular velocity equal to that of the orbit $\Omega=\omega_{\mathrm{K}}\left(r_{0}\right)=\dot{\varphi}$. In such frame of reference the unperturbed particle is at rest, because the gravitational pull is balanced by the centrifugal force

$$
\begin{equation*}
m \frac{d \boldsymbol{v}}{d t}=-G \frac{m M}{r_{0}^{2}} \frac{\boldsymbol{r}_{0}}{r_{0}}+m \boldsymbol{\Omega} \times\left(\boldsymbol{r}_{\mathbf{0}} \times \boldsymbol{\Omega}\right)=0 . \tag{4.36}
\end{equation*}
$$

Lets now perturb the particle slightly in the $\hat{r} \hat{\varphi}$-plane so that it acquires some non-zero velocity. As a result unperturbed motion $\boldsymbol{r}_{\mathbf{0}}, \varphi_{0}=0$ changes into perturbed motion $\boldsymbol{r}=$ $r_{0}+\delta \boldsymbol{r}(t), \varphi=\varphi_{0}+\delta \varphi(t)$. Equation of motion for the particle $m$ becomes

$$
\begin{equation*}
m \frac{d \boldsymbol{v}}{d t}=-G \frac{m M}{r^{2}} \frac{\boldsymbol{r}}{r}+m \boldsymbol{\Omega} \times(\boldsymbol{r} \times \boldsymbol{\Omega})+2 m \boldsymbol{v} \times \boldsymbol{\Omega} \tag{4.37}
\end{equation*}
$$

where the last term on the right-hand side is the Coriolis force, which now appeared because particle picked up some velocity as a result of perturbation. Cancelling out $m$ and writing down $r$ and $\varphi$ components of (4.37) we have

$$
\left\{\begin{array}{l}
\dot{v}_{r}=-\frac{G M}{r^{2}}+(\boldsymbol{\Omega} \times(\boldsymbol{r} \times \boldsymbol{\Omega}))_{r}+2(\boldsymbol{v} \times \boldsymbol{\Omega})_{r}  \tag{4.38}\\
\dot{v}_{\varphi}=(\boldsymbol{\Omega} \times(\boldsymbol{r} \times \boldsymbol{\Omega}))_{\varphi}+2(\boldsymbol{v} \times \boldsymbol{\Omega})_{\varphi}
\end{array}\right.
$$

We can use the Taylor series to expand $1 / r^{2}$ around the small parameter:

$$
\frac{1}{\left(r_{0}+\delta r\right)^{2}} \approx \frac{1}{r_{0}^{2}}-\frac{2}{r_{0}^{3}} \delta r
$$

After calculating $r$ and $\varphi$ components of the vector products in (4.38) and (4.39) we obtain

$$
\left\{\begin{array}{l}
\ddot{\delta} r=-\frac{G M}{r_{0}^{2}}+\frac{2 G M}{r_{0}^{3}} \delta r+\Omega^{2} r_{0}+\Omega^{2} \delta r+2 \Omega r_{0} \dot{\delta} \varphi  \tag{4.40}\\
r_{0} \ddot{\delta} \varphi=-2 \Omega \dot{\delta} r
\end{array}\right.
$$

where $\dot{v}_{r}=\ddot{\delta} r, \dot{v}_{\varphi}=r_{0} \ddot{\delta} \varphi$. First and third terms on the right-hand side of equation (4.40) represent the unperturbed orbit and balance each other out, therefore yielding

$$
\left\{\begin{array}{l}
\ddot{\delta} r=k \delta r+a r_{0} \dot{\delta} \varphi  \tag{4.42}\\
r_{0} \ddot{\delta} \varphi=-a \dot{\delta} r
\end{array}\right.
$$

with $k=2 G M / r_{0}^{3}+\Omega^{2}, a=2 \Omega$. This system of equations can be solved by differentiating (4.42) with respect to time, followed by substitution of (4.43):

$$
\begin{equation*}
\ddot{v}_{r}=k \dot{\delta} r+a r_{0} \ddot{\delta} \varphi=\left(k-a^{2}\right) \dot{\delta} r=\left(k-a^{2}\right) v_{r} \tag{4.44}
\end{equation*}
$$

Or, after substituting back expressions for $k$ and $a$

$$
\begin{align*}
& \ddot{v}_{r}+\omega_{\mathrm{osc}}^{2} v_{r}=0,  \tag{4.45}\\
& \omega_{\mathrm{osc}}^{2}=4 \Omega^{2}-\frac{2 G M}{r_{0}^{3}}-\Omega^{2} . \tag{4.46}
\end{align*}
$$

Recalling that our frame of reference is uniformly-rotating with $\Omega=\omega_{\mathrm{K}}\left(r_{0}\right)=\left(G M / r_{0}^{3}\right)^{1 / 2}$, we have from (4.46)

$$
\begin{equation*}
\omega_{\mathrm{osc}}=\sqrt{\frac{G M}{r_{0}^{3}}}=\omega_{\mathrm{K}}\left(r_{0}\right) \tag{4.47}
\end{equation*}
$$

Equation of free oscillations (4.45) has solution

$$
\begin{equation*}
v_{r}=C_{1} \sin \left(\omega_{\mathrm{osc}} t\right)+C_{2} \cos \left(\omega_{\mathrm{osc}} t\right)=\dot{\delta} r . \tag{4.48}
\end{equation*}
$$

$\delta r$ is obtained by integration of the last equation

$$
\begin{equation*}
\delta r(t)=\frac{1}{\omega_{\mathrm{osc}}}\left[-C_{1} \cos \left(\omega_{\mathrm{osc}} t\right)+C_{2} \sin \left(\omega_{\mathrm{osc}} t\right)\right]+C_{3} \tag{4.49}
\end{equation*}
$$

Solution for $\delta \varphi(t)$ can be found by integrating equation (4.43)

$$
\begin{equation*}
\ddot{\delta} \varphi=-\frac{2 \Omega}{r_{0}}\left[C_{1} \sin \left(\omega_{\mathrm{osc}} t\right)+C_{2} \cos \left(\omega_{\mathrm{osc}} t\right)\right] \tag{4.50}
\end{equation*}
$$

twice, giving

$$
\begin{equation*}
\delta \varphi(t)=\frac{2}{\omega_{\mathrm{osc}} r_{0}}\left[C_{1} \sin \left(\omega_{\mathrm{osc}} t\right)+C_{2} \cos \left(\omega_{\mathrm{osc}} t\right)\right] . \tag{4.51}
\end{equation*}
$$

Solutions for small oscillations $\delta r$ and $\delta \varphi$ (4.49) and (4.51) show that both are stable (sine and cosine have bounded variations) and occur at the same angular frequency equal to
the Kepler angular frequency of the unperturbed orbit, i.e. $\omega_{\mathrm{osc}}=\omega_{\mathrm{K}}\left(r_{0}\right)$ as follows from equation (4.47).

Let us now analyze a small perturbation $\delta z$ in $z$-direction that rotates the original orbital plane by a small angle $\theta$, but leaves the radius of the orbit $r=r_{0}$ unaltered. Obviously, such perturbation changes the angular momentum of the system. Because of the angle, attractive gravitational force acquires an additional $z$-component

$$
\begin{equation*}
F_{z}=-\frac{G m M}{r_{0}^{2}} \cdot \sin \theta=-\frac{G m M}{r_{0}^{2}} \cdot \frac{\delta z}{r_{0}} . \tag{4.52}
\end{equation*}
$$

This is the only non-zero force component acting in $z$, therefore the equation of motion is simply

$$
\begin{equation*}
m \ddot{\delta} z=-\frac{G m M}{r_{0}^{3}} \delta z \tag{4.53}
\end{equation*}
$$

After dividing out $m$, we have

$$
\begin{equation*}
\ddot{\delta} z+\frac{G M}{r_{0}^{3}} \delta z=0 . \tag{4.54}
\end{equation*}
$$

As in case of $r \varphi$ - perturbations, we obtained equation of free oscillations, which solutions are sines and cosines (and therefore bounded) with the angular frequency equal to the Keplerian angular frequency $\omega_{\mathrm{K}}(r)$ (4.47) of the unperturbed orbit.

It is also possible to show that in more general case of Keplerian elliptical orbits, the same result will hold, but we would have to deal with the time dependence of the coefficients in linearized equations of motion (4.42), (4.43), (4.54).

### 4.2.2 One-time/continuous Damped Harmonic Oscillator source

From the above considerations it is natural to assume, that for each annulus of the disk the driving force can be modeled by the Damped Harmonic Oscillator (DHO) with the initial (undamped) angular velocity equal to the Keplerian angular velocity $\omega_{\mathrm{K}}$ (2.1) at
given radius. For mass $m$ and spring constant $k$ the unforced DHO satisfies equation

$$
\begin{equation*}
\ddot{x}+\frac{1}{Q} \omega_{0} \dot{x}+\omega_{0}^{2} x=0, \tag{4.55}
\end{equation*}
$$

where $\omega_{0}^{2}=k / m$ and the quality factor $Q=\omega_{0} / \Delta \omega$ characterizes decay time of the oscillator. Free DHO equation can be solved with the help of an auxiliary equation $q^{2}+$ $\left(\omega_{0} / Q\right) q+\omega_{0}^{2}=0$ and the substitution $x=e^{q t}$. For $Q>1 / 2$ (underdamped case) the DHO will experience oscillations with the amplitude gradually decreasing to zero and at a slightly modified frequency than the undamped case

$$
\begin{equation*}
x(t)=A \sin \left(\omega_{0} t \sqrt{1-\frac{1}{4 Q^{2}}}+\varphi\right) e^{-\frac{\Gamma}{2} t} \tag{4.56}
\end{equation*}
$$

where the dampening factor $\Gamma \equiv \Delta \omega=\omega_{0} / Q$. The solution of the free underdamped oscillator with $f_{0}=100 \mathrm{~Hz}$, and $Q=5$ is shown on Figure 4.1. The characteristic decay time (after which the amplitude drops by factor $e$ ) is determined from $e^{-t / t_{0}}$ where $t_{0}=$ $2 Q / \omega_{0}$. We will use the source term defined by the solution (4.56) (" one-time" source) of the unforced equation in order to study the solution of the disk diffusion equation and behavior of the power density spectrum on a "characteristic" timescale of the system:

$$
\begin{equation*}
f_{\mathrm{dr}}^{1 \mathrm{t}}(R, t)=A(R) \sin \left[\omega_{\mathrm{K}}(R) t \sqrt{1-\frac{1}{4 Q^{2}}}+\varphi\right] e^{-\frac{\Gamma(R)}{2} t} \tag{4.57}
\end{equation*}
$$

where superscript " 1 t " stands form "one-time". Notice that $f_{\mathrm{dr}}^{1 \mathrm{t}}(R, t)$ is not a separable source, i.e. it can not be written as a product of space and time parts $\mathcal{R}(R) \varphi(t)$. Driving perturbations described by (4.57) once started at time instant $t=0$ gradually decay to zero as time goes to infinity, without ever being re-excited. Even though not realistic, such a source allows to study all the principal properties of the solution $\triangle \dot{M}(R, t)$ and
reproduces the key power spectrum features. It is clear from relations $\omega_{K}(R) \propto R^{-3 / 2}$ and $Q=\omega_{\mathrm{K}}(R) / \Delta \omega$ that the fastest oscillations occur at the smallest radii also have shortest decay time, while the slowest ones, that occur at the largest radii, decay on longer timescales. The obvious shortfall of such a source is that at long-enough times only the slowest oscillation modes survive, which of course is never the case in reality. However, (4.57) is sufficient to produce the principal solution on a characteristic/local (for a given radius) timescale. Therefore, we labeled it the "one-time" perturbation source. In order to simulate a more realistic variable source of perturbations, that gets randomly re-excited over time, with each following state based only on the previous state, one usually uses the autoregressive scheme, called the Markov chain [21]. A Markov chain produces a stationary series resulting from a stochastic process. In our case, with both dampening and oscillating terms present in the perturbation source, the quadratic autoregressive scheme should be implemented, where the $n$-th member of the series is given in terms of the previous. Such a process can be modeled using the numerical solution of the forced DHO equation, where the external force on the right-hand side is assigned a random value (for instance a Gaussian deviate with zero mean and variance $\sigma^{2}$ ) at each time instant

$$
\begin{equation*}
\ddot{f}_{\mathrm{dr}}^{\mathrm{st}}+\frac{1}{Q} \omega_{\mathrm{K}} \dot{f}_{\mathrm{dr}}^{\mathrm{st}}+\omega_{\mathrm{K}}^{2} f_{\mathrm{dr}}^{\mathrm{st}}=F_{\mathrm{ext}}(t) \tag{4.58}
\end{equation*}
$$

where superscript "st" stands for stochastic, $F_{\text {ext }}$ is a random deviate distributed as

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{2 \pi} \sigma} e^{-x^{2} / 2 \sigma^{2}} \tag{4.59}
\end{equation*}
$$

The variance of the distribution determines the resulting amplitude $f_{\mathrm{dr}}^{\mathrm{st}}(R, t)$, which is a function of radius. The radial dependence of $f_{\mathrm{dr}}^{\text {st }}$ can be understood from the following argument. In order to have a detectable effect on the oscillator $m$ attached to a spring with a spring constant $k$, the external force $F_{\text {ext }}$ should have an amplitude comparable to the
product of the spring constant and displacement

$$
\begin{equation*}
\frac{F_{\mathrm{ext}}}{m} \sim \frac{k}{m} \triangle x=\omega_{0}^{2} \triangle x . \tag{4.60}
\end{equation*}
$$

I.e., in order to achieve a unit displacement $F_{\text {ext }} / m$ should be of order $\omega_{0}^{2}$. In the case of excitation of Keplerian orbits displacement $\Delta x$ is replaced with the amplitude of the resulting driving force $f_{\mathrm{dr}}^{\mathrm{st}}$ and $\omega_{0}$ is replaced with $\omega_{\mathrm{K}}(R)$. Thus, for a given excitation amplitude $F_{\text {ext }}$ in (4.58), the response $f_{\mathrm{dr}}^{\mathrm{st}}(R, t)$ will be a growing function of radius (as angular velocity $\omega_{\mathrm{K}}$ decreases with radius as $\left.R^{-3 / 2}\right)$. Introducing new variables $u=f_{\mathrm{dr}}^{\mathrm{st}}(t)$, and $v=\dot{f}_{\mathrm{dr}}^{\mathrm{st}}(t)$ a second-order equation (4.58) can be written as a system of the two firstorder equations

$$
\left\{\begin{array}{l}
\dot{u}=v,  \tag{4.61}\\
\dot{v}=F_{\mathrm{ext}}(t)-\alpha v-\beta u,
\end{array}\right.
$$

where $\alpha=\omega_{\mathrm{K}} / Q, \beta=\omega_{\mathrm{K}}^{2}$. One can write a solution of the system (4.61), (4.62) in terms of the first-order forward finite difference (forward Euler method):

$$
\left\{\begin{array}{l}
v_{k+1}=v_{k}+\Delta t\left[F_{\mathrm{ext}}\left(t_{k}\right)-\alpha v_{k}-\beta u_{k}\right]  \tag{4.63}\\
u_{k+1}=u_{k}+v_{k} \triangle t
\end{array}\right.
$$

A system of equations (4.63), (4.64) can be easily written in a computer code. A continuous stochastic perturbation source $f_{\mathrm{dr}}^{\mathrm{st}}\left(R, t_{k}\right)$ (shown on Figure 4.2) as a solution of (4.58), (4.63) and (4.64) allows to solve for a much more realistic time signal, which exhibits life-like fluctuations in phase and amplitude, aka the photon "lightcurve", with all disk oscillation frequencies present at any arbitrarily-remote instant of time.


Figure 4.1: Unforced damped harmonic oscillator (4.56) with $Q=5$.


Figure 4.2: Damped harmonic oscillator (4.58) with random external force term (4.59).

### 4.3 Validation of the numerical method

Implementation of the numerical algorithm in the computer code must be checked to ensure its functionality and tested to produce the anticipated output. Such testing can be done by solving numerically a problem, the answer for which is known from exact analytical calculation, and comparing the results. It turns out that the diffusion problem for the extended Keplerian disk (4.3) can be solved analytically for certain values of parameters in the equation. Considered below are

- exact analytical solution of the initial-value problem for Keplerian configuration and comparison of the analytical result against numerical,
- exact analytical solution of the diffusion problem with the time-dependent perturbation source for Keplerian configuration and comparison of the analytical result against numerical.


### 4.3.1 Initial-value problem for Keplerian configuration

The simplest relevant problem can be formulated as the initial-value problem for the diffusion equation (4.3) with no source term, and with the appropriate initial and boundary conditions. In order to simultaneously test both Dirichlet and Neumann boundary conditions consider the following initial-value problem specified over arbitrary spatial region $r_{\text {min }} \leq r \leq r_{\text {max }}:$

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}=3 \nu_{0} R_{0}^{-2} r_{\mathrm{out}}^{-\psi} r^{1 / 2} \frac{\partial}{\partial r}\left[r^{\psi-1 / 2} \frac{\partial u}{\partial r}\right], \quad r_{\min } \leq r \leq r_{\max }  \tag{4.65}\\
u\left(r_{\min }\right)=0 \\
\frac{\partial u}{\partial r}\left(r_{\max }\right)=0 \\
u(r, 0)=f(r)
\end{array}\right.
$$

where the initial condition is defined as

$$
f(r)= \begin{cases}-A\left(r-r_{\min }-\epsilon\right)^{2}+B, & \text { for } r_{\min } \leq r \leq r_{\min }+\epsilon  \tag{4.69}\\ u_{0}=\text { const }, & \text { for } r_{\min }+\epsilon \leq r \leq r_{\max }\end{cases}
$$

and coefficients $A$ and $B$ are determined from

$$
\begin{cases}-A\left(r-r_{\min }-\epsilon\right)^{2}+B=0, & \text { at } r=r_{\min }  \tag{4.71}\\ -A\left(r-r_{\min }-\epsilon\right)^{2}+B=u_{0}, & \text { at } r_{\min }+\epsilon\end{cases}
$$

Example of such initial condition is demonstrated on the Figure 4.3 for $r_{\min }=1, \epsilon=0.1$, and $u_{0}=1$. Defined above function $f(r)$ is a parabola on the interval $r_{\min } \leq r \leq r_{\min }+\epsilon$, which then smoothly turns into a constant at radius $r_{\text {min }}+\epsilon$. Let us now assume that


Figure 4.3: Example of the initial condition (4.69), (4.70) with $r_{\min }=1, \epsilon=0.1$, and $u_{0}=1$.
viscosity is a linear function of radius, i.e. $\psi=1$. Applying method of separation of variables $u=X(r) T(t)$ we obtain

$$
\left\{\begin{array}{l}
\frac{d}{d r}\left(r^{1 / 2} \frac{d X}{d r}\right)+\lambda^{2} \rho(r) X=0  \tag{4.73}\\
T(t)=A e^{-\lambda^{2} t}
\end{array}\right.
$$

where $\rho(r)=\beta r^{-1 / 2}, \beta=R_{0}^{2} / 3 \nu_{0} r_{\text {out }}^{\psi}$. Radial equation

$$
\begin{equation*}
r X^{\prime \prime}+\frac{1}{2} X^{\prime}+\lambda^{2} \beta X=0 \tag{4.75}
\end{equation*}
$$

is a type of Bessel equation and has solution

$$
\begin{equation*}
X(r)=r^{1 / 4}\left[C_{1} J_{1 / 2}(2 \lambda \sqrt{\beta r})+C_{2} Y_{1 / 2}(2 \lambda \sqrt{\beta r)}] .\right. \tag{4.76}
\end{equation*}
$$

Asymptotic forms of Bessel functions of order $1 / 2$ have exact representation in trigonometric functions (this is the reason for choosing viscosity index $\psi=1$ )

$$
\left\{\begin{array}{l}
J_{1 / 2}(x)=\sqrt{\frac{2}{\pi x}} \cos (x-\pi / 2),  \tag{4.77}\\
Y_{1 / 2}(x)=\sqrt{\frac{2}{\pi x}} \sin (x-\pi / 2)
\end{array}\right.
$$

Thus, using (4.77) and (4.78), the solution of the radial solution becomes

$$
\begin{equation*}
X(r)=(\pi \lambda)^{-1 / 2} \beta^{-1 / 4}\left[C_{1} \sin (2 \lambda \sqrt{\beta r})-C_{2} \cos (2 \lambda \sqrt{\beta r})\right] . \tag{4.79}
\end{equation*}
$$

Dropping a constant factor (which is canceled out in the final solution anyway), and introducing $\omega=2 \lambda \sqrt{\beta}, \xi=\sqrt{r}$, expression for $X(r)$ can be reduced to

$$
\begin{equation*}
X(r) \propto C_{1} \sin (\omega \xi)-C_{2} \cos (\omega \xi)=C_{0} \sin \left(\omega \xi+\phi_{0}\right)=C_{0} \sin \left[\omega\left(\xi-\xi_{0}\right)\right] \tag{4.80}
\end{equation*}
$$

where $C_{0}=\sqrt{C_{1}^{2}+C_{2}^{2}}, \operatorname{tg}\left(\phi_{0}\right)=-C_{2} / C_{1}$. The boundary conditions (4.66), (4.67) give us equations for $X(r)$ and $\lambda$

$$
\begin{cases}\sin \left[\omega\left(\xi_{\min }-\xi_{0}\right)\right]=0, & \xi_{0}=\xi_{\min },  \tag{4.81}\\ \cos \left[\omega\left(\xi_{\max }-\xi_{0}\right)\right]=0, & \omega_{k}\left(\xi_{\max }-\xi_{0}\right)=\pi / 2+k \pi, \quad k=0,1,2 \cdots\end{cases}
$$

Switching back from $\omega$ and $\xi$ to our regular variables, we find the spatial eigenfunctions

$$
\begin{equation*}
X_{k}(r) \propto \sin \left[2 \lambda_{k} \sqrt{\beta}\left(\sqrt{r}-\sqrt{r_{\min }}\right)\right], \tag{4.83}
\end{equation*}
$$

where the eigenvalues are given by

$$
\begin{equation*}
\lambda_{k}=\frac{(k / 2-1 / 4) \pi \beta^{-1 / 2}}{\sqrt{r_{\max }}-\sqrt{r_{\min }}}, \quad k=1,2, \cdots \tag{4.84}
\end{equation*}
$$

The full solution of the initial-value problem (4.65)-(4.68) is then written as a series

$$
\begin{equation*}
u(r, t)=\sum_{k=1}^{\infty} X_{k} A_{k} e^{-\lambda_{k}^{2} t}, \quad k=1,2, \cdots \tag{4.85}
\end{equation*}
$$

Consider now time instant $t=0$ :

$$
\begin{equation*}
u(r, 0)=\sum_{k=1}^{\infty} X_{k} A_{k}, \quad k=1,2, \cdots \tag{4.86}
\end{equation*}
$$

It is easy to show that the differential operator in equation (4.73) is self-adjoint, and hence its eigenfunctions $\left\{X_{k}\right\}$ form a complete orthonormal basis, such that (4.86) represents an eigenfunction expansion of $u(r, 0)$ in $\left\{X_{k}\right\}$, with the Fourier coefficients

$$
\begin{equation*}
A_{k}=\int_{r_{\min }}^{r_{\max }} u(r, 0) \frac{X_{k}(r)}{\left\|X_{k}\right\|^{2}} \rho(r) d r \tag{4.87}
\end{equation*}
$$

where the eigenfunction's norm is

$$
\begin{equation*}
\left\|X_{k}\right\|^{2}=\int_{r_{\min }}^{r_{\max }} X_{k}(r)^{2} \rho(r) d r . \tag{4.88}
\end{equation*}
$$

Therefore, we obtain

$$
\begin{equation*}
u(r, t)=\sum_{k=1}^{\infty} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} X_{k} e^{-\lambda_{k}^{2} t}, \quad k=1,2, \cdots, \tag{4.89}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{k}=\int_{r_{\min }}^{r_{\max }} u(r, 0) X_{k}(r) \rho(r) d r \tag{4.90}
\end{equation*}
$$

For the purpose of calculation of expansion coefficients $c_{k}$ one can always set $\epsilon \rightarrow 0$ in (4.69), (4.70), approximating function $u(r, 0)$ in the integral (4.90) with $u_{0}$. This is a valid approximation, because in the limit of infinitely small $\epsilon$ contribution of the region $r_{\text {min }} \leq r \leq r_{\text {min }}+\epsilon$ to the integral (4.90) will be infinitely small. This provides for a simple calculation of $c_{k}$ :

$$
\begin{equation*}
c_{k}=u_{0} \int_{r_{\min }}^{r_{\max }} X_{k}(r) \rho(r) d r . \tag{4.91}
\end{equation*}
$$

Using formulas (4.83), (4.88), and (4.91) we find

$$
\begin{equation*}
\left\|X_{k}\right\|^{2}=\beta \int_{r_{\min }}^{r_{\max }} \sin ^{2}\left[2 \lambda_{k} \sqrt{\beta}\left(\sqrt{r}-\sqrt{r_{\min }}\right)\right] r^{-1 / 2} d r . \tag{4.92}
\end{equation*}
$$

Likewise

$$
\begin{equation*}
c_{k}=\beta u_{0} \int_{r_{\min }}^{r_{\max }} \sin \left[2 \lambda_{k} \sqrt{\beta}\left(\sqrt{r}-\sqrt{r_{\min }}\right)\right] r^{-1 / 2} d r . \tag{4.93}
\end{equation*}
$$

A constant factor $\beta$ is divided out in the final sum, so it can be neglected. Thus, after taking definite integrals, we obtain

$$
\begin{align*}
\left\|X_{k}\right\|^{2} & \propto r^{1 / 2}-\left.\frac{1}{4 \lambda_{k} \sqrt{\beta}} \sin \left[4 \lambda_{k} \sqrt{\beta}\left(\sqrt{r}-\sqrt{r_{\min }}\right)\right]\right|_{r_{\min }} ^{r_{\max }}  \tag{4.94}\\
c_{k} & \propto-\left.\frac{1}{\lambda_{k} \sqrt{\beta}} \cos \left[2 \lambda_{k} \sqrt{\beta}\left(\sqrt{r_{\min }}-\sqrt{r}\right)\right]\right|_{r_{\min }} ^{r_{\max }} \tag{4.95}
\end{align*}
$$

Finally, the solution of the initial-value problem (4.65)-(4.68) is given by formulas (4.83), (4.84), (4.89), (4.94), (4.95). For this "validation run" we assume a $10 M_{\odot}$ black hole, where characteristic viscosity in the disk is $\nu_{0}=\lambda_{t} v_{t} \approx 0.1 r_{\text {out }} R_{0} v_{\mathrm{t}}$, with $v_{\mathrm{t}} \approx 4 \times 10^{6} \mathrm{~cm} / \mathrm{s}$. A solution for region $1 \leq r \leq 2$ with the initial condition $u_{0}=10$ is shown on Figure 4.4 (top panel). Color depth represents the value of $u(r, t)$ at each point $\left(r_{i}, t_{j}\right)$. Calculation was done on a $500 \times 2000$ space-time grid with a time step $\triangle t=1 \times 10^{-5} \mathrm{~s}$. It is clear that presented solution satisfies initial and boundary conditions and that uniform distribution $u(t=0)$ diffuses out to zero as time goes to infinity. In order to compare the results of analytical and numerical calculations, I used the scatter plot visual representation, on which a single dot is drawn for a pair of solutions $u^{\mathrm{a}}, u^{\mathrm{n}}$ at node $\left(r_{i}, t_{j}\right)$. X/Y axes scales correspond to the amplitude $u^{\mathrm{a} / \mathrm{n}}$ computed by analytical/numerical methods. This way of comparing the results gives a clear and comprehensive picture on how the two solutions


Figure 4.4: Analytical solution of the initial-value problem (4.65)-(4.68) (top panel), Numerical solution plotted against analytical as a scatter plot (bottom panel).
agree (or disagree) with each other. Drawing just the relative error $\left(u^{\mathrm{a}}-u^{\mathrm{n}}\right) / u^{\mathrm{a}}$ can often lead to false perception, when both quantities are much less than unity. If the numerical result perfectly reproduces the analytical one, the scatter plot would look like an infinitely narrow straight line with the slope of 1 . The width of the line (width of the transverse "profile" of the scatter) indicates how well the two solutions agree. The scatter plot shown on the bottom panel of Figure 4.4 demonstrates a remarkable agreement between our solutions, as all points are grouped in a very tight band, with no a single outlier, resembling a straight line with the slope equal 1 . This result was anticipated for a highly-accurate Crank-Nicolson scheme with the small time step used. This test, therefore, establishes the validity of the numerical scheme, when solving the initial-boundary-value problem without the source term. In this study we intend to solve a diffusion equation with the perturbation source varying in both space and time, therefore such configuration should also be tested. The unforced damped harmonic oscillator with angular frequency, varying as a function of radius as Keplerian angular velocity, is a good model for distributed perturbations in the accretion disk:

$$
\begin{equation*}
f_{\mathrm{dr}}(R, t)=A(R) \cos \left[\omega_{\mathrm{K}}(R) t \sqrt{1-\frac{1}{4 Q^{2}}}+\varphi\right] e^{-\frac{\Gamma(R)}{2} t} \tag{4.96}
\end{equation*}
$$

where $\omega_{\mathrm{K}}$ is given by (2.46) [via $\omega(r)=\omega_{0} \theta(r)$ ], damping coefficient $\Gamma=\omega_{\mathrm{K}} / Q$, and Q is the coherence factor. Solution of the diffusion problem with the space and time dependent source term can be obtained as a time integral

$$
\begin{equation*}
u(r, t)=\int_{0}^{t} u^{\dagger}\left(r, t-t^{\prime}\right) d t^{\prime} \tag{4.97}
\end{equation*}
$$

where $u^{\dagger}(r, t)$ is the solution of the homogeneous equation with the initial condition $u^{\dagger}(r, 0)=$ $S(r, t)$. It is not hard to see that solving the homogeneous equation will inevitably result in a non-trivial integral, when calculating Fourier coefficients in the expansion (4.89) (due to non-trivial initial condition at each time instant). This makes it difficult to write down
the exact analytic formula, which can be used to verify the numerical result. Our goal is to check that numerical method solves the non-homogeneous equation. For the source function, which depends only on time, the solution of the non-homogeneous equation is considerably more manageable. Setting $\omega_{\mathrm{K}}=\omega=$ const, $A=$ const throughout $r_{\min } \leq r \leq r_{\max }$, allows us to write $u(r, t)$ as a convolution of the time-dependent source $S(t)=A \cos (\omega t) e^{-(\Gamma / 2) t}$ with the (already known) solution of the homogeneous problem (4.65)-(4.68)

$$
\begin{equation*}
u(r, t)=\int_{0}^{t} u^{*}\left(r, t-t^{\prime}\right) S\left(t^{\prime}\right) d t^{\prime} \tag{4.98}
\end{equation*}
$$

where $u^{*}(r, t)$ is given by (4.89). Integral (4.98) is taken in elementary functions, such that

$$
\begin{equation*}
u(r, t)=\left.A \sum_{k=1}^{\infty} \frac{c_{k}}{\left\|X_{k}\right\|^{2}} X_{k} e^{-\lambda_{k}^{2} \tau-\Gamma / 2(t-\tau)}\left\{\frac{\left(\Gamma / 2-\lambda_{k}^{2}\right) \cos [\omega(t-\tau)]-\omega \sin [\omega(t-\tau)]}{\lambda_{k}^{4}-\Gamma \lambda_{k}^{2}+\Gamma^{2} / 4+\omega^{2}}\right\}\right|_{0} ^{t} \tag{4.99}
\end{equation*}
$$

where we changed the variable of integration to $\tau=t-t^{\prime}$. The top panel of Figure 4.5 displays the solution (4.99) of the boundary-value problem with the time-varying source term

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}=3 \nu_{0} R_{0}^{-2} r_{\text {out }}^{-\psi} r^{1 / 2} \frac{\partial}{\partial r}\left[r^{\psi-1 / 2} \frac{\partial u}{\partial r}\right]+A \cos (\omega t) e^{-\frac{\Gamma}{2} t}, \quad r_{\min } \leq r \leq r_{\max },  \tag{4.100}\\
u\left(r_{\text {min }}\right)=0 \\
\frac{\partial u}{\partial r}\left(r_{\max }\right)=0
\end{array}\right.
$$

as a color-coded plot on the same grid, with same space/time steps, as were used for the initial-value problem. Angular frequency was chosen to be $\omega=1370 \mathrm{rad} / \mathrm{s}$, as the highest Keplerian frequency $\omega_{\mathrm{K}}^{\max }$ at $r_{\text {min }}=1$. With the time step $\Delta t=1 \times 10^{-5} \mathrm{~s}$, one period of such oscillations is sampled $\sim 460$ times, which is more than enough for smooth rendering.


Figure 4.5: Analytical solution of the boundary-value problem with the oscillating source (top panel), Numerical solution plotted against analytical as a scatter plot (bottom panel).

Comparison against the numerical result, in form of the scatter plot, is shown on the bottom panel of Figure 4.5. We notice the same exceptional consistency between the two solutions as we observed for the homogeneous problem. This result completes validation of the numerical method.

### 4.4 Solution of the diffusion propagation problem in transition layer

Diffusion equation for accretion rate perturbations $u=\triangle \dot{M}$ in the non-Keplerian transition layer is given by

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\nu_{0} R_{0}^{-2} r_{\mathrm{adj}}^{-\psi} r^{\gamma-1} \frac{\partial}{\partial r}\left[\left(\frac{\gamma}{2-\gamma} r^{\psi+1-\gamma}+\frac{2}{2-\gamma}\left(D_{1}^{-1}-1\right) r^{\psi-1}\right) \frac{\partial u}{\partial r}\right]+f_{\mathrm{dr}}(r, t) . \tag{4.103}
\end{equation*}
$$

In order to proceed with the numerical solution we have to assign numerical values to its coefficients. All the quantities in (4.103), besides the value of kinematic viscosity $\nu_{0}$ at radius $r_{\text {adj }}$, are defined by the two parameters of the accreting compact object: its mass and its rotational frequency. Estimate for viscosity can be obtained from the following reasoning. The standard theory of accretion disks by Shakura \& Sunyaev, 1973, [47] states that viscosity is defined by characteristic turbulent size and turbulent velocity

$$
\begin{equation*}
\nu \sim \lambda_{\mathrm{t}} v_{\mathrm{t}} \tag{4.104}
\end{equation*}
$$

where $\lambda_{\mathrm{t}}$ can not exceed the characteristic size of the configuration, $v_{\mathrm{t}}$ can not be greater than speed of sound. For the transition layer it is typically assumed that its height is comparable to its radial size [55]. This gives us an estimate for the size of the configuration: $\delta_{\mathrm{TL}}=R_{\mathrm{adj}}-R_{\mathrm{max}}$. Speed of sound in the region is determined by the photon temperature

$$
\begin{equation*}
v_{\mathrm{S}} \approx 0.001 c \sqrt{\frac{k_{\mathrm{B}} T}{0.938}} \tag{4.105}
\end{equation*}
$$

where $k_{\mathrm{B}} T$ is in units of keV . For this modeling we will suppose that photon energies vary approximately from $\sim 5 \mathrm{keV}$ in a high-soft state to $\sim 50 \mathrm{keV}$ in a low-hard state (a typical range observed in spectral transitions of X-ray binaries) and that $v_{\mathrm{t}} \approx 0.2 v_{\mathrm{S}}$, and $\lambda_{\mathrm{t}} \approx 0.2 \delta_{\mathrm{TL}}$. We impose the same boundary conditions, as were used in the analytical solution, i.e. $\partial u / \partial r=0$, at $r=r_{\max }$, and $u=0$ at $r=r_{\text {adj. Steps }} \Delta t$ and $\Delta r$ in the finite difference scheme (4.10) were determined from the conditions that the fastest oscillation period in the system (the inverse of the Keplerian frequency at the inner boundary) does not vary significantly over one step in time, as well as, the solution, for a given time step, does not change significantly over one step in radius. Specifically, the time step $\Delta t=10^{-4} \mathrm{~s}$ and 200 spatial intervals we used to solve the diffusion problem for the transition layer. Figure 4.6 displays the solution of the diffusion equation (4.103) for viscosity index $\psi=0$, parameter $\gamma=3$, and the quality factor $Q=4$ in the driving source. The top panel corresponds to the "one-time" perturbation source (4.57). The bottom panel corresponds to the continuous source term, determined by equations (4.58), (4.63), (4.64). The horizontal scale represents the dimensionless radius $r=R / R_{0}$, where $R_{0}=3 R_{\mathrm{S}}$, the vertical scale is time in seconds. Color depth measures the amplitude of the solution. The merits of the "one-time" source are now clearly seen. Both oscillating and decaying behavior of the solution is obvious. We notice that the outer parts of the region indeed oscillate at a lower frequency than do the inner parts, since the perturbation source is distributed according to $\omega_{\mathrm{K}} \propto r^{-3 / 2}$. Likewise, the oscillations at the smaller radii are also decaying at a faster rate. It is even possible to envision the shape of the power spectrum of the integrated solution. There will be less or more defined periodic component, associated with the solution oscillations. For the signal integrated over radius, the centroid of the quasi-periodic variability peak will correspond to the radius-averaged frequency. Power spectrum should also contain a distinctive aperiodic component, reflecting the exponential decay of the solution with time. From our previous analysis, we anticipate its power spectrum to be given by a zero-centered Lorentzian with the "break" defined by the diffusion timescale, which is determined by the characteristic time $\nu_{0} R_{0}^{-2}$ and the diffusion coefficient, which is a function of $\gamma$ and $\psi$. None of these


Figure 4.6: Solution $\triangle \dot{M}(r, t)$ for a "one-time" perturbation source (4.57) (top panel), and for a continuous stochastic source defined as a numerical solution of equations (4.58), (4.59).
conclusions could be drawn from the solution of the equation with the stochastic perturbation source. In fact, it may seem like it exhibits completely random variation in both space and time, with the only little clue in that the typical size of the "blobs" on the plot tend to be smaller for the smaller radii, where the oscillation frequency of the driving source is higher. This is where Fourier power spectral analysis comes in very helpful.

### 4.4.1 Power spectrum from transition layer

From (2.24), (2.25) we obtain that the total X-ray signal, emerging from the transition layer region, is given by the integral

$$
\begin{equation*}
\Delta L(t) \propto \int_{r_{\max }}^{r_{a d j}} \Delta \dot{M}(r, t)\left[r^{-2}-r^{-5 / 2}\right] d r . \tag{4.106}
\end{equation*}
$$

Figures 4.7 and 4.8 show examples of the integral solution, for the" one-time" and the stochastic perturbations sources in the diffusion equation. These can be thought of as the time profiles of two the dimensional solutions, shown on Figure 4.6. Power spectrum of the integrated signal can be calculated, using standard Fast Fourier Transform libraries. I used the popular FFTW C library. The FFT computes the discrete Fourier transform (DFT) of a uniformly-spaced time series $X_{k}$ of length $N$

$$
\begin{equation*}
Y_{k}=\sum_{j=0}^{N-1} X_{j} e^{-i 2 \pi j k / N} \tag{4.107}
\end{equation*}
$$

The power spectrum for the $k-t h$ harmonic is defined as $Y_{k}^{*} Y_{k}$, where the asterisk symbol stands for complex conjugation. The Nyquist frequency is determined by the sampling time interval: for $\Delta t=10^{-4} \mathrm{~s}, \nu_{\mathrm{NYQ}}=1 /(2 \triangle t)=5000 \mathrm{~Hz}$. We do not anticipate to see any features in the power spectrum far above the highest Keplerian frequency. For the largest value of Reynolds number, $\gamma=80$, the inner radius of the transition layer $r_{\max } \approx 1.046$ corresponds to the Keplerian frequency $\left.\nu_{\mathrm{K}}\right|_{r=r_{\max }} \approx 206 \mathrm{~Hz}$. Thus, for plotting, we limit


Figure 4.7: Integrated over transition layer solution of the diffusion equation with "onetime" perturbation source.


Figure 4.8: Integrated over transition layer solution of the diffusion equation with continuous perturbation source.
the upper boundary of the spectrum at 500 Hz . The solution time series was calculated on 200,000 -point time grid, so the low frequency limit in the power spectrum is given by $\nu_{\min }=1 /(N \triangle t)=0.05 \mathrm{~Hz}$. This also determines the frequency resolution of the spectrum $\triangle f=0.05 \mathrm{~Hz}$. The diffusion propagation problem was solved for the three values of Reynolds number: $\gamma=3,10$, and 80. According to [60], [61] the range $3 \lesssim \gamma \lesssim 80$, inferred from observations, approximately corresponds to evolution of X-ray source from low-hard to high-soft spectral state. Radial independence of viscosity was assumed within the transition layer $(\psi=0)$. In fact, solution with viscosity indexes $\psi=1,2$, that imply linear or quadratic dependence of viscosity on radius, resulted in almost identical solutions to the one with $\psi=0$, showing that for the tenuous transition region uniform viscosity distribution may be a good approximation. The power spectra, calculated for the "onetime" and continuous perturbation sources in the diffusion equation, are shown on Figures 4.9 and 4.10 correspondingly. The power spectra are presented in a log-log scale and are normalized to the maximum value. One can clearly identify the two main features of the spectra: the aperiodic "white-red" noise component, given by the sum of zero-centered Lorentzians, with the characteristic "break", corresponding to the diffusion timescale, and the quasi-periodic peak in the hecto-Hertz range, corresponding to the mean frequency of driving perturbations. A very strong dependence of the power spectrum normalization on Reynolds number is obvious. This behavior is anticipated and was predicted by the analytical solution. It is worth noting that both features, the "break" and the QPO peak are shifted into higher frequencies as $\gamma$ increases. Power spectrum decay, accompanied by a high-frequency shift of its features, as an X-ray source makes a transition from low-hard to high-soft state, is a well-established observational result. That gives us some confidence in that our simple model not only does not contradict the observational facts, but, in fact, is capable of explaining and predicting them, to some extension. The striking similarity of the power spectra, calculated using the realistic continuous source term in the diffusion equation, to the real power spectra, calculated from the photon lightcurves, is also remarkable. It has to be emphasized, however, that presented spectra were created for the specific values of


Figure 4.9: Power spectrum of the integrated signal for different values of Reynolds number. "One-time" perturbation source.


Figure 4.10: Power spectrum of the integrated signal for different values of Reynolds number. Continuous perturbation source.
parameters, and can vary wildly, depending on the way viscosity $\nu_{0}$ is defined, coherence of the QPO peak, amplitude of the driving force, relation between the photon's temperature and Reynolds number of the flow, distribution of the driving perturbations in the disk, etc. Physical processes, not included in the model, for instance, hot plasma outflow in the centrifugal barrier (the outer boundary of the transition layer), particularly important in the high-soft state, can significantly diminish the presence of the QPO feature, to the degree that it may not be detectable at all [55].

### 4.4.2 Fractional variability amplitude of the lightcurve as a function of Reynolds number

Variability of the photon lightcurve can be characterized by the quantity, known as the fractional root-mean square (RMS) amplitude. In the analysis of real-life lightcurves the intrinsic variability amplitude relative to the mean photon count rate and in excess of measurement noise is defined as follows

$$
\begin{equation*}
F_{\mathrm{var}}=\sqrt{\frac{S^{2}-\left\langle\sigma_{\mathrm{er}}^{2}\right\rangle}{\langle X\rangle^{2}}} \tag{4.108}
\end{equation*}
$$

Here $S^{2}$ is the total variance of the light curve, $\left\langle\sigma_{\text {err }}^{2}\right\rangle$ is the mean square error accounting for the noise, instrumental and statistical errors, $\langle X\rangle$ is the mean count rate. In our case, where we do not model individual photons and no instrument errors exist, $\left\langle\sigma_{\text {err }}^{2}\right\rangle=0$. If we assume that the diffusion equation (4.103) is written for the quantity $\triangle \dot{M} / \dot{M}$, where $\dot{M}$ is a steady-state accretion rate (analogous to mean count rate), then the expression (4.108) reduces to

$$
\begin{equation*}
F_{\mathrm{var}}=\sqrt{\frac{1}{N} \sum_{n=1}^{N}\left[u\left(t_{n}\right)\right]^{2}} \tag{4.109}
\end{equation*}
$$

where $u\left(t_{n}\right)$ is the solution time series of length $N$. Figure 4.11 shows the RMS amplitude (4.109) computed for the space-integrated solution of the diffusion equation (4.103) with the continuous perturbation source, for Reynolds number $\gamma$ varying from 3 to 30. The RMS amplitude on the plot should not be considered as a meaningful quantity by itself, since it is directly proportional to the amplitude of the driving oscillations and depends on the value of $\dot{M}$, which we do not model. The general trend, however, has the meaning, as well as the relative change in the RMS with varying $\gamma$. It is easy to see from the plot that as $\gamma$ increases by a factor of two (from 3 to $\sim 6$ ) the RMS amplitude drops by a factor of 10. For $\gamma=30$ the RMS amplitude is $\sim 320$ times smaller than for $\gamma=3$. Our modeling predicts a very rapid decay of variability amplitude with growing Reynolds number. It can be shown that this is a direct consequence of the way the diffusion coefficient in equation (4.103) and size of the transition region depend on Reynolds number. Qualitatively this can be explained using the following reasoning: $\gamma$-dependent part of the diffusion operator in equation (4.103) can be expanded as

$$
\begin{equation*}
r^{\gamma-1} \frac{\partial}{\partial r}\left[\left(\mu r^{1-\gamma}+\xi r^{-1}\right) \frac{\partial u}{\partial r}\right]=\left(\mu+\xi r^{\gamma-2}\right) u^{\prime \prime}+\left[\mu(1-\gamma) r^{-1}-\xi r^{\gamma-3}\right] u^{\prime} . \tag{4.110}
\end{equation*}
$$

Evaluating coefficients and taking characteristic radius to be the average radius in the region, yields $\mu=-3, \xi \approx 2.11,\langle r\rangle \approx 2.13$ for $\gamma=3$, and $\mu=-1.07, \xi \approx 0.082,\langle r\rangle \approx 1.12$, for $\gamma=30$. I.e. the right-hand side of the diffusion equation proportional (neglecting the constant factor in front of the diffusion coefficient and dropping the source term) to

$$
\text { RHS(Eqn.(4.103)) } \propto \begin{cases}1.5 u^{\prime \prime}+0.7 u^{\prime}, & \text { for } \gamma=3,  \tag{4.111}\\ 0.9 u^{\prime \prime}+26 u^{\prime}, & \text { for } \gamma=30\end{cases}
$$

Notice that the multiplying factor at $u^{\prime}$ increased by a factor $\sim 37$, while the coefficient at $u^{\prime \prime}$ has not changed substantially, as Reynolds number changed from $\gamma=3$ to $\gamma=30$. It is
not hard to show that the large factor at $u^{\prime}$ will result in a very weak dependence of solution on radius. Keeping in mind that at the same time the size of the transition layer shrank by a factor 27 , and the boundary condition $u=0$ at $r=r_{\mathrm{adj}}$, it becomes apparent that combination of the above conditions will result in significant suppression of the amplitude of the solution in the region. It is interesting to notice that for real X-ray sources the RMS amplitude decays from roughly $\sim 40 \%$ to a few percent, as the source makes a transition from hard state to soft state. Clearly, our calculation produces the correct trend, though the rate of amplitude decay seems to be exponential.


Figure 4.11: RMS amplitude as a function of Reynolds number.

### 4.4.3 Magnetoacoustic oscillations of the transition layer

Titarchuk, Bradshaw, and Wood, 2001, [63] developed a general formalism of magnetoacoustic (magnetosonic) radial oscillations in the transition layer. In their model the magnetoacoustic frequency is derived as the eigenfrequency of the boundary-value problem, which results from an MHD treatment of the interaction of the transition layer with the magnetic field. The principal idea behind their model is following. Introducing small perturbations to the parameters (matter density, velocity, and magnetic field) of the transition layer

$$
\begin{equation*}
\rho=\rho_{0}+\rho_{1}, \quad \boldsymbol{V}=\boldsymbol{V}_{0}+\boldsymbol{V}_{1}, \quad \boldsymbol{B}=\boldsymbol{B}_{0}+\boldsymbol{B}_{1}, \tag{4.113}
\end{equation*}
$$

and assuming that $\boldsymbol{B}_{0}=B_{0} \boldsymbol{e}_{z}$ and $\boldsymbol{V}_{1}=V_{1} \boldsymbol{e}_{r}$, they combined the continuity equation, the equation of motion, and the ideal gas equation to obtain the fundamental equation for magnetoacoustic oscillations in the transition layer. Its radial component reads

$$
\begin{equation*}
\frac{\partial^{2} V_{1}}{\partial t^{2}}-v_{\mathrm{S}}^{2} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial r V_{1}}{\partial r}\right)-V_{\mathrm{A}} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial r V_{\mathrm{A}} V_{1}}{\partial r}\right)=0 . \tag{4.114}
\end{equation*}
$$

Combined with the boundary conditions (both stiff and free boundary conditions on $V_{1}$ were considered) this equation allows to solve for the eigenfrequencies of the magnetoacoustic oscillations in the transition layer. The solution yields a resulting velocity as a mixture of speed of sound and the Alfven velocity, becoming either one at the appropriate limit (pure acoustic or pure magnetic cases). In the case of the weak magnetic field the magnetoacoustic oscillation frequency is given by

$$
\begin{equation*}
\nu_{\mathrm{S}}=\frac{v_{\mathrm{S}}}{2 \pi L}, \tag{4.115}
\end{equation*}
$$

where speed of sound $v_{\mathrm{S}}$ is determined by plasma temperature $T$ and $L=R_{\mathrm{adj}}-R_{0}$ is the radial size of the transition layer. We recall that for a given Reynolds number $\gamma$ the radial size $L$ of the transition region is determined by (2.50). Using our phenomological relation (see section "Solution of the diffusion propagation problem in transition layer")
between temperature and spectral state (which is determined by the Reynolds number) one can obtain the temperature of plasma. Therefore, using $L$ and $T$, the magnetoacoustic oscillation frequency $\nu_{\mathrm{S}}$ can be found immediately. Figure 4.12 shows the power spectra for $\gamma=3,10$, and 80 with the transition layer oscillation frequencies (low-frequency quasiperiodic oscillation) added to the continuum. The normalization and coherence parameters of the peaks were chosen to reproduce the ones observed in the data. Even though, the lowfrequency magnetoacoustic QPO feature is not a direct outcome of our diffusion model, its location on a frequency scale in the power spectrum, as a function of spectral state, certainly is. One can clearly see that the position of the low-frequency QPO peak is drifting towards high frequency, as do other power-spectral components (high frequency QPO and "break" in the continuum), as $\gamma$ increases from 3 to 80 and the transition zone becomes more compact.


Figure 4.12: Power spectrum of the integrated solution of the diffusion equation (4.103) with the "one-time" perturbation source (4.57), with added magnetoacoustic quasi-periodic oscillation low-frequency peak (4.115) for $10 M_{\odot}$ black hole with $f_{0}=30 \mathrm{~Hz}$.

### 4.4.4 Observational data

In order to draw a connection between the model predictions and the observational data, we chose a few observations of the microquasar XTE J1550-564 by the Rossi X-ray Timing Explorer (RXTE). XTE J1550-564 is a binary system, involving $\sim 10 M_{\odot}$ black hole, with the spin parameter $a_{*} \approx 0.5$, powered by the accretion disk, resulting from Roche Lobe overflow of the massive evolved companion star. In 1998 XTE J1550-564 underwent a few powerful X-ray flares, in one of which its X-ray flux exceeded the flux from the brightest currently known X-ray source, Crab Nebula, by $\sim 1.5$ times. During these flares the source changed its spectral states several times. We selected the four representative observations, namely 30188-06-09-00, 30188-06-01-03, 30188-06-01-00, 3019101-03-00 for comparison. Both photon energy spectra and power density spectra were produced. Energy spectrum in $3-50$ keV energy range was fitted by the combination of the Bulk Motion Comptonization model (BMC), from the HEASOFT spectral fitting package XSPEC, to fit the spectrum continuum, and the Gaussian model to account for the iron $K_{\alpha}$ fluorescent emission line. The photon spectra are presented on Figure 4.13. One can clearly see the source's evolution from low-hard to high-soft state with the increasing strength of the soft thermal, black-body like component and decaying hard power-law component. BMC photon index $\Gamma$ changes through values $1.6,1.8,2.4,2.7$ during the source transition. Figure 4.14 demonstrates the simultaneous evolution of the power density spectrum. We used $\mathrm{RMS}^{2} / \mathrm{Hz}$ normalization for the power spectrum. One can see that the total normalization of the PDS continuum decays, during the source's transition to high-soft state, accompanied by the characteristic shift of the power-spectral features, namely the location of the "break" frequency and the position of the low-frequency quasi-periodic oscillation (QPO) peak. However, within the framework of our diffusion propagation model, we cannot provide a definitive conclusion on the (quantitative) magnitude of the power spectrum normalization decay, i.e. predict the amount by which the normalization will change from one state to another, because we operate with the absolute (arbitrary) units of $\triangle \dot{M}$ in the diffusion equation, unlike the observational data, where the relative flux variability is studied. We notice that the driving,
high-frequency (HF) QPO is not seen in the PDS. There is clearly some power present in the hecto-Hertz range of the power spectrum, which could be related to the distributed driving source, but large error bars hinder appearance of the features. We associate the prominent low-frequency QPO peak with the frequency of the magnetoacoustic oscillations in the transition layer. One can estimate characteristic parameters of the model for a particular state and overlay model solution with the data, to demonstrate that it produces the expected shape. This is shown on Figure 4.15 for the low-hard state. Using the location of the significant low-frequency magnetoacoustic QPO peak as the reference point, Shakura-Sunyaev parametrization of the viscosity, and phenomenological relation between plasma temperature and the effective Reynolds number we calculate the PDS continuum shape for the given compact object. Taking into account the fact that this is just the shape produced by the solution of the diffusion propagation equation, and not the full-fledge fitting model, one can conclude that the simulated continuum describes observational power spectrum continuum pretty good. As was noticed before, the strong high-frequency driving QPO predicted by the model, is not seen in the data, which is a rather common observational fact for the black hole X-ray sources. This can be attributed to several possibles reasons: the weak QPO feature itself, the matter outflow (wind) from the accreter, which smears out the narrow features into a broader frequency range, low photon flux, resulting in higher uncertainty and larger errors. It should be noted that there are a few observations of XTE J1550-564, in which the high-frequency, driving QPO feature is present with good significance. Despite the fact that our simple diffusion propagation formalism did not originally aim at direct comparison to real data, but was developed as a model description of the power spectrum shape, based on the first physical principles, it is capable of doing the correct predictions, at least in a qualitative way. In order to have a complete model one has to make a connection to the mass accretion rate $\dot{M}(R, t)$ in the accretion disk, when formulating the diffusion problem. This would require a substantially more involved treatment, which includes energy balance in the disk, accounts for gas pressure and the disk structure in radial direction.


Figure 4.13: Photon energy spectra of XTE J1550-564 during state transition.


Figure 4.14: Power density spectra of XTE J1550-564 during state transition.


Figure 4.15: Power spectrum of XTE J1550-564 overlayed with the model.

### 4.5 Simulation of broadband power spectrum from accreting X-ray sources

A few works [13], [44], [61], present power spectra of X-ray binary systems, in which variability timescales vary from milliseconds to a few years. Such data carries unique timing information about the emitting system as a whole. Analysis of mechanisms that cause such extremely broad range variability can help to understand how the different components in the accretion flow interact with each other. A broadband power spectrum of the neutron star Cyg X-2, produced by [13], clearly exhibits the two (presumably independent) whitered noise components, one with the characteristic timescale $\sim 1 \mathrm{~Hz}$, the other with the characteristic timescale $\sim 10^{-7} \mathrm{~Hz}$. Employing the formalism of the transition layer model of TLM98, one can argue that these two components may occur independently from the two weakly correlated X-ray emitting configurations: a compact non-Keplerian transition
region contained within a few $R_{\mathrm{S}}$, and, separated by a centrifugal barrier [57], [55], extended Keplerian accretion disk, which can have size of order of a few thousands $R_{\mathrm{S}}$. We attempt to model these two configurations, using our perturbation propagation model, to find out if it is possible to reproduce the observed two-component power spectrum, and understand why sometimes the two components are seen in the spectrum, and sometimes only one or the other. An obvious challenge one faces when trying to model numerically a power spectrum in a broad frequency range is the size and number of the time steps. For instance, if we were to sample the frequency range from 1 ms to 1 month with a single "sweep", the sampling time interval would have to be at least $\Delta t_{\mathrm{NYQ}}=5 \times 10^{-4} \mathrm{~ms}$ (or shorter), thus requiring $\geq 5 \times 10^{9}$ points. Generating and processing a time sequence of this length is prohibitively time consuming task, even on a modern computer, using the most swift fast Fourier transform (FFT) libraries. One way around this difficulty is to sample different parts of the spectrum with shortest time intervals used to produce high-frequency part of the spectrum) to longest (for the low-frequency part). The different pieces can than be merged together to yield a power spectrum that subtends the entire range. However, for our diffusion problem with the "one-time" perturbation source (4.57), where the rate of decay of perturbations is proportional to the oscillation frequency, only "slow" oscillations survive to the very long times, it may seem natural to use logarithmic time binning, where the signal $u(t)$ is spaced uniformly in logarithm of $t$. In this case $d(\ln t)=d t / t=$ const, i.e. the time bin $d t$ becomes broader with increasing time. Conversion from $t$ to $\ln t$ will result in tremendous saving in the number of points on a time grid. For example, to span the frequency interval $1 \times 10^{-8} \mathrm{~Hz}-1 \times 10^{3} \mathrm{~Hz}$ with 1000 nodes per decade we would have to generate just 11000 samples. That is 7 orders of magnitude difference with the number of points we would need if we used regular binning. Modeling of the accretion disk that extends to thousands of Schwarzschild radii may also be made more efficient if one converts to logarithm of the space coordinate.

### 4.5.1 Finite difference approximation for a diffusion equation on a uniform grid $\ln t$ and $\ln r$

Transition from the finite difference approximation (4.11), (4.12)-(4.14) of the diffusion equation (4.4), defined over a grid uniformly spaced in $r$ and $t$ to the finite difference defined over a grid uniformly spaced in $\ln r, \ln t$ requires only minor changes. If the substitution $\tau=\ln t$ is made in equation (4.4), one obtains

$$
\begin{equation*}
e^{-\tau} \frac{\partial u}{\partial \tau}=f(r) \frac{\partial}{\partial r}\left[D(r) \frac{\partial u}{\partial r}\right]+S(r, \tau) . \tag{4.116}
\end{equation*}
$$

The Crank-Nicolson scheme for equation (4.116) is

$$
\begin{align*}
& \frac{u_{i}^{k+1}-u_{i}^{k}}{\Delta \tau}=e^{\tau_{k}} \frac{f_{i}}{2 \triangle r^{2}}\left[D_{i+1 / 2}\left(u_{i+1}^{k}-u_{i}^{k}\right)-D_{i-1 / 2}\left(u_{i}^{k}-u_{i-1}^{k}\right)\right]+\frac{e^{\tau_{k}}}{2} S_{i}^{k}+  \tag{4.117}\\
& \quad e^{\tau_{k+1}} \frac{f_{i}}{2 \triangle r^{2}}\left[D_{i+1 / 2}\left(u_{i+1}^{k+1}-u_{i}^{k+1}\right)-D_{i-1 / 2}\left(u_{i}^{k+1}-u_{i-1}^{k+1}\right)\right]+\frac{e^{\tau_{k+1}}}{2} S_{i}^{k+1}
\end{align*}
$$

After regrouping terms, one obtains a tridiagonal system of equations suitable for solving numerically

$$
\begin{equation*}
a_{i} u_{i-1}^{k+1}+b_{i} u_{i}^{k+1}+c_{i} u_{i+1}^{k+1}=\tilde{a}_{i} u_{i-1}^{k}+\tilde{b}_{i} u_{i}^{k}+\tilde{c}_{i} u_{i+1}^{k}+\frac{\triangle \tau}{2}\left(e^{\tau_{k}} S_{i}^{k}+e^{\tau_{k+1}} S_{i}^{k+1}\right), \tag{4.118}
\end{equation*}
$$

where the coefficients on the left-hand side of equation (4.118) are

$$
\left\{\begin{array}{l}
a_{i}=-f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k+1}} D_{i-1 / 2}  \tag{4.119}\\
b_{i}=1+f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k+1}}\left(D_{i-1 / 2}+D_{i+1 / 2}\right) \\
c_{i}=-f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k+1}} D_{i+1 / 2}
\end{array}\right.
$$

the coefficients on the right-hand side are

$$
\left\{\begin{array}{l}
\tilde{a}_{i}=f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k}} D_{i-1 / 2}  \tag{4.122}\\
\tilde{b}_{i}=1-f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k}}\left(D_{i-1 / 2}+D_{i+1 / 2}\right) \\
\tilde{c}_{i}=f_{i} \frac{\triangle \tau}{2 \triangle r^{2}} e^{\tau_{k}} D_{i+1 / 2}
\end{array}\right.
$$

For a non-Keplerian transition layer the diffusion coefficient in (4.116) is determined by

$$
\begin{align*}
& f(r)=\nu_{0} R_{0}^{-2} r_{\mathrm{adj}}^{-\psi} r^{\gamma-1}  \tag{4.125}\\
& D(r)=\mu r^{\psi+1-\gamma}+\xi r^{\psi-1}  \tag{4.126}\\
& \mu=\gamma /(2-\gamma)  \tag{4.127}\\
& \xi=2 /(2-\gamma)\left(D_{1}^{-1}-1\right) \tag{4.128}
\end{align*}
$$

Equation (4.116) can be used to solve for accretion rate fluctuations in the compact configuration on an extra broad range of times, using logarithmic time step $d \tau=d(\ln t)=d t / t$. For an extended Keplerian disk the logarithmic change of variables should also be made for spatial coordinate $x=\ln r$

$$
\begin{equation*}
e^{-\tau} \frac{\partial u}{\partial \tau}=f(x) \frac{\partial}{\partial x}\left[D(x) \frac{\partial u}{\partial r}\right]+S(x, \tau) \tag{4.129}
\end{equation*}
$$

Taking into account that the angular velocity in a Keplerian disk is given by (2.47), we find that

$$
\begin{align*}
& f(x)=3 \nu_{0} R_{0}^{-2} e^{-\psi x_{\mathrm{out}}} e^{-x / 2}  \tag{4.130}\\
& D(x)=e^{(\psi-3 / 2) x} \tag{4.131}
\end{align*}
$$

where the above assumption for viscosity was made. The tridiagonal system for the extended configuration remains unchanged (4.118), as do the matrix elements (4.119) - (4.121) and the right-hand side coefficients $(4.122)-(4.124)$, with $\Delta r$ replaced by $\triangle x$. Of course, in both cases the source term has to be expressed in terms of new variables $x$ and $\tau$. An obvious question now is how to calculate the Fourier transform (and the power spectrum) of the logarithmically-spaced signal? The answer is the FFTLog algorithm.

### 4.5.2 (Fast) Fourier Transform of a logarithmically spaced periodic sequence

FFTLog algorithm was originally proposed in 1978 by [53], however its actual implementation [15], as a set of Fortran subroutines, was done in 1999. "FFTLog can be regarded as a natural analogue to the fast Fourier transform (FFT) of a linearly spaced periodic sequence, in the sense that, just as the normal FFT gives the exact (to machine precision) Fourier transform of a linearly spaced periodic sequence, so also FFTLog gives the exact Fourier or Hankel transform of arbitrary order, of a logarithmically spaced periodic sequence", (Hamilton, 2000) [15]. FFTLog belongs to the family of fast Hankel transform (FHT) algorithms. One of the advantages of FFTlog is that the Bessel function order $\nu$ in the Hankel transform may be any real number. This includes half-integer $\nu$, and hence Fourier sine, cosine and spherical Hankel transforms.

### 4.5.3 Power spectrum expressed via Fourier sine/cosine transforms

Our goal is to obtain the power density spectrum of the solution of the diffusion equation, which is logarithmically spaced in time. The FFTLog algorithm does not produce the Fourier transform of the form

$$
\begin{equation*}
\mathcal{F}_{f}(\omega)=\int_{-\infty}^{\infty} f(t) e^{-i \omega t} d t \tag{4.132}
\end{equation*}
$$

suitable for direct computation of power spectrum, but rather gives discrete approximations of the Fourier sine and cosine transforms (FST/FCT). Let us demonstrate that the power density spectrum, as defined by relation (B.8) (Appendix B), of a real-valued function

$$
f= \begin{cases}f(t) & \text { for } t \geq 0  \tag{4.133}\\ 0 & \text { for } t<0\end{cases}
$$

can be expressed in terms of FST and FCT. Substituting the definition of the continuous Fourier transform (4.132) of the function $f(t)$ into (B.8), we have

$$
\begin{equation*}
\left\|\mathcal{F}_{f}(\omega)\right\|^{2}=\frac{1}{2 \pi}\left\{\left[\int_{0}^{\infty} f(t) \cos (\omega t) d t\right]^{2}+\left[\int_{0}^{\infty} f(t) \sin (\omega t) d t\right]^{2}\right\} \tag{4.135}
\end{equation*}
$$

On the other hand, sum of the squares of the Fourier sine and cosine transforms of $f(t)$ is given by

$$
\begin{equation*}
\mathcal{F}_{\mathrm{S} f}^{2}(\omega)+\mathcal{F}_{\mathrm{C} f}^{2}(\omega)=\frac{2}{\pi}\left\{\left[\int_{0}^{\infty} f(t) \cos (\omega t) d t\right]^{2}+\left[\int_{0}^{\infty} f(t) \sin (\omega t) d t\right]^{2}\right\} \tag{4.136}
\end{equation*}
$$

Comparing (4.135) and (4.136), we have

$$
\begin{equation*}
\left\|\mathcal{F}_{f}(\omega)\right\|^{2}=\frac{1}{4}\left[\mathcal{F}_{\mathrm{S} f}^{2}(\omega)+\mathcal{F}_{\mathrm{C} f}^{2}(\omega)\right]^{2} \tag{4.137}
\end{equation*}
$$

Thus, the power spectrum of $f(t)$ is equal to the sum of the squares of the Fourier sine and cosine transforms times a constant factor.

Below we consider the cases of $\nu= \pm 1 / 2$, for which the Hankel transform is equivalent to the Fourier sine/cosine transform, as well as, outline the key principles of the FFTLog algorithm.

### 4.5.4 Fourier Sine/Cosine Transform as a special case of Hankel Transform

The continuous Hankel (Fourier-Bessel) transform of a function $a(t)$ with the power law bias $(\omega t)^{ \pm q}$ is defined as

$$
\begin{equation*}
\tilde{a}(\omega)=\int_{0}^{\infty} a(t)(\omega t)^{q} J_{\nu}(\omega t) \omega d t \tag{4.138}
\end{equation*}
$$

If the substitutions

$$
\left\{\begin{array}{l}
a(t)=A(t) t^{1 / 2-q},  \tag{4.139}\\
\tilde{a}(\omega)=\tilde{A}(\omega) \omega^{1 / 2+q},
\end{array}\right.
$$

are made, then (4.138) becomes

$$
\begin{equation*}
\tilde{A}(\omega)=\int_{0}^{\infty} \omega^{1 / 2} t^{1 / 2} A(t) J_{\nu}(\omega t) d t \tag{4.141}
\end{equation*}
$$

It is easy to see that the Fourier sine and cosine transforms $\mathcal{F}_{\mathrm{S}}(\omega), \mathcal{F}_{\mathrm{C}}(\omega)$ of the function $A(t)$ are special cases of the Hankel transform (4.141) with the Bessel function index $\nu= \pm 1 / 2$, since

$$
\begin{align*}
& J_{1 / 2}(x)=\sqrt{\frac{2}{\pi x}} \sin (x)  \tag{4.142}\\
& J_{-1 / 2}(x)=\sqrt{\frac{2}{\pi x}} \cos (x) . \tag{4.143}
\end{align*}
$$

Using identities (4.142), (4.143), we obtain

$$
\tilde{A}(\omega)= \begin{cases}\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} A(t) \sin (\omega t) d t \equiv \mathcal{F}_{\mathrm{S}}(\omega), & \text { for } \nu=1 / 2  \tag{4.144}\\ \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} A(t) \cos (\omega t) d t \equiv \mathcal{F}_{\mathrm{C}}(\omega), & \text { for } \nu=-1 / 2\end{cases}
$$

An efficient way of calculating the Discrete Hankel Transform is realized in the the Fortran library FFTLog (A. J. S. Hamilton, "Uncorrelated modes of the nonlinear power spectrum" 2000) [15]. One can refer to this publication for the detailed explanation of the algorithm.

### 4.5.5 Parseval's identity for the Discrete Hankel Transform

Bessel functions $J_{\nu}(\omega t)$ in the kernel of Hankel transform form an orthogonal basis with respect to the weighting factor $t$. Thus the Parseval's theorem must also hold. In the case of the continuous Hankel transform with no power law bias ( $q=0$ )

$$
\begin{align*}
& \tilde{a}(\omega)=\int_{0}^{\infty} a(t) J_{\nu}(\omega t) t d t,  \tag{4.146}\\
& a(t)=\int_{0}^{\infty} \tilde{a}(\omega) J_{\nu}(\omega t) \omega d \omega, \tag{4.147}
\end{align*}
$$

[obtained from the definition (4.138) by the substitutions $a(t) \rightarrow a(t) t, \tilde{a}(w) \rightarrow \tilde{a}(\omega) \omega$ ] the Parseval's identity can be derived, using the orthogonality relationship

$$
\begin{equation*}
\int_{0}^{\infty} J_{\nu}(\omega t) J_{\nu}\left(\omega^{\prime} t\right) t d t=\frac{1}{\omega} \delta\left(\omega-\omega^{\prime}\right) . \tag{4.148}
\end{equation*}
$$

Consider the integral of the product of the two inverse Hankel transforms

$$
\begin{align*}
& a(t)=\int_{0}^{\infty} \tilde{a}(\omega) J_{\nu}(\omega t) \omega d \omega  \tag{4.149}\\
& b(t)=\int_{0}^{\infty} \tilde{b}\left(\omega^{\prime}\right) J_{\nu}\left(\omega^{\prime} t\right) \omega^{\prime} d \omega^{\prime}, \tag{4.150}
\end{align*}
$$

over $t$ with the weight $t$

$$
\begin{equation*}
\int_{0}^{\infty} a(t) b(t) t d t=\int_{0}^{\infty}\left[\int_{0}^{\infty} \tilde{a}(\omega) J_{\nu}(\omega t) \omega d \omega \int_{0}^{\infty} \tilde{b}\left(\omega^{\prime}\right) J_{\nu}\left(\omega^{\prime} t\right) \omega^{\prime} d \omega^{\prime}\right] t d t \tag{4.151}
\end{equation*}
$$

After regrouping integrals on the right-hand side of (4.151), we have

$$
\begin{equation*}
\int_{0}^{\infty} a(t) b(t) t d t=\int_{0}^{\infty} \int_{0}^{\infty} \tilde{a}(\omega) \tilde{b}\left(\omega^{\prime}\right)\left[\int_{0}^{\infty} J_{\nu}(\omega t) J_{\nu}\left(\omega^{\prime} t\right) t d t\right] \omega d \omega \omega^{\prime} d \omega^{\prime} . \tag{4.152}
\end{equation*}
$$

The expression in the square brackets, according to (4.148), is $\omega^{-1} \delta\left(\omega-\omega^{\prime}\right)$, yielding the Plancherel's identity for the continuous Hankel transform

$$
\begin{equation*}
\int_{0}^{\infty} a(t) b(t) t d t=\int_{0}^{\infty} \tilde{a}(\omega) \tilde{b}(\omega) \omega d \omega . \tag{4.153}
\end{equation*}
$$

The Parseval's identity is a special case of (4.153), with $a=b$

$$
\begin{equation*}
\int_{0}^{\infty}|a(t)|^{2} t d t=\int_{0}^{\infty}|\tilde{a}(\omega)|^{2} \omega d \omega . \tag{4.154}
\end{equation*}
$$

Utilizing the orthogonality property of the basis functions of the discrete transform, one obtains the discrete analog of (4.154)

$$
\begin{equation*}
\sum_{n=-N / 2}^{N / 2}\left|a_{n}\right|^{2}=\sum_{m=-N / 2}^{N / 2}\left|\tilde{a}_{m}\right|^{2} \tag{4.155}
\end{equation*}
$$

Recalling that for the unbiased Hankel transform with the Bessel function index $\nu \pm 1 / 2$ (to get the Fourier sine/cosine transform), we made substitutions (4.139), (4.140), we obtain

$$
\begin{equation*}
\sum_{n=-N / 2}^{N / 2}\left|A_{n}\right|^{2} t_{n}=\sum_{m=-N / 2}^{N / 2}\left|\tilde{A}_{m}\right|^{2} \omega_{m} . \tag{4.156}
\end{equation*}
$$

### 4.5.6 FFTLog validation

A self-consistent way to validate the FFTLog algorithm is to demonstrate that the Parseval's theorem (4.156), where $A_{n}=A\left(t_{n}\right)$ is the logarithmically-sampled solution of the diffusion equation, $\tilde{A}_{m}=\tilde{A}\left(\omega_{m}\right)$ its discrete Hankel transform, is indeed satisfied. In order to do this we can solve the diffusion equation (4.116) for both disk configurations (compact transition layer and extended Keplerian disk), calculate the discrete Hankel transform, using FFTLog, and compare total energy in time and frequency domains at each radius. If the algorithm works correctly and no errors were made in using it, the two ways of computing energy must yield the identical result. Shown on the top panel of Figure 4.16 is the total energy contained in a solution $u(r, t)$ of the diffusion problem for transition layer configuration overlayed with the total energy contained in the waveform $\mathcal{F}_{u}(\omega)$, calculated using FFTLog, at each spatial point. A quite remarkable agreement between the two lines assures that the method does work correctly. The bottom panel of the same Figure displays a similar comparison for the extended disk configuration, where the logarithm of radius $x=\ln r$ is used as a scale on the horizontal axis.


Figure 4.16: Validation of FFTLog algorithm: transition layer (top panel), extended disk (bottom panel).

### 4.5.7 Combined power spectra of the two configurations

The diffusion propagation problem for the extended Keplerian disk is formulated as the equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=3 \nu_{0} R_{0}^{-2} r_{\text {out }}^{-\psi} r^{1 / 2} \frac{\partial}{\partial r}\left[r^{\psi-1 / 2} \frac{\partial u}{\partial r}\right]+f_{\mathrm{dr}}(r, t), \tag{4.157}
\end{equation*}
$$

defined over region $r_{\text {adj }} \leq r \leq r_{\text {out }}$, with the boundary conditions $u=0$ at $r=r_{\text {adj }}$ and $\partial u / \partial r=0$ at $r=r_{\text {out }}$. The inner disk radius is set to $r_{\text {adj }}$, under assumption that the large disk begins at the outer edge of the transition layer, and thus is determined by the Reynolds number. Outer radius of the Keplerian disk is taken to be $r_{\text {out }}=10^{4}$. The equation is solved for a $10 M_{\odot}$ black hole rotating with $f_{0}=30 \mathrm{~Hz}$. The same parametrization for kinematic viscosity $\nu_{0}$ was made as in the the problem for the transition layer, i.e. $\nu \sim \lambda_{\mathrm{t}} v_{\mathrm{t}}$, where $\lambda_{\mathrm{t}} \lesssim 2 H$, where the height of the accretion disk at given radius is $2 H \sim 0.1 R[47] ; v_{\mathrm{t}} \lesssim v_{\mathrm{S}}$, where speed of sound is given by (4.105). We assumed that the characteristic temperature at the outer edge of the disk is $\sim 1 \mathrm{eV}$. A "one-time" perturbation source (4.57) was used as $f_{\mathrm{dr}}(r, t)$ on the right side of the diffusion equation. Since in reality only the innermost part of the Keplerian disk contributes to production of photons with energies $\gtrsim 1 \mathrm{keV}$, no radial integration of the solution is done, but rather the signal from the inner edge of the disk is used for computing the power density spectrum. In order to eliminate undesirable artifacts in the power spectrum, caused by ringing and aliasing in FFTLog, a much broader then needed time range was used, with 32000 points on a logarithmic time grid. Logarithmically spaced spatial grid has 1000 points. The simplest model with viscosity index $\psi=0$ was first considered. Figure 4.17 shows the two power spectra, calculated for the transition layer and the extended disk, overlayed on one plot. It should be pointed out that in order to have comparable normalizations for power spectra of the two (vastly different in sizes) configurations, the amplitude of the driving perturbations in the large disk should be $\sim 5$ orders of magnitude smaller than the amplitude of the driving perturbations in the transition region. It is easy to explain, if one recalls that the driving source term in the
diffusion equation is defined per unit area of the disk, and thus scales with radius as $\propto r$. As anticipated, the power spectrum from the extended Keplerian disk (red line on the plot) appears as the white-red noise with characteristic "knee" in the continuum, corresponding to the viscous timescale of the disk. Notice that this break occurs at $\sim 10^{-6} \mathrm{~Hz}$, while the high-frequency component has break at $\sim 1 \mathrm{~Hz}$. One can also see a weak high-frequency quasi-periodic feature, approximately lined up with the QPO peak from the transition layer. This peak reflects the presence of the driving sources in the emission area (the inner edge of the large disk). As was shown in Chapter 2, the resulting power spectrum of a sum of two weakly correlated signals is given by the sum of power spectra of the two signals (2.53). Assuming that the two configurations, separated by a shock-like centrifugal barrier, are independent, we can combine the two power spectra, by adding together high-frequency and low-frequency components. The combined spectra, calculated for the Reynolds number $\gamma=3,10$, and 80 are presented of Figure 4.18. From the plot one can infer, that observed in data, power spectrum patterns, when only one of the components is seen, or both present simultaneously, can be realized when an X-ray source experiences a spectral transition. During a change from $\gamma=3$ to $\gamma=80$ (what presumably corresponds to transition from low-hard to high-soft state) the HF component of the spectrum undergoes significant change in terms of its normalization with respect to a LF component and frequency location of its features (break, QPOs). For instance, the top red line on Figure 4.18 exhibits strong HF power spectrum from the compact configuration, with barely detectable presence of the LF component from the accretion disk. At contrast, the bottom (blue) line, on the same plot, shows much weaker HF component as compared to a LF one. In real observational power spectra the high-frequency QPO peak from the transition region may not be seen at al, for the source in high-soft or very high state, so that the only visible component is white-red noise from the extended disk. Due long collection time needed in order to detect the break in the low-frequency power spectrum, the real data spectrum is often truncated at low frequency, resulting in that the only LF red noise (power) component seen. In between the two extremities, there is the case where both HF and LF parts of the power spectrum can


Figure 4.17: Two power spectrum components in extra broad frequency range. Blue line represents the power spectrum from the transition layer.


Figure 4.18: Combined power spectrum as a sum of LF and HF components for different values of Reynolds number.
be observed as distinct components. This situation is represented by the green line on the same figure. Of course, presented plots reflect only a few very particular scenarios for some fixed values of parameters. As was mentioned before, the shape and normalization of power spectrum can vary wildly even within the framework of our simplistic model, depending on multiple variables, including the parametrization of kinematic viscosity, distribution of perturbation sources in the disk, relationship between Reynolds number and photon temperature, coherence factor of the driving sources, etc. So far we only studied the cases where kinematic viscosity in the disk was assumed to be uniform throughout the disk, $\nu_{0}=$ const. We found earlier that for the compact transition layer zone this is indeed a good approximation, since neither the solution of the diffusion problem, nor the calculated from it power spectrum did not show any changes with introduction of linear and quadratic dependence of viscosity as a function of radius. For a Keplerian accretion disk, extended over thousands of Schwarzschild radii, such dependence might be significant. In order to find out in what way, we simulated the extended configuration for the three values of viscosity index $\psi=0,1 / 2$, and 1 , keeping all other parameters of the problem unaltered. The results are presented on Figure 4.19. Eye analysis of the power spectra for varying viscosity distribution law suggests that there is rather strong dependence, in the way that "undistorted" power spectrum for the case $\psi=0$ experiences a dramatic change as $\psi$ evolves to values 0.5 , and 1. Straight power law part seems to be replaced by the two power laws with different slopes for $\psi=1$, accompanied by a huge growth ( $\sim 7$ orders of magnitude for the low-frequency part of power spectrum) of the continuum normalization. The high-frequency part, also undergoes a lift up in normalization, though to a less degree. We do not provide physical interpretation of these results here. However, these predictions, might be of a potential value in analysis of real data. For instance, if one observes a significant deviation from the straight power law line (in a log-log scale) of the low-frequency power spectrum component, this may be a signal to that there may be some other than uniform distribution of viscosity in the accretion disk.


Figure 4.19: Power spectrum from the extended Keplerian disk as a function of the viscosity distribution law (4.1).

## Chapter 5: Summary

We present analytical and numerical treatment of a problem of formation of the Fourier power density spectrum of the X-ray luminosity fluctuations in the non-Keplerian adjustment region near the compact object and in the extended Keplerian accretion disk. Our approach is based on the diffusion model in which the local perturbations of the mass accretion rate occur at each radius in the disk and then diffusively propagate outward. The equation describing the evolution of such driving oscillations can be derived from the laws of mass continuity and angular momentum balance in the disk geometry. We write this equation directly in terms of accretion rate fluctuations $\triangle \dot{M}$. The Centrifugal Barrier Model (CBM), by Titarchuk, Lapidus, and Muslimov, 1998, was adopted for the analysis. CBM predicts the existence of the compact coronal region around an accreting object, where the Keplerian accretion flow dynamically adjusts to the inner sub-Keplerian boundary condition. Applying CBM to the problem of diffuse perturbation propagation resulted in the diffusion coefficient that changes its sign at "turnover" radius $R_{\max }$, where the angular velocity reaches maximum. The negative diffusion coefficient in the equation leads to the unbounded exponential solution. We conclude that the discovered instability is an intrinsic property of any problem with the adjustment of the Keplerian motion to the sub-Keplerian inner boundary. Whenever the accretion disk extends all the way down to the surface of the slowly-rotating neutron star or the last stable circular orbit around a black hole this type of phenomena will take place. Analytical solution of the diffusion problem for the stable zone of the compact transition layer showed that the emergent power spectrum of the accretion rate variations is given by the product of the power spectrum of the solution of the initial-value problem for the distributed initial condition and the power spectrum of the driving perturbation. We demonstrate that the power spectrum continuum of the solution of the initial-value problem is the white-red noise. This shape is produced by the weighted
sum of the zero-centered Lorentzians. Its characteristic feature, the "break" frequency is associated with the system's diffusion timescale, and thus have immediate application in data analysis. Below this break the power spectrum is frequency-independent, while above it the power spectrum behaves as the power law with index -2 . One of the consequences of the CBM is the strong dependence of the size of the transition layer on the Reynolds number $\gamma$ of the flow, which is directly proportional to mass accretion rate. We found that this dependence significantly affects the power spectrum normalization and frequency location of its features: power spectrum variability rapidly decays with increasing $\gamma$, while the spectrum features drift into higher frequencies. This finding is in a good agreement with observational results. The unstable region of the adjustment layer also produces the white-red noise-like power spectrum, which is, however, unlikely to be detected. The diffusion problem with the damped harmonic oscillator source term was solved numerically for both the compact transition layer zone and the extended Keplerian disk. Main conclusions drawn from the analytical solution were confirmed by the numerical solution. We obtained the similarly-shaped (white-red noise) power spectrum which also has the quasi-periodic feature in the hecto-Hertz range, associated the distributed perturbation sources. These sources of fluctuations occur on a dynamical timescale $t_{\varphi}$ which is inverse-proportional to the frequency of the Keplerian orbital motion $\nu_{\mathrm{K}}$. The question of dependence of the RMS variability amplitude of the lightcurve as a function of spectral state was considered. The observed rapid variability decay with increasing Reynolds number $\gamma$ is explained by the compression of the emission zone and properties of the diffusion operator, which is a strong function of $\gamma$. A solution of the diffusion propagation equation and its power spectrum were also obtained on an extra broad frequency range. Our model is capable of reproducing the two-component power spectra, seen in real data. According the the CBM formalism these components can be attributed to the compact non-Keplerian adjustment region and independent extended Keplerian disk. We found out that the viscosity distribution in the transition layer is likely to be uniform throughout the region. Power spectrum from the Keplerian disk, however, exhibits a strong dependence on the viscosity distribution law.

## Appendix A: Break-up speed of a rotating sphere

Consider a massive spherical object of mass $M$, radius $R$ composed of particles held together by gravitational force. Suppose it is rotating with angular velocity $\omega$. There are two competing forces acting on a fragment of mass $m$ located at $R$ : gravitational pull towards the center

$$
\begin{equation*}
F_{\mathrm{grv}}=G \frac{m M}{R^{2}} \tag{A.1}
\end{equation*}
$$

and centrifugal force in opposite direction of magnitude

$$
\begin{equation*}
F_{\mathrm{cfg}}=m \frac{v_{\varphi}^{2}}{R}=m \omega^{2} R . \tag{A.2}
\end{equation*}
$$

The rotational break-up will occur if the centrifugal acceleration exceeds the gravitational acceleration

$$
\begin{equation*}
G \frac{M}{R^{2}}<\omega^{2} R, \tag{A.3}
\end{equation*}
$$

i.e. the sphere's break-up angular velocity is determined by the condition

$$
\begin{equation*}
\omega_{\mathrm{br}}=\sqrt{\frac{G M}{R^{3}}}=\omega_{\mathrm{K}} \tag{A.4}
\end{equation*}
$$

where $\omega_{\mathrm{K}}$ is the angular velocity of the Keplerian orbit at radius $R$.

## Appendix B: Parseval's theorem, power density spectrum, and convolution theorem

In order to understand why the squared Fourier amplitude is associated with power, let us recall the useful relationship, known as Parseval's theorem. Consider the product of the continuous Fourier transform $\mathcal{F}(\omega)$ of some function $f(t)$ and its complex-conjugate $\mathcal{F}^{*}(\omega)$

$$
\begin{equation*}
\mathcal{F}^{*}(\omega) \mathcal{F}(\omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} f^{*}\left(t^{\prime}\right) e^{\mathrm{i} \omega t^{\prime}} d t^{\prime} \cdot \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i} \omega t} d t . \tag{B.1}
\end{equation*}
$$

Regrouping the integrals on the right-hand side of (B.1), followed by integration of the entire equation over $\omega$ from $-\infty$ to $\infty$ gives

$$
\begin{align*}
\int_{-\infty}^{\infty} \mathcal{F}^{*}(\omega) \mathcal{F}(\omega) d \omega= & \frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^{*}\left(t^{\prime}\right) f(t) e^{\mathrm{i} \omega\left(t^{\prime}-t\right)} d t^{\prime} d t d \omega= \\
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^{*}\left(t^{\prime}\right) f(t)\left[\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{\mathrm{i} \omega\left(t^{\prime}-t\right)} d \omega\right] d t^{\prime} d t . \tag{B.2}
\end{align*}
$$

The Fourier transform of a shifted Dirac $\delta$-function is

$$
\begin{equation*}
\mathcal{F}\left(\delta\left(t-t^{\prime}\right)\right)(\omega)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \delta\left(t-t^{\prime}\right) e^{-\mathrm{i} \omega t} d t=\frac{1}{\sqrt{2 \pi}} e^{-\mathrm{i} \omega t^{\prime}}, \tag{B.3}
\end{equation*}
$$

Taking the inverse transform yields

$$
\begin{equation*}
\delta\left(t-t^{\prime}\right)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \mathcal{F}\left(\delta\left(t-t^{\prime}\right)\right)(\omega) e^{\mathrm{i} \omega t} d \omega=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{\mathrm{i} \omega\left(t-t^{\prime}\right)} d \omega . \tag{B.4}
\end{equation*}
$$

Formula (B.4) expresses the orthogonality property of the Fourier transform basis functions $\exp (i \omega t)$. Since the $\delta$-function is symmetric, the expression in the square brackets in
equation (B.2) is $\delta\left(t^{\prime}-t\right.$ ), thus reducing it to

$$
\begin{equation*}
\int_{-\infty}^{\infty} \mathcal{F}^{*}(\omega) \mathcal{F}(\omega) d \omega=\int_{-\infty}^{\infty} f^{*}(t) f(t) d t \tag{B.5}
\end{equation*}
$$

Relationship (B.5) is called the Parseval's identity for the continuous Fourier transform. For a real-valued function $f^{*}(t) f(t)=f^{2}(t)$. If a potential difference $U=U(t)$ is applied across a resistor $R$, the dissipated power is given by

$$
\begin{equation*}
P=\frac{1}{R} \int_{-\infty}^{\infty} U^{2}(t) d t . \tag{B.6}
\end{equation*}
$$

For a 1 Ohm load, according to (B.5), it is equal to the integral over angular frequency of the square of the Fourier transform of $U(t)$

$$
\begin{equation*}
P(\text { on } 1 \mathrm{Ohm})=\int_{-\infty}^{\infty} U^{2}(t) d t=\int_{-\infty}^{\infty} \mathcal{F}_{U}^{*}(\omega) \mathcal{F}_{U}(\omega) d \omega . \tag{B.7}
\end{equation*}
$$

The quantity

$$
\begin{equation*}
\left\|\mathcal{F}_{U}(\omega)\right\|^{2}=\mathcal{F}_{U}^{*}(\omega) \mathcal{F}_{U}(\omega), \tag{B.8}
\end{equation*}
$$

characterizing how much energy of a signal $U(t)$ is contained in a frequency interval $\omega$ $\omega+d \omega$, is called the (unnormalized) power density spectrum, or just the power spectrum of the signal $U(t)$. For our analysis we will adopt this definition and will not care about normalization for now.

Another important property of the Fourier transform is the convolution theorem. Consider the continuous Fourier transform $\mathcal{F}(\omega)$ of a function $f(t)$ and it inverse

$$
\begin{equation*}
\mathcal{F}(w)=\int_{-\infty}^{\infty} f(t) e^{-i \omega t} d t, \quad f(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i \omega t} d w, \tag{B.9}
\end{equation*}
$$

where non-unitary normalization is used. Let functions $f(t), g(t), h(t)$ have their Fourier transforms $\mathcal{F}(\omega), \mathcal{G}(\omega), \mathcal{H}(\omega)$. The convolution theorem states that if $\mathcal{H}(\omega)=\mathcal{F}(\omega) \mathcal{G}(\omega)$, then

$$
\begin{equation*}
h(t)=\int_{-\infty}^{\infty} f(\xi) g(t-\xi) d \xi \equiv f * g . \tag{B.10}
\end{equation*}
$$

The function $h(t)$ defined by (B.10) is called the convolution of $f(t)$ and $g(t)$. Convolution theorem can be proved, using the inverse Fourier transform of $h(t)$

$$
\begin{equation*}
h(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathcal{H}(\omega) e^{i \omega t} d \omega=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) \mathcal{G}(\omega) e^{i \omega t} d \omega \tag{B.11}
\end{equation*}
$$

Substituting the definition of $\mathcal{F}(\omega)$ we get

$$
\begin{equation*}
h(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi) e^{-i \omega \xi} d \xi\right] \mathcal{G}(\omega) e^{i \omega t} d \omega \tag{B.12}
\end{equation*}
$$

We can interchange the order of integration to obtain

$$
\begin{equation*}
h(t)=\int_{-\infty}^{\infty} f(\xi)\left[\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathcal{G}(\omega) e^{i \omega(t-\xi)} d \omega\right] d \xi \tag{B.13}
\end{equation*}
$$

But from definition (B.9) we have

$$
\begin{equation*}
g(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathcal{G}(\omega) e^{i \omega t} d \omega, \tag{B.14}
\end{equation*}
$$

and by replacing $t$ by $t-\xi$ in (B.14) we see that the expression in the square brackets in (B.13) is $g(t-\xi)$, i.e.

$$
\begin{equation*}
h(t)=\int_{-\infty}^{\infty} f(\xi) g(t-\xi) d \xi \equiv f * g . \tag{B.15}
\end{equation*}
$$

Thus (B.10) holds. This result is immediately applicable to our problem of finding the power spectrum of a function $h(t)=\int_{-\infty}^{\infty} f(\xi) g(t-\xi) d \xi$. Using the convolution theorem, we have

$$
\left\{\begin{array}{l}
\mathcal{H}(\omega)=\mathcal{F}(\omega) \mathcal{G}(\omega),  \tag{B.16}\\
\mathcal{H}^{*}(\omega)=\mathcal{F}^{*}(\omega) \mathcal{G}^{*}(\omega),
\end{array}\right.
$$

where the asterisk symbol stands for complex conjugate. Multiplying equation (B.16) by $\mathcal{H}^{*}(\omega)$, and employing equality (B.17), we obtain

$$
\begin{equation*}
\mathcal{H}^{*} \mathcal{H}=\mathcal{F}^{*} \mathcal{F} \cdot \mathcal{G}^{*} \mathcal{G} \tag{B.18}
\end{equation*}
$$

i.e. the power spectrum of a function $h(t)$, presented as a convolution of $f(t)$ and $g(t)$ is given by the product of power spectra of $f(t)$ and $g(t)$

$$
\begin{equation*}
\|\mathcal{H}\|^{2}=\|\mathcal{F}\|^{2}\|\mathcal{G}\|^{2} \tag{B.19}
\end{equation*}
$$

## Appendix C: Reynolds number and viscous torques in the disk

Throughout this work we regard the accretion flow as the Newtonian fluid, i.e. the one in which the stress (the force per unit area) is proportional to the velocity gradient

$$
\begin{equation*}
\tau=\eta \frac{\partial u}{\partial x} \tag{C.1}
\end{equation*}
$$

where $\eta\left[\mathrm{g} \mathrm{cm}^{-1} \mathrm{~s}^{-1}\right]$ is the coefficient of viscosity of the fluid, called the dynamic or turbulent viscosity. The basic equation, which expresses the Newton's second law for a fluid of constant density, is the Navier-Stokes equation [66]

$$
\begin{equation*}
\rho \frac{\mathrm{D} \mathbf{u}}{\mathrm{D} t}=-\nabla p+\eta \nabla^{2} \mathbf{u}+\mathbf{F}, \tag{C.2}
\end{equation*}
$$

where $\mathrm{D} / \mathrm{D} t=\partial / \partial t+\mathbf{u} \cdot \nabla$, and the velocity of the particle is $\mathbf{u}=(u, v, w)$. The Reynolds number is defined as a dimensionless parameter combined of the following four quantities: density $\rho$, viscosity $\eta$, velocity and length scales $U$ and $L$ correspondingly. Thus

$$
\begin{equation*}
\operatorname{Re}=\frac{\rho U L}{\eta}=\frac{U L}{\nu}, \tag{C.3}
\end{equation*}
$$

where we introduced the kinematic viscosity $\nu=\eta / \rho\left[\mathrm{cm}^{2} \mathrm{~s}^{-1}\right]$. One can show, by considering the dynamical equation of a steady incompressible flow, that the Reynolds number physical meaning is expressed by the ratio

$$
\begin{equation*}
\operatorname{Re} \sim \frac{\text { inertia forces }}{\text { viscous forces }} \tag{C.4}
\end{equation*}
$$

I.e. the Reynolds number indicates the relative importance of two dynamical processes. It is an experimental fact that for higher values of Reynolds number the flow becomes turbulent, while for low values Reynolds number flow tends to be laminar.
"The physical idea of the viscous torque $G$ [in a differentially-rotating gas] is to assume that, if the gas flow is turbulent to some extent, gas particles of adjacent layers will be exchanged in the radial direction. Since the two different radial flows of material originating in both layers will have different specific angular momenta, this will cause a net transfer of angular momentum between them." (O. Toledano, E. Moreno, G. Koenigsberger, R. Detmers and N. Langer "Tides in asynchronous binary systems" 2007) [64]. One can calculate the viscous torque on the annulus of the disk in terms of kinematic viscosity $\nu=\eta / \rho$. Here $\rho$ is the local density of the gas, and the turbulent viscosity $\eta$ (the coefficient of shear viscosity) is related to the component $\tau_{r \varphi}$ of the viscous stress tensor by [26]

$$
\begin{equation*}
\tau_{r \varphi}=\eta\left(\frac{1}{R} \frac{\partial v_{R}}{\partial \varphi}+\frac{\partial v_{\varphi}}{\partial R}-\frac{v_{\varphi}}{R}\right) \tag{C.5}
\end{equation*}
$$

where $v_{R}, v_{\varphi}$ are the radial and azimuthal components of the velocity. Inserting $v_{\varphi}=\omega R$, with $\omega$ as the rotational angular velocity of the shear flow,

$$
\frac{\partial v_{\varphi}}{\partial R}=\frac{\partial(\omega R)}{\partial R}=\omega+R \frac{\partial \omega}{\partial R}
$$

and neglecting the shear effects in the radial direction, the viscous stress tensor becomes

$$
\begin{equation*}
\tau_{r \varphi}=\eta R \frac{\partial \omega}{\partial R} \tag{C.6}
\end{equation*}
$$

It has the meaning of the tangential (shear) force per unit area acting in $\varphi$-direction on the $r-t h$ face of the fluid element, and has dimension of $\left[\mathrm{g} \mathrm{cm}^{-1} \mathrm{~s}^{-2}\right]$. Hence, the total viscous torque $G$ exerted by the ring-shaped outer layer of gas on the inner one is obtained by
multiplying $\tau_{r \varphi}$ by the lever arm $R$ and integrating over the area of interaction $d A=R d \varphi d z$

$$
\begin{equation*}
G=2 \pi R^{2} \int_{0}^{2 H} \eta R \frac{\partial \omega}{\partial R} d z=2 \pi R^{2} \nu \Sigma R \frac{\partial \omega}{\partial R} . \tag{C.7}
\end{equation*}
$$

## Appendix D: Diffusion equation for small perturbations

Suppose that equation (3.6) is written for the unperturbed quantities $\dot{M}_{0}(R, t), \omega_{0}(R)$, $\nu_{0}(R)$, and $Q_{0}(R, t)$. It is reasonable to assume that if we introduce a small perturbation in the accretion rate

$$
\triangle \dot{M}(R, t) \ll \dot{M}_{0}(R, t)
$$

it will induce small variations in other physical parameters about their unperturbed values:

$$
\left\{\begin{array}{l}
\dot{M}_{0} \rightarrow \dot{M}_{0}(R, t)+\triangle \dot{M}(R, t),  \tag{D.1}\\
\omega_{0} \rightarrow \omega_{0}(R)+\triangle \omega(R, t), \\
\nu_{0} \rightarrow \nu_{0}(R)+\triangle \nu(R, t), \\
Q_{0} \rightarrow Q_{0}(R, t)+\triangle Q(R, t) .
\end{array}\right.
$$

Substitution of (D.1) into (3.6) gives

$$
\begin{array}{r}
\frac{\partial \dot{M}_{0}}{\partial t}+\frac{\partial \triangle \dot{M}}{\partial t}+\frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}+\left(\triangle \omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left[R^{2}\left(\nu_{0}+\triangle \nu\right)\left(\omega_{0}+\triangle \omega\right)^{\prime}\left(\frac{\partial \dot{M}_{0}}{\partial R}+\frac{\partial \triangle \dot{M}}{\partial R}\right)\right]+ \\
\frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}+\left(\triangle \omega R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left[2 \pi R^{3}\left(\nu_{0}+\triangle \nu\right)\left(\omega_{0}+\triangle \omega\right)^{\prime}\left(Q_{0}+\triangle Q\right)\right]=0 .
\end{array}
$$

Keeping only terms of first-order smallness, and using the Taylor series expansion for

$$
\frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}+\left(\Delta \omega R^{2}\right)^{\prime}} \approx \frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}}-\frac{\left(\Delta \omega R^{2}\right)^{\prime}}{\left(\left(\omega_{0} R^{2}\right)^{\prime}\right)^{2}},
$$

equation (3.6) can be rewritten as

$$
\begin{equation*}
\frac{\partial \triangle \dot{M}}{\partial t}+\frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left[R^{2} \nu_{0} \omega_{0}^{\prime} \frac{\partial \triangle \dot{M}}{\partial R}\right]+f(R, t)=0 \tag{D.2}
\end{equation*}
$$

where the source term $f(R, t)=$

$$
\begin{aligned}
& \frac{1}{\left(\omega_{0} R^{2}\right)^{\prime}} \frac{\partial}{\partial R}\left[R^{2}\left(\nu_{0} \triangle \omega^{\prime} \frac{\partial \dot{M}_{0}}{\partial R}+\triangle \nu \omega_{0}^{\prime} \frac{\partial \dot{M}_{0}}{\partial R}\right)+2 \pi R^{3}\left(\nu_{0} \triangle \omega^{\prime} Q_{0}+\triangle \nu \omega_{0}^{\prime} Q_{0}+\nu_{0} \omega_{0}^{\prime} \triangle Q\right)\right]- \\
& \frac{\left(\triangle \omega R^{2}\right)^{\prime}}{\left(\left(\omega_{0} R^{2}\right)^{\prime}\right)^{2}} \frac{\partial}{\partial R}\left[R^{2} \nu_{0} \omega_{0}^{\prime} \frac{\partial \dot{M}_{0}}{\partial R}+2 \pi R^{3} \nu_{0} \omega_{0}^{\prime} Q_{0}\right],
\end{aligned}
$$

is determined by the perturbation terms $\triangle \omega(R, t), \triangle \nu(R, t)$, and $\triangle Q(R, t)$. We now have to justify our assumption about smallness of these "response" fluctuations. First, kinematic viscosity $\nu$ depends on properties of accreting plasma, which are unknown, and therefore we simply assume that the small variations in the accretion rate $\triangle \dot{M} \ll \dot{M}_{0}$ will cause the small variations in viscosity $\Delta \nu \ll \nu_{0}$. Secondly, we are free to choose the mass accretion input term $Q(R, t)$ to be such that its response to the small fluctuations in $\dot{M}$ results in the small fluctuations in $\mathrm{Q}, \triangle Q \ll Q_{0}$. We have to verify, however, that the small fluctuations in $\dot{M}$ lead to the small fluctuations in $\omega, \Delta \omega \ll \omega_{0}$. According to (2.44), the unperturbed angular velocity radial profile is

$$
\omega_{0} \propto\left[D_{1} r^{-\gamma}+\left(1-D_{1}\right) r^{-2}\right] .
$$

Dropping subscripts for simplicity and taking the differential, gives

$$
\begin{equation*}
d \omega \propto d\left(D r^{-\gamma}\right)+d\left[(1-D) r^{-2}\right]=-D r^{-\gamma} \ln r d \gamma+\left(r^{-\gamma}-r^{-2}\right) d D \tag{D.3}
\end{equation*}
$$

where we used $r^{-\gamma}=e^{-\gamma \ln r}$. Formula for $D$ is given by equation (2.45). Employing relation (2.46) we can write $D$ in terms of $r_{\text {adj }}$ only:

$$
D=\frac{\alpha r_{\mathrm{adj}}^{1 / 2}-1}{r_{\mathrm{adj}}^{2-\gamma}-1}, \quad \text { where } \alpha=\frac{6}{m\left(\frac{f_{0}}{363}\right)}
$$

Correspondingly, after taking the differential, we obtain

$$
\begin{align*}
& d D=\frac{d\left(\alpha r_{\mathrm{adj}}^{1 / 2}\right)\left(r_{\mathrm{adj}}^{2-\gamma}-1\right)-\left(\alpha r_{\mathrm{adj}}^{1 / 2}-1\right) d\left(r_{\mathrm{adj}}^{2-\gamma}\right)}{\left(r_{\mathrm{adj}}^{2-\gamma}-1\right)^{2}}=  \tag{D.4}\\
& =\frac{\frac{\alpha}{2} r_{\mathrm{adj}}^{-1 / 2} d r_{\mathrm{adj}}\left(r_{\mathrm{adj}}^{2-\gamma}-1\right)-\left(\alpha r_{\mathrm{adj}}^{1 / 2}-1\right) r_{\mathrm{adj}}^{2-\gamma}\left[-\ln r_{\mathrm{adj}} d \gamma+(2-\gamma) \frac{d r_{\mathrm{adj}}}{r_{\mathrm{adj}}}\right]}{\left(r_{\mathrm{adj}}^{2-\gamma}-1\right)^{2}} .
\end{align*}
$$

Now, relationship between $r_{\text {adj }}$ and $\gamma$ is determined by the boundary condition (2.48).
Taking the differential of equation (2.48), we have

$$
\frac{3}{2} \alpha d\left(r_{\mathrm{adj}}^{-3 / 2}\right)=d\left(D \gamma r_{\mathrm{adj}}^{-\gamma}\right)+2 d\left[(1-D) r_{\mathrm{adj}}^{-2}\right]
$$

Proceeding, we find that $d r_{\mathrm{adj}}, d D$, and $d \gamma$ are related to each other via

$$
\begin{equation*}
d r_{\mathrm{adj}}=\frac{\left(\gamma r_{\mathrm{adj}}^{-\gamma}-2 r_{\mathrm{adj}}^{-2}\right) d D+\left(D r_{\mathrm{adj}}^{-\gamma}-\gamma D r_{\mathrm{adj}}^{-\gamma} \ln r_{\mathrm{adj}}\right) d \gamma}{4(1-D) r_{\mathrm{adj}}^{-3}+\gamma^{2} D r_{\mathrm{adj}}^{-\gamma-1}-\frac{9}{4} \alpha r_{\mathrm{adj}}^{-5 / 2}} \tag{D.5}
\end{equation*}
$$

Combining (D.4) and (D.5) together to eliminate $d r_{\text {adj }}$ we obtain

$$
d D=\frac{\frac{D r_{\mathrm{adj}}^{-\gamma}-\gamma D r_{\mathrm{adj}}^{-\gamma} \ln r_{\mathrm{adj}}}{4(1-D) r_{\mathrm{adj}}^{-3}+\gamma^{2} D r_{\mathrm{adj}}^{--1}-\frac{9}{4} \alpha r_{\mathrm{adj}}^{-5 / 2}}\left[(2-\gamma) r_{\mathrm{adj}}^{1-\gamma}\right] d \gamma-r_{\mathrm{adj}}^{2-\gamma} \ln r_{\mathrm{adj}} d \gamma}{\left[1-\frac{\gamma r_{\mathrm{adj}}^{-\gamma}-2 r_{\mathrm{adj}}^{-2}}{4(1-D) r_{\mathrm{adj}}^{-3}+\gamma^{2} D r_{\mathrm{adj}}^{-\gamma-1}-\frac{9}{4} \alpha r_{\mathrm{adj}}^{-5 / 2}}(2-\gamma) r_{\mathrm{adj}}^{1-\gamma}\right]}
$$

Pulling the differential of $\gamma$ out and denoting the rest by $\psi(\gamma)$ we can write

$$
\begin{equation*}
d D=\psi(\gamma) d \gamma \tag{D.6}
\end{equation*}
$$

which upon substitution into (D.3) results in

$$
\begin{equation*}
\Delta \omega \propto\left\{-D r^{-\gamma} \ln r+\left(r^{-\gamma}-r^{-2}\right) \psi(\gamma)\right\} \triangle \gamma \tag{D.7}
\end{equation*}
$$

I.e. the variations $\Delta \omega$ of the angular velocity are determined by the variations $\Delta \gamma$ of the Reynolds number, but according to equation (2.37)

$$
\begin{equation*}
\Delta \gamma \propto \Delta \dot{M}, \quad \rightarrow \quad \Delta \omega \propto \triangle \dot{M} \tag{D.8}
\end{equation*}
$$

Therefore, small perturbations $\triangle \dot{M}(R, t)$ in accretion rate, will indeed result in the small perturbations $\triangle \omega(R, t)$ in angular velocity, thus supporting our assumption about "smallness" of induced fluctuations, and justifying the transition from equation (3.6), for accretion rate, to equation (3.7), for the small perturbations of accretion rate.

## Appendix E: Convolution solution of the diffusion problem

Solution $u=u(R, t)$ of the equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\Lambda_{R} u+\mathcal{R}(R) \varphi(t) \tag{E.1}
\end{equation*}
$$

( $\Lambda_{R}$ is the space diffusion operator) combined with the appropriate boundary conditions at $R_{\mathrm{in}}$ and $R_{\text {out }}$ and homogeneous initial condition $u(R, 0)=0$, at any $R$ and $t$ can be written as the convolution

$$
\begin{equation*}
u(R, t)=\int_{0}^{t} \varphi\left(t^{\prime}\right) G\left(R, t-t^{\prime}\right) d t^{\prime} \tag{E.2}
\end{equation*}
$$

with the kernel $G\left(R, t-t^{\prime}\right)$ being a solution of the initial-value problem for the homogeneous equation

$$
\left\{\begin{array}{l}
\frac{\partial G}{\partial t}=\Lambda_{R} G  \tag{E.3}\\
G\left(R, t-t^{\prime}\right)_{t=t^{\prime}}=\mathcal{R}(R)
\end{array}\right.
$$

with the same boundary conditions as that for $u(R, t)$. This can be easily verified by substitution (E.2) in (E.1), keeping in mind equations (E.3), (E.4). Differentiation with respect to the limit of integration can be done using the identity

$$
\frac{d}{d t} \int_{0}^{t} f\left(t, t^{\prime}\right) d t^{\prime}=f\left(t^{\prime}=t\right)+\int_{0}^{t} f_{t}^{\prime}\left(t, t^{\prime}\right) d t^{\prime}
$$

Therefore for the left-hand side of (E.1) we have

$$
\begin{gathered}
\operatorname{LHS}(\text { E. } 1)=\frac{\partial}{\partial t}\left[\int_{0}^{t} \varphi\left(t^{\prime}\right) G\left(R, t-t^{\prime}\right) d t^{\prime}\right]=\varphi(t) G(R, 0)+\int_{0}^{t} \varphi\left(t^{\prime}\right) G_{t}^{\prime}\left(R, t-t^{\prime}\right) d t^{\prime}= \\
=\varphi(t) G(R, 0)+\int_{0}^{t} \varphi\left(t^{\prime}\right) \Lambda_{R} G\left(R, t-t^{\prime}\right) d t^{\prime}
\end{gathered}
$$

The right-hand side of (E.1) becomes after substitution of (E.2)

$$
\operatorname{RHS}(\mathrm{E} .1)=\Lambda_{R} \int_{0}^{t} \varphi\left(t^{\prime}\right) G\left(R, t-t^{\prime}\right) d t^{\prime}+\mathcal{R}(R) \varphi(t)
$$

Given that $G(R, 0)=\mathcal{R}(R)$, comparison of the LHS(E.1) with the RHS(E.1), yields their equivalence, i.e. proves the correctness of the solution (E.2).

## Appendix F: Bessel function asymptotic form

To justify the applicability of the approximate formula

$$
\begin{equation*}
J_{\nu}(z) \approx \sqrt{2 /(\pi z)} \cos \left(z-\frac{\nu \pi}{2}-\frac{\pi}{4}\right), \text { for } z \gg 1 . \tag{F.1}
\end{equation*}
$$

for finding roots of $J_{0}$ we should compare it against the exact expression for Bessel function represented by an infinite series

$$
J_{\nu}(z)=\left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{-z^{2}}{4}\right)^{k}}{k!\Gamma(\nu+k+1)} .
$$

Figure F. 1 shows $J_{0}(x)$ plotted using both infinite series representation and asymptotic expression. It is clear that the asymptotic form reproduces the exact series very closely


Figure F.1: Bessel function $J_{0}$
starting approximately from argument values $\geq 1$. For the purpose of root finding the asymptotic representation is accurate to within $2 \%$ for the first root, which is well sufficient in our approximation.

## Appendix G: Expansion coefficients and eigenfunction's norm for uniform initial perturbation

The weight function $\rho$ (3.36) in the spatial equation (3.35) is independent of $x$, so it can be pulled outside the integral (3.29) along with $\mathcal{R}_{0}=$ const:

$$
c_{k}=\rho \mathcal{R}_{0} \int_{0}^{x_{\mathrm{adj}}} X_{k}(x) d x
$$

where $X_{k}(x)=J_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right)$ is the eigenfunction of (3.35). Substitution gives

$$
c_{k}=\rho \mathcal{R}_{0} \int_{x_{0}}^{x_{\mathrm{adj}}} J_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right) d x=\frac{\rho \mathcal{R}_{0}}{2 \lambda_{k}^{2} \rho} \int_{x_{0}}^{x_{\mathrm{adj}}} J_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right) 2 \lambda_{k} \sqrt{\rho x} d\left(2 \lambda_{k} \sqrt{\rho x}\right) .
$$

Employing the derivative identity for Bessel functions

$$
y^{n} J_{n-1}(y)=\frac{d}{d y}\left[y^{n} J_{n}(y)\right],
$$

we find that $\int y J_{0}(y) d y=y J_{1}(y)$, and hence

$$
c_{k}=\rho \mathcal{R}_{0} \lambda_{k}^{-1} \sqrt{\frac{x_{\mathrm{adj}}}{\rho}} J_{1}\left(2 \lambda_{k} \sqrt{\rho x_{\mathrm{adj}}}\right) .
$$

The eigenfunction's norm $\left\|X_{k}\right\|^{2}$, is given by (3.27):

$$
\left\|X_{k}\right\|^{2}=\rho \int_{0}^{x_{\mathrm{adj}}} J_{0}^{2}\left(2 \lambda_{k} \sqrt{\rho x}\right) d x=\frac{\rho}{2 \lambda_{k}^{2} \rho} \int_{0}^{x_{\mathrm{adj}}} J_{0}^{2}\left(2 \lambda_{k} \sqrt{\rho x}\right) 2 \lambda_{k} \sqrt{\rho x} d\left(2 \lambda_{k} \sqrt{\rho x}\right) .
$$

This integral can be taken by integrating by parts, using the recurrent relation for Bessel functions. We have

$$
\int_{0}^{a} J_{0}^{2}(y) y d y=\frac{a^{2}}{2} J_{0}^{2}(a)-\frac{1}{2} \int_{0}^{a} y^{2} 2 J_{0}(y) J_{0}^{\prime}(y) d y .
$$

Using relation $J_{0}^{\prime}(y)=-J_{1}(y)$, the second term on the right-hand side can be rewritten as

$$
\frac{a^{2}}{2} J_{0}^{2}(a)+\int_{0}^{a} y^{2} J_{0}(y) J_{1}(y) d y
$$

Invoking the derivative identity for $n=1, y J_{0}(y)=\left[y J_{1}(y)\right]_{y}^{\prime}$ and substituting it into the equation above gives

$$
\frac{a^{2}}{2} J_{0}^{2}(a)+\int_{0}^{a} \frac{d}{d y}\left[y J_{1}(y)\right] y J_{1}(y) d y=\frac{a^{2}}{2} J_{0}^{2}(a)+\frac{a^{2}}{2} J_{1}^{2}(a) .
$$

We obtain

$$
\left\|X_{k}\right\|^{2}=\rho x_{\mathrm{adj}}\left[J_{0}^{2}\left(2 \lambda_{k} \sqrt{\rho x_{\mathrm{adj}}}\right)+J_{1}^{2}\left(2 \lambda_{k} \sqrt{\rho x_{\mathrm{adj}}}\right)\right],
$$

but from the boundary condition $X_{k}\left(x_{\text {adj }}\right)=0$, it follows that $J_{0}\left(2 \lambda_{k} \sqrt{\rho x_{\text {adj }}}\right)=0$, and correspondingly

$$
\left\|X_{k}\right\|^{2}=\rho x_{\mathrm{adj}} J_{1}^{2}\left(2 \lambda_{k} \sqrt{\rho x_{\mathrm{adj}}}\right) .
$$

## Appendix H: Approximations for $r_{\text {max }}, r_{\text {adj }}$ in the limit $\gamma \gg 1$

For large Reynolds number, $\gamma \gg 1$, a simple approximate formula for calculating $r_{\text {max }}$ and $r_{\text {adj }}$ may be practical. Notice that expression in square brackets in formula (2.49) for $r_{\text {max }}$ is slightly less than unity

$$
\begin{equation*}
\xi=\left[\frac{\theta_{\text {adj }}-r_{\text {adj }}^{-2}}{\theta_{\text {adj }}-r_{\text {adj }}^{-\gamma}}\right]^{\frac{1}{\gamma-2}} \lesssim 1 . \tag{H.1}
\end{equation*}
$$

Thus, when raised into power $1 /(\gamma-2)$, where $\gamma \gg 1, \xi \rightarrow 1$. For large $\gamma$ the quantity $\xi$ asymptotically approaches unity. Therefore, neglecting $\xi$, we have

$$
\begin{equation*}
r_{\max }=\left(\frac{\gamma}{2}\right)^{\frac{1}{\gamma-2}} \xi \approx\left(\frac{\gamma}{2}\right)^{\frac{1}{\gamma-2}} \tag{H.2}
\end{equation*}
$$

Likewise, neglecting term $r_{\text {adj }}^{-\gamma}$ in formula (2.45) for coefficient $D_{1}$, we have

$$
\begin{equation*}
D_{1} \approx 1-\theta_{\mathrm{adj}} r_{\mathrm{adj}}^{2} \tag{H.3}
\end{equation*}
$$

Substitution of (H.3) into the boundary condition (2.48) gives

$$
3 / 2 \theta_{\text {adj }} \approx 2 \theta_{\text {adj }}+\gamma\left(r_{\text {adj }}^{-\gamma}-\theta_{\text {adj }} r_{\text {adj }}^{2-\gamma}\right),
$$

or after dividing through by $\theta_{\text {adj }}$

$$
3 / 2 \approx 2+\gamma r_{\mathrm{adj}}^{-\gamma}\left(\theta_{\mathrm{adj}}^{-1}-r_{\mathrm{adj}}^{2}\right) .
$$

For $10 \lesssim \gamma \lesssim 100, r_{\text {adj }}^{2}$ is $\sim 7-9$ times larger than $\theta_{\text {adj }}^{-1}$, i.e. one can neglect the latter to obtain

$$
\begin{equation*}
r_{\mathrm{adj}} \approx(2 \gamma)^{1 /(\gamma-2)} \tag{H.4}
\end{equation*}
$$

## Appendix I: Test of a negative separation constant in the analytical solution for region 2

The separation constant $-\lambda^{2}$ results in the spatial equation

$$
\begin{equation*}
\frac{d}{d R}\left(\nu \omega^{\prime} R^{2} \frac{d X}{d R}\right)-\lambda^{2}\left(\omega R^{2}\right)^{\prime} X=0 \tag{I.1}
\end{equation*}
$$

Using approximation $\omega^{\prime}(x) \approx-a_{0} x$ (see solution for region 1 for details) for the angular velocity derivative, where $a_{0}=-\omega^{\prime \prime}\left(R_{\max }\right)>0, x=R-R_{\max }$, such that $-x_{0} \leq x \leq 0$, and replacing $\left(\omega R^{2}\right)^{\prime}$ with $\alpha R^{1-\gamma}$ we have

$$
\begin{equation*}
\frac{d}{d x}\left(\nu a_{0} x R^{2} \frac{d X}{d x}\right)+\lambda^{2} \alpha R^{1-\gamma} X=0 \tag{I.2}
\end{equation*}
$$

After setting $R \approx R_{\max }$ and pulling the quantity $\nu a_{0} R_{\max }^{2}$ outside the derivative, we have

$$
\begin{equation*}
\frac{d}{d x}\left(x \frac{d X}{d x}\right)+\lambda^{2} \rho X=0 \tag{I.3}
\end{equation*}
$$

where $\rho=\alpha /\left(\nu a_{0}\right) R_{\max }^{-(\gamma+1)}$. As was shown this equation can be rewritten as Bessel's equation, which has the solution

$$
\begin{equation*}
X_{k}(x)=C_{1} J_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right)+C_{2} Y_{0}\left(2 \lambda_{k} \sqrt{\rho x}\right) . \tag{I.4}
\end{equation*}
$$

Since $x \leq 0$, a negative expression appears under the square root

$$
X_{k}(x)=C_{1} J_{0}\left(2 \lambda_{k} i \sqrt{\rho|x|}\right)+C_{2} Y_{0}\left(2 \lambda_{k} i \sqrt{\rho|x|}\right) .
$$

The Bessel's function identity $I_{\xi}(x)=i^{-\xi} J_{\xi}(i x)$ gives that $J_{0}(i x)=I_{0}(x)$, thus yielding

$$
X_{k}(x)=C_{1} I_{0}\left(2 \lambda_{k} \sqrt{\rho|x|}\right)+C_{2} Y_{0}\left(2 \lambda_{k} i \sqrt{\rho|x|}\right) .
$$

The outer boundary condition requires that $X_{k}(x=0)< \pm \infty$, but the function $Y_{0}(0)$ diverges to negative infinity, so we have to put $C_{2}=0$. The inner boundary condition requires that $X_{k}\left(x=-x_{0}\right)=0$, but the function $I_{0}(x)$ is always nonnegative, so we have to set $C_{1}=0$ as well. I.e. the only solution of the spatial problem, given the boundary conditions, is the trivial solution $X_{k}(x)=0$, which can not be a solution of the eigenvalue problem. We must, therefore, consider a positive separation constant.

# Appendix J: The "sweep" method for tridiagonal linear systems 

The sweep method (Thomas method) was developed for solving systems of linear algebraic equations with sparse, tridiagonal or aperiodic coefficient matrix. It is perfectly applicable for the diffusion PDE, which can be approximated in finite differences and written as a system of linear equations with tridiagonal coefficient matrix. The method is based on that a given system of $n$ linear algebraic equations gets transformed to a form

$$
x_{i}=\alpha_{i}+\beta_{i} x_{i+1}, \quad i=1,2, \cdots, n-1,
$$

where the sweep coefficients $\alpha_{i}, \beta_{i}$ are found consequently in a forward sweep, $x_{n}$ are found in a backward sweep, and then $x_{i}$ are determined $(i=n-1, n-2, \cdots, 1)$ by consecutive application of the recurrent relation $x_{i}=\alpha_{i}+\beta_{i} x_{i+1}$.

Lets consider a system of linear equations $A y=f$, where $A=\left\|a_{i j}\right\|$ is tridiagonal, i.e. such that $a_{i j}=0$, if $j>i+1$ or $j<i-1$. In general, a system of linear algebraic equations with tridiagonal matrix can be written as

$$
\left\{\begin{array}{l}
a_{j} y_{j-1}-c_{j} y_{j}+b_{j} y_{j+1}=-f_{j}, \quad j=1, \cdots, N-1  \tag{J.1}\\
y_{0}=\chi_{1} y_{1}+\mu_{1} \\
y_{N}=\chi_{2} y_{N-1}+\mu_{2}
\end{array}\right.
$$

For solving such a system numerically the method of successive elimination of unknowns (a Gauss method) can be applied. Our coefficient matrix A , written explicitly, has a form $A=$

$$
\left(\begin{array}{ccccccccc}
1 & -\chi_{1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
a_{1} & -c_{1} & b_{1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{2} & -c_{2} & b_{2} & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \cdots & a_{N-1} & -c_{N-1} & b_{N-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\chi_{2} & 1
\end{array}\right)
$$

We will seek the solution of the system (J.1) in form

$$
\begin{equation*}
y_{j}=\alpha_{j+1} y_{j+1}+\beta_{j+1}, \quad \text { where } j=0, \cdots, N-1, \tag{J.4}
\end{equation*}
$$

where $\alpha_{j+1}, \beta_{j+1}$ are unknown coefficients, which are found successively from $\alpha_{1}, \beta_{1}$ to $\alpha_{N}, \beta_{N}$ (forward sweep), then $y_{N}, y_{N-1}, \cdots, y_{0}$ are found successively (backward sweep). Formulas for calculation $\alpha_{j+1}, \beta_{j+1}$ can be derived from (J.4). Proceeding, we have

$$
\begin{gathered}
y_{j-1}=\alpha_{j} y_{j}+\beta_{j}=\alpha_{j}\left(\alpha_{j+1} y_{j+1}+\beta_{j+1}\right)+\beta_{j}=\alpha_{j} \alpha_{j+1} y_{j+1}+\left(\alpha_{j} \beta_{j+1}+\beta_{j}\right), \quad \text { where } \\
j=1, \cdots, N-1 .
\end{gathered}
$$

Substituting existing expression for $y_{j}, y_{j-1}$ into equation (J.1) for $j=1, \cdots, N-1$ we obtain

$$
\left[\alpha_{j+1}\left(a_{j} \alpha_{j}-c_{j}\right)+b_{j}\right] y_{j+1}+\left[\beta_{j+1}\left(a_{j} \alpha_{j}-c_{j}\right)+a_{j} \beta_{j}+f_{j}\right]=0
$$

This equation will be satisfied if the coefficients $\alpha_{j+1}, \beta_{j+1}$ are chosen to make expressions in square brackets vanishing, i.e.

$$
\begin{equation*}
\alpha_{j+1}=\frac{b_{j}}{c_{j}-\alpha_{j} a_{j}}, \quad \beta_{j+1}=\frac{a_{j} b_{j}+f_{j}}{c_{j}-\alpha_{j} a_{j}}, \quad j=1, \cdots, N-1 . \tag{J.5}
\end{equation*}
$$

In order to find all $\alpha_{j}, \beta_{j}$ it is sufficient to set $\alpha_{1}, \beta_{1}$, which can be found from the requirement of equivalence of (J.4), at $j=0$, i.e. $y_{0}=\alpha_{1} y_{1}+\beta_{1}$, and (J.2). Thus we obtain

$$
\begin{equation*}
\alpha_{1}=\chi_{1}, \quad \beta_{1}=\mu_{1} . \tag{J.6}
\end{equation*}
$$

Finding coefficients $\alpha_{j+1}, \beta_{j+1}$, using (J.5), (J.6) is called a forward sweep. Once sweep coefficients $\alpha_{j+1}, \beta_{j+1}, j=0, \cdots, N-1$ have been determined the solution of system (J.1)-(J.3) can be found using recurrent relation (J.4), starting from $j=N-1$. But in order to start calculations using this formula we need to know $y_{N}$, which can be determined from equations

$$
y_{N}=\chi_{2} y_{N-1}+\mu_{2}, \quad y_{N-1}=\alpha_{N} y_{N}+\beta_{N}, \quad y_{N}=\frac{\chi_{2} \beta_{N}+\mu_{2}}{1-\chi_{2} \alpha_{N}} .
$$

The process of finding $y_{i}$, using formulas

$$
\begin{equation*}
y_{j}=\alpha_{j+1} y_{j+1}+\beta_{j+1}, \quad j=N-1, \cdots, 0, \quad y_{N}=\frac{\chi_{2} \beta_{N}+\mu_{2}}{1-\chi_{2} \alpha_{N}} . \tag{J.7}
\end{equation*}
$$

is called a backward sweep. The algorithm of solving a system (J.1)-(J.3) using formulas (J.5)-(J.7) defines the sweep method. We notice that this method is valid if denominators of (J.5) and (J.7) are not equal to zero. It can be shown that it is sufficient to impose the following conditions, in order to ensure the method's applicability

$$
\begin{cases}\alpha_{j} \neq 0, & b_{j} \neq 0, \quad\left|c_{j}\right| \geq\left|a_{j}\right|+\left|b_{j}\right|, \quad \text { for } j=1, \cdots, N-1, \\ \left|\chi_{1}\right| \leq 1, & \left|\chi_{2}\right|<1 .\end{cases}
$$

If these conditions are satisfied the system (J.1)-(J.3) is equivalent to the system (J.5)-(J.7), and therefore existence and uniqueness of solution are guaranteed. Moreover, inequalities
$\left|\alpha_{j}\right| \leq 1$, for $j=1, \cdots, N$ provide stability of the calculation using recurrent relations (J.7), i.e. the error introduced at some step in the calculation would not grow larger in the following steps. Lets suppose that in formula (J.7) at $j=j_{0}+1$ instead of $y_{j_{0}+1}$, $\hat{y}_{j_{0}+1}=y_{j_{0}+1}+\delta_{j_{0}+1}$ has been computed. Then in the next step, i.e. at $j=j_{0}$, instead of $y_{j_{0}}=\alpha_{j_{0}+1} y_{j_{0}+1}+\beta_{j_{0}+1}$ we will get $\hat{y}_{j_{0}}=\alpha_{j_{0}+1}\left(y_{j_{0}+1}+\delta_{j_{0}+1}\right)+\beta_{j_{0}+1}$ and the error would be equal $\delta_{j_{0}}=\hat{y}_{j_{0}}-y_{j_{0}}=\alpha_{j_{0}+1} \delta_{j_{0}+1}$. It follows that $\left|\delta_{j_{0}}\right|=\left|\alpha_{j_{0}+1}\right|\left|\delta_{j_{0}+1}\right| \leq\left|\delta_{j_{0}+1}\right|$, i.e. error of computation is limited and is not growing.

It is important to point out that the negative diffusion coefficient in the equation (3.7) results in an unbounded evolution operator, causing an arbitrary small error in the initial data result in vast differences of the solutions after arbitrary short amount of time, if we consider the exact solution. Thus the diffusion equation with a negative diffusion coefficient is not a well-posed problem in the beginning. Let us show explicitly that the sweep method described above is not applicable for solving such a problem numerically, due to its intrinsic instability in the region where diffusion coefficient is negative. The diffusion equation of type (4.4) can be expressed in a matrix form

$$
a_{i} u_{i-1}^{k+1}+b_{i} u_{i}^{k+1}+c_{i} u_{i+1}^{k+1}=f\left(u^{k}\right),
$$

with tridiagonal matrix elements given by

$$
\left\{\begin{array}{l}
a_{i}=-\frac{\Delta t}{(\Delta r)^{2}} f_{i} D_{i-1 / 2}  \tag{J.8}\\
b_{i}=1+\frac{\triangle t}{(\triangle r)^{2}} f_{i}\left[D_{i-1 / 2}+D_{i+1 / 2}\right] \\
c_{i}=-\frac{\Delta t}{(\triangle r)^{2}} f_{i} D_{i+1 / 2}
\end{array}\right.
$$

The diffusion coefficient alternates its sign in the transition layer (Figure 3.1), and becomes negative in the region $r<r_{\text {max }}$. In order for the sweep method to remain stable the
following inequalities must hold

$$
\left\{\begin{array}{l}
\left|b_{i}\right| \geq\left|a_{i}\right|+\left|c_{i}\right|, \text { for } i=1, \cdots, N  \tag{J.11}\\
a_{i} \neq 0, \quad c_{i} \neq 0, \text { for } i=2, \cdots, N-1
\end{array}\right.
$$

Lets check whether these requirements are satisfied for the cases $D>0$ and $D<0$. For $D>0$ we have

$$
\begin{gather*}
\left|b_{i}\right|=\left|1+\frac{\Delta t}{(\triangle r)^{2}} f_{i}\left[D_{i-1 / 2}+D_{i+1 / 2}\right]\right|, \text { while }  \tag{J.13}\\
\left|a_{i}\right|+\left|c_{i}\right|=\left|-\frac{\triangle t}{(\triangle r)^{2}} f_{i} D_{i-1 / 2}\right|+\left|-\frac{\triangle t}{(\triangle r)^{2}} f_{i} D_{i+1 / 2}\right| \tag{J.14}
\end{gather*}
$$

Noting that quantities $\Delta t,(\Delta r)^{2}$, and $f_{i}$ are strictly positive, it is clear that conditions (J.11), (J.12) indeed hold true. However, for $D<0$ we have

$$
\begin{gather*}
\left|b_{i}\right|=\left|\frac{\triangle t}{(\triangle r)^{2}} f_{i}\left[\left|D_{i-1 / 2}\right|+\left|D_{i+1 / 2}\right|\right]-1\right|, \text { while }  \tag{J.15}\\
\left|a_{i}\right|+\left|c_{i}\right|=\frac{\triangle t}{(\triangle r)^{2}} f_{i}\left[\left|D_{i-1 / 2}\right|+\left|D_{i+1 / 2}\right|\right] \tag{J.16}
\end{gather*}
$$

yielding $\left|b_{i}\right|<\left|a_{i}\right|+\left|c_{i}\right|$, which is a violation of condition (J.11). Thus the sweep method becomes unstable in the radial region where the diffusion coefficient is negative.

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## Curriculum Vitae

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