On Extremal Coin Graphs, Flowers, and Their Rational Representations

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Dedication

For Dan.

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Hálfnað er verk þá hafið er.

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Abstract

ON EXTREMAL COIN GRAPHS, FLOWERS, AND THEIR RATIONAL REPRESENTATIONS

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We study extremal coin graphs in the Euclidean plane on n vertices with the maximum number of edges. This is related to the *unit coin graph problem* first posed by Erdős in 1946, and considers coin graphs that satisfy certain conditions relating to the ratios of the possible radii of the coins in the graph. A motivating problem is a special case of a coin graph with multiple radii.

We explore the algebraic equations describing a *flower*, the coin graph presentation of a wheel graph, and present a class of irreducible symmetric polynomials that describe the relation of the radii of each flower. These polynomials are then used to fully characterize flowers on four coins, also known as *Soddy circles*, with rational radii. This yields a free parametrization of all flowers on four coins with rational radii. A similar method is used to characterize all flowers on five coins with rational radii and to describe a large class of solutions for flowers on n coins.

Chapter 1: Introduction

1.1 Background and History

In this chapter, we begin our investigation of coin graphs. We will revisit a known result of Harborth's and use an explicit construction to prove the lower bound on the maximum number of edges of a special coin graph on two radii. We will also discuss background and history of the problem and introduce terminology that will be used throughout this dissertation.

We start by defining our fundamental object:

Definition 1.1.1. A *coin graph* G is a graph whose vertices can be represented as closed, non-overlapping disks in the Euclidean plane such that two vertices are adjacent if and only if their corresponding disks intersect at their boundaries, i.e. they touch.

Further, a *unit coin graph* is a coin graph represented by disks of the same radius.

The problem of determining the maximum number of edges of a unit coin graph on n vertices in the Euclidean plane was posed by Erdős [10] in 1946 and again in its current form by Reutter [11] in 1972. This problem was completely solved by Harborth [6] in 1974, who showed that the maximum number of edges is given by $T(n) := \left\lfloor 3n - \sqrt{12n-3} \right\rfloor$. The configuration which achieves this maximum is a "hexagonal spiral" or honeycomb. Figure 1.1 shows an example of such a maximal configuration for $n = 7$.

Harborth's proof shows that the expression $T(n)$ is both an upper and a lower bound. The lower bound is proved constructively using the hexagonal spiral configuration shown in Figure 1.1. The main idea of the proof of the upper bound is to take such a maximal coin graph, remove the vertices bounding the infinite face, and use induction on n.

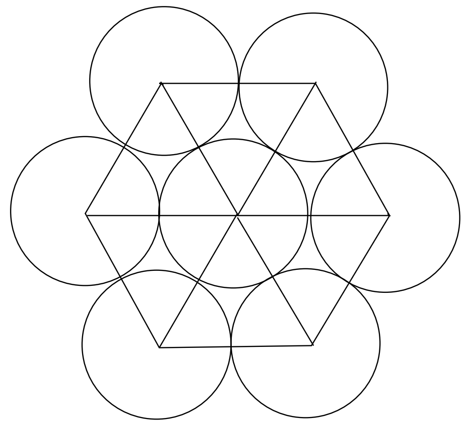

Figure 1.1: A unit coin graph with the maximum number of edges for $n = 7$.

This problem can be generalized in many ways, as suggested in [3] starting on page 222: (i) it can be generalized to graphs embedded in other surfaces such as the sphere or g-torus or to graphs embedded in n-dimensional Euclidean space for $n \geq 3$, where the definition of a coin graph is modified appropriately to an n-dimensional sphere graph [3]. (ii) The additional constraint that no three vertices of the graph can be collinear forces the maximum degree of any vertex to be 5, leading to a different upper bound [3]. (iii) A similar class of graphs can be defined by connecting two vertices if and only if their distance d satisfies $1 \leq d \leq 1 + \epsilon$ for some given small $\epsilon > 0$. This structure can be pictured as a unit coin graph using elastic disks that can stretch some small amount. It is conjectured that for small ϵ (less than 0.15 times the defined unit distance) the maximum number of edges is still $T(n)$ as in the case of the unit coin graph [3]. (iv) A related problem, also posed by Paul Erdős, asks for the minimum independence number for smallest-distance graphs [3].

For a finite set of points in the plane, there is a smallest distance among all pairs of points. The smallest-distance graph is obtained by connecting two points as vertices of the graph if and only if their distance is equal to this smallest distance. This is the original formulation of the coin graph problem as posed by Erdős [10]. For a unit coin graph G with radius $r = 1$, the smallest distance is $d = 2$, and the number of occurrences of this smallest distance is precisely the number of edges of G. In this case of unit coin graphs, the minimum independence number problem asks: what is the largest number of vertices one can select such that none of their corresponding disks touch? (v) Swanepoel recently conjectured that the largest number of edges in a coin graph with no triangular faces is given by $\lfloor 2n - 2\sqrt{n} \rfloor$ [15].

Another natural generalization of the unit coin graph problem is to allow coins of more than one possible radius. Brightwell and Scheinerman [4] explored integral representations of coin graphs, where the radii of the coins can take arbitrary positive integer values. Specifically, they hoped to use coin graphs to answer a conjecture of Harborth's, namely whether all planar graphs admit a straight-line embedding where each edge has integer length. Brightwell and Scheinerman instead proved that there exist planar graphs which cannot be constructed as coin graphs with edge lengths among the constructible algebraic numbers [7]. This leaves Harborth's conjecture unanswered.

This research will explore and attempt to characterize certain types of coin graphs that can be represented by disks of integral radius. To the best of our knowledge, this has not been considered in the discrete geometry literature.

Theorem 1.1.2 (W. Thurston [16])**.** *A graph G is a coin graph if and only if it is planar.*

That a coin graph is planar is fairly clear. In fact, by connecting the centers of touching disks, we see that each coin graph is a plane graph where each edge is a straight line segment. The converse involves some nontrivial results from the theory of orbifolds (a generalization of manifolds) and will not be presented here. Given this result, we see that any planar graph can be embedded in the plane as a coin graph.

This theorem can also be attributed to Koebe [8] and Andreev [1]. Koebe's original proof covered only the case of fully-triangulated planar graphs. Thurston reduced the proof to the previous theorem of Andreev. Thurston's proof is of the more general case of all planar graphs.

The present research has three parts. In the first part we will explore upper and lower bounds on the maximum number of edges of coin graphs using mostly combinatorial arguments. In the second part we will explore the algebraic equations describing wheel graphs, a fundamental substructure of certain coin graphs. Finally, we will characterize all rational solutions to the coin graph problem for flowers (coin graph representations of wheel graphs) with $n = 3$ and $n = 4$ petals. The parametrization for $n = 4$ petals generalizes, giving large classes of solutions for coin graphs with higher numbers of petals, for all $n > 4$.

1.2 The Unit Coin Graph Problem

The original proof of the unit coin graph problem is due to Harborth [6], where it is shown that the maximum number of edges in a unit coin graph on n vertices is $E(n) = T(n)$. As his published proof of the lower bound is somewhat brief, we will give here a detailed verification that this is the case for all n.

What is remarkable here is that the upper bound, relatively easily obtained by induction, matches the lower bound and is expressible in such simple terms.

Proposition 1.2.1. *The lower bound on the maximum number of edges $E(n)$ of a unit coin graph G on n coins is given by $T(n) = \left\lfloor 3n - \sqrt{12n-3} \right\rfloor$.*

Definition 1.2.2. For any $k \in \mathbb{N}$, the *centered hexagonal number* $\mathrm{ch}(k) := 3k^2 - 3k + 1$ is the number of vertices in the hexagonal-configuration coin graph with k layers. The corresponding number of edges in this graph is then given by $\mathrm{ech}(k) := 3(k-1)(3(k-1)+1) = 3(k-1)(3k-2)$.

Proof. **Lower bound for centered hexagonal numbers:**
Arranging hexagonally, we obtain a coin graph on $n = \mathrm{ch}(k+1)$ vertices and $m = \mathrm{ech}(k+1) = 3k(3k+1)$ edges (see Figure 1.2.) Solving for k, we get:

$$k \quad = \quad \frac{\sqrt{\frac{4n-1}{3}} - 1}{2}.$$

Substituting this expression of k into the expression for m yields the number m of edges of

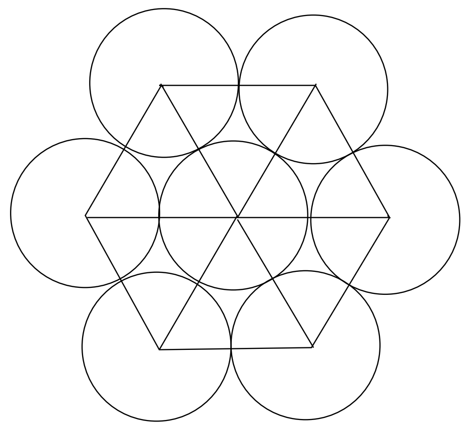

Figure 1.2: When $k = 2$, $n = 7$.

this graph, as a function of n:

$$\begin{aligned} m &= 3\left(\frac{\sqrt{\frac{4n-1}{3}}-1}{2}\right)\left(3\left(\frac{\sqrt{\frac{4n-1}{3}}-1}{2}\right)+1\right) \\ &= 3n-\sqrt{12n-3}. \end{aligned}$$

Thus for centered hexagonal numbers, we have that the maximum number of edges $E(n)$ satisfies $E(n) \geq T(n)$.

Lower bound for $\mathrm{ch}(k) < n < \mathrm{ch}(k+1)$**:**

First note that $\mathrm{ch}(k+1) - \mathrm{ch}(k) = 6k$ and hence $n = \mathrm{ch}(k) + l$ where $l \in \{1, \ldots, 6k-1\}$.

Here the task is to show that for some configuration of coins, the lower bound holds for n in the stated interval. The configuration we use starts with the hexagonal configuration that is optimal for $n = \mathrm{ch}(k)$, then adds each additional vertex around the outside in a spiral fashion. See figure 1.3 for an example where $k = 3$ and $\mathrm{ch}(3) < n < \mathrm{ch}(4)$. We see that the pattern of edges for each vertex added around the hexagon in the $k+1^{st}$ layer is as follows: 1 vertex with 2 edges, $k-2$ vertices with 3 additional edges, 1 vertex with 2 edges, $k-1$ vertices with 3 edges, 1 vertex with 2 edges, $k-1$ vertices with 3 edges, 1 vertex with 2 edges, $k-1$ vertices with 3 edges, 1 vertex with 2 edges, $k-1$ vertices with 3 edges, 1 vertex

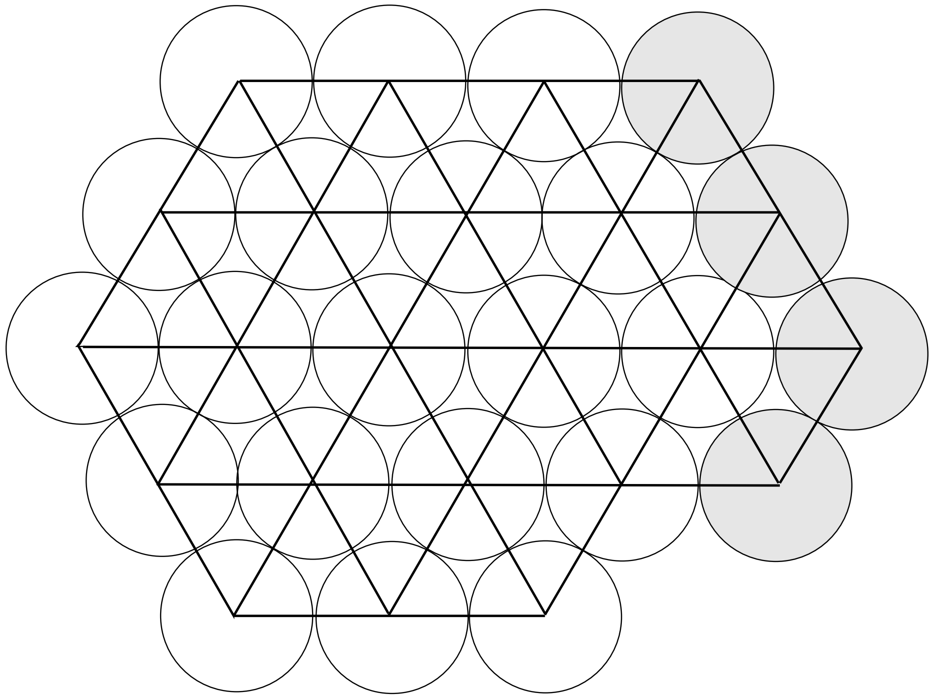

Figure 1.3: Adding additional vertices around the edges of the hexagonal configuration.

with 2 edges, then k vertices with 3 edges each, for a total of $6k$ vertices and an additional $18k-6$ edges, as calculated above. This pattern is illustrated in Figure 1.4.

Hence, for $n = \text{ch}(k) + l$ where $l \in \{1, \ldots, 6k-1\}$ the number of edges obtained in this spiral configuration is as follows:

For $1 \le l \le k-1$ we have a total of $\text{ech}(k) + 2 + 3(l-1)$ edges.

For $ik \le l \le (i+1)k - 1$ where $i \in \{1, 2, 3, 4, 5\}$ we have a total of

$$\text{ech}(k) + 2 + 3(k-2) + 3(i-1)(k-1) + 3(l-ik) \quad = \quad \text{ech}(k) + 3l - i - 1$$

edges.

We now compare the numbers above with $T(n) = \lfloor 3n - \sqrt{12n-3} \rfloor = 3n - \lceil \sqrt{12n-3} \rceil$ in each of the two cases:

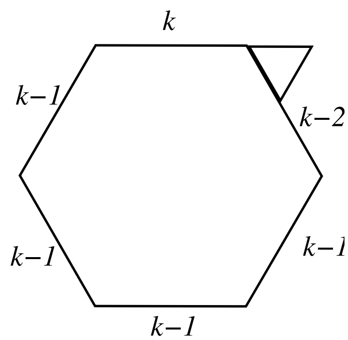

Figure 1.4: The pattern of 3-edge vertices around the hexagon.

For $1 \leq l \leq k-1$ we have for $n = \mathrm{ch}(k) + l$ that

$$\begin{aligned}
(6k-3)^2 + 1 &< 36k^2 - 36k + 21 \\
&= 12(\mathrm{ch}(k) + 1) - 3 \\
&\leq 12n - 3 \\
&\leq 12(\mathrm{ch}(k) + k - 1) - 3 \\
&= 36k^2 - 24k - 3 \\
&< (6k-2)^2.
\end{aligned}$$

Applying the increasing function $x \to \lceil \sqrt{x} \rceil$ throughout the above inequality we obtain

$$\lceil \sqrt{12n-3} \rceil = 6k - 2$$

in this case, and hence

$$\begin{aligned}
3n - \lceil \sqrt{12n-3} \rceil &= 3(\mathrm{ch}(k) + l) - (6k-2) \\
&= 9k^2 - 15k + 5 + 3l \\
&= \mathrm{ech}(k) + 2 + 3(l-1),
\end{aligned}$$

which agrees with the number of edges in the hexagonal spiral coin configuration.

For $ik \leq l \leq (i+1)k-1$ where $i \in \{1,2,3,4,5\}$, we similarly compare the above number of constructed edges with $T(n)$. Here $n = \mathrm{ch}(k) + l$ and hence

$$\begin{aligned} (6k+i-3)^2+1 &= 36k^2+12(i-3)k+(i-3)^2+1 \\ &< 36k^2+12(i-3)k+9 \end{aligned}$$

since $i \in \{1,2,3,4,5\}$ and hence $|i-3| \leq 2$. Also, this last expression

$$\begin{aligned} 36k^2+12(i-3)k+9 &= 12(\mathrm{ch}(k)+ik)-3 \\ &\leq 12n-3 \\ &\leq 12(\mathrm{ch}(k)+(i+1)k-1)-3 \\ &< (6k+i-2)^2. \end{aligned}$$

Again, applying $x \to \lceil\sqrt{x}\rceil$ we obtain

$$\lceil\sqrt{12n-3}\rceil = 6k+i-2$$

and hence

$$\begin{aligned} 3n-\lceil\sqrt{12n-3}\rceil &= 3(\mathrm{ch}(k)+l)-(6k+i-2) \\ &= 9k^2-15k+5+3l-i \\ &= \mathrm{ech}(k)+3l-i-1, \end{aligned}$$

which agrees with the constructed number of edges in this case. This completes the proof of Proposition 1.2.1. $\square$

We now present a proof of the upper bound, a variant of Harborth's proof [6]. This proof uses fewer variables and includes more exposition on the procedure.

Proposition 1.2.3. *The upper bound on the maximum possible number of edges $E(n)$ is given by $E(n) \leq 3n - \sqrt{12n-3}$.*

Proof. First, we can assume that the coin graph is a single connected component. If the coin graph were made up of disconnected components, a coin graph could always be created with more edges by moving the components in the plane until they are touching. In addition, by suitable rotation we may also assume that the coin graph has no cut-points. This will ensure that the infinite face is bounded by a simple cycle.

For a general unit coin graph, consider the edges bounding the infinite face. We will be removing the m coins incident on these edges, where $m \in \mathbb{N}$, so the edges will form an m-gon. We must determine how many edges are removed by this operation. Each of the m outer coins has a corresponding interior angle, which we can call α_i for $i = 1 \ldots m$. For an m-gon, these angles sum to $(m-2)\pi$. It is clear that the angle between two edges in a unit coin graph must be at least $60°$, or $\frac{\pi}{3}$ radians. The maximum number of connections to other coins is therefore given by $\left\lfloor \frac{3\alpha_i}{\pi} \right\rfloor + 1$. The upper bound on the number e' of edges removed with the m exterior coins is given by:

$$
\begin{aligned}
e' &\leq \sum_{i=1}^{m} \left(\left\lfloor \frac{3\alpha_i}{\pi} \right\rfloor + 1 \right) - m \\
&\leq \frac{3}{\pi} \sum_{i=1}^{m} \alpha_i \\
&= \frac{3}{\pi}(m-2)\pi \\
&= 3m - 6.
\end{aligned}
$$

The remaining graph has its number of edges given by $E(n-m)$, which by the inductive hypothesis satisfies $E(n-m) \leq 3(n-m) - \sqrt{12(n-m)-3}$. So by definition, the original graph has the upper bound of its number of edges given by

$$
\begin{aligned}
E(n) &\leq 3m - 6 + E(n-m) \leq 3m - 6 + 3(n-m) - \sqrt{12(n-m)-3} \\
&= 3n - 6 - \sqrt{12(n-m)-3}.
\end{aligned}
$$

We see that if $m \leq \sqrt{12n-3} - 3$ then the right hand side of this expression is less than or equal to $3n - \sqrt{12n-3}$ and thus the proposition holds when $m \leq \sqrt{12n-3} - 3$.

Now assume $m > \sqrt{12n-3} - 3$. This is the case when the number of boundary coins m is "large". For this case, we will show that the claim holds by first adding an additional vertex in the infinite face and connecting it to all m outer coins, and then we add more edges to fully triangulate this plane graph on $n+1$ vertices. (This resulting graph is no longer a unit coin graph.) If the original graph had e edges, this new graph has $e + m'$ edges for some $m' \geq m$, so by Euler's Formula, we have:

$$
e + m' = 3(n+1) - 6 = 3n - 3.
$$

Solving for e and using the assumption that $m' > m > \sqrt{12n-3} - 3$, we obtain:

$$
\begin{aligned}
e &= 3n - 3 - m' \\
&< 3n - 3 - (\sqrt{12n-3} - 3) \\
&= 3n - \sqrt{12n-3}.
\end{aligned}
$$

By induction the claim holds for all values of m. Since $E(n)$ is an integer we have the upper bound for the maximum number of edges in a unit coin graph on n vertices satisfies $E(n) \leq T(n) = \left\lfloor 3n - \sqrt{12n-3} \right\rfloor$. □

Before discussing further, we need to introduce some new terminology.

1.3 Terminology

In what follows, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers.

Definition 1.3.1. A *multiset* $M = (1 \cdot M, \alpha)$ is a generalization of a set where $1 \cdot M$ is the underlying set and $\alpha : 1 \cdot M \to \mathbb{N}$ is a map indicating multiplicity of each element of the multiset.

A multiset $S = (1 \cdot S, \beta)$ is a *submultiset* of M if and only if $1 \cdot S \subseteq 1 \cdot M$ and $\beta(s) \leq \alpha(s)$ for all $s \in 1 \cdot S$.

By $\infty \cdot M$ we mean the multiset obtained by taking infinitely many copies of each element in M (or $1 \cdot M$, if M is a multiset):

$$\infty \cdot M \quad = \quad \bigcup_{x \in 1 \cdot M} \{\infty \cdot x\}.$$

The cardinality of a multiset $M = (1 \cdot M, a)$ is given by

$$|M| \quad = \quad \sum_{x \in 1 \cdot M} a(x).$$

$|1 \cdot M|$ indicates the cardinality of the underlying set.

Definition 1.3.2. For $n \in \mathbb{N}$ and a multiset of positive real numbers R, let $\mathcal{G}(R; n)$ be the collection of coin graphs on n vertices whose radii are a submultiset of the multiset R. The maximum number of edges of a graph in $\mathcal{G}(R; n)$ is denoted $T(R; n)$.

As a matter of convenience, when $|1 \cdot R|$ is small it will be written out. For example, if the coin graph has only two radii 1 and $r = 3 + 2\sqrt{3}$, it is in the set $\mathcal{G}(1, r; n)$, actually meaning $\mathcal{G}(\infty \cdot R; n)$ where the underlying set is $1 \cdot R = \{1, 3+2\sqrt{3}\}$. The maximum number of edges of a graph in this collection is denoted by $T(1, r; n)$.

We introduce some new terminology here, consistent with what can be found in [14].

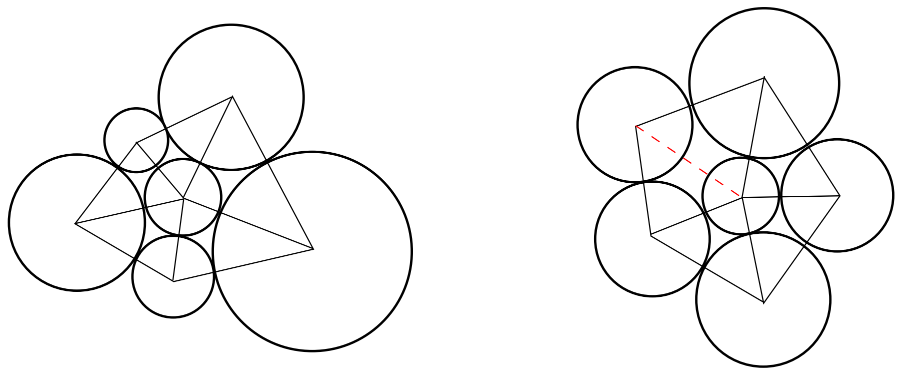

Figure 1.5: Examples of a flower and a non-flower.

Definition 1.3.3. A vertex of a planar graph G is called a *hub* if all the faces it bounds are triangular. If G is a coin graph, the coin representing the hub is called an *eye*.

A neighbor of a hub is always called a *petal*. The closed neighborhood of a hub is called a *wheel*. A wheel with k petals is denoted by W_k. If G is a coin graph, the closed neighborhood of an eye is called a *flower*. So a flower is a coin graph representation of a wheel. An edge from a hub to a petal is called a *spoke*.

A vertex v is *flowered* if v is bounded by only triangular faces.

Definition 1.3.4. A coin graph with no eyes is called *non-flowered*. A coin graph in which every flower is formed from 7 coins of equal radius is called *unit-flowered*.

Examples of a flower and a non-flower are shown in figure 1.5.

1.4 A Special Coin Graph on Two Radii

K_4, the complete graph on 4 vertices, is planar but cannot be represented as a unit coin graph. In fact, it is the smallest complete graph K_n that cannot be represented as a unit coin graph, but the largest complete graph that is planar and thus representable as a coin graph.

If we wish to use two radii to represent K_4, it is easy to show that the radii of the outer three coins must all be equal. In fact, it can be shown that the radii have a ratio of $3 + 2\sqrt{3}$

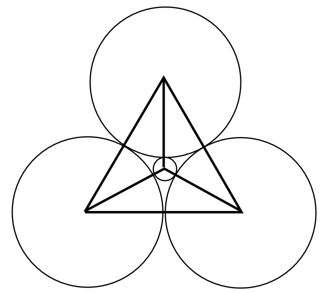

Figure 1.6: The complete graph K_4 represented as a coin graph with two possible radii.

to 1. Figure 1.6 shows K_4 represented as a coin graph as just described.

Observation 1.4.1. Up to scaling, there is only one way to represent K_4 as a coin graph with at most two radii.

For this case with two radii, the general method for finding a lower bound on the maximum number of edges of a coin graph on n vertices begins constructively, as with the unit coin graph problem. The construction used for $\mathcal{G}(1, r; n)$ using radii 1 and $r = 3 + 2\sqrt{3}$ starts with the same hexagonal spiral configuration used for the unit coin graph. The space between any 3 larger coins is then filled with a smaller coin (see Figure 1.7). Recalling the definition of centered hexagonal numbers (Definition 1.2.2,) the construction with k layers in this case will have $n_k := \mathrm{ch}(k) + 6k^2 = 9k^2 - 3k + 1$ vertices and $m_k := \mathrm{ech}(k) + 3(6k^2) = 3(k-1)(3k-2) + 3(6k^2) = 27k^2 - 15k + 6$ edges. This yields the following:

Proposition 1.4.2. *The maximum number of edges $T(1, r; n)$ of a coin graph G on n vertices of radii $r_1 = 3 + 2\sqrt{3}$ and $r_2 = 1$ satisfies $T(1, r; n) \geq \lfloor 3n + 2 - \sqrt{4n-3} \rfloor$.*

Proof. **First case: $n = n_k$ for some $k \geq 1$:**

Arranging hexagonally as explained above, we obtain a coin graph on n_k vertices and m_k edges. Solving n_k for k, we have:

$$k = \frac{\sqrt{4n_k - 3} + 1}{6}.$$

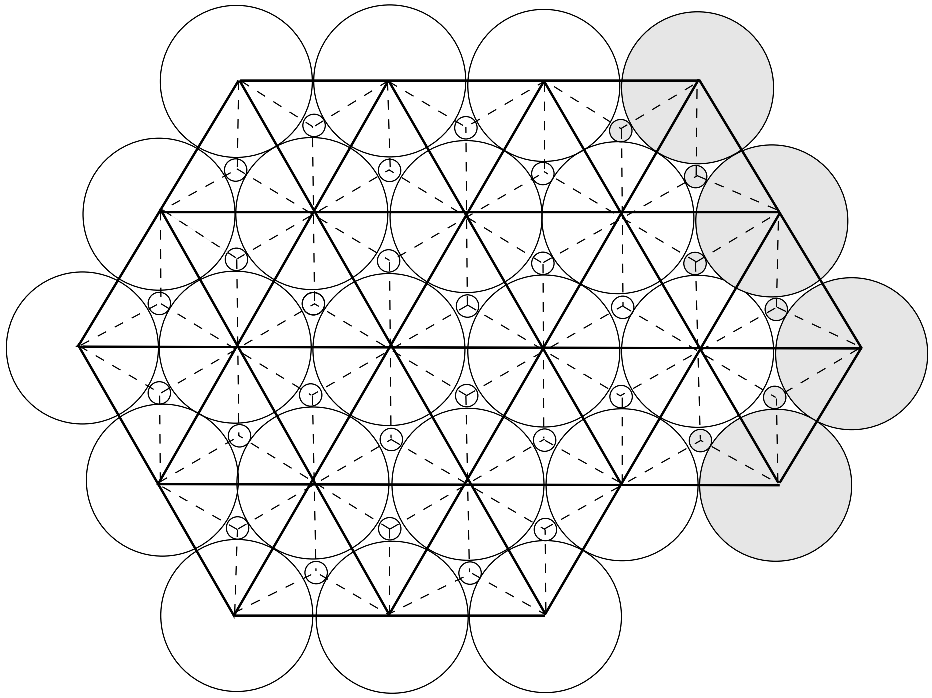

Figure 1.7: The modified hexagonal configuration for the case with 2 radii.

Substituting this expression for k into the expression for m_k yields the number of edges of this graph as a function of $n = n_k$:

$$
\begin{aligned}
m_k &= 3\left(\frac{\sqrt{4n-3}+1}{6} - 1\right)\left(3\left(\frac{\sqrt{4n-3}+1}{6}\right) - 2\right) + 18\left(\frac{\sqrt{4n-3}+1}{6}\right)^2 \\
&= 3n + 2 - \sqrt{4n-3}.
\end{aligned}
$$

Thus for $n = n_k = 9k^2 - 3k + 1$, we have that the maximum number of edges $E(n)$ satisfies $T(1, r; n) \geq 3n + 2 - \sqrt{4n-3} = f(n)$.

Second case: $n_{k-1} < n < n_k$**:**

Now the task is to show that for some configuration of coins, the lower bound holds for all those mentioned n. The configuration we use starts with the modified hexagonal configuration that is optimal for $n_{k-1} = 9k^2 - 21k + 13$, then adds each additional vertex in a layer around the outside, filling in small coins whenever possible. Since $n_k - n_{k-1} =$

$9k^2 - 3k + 1 - (9k^2 - 21k + 13) = 18k - 12$, a total of $18k - 12$ coins are added in each layer: $6k$ coins of radius $3 + 2\sqrt{3}$ as in the unit coin graph case, and $12(k-1)$ smaller coins of radius 1.

Let $f(n) = \lfloor 3n + 2 - \sqrt{4n-3} \rfloor$ and $C(n)$ be the number of edges in the construction. Note that $C(n) = C(n_{k-1} + l)$ where l is the number of coins added around the outside of the previous configuration. As in the proof of Proposition 1.2.1, we consider two cases:

For $n = n_{k-1} + l$ where $l \in \{1, \ldots, 18k - 13\}$ the number of edges obtained in this spiral configuration is as follows:

For $1 \leq l \leq 3(k-1)$, the number of edges in the construction is

$$C(n_{k-1} + l) = m_{k-1} + 3l - 1 = 27k^2 - 69k + 47 + 3l.$$

Similarly, for the remaining cases where $i(3k-2) \leq l \leq (i+1)(3k-2) - 1$ for $i \in \{1, 2, 3, 4, 5\}$, the number of edges in the construction is

$$C(n_{k-1} + l) = m_{k-1} + 3l - (i+1) = 27k^2 - 69k + 47 + 3l - i.$$

We now compare the numbers above with $f(n) = \lfloor 3n + 2 - \sqrt{4n-3} \rfloor = 3n + 2 - \lceil \sqrt{4n-3} \rceil$ in each of the two cases:

For $1 \leq l \leq 3(k-1)$ we have that

$$\begin{aligned} 4(n_{k-1} + l) - 3 &\geq (6k-7)^2 + 1 \\ \left\lceil \sqrt{4(n_{k-1} + l) - 3} \right\rceil &\geq 6k - 6 \\ \left\lceil \sqrt{36k^2 - 84k + 49 + 4l} \right\rceil &\geq 6k - 6, \end{aligned}$$

so for $n = n_{k-1} + l$,

$$\begin{aligned} f(n_{k-1}+l) &= 3(n_{k-1}+l) + 2 - \left\lceil \sqrt{4(n_{k-1}+l)-3} \right\rceil \\ &= 27k^2 - 63k + 41 + 3l - \left\lceil \sqrt{36k^2 - 84k + 49 + 4l} \right\rceil \\ &\le 27k^2 - 69k + 47 + 3l \\ &= C(n_{k-1}+l). \end{aligned}$$

For $i(3k-2) \le l \le (i+1)(3k-2) - 1$ where $i \in \{1,2,3,4,5\}$ we similarly compare the above number of constructed edges with $f(n)$. Here we have

$$\begin{aligned} 4(n_{k-1}+l) - 3 &\ge (6k-7+i)^2 + 1 \\ \left\lceil \sqrt{4(n_{k-1}+l)-3} \right\rceil &\ge 6k - 6 + i \\ \left\lceil \sqrt{36k^2 - 84k + 49 + 4l} \right\rceil &\ge 6k - 6 + i, \end{aligned}$$

so for $n = n_{k-1} + l$,

$$\begin{aligned} f(n_k+l) &= 3(n_{k-1}+l) + 2 - \left\lceil \sqrt{4(n_{k-1}+l)-3} \right\rceil \\ &= 27k^2 - 63k + 41 + 3l - \left\lceil \sqrt{36k^2 - 84k + 49 + 4l} \right\rceil \\ &\le 27k^2 - 69k + 47 + 3l - i \\ &= C(n_{k-1}+l). \end{aligned}$$

Thus the number of edges in the construction is greater than or equal to the number of edges given by the function $f(n)$ for all cases, we have that for any n the maximum number of edges $T(1,r;n)$ satisfies $T(1,r;n) \ge f(n) = \left\lfloor 3n + 2 - \sqrt{4n-3} \right\rfloor$.

$\square$

An asymptotic consequence of Proposition 1.4.2 can be stated in terms of the limsup:

Corollary 1.4.3. *The maximum number of edges $T(1, r; n)$ of a coin graph $G \in \mathcal{G}(1, r; n)$ on n vertices of radii 1 and $r = 3 + 2\sqrt{3}$ satisfies*

$$\limsup_{n\to\infty} \left\{ \frac{3n - T(1, r; n)}{\sqrt{n}} \right\} \leq 2.$$

We do believe that asymptotically the limsup is tight:

Conjecture 1.4.4. The maximum number of edges $T(1, r; n)$ of a coin graph $G \in \mathcal{G}(1, r; n)$ on n vertices of radii 1 and $r = 3 + 2\sqrt{3}$ satisfies

$$\lim_{n\to\infty} \left\{ \frac{3n - T(1, r; n)}{\sqrt{n}} \right\} = 2.$$

However, obtaining an upper bound for $T(1, r; n)$ is not as straightforward as in the unit coin graph problem. The direct inductive approach of Harborth's proof [6] for determining the maximum number of edges of a unit coin graph on n vertices cannot be used in this case with two or more radii to determine $T(1, r; n)$. The inductive step assumes that removing the vertices bounding the infinite face will leave a configuration that is still maximal, which is not guaranteed in the general case. More is needed.

Chapter 2: On the Maximum Number of Edges of Certain Plane Graphs

2.1 Motivation

In this chapter, we will prove a general result which gives the maximum number of edges in a plane graph on n vertices, where each vertex bounds some l-gon for $l \geq k$. The special case where $k = 4$ gives rise to a conjecture about the maximum number of edges of a non-flowered coin graph.

Recall Definitions 1.3.2 and 1.3.4.

Definition 2.1.1. A collection $\mathcal{G}(R; n)$ is *non-flowerable* if every coin graph in $\mathcal{G}(R; n)$ is non-flowered. Denote by $\mathcal{G}_0(n)$ the collection of all non-flowered coin graphs on n vertices:

$$\mathcal{G}_0(n) = \bigcup_{\substack{R:\mathcal{G}(R;n) \text{ is} \\ \text{non-flowerable}}} \mathcal{G}(R; n).$$

For $n \geq 3$, let $T_0(n) = \max\{|E(G)| : G \in \mathcal{G}_0(n)\}$.

$\mathcal{G}_0(n)$ is a collection of coins from which no flower can be formed. In particular, the multiset R does not contain seven coins of the same size.

Definition 2.1.2. A collection $\mathcal{G}(R; n)$ is *unit-flowerable* if every coin graph in $\mathcal{G}(R; n)$ is unit-flowered. Denote by $\mathcal{G}_1(n)$ the collection of all unit-flowered coin graphs on n vertices:

$$\mathcal{G}_1(n) = \bigcup_{\substack{R:\mathcal{G}(R;n) \text{ is} \\ \text{unit-flowerable}}} \mathcal{G}(R; n).$$

For $n \geq 3$, let $T_1(n) = \max\{|E(G)| : G \in \mathcal{G}_1(n)\}$.

Similarly if $\mathcal{G}(R;n)$ is unit-flowerable, this corresponds to having a given collection of coins on the table such that the only way to form a flower is to pick seven coins of the same radius.

Conjecture 2.1.3. The maximum number of edges $T_1(n)$ of a unit-flowered coin graph $G \in \mathcal{G}$ on n vertices is given by $T_1(n) = \lfloor 3n - \sqrt{12n-3} \rfloor$.

Observation 2.1.4. If a coin graph G is non-flowered, then every vertex must be bounded by an l-gon for $l \geq 4$.

Hence, a natural question is whether we can use this information about the structure of this type of coin graph to answer the question of the maximum number of edges.

2.2 Plane graphs where each vertex bounds an l-gon

For a moment, let us step back from coin graphs and consider plane graphs in general. The following theorem applies to all plane graphs, and therefore by Thurston's Theorem (Theorem 1.1.2) applies to all coin graphs. It generalizes the conditions in Observation 2.1.4.

Theorem 2.2.1. *Let $k \geq 4$ be fixed. The maximum number of edges $E_k(n)$ of a plane graph on n vertices, where each vertex bounds some l-gon for $l \geq k$, is given by*

$$E_k(n) = T_k(n) := \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor - \alpha$$

where

$$\alpha = \begin{cases} 0 & \text{if } n \equiv k-1 \pmod{k} \\ \left\lfloor 2 - \frac{6}{k} \right\rfloor & \text{if } n \equiv k-2 \pmod{k} \\ \left\lfloor \frac{3\beta}{k} \right\rfloor & \text{if } n \equiv \beta \pmod{k} \text{ for } 0 \leq \beta \leq k-3. \end{cases}$$

We will show that $T_k(n)$ is both a lower bound and an upper bound for $E_k(n)$. We first prove $T_k(n)$ is a lower bound by explicit construction. We consider each of the three cases, $n \equiv k-1, k-2, \beta \pmod{k}$ where $0 \leq \beta \leq k-3$, separately, since each case has a unique construction.

Proof. **The lower bound**

First case: $n \equiv k-1 \pmod{k}$**:**

Let $n = k(j+1) - 1$ and form $j-1$ disjoint copies of C_k and one copy of C_{2k-1} in the plane, no cycle containing another cycle, consisting of n edges altogether. We need $3(j-1)$ edges to connect the cycles into one connected component such that (i) the infinite face is bounded by a simple n-cycle and (ii) the internal faces of this n-cycle other than the C_ks and the C_{2k-1} are triangular. Then we add $n-3$ edges to fully triangulate the infinite face. Now, note that two additional edges can be added to the interior of the cycle C_{2k-1} to create 3 regions, 2 bounded by k-gons and one by a triangle such that every vertex is bounded by a k-gon. Add these additional edges between appropriate vertices of the cycle C_{2k-1}. The total number of edges is then given by

$$\begin{aligned} n + 3(j-1) + (n-3) + 2 &= 2n + 3j + 2 - 6 \\ &= 2n + 3(j+1) - 6 - 1 \\ &= 2n + 3\left(\frac{n+1}{k}\right) - 6 - 1 \\ &= \frac{(2k+3)n}{k} - 6 - \left(1 - \frac{3}{k}\right) \\ &= \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor. \end{aligned}$$

Second case: $n \equiv k-2 \pmod{k}$**:**

Let $n = k(j+1) - 2$ and form $j-1$ disjoint copies of C_k and one copy of C_{2k-2} in the plane, no cycle containing another cycle, consisting of n edges altogether. Again, $3(j-1)$ edges are needed to connect the cycles into one connected component such that (i) the infinite face is bounded by a simple n-cycle and (ii) the internal faces of this n-cycle other than the C_ks and the C_{2k-2} are triangular. Then add $n-3$ edges that fully triangulate the infinite face. Now, note that one additional edge can be added to the interior of the cycle C_{2k-2} to create 2 regions bounded by k-gons. Add this additional edge between appropriate vertices of the cycle C_{2k-2}. The number of edges is given by

$$\begin{aligned} n+3(j-1)+(n-3)+1 &= 2n+3j+1-6 \\ &= 2n+3(j+1)-6-2 \\ &= 2n+3\left(\frac{n+2}{k}\right)-6-2 \\ &= \frac{(2k+3)n}{k}-6-\left(2-\frac{6}{k}\right). \end{aligned}$$

Since for any real numbers x, y with $x-y$ a positive integer we have $x-y=\lfloor x\rfloor-\lfloor y\rfloor$, then this last expression equals $\left\lfloor \frac{(2k+3)n}{k}-6\right\rfloor-\lfloor 2-\frac{6}{k}\rfloor$.

Third case: $n \equiv \beta \pmod{k}$ **for** $0 \leq \beta \leq k-3$**:**

Let $n = kj+\beta$, where $\beta \leq k-3$, and form $j-1$ disjoint copies of C_k and one copy of $C_{k+\beta}$ in the plane, no cycle containing another cycle, consisting of n edges altogether. Once again, $3(j-1)$ edges are needed to connect the cycles into one connected component such that (i) the infinite face is bounded by a simple n-cycle and (ii) the internal faces of this n-cycle other than the C_ks and the $C_{k+\beta}$ are triangular. Then add $n-3$ edges that fully triangulate the infinite face. The number of edges is given by

$$\begin{aligned}
n+3(j-1)+(n-3) &= 2n+3j-6 \\
&= 2n+3\left(\frac{n-\beta}{k}\right)-6 \\
&= 2n+3\left(\frac{n}{k}-\frac{\beta}{k}\right)-6 \\
&= \frac{(2k+3)n}{k}-6-\frac{3\beta}{k} \\
&= \left\lfloor\frac{(2k+3)n}{k}-6\right\rfloor-\left\lfloor\frac{3\beta}{k}\right\rfloor,
\end{aligned}$$

the last step just as in the previous case.

These three cases show that the mentioned bound $T_k(n)$ can always be reached.

The upper bound

We will derive the upper bound using integer programming.

Assume we have a plane graph G on n vertices with the property mentioned in the theorem. The number of edges is m and the number of faces is f. Form a new graph G' by adding a vertex inside each l-gon, where $l \geq k$ and connect that vertex with all the vertices bounding the l-gon. Let n', m', and f' be the number of vertices, edges, and faces of G'. Note that G' is planar and fully triangulated. For $i \in \{3, \ldots, k-1\}$, let f_i denote the number of i-sided faces of G and f_k be the number of all l-sided faces where $l \geq k$. Then $f = f_3 + f_4 + \cdots + f_{k-1} + f_k$. By assumption we have $n' = n + f_4 + \cdots + f_{k-1} + f_k$ and $m' = 3n' - 6$.

Let d be the sum of the degrees of all the vertices that were added above, so d also equals the number of edges added to G to obtain G'. Hence $m' = m + d = 3(n + f_4 + \cdots + f_k) - 6$, so $m = 3n - 6 - (d - 3(f_4 + \cdots + f_{k-1} + f_k))$. Note that $d = d_4 + d_5 + \cdots + d_{k-1} + d_k$ where for each $i \in \{4, \ldots, k-1\}$, d_i is the sum of the degrees of the vertices of degree i added to

G and d_k is the sum of degrees of vertices of degree greater than or equal to k added to G. Therefore we have $d_i = if_i$ for each $i \in \{4, \ldots, k-1\}$ and so $d = 4f_4 + \cdots + (k-1)f_{k-1} + d_k$ and hence

$$m = 3n - 6 - (f_4 + 2f_5 + \cdots + (k-3)f_{k-1} + d_k - 3f_k).$$

Note that m is maximized if $f_4 + 2f_5 + \cdots + (k-3)f_{k-1} + d_k - 3f_k$ is minimized. Since the conditions are (1) $n \leq d_k$, (2) $f_i \geq 0$ for $i \in \{4, \ldots, k\}$, and (3) $kf_k \leq d_k$, we can simplify this optimization problem by setting $f_i = 0$ for $i = 4, \ldots, k-1$ and the problem reduces to minimizing the value of $d_k - 3f_k$ over nonnegative integers, given the constraints $d_k \geq n$ and $kf_k \leq d_k$.

Lemma 2.2.2. *Let $k \geq 4$ and $n \geq k$. If*

$$\mu(n;k) := \min\{x - 3y : x, y \in \mathbb{N} \cup \{0\}, x \geq n, ky \leq x\}$$

then

$$\mu(n;k) \quad = \quad n + \gamma - 3\left\lfloor \frac{n+\gamma}{k} \right\rfloor$$

where

$$\gamma \quad = \quad \begin{cases} 1 & \textit{if } n \equiv k-1 \pmod{k} \\ 2 & \textit{if } n \equiv k-2 \pmod{k} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using integer programming, we see that the integer point minimizing the function $x - 3y = (1, -3) \cdot (x, y)$ will either be $x = n$, $y = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv i \pmod{k}$ where $i = 0, 1, \ldots, k-3$, or $x = k\left\lceil \frac{n}{k} \right\rceil$, $y = \left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{x}{k} \right\rfloor$ otherwise. See Figures 2.1 and 2.2 for examples, the pattern of which remains the same for all other values of n and k. Using the

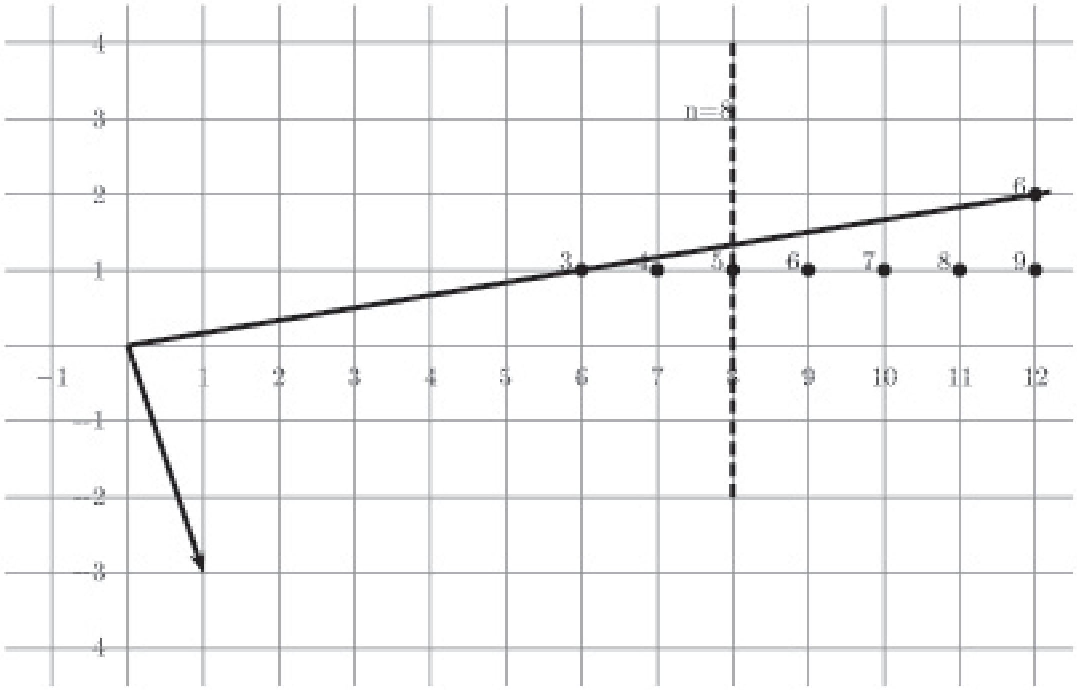

Figure 2.1: When $k = 6$ and $n = 8$, the function is minimized at $x = 8$, $y = 1$.

definition of γ from the statement of the lemma, we can write $x = n + \gamma$ as the x-value that will always minimize the function. Then we have $y = \left\lfloor \frac{x+\gamma}{k} \right\rfloor$ as the y-value that will always minimize the function. □

We obtain by this lemma that $d_k - 3f_k$ is minimized when $d_k = n + \gamma$ and $f_k = \left\lfloor \frac{n+\gamma}{k} \right\rfloor$, hence:

$$\begin{aligned} m &= 3n - 6 - (d_k - 3f_k) \\ &\geq 3n - 6 - n - \gamma + 3 \left\lfloor \frac{n+\gamma}{k} \right\rfloor \\ &= 2n - 6 + 3 \left\lfloor \frac{n+\gamma}{k} \right\rfloor - \gamma. \end{aligned}$$

If $n \equiv k-1 \pmod{k}$, then $\gamma = 1$ and

$$\begin{aligned} m &\geq 2n - 6 + 3\left\lfloor \frac{n+1}{k} \right\rfloor - 1 \\ &= 2n - 6 + 3\left(\frac{n+1}{k}\right) - 1 \\ &= \frac{(2k+3)n}{k} - 6 - \left(1 - \frac{3}{k}\right) \\ &= \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor. \end{aligned}$$

If $n \equiv k-2 \pmod{k}$ then $\gamma = 2$ and

$$\begin{aligned} m &\geq 2n - 6 + 3\left\lfloor \frac{n+2}{k} \right\rfloor - 2 \\ &= 2n - 6 + 3\left(\frac{n+2}{k}\right) - 2 \\ &= \frac{(2k+3)n}{k} - 6 + \left(2 - \frac{6}{k}\right) \\ &= \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor + \left\lfloor 2 - \frac{6}{k} \right\rfloor. \end{aligned}$$

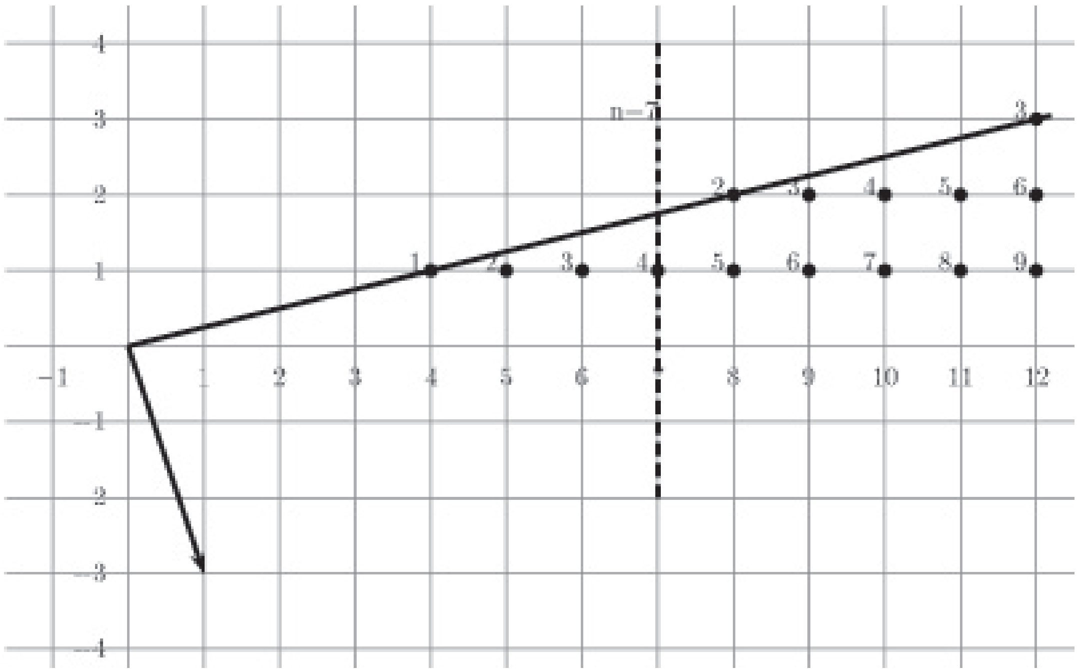

Figure 2.2: When $k = 4$ and $n = 7$, the function is minimized at $x = 8$, $y = 2$.

If $n \equiv \beta$ where $\beta \in \{0, 1, \ldots, k-3\}$ then $\gamma = 0$ and

$$\begin{aligned} m &\geq 2n - 6 + 3\left\lfloor \frac{n}{k} \right\rfloor \\ &= 2n - 6 + \left(\frac{n-\beta}{k} \right) \\ &= \frac{(2k+3)n}{k} - 6 - \left(\frac{3\beta}{k} \right) \\ &= \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor - \left\lfloor \frac{3\beta}{k} \right\rfloor, \end{aligned}$$

all as stated in the theorem. $\square$

In the especially interesting case where $k = 4$, the discrepancy term $\alpha \in \{0, \lfloor 2 - \frac{6}{k} \rfloor, \lfloor \frac{3\beta}{k} \rfloor\}$ for $0 \leq \beta \leq k - 3$ will be 0 in all cases, and hence we obtain the following:

Corollary 2.2.3. *The maximum number of edges $E_4(n)$ of a plane graph on n vertices, where each vertex bounds some l-gon for $l \geq 4$, is given by*

$$E_4(n) \quad = \quad \left\lfloor \frac{11}{4}n - 6 \right\rfloor.$$

By Observation 2.1.4 we have in particular the following

Corollary 2.2.4. *The maximum number of edges $T_0(n)$ of a non-flowered coin graph $G \in \mathcal{G}_0(n)$ on n vertices satisfies*

$$E_4(n) = T_0(n) \leq \left\lfloor \frac{11}{4}n - 6 \right\rfloor.$$

Whether or not the bound $E_4(n)$ can be reached for a graph $G \in \mathcal{G}_0(n)$ is currently unknown.

Conjecture 2.2.5. The maximum number of edges $T_0(n)$ of a non-flowered coin graph $G \in \mathcal{G}_0(n)$ on n vertices is given by

$$T_0(n) = \left\lfloor \frac{11}{4}n - 6 \right\rfloor.$$

We know $E_4(n)$ is a tight bound for general plane graphs, because the construction used in the proof is a plane graph that achieves the bound, so by Thurston's Theorem, there is some embedding of this planar graph as a coin graph. However, we do not know if that coin graph is in $\mathcal{G}_0(n)$, i.e. whether the coins in that coin graph are non-flowerable, or whether there exists a graph in $\mathcal{G}_0(n)$ achieving this bound. We conjecture that there does exist such a coin graph: a construction that achieves this bound, the radii of which belong to the collection of non-flowerable coin graphs. We suspect that a proof of Conjecture 2.2.5 might use some complex analysis together with appropriate inversion about the unit circle.

Chapter 3: Algebraic Equations Describing the Wheel Graph

3.1 Deriving Algebraic Equations to Describe the Wheel Graph

In this chapter, we first investigate equations describing the cosines of the internal angles of a flower in terms of what rational radii could satisfy them. We show that for each n-petaled flower, there is one rational equation that must be satisfied. These equations correspond to polynomial equations. We find the smallest such polynomial equations describing this relationship. Using Galois theory, we then show that these polynomials are symmetric and irreducible. We also establish a recursion that these minimal polynomials satisfy.

Every flower imposes a relation on the radii of its coins. Hence, a multiset R of radii is non-flowerable if the radii do not satisfy any of the relations imposed by any flower. Questions like "Is a collection of coins with distinct integer radii non-flowerable?" are legitimate. Hence it is worthwhile to study these relations on their own.

Assume we have n disks of radii $r_1, \ldots, r_n$. We can view the radii r_i as variables and consider the equations that determine flowering conditions in terms of the r_i. Each flower with k petals, $k \in \{3, \ldots, n-1\}$, is determined by one equation. In order for the coin graphs with any k of these n radii to be non-flowerable, we would like the $r_1, \ldots, r_n$ to avoid satisfying all these equations. We can show there are finitely many equations by counting them in the following way:

Proposition 3.1.1. *The number of equations in $r_1, \ldots, r_n$ that determine some flowering condition is*

$$\mathrm{fl}(n) = \sum_{k=3}^{n-1} \left(n \binom{n-1}{k} \frac{(k-1)!}{2} \right).$$

Proof. For each $k \in \{3, \dots, n-1\}$, the radius of the eye can be chosen in n ways. The k petals are then chosen from the remaining $n-1$ radii: $\binom{n-1}{k}$. The petals are arranged around the eye in $\frac{(k-1)!}{2}$ possible ways (arrangements which differ only by clockwise versus counterclockwise orientation are considered identical and are not counted twice.) $\square$

We will show that each of these $\mathrm{fl}(n)$ equations corresponds to an element F of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. For each pair of radii r_i and r_{i+1} of consecutive petals around an eye of radius r we obtain a triangle with sides of length $r + r_i$, $r + r_{i+1}$, and $r_i + r_{i+1}$ and the angle θ_i at the eye is given by

$$\theta_i \quad = \quad \arccos\left(\frac{(r+r_i)^2 + (r+r_{i+1})^2 - (r_i + r_{i+1})^2}{2(r+r_i)(r+r_{i+1})}\right).$$

The equation that determines a flower with petals of radii $r_1, \dots, r_k$ is

$$\sum_{i=1}^{k} \theta_i = 2\pi. \tag{3.1}$$

For $G \subseteq S_k$, a polynomial f is *G-symmetric* if $f(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ for all $\sigma \in G$. We see that (3.1) is a D_k-symmetric function in terms of $r_1/r, \dots, r_k/r$, where D_k is the dihedral group of symmetries on the regular polygon with k sides. In [13] it is shown that for *reflection groups* like the dihedral group D_k there is a basis of polynomials just like the elementary symmetric functions for the symmetric group S_k.

Claim 3.1.2. *Let $k \in \{1, \dots, n\}$ denote the number of petals.*

1. *If $x_i = \cos\theta_i$ for each $i \in \{1, \dots, k\}$, then (3.1) corresponds to a symmetric polynomial $f \in \mathbb{Q}[x_1, \dots, x_k]$.*

2. *If the eye has radius $r = 1$ so $\theta_i = \theta_i(1, r_i, r_{i+1})$, then (3.1) corresponds to a D_k-symmetric polynomial $g \in \mathbb{Q}[r_1, \dots, r_k]$. In particular, for general radius r of the*

eye (replacing r_i with r_i/r), if $d = \deg(g)$, which we define as the sum degree, then $r^d g\left(\frac{r_1}{r}, \ldots, \frac{r_k}{r}\right) \in \mathbb{Q}[r, r_1, \ldots, r_k]$ is a homogeneous element and

$$r^d g\left(\frac{r_1}{r}, \ldots, \frac{r_k}{r}\right) = \sum_{i=0}^{d} g_i r^i \in \mathbb{Q}[r_1, \ldots, r_k][r],$$

where each $g_i \in \mathbb{Q}[r_1, \ldots, r_k]$ is a D_k-symmetric polynomial.

Although intuitively clear, we will in what follows demonstrate this claim. To obtain a symmetric function $f = f(x_1, \ldots, x_k)$ we will take the cosine of both sides of (3.1). For this, we will need generalized addition formulae for cosines. In order to prove this, we will show the generalized addition formulae for both sine and cosine in the following technical lemma:

Lemma 3.1.3. *For $n \geq 1$ we have the following generalized addition formulae for* cos *and* sin*:*

$$\cos\left(\sum_{i=1}^{n} \theta_i\right) = \sum_{2^{n-1} \text{ terms}} \pm \mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_n),$$

where the sum is taken over the 2^{n-1} possible terms where (i) each cs*-function represents either* sin *or* cos *and (ii) each term has an even number $2e$ of* sin*-functions and the sign of the term is given by $(-1)^e$.*

Similarly for sin *we have*

$$\sin\left(\sum_{i=1}^{n} \theta_i\right) = \sum_{2^{n-1} \text{ terms}} \pm \mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_n),$$

where the sum is taken over the 2^{n-1} possible terms where (i) each cs*-function represents either* sin *or* cos *and (ii) each term has an odd number $2e+1$ of* sin*-functions and the sign of the term is given by $(-1)^e$.*

Proof. For $n = 1$ the claim is trivial, and for $n = 2$ the addition rules for sine and cosine are well-known and satisfy the claim.

Let $n > 2$ and assume the inductive hypothesis. Then we have

$$\begin{aligned}
\cos\left(\sum_{i=1}^{n}\theta_i\right) &= \cos\left(\left(\sum_{i=1}^{n-1}\theta_i\right) + \theta_n\right) \\
&= \cos\left(\sum_{i=1}^{n-1}\theta_i\right)\cos\theta_n - \sin\left(\sum_{i=1}^{n-1}\theta_i\right)\sin\theta_n \\
&= \left(\sum_{2^{n-2}\ terms} \pm\mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_{n-1})\right)\cos\theta_n \\
&\quad - \left(\sum_{2^{n-2}\ terms} \pm\mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_{n-1})\right)\sin\theta_n,
\end{aligned}$$

where in the first summation, there are an even number $2e$ of sin-functions and the sign of the term is given by $(-1)^e$, and in the second summation, there are an odd number $2e+1$ of sin-functions and the sign of the term is given by $(-1)^e$, by assumption. Then we have

$$\cos\left(\sum_{i=1}^{n}\theta_i\right) = \sum_{2^{n-1}\ terms} \pm\mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_n),$$

where the summation has the properties mentioned above, which gives the statement of the proposition for cosine. Similarly for sine we have

$$
\begin{aligned}
\sin\left(\sum_{i=1}^{n}\theta_i\right) &= \sin\left(\left(\sum_{i=1}^{n-1}\theta_i\right)+\theta_n\right) \\
&= \cos\left(\sum_{i=1}^{n-1}\theta_i\right)\sin\theta_n + \sin\left(\sum_{i=1}^{n-1}\theta_i\right)\cos\theta_n \\
&= \left(\sum_{2^{n-2}\ terms} \pm \mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_{n-1})\right)\sin\theta_n \\
&\quad + \left(\sum_{2^{n-2}\ terms} \pm \mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_{n-1})\right)\cos\theta_n,
\end{aligned}
$$

where in the first summation, there are an even number $2e$ of sin-functions and the sign of the term is given by $(-1)^e$, and in the second summation, there are an odd number $2e+1$ of sin-functions and the sign of the term is given by $(-1)^e$, by assumption. Then we have

$$
\sin\left(\sum_{i=1}^{n}\theta_i\right) = \sum_{2^{n-1}\ terms} \pm \mathrm{cs}(\theta_1)\mathrm{cs}(\theta_2)\cdots\mathrm{cs}(\theta_n),
$$

where the summation has the properties mentioned above, which gives the statement of the proposition for sine.

$\square$

Alternately, we can use the relation $e^{i\theta} = \cos\theta + i\sin\theta$ to obtain these addition formulae by taking the real and imaginary parts of $e^{i(\theta_1+\ldots+\theta_n)} = e^{i\theta_1}\cdots e^{i\theta_n}$.

For example, the familiar sum rules for sin and cos follow this pattern:

$$
\begin{aligned}
\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\
\sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.
\end{aligned}
$$

The sum rules for 3 and 4 terms can then be obtained similarly from the formula:

$$\begin{aligned}
\cos(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3 \\
&- \sin\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 \\
\sin(\theta_1+\theta_2+\theta_3) &= \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_2\ \cos\theta_3 \\
&+ \cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3
\end{aligned}$$

$$\begin{aligned}
\cos(\theta_1+\theta_2+\theta_3+\theta_4) &= \cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_1\cos\theta_2\sin\theta_3\sin\theta_4 \\
&- \cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 - \sin\theta_1\sin\theta_2\cos\theta_3\cos\theta_4 \\
&- \cos\theta_1\sin\theta_2\cos\theta_3\sin\theta_4 - \sin\theta_1\cos\theta_2\sin\theta_3\cos\theta_4 \\
&- \sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 + \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4
\end{aligned}$$

$$\begin{aligned}
\sin(\theta_1+\theta_2+\theta_3+\theta_4) &= \sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4 + \cos\theta_1\sin\theta_2\ \cos\theta_3\cos\theta_4 \\
&+ \cos\theta_1\cos\theta_2\sin\theta_3\cos\theta_4 + \cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 \\
&- \sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 - \sin\theta_1\sin\theta_2\cos\theta_3\sin\theta_4 \\
&- \sin\theta_1\cos\theta_2\sin\theta_3\sin\theta_4 - \cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4.
\end{aligned}$$

Letting $x_i = \cos\theta_i$ for each i, then $y_i = \sin\theta_i$ satisfies the equation $x_i^2 + y_i^2 = 1$ and hence $y_i = \pm\sqrt{1-x_i^2}$. The geometric properties of the coin graph determine that for the interior angles, $\theta_i < \pi$ and so $\sin\theta_i > 0$ and we can disregard the negative root, letting $y_i = \sqrt{1-x_i^2}$.

Definition 3.1.4. We define the algebraic equations obtained by taking the sine or cosine of (3.1) by

$$\begin{aligned}
\mathrm{EC}_n(x_1,\ldots,x_n) &= \cos\left(\sum_{i=1}^{n}\theta_i\right),\\
\mathrm{ES}_n(x_1,\ldots,x_n) &= \sin\left(\sum_{i=1}^{n}\theta_i\right).
\end{aligned}$$

Each expression is in terms of the variables $x_1,\ldots,x_n$.

Example 3.1.5.

$$\begin{aligned}
\mathrm{EC}_1(x_1) &= x_1\\
\mathrm{ES}_1(x_1) &= y_1 = \sqrt{1-x_1^2}\\
\mathrm{EC}_2(x_1,x_2) &= x_1x_2 - y_1y_2 = x_1x_2 - \sqrt{1-x_1^2}\sqrt{1-x_2^2}\\
\mathrm{ES}_2(x_1,x_2) &= y_1x_2 + x_1y_2 = x_2\sqrt{1-x_1^2} + x_1\sqrt{1-x_2^2}\\
\mathrm{EC}_3(x_1,x_2,x_3) &= x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3\\
&= x_1x_2x_3 - x_1\sqrt{1-x_2^2}\sqrt{1-x_3^2}\\
&- x_2\sqrt{1-x_1^2}\sqrt{1-x_3^2} - x_3\sqrt{1-x_1^2}\sqrt{1-x_2^2}\\
\mathrm{ES}_3(x_1+x_2+x_3) &= y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 - y_1y_2y_3\\
&= x_2x_3\sqrt{1-x_1^2} + x_1x_3\sqrt{1-x_2^2}\\
&+ x_1x_2\sqrt{1-x_3^2} - \sqrt{1-x_1^2}\sqrt{1-x_2^2}\sqrt{1-x_3^2}.
\end{aligned}$$

Lemma 3.1.6. *For each $i \in \{1, \ldots, n\}$ we have*

$$\begin{aligned}
\mathrm{EC}_n(x_1, \ldots, x_n) &= x_i \mathrm{EC}_{n-1}(\widehat{x_i}) - y_i \mathrm{ES}_{n-1}(\widehat{x_i}) \\
\mathrm{ES}_n(x_1, \ldots, x_n) &= y_i \mathrm{EC}_{n-1}(\widehat{x_i}) + x_i \mathrm{ES}_{n-1}(\widehat{x_i})
\end{aligned}$$

where $y_i = \sqrt{1 - x_i^2}$ and $(\widehat{x_i}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. In particular for $i = 1$ we have

$$\begin{aligned}
\mathrm{EC}_n(x_1, \ldots, x_n) &= x_1 \mathrm{EC}_{n-1}(\widehat{x_1}) - y_1 \mathrm{ES}_{n-1}(\widehat{x_1}) \\
\mathrm{ES}_n(x_1, \ldots, x_n) &= y_1 \mathrm{EC}_{n-1}(\widehat{x_1}) + x_1 \mathrm{ES}_{n-1}(\widehat{x_1}).
\end{aligned}$$

Proof. Since $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we get

$$\begin{aligned}
\mathrm{EC}_n &= \mathrm{EC}_n(x_1, \ldots, x_n) \\
&= \cos\left(\sum_{j=1}^{n} \theta_j\right) \\
&= \cos\left(\theta_i + \sum_{j \in \{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) \\
&= \cos\theta_i \cos\left(\sum_{j \in \{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) - \sin\theta_i \sin\left(\sum_{j \in \{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) \\
&= x_i \mathrm{EC}_{n-1}(\widehat{x_i}) - y_i \mathrm{ES}_{n-1}(\widehat{x_i}).
\end{aligned}$$

Similarly for ES:

$$\begin{aligned}
\mathrm{ES}_n &= \mathrm{ES}_n(x_1, \ldots, x_n) \\
&= \sin\left(\sum_{j=1}^{n} \theta_j\right) \\
&= \sin\left(\theta_i + \sum_{j\in\{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) \\
&= \cos\theta_i \sin\left(\sum_{j\in\{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) + \sin\theta_i \cos\left(\sum_{j\in\{1,\ldots,i-1,i+1,\ldots,n\}} \theta_j\right) \\
&= y_i \mathrm{EC}_{n-1}(\widehat{x_i}) + x_i \mathrm{ES}_{n-1}(\widehat{x_i}).
\end{aligned}$$

$\square$

Note that the sum, and thus its cosine and sine, are symmetric in $x_1, \ldots, x_k$. See [7] page 252 for more information on symmetric polynomials.

Each of these algebraic equations EC_n yields a polynomial equation $C_k = 0$, which we obtain by repeatedly squaring and rearranging terms in order to eliminate any y_i terms. For example,

$$\begin{aligned}
C_1(x_1) &= x_1 - 1 \\
C_2(x_1, x_2) &= (x_1 - x_2)^2 \\
C_3(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1)^2 \\
C_4(x_1, x_2, x_3, x_4) &= (x_1^4 + x_2^4 + x_3^4 + x_4^4 \\
&\quad -2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2 + x_1^2x_3^2 + x_2^2x_4^2) \\
&\quad +4(x_1^2x_2^2x_3^2 + x_2^2x_3^2x_4^2 + x_1^2x_3^2x_4^2 + x_1^2x_2^2x_4^2) \\
&\quad +4x_1x_2x_3x_4(2 - x_1^2 - x_2^2 - x_3^2 - x_4^2))^2.
\end{aligned}$$

As mentioned above, we have the familiar Law of Cosines relationship

$$x_i = \frac{(r+r_i)^2 + (r+r_{i+1})^2 - (r_i + r_{i+1})^2}{2(r+r_i)(r+r_{i+1})}.$$

Allowing the eye to have radius $r = 1$, which we can do by scaling appropriately, we have for each x_i

$$x_i = \frac{(1+r_i)^2 + (1+r_{i+1})^2 - (r_i + r_{i+1})^2}{2(1+r_i)(1+r_{i+1})}.$$

Substituting these expressions for the x_i into the C_n polynomial yields a rational expression in $r_1, \ldots, r_k$. This rational expression can then be transformed into a polynomial $g \in \mathbb{Q}[r_1, \ldots, r_k]$. That the polynomial will be D_k-symmetric is clear from geometry: it does not matter which angle we label θ_1 (rotation) or whether we do our numbering clockwise or counter-clockwise (reflection.)

Example 3.1.7. For $n = 3$ we have $f = C_3 = (x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1)^2$ and we can calculate

$$\begin{aligned}
C_3 &= \Bigg(\left(\frac{(r+r_1)^2 + (r+r_2)^2 - (r_1+r_2)^2}{2(r+r_1)(r+r_2)}\right)^2 + \left(\frac{(r+r_2)^2 + (r+r_3)^2 - (r_2+r_3)^2}{2(r+r_2)(r+r_3)}\right)^2 \\
&+ \left(\frac{(r+r_3)^2 + (r+r_1)^2 - (r_3+r_1)^2}{2(r+r_3)(r+r_1)}\right)^2 - 2\left(\frac{(r+r_1)^2 + (r+r_2)^2 - (r_1+r_2)^2}{2(r+r_1)(r+r_2)}\right) \\
&\quad \left(\frac{(r+r_2)^2 + (r+r_3)^2 - (r_2+r_3)^2}{2(r+r_2)(r+r_3)}\right)\left(\frac{(r+r_3)^2 + (r+r_1)^2 - (r_3+r_1)^2}{2(r+r_3)(r+r_1)}\right) - 1\Bigg)^2 \\
&= \frac{16}{(r+r_1)^4(r+r_2)^4(r+r_3)^4}\left(-2r_1^2r_2r_3r^2 + r_1^2r_3^2r^2 + r_2^2r_3^2r^2 + r_1^2r_2^2r^2 - 2r_1r_2r_3^2r^2\right. \\
&- \left. 2r_1r_2^2r_3r^2 - 2r_1^2r_2^2r_3r - 2r_1r_2^2r_3^2r - 2r_1^2r_2r_3^2r + r_1^2r_2^2r_3^2\right)^2 \\
&= 0.
\end{aligned}$$

Then we have as our polynomial

$$
\begin{aligned}
g &= 16\left(-2r_1^2r_2r_3r^2 + r_1^2r_3^2r^2 + r_2^2r_3^2r^2 + r_1^2r_2^2r^2 - 2r_1r_2r_3^2r^2\right. \\
&- \left. 2r_1r_2^2r_3r^2 - 2r_1^2r_2^2r_3r - 2r_1r_2^2r_3^2r - 2r_1^2r_2r_3^2r + r_1^2r_2^2r_3^2\right)^2
\end{aligned}
$$

and we can write

$$
\begin{aligned}
r^{12}g\left(\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}\right) &= r^8g_8 + r^6g_6 + r^4g_4 - r^2g_2 + g_0 \\
&\in \mathbb{Q}[r_1, r_2, r_3][r],
\end{aligned}
$$

where

$$
\begin{aligned}
g_8 &= 16(r_2^4r_3^4 + r_1^4r_2^4 + r_1^4r_3^4 + 4r_1^3r_2^3r_3^2 - 4r_1^4r_2r_3^3 + 4r_1^2r_2^3r_3^3 - 4r_1^4r_2^3r_3 - 4r_1^3r_2^4r_3 \\
&+ 6r_1^2r_2^2r_3^4 - 4r_1^3r_2r_3^4 - 4r_1r_2^3r_3^4 - 4r_1r_2^4r_3^3 + 6r_1^4r_2^2r_3^2 + 6r_1^2r_2^4r_3^2 + 4r_1^3r_2^2r_3^3), \\
g_6 &= 64(r_1^3r_2^2r_3^4 - r_1^4r_2r_3^4 + 6r_1^3r_2^3r_3^3 + r_1^4r_2^2r_3^3 + r_1^4r_2^3r_3^2 + r_1^3r_2^4r_3^2 + r_1^2r_2^3r_3^4 + r_1^2r_2^4r_3^3 - r_1r_2^4r_3^4 - r_1^4r_2^4r_3), \\
g_4 &= 32(3r_1^4r_2^2r_3^4 + 2r_1^4r_2^3r_3^3 + 2r_1^3r_2^3r_3^4 + 3r_1^4r_2^4r_3^2 + 3r_1^2r_2^4r_3^4 + 2r_1^3r_2^4r_3^3), \\
g_2 &= 64(r_1^3r_2^4r_3^4 - r_1^4r_2^3r_3^4 - r_1^4r_2^4r_3^3), \\
g_0 &= 16r_1^4r_2^4r_3^4
\end{aligned}
$$

and each of these $g_i \in \mathbb{Q}[r_1, r_2, r_3]$ is a D_3-symmetric polynomial.

For the terms with the maximum degree d, r^d will cancel out all the denominators and we will be left with g_0, a polynomial element of $\mathbb{Q}[r_1, ..., r_k]$. That g_0 will be D_k-symmetric follows from the D_k-symmetry of the polynomials and (3.1).

For the terms with degree $0 \leq \delta < d$, r^d will cancel out all the denominators and we will be left with $r^{d-\delta}g_{d-\delta}$, a polynomial element of $\mathbb{Q}[r_1, ..., r_k]$. That $g_{d-\delta}$ will be D_k-symmetric again follows from the D_k-symmetry of the polynomials and (3.1).

The sum of all these terms will give us the statement of the claim. In particular, we have that for general radius r of the eye, we do not have a symmetric polynomial in $\mathbb{Q}[r_1, \ldots, r_k, r]$, but we do have a way to write it as a polynomial in $\mathbb{Q}[r_1, \ldots, r_k][r]$ with coefficients that are D_k-symmetric in $\mathbb{Q}[r_1, \ldots, r_k]$.

3.2 Using Galois Theory to Describe the Polynomials

From now on we will use n in the role of k in Equation (3.1) to denote the number of petals of the flower, since we are considering the polynomials themselves.

We will show that for each $n \geq 2$, $C_n = P_n^2$, where P_n is an irreducible polynomial for $n \geq 2$ and is in fact symmetric for $n \geq 3$. First, we will need some tools from Galois theory to describe the polynomials C_n and P_n.

Definition 3.2.1. For $n \geq 1$, let G_n^* be the Galois group (group of automorphisms) on $\mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ that fixes the field $\mathbb{Q}(x_1, \ldots, x_n)$, and G_n be the Galois group on $\mathbb{Q}(x_1, \ldots, x_n, y_i y_j : i < j)$ that also fixes the field $\mathbb{Q}(x_1, \ldots, x_n)$. They are denoted:

$$G_n^* = \mathrm{Gal}(\mathbb{Q}(x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)/\mathbb{Q}(x_1, \ldots, x_n)),$$
$$G_n = \mathrm{Gal}(\mathbb{Q}(x_1, x_2, \ldots, x_n, y_i y_j : i < j)/\mathbb{Q}(x_1, \ldots, x_n)),$$

where the x_i are algebraically independent indeterminates and $y_i = \sqrt{1 - x_i^2}$, as discussed above, i.e. y_i satisfies $X^2 + x_i^2 - 1 = 0 \in \mathbb{Q}(x_1, \ldots, x_n)[X]$.

Lemma 3.2.2. *For $n \geq 1$ we have $G_n^* \cong \mathbb{Z}_2^n$ and $G_n \cong \mathbb{Z}_2^{n-1}$.*

Proof. For G_n^*, each y_i is the root of an irreducible quadratic polynomial $X^2 - (1 - x_i^2) \in \mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_{i-1})[X]$, which is the minimum polynomial of y_i over $\mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_{i-1})$ for each i. Hence we have $G_n^* \cong \mathbb{Z}_2^n$.

For G_n, each y_iy_j with $i < j$ is also the root of an irreducible quadratic polynomial $X^2 - (1-x_i^2)(1-x_j^2) \in \mathbb{Q}(x_1, \ldots, x_n)[X]$. However, every element of $\mathbb{Q}(x_1, x_2, \ldots, x_n, y_iy_j : i < j)$ can be written as a rational function in terms of only elements of the form y_iy_{i+1} as follows:

$$\begin{aligned} y_iy_j &= \frac{(y_iy_{i+1})(y_{i+1}y_{i+2})\cdots(y_{j-1}y_j)}{y_{i+1}^2\cdots y_{j-1}^2} \\ &= \frac{(y_iy_{i+1})(y_{i+1}y_{i+2})\cdots(y_{j-1}y_j)}{(1-x_{i+1}^2)\cdots\left(1-x_{j-1}^2\right)}. \end{aligned}$$

So we have that

$$\mathbb{Q}(x_1, x_2, \ldots, x_n, y_iy_j : i < j) = \mathbb{Q}(x_1, x_2, \ldots, x_n, y_iy_{i+1} : 1 \le i < n). \tag{3.2}$$

Each of the $n-1$ terms y_iy_{i+1} is the root of an irreducible quadratic polynomial $X^2 - (1-x_i^2)(1-x_{i+1}^2) \in \mathbb{Q}(x_1, \ldots, x_n, y_1y_2, \ldots, y_{i-1}y_i)[X]$, which is the minimum polynomial of y_iy_{i+1} over $\mathbb{Q}(x_1, \ldots, x_n, y_1y_2, \ldots, y_{i-1}y_i)$ for each $i \in \{1, \ldots, n-1\}$. Therefore we have that $G_n \cong \mathbb{Z}_2^{n-1}$. $\square$

Lemma 3.2.3. *For $n \in \mathbb{N}$, the group $G_n \cong \mathbb{Z}_2^{n-1}$ can be presented as*

$$G_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i^2 = e, \sigma_i\sigma_j = \sigma_j\sigma_i \right\rangle,$$

where each σ_i is an automorphism fixing $\mathbb{Q}(x_1, \ldots x_n)$ and

$$\sigma_i(y_jy_{j+1}) = \begin{cases} -y_jy_{j+1} & \text{if } i = j \\ y_jy_{j+1} & \text{if } i \neq j. \end{cases}$$

Proof. Since $(y_iy_{i+1})^2 = (1-x_i^2)(1-x_{i+1}^2)$ and the Galois group G_n is fixing the x_i, the only possible automorphisms are $\sigma(y_iy_{i+1}) = -y_iy_{i+1}$ and $\sigma(y_iy_{i+1}) = y_iy_{i+1}$. We can then generate the group as in the statement of the theorem with $n-1$ generators σ_i. □

Corollary 3.2.4. *For every $\sigma \in G_n$, let $s_{\sigma;j} \in \{-1, 1\}$ be such that $\sigma(y_jy_{j+1}) = s_{\sigma;j}y_jy_{j+1}$. Then for every $i < j$ we have*

$$\sigma(y_iy_j) \quad = \quad s_{\sigma;i}s_{\sigma;i+1}\cdots s_{\sigma;j}y_iy_j.$$

In particular, we have

1. *If $i < n$ then $\sigma_{n-1}(y_iy_n) = -y_iy_n$.*

2. *If $i > 1$ then $\sigma_1(y_1y_i) = -y_1y_i$.*

As a simple example, $G_2 \cong \mathbb{Z}_2$ is generated by the single element σ such that $\sigma(y_1y_2) = -y_1y_2$ and $\sigma^2 = e$. So we have

$$\begin{aligned}\prod_{\sigma\in G_2}(\sigma(\mathrm{EC}_2) - 1) \quad &= \quad (x_1x_2 - y_1y_2 - 1)(x_1x_2 - \sigma(y_1y_2) - 1) \\ &= \quad (x_1 - x_2)^2 = C_2(x_1, x_2).\end{aligned}$$

For $n = 3$, we have $G_3 = \langle\sigma_1, \sigma_2\rangle$ where $\sigma_1(y_1y_2) = -y_1y_2$ and $\sigma_2(y_2y_3) = -y_2y_3$. Then we have

$$
\begin{aligned}
\prod_{\sigma\in G_3}(\sigma(\mathrm{EC}_3)-1) &= (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3 - 1)\\
&\quad (x_1x_2x_3 - x_1y_2y_3 - \sigma_1(y_1x_2y_3) - \sigma_1(y_1y_2x_3) - 1)\\
&\quad (x_1x_2x_3 - \sigma_2(x_1y_2y_3) - \sigma_2(y_1x_2y_3) - y_1y_2x_3 - 1)\\
&\quad (x_1x_2x_3 - \sigma_1\sigma_2(x_1y_2y_3) - \sigma_1\sigma_2(y_1x_2y_3) - \sigma_1\sigma_2(y_1y_2x_3) - 1)\\
&= (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3 - 1)\\
&\quad (x_1x_2x_3 - x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 - 1)\\
&\quad (x_1x_2x_3 + x_1y_2y_3 + y_1x_2y_3 - y_1y_2x_3 - 1)\\
&\quad (x_1x_2x_3 - x_1y_2y_3 + y_1x_2y_3 - y_1y_2x_3 - 1)\\
&= (x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1)^2 = C_3(x_1, x_2, x_3).
\end{aligned}
$$

We can use this as a precise definition of C_n for each $n \in \mathbb{N}$:

Definition 3.2.5. The polynomial C_n, corresponding to a flower with n petals, can be defined as:

$$C_n(x_1,\ldots,x_n) \;=\; \prod_{\sigma\in G_n}(\sigma(\mathrm{EC}_n)-1) \in \mathbb{Q}[x_1,\ldots,x_n].$$

From Definition 3.2.5 we see that C_n is symmetric in $x_1,\ldots,x_n$. By Lemma 3.1.3, each of the 2^{n-2} terms of ES_{n-1} in terms of $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}$ contains positive odd factors of y_i for $i \le n-1$. Hence $\sigma_{n-1} \in G_n$ fixes $\mathbb{Q}(x_1,\ldots,x_n,y_1y_2,\ldots,y_{n-2}y_{n-1})$ and $\sigma_{n-1}(y_{n-1}y_n) = -y_{n-1}y_n$. Then we have by Corollary 3.2.4:

Claim 3.2.6. *For $n \ge 2$ we have*

$$G_n = G_{n-1} \cup G_{n-1}\sigma_{n-1} = G_{n-1} \cup \sigma_{n-1}G_{n-1}$$

and

$$\sigma_{n-1}(y_n \mathrm{ES}_{n-1}) = -y_n \mathrm{ES}_{n-1}.$$

Lemma 3.2.7. *If* $\sigma_{n-1} \in G_n$ *fixes* $\mathbb{Q}(x_1, \ldots, x_n, y_1y_2, \ldots, y_{n-2}y_{n-1})$ *and* $\sigma_{n-1}(y_{n-1}y_n) = -y_{n-1}y_n$ *then*

$$(\mathrm{EC}_n - 1)\left(\sigma_{n-1}\left(\mathrm{EC}_n\right) - 1\right) \quad = \quad \left(x_n - \mathrm{EC}_{n-1}\right)^2.$$

In particular, $\mathrm{EC}_n = 1$ *implies* $x_n = \mathrm{EC}_{n-1}$.

Proof. By Claim 3.2.6, $\sigma_{n-1}(y_n \mathrm{ES}_{n-1}) - -y_n \mathrm{ES}_{n-1}$ and hence

$$\begin{aligned}
(\mathrm{EC}_n - 1)(\sigma_{n-1}(\mathrm{EC}_n) - 1) &= (x_n \mathrm{EC}_{n-1} - y_n \mathrm{ES}_{n-1} - 1)(\sigma_{n-1}(x_n \mathrm{EC}_{n-1} - y_n \mathrm{ES}_{n-1}) - 1) \\
&= (x_n \mathrm{EC}_{n-1} - y_n \mathrm{ES}_{n-1} - 1)(x_n \mathrm{EC}_{n-1} + y_n \mathrm{ES}_{n-1} - 1) \\
&= (x_n \mathrm{EC}_{n-1} - 1)^2 - y_n^2 \mathrm{ES}_{n-1}^2 \\
&= (x_n \mathrm{EC}_{n-1} - 1)^2 - (1 - x_n^2)(1 - \mathrm{EC}_{n-1}^2) \\
&= (x_n - \mathrm{EC}_{n-1})^2.
\end{aligned}$$

□

Remark 3.2.8. In fact, (3.1) implies directly that

$$\cos(\theta_1 + \cdots + \theta_h) = \cos(\theta_{h+1} + \cdots + \theta_n)$$

where $h + k = n$ and hence $\mathrm{EC}_h(x_1, \ldots, x_h) = \mathrm{EC}_k(x_{h+1}, \ldots, x_n)$, which itself implies $x_n = \mathrm{EC}_{n-1}$ by letting $h = n - 1$ and $k = 1$.

Figure 3.1 gives a visual representation which makes this intuitively clear, since this result is based on angles adding up to 2π.

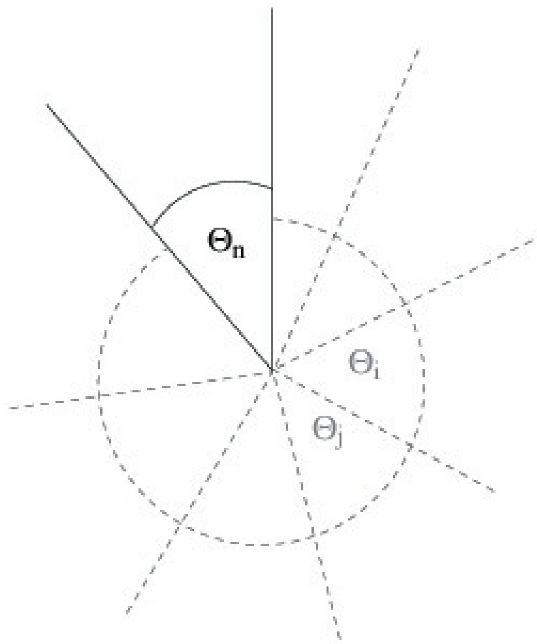

Figure 3.1: $\cos(\theta_n) = \cos \sum_{i=1}^{n-1} \theta_i$

Corollary 3.2.9. *For $n \geq 2$ we have $C_n = P_n^2$ where*

$$P_n := \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1})).$$

Proof. By Lemma 3.2.7 we obtain:

$$
\begin{aligned}
C_n &= \prod_{\sigma \in G_n} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in \sigma_n G_{n-1} \cup G_{n-1}} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} (\sigma(\mathrm{EC}_n) - 1)(\sigma_n \sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((\mathrm{EC}_n - 1)(\sigma_n(\mathrm{EC}_n) - 1)) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((x_n - \mathrm{EC}_{n-1})^2) \\
&= \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1}))^2 \\
&= P_n^2
\end{aligned}
$$

where $P_n = \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1}))$. □

By exactly the same token as Claim 3.2.6, Lemma 3.2.7, and Corollary 3.2.9, we obtain analogous results by reordering the variables $y_1, \ldots, y_n$ in the reverse order: $y_n, y_{n-1}, \ldots, y_1$. Namely, if $\sigma_i \in G_n$ is the field automorphism of $\mathbb{Q}(x_1, \ldots, x_n, y_1y_2, y_2y_3, \ldots, y_{n-1}y_n)$ with $\sigma_i(y_iy_{i+1}) = -y_iy_{i+1}$ fixing $\mathbb{Q}(x_1, \ldots, x_n)$ and each y_jy_{j+1} for $j \neq i$ (as in Lemma 3.2.3) then:

Claim 3.2.10.

$$
G_n = G'_{n-1} \cup \sigma_1 G'_{n-1} = G'_{n-1} \cup G'_{n-1}\sigma_1
$$

where $G'_{n-1} = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle \subseteq G_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ *and*

$$\sigma_1(y_1 \mathrm{ES}_{n-1}(x_2, \ldots, x_n)) = -y_1 \mathrm{ES}_{n-1}(x_2, \ldots, x_n).$$

Proof. By Lemma 3.1.3, each of the 2^{n-2} terms of $\mathrm{ES}_{n-1}(x_2, \ldots, x_n) = \mathrm{ES}_{n-1}(x_2, \ldots, x_n, y_2, \ldots, y_n)$ (by substituting $y_i = \sqrt{1 - x_i^2}$ for each $i = 2, \ldots, n$) has positive odd factors of y_i for $i \geq 2$. hence the claim follows by Corollary 3.2.4. $\square$

Lemma 3.2.11. *If* $\sigma_1 \in G_n$ *is as above then*

$$(\mathrm{EC}_n - 1)(\sigma_1(\mathrm{EC}_n - 1)) = (x_1 - \mathrm{EC}_{n-1}(x_2, \ldots, x_n))^2.$$

Proof. By Claim 3.2.10 we obtain

$$\begin{aligned}
(\mathrm{EC}_n - 1)(\sigma_1(\mathrm{EC}_n - 1)) &= (x_1 \mathrm{EC}_{n-1}(x_2, \ldots, x_n) - y_1 \mathrm{ES}_{n-1}(x_2, \ldots, x_n) - 1) \\
&\quad (\sigma_1(x_1 \mathrm{EC}_{n-1}(x_2, \ldots, x_n) - y_1 \mathrm{ES}_{n-1}(x_2, \ldots, x_n)) - 1) \\
&= (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - y_1 \mathrm{ES}_{n-1}(\widehat{x}_1) - 1)(x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) + y_1 \mathrm{ES}_{n-1}(\widehat{x}_1) - 1) \\
&= (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - 1)^2 - y_1^2 \mathrm{ES}_{n-1}(\widehat{x}_1)^2 \\
&= (x_1 \mathrm{EC}_{n-1}(\widehat{x}_1) - 1)^2 - (1 - x_1^2)(1 - \mathrm{EC}_{n-1}(\widehat{x}_1)^2 \\
&= (x_1 - \mathrm{EC}_{n-1}(\widehat{x}_1))^2,
\end{aligned}$$

where $(\widehat{x_1}) = (x_2, \ldots, x_n)$ as above. $\square$

By symmetry of C_n we have the following:

Corollary 3.2.12. *For* $n \geq 3$ *we have*

$$C_n = \prod_{\sigma \in G'_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1)))^2.$$

Remark 3.2.13. For $n = 1$ we have $P_1 = C_1 = x_1 - 1$. For $n = 2$ we have (as defined in Corollary 3.2.9) $P_2 = x_2 - x_1$. However, this is a matter of taste, since we could have set $P_2 = x_1 - x_2$. The case $n = 2$ is the only one where $C_2(x_1, x_2)$ is symmetric while P_2 is not. (See below.)

Proof of Corollary 3.2.12. By Lemma 3.2.11 we obtain as in the proof of Corollary 3.2.9

$$
\begin{aligned}
C_n &= \prod_{\sigma \in G_n} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G'_{n-1} \cup \sigma_1 G'_{n-1}} (\sigma(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} (\sigma(\mathrm{EC}_n) - 1)(\sigma\sigma_1(\mathrm{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} \sigma\left(((\mathrm{EC}_n) - 1)(\sigma_1(\mathrm{EC}_n) - 1)\right) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((x_1 - \mathrm{EC}_{n-1}(\widehat{x}_1))^2) \\
&= \prod_{\sigma \in G'_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1)))^2.
\end{aligned}
$$

□

Hence, by Corollary 3.2.12 we obtain $C_n = Q_n^2$ where

$$Q_n = \prod_{\sigma \in G_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1))).$$

Since $P_n^2 = C_n = Q_n^2$, then as elements in a polynomial ring over a field which is an integer domain we get

$$0 = P_n^2 - Q_n^2 = (P_n - Q_n)(P_n + Q_n)$$

and therefore for each $n \geq 2$, $Q_n = P_n$ or $Q_n = -P_n$.

For $n = 2$ we obtain $P_2 = x_2 - x_1$ and $Q_2 = x_1 - x_2$ so $Q_2 = -P_2$.

For $n \geq 3$ we first note that by evaluating $\mathrm{EC}_{n-1}(\widehat{x}_n)$ and $\mathrm{EC}_{n-1}(\widehat{x}_1)$ at $x_2 = \cdots = x_{n-1} = 1$ yields

$$\begin{aligned}
\mathrm{EC}_{n-1}(\widehat{x}_n)|_{x_2=\cdots=x_{n-1}=1} &= x_1 \\
\mathrm{EC}_{n-1}(\widehat{x}_1)|_{x_2=\cdots=x_{n-1}=1} &= x_n
\end{aligned}$$

and hence we obtain

$$\begin{aligned}
P_n(x_1, 1, \ldots, 1, x_n) &= \prod_{\sigma \in G_{n-1}} (x_n - x_1) = (x_n - x_1)^{2^{n-2}} \\
Q_n(x_1, 1, \ldots, 1, x_n) &= \prod_{\sigma \in G_{n-1}} (x_1 - x_n) = (x_1 - x_n)^{2^{n-2}}.
\end{aligned}$$

Since $n \geq 3$ we have 2^{n-2} is even and so $(x_n - x_1)^{2^{n-2}} = (x_1 - x_n)^{2^{n-2}}$ and therefore

$$P_n(x_1, 1, \ldots, 1, x_n) = Q_n(x_1, 1, \ldots, 1, x_n).$$

From this we have:

Corollary 3.2.14. *For $n \geq 3$, $Q_n = P_n$ and hence*

$$P_n = \prod_{\sigma \in G_{n-1}} (x_1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x}_1))).$$

Let $n \geq 3$. If $\pi \in S_n$ is a permutation on $\{1, \ldots, n\}$ then π acts naturally on $(x_1, \ldots, x_n)$ by $\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. By definition of P_n in Corollary 3.2.9 we have

$$(P_n \circ \pi)(x_1, \ldots, x_n) = P_n(x_{\pi(1)}, \ldots, x_{\pi(n)}) = P_n(x_1, \ldots, x_n)$$

or $P_n \circ \pi = P_n\pi = P_n$ for all $\pi \in S_n$ with $\pi(n) = n$. Likewise by Corollary 3.2.14 we have $P_n\pi = P_n$ for all $\pi \in S_n$ with $\pi(1) = 1$.

Let $\tau \in S_n$ be an arbitrary transposition $\tau = (i, j)$. If $\{i, j\} \subseteq \{1, \ldots, n-1\}$ or $\{i, j\} \subseteq \{2, \ldots, n\}$ then by the above, $P_n\tau = P_n$. Otherwise if $\tau = (1, n)$ then since $n \geq 3$ there is an $l \in \{2, \ldots, n-1\}$ such that we can write $\tau = (1, n) = (1, l)(l, n)(1, l)$ where $\{1, l\} \subseteq \{2, \ldots, n\}$. From the above, we therefore have

$$\begin{aligned} P_n\tau = P_n(1, n) &= P_n(1, l)(l, n)(1, l) \\ &= P_n(l, n)(1, l) \\ &= P_n(1, l) \\ &= P_n. \end{aligned}$$

Since each permutation $\pi \in S_n$ is a composition of transpositions then we have $P_n\pi = P_n$ for each $\pi \in S_n$.

Theorem 3.2.15. *For $n \geq 3$ the polynomial $P_n = P_n(x_1, \ldots, x_n)$ is symmetric.*

Corollary 3.2.16. *For $n \geq 3$ and any $i \in \{1, \ldots, n\}$ we have*

$$P_n = \prod_{\sigma \in G_{n-1}} \sigma(x_i - \mathrm{EC}_{n-1}(\widehat{x_i})).$$

In particular as a polynomial in x_i, then P_n is monic of degree 2^{n-2} in each x_i.

Observation 3.2.17. For $n \geq 3$

$$P_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = P_{n-1}(\widehat{x_i})^2 = C_{n-1}(\widehat{x_i}).$$

and hence also $C_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = C_{n-1}(\widehat{x_i})^2$.

Proof. Since

$$P_n = \prod_{\sigma \in G_{n-1}} (x_i - \sigma(\mathrm{EC}_{n-1}(\widehat{x_i}))),$$

then we obtain

$$P_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = \prod_{\sigma \in G_{n-1}} (1 - \sigma(\mathrm{EC}_{n-1}(\widehat{x_i}))) = C_{n-1}(\widehat{x_i})$$

by definition.

In particular, we then obtain

$$\begin{aligned} C_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) &= P_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)^2 \\ &= C_{n-1}(\widehat{x_i})^2. \end{aligned}$$

□

Proposition 3.2.18. *Let $n \geq 1$ and $n_1 + \cdots + n_k = n$. If $\sum_{i=1}^{n} \theta_i = 2\pi$ and $x_i = \cos\theta_i$ for $i \in \{1, \ldots, n\}$ then $P_k(\mathrm{EC}_{n_1}, \ldots, \mathrm{EC}_{n_k}) = 0$ where for each $l \in \{1, \ldots, k\}$ $\mathrm{EC}_{n_l} = \mathrm{EC}_{n_l}(x_{n_1+\cdots+n_{l-1}+1}, \ldots x_{n_1+\cdots+n_l})$. In particular for $k = n-1$ and $n_1 = \cdots = n_{n-2} = 1$ and $n_{n-1} = 2$, we have $P_{n-1}(x_1, \ldots, x_{n-2}, \mathrm{EC}_2(x_{n-1}, x_n)) = 0$.*

Proof. Letting $\phi_l = \theta_{n_1+\cdots+n_{l-1}+1} + \cdots + \theta_{n_1+\cdots+n_l}$ for each $l \in \{1, \ldots, k\}$, then

$$\sum_{l=1}^{k} \phi_l = 2\pi$$

and hence if $t_l = \cos(\phi_l)$ then by Corollary 3.2.16,

$$0 = P_k(t_1, \ldots, t_l) = P_k(\mathrm{EC}_{n_1}, \ldots, \mathrm{EC}_{n_k}),$$

where for each $l \in \{1, \ldots, k\}$, $\mathrm{EC}_{n_l} = \mathrm{EC}_{n_l}(x_{n_1+\cdots+n_{l-1}+1}, \ldots, x_{n_1+\cdots+n_l})$. $\square$

We now obtain a recursive method for determining P_n. Let $\overline{\mathrm{EC}_2}(x_j, x_{j+1}) = x_j x_{j+1} + y_j y_{j+1}$ be the conjugate of $\mathrm{EC}_2(x_j, x_{j+1})$. Recall that by Claim 3.2.6 we have for $n-1$ that

$$G_{n-1} = G_{n-2} \cup \sigma_{n-1} G_{n-2} = G_{n-2} \cup G_{n-2}\sigma_{n-1}$$

and

$$\sigma_{n-2}(y_{n-1}\mathrm{ES}_{n-2}) = -y_{n-1}\mathrm{ES}_{n-2}.$$

Lemma 3.2.19. *For $n \geq 3$ we have*

$$(x_n - \mathrm{EC}_{n-1})(x_n - \sigma_{n-2}(\mathrm{EC}_{n-1})) = x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2.$$

Proof. Since $\mathrm{EC}_{n-1} = x_{n-1}\mathrm{EC}_{n-2} - y_{n-1}\mathrm{ES}_{n-2}$, we obtain by above

$$\begin{aligned}
(x_n - \mathrm{EC}_{n-1})(x_n - \sigma_{n-2}(\mathrm{EC}_{n-1})) &= (x_n - x_{n-1}\mathrm{EC}_{n-2} + y_{n-1}\mathrm{ES}_{n-2}) \\
&\cdot (x_n - x_{n-1}\mathrm{EC}_{n-2} - y_{n-1}\mathrm{ES}_{n-2}) \\
&= (x_n - x_{n-1}\mathrm{EC}_{n-2})^2 - y_{n-1}^2\mathrm{ES}_{n-2}^2 \\
&= (x_n - x_{n-1}\mathrm{EC}_{n-2})^2 - (1 - x_{n-1}^2)(1 - \mathrm{EC}_{n-2}^2) \\
&= x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2.
\end{aligned}$$

$\square$

By direct computation and the definition of P_{n-1}, since $\mathrm{EC}_2(x_i, x_{i+1}) = x_i x_{i+1} - y_i y_{i+1}$

we get

$$
\begin{aligned}
P_{n-1}(x_1, \quad \ldots, \quad & x_{n-2}, \mathrm{EC}_2(x_{n-1}x_n))P_{n-1}(x_1, \ldots, x_{n-2}, \overline{\mathrm{EC}_2}(x_{n-1}x_n)) \\
&= \prod_{\sigma \in G_{n-2}} (\mathrm{EC}_2(x_{n-1}, x_n) - \sigma(\mathrm{EC}_{n-2})) \cdot \prod_{\sigma \in G_{n-2}} (\overline{\mathrm{EC}_2}(x_{n-1}, x_n) - \sigma(\mathrm{EC}_{n-2})) \\
&= \prod_{\sigma \in G_{n-2}} (x_{n-1}x_n - y_{n-1}y_n - \sigma(\mathrm{EC}_{n-2})) \cdot \prod_{\sigma \in G_{n-2}} (x_{n-1}x_n + y_{n-1}y_n - \sigma(\mathrm{EC}_{n-2})) \\
&= \prod_{\sigma \in G_{n-2}} \left((x_{n-1}x_n - \sigma(\mathrm{EC}_{n-2}))^2 - y_{n-1}^2 y_n^2\right) \\
&= \prod_{\sigma \in G_{n-2}} \left((x_{n-1}x_n - \sigma(\mathrm{EC}_{n-2}))^2 - (1 - x_{n-1}^2)(1 - x_n^2)\right) \\
&= \prod_{\sigma \in G_{n-2}} \left(x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\sigma(\mathrm{EC}_{n-2}^2) + \sigma(\mathrm{EC}_{n-2}^2)\right) \\
&= \prod_{\sigma \in G_{n-2}} \sigma\left(x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2\right).
\end{aligned}
$$

From this we can prove the following:

Theorem 3.2.20. *The polynomials P_n are completely determined by the following recursion*

$$
\begin{aligned}
P_1 &= x_1 - 1, \\
P_2 &= x_2 - x_1,
\end{aligned}
$$

and for $n \geq 3$

$$
P_n = P_{n-1}(x_1, \ldots, x_{n-2}, \mathrm{EC}_2(x_{n-1}, x_n)) \cdot P_{n-1}(x_1, \ldots, x_{n-2}, \overline{\mathrm{EC}_2}(x_{n-1}, x_n)).
$$

Proof. By Lemma 3.2.19 and the preceding paragraph we get

$$\begin{aligned}
P_n &= \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\mathrm{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2} \cup \sigma_{n-1} G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-1}))(x_n - \sigma\sigma_{n-1}(\mathrm{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2}} \sigma\left((x_n - (\mathrm{EC}_{n-1}))(x_n - \sigma_{n-1}(\mathrm{EC}_{n-1}))\right) \\
&= \prod_{\sigma \in G_{n-2}} \sigma\left(x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\mathrm{EC}_{n-2}^2 + \mathrm{EC}_{n-2}^2\right) \\
&= P_{n-1}(x_1, \ldots, x_{n-2}, \mathrm{EC}_2(x_{n-1}, x_n)) \cdot P_{n-1}(x_1, \ldots, x_{n-2}, \overline{\mathrm{EC}_2}(x_{n-1}, x_n)).
\end{aligned}$$

□

Example 3.2.21. The first 5 polynomials P_n can now be computed easily by the recursion in Theorem 3.2.20.

$$P_1 = x_1 - 1$$

$$P_2 = x_2 - x_1$$

$$\begin{aligned}
P_3 &= P_2(x_1, \mathrm{EC}_2(x_2, x_3)) \cdot P_2(x_1, \overline{\mathrm{EC}_2}(x_2, x_3)) \\
&= (x_2x_3 - y_2y_3 - x_1) \cdot (x_2x_3 + y_2y_3 - x_1) \\
&= (x_2x_3 - x_1)^2 - y_2^2y_3^2 \\
&= x_2^2x_3^2 - 2x_1x_2x_3 + x_1^2 - (1 - x_2^2)(1 - x_3^2) \\
&= x_2^2x_3^2 - 2x_1x_2x_3 + x_1^2 - 1 + x_2^2 + x_3^2 - x_2^2x_3^2 \\
&= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1.
\end{aligned}$$

$$\begin{aligned}
P_4 &= P_3(x_1, x_2, \mathrm{EC}_2(x_3, x_4)) \cdot P_3(x_1, x_2, \overline{\mathrm{EC}_2}(x_3, x_4)) \\
&= (x_1^2 + x_2^2 + (x_3x_4 - y_3y_4)^2 - 2x_1x_2(x_3x_4 - y_3y_4) - 1) \\
&\cdot\ (x_1^2 + x_2^2 + (x_3x_4 + y_3y_4)^2 - 2x_1x_2(x_3x_4 + y_3y_4) - 1) \\
&= x_1^4 + x_2^4 + x_3^4 + x_4^4 - 2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2 + x_1^2x_3^2 + x_2^2x_4^2) \\
&+\ 4(x_1^2x_2^2x_3^2 + x_2^2x_3^2x_4^2 + x_1^2x_3^2x_4^2 + x_1^2x_2^2x_4^2) \\
&+\ 4x_1x_2x_3x_4(2 - x_1^2 - x_2^2 - x_3^2 - x_4^2).
\end{aligned}$$

$$
\begin{aligned}
P_5 &= P_4(x_1,x_2,x_3,\mathrm{EC}_2(x_4,x_5))\cdot P_4(x_1,x_2,x_3,\overline{\mathrm{EC}_2}(x_4,x_5))\\
&= x_5^8-8x_1x_2x_3x_4x_5^7-8x_3^2x_4^2x_5^6+4x_2^2x_5^6-4x_5^6+4x_3^2x_5^6+16x_1^2x_2^2x_3^2x_5^6\\
&- 8x_2^2x_3^2x_5^6-8x_1^2x_4^2x_5^6-8x_1^2x_3^2x_5^6-8x_1^2x_2^2x_5^6+4x_4^2x_5^6-8x_2^2x_4^2x_5^6\\
&+ 16x_1^2x_3^2x_4^2x_5^6+16x_2^2x_3^2x_4^2x_5^6+4x_1^2x_5^6+16x_1^2x_2^2x_4^2x_5^6+40x_1x_2^3x_3x_4x_5^5\\
&+ 40x_1x_2x_3x_4^3x_5^5-32x_1^3x_2x_3x_4^3x_5^5+40x_1^3x_2x_3x_4x_5^5-32x_1^3x_2x_3^3x_4x_5^5\\
&- 32x_1x_2^3x_3^3x_4x_5^5-32x_1x_2^3x_3x_4^3x_5^5-24x_1x_2x_3x_4x_5^5-32x_1^3x_2^3x_3x_4x_5^5\\
&- 32x_1x_2x_3^3x_4^3x_5^5+40x_1x_2x_3^3x_4x_5^5+64x_1^2x_2^4x_3^2x_4^2x_5^4-16x_1^4x_4^2x_5^4\\
&+ 28x_2^2x_4^2x_5^4-16x_3^2x_4^4x_5^4-24x_1^2x_2^2x_3^2x_5^4+28x_1^2x_4^2x_5^4-12x_3^2x_5^4\\
&+ 28x_2^2x_3^2x_5^4-16x_2^2x_3^4x_5^4-16x_2^2x_4^4x_5^4+64x_1^2x_2^2x_3^4x_4^2x_5^4-24x_2^2x_3^2x_4^2x_5^4\\
&+ 16x_1^4x_4^4x_5^4-12x_4^2x_5^4-24x_1^2x_3^2x_4^2x_5^4+6x_5^4+6x_2^4x_5^4-16x_1^2x_3^4x_5^4\\
&- 16x_1^2x_4^4x_5^4+6x_4^4x_5^4+64x_1^4x_2^2x_3^2x_4^2x_5^4+16x_1^4x_3^4x_5^4+16x_2^4x_3^4x_5^4\\
&- 16x_1^4x_2^2x_5^4-24x_1^2x_2^2x_4^2x_5^4+16x_1^4x_2^4x_5^4+16x_3^4x_4^4x_5^4+6x_3^4x_5^4-12x_2^2x_5^4\\
&- 144x_1^2x_2^2x_3^2x_4^2x_5^4+64x_1^2x_2^2x_3^2x_4^4x_5^4+28x_1^2x_3^2x_5^4-16x_2^4x_4^2x_5^4\\
&- 16x_2^4x_3^2x_5^4-16x_3^4x_4^2x_5^4+28x_3^2x_4^2x_5^4+28x_1^2x_2^2x_5^4-12x_1^2x_5^4-16x_1^4x_3^2x_5^4\\
&+ 6x_1^4x_5^4-16x_1^2x_2^4x_5^4+16x_2^4x_4^4x_5^4+112x_1^3x_2x_3^3x_4x_5^3+112x_1^3x_2x_3x_4^3x_5^3\\
&+ 40x_1x_2x_3^5x_4x_5^3-32x_1^5x_2x_3x_4^3x_5^3-32x_1x_2^5x_3x_4^3x_5^3+112x_1^3x_2^3x_3x_4x_5^3\\
&+ x_1x_2^5x_3x_4x_5^3+40x_1^5x_2x_3x_4x_5^3+40x_1x_2x_3x_4^5x_5^3-32x_1^3x_2x_3x_4^5x_5^3\\
&- 112x_1x_2^3x_3x_4x_5^3-32x_1x_2x_3^5x_4^3x_5^3-32x_1^5x_2x_3^3x_4x_5^3-112x_1^3x_2x_3x_4x_5^3\\
&+ 112x_1x_2^3x_3^3x_4x_5^3-32x_1x_2^3x_3^5x_4x_5^3-32x_1x_2^5x_3^3x_4x_5^3-32x_1x_2^3x_3x_4^5x_5^3\\
&- 32x_1^3x_2^5x_3x_4x_5^3+112x_1x_2x_3^3x_4^3x_5^3+112x_1x_2^3x_3x_4^3x_5^3-112x_1x_2x_3^3x_4x_5^3\\
&- 112x_1x_2x_3x_4^3x_5^3+72x_1x_2x_3x_4x_5^3-128x_1^3x_2^3x_3^3x_4^3x_5^3-32x_1^5x_2^3x_3x_4x_5^3\\
&- 32x_1^3x_2x_3^5x_4x_5^3-32x_1x_2x_3^3x_4^5x_5^3+16x_1^2x_2^6x_4^2x_5^2+28x_3^2x_4^4x_5^2+28x_1^4x_3^2x_5^2\\
&- 8x_2^2x_4^6x_5^2-12x_4^4x_5^2-16x_1^4x_4^4x_5^2+16x_1^2x_3^2x_4^6x_5^2-24x_1^2x_2^2x_3^4x_5^2-8x_1^6x_3^2x_5^2
\end{aligned}
$$

$$- \quad 32x_2^2x_3^2x_5^2 - 24x_2^2x_3^2x_4^4x_5^2 - 16x_1^4x_2^4x_5^2 - 8x_1^6x_4^2x_5^2 + 40x_2^2x_3^2x_4^2x_5^2$$

$$- \quad 16x_1^4x_3^4x_5^2 - 24x_1^4x_2^2x_4^2x_5^2 + 28x_3^4x_4^2x_5^2 - 8x_1^2x_2^6x_5^2 + 64x_1^2x_2^4x_3^4x_4^2x_5^2$$

$$- \quad 32x_1^2x_4^2x_5^2 - 16x_3^4x_4^4x_5^2 - 12x_3^4x_5^2 + 28x_2^2x_3^4x_5^2 - 144x_1^2x_2^2x_3^4x_4^2x_5^2$$

$$+ \quad 16x_2^2x_3^6x_4^2x_5^2 + 28x_1^2x_4^4x_5^2 + 4x_4^6x_5^2 + 64x_1^2x_2^2x_3^4x_4^4x_5^2 - 24x_2^4x_3^2x_4^2x_5^2$$

$$+ \quad 28x_2^4x_3^2x_5^2 - 8x_1^6x_2^2x_5^2 + 16x_1^6x_2^2x_3^2x_5^2 + 16x_1^2x_2^2x_4^6x_5^2 + 16x_2^2x_3^2x_4^6x_5^2$$

$$+ \quad 12x_4^2x_5^2 - 24x_1^4x_2^2x_3^2x_5^2 + 16x_1^2x_3^6x_4^2x_5^2 - 24x_1^2x_2^4x_3^2x_5^2 - 8x_2^2x_3^6x_5^2$$

$$+ \quad 64x_1^4x_2^2x_3^2x_4^4x_5^2 + 192x_1^2x_2^2x_3^2x_4^2x_5^2 - 12x_2^4x_5^2 - 24x_1^2x_2^4x_4^2x_5^2 + 12x_2^2x_5^2$$

$$- \quad 8x_1^2x_3^6x_5^2 + 40x_1^2x_3^2x_4^2x_5^2 - 24x_1^2x_2^2x_4^4x_5^2 - 32x_1^2x_2^2x_5^2 + 64x_1^4x_2^2x_3^4x_4^2x_5^2$$

$$+ \quad 28x_2^2x_4^4x_5^2 - 8x_1^2x_4^6x_5^2 - 4x_5^2 + 4x_1^6x_5^2 + 12x_1^2x_5^2 + 28x_1^4x_4^2x_5^2 - 16x_2^4x_3^4x_5^2$$

$$+ \quad 16x_1^6x_3^2x_4^2x_5^2 - 8x_2^6x_4^2x_5^2 + 16x_1^6x_2^2x_4^2x_5^2 + 64x_1^4x_2^4x_3^2x_4^2x_5^2$$

$$- \quad 24x_1^2x_3^2x_4^4x_5^2 + 12x_3^2x_5^2 + 16x_1^2x_2^2x_3^6x_5^2 + 16x_2^6x_3^2x_4^2x_5^2 + 16x_1^2x_2^6x_3^2x_5^2$$

$$+ \quad 40x_1^2x_2^2x_4^2x_5^2 - 8x_3^2x_4^6x_5^2 - 24x_1^2x_3^4x_4^2x_5^2 - 16x_2^4x_4^4x_5^2 + 28x_1^2x_3^4x_5^2$$

$$- \quad 144x_1^2x_2^2x_3^2x_4^4x_5^2 + 28x_2^4x_4^2x_5^2 - 8x_3^6x_4^2x_5^2 - 32x_2^2x_4^2x_5^2 + 64x_1^2x_2^4x_3^2x_4^4x_5^2$$

$$+ \quad 40x_1^2x_2^2x_3^2x_5^2 - 24x_2^2x_3^4x_4^2x_5^2 - 144x_1^2x_2^4x_3^2x_4^2x_5^2 - 12x_1^4x_5^2 + 4x_3^6x_5^2$$

$$+ \quad 28x_1^2x_2^4x_5^2 - 144x_1^4x_2^2x_3^2x_4^2x_5^2 - 8x_2^6x_3^2x_5^2 + 28x_1^4x_2^2x_5^2 - 32x_3^2x_4^2x_5^2$$

$$- \quad 32x_1^2x_3^2x_5^2 + 4x_2^6x_5^2 - 24x_1^4x_3^2x_4^2x_5^2 - 24x_1x_2^5x_3x_4x_5 - 112x_1^3x_2x_3x_4^3x_5$$

$$- \quad 112x_1^3x_2^3x_3x_4x_5 + 40x_1^3x_2x_3^5x_4x_5 - 112x_1x_2x_3^3x_4^3x_5 - 32x_1x_2^3x_3^5x_4^3x_5$$

$$- \quad 32x_1^3x_2x_3^5x_4^3x_5 - 8x_1x_2x_3^7x_4x_5 + 40x_1^3x_2x_3x_4^5x_5 - 24x_1x_2x_3x_4^5x_5$$

$$- \quad 112x_1x_2^3x_3x_4^3x_5 + 40x_1x_2^5x_3^3x_4x_5 + 40x_1x_2^3x_3x_4^5x_5 - 32x_1^5x_2x_3^3x_4^3x_5$$

$$- \quad 32x_1^3x_2^3x_3^5x_4x_5 + 40x_1^3x_2^5x_3x_4x_5 - 8x_1x_2^7x_3x_4x_5 - 8x_1x_2x_3x_4^7x_5 + 112x_1x_2^3x_3^3x_4^3x_5$$

$$+ \quad 40x_1x_2^5x_3x_4^3x_5 - 112x_1^3x_2x_3^3x_4x_5 + 40x_1^5x_2x_3^3x_4x_5 + 72x_1x_2x_3^3x_4x_5$$

$$- \quad 32x_1x_2^3x_3^3x_4^5x_5 + 72x_1x_2^3x_3x_4x_5 + 40x_1^5x_2^3x_3x_4x_5 + 40x_1^5x_2x_3x_4^3x_5$$

$$- \quad 32x_1^3x_2^3x_3x_4^5x_5 - 24x_1x_2x_3^5x_4x_5 - 32x_1^5x_2^3x_3x_4^3x_5 - 32x_1^3x_2^5x_3x_4^3x_5$$

$$
\begin{aligned}
&- && 32x_1x_2^5x_3^3x_4^3x_5 + 112x_1^3x_2^3x_3x_4^3x_5 + 40x_1x_2x_3^5x_4^3x_5 - 40x_1x_2x_3x_4x_5 \\
&+ && 40x_1x_2x_3^3x_4^5x_5 - 32x_1^5x_2^3x_3^3x_4x_5 - 112x_1x_2^3x_3^3x_4x_5 - 24x_1^5x_2x_3x_4x_5 \\
&- && 32x_1^3x_2^5x_3^3x_4x_5 + 72x_1x_2x_3x_4^3x_5 - 8x_1^7x_2x_3x_4x_5 + 112x_1^3x_2^3x_3^3x_4x_5 \\
&+ && 112x_1^3x_2x_3^3x_4^3x_5 - 32x_1^3x_2x_3^3x_4^5x_5 + 40x_1x_2^3x_3^5x_4x_5 + 72x_1^3x_2x_3x_4x_5 \\
&+ && 28x_1^4x_3^2x_4^2 + 16x_1^2x_2^6x_3^2x_4^2 - 24x_1^2x_2^2x_3^2x_4^4 + 28x_1^2x_2^2x_4^4 - 8x_1^2x_3^2x_4^6 \\
&- && 16x_1^2x_2^4x_4^4 + 28x_2^2x_3^4x_4^2 + 28x_1^4x_2^2x_4^2 - 32x_1^2x_2^2x_4^2 - 8x_1^2x_2^2x_4^6 + 4x_3^2x_4^6 \\
&- && 24x_1^2x_2^2x_3^4x_4^2 - 12x_3^2x_4^4 + 16x_2^4x_3^4x_4^4 + 16x_1^2x_2^2x_3^6x_4^2 + 4x_2^2x_4^6 - 32x_1^2x_3^2x_4^2 \\
&+ && 4x_2^2x_3^6 - 24x_1^2x_2^4x_3^2x_4^2 - 12x_2^4x_4^2 + 28x_1^2x_3^4x_4^2 + 6x_1^4x_4^4 - 16x_1^4x_3^2x_4^4 + 28x_1^2x_2^4x_4^2 \\
&+ && 12x_1^2x_3^2 + 6x_3^4x_4^4 + 16x_1^4x_2^4x_4^4 - 8x_2^2x_3^2x_4^6 - 12x_2^2x_4^4 - 32x_2^2x_3^2x_4^2 - 8x_2^2x_3^6x_4^2 \\
&+ && 12x_2^2x_4^2 + 4x_2^6x_3^2 - 16x_1^4x_2^2x_3^4 - 16x_2^4x_3^4x_4^2 + 40x_1^2x_2^2x_3^2x_4^2 + 6x_3^4 - 8x_2^6x_3^2x_4^2 \\
&- && 4x_2^2 - 16x_1^2x_3^4x_4^4 + x_3^8 - 12x_1^2x_4^4 - 16x_2^4x_3^2x_4^4 + 6x_1^4x_3^4 - 16x_1^4x_2^4x_4^2 - 12x_1^4x_2^2 \\
&+ && 16x_1^4x_3^4x_4^4 - 4x_4^2 - 8x_1^2x_3^6x_4^2 + x_4^8 - 8x_1^2x_2^6x_3^2 - 16x_1^4x_3^4x_4^2 + 28x_1^2x_2^2x_3^4 \\
&+ && 4x_3^6x_4^2 + 16x_1^4x_2^4x_3^4 + 16x_1^6x_2^2x_3^2x_4^2 + 12x_2^2x_3^2 + 4x_1^2x_3^6 + 4x_1^2x_4^6 + 4x_2^6x_4^2 \\
&- && 8x_1^2x_2^2x_3^6 + 16x_1^2x_2^2x_3^2x_4^6 - 4x_4^6 - 8x_1^6x_2^2x_3^2 - 12x_1^2x_2^4 - 16x_1^4x_2^4x_3^2 - 12x_1^2x_3^4 \\
&- && 12x_2^4x_3^2 - 16x_2^2x_3^4x_4^4 + 12x_1^2x_4^2 + x_1^8 + 4x_1^6x_4^2 - 24x_1^4x_2^2x_3^2x_4^2 - 8x_1^2x_2^6x_4^2 \\
&+ && 6x_4^4 + 12x_3^2x_4^2 + 28x_2^2x_3^2x_4^4 + 6x_1^4x_2^4 + 6x_1^4 + 28x_1^4x_2^2x_3^2 + 28x_2^4x_3^2x_4^2 + 6x_2^4x_3^4 \\
&- && 32x_1^2x_2^2x_3^2 + 4x_1^2x_2^6 - 4x_3^2 - 4x_1^6 - 4x_1^2 - 8x_1^6x_3^2x_4^2 + x_2^8 - 16x_1^4x_2^2x_4^4 - 16x_1^2x_2^4x_3^4 \\
&+ && 4x_1^6x_2^2 + 6x_2^4x_4^4 - 4x_3^6 - 8x_1^6x_2^2x_4^2 - 12x_3^4x_4^2 + 12x_1^2x_2^2 - 12x_2^2x_3^4 + 28x_1^2x_3^2x_4^4 \\
&- && 12x_1^4x_4^2 + 28x_1^2x_2^4x_3^2 + 1 - 4x_2^6 - 12x_1^4x_3^2 + 6x_2^4 + 4x_1^6x_3^2.
\end{aligned}
$$

The recursion given in Theorem 3.2.20, although fundamental, is a special case of a more general recursion that P_n satisfies:

Claim 3.2.22. *Let $P_1 = x_1 - 1$ and $n \geq 2$. For $n_1 + \cdots + n_k = n$ we have (with some abuse* *of notation) that*

$$P_n(x_1,\ldots,x_n) = \prod_{\substack{\sigma_i\in G_{n_i-1}\\ i=1,\ldots,k}} P_k(\sigma_1(\mathrm{EC}_{n_1}(x_1,\ldots,x_{n_1}),\sigma_2(\mathrm{EC}_{n_2}(x_{n_1+1},\ldots,x_{n_1+n_2}),$$

$$\ldots,\sigma_k(\mathrm{EC}_{n_k}(x_{n-n_k+1},\ldots,x_n)).$$

Example 3.2.23. P_5 is the smallest example that can be generated using a recurrence that is not an example of the special recurrence from Theorem 3.2.20:

$$\begin{aligned}
P_5(x_1,\ldots,x_5) &= \prod_{\substack{\sigma_1'\in G_1=\langle\sigma_1\rangle\\ \sigma_2'\in G_2=\{e\}\\ \sigma_3'\in G_3=\langle\sigma_4\rangle}} P_3(\sigma_1'(\mathrm{EC}_2(x_1,x_2)),\sigma_2'(\mathrm{EC}_1(x_3)),\sigma_3'(\mathrm{EC}_2(x_4,x_5)))\\
&= \prod_{\substack{\sigma_1'\in G_1=\langle\sigma_1\rangle\\ \sigma_2'\in G_2=\{e\}\\ \sigma_3'\in G_3=\langle\sigma_4\rangle}} P_3(\sigma_1'(x_1x_2-y_1y_2),\sigma_2'(x_3),\sigma_3'(x_4x_5-y_4y_5))\\
&= P_3(x_1x_2-y_1y_2,x_3,x_4x_5-y_4y_5)\cdot P_3(x_1x_2+y_1y_2,x_3,x_4x_5-y_4y_5)\\
&\cdot\ P_3(x_1x_2-y_1y_2,x_3,x_4x_5+y_4y_5)\cdot P_3(x_1x_2+y_1y_2,x_3,x_4x_5+y_4y_5)\\
&= ((x_1x_2-y_1y_2)^2+x_3^2+(x_4x_5-y_4y_5)^2-2(x_1x_2-y_1y_2)x_3(x_4x_5-y_4y_5)-1)\\
&\cdot\ ((x_1x_2+y_1y_2)^2+x_3^2+(x_4x_5-y_4y_5)^2-2(x_1x_2+y_1y_2)x_3(x_4x_5-y_4y_5)-1)\\
&\cdot\ ((x_1x_2-y_1y_2)^2+x_3^2+(x_4x_5+y_4y_5)^2-2(x_1x_2-y_1y_2)x_3(x_4x_5+y_4y_5)-1)\\
&\cdot\ ((x_1x_2+y_1y_2)^2+x_3^2+(x_4x_5+y_4y_5)^2-2(x_1x_2+y_1y_2)x_3(x_4x_5+y_4y_5)-1),
\end{aligned}$$

Expanded, this yields the same expression for P_5 as given in Example 3.2.21.

Our final goal in this chapter is to prove the irreducibility of P_n. To illuminate our approach we state and prove the following simplest case, that $P_3 = P_3(x_1,x_2,x_3)$ is irreducible.

Suppose $P_3 = fg$ with $f,g\in\mathbb{Q}[x_1,x_2,x_3]$. Since P_3 is monic in x_3, both f and g contain the variable x_3, and hence both f and g are of degree 1 in x_3 (unless f or $g = P_3$.) Since

P_3 factors in $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3]$ as

$$P_3 = (x_3 - x_1x_2 - y_1y_2)(x_3 - x_1x_2 + y_1y_2)$$

by definition of P_3, then since $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3]$ is a UFD we must have

$$\{f, g\} = \{x_3 - x_1x_2 - y_1y_2, x_3 - x_1x_2 + y_1y_2\}$$

which contradicts the assumption that $f, g \in \mathbb{Q}[x_1, x_2, x_3]$. Hence we have the following observation:

Observation 3.2.24. The polynomial $P_3(x_1, x_2, x_3)$ is irreducible over $\mathbb{Q}$.

We now want to use this same approach to prove the following.

Theorem 3.2.25. *For $n \geq 3$ the polynomials $P_n(x_1, \ldots, x_n)$ are irreducible over $\mathbb{Q}$.*

Before proving Theorem 3.2.25, we need to prove the following:

Claim 3.2.26. *For $n \geq 3$ we have*

$$P_{n-1}(\mathrm{EC}_2(x_1, x_2), x_3, \ldots, x_n) \quad = \quad P_{n-1}(x_1x_2 - y_1y_2, x_3, \ldots, x_n)$$

and

$$P_{n-1}(\overline{\mathrm{EC}_2}(x_1, x_2), x_3, \ldots, x_n) \quad = \quad P_{n-1}(x_1x_2 + y_1y_2, x_3, \ldots, x_n)$$

are irreducible in $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \ldots, x_n]$.

Proof. Assume $P^*_{n-1} := P_{n-1}(x_1x_2 - y_1y_2, x_3, \ldots, x_n)$ factors $P^*_{n-1} = h^* \cdot k^* \in \mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \ldots, x_n]$, where both h^* and k^* involve x_n. Since

$$P_{n-1} \quad = \quad \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-2})),$$

we see that

$$P_{n-1}^* \quad = \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\mathrm{EC}_{n-2}(x_1x_2 - y_1y_2, x_3, \ldots, x_n))),$$

and hence both h^* and k^* must be products of these linear factors.

In particular, we can evaluate $P_{n-1}^* = h^* \cdot k^*$ at $x_1 = 1$ and obtain

$$P_{n-1}(x_2, \ldots, x_n) \quad = \quad \left(P_{n-1}^*\right)|_{x_1=1} = (h^*|_{x_1-1})\,(k^*|_{x_1-1}) \;=\; h \cdot k$$

in $\mathbb{Q}(x_2)[x_3, \ldots, x_n]$, which is a UFD.

Since by the inductive hypothesis, $P_{n-1}(x_2, \ldots, x_n)$ is irreducible in $\mathbb{Q}(x_2)[x_3, \ldots, x_n]$, either h or k equals $P_{n-1}(x_2, \ldots, x_n)$, which contradicts the fact that both h^* and k^* involve x_n. Hence P_{n-1}^* is irreducible. In the same way we obtain that $P_{n-1}(x_1x_2 + y_1y_2, x_3, \ldots, x_n)$ is irreducible. □

Proof of Theorem 3.2.25. Assume $P_n = fg$ with $f, g \in \mathbb{Q}[x_1, \ldots, x_n]$. We may assume f is irreducible. Let

$$\begin{aligned} \phi_i : \mathbb{Q}[x_1, \ldots, x_n] \quad &\longrightarrow \quad \mathbb{Q}[\widehat{x_i}] \\ \phi_i(F) \quad &= \quad F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n). \end{aligned}$$

ϕ_i is a $\mathbb{Q}$-algebra homomorphism for each $i \in \{1, \ldots, n\}$ and hence a ring homomorphism. Therefore

$$\phi_1(P_n) \quad = \quad \phi_1(fg) = \phi_1(f)\phi_1(g) \in \mathbb{Q}[x_2, \ldots, x_n].$$

But $\phi_1(P_n) = P_{n-1}(x_2, \ldots, x_n)^2 \in \mathbb{Q}[x_2, \ldots, x_n]$, which is a UFD. By the inductive hypothesis, P_{n-1} is irreducible in $\mathbb{Q}[x_2, \ldots, x_n]$. Therefore, $\phi_1(f) = P_{n-1} = \phi_1(g)$ (unless $f = P_n$.)

Viewing $f, g \in \mathbb{Q}[x_1, \ldots, x_{n-1}][x_n]$, then since P_n and P_{n-1} are monic in every variable x_i (and hence also in x_n,) we have

$$\deg_{x_n}(f) = \deg_{x_n}(g) = \frac{\deg_{x_n}(P_n)}{2} = 2^{n-3}.$$

By symmetry of P_n for $n \geq 3$, from Theorem 3.2.15 and Theorem 3.2.20 we have

$$P_n \quad = \quad P_{n-1}(\mathrm{EC}_2(x_1, x_2), x_3, \ldots, x_n) \cdot P_{n-1}(\overline{\mathrm{EC}_2}(x_1, x_2), x_3, \ldots, x_n)$$

in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \ldots, x_n]$, which is a UFD.

Since by assumption $P_n = f \cdot g$ where $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is irreducible and $f|_{x_1=1} = P_{n-1}(x_2, \ldots, x_n)$, we obtain in the same way as in the proof of Claim 3.2.26 that f is irreducible in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \ldots, x_n]$.

So

$$P_n = f \cdot g = P_{n-1}(\mathrm{EC}_2, x_3, \ldots, x_n) \cdot P_{n-1}(\overline{\mathrm{EC}_2}, x_3, \ldots, x_n)$$

in $\mathbb{Q}(x_1, x_2, y_1 y_2)[x_3, \ldots, x_n]$, which is a UFD. Hence $f \in \{P_{n-1}(\mathrm{EC}_2, x_3, \ldots, x_n), P_{n-1}(\overline{\mathrm{EC}_2}, x_3, \ldots, x_n)\}$, which is a contradiction, since $f \in \mathbb{Q}[x_1, \ldots, x_n]$. $\square$

As a corollary we obtain the following, which in fact equivalent to Theorem 3.2.25:

Corollary 3.2.27. *For $n \geq 1$ we have*

$$[\mathbb{Q}(x_1, \ldots, x_n, \mathrm{EC}_n) : \mathbb{Q}(x_1, \ldots, x_n)] \quad = \quad 2^{n-1}.$$

Since for $m \leq n$, $\mathrm{EC}_m = \mathrm{EC}_m(x_1, \ldots, x_m)$ is only in terms of the first m of n variables, then for any $m \leq n$ we have

$$[\mathbb{Q}(x_1, \ldots, x_n, \mathrm{EC}_m) : \mathbb{Q}(x_1, \ldots, x_n)] \quad = \quad 2^{m-1}.$$

So we obtain a summarizing result:

Corollary 3.2.28. *For $1 \leq m \leq n$ we have*

- $\mathbb{Q}(x_1, \ldots, x_n, \mathrm{EC}_m) = \mathbb{Q}(x_1, \ldots, x_n, y_1 y_2, \ldots, y_{m-1} y_m)$

- $$\begin{aligned} &\mathrm{Gal}(\mathbb{Q}(x_1, \ldots, x_n, \mathrm{EC}_m)/\mathbb{Q}(x_1, \ldots, x_n)) \\ &= \mathrm{Gal}(\mathbb{Q}(x_1, \ldots, x_n, y_1 y_2, \ldots, y_{m-1} y_m)/\mathbb{Q}(x_1, \ldots, x_n)) \\ &\cong \mathbb{Z}_2^{m-1}. \end{aligned}$$

- $P_{m+1}(x_1, \ldots, x_m, X) \in \mathbb{Q}(x_1, \ldots, x_m)[X]$ *is the minimal polynomial of* $\mathrm{EC}_m = \mathrm{EC}_m(x_1, \ldots, x_m)$ *over* $\mathbb{Q}(x_1, \ldots, x_m)$.

Chapter 4: Results from Elementary Number Theory

4.1 Generalizations of the Pythagorean Triples

These generalizations of the well-known Pythagorean Triples problem will be needed in the next chapter. In the following, a primitive solution is a solution where x, y, and z are pairwise relatively prime.

Theorem 4.1.1. *Let β be a square-free integer. The integers x, y, z form a primitive solution to the Diophantine equation $x^2 + \beta y^2 = z^2$ if and only if there are positive integers m and n and a factorization $\beta = bc$ where bm^2 and cn^2 are relatively prime such that*

$$x = \frac{bm^2 - cn^2}{2}, \quad y = mn, \quad z = \frac{bm^2 + cn^2}{2},$$

where both m and n are odd or both are even, or

$$x = bm^2 - cn^2, \quad y = 2mn, \quad z = bm^2 + cn^2$$

otherwise.

To prove Theorem 4.1.1, we need the following:

Claim 4.1.2. *If r,s,t are positive integers such that r and s are relatively prime and $rs = t^2$ then there are relatively prime integers m and n such that $r = m^2$ and $s = n^2$.*

Proof of Theorem 4.1.1. Note that $\gcd(b, c) = 1$. This proof follows and extends the exposition in [12].

Assume x, y, z form a primitive solution. In this case, x and y cannot both be even.

Case 1: x, y are both odd. Then $x^2 \equiv 1 \pmod 4$ and $y^2 \equiv 1 \pmod 4$, giving $z^2 \equiv 1 + \beta \pmod 4$. Since $z^2 \equiv 0, 1 \pmod 4$, then $\beta \equiv 0$ or $\beta \equiv 3 \pmod 4$ must hold. However, $\beta \equiv 0 \pmod 4$ implies that 4 divides β, contradicting the assumption that β is square-free. So the only case to consider here is the case where z is even and $\beta \equiv 3 \pmod 4$.

$$\beta y^2 \quad = \quad z^2 - x^2 = (z+x)(z-x). \tag{4.1}$$

Letting $\gcd(z+x, z-x) = d$ we get d divides both $z+x+z-x = 2z$ and $z+x-(z-x) = 2x$. Since x and z are relatively prime, $d = 1$ or 2. Since both $z+x$ and $z-x$ are odd, then $d = 1$ must hold. Since now $\gcd(z+x, z-x) = 1$ we have from (4.1) that for some factorization $\beta = bc$ then $r = z + x$ is divisible by b and $s = z - x$ is divisible by c. Since $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$, we have by Claim 4.1.2 that $m^2 = \frac{r}{b}$ and $n^2 = \frac{s}{c}$, and hence $y = mn$, $x = \frac{r-s}{2} = \frac{bm^2-cn^2}{2}$, and $z = \frac{r+s}{2} = \frac{bm^2+cn^2}{2}$.

Case 2: x is even and y is odd. Then $x^2 \equiv 0 \pmod 4$ and $y^2 \equiv 1 \pmod 4$, giving $z^2 \equiv \beta \pmod 4$. Therefore $\beta \equiv 0$ or $\beta \equiv 1 \pmod 4$. However, $\beta \equiv 0 \pmod 4$ implies that 4 divides β, again contradicting the assumption that β is square-free. So the only case to consider here is the case where z is odd and $\beta \equiv 1 \pmod 4$, which proceeds exactly as in case 1.

Case 3: x is odd and y is even. Then $x^2 \equiv 1 \pmod 4$ and $y^2 \equiv 0 \pmod 4$, giving $z^2 \equiv 1 \pmod 4$, and so z is odd.

Unlike cases 2 and 3, $z + x$ and $z - x$ are both even. Letting $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = d$ we get that d divides $\frac{z+x+z-x}{2} = z$ and $\frac{z+x-(z-x)}{2} = x$. Since x and z are relatively prime, $d = 1$. Now we have $\frac{\beta y^2}{4} = rs$ where $r = \frac{z+x}{2}$ and $s = \frac{z-x}{2}$. Hence b divides r and c divides s for some appropriate factorization $\beta = bc$. Since $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$, so we have by Claim 4.1.2 that $m^2 = \frac{r}{b}$ and $n^2 = \frac{s}{c}$ and hence $y = 2mn$, $x = r - s = bm^2 - cn^2$, and $z = r + s = bm^2 + cn^2$.

For the other direction, first we show that x, y, z as given in cases 1 and 2 do form a solution:

$$\begin{aligned}
x^2+\beta y^2 &= \left(\frac{bm^2-cn^2}{2}\right)^2+\beta(mn)^2\\
&= \frac{(bm^2)^2-2bm^2cn^2+(cn^2)^2}{4}+\beta(mn)^2\\
&= \frac{(bm^2)^2+2\beta m^2n^2+(cn^2)^2}{4}\\
&= \left(\frac{bm^2+cn^2}{2}\right)^2.
\end{aligned}$$

Also for case 3 we get:

$$\begin{aligned}
x^2+\beta y^2 &= \left(bm^2-cn^2\right)^2+\beta(2mn)^2\\
&= (bm^2)^2-2bm^2cn^2+(cn^2)^2+\beta(2mn)^2\\
&= (bm^2)^2+2\beta m^2n^2+(cn^2)^2\\
&= \left(bm^2+cn^2\right)^2.
\end{aligned}$$

To show that the triple is primitive for cases 1 and 2, assume on the contrary that $\gcd(x,y,z)=d>1$. Then there is a prime p that divides d. This p divides x and z and also their sum and difference: $x+z=\frac{bm^2-cn^2}{2}+\frac{bm^2+cn^2}{2}=bm^2$ and $x-z=\frac{bm^2-cn^2}{2}-\frac{bm^2+cn^2}{2}=cn^2$. This contradicts the assumption that bm^2 and cn^2 are relatively prime.

For case 3, again assume on the contrary that $(x,y,z)=d>1$. Then there is an odd prime p that divides d. $p\neq 2$ because x and z are both odd. This p divides x and z and also their sum and difference: $x+z=2bm^2$ and $x-z=2cn^2$. Again, this contradicts the assumption that bm^2 and cn^2 are relatively prime.

$\square$

Lemma 4.1.3. *The solution to the Diophantine equation $x^2+y^2=\gamma z^2$, where γ is square-free, is given by*

$$\begin{aligned} x &= \frac{c}{\gamma}(am^2 - an^2 + 2mnb) \\ y &= \frac{c}{\gamma}(-bm^2 + bn^2 + 2mna) \\ z &= \frac{c(m^2+n^2)}{\gamma}, \end{aligned}$$

where $\gamma = a^2+b^2$ and $m, n, c \in \mathbb{Z}$.

Proof. The existence of an integer solution to $x^2+y^2=\gamma z^2$ implies that each prime factor of γ is congruent to 1 modulo 4. Therefore, for some appropriate choice of non-negative integers a, b, we can write $\gamma = a^2+b^2$. Then we have:

$$(ax-by)^2 + (bx+ay)^2 = (a^2+b^2)(x^2+y^2) = \gamma(\gamma z^2) = (\gamma z)^2.$$

Then using the well-known formula for Pythagorean triples, let $ax - by = c(m^2-n^2)$, $bx + ay = 2cmn$, and $\gamma z = c(m^2+n^2)$. Solving for x, y, and z, we get the statement of the lemma. $\square$

Chapter 5: Rational Solutions when $n = 3$

5.1 A Necessary Condition for Rational Radii

In this chapter, we will characterize all rational solutions for flowers with three petals. We then compare our parametrization to an existing parametrization of the curvatures of four mutually tangent circles and show how our equation-free parameterization is an improvement on the existing one.

Recall coin graphs formed by n petals in the Euclidean plane. By Definition 1.3.2 we have by scaling the following observation:

Observation 5.1.1. For positive integers n we have $T(\mathbb{N}; n) = T(\mathbb{Q}; n)$ when both $\mathbb{N}$ and $\mathbb{Q}$ are viewed as multisets.

Our goal is to characterize all flowers with three petals and integral radii. By the observation, we can look at rational radii and then scale as necessary. When the lengths of the sides of a triangle are rational, the cosine will be rational. The converse is not necessarily true, however. We would like to find rational radii that create a flower configuration with 3 petals. For a necessary first step, we will determine what the cosines must be.

First, we have the irreducible polynomial $P_3 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1$. We can solve for any one of the variables, say x_3, by definition of P_3 and Lemma 3.2.7:

$$x_3 \quad = \quad \mathrm{EC}_2(x_1, x_2) = x_1x_2 - \sqrt{(1 - x_1^2)(1 - x_2^2)}.$$

Now it is clear that x_3 will be rational if and only if x_1, x_2 are rational, and the term under the radical is the square of a rational number. Since we want the x_i to be rational, let

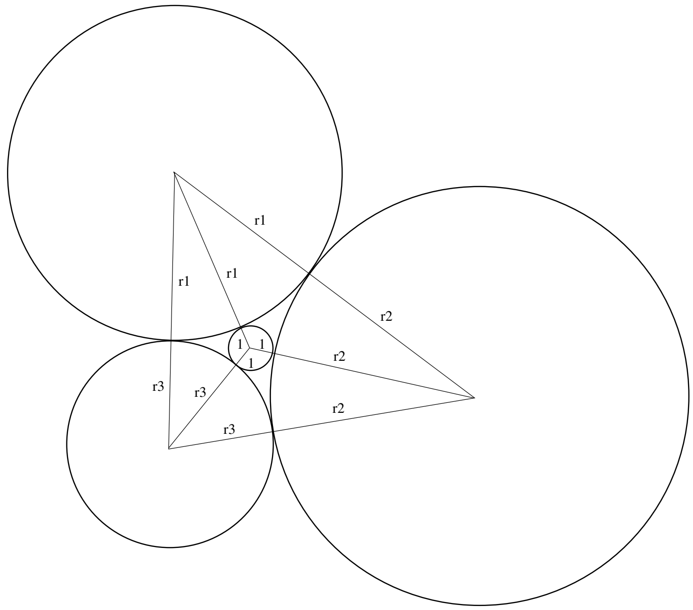

Figure 5.1: A 3-petaled flower.

$x_i = \frac{p_i}{q_i}$ for $i = 1, 2$ with $p_i, q_i \in \mathbb{Z}$. Then we can transform the term under the radical:

$$x_3 = x_1 x_2 - \frac{1}{q_1 q_2}\sqrt{(q_1^2 - p_1^2)(q_2^2 - p_2^2)}.$$

When $(q_1^2 - p_1^2)(q_2^2 - p_2^2)$ is a square, we can write $q_i^2 - p_i^2 = s_i^2 \beta$ for $i = 1, 2$, where β is square-free.

5.2 Parametrization of the Radii

Suppose we have a flower with 3 petals (see figure 5.1.) By scaling, we can assume the radius of the eye (center coin) is 1. Then we can denote the other 3 radii by r_1, r_2, r_3 and the angles between the edges incident on the center of the eye by $\theta_1, \theta_2, \theta_3$. Then applying the law of cosines, we obtain

$$\begin{aligned}
x_1 &= \cos\theta_1 = \frac{(r_1+1)^2+(r_2+1)^2-(r_1+r_2)^2}{2(r_1+1)(r_2+1)}, \\
x_2 &= \cos\theta_2 = \frac{(r_2+1)^2+(r_3+1)^2-(r_2+r_3)^2}{2(r_2+1)(r_3+1)}, \\
x_3 &= \cos\theta_3 = \frac{(r_3+1)^2+(r_1+1)^2-(r_3+r_1)^2}{2(r_3+1)(r_1+1)},
\end{aligned}$$

where for convenience of notation, we will denote $\cos\theta_i$ as x_i. By expanding both the numerator and denominator of x_1, we obtain

$$x_1 = \frac{r_1+r_2-r_1r_2+1}{r_1+r_2+r_1r_2+1}.$$

Rewriting this as a polynomial equation in terms of r_1 and r_2 we obtain

$$r_1r_2+\left(\frac{x_1-1}{x_1+1}\right)(r_1+r_2) = \left(\frac{1-x_1}{x_1+1}\right).$$

Factoring in terms of r_1 and r_2, we get

$$\left(r_1+\frac{x_1-1}{x_1+1}\right)\left(r_2+\frac{x_1-1}{x_1+1}\right) = \frac{2(1-x_1)}{(x_1+1)^2}.$$

And similarly for x_2 and x_3 we get

$$\begin{aligned}
\left(r_2+\frac{x_2-1}{x_2+1}\right)\left(r_3+\frac{x_2-1}{x_2+1}\right) &= \frac{2(1-x_2)}{(x_2+1)^2}, \\
\left(r_3+\frac{x_3-1}{x_3+1}\right)\left(r_1+\frac{x_3-1}{x_3+1}\right) &= \frac{2(1-x_3)}{(x_3+1)^2}.
\end{aligned}$$

Now we can solve the first and third equations for r_2 and r_3 respectively in terms of r_1, x_1, x_3. Substituting these into the second equation, we can then solve it for r_1 in terms of x_1, x_2, x_3, obtaining

$$r_1 \quad = \quad \frac{-1 - x_1x_3 + x_3 + x_1 \pm \sqrt{2(1-x_1)(1-x_2)(1-x_3)}}{2x_2 - x_1 + x_1x_3 - 1 - x_3}. \tag{5.1}$$

Since $x_i = \frac{p_i}{q_i}$ for $i = 1, 2$ we have by Theorem 4.1.1 that

$$x_1 = \frac{b_1m_1^2 - c_1n_1^2}{b_1m_1^2 + c_1n_1^2}, \quad x_2 = \frac{b_2m_2^2 - c_2n_2^2}{b_2m_2^2 + c_2n_2^2},$$

where $\beta = b_1c_1 = b_2c_2$ are two factorizations of the square-free integer β, and where m_i, n_i can be chosen from the non-negative integers. Hence

$$
\begin{aligned}
x_3 &= x_1x_2 - \sqrt{(1-x_1^2)(1-x_2^2)} \\
&= \frac{(b_1m_1^2 - c_1n_1^2)(b_2m_2^2 - c_2n_2^2)}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} - \sqrt{\left(1 - \left(\frac{b_1m_1^2 - c_1n_1^2}{b_1m_1^2 + c_1n_1^2}\right)^2\right)\left(1 - \left(\frac{b_2m_2^2 - c_2n_2^2}{b_2m_2^2 + c_2n_2^2}\right)^2\right)} \\
&= \frac{(b_1m_1^2 - c_1n_1^2)(b_2m_2^2 - c_2n_2^2)}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} \\
&- \frac{\sqrt{\left((b_1m_1^2 + c_1n_1^2)^2 - (b_1m_1^2 - c_1n_1^2)^2\right)\left((b_2m_2^2 + c_2n_2^2)^2 - (b_2m_2^2 - c_2n_2^2)^2\right)}}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} \\
&= \frac{(b_1m_1^2 - c_1n_1^2)(b_2m_2^2 - c_2n_2^2)}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} - \frac{\sqrt{\left(4b_1c_1m_1^2n_1^2\right)\left(4b_2c_2m_2^2n_2^2\right)}}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} \\
&= \frac{(b_1m_1^2 - c_1n_1^2)(b_2m_2^2 - c_2n_2^2)}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} - \frac{4m_1m_2n_1n_2\sqrt{b_1c_1b_2c_2}}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)} \\
&= \frac{\left(b_1m_1^2 - c_1n_1^2\right)\left(b_2m_2^2 - c_2n_2^2\right) - 4m_1m_2n_1n_2\beta}{(b_1m_1^2 + c_1n_1^2)(b_2m_2^2 + c_2n_2^2)}.
\end{aligned}
$$

Substituting these expressions for x_1, x_2, x_3 into Equation 5.1, we get an expression for r_1 in terms of $b_1, b_2, c_1, c_2, m_1, m_2, n_1, n_2$:

$$
\begin{aligned}
r_1 &= \frac{n_1(b_2c_1^2m_2^2n_1^3 + 2\beta c_1m_1m_2n_1^2n_2 + b_1c_1c_2m_1^2n_1n_2^2)}{b_1c_1c_2m_1^2n_1^2n_2^2 - b_2c_1^2m_2^2n_1^4 + c_1^2c_2n_1^4n_2^2 - 2\beta c_1m_1m_2n_1^3n_2 + b_1^2c_2m_1^4n_2^2} \\
&\pm \frac{n_1n_2(b_1m_1^2 + c_1n_1^2)\sqrt{c_1c_2(b_1c_2m_1^2n_2^2 + 2\beta m_1m_2n_1n_2 + b_2c_1m_2^2n_1^2)}}{b_1c_1c_2m_1^2n_1^2n_2^2 - b_2c_1^2m_2^2n_1^4 + c_1^2c_2n_1^4n_2^2 - 2\beta c_1m_1m_2n_1^3n_2 + b_1^2c_2m_1^4n_2^2}.
\end{aligned}
$$

This expression has one term to the one-half power, so in order for r_1 to be rational, this term must be a perfect square. The term under the radical is $c_1c_2(b_1m_1^2c_2n_2^2 + 2m_1n_1m_2n_2\beta + c_1n_1^2b_2m_2^2)$. Thus we must find conditions for when this term is a square. Using the fact that $\beta = b_1c_1 = b_2c_2$, we can reduce this expression:

$$\begin{aligned} c_1c_2(b_1c_2m_1^2n_2^2 + 2\beta m_1m_2n_1n_2 + b_2c_1m_2^2n_1^2) &= \beta c_2^2m_1^2n_2^2 + 2\beta c_1c_2m_1m_2n_1n_2 + \beta c_1^2m_2^2n_1^2 \\ &= \beta(c_2m_1n_2 + c_1m_2n_1)^2. \end{aligned}$$

Thus this expression will only yield a perfect square when $\beta = 1$. Going back up to the first section of this chapter, r_1 is therefore rational only when $q_i^2 - p_i^2 = s_i^2$, or in other words when $1 - x_i^2$ is a perfect square for $i = 1, 2$.

Proposition 5.2.1. *The 3-petaled flower where the radii r_1, r_2, r_3 are parametrized by the method given here has rational radii only when $\beta = 1$, that is when $1 - x_i^2$ is the square of a rational number.*

Now we can write a parametrization for the cosines x_i and the radii r_i in the case where $n = 3$. Let $m_1, n_1, m_2, n_2 \in \mathbb{N}$. Then

$$\begin{aligned} x_1 &= \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \\ x_2 &= \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2}, \\ x_3 &= \frac{\left(m_1^2 - n_1^2\right)\left(m_2^2 - n_2^2\right) - 4m_1m_2n_1n_2}{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}. \end{aligned} \tag{5.2}$$

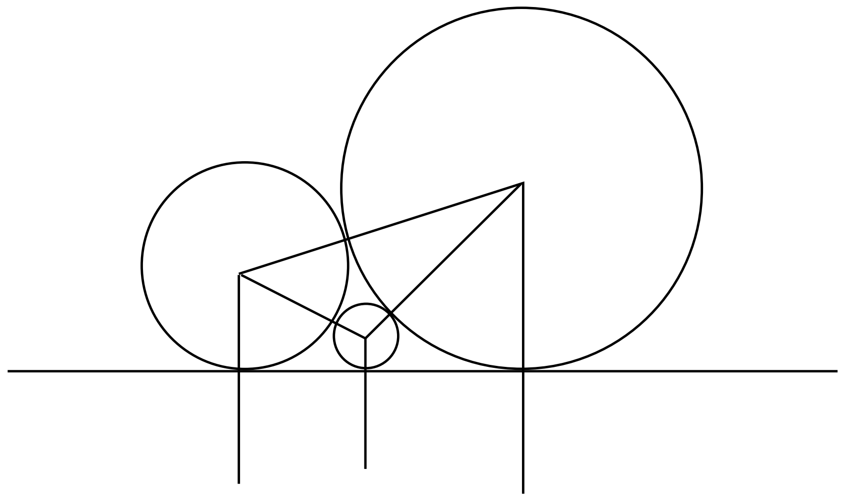

Figure 5.2: A 3-petaled flower where the radius of one petal is increased to infinity.

$$
\begin{aligned}
r_1 &= \frac{n_1(m_1n_2+m_2n_1)}{-m_1^2n_2-n_1^2n_2\pm(m_1n_1n_2+m_2n_1^2)}\\
r_2 &= \frac{n_1n_2}{-n_1n_2\pm(m_2n_1+m_1n_2)}\\
r_3 &= \frac{n_2(m_1n_2+m_2n_1)}{-m_1n_2^2-m_2n_1n_2\pm(n_1n_2^2+m_2^2n_1)}.
\end{aligned}
$$

We will determine the signs of the terms in the denominator in what follows.

5.3 Obtaining Meaningful Solutions

Proposition 5.3.1. *If $\theta_1, \theta_2, \theta_3$ are the angles incident on the center of the eye of a 3-petaled flower, then $90° < \theta_i < 180°$ for each i. These three inequalities are all sharp.*

Proof. Consider a 3-petaled flower. Keep the radii r_1 and r_2 fixed and let $r_3 \to \infty$. Then the radius r of the central coin will increase and θ_1, the angle between the first and second coins, will decrease. Figure 5.2 illustrates this situation.

By symmetry it suffices to show that $\theta_1 > 90°$. If we start with Figure 5.2 and draw a line parallel to the infinite circle that goes through the center of the central coin, we have 2 right triangles with side lengths $r_i - r, r_i + r$, and using the Pythagorean theorem, $2\sqrt{r_i r}$. Therefore, the length of the segment forming the bottom of the rhombus we are investigating is $2\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right)$.

We can now draw a segment parallel to this segment and passing through the center of the coin with the smaller radius. Without loss of generality, say $r_1 \leq r_2$. Now we have a right triangle with side lengths $2\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right), r_2 - r_1$ and $r_1 + r_2$. Then the Pythagorean theorem gives us:

$$4\left(\sqrt{r_1 r} + \sqrt{r_2 r}\right)^2 + (r_2 - r_1)^2 \quad = \quad (r_1 + r_2)^2$$

which can be solved for r, obtaining

$$r = \frac{r_1 r_2}{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}.$$

With this expression for r, we can show that $(r_1 + r_2)^2 > (r + r_1)^2 + (r + r_2)^2$ which implies $\theta_1 > 90°$:

$$
\begin{aligned}
(r_1+r_2)^2 &= \frac{(r_1+r_2)^2\left(\sqrt{r_1}+\sqrt{r_2}\right)^4}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^4} \\
&= \frac{r_1^4+r_2^4+14r_1^2r_2^2+8\left(r_1r_2^3+r_1^3r_2\right)+12r_1^{5/2}r_2^{3/2}+12r_1^{3/2}r_2^{5/2}+4\left(r_1^{7/2}\sqrt{r_2}+\sqrt{r_1}r_2^{7/2}\right)}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^4} \\
&> \frac{r_1^4+r_2^4+8\left(r_1^2r_2^2+r_1^3r_2+r_1^{5/2}r_2^{3/2}+r_1^{3/2}r_2^{5/2}+r_1r_2^3\right)+4\left(r_1^{7/2}\sqrt{r_2}+\sqrt{r_1}r_2^{7/2}\right)}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^4} \\
&= \frac{2r_1^2r_2^2+(2r_1^2r_2+2r_1r_2^2)\left(\sqrt{r_1}+\sqrt{r_2}\right)^2+(r_1^2+r_2^2)\left(\sqrt{r_1}+\sqrt{r_2}\right)^4}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^4} \\
&= \left(\frac{r_1r_2+r_1\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}\right)^2+\left(\frac{r_1r_2+r_2\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}\right)^2 \\
&= \left(\frac{r_1r_2}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}+r_1\right)^2+\left(\frac{r_1r_2}{\left(\sqrt{r_1}+\sqrt{r_2}\right)^2}+r_2\right)^2 \\
&= (r+r_1)^2+(r+r_2)^2.
\end{aligned}
$$

□

We now know that for all the angles θ_i, we have $90^\circ < \theta_i < 180^\circ$, and hence $-1 < \cos\theta_i < 0$. So in the parameterization of x_1 and x_2

$$x_1 = \frac{m_1^2-n_1^2}{m_1^2+n_1^2}, \quad x_2 = \frac{m_2^2-n_2^2}{m_2^2+n_2^2},$$

we must choose $n_i > m_i$.

Then, by 5.2, for x_3 we must have $\left(m_1^2-n_1^2\right)\left(m_2^2-n_2^2\right)-4m_1m_2n_1n_2 < 0$, but we see that this is equivalent to $(m_1n_2+m_2n_1)^2 > (m_1m_2-n_1n_2)^2$.

Since $m_i < n_i$ this is equivalent to $m_1n_2 + m_2n_1 > n_1n_2 - m_1m_2$, or equivalently

$$m_1n_2 + m_2n_1 + m_1m_2 \quad > \quad n_1n_2. \tag{5.3}$$

There is also a choice between taking the positive or negative term in the equations for the radii. For example, in the equation for r_1:

$$r_1 \quad = \quad \frac{n_1(m_1n_2 + m_2n_1)}{-m_1^2n_2 - n_1^2n_2 \pm (m_1n_1n_2 + m_2n_1^2)},$$

the terms in the numerator will always be positive, and we can see that in order for the radius to be positive, we must necessarily take the positive term and we get the additional constraint that $n_1(m_1n_2 + m_2n_1) > n_2(m_1^2 + n_1^2)$. Then re-solving for the radii r_2 and r_3 using the positive term in the equation for r_1, we have:

$$\begin{aligned} r_2 \quad &= \quad \frac{n_1n_2}{m_2n_1 + m_1n_2 - n_1n_2} \\ r_3 \quad &= \quad \frac{n_2(m_1n_2 + m_2n_1)}{n_1n_2^2 + m_2^2n_1 - m_1n_2^2 - m_2n_1n_2}, \end{aligned}$$

which give us two additional constraints in order to assure positive radii: $m_2n_1 + m_1n_2 > n_1n_2$ and $n_1(m_2^2+n_2^2) > n_2(m_1n_2+m_2n_1)$. Note that the first of these constraints is stronger than (5.3), and so will replace it in the following summarizing theorem:

Theorem 5.3.2. *Let $m_1, n_1, m_2, n_2 \in \mathbb{N}$ with $n_1 > m_1$ and $n_2 > m_2$, $m_1n_2 + m_2n_1 > n_1n_2$, $n_1(m_1n_2 + m_2n_1) > n_2(m_1^2 + n_1^2)$, and $n_1(m_2^2 + n_2^2) > n_2(m_1n_2 + m_2n_1)$. Then the cosines*

x_i and the radii r_i are parametrized by:

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2},$$

$$x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2},$$

$$x_3 = \frac{\left(m_1^2 - n_1^2\right)\left(m_2^2 - n_2^2\right) - 4m_1m_2n_1n_2}{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}$$

and

$$r_1 = \frac{n_1(m_1n_2 + m_2n_1)}{m_1n_1n_2 + m_2n_1^2 - m_1^2n_2 - n_1^2n_2}$$

$$r_2 = \frac{n_1n_2}{m_2n_1 + m_1n_2 - n_1n_2}$$

$$r_3 = \frac{n_2(m_1n_2 + m_2n_1)}{n_1n_2^2 + m_2^2n_1 - m_1n_2^2 - m_2n_1n_2}.$$

Example 5.3.3. Let $m_1 = 1$, $n_1 = 2$, $m_2 = 4$, and $n_2 = 5$. We can see that the constraints will be satisfied:

$$\begin{aligned} m_1n_2 + m_2n_1 = 1 \cdot 5 + 4 \cdot 2 = 13 &> 10 = 2 \cdot 5 = n_1n_2 \\ n_1(m_1n_2 + m_2n_1) = 2(1 \cdot 5 + 4 \cdot 2) = 26 &> 25 = 5(1+4) = n_2(m_1^2 + n_1^2) \\ n_1(m_2^2 + n_2^2) = 2(16 + 25) = 82 &> 65 = 5(1 \cdot 5 + 4 \cdot 2) = n_2(m_1n_2 + m_2n_1). \end{aligned}$$

Then we have:

$$x_1 = -\frac{3}{5}, \quad x_2 = -\frac{9}{41}, \quad x_3 = -\frac{133}{205}.$$

And the radii:

$$r_1 = 26, \quad r_2 = \frac{54}{11}, \quad r_3 = \frac{351}{59}.$$

Or, if we chose to scale to make an integral flower:

$$r = 649, \quad r_1 = 16874, \quad r_2 = 3186, \quad r_3 = 3861.$$

5.4 Descartes' Circle Theorem

Theorem 5.4.1 (Descartes' Circle Theorem [2]). *A collection of four mutually tangent circles in the plane, where* $b_i = \frac{1}{r_i}$ *denotes the curvatures of the circles, satisfies the relation:*

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 \;=\; \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Four mutually tangent circles in the plane are sometimes referred to as *Soddy circles* for Frederick Soddy, an English chemist who rediscovered Descartes' Circle Theorem in 1936 [2].

Theorem 5.4.2 (Graham et al. [5]). *The following parametrization characterizes the integral curvatures of a set of Soddy circles:*

$$\begin{aligned} b_1 &= x \\ b_2 &= d_1 - x \\ b_3 &= d_2 - x \\ b_4 &= -2m + d_1 + d_2 - x \end{aligned}$$

where $x^2 + m^2 = d_1 d_2$ *and* $0 \leq 2m \leq d_1 \leq d_2$.

Theorem 5.4.3. *The parameterization given in Theorem 5.3.2 characterizes all sets of four* *mutually tangent circles of rational radius in the plane.*

Proof. It is straightforward to check that the parameterization from Theorem 5.3.2 satisfies Descartes' Circle Theorem.

To compare with the parametrization in Theorem 5.4.2, suppose we have some set of integral curvatures (b_1, b_2, b_3, b_4) where b_1 is the curvature of the fourth, either interior or exterior, circle. We can constrain $b_1 > 0$ to ensure that it is the configuration with the fourth coin interior. If we scale so that the radius of the inner circle is 1, the radii are then given by

$$r_1 = \frac{b_1}{b_2}, \quad r_2 = \frac{b_1}{b_3}, \quad r_3 = \frac{b_1}{b_4}.$$

If we replace b_1, b_2, b_3, b_4 in these equations with the parametrization in Theorem 5.4.2, we can then solve

$$\begin{aligned}
\frac{b_1}{b_2} = r_1 \quad &= \quad \frac{n_1(m_1n_2 + m_2n_1)}{-m_1^2n_2 - n_1^2n_2 + m_1n_1n_2 + m_2n_1^2} \\
\frac{b_1}{b_3} = r_3 \quad &= \quad \frac{n_2(m_1n_2 + m_2n_1)}{-m_1n_2^2 - m_2n_1n_2 + n_1n_2^2 + m_2^2n_1} \\
\frac{b_1}{b_4} = r_2 \quad &= \quad \frac{n_1n_2}{-n_1n_2 + m_2n_1 + m_1n_2}.
\end{aligned}$$

for d_1, d_2, m in terms of m_1, m_2, n_1, n_2, and x, which will be a free variable to be chosen from the positive integers:

$$
\begin{aligned}
m &= \frac{x(n_1 n_2 - m_1 m_2)}{m_1 n_2 + m_2 n_1} \\
d_1 &= \frac{x n_2 (m_1^2 + n_1^2)}{n_1(m_1 n_2 + m_2 n_1)} \\
d_2 &= \frac{x n_1 (m_2^2 + n_2^2)}{n_2(m_1 n_2 + m_2 n_1)}.
\end{aligned}
$$

We can see that all the conditions in Theorem 5.4.2, except one, will be satisfied:

$$
\begin{aligned}
x^2 + m^2 - d_1 d_2 &= x^2 + x^2 \frac{(n_1 n_2 - m_1 m_2)^2}{(m_1 n_2 + m_2 n_1)^2} - x^2 \frac{n_2(m_1^2 + n_1^2)}{n_1(m_1 n_2 + m_2 n_1)} \frac{x n_1 (m_2^2 + n_2^2)}{n_2(m_1 n_2 + m_2 n_1)} \\
&= 0.
\end{aligned}
$$

The first inequality, $0 \leq 2m$, will clearly hold when we choose $x, m_1 < n_1, m_2 < n_2$ to be positive integers, as we have previously stated.

Remark 5.4.4. The second inequality, $2m \leq d_1$, is not always satisfied due to a difference in the range of the parameterizations. If we were to insist upon this inequality being satisfied, we still might obtain all solutions. However, this is not necessary, because no additional solutions are obtained by forcing this inequality to be satisfied.

The third inequality is $d_1 \leq d_2$. This can also be shown using the inequality constraints from Theorem 5.3.2:

$$
\begin{aligned}
d_1 &= \frac{xn_2(m_1^2+n_1^2)}{n_1(m_1n_2+m_2n_1)} \\
&\leq \frac{xn_1n_2(m_1n_2+m_2n_1)}{n_1n_2(m_1n_2+m_2n_1)} \\
&\leq \frac{xn_1(m_2^2+n_2^2)}{n_2(m_1n_2+m_2n_1)} \\
&= d_2.
\end{aligned}
$$

Therefore we see that Graham et al.'s characterization of Soddy circles is implied by our parametrization of wheel graphs with $n = 3$ petals and rational radii given in Theorem 5.3.2. □

Remark 5.4.5. The parametrization given by Graham et al in Theorem 5.4.2 relies on solving the Diophantine equation $x^2 + m^2 = d_1d_2$, while the parametrization developed here and given in Theorem 5.3.2 does not rely on satisfying any such equation, only inequalities.

Chapter 6: Rational Solutions when $n \geq 4$

6.1 Motivation

In this chapter, we will characterize all rational solutions when $n = 4$, for flowers with four petals. We then show how this parametrization generalizes to give a large class of solutions for flowers with $n > 4$ petals. We also demonstrate how this parametrization of the radii of flowers also gives us a parametrization of the radii in a related, inverted problem.

The case of the wheel graph with $n = 3$ petals is a special one. With the four coins all mutually tangent, the structure is rigid: once two of the three interior angles are specified, the third interior angle and all the outer radii are then fixed. We can describe this system as having two degrees of freedom: two of the angles.

When we move up to the case of the wheel graph with $n = 4$ petals, we find that this is not the case. Specifying three of the four interior angles does fix the fourth interior angle, but the radii can still vary. For example, Figure 6.1 shows two different valid configurations with the same interior angles. The first three interior angles are chosen to be right angles in this case, giving us a fourth right angle. But in the first configuration, we choose $r_1 = 3$, and the rest of the radii are then determined by the Pythagorean Theorem: $r_2 = r_4 = 2$ and $r_3 = r_1 = 3$. In the second configuration, we choose $r_1 = 13$, which then determines a different but equally valid configuration: $r_2 = r_4 = \frac{15}{13}$ and $r_3 = r_1 = 13$. This system can be described as having four degrees of freedom: three of the angles plus the radius of one circle. In our parametrization, we will choose three of the four interior angles and one radius, as described.

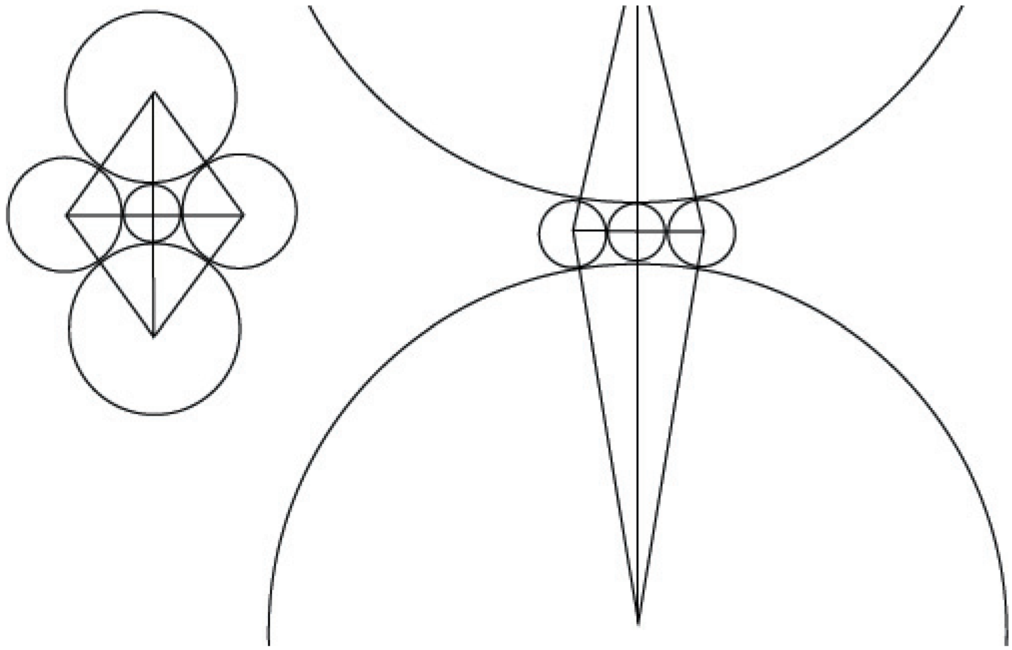

Figure 6.1: Two different valid configurations with the same interior angles.

6.2 Parametrization when $n = 4$

When the lengths of the sides of a triangle are rational, the cosine will be rational. The converse is not necessarily true, however. We would like to find rational radii that create a flower configuration with 4 petals. For a necessary first step, we will determine what the cosines must be.

For $n = 4$, Equation (3.1) yields

$$\mathrm{EC}_2(x_1, x_2) = \mathrm{EC}_2(x_3, x_4) \tag{6.1}$$

and hence

$$x_3x_4 - \sqrt{(1-x_3^2)(1-x_4^2)} = x_1x_2 - \sqrt{(1-x_1^2)(1-x_2^2)}.$$

Observation 6.2.1. For rationals a_1, a_2, b_1, b_2 with

$$a_1 + \sqrt{b_1} = a_2 + \sqrt{b_2}$$

then either

1. $a_1 = a_2$ and $b_1 = b_2$, or

2. $\sqrt{b_1}, \sqrt{b_2} \in \mathbb{Q}$.

We can apply the observation to obtain two cases:

Case 1: $x_3x_4 = x_1x_2$ and $(1-x_3^2)(1-x_4^2) = (1-x_1^2)(1-x_2^2)$.

Expanding the second equation and then applying the first, we have:

$$\begin{aligned} 1 - x_3^2 - x_4^2 + (x_3x_4)^2 &= 1 - x_1^2 - x_2^2 + (x_1x_2)^2 \\ x_3^2 + x_4^2 &= x_1^2 + x_2^2. \end{aligned}$$

Hence (x_3, x_4) and (x_1, x_2) represent points on the same circle. Since $x_1x_2 = x_3x_4$ and $x_1^2 + x_2^2 = x_3^2 + x_4^2$ then $(x_1 + x_2)^2 = (x_3 + x_4)^2$ and hence $x_3 + x_4 = \pm(x_1 + x_2)$.

Solving for x_4, we have $x_4 = \frac{x_1x_2}{x_3}$ and $x_4 = \pm(x_1 + x_2) - x_3$. Therefore, $\frac{x_1x_2}{x_3} = \pm(x_1 + x_2) - x_3$. Now we have $(x_3 - x_1)(x_3 - x_2) = 0$ or $(x_3 + x_1)(x_3 + x_2) = 0$. By symmetry we may also state $x_3 = \pm x_1$, and therefore $x_4 = \pm x_2$.

Example: Let $x_1 = -\frac{3}{5}$ and $x_2 = -\frac{4}{5}$. Then $x_3 = \pm\frac{3}{5}, x_4 = \pm\frac{4}{5}$. Selecting a reasonable combination of signs (for example, $x_3 = \frac{3}{5}, x_4 = \frac{4}{5}$) and putting these values into the original Law of Cosines equations along with a seed value for one of the radii (in this case $r_1 = 5$,) we obtain

$$r_1 = 5, r_2 = 24, r_3 = 15, r_4 = \frac{16}{59}.$$

Case 2: $\sqrt{(1-x_3^2)(1-x_4^2)}, \sqrt{(1-x_1^2)(1-x_2^2)} \in \mathbb{Q}$. So as we saw in the case where $n = 3$, we can write this with $x_i = \frac{p_i}{q_i}$ and then re-write them as $\frac{1}{q_3q_4}\sqrt{(q_3^2 - p_3^2)(q_4^2 - p_4^2)}$ and $\frac{1}{q_1q_2}\sqrt{(q_1^2 - p_1^2)(q_2^2 - p_2^2)}$. When $(q_i^2 - p_i^2)(q_{i+1}^2 - p_{i+1}^2)$ is a square, we can write $q_i^2 - p_i^2 = s_i^2\beta$ for $i = 1, 2$ and $q_i^2 - p_i^2 = s_i^2\alpha$ for $i = 3, 4$, where α, β are square-free. Then we have

$q_1^2 - p_1^2 = s_1^2\beta$, $q_2^2 - p_2^2 = s_2^2\beta$, $q_3^2 - p_3^2 = s_3^2\alpha$, and $q_4^2 - p_4^2 = s_4^2\alpha$.

Recalling the solution of the Diophantine equation $p^2 + \beta s^2 = q^2$ where β is square-free from Theorem 4.1.1, and since $x_i = \frac{p_i}{q_i}$ for $i = 1, 2$ we have in either case that

$$
\begin{aligned}
x_1 &= \frac{b_1 m_1^2 - c_1 n_1^2}{b_1 m_1^2 + c_1 n_1^2}, \\
x_2 &= \frac{b_2 m_2^2 - c_2 n_2^2}{b_2 m_2^2 + c_2 n_2^2}, \\
x_3 &= \frac{b_3 m_3^2 - c_3 n_3^2}{b_3 m_3^2 + c_3 n_3^2}, \\
x_4 &= \frac{1}{(b_3 m_3^2 + c_3 n_3^2)(b_2 m_2^2 + c_2 n_2^2)(b_1 m_1^2 + c_1 n_1^2)} (b_3 c_1 c_2 m_3^2 n_1^2 n_2^2 + b_1 c_2 c_3 m_1^2 n_2^2 n_3^2 \qquad (6.2) \\
&+ 4\beta c_3 m_1 m_2 n_1 n_2 n_3^2 + b_1 b_2 b_3 m_1^2 m_2^2 m_3^2 + b_2 c_1 c_3 m_2^2 n_1^2 n_3^2 \\
&- (b_1 b_3 c_2 m_1^2 m_3^2 n_2^2 + c_1 c_2 c_3 n_1^2 n_2^2 n_3^2 + 4\beta b_3 m_1 m_2 m_3^2 n_1 n_2 + b_2 b_3 c_1 m_2^2 m_3^2 n_1^2 + b_1 b_2 c_3 m_1^2 m_2^2 n_3^2) \\
&\pm (4\sqrt{\alpha\beta}(b_2 m_1 m_2^2 m_3 n_1 n_3 + b_1 m_1^2 m_2 m_3 n_2 n_3 - c_1 m_2 m_3 n_1^2 n_2 n_3 - c_2 m_1 m_3 n_1 n_2^2 n_3))
\end{aligned}
$$

where x_4 is derived by solving Equation (6.1) for x_4 and as in the parametrization from Chapter 5, $\beta = b_1 c_1 = b_2 c_2$ and $\alpha = b_3 c_3$ are factorizations of the square-free integers α and β and m_i, n_i can be chosen from the non-negative integers.

In order to guarantee that the fourth cosine x_4 is rational, the product $\alpha\beta$ must equal a square. Since both are square-free integers, this implies $\alpha = \beta$.

Observation 6.2.2. Since $x_4 \in \mathbb{Q}$, we have that $\alpha = \beta$ must hold.

As we saw in the previous chapter, we can manipulate the equations for the x_i to obtain:

$$\begin{aligned}
\left(r_1 + \frac{x_1 - 1}{x_1 + 1}\right)\left(r_2 + \frac{x_1 - 1}{x_1 + 1}\right) &= \frac{2(1 - x_1)}{(x_1 + 1)^2}, \\
\left(r_2 + \frac{x_2 - 1}{x_2 + 1}\right)\left(r_3 + \frac{x_2 - 1}{x_2 + 1}\right) &= \frac{2(1 - x_2)}{(x_2 + 1)^2}, \\
\left(r_3 + \frac{x_3 - 1}{x_3 + 1}\right)\left(r_4 + \frac{x_3 - 1}{x_3 + 1}\right) &= \frac{2(1 - x_3)}{(x_3 + 1)^2}, \\
\left(r_4 + \frac{x_4 - 1}{x_4 + 1}\right)\left(r_1 + \frac{x_4 - 1}{x_4 + 1}\right) &= \frac{2(1 - x_4)}{(x_4 + 1)^2}.
\end{aligned} \tag{6.3}$$

Instead of solving this whole system, we can simply let $r_1 = \frac{p}{q}$ for some $p, q \in \mathbb{Q} - \{0\}$. Then we can substitute the expressions for r_1 and x_1 into the first of the manipulated Law of Cosines equations (6.3) and solve for r_2. Continuing down the list, we then get expressions for all the radii:

$$\begin{aligned}
r_2 &= \frac{c_1 n_1^2 (p + q)}{b_1 p m_1^2 - c_1 q n_1^2} \\
r_3 &= \frac{c_2 p n_2^2 (b_1 m_1^2 + c_1 n_1^2)}{c_1 n_1^2 (b_2 p m_2^2 + b_2 q m_2^2 + c_2 q n_2^2) - b_1 c_2 p m_1^2 n_2^2} \\
r_4 &= \frac{c_1 c_3 n_1^2 n_3^2 (p + q)(b_2 m_2^2 + c_2 n_2^2)}{X} \\
X &= c_2 p n_2^2 (b_1 b_3 m_1^2 m_3^2 + b_1 c_3 m_1^2 n_3^2 + b_3 c_1 m_3^2 n_1^2) \\
&- c_1 c_3 n_1^2 n_3^2 (b_2 p m_2^2 + b_2 q m_2^2 + c_2 q n_2^2).
\end{aligned}$$

6.3 Obtaining Meaningful Solutions

In the $n = 3$ case, we have from Proposition 5.3.1 that for all the angles θ_i, $90° < \theta_i < 180°$, so $-1 < \cos\theta_i < 0$. When $n = 4$, we don't have such a neat description. However, we do know that for all the angles θ_i, $0° < \theta_i < 180°$, and so $0 < \sin\theta_i < 1$.

At the very least we know that the sum of the first 3 angles must be less than 2π. Of the 3 angle measures chosen, at least one must have the property $\theta_i > \frac{\pi}{3}$ (thus $x_i < 0.5$) and at least one must have the property $\theta_j < \frac{2\pi}{3}$ (thus $x_j > -0.5$.)

Additionally, we must choose the sign in the denominator of x_4. Not only does x_4 need to satisfy Equation 6.1, but also

$$\begin{aligned}
x_4 &= x_1x_2x_3 - x_1\sqrt{1-x_2^2}\sqrt{1-x_3^2} - x_2\sqrt{1-x_1^2}\sqrt{1-x_3^2} - x_3\sqrt{1-x_1^2}\sqrt{1-x_2^2} \\
&= \frac{1}{(b_1m_1^2+c_1n_1^2)(b_2m_2^2+c_2n_2^2)(b_3m_3^2+c_3n_3^2)}(b_1b_2b_3m_1^2m_2^2m_3^2 + b_1c_2c_3m_1^2n_2^2n_3^2 \\
&+ b_2c_1c_3m_2^2n_1^2n_3^2 + b_3c_1c_2m_3^2n_1^2n_2^2 + 4\beta c_2m_1m_3n_1n_2^2n_3 + 4\beta c_3m_1m_2n_1n_2n_3^2 \\
&+ 4\beta c_1m_2m_3n_1^2n_2n_3 - (b_1b_2c_3m_1^2m_2^2n_3^2 + b_1b_3c_2m_1^2m_3^2n_2^2 + b_2b_3c_1m_2^2m_3^2n_1^2 \\
&+ c_1c_2c_3n_1^2n_2^2n_3^2 + 4\beta b_1m_1^2m_2m_3n_2n_3 + 4\beta b_2m_1m_2^2m_3n_1n_3 + 4\beta b_3m_1m_2m_3^2n_1n_2)),
\end{aligned}$$

which corresponds to taking the negative sign in the denominator of (6.2).

In order for the radii to be positive, we must have $b_1pm_1^2 > c_1qn_1^2$, $c_1n_1^2(b_2pm_2^2+b_2qm_2^2+c_2qn_2^2) > b_1c_2pm_1^2n_2^2$, and $c_2pn_2^2(b_1b_3m_1^2m_3^2 + b_1c_3m_1^2n_3^2 + b_3c_1m_3^2n_1^2) > c_1c_3n_1^2n_3^2(b_2pm_2^2 + b_2qm_2^2 + c_2qn_2^2)$.

Theorem 6.3.1. *Let $m_1, n_1, m_2, n_2, m_3, n_3 \in \mathbb{N}$, $b_1, c_1, b_2, c_2, b_3, c_3 \in \mathbb{N}$ with the properties that $\beta = b_1c_2 = b_2c_2 = b_3c_3$, and β is square-free. Constrain further that $b_1pm_1^2 > c_1qn_1^2$, $c_1n_1^2(b_2pm_2^2+b_2qm_2^2+c_2qn_2^2) > b_1c_2pm_1^2n_2^2$, and $c_2pn_2^2(b_1b_3m_1^2m_3^2+b_1c_3m_1^2n_3^2+c_1b_3m_3^2n_1^2) > c_1c_3n_1^2n_3^2(b_2pm_2^2+b_2qm_2^2+c_2qn_2^2)$ and that at least one of x_1, x_2, x_3 has the property $x_i < 0.5$ and at least one has the property $x_j > -0.5$. Then the cosines x_i and the radii r_i are* *parametrized by:*

$$
\begin{aligned}
x_1 &= \frac{b_1m_1^2 - c_1n_1^2}{b_1m_1^2 + c_1n_1^2}, \\
x_2 &= \frac{b_2m_2^2 - c_2n_2^2}{b_2m_2^2 + c_2n_2^2}, \\
x_3 &= \frac{b_3m_3^2 - c_3n_3^2}{b_3m_3^2 + c_3n_3^2} \\
x_4 &= \frac{1}{(b_3m_3^2 + c_3n_3^2)(b_2m_2^2 + c_2n_2^2)(b_1m_1^2 + c_1n_1^2)}(b_3c_1c_2m_3^2n_1^2n_2^2 + b_1c_2c_3m_1^2n_2^2n_3^2 \\
&+ 4\beta c_3m_1m_2n_1n_2n_3^2 + b_1b_2b_3m_1^2m_2^2m_3^2 + b_2c_1c_3m_2^2n_1^2n_3^2 \\
&- (b_1b_3c_2m_1^2m_3^2n_2^2 + c_1c_2c_3n_1^2n_2^2n_3^2 + 4\beta b_3m_1m_2m_3^2n_1n_2 + b_2b_3c_1m_2^2m_3^2n_1^2 + b_1b_2c_3m_1^2m_2^2n_3^2) \\
&- (4\beta b_2m_1m_2^2m_3n_1n_3 + 4\beta b_1m_1^2m_2m_3n_2n_3 - 4\beta c_1m_2m_3n_1^2n_2n_3 - 4\beta c_2m_1m_3n_1n_2^2n_3))
\end{aligned}
$$

and

$$
\begin{aligned}
r_2 &= \frac{c_1n_1^2(p+q)}{b_1pm_1^2 - c_1qn_1^2} \\
r_3 &= \frac{c_2pn_2^2(b_1m_1^2 + c_1n_1^2)}{b_2c_1pm_2^2n_1^2 + b_2c_1qm_2^2n_1^2 + c_1c_2qn_1^2n_2^2 - b_1c_2pm_1^2n_2^2} \\
r_4 &= \frac{c_1c_3n_1^2n_3^2(p+q)(b_2m_2^2 + c_2n_2^2)}{X} \\
X &= c_2pn_2^2(b_1b_3m_1^2m_3^2 + b_1c_3m_1^2n_3^2 + b_3c_1m_3^2n_1^2) \\
&- c_1c_3n_1^2n_3^2(b_2pm_2^2 + b_2qm_2^2 + c_2qn_2^2).
\end{aligned}
$$

This parametrization characterizes all 4-petaled flowers with rational radii.

Example 6.3.2. Let $\beta = 1$. Then $b_i = c_i = 1$ for $i = 1, 2, 3$. Choose $m_1 = 1, n_1 = 2, m_2 =$

$1, n_2 = 3, m_3 = 2, n_3 = 1, p = 5, q = 1$. We see that the constraints are satisfied:

$$b_1 p m_1^2 = 5 \quad > \quad 4 = c_1 q n_1^2.$$

$$c_1 n_1^2 (b_2 p m_2^2 + b_2 q m_2^2 + c_2 q n_2^2) = 60 \quad > \quad 45 = b_1 c_2 p m_1^2 n_2^2.$$

$$c_2 p n_2^2 (b_1 b_3 m_1^2 m_3^2 + b_1 c_3 m_1^2 n_3^2 + b_3 c_1 m_3^2 n_1^2) = 945 \quad > \quad 60 = c_1 c_3 n_1^2 n_3^2 (b_2 p m_2^2 + b_2 q m_2^2 + c_2 q n_2^2).$$

Then using the parametrization, we have:

$$x_1 = -\frac{3}{5},\ x_2 = -\frac{4}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5},$$

$$r_1 = 5,\ r_2 = 24,\ r_3 = 15,\ r_4 = \frac{16}{59},$$

or, if we wish to have an integral solution

$$r = 59,\ r_1 = 295,\ r_2 = 1416,\ r_3 = 885,\ r_4 = 16.$$

Finally, we have in this case that x_1 and x_2 are less than 0.5, while x_3 and x_4 are greater than -0.5.

6.4 Generalization for $n > 4$

This method of parameterizing the radii of wheel graphs with $n = 4$ petals can be generalized for wheel graphs with $n > 4$ petals. The parametrization will simply have n degrees of freedom, which we will constrain by choosing $n-1$ of the interior angles and one of the radii in the same manner as for the $n = 4$ case.

The first $n-1$ cosines then can be given by $x_i = \frac{b_i m_i^2 - c_i n_i^2}{b_i m_i^2 + c_i n_i^2}$, the n^{th} cosine using an equation in the style of Remark 3.2.8, and the radii will be determined by solving the manipulated Law of Cosines equations, so we will have

$$r_{i+1} = \frac{-r_i x_i - x_i + r_i + 1}{r_i x_i + r_i + x_i - 1}$$

for $1 < i \leq n - 1$.

Remark 6.4.1. For $n \geq 5$ this will give a family of rational solutions, but not necessarily all solutions (as in the cases when $n = 3$ and $n = 4$.)

As we increase the number of petals, it becomes increasingly hard to characterize when we will obtain meaningful solutions. We know that the sum of the first $n - 1$ angles must be less than 2π, so we can say that of the $n - 1$ angle measures chosen, at least one must have the property $\theta_i > \frac{\pi}{n}$ and at least one must have the property $\theta_j < \frac{2\pi}{n}$.

6.5 Inversion about a Circle

Definition 6.5.1. The *inversion about a circle* C centered at point O sends a point P to a point P' such that the points O, P, P' are collinear and the product of the distances $|OP| \cdot |OP'|$ equals the square of the radius of the circle.

Inversion about a circle C maps lines and circles to lines and circles. More specifically, (i) a circle that does not pass through the center of C is inverted to another circle that also does not pass through the center of C, and (ii) a circle is mapped to itself if and only if it intersects the circle C at two right angles.

Now that we have a characterization of all rational solutions for flowers with $n = 3, 4$ petals and presented a class of rational solutions for each $n \geq 4$, we can use inversion about a circle to obtain all corresponding solutions to a related problem:

If we invert about the center coin of the flower, which has radius 1, each petal of radius r_i will be sent to a circle of radius $\frac{1}{r_i}$ tangent to the central circle from the interior. Figure

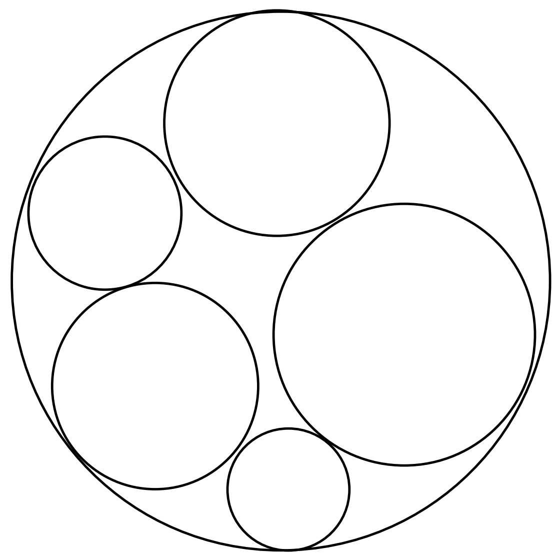

Figure 6.2: A 5-petaled flower inverted about the center coin.

6.2 shows a 5-petaled flower inverted about its center coin. Inversion about a circle preserves tangencies, so the result will be an arrangement of n circles around the inside of a circle of radius 1 such that the n circles fit perfectly. By taking the reciprocal of any set of solutions obtained from the parametrization described above, we get a solution to this inverted problem. Thus we have also characterized the solutions to this related problem for $n = 3, 4$.

Chapter 7: Summary and Future Work

7.1 Summary of Major Results

In chapter one, we revisit Harborth's result on the maximum number of edges of a unit coin graph (Propositions 1.2.1 and 1.2.3.) The proof given here expands the exposition for the lower bound and simplifies the argument for the upper bound. We also use a similar explicit construction method to establish a lower bound on the maximum number of edges in a special case of a coin graph on two radii in Proposition 1.4.2.

In chapter two, we proved Theorem 2.2.1, which gives the exact maximum number of edges in a plane graph on n vertices, where each vertex bounds some l-gon for $l \geq k$. From here we get the specific case when $k = 4$, given in Corollary 2.2.3.

In chapter three, we first investigated equations describing the cosines of the internal angles of a flower in terms of what rational radii could satisfy them. We showed that for each n-petaled flower, there is one radical equation that must be satisfied. These equations correspond to polynomial equations. We found the smallest such polynomial equations describing this relationship. Using Galois theory, we then showed that these polynomials are symmetric (Theorem 3.2.15) and irreducible (Theorem 3.2.25.) They are also presented recursively in Theorem 3.2.20 and Claim 3.2.22.

Chapter four detailed the proof of two generalizations of the Pythagorean Triples, Theorem 4.1.1 and Lemma 4.1.3.

In chapter five, Theorem 5.3.2 characterizes all rational solutions for flowers with three petals. We then compared this parametrization to an existing parametrization of the curvatures of four mutually tangent circles and showed how our equation-free parametrization implies the existing one.

In chapter six, Theorem 6.3.1 characterizes all rational solutions for flowers with four petals. We then showed how this parametrization generalizes to give a large class of solutions for flowers with $n > 4$ petals. We also demonstrated how the parametrization of the radii of flowers also gives us a parametrization of the radii in a related, inverted problem.

7.2 Future Work

Several open questions remain, and some additional questions have been suggested by this research. The following are a few of these questions for future research:

- Establish an upper bound on $T(1, r; n)$ (the 2-radii case) and prove Conjecture 1.4.4.
- Prove Conjecture 2.1.3 for the collection of unit-flowered coin graphs.
- Find a complete characterization of rational n-wheels for $n > 4$, not just a large class of solutions.
- Investigate the existence of a non-flowerable configuration which yields the upper bound $T(n) = \left\lfloor \frac{11}{4}n - 6 \right\rfloor$ given in Corollary 2.2.3.
- Investigate the existence of a suitable set of n radii which yield no flowers.
- Use the characterization of flowers to bound the number of edges in various coin graphs.
- Investigate $T(\mathbb{N}; n)$ where the radii are drawn from the set $\mathbb{N}$ of natural numbers with one copy of each element.
- Investigate Swanepoel's Conjecture that the largest number of edges in a coin graph with no triangular faces is given by $\lfloor 2n - 2\sqrt{n} \rfloor$.
- Investigate a problem of Scheinerman's: characterize which bipartite coin graphs function as systems of gears.

Appendix A: Further Results

A.1 Further Results from Elementary Number Theory

In the early stage of this investigation, we were led to the question of of when

$$(bm_1^2+cn_1^2)(b^2m_1^2+c^2n_1^2)(bm_2^2+cn_2^2)(b^2m_2^2+c^2n_2^2)$$

will be a perfect square. This is equivalent to $(bm_i^2+cn_i^2)(b^2m_i^2+c^2n_i^2)=\beta z_i^2$ for $i=1,2$ and β some square-free integer. As the first step to solve this latter version, we establish the following:

Lemma A.1.1. *For square-free relatively prime integers b and c and relatively prime integers x and y we have*

$$\gcd(bx^2+cy^2,b^2x^2+c^2y^2)=\operatorname{lcm}(\gcd(b,y^2)\cdot\gcd(c,x^2),\gcd(b-c,bx^2+cy^2)).$$

Proof. By applying the Euclidean algorithm one time, we can obtain $\gcd(bx^2+cy^2,b^2x^2+c^2y^2)=\gcd((b-c)cy^2,bx^2+cy^2)$. Suppose p^α divides c. Then p^α divides bx^2, but we know that c and b are relatively prime, so p^α divides c and x^2. Now suppose q^β divides y^2. Then q^β divides bx^2, but we know that x^2 and y^2 are relatively prime, so q^β divides y^2 and b.

Therefore, any factor of $\gcd(bx^2+cy^2,b^2x^2+c^2y^2)$ is a factor of $\gcd(b,y^2)$, $\gcd(c,x^2)$, or $\gcd(b-c,bx^2+cy^2)$. Since $\gcd(b,c)=1$ and $\gcd(x,y)=1$, $\gcd(\gcd(c,x^2),\gcd(b,y^2))=1$. Therefore,

$$\begin{aligned}\gcd(bx^2+cy^2,b^2x^2+c^2y^2) &\mid \operatorname{lcm}(\gcd(b,y^2),\gcd(c,x^2),\gcd(b-c,bx^2+cy^2))\\ &= \operatorname{lcm}(\gcd(b,y^2)\cdot\gcd(c,x^2),\gcd(b-c,bx^2+cy^2)).\end{aligned}$$

For the other inclusion, suppose p^α divides $\operatorname{lcm}(\gcd(b,y^2)\cdot\gcd(c,x^2),\gcd(b-c,bx^2+$

cy^2)). Then p^α divides either $\gcd(b, y^2) \cdot \gcd(c, x^2)$ or $\gcd(b-c, bx^2+cy^2)$ (or both.) As mentioned above, $\gcd(b, y^2)$ and $\gcd(c, x^2)$ have no factors in common. Thus, p^α divides $\gcd((b-c)cy^2, bx^2+cy^2)$. $\square$

In particular, we have that $\gcd(bx^2+cy^2, b^2x^2+c^2y^2)$ divides $\operatorname{lcm}(bc, b-c)$.

Theorem A.1.2 (Legendre's Equation [9])**.** *The Diophantine equation*

$$ax^2 + by^2 + cz^2 = 0$$

has a nontrivial solution in the integers if and only if

$$-bc \pmod{a}, \ -ca \pmod{b}, \ -ab \pmod{c}$$

are quadratic residues, where $a, b,$ and c are nonzero, square-free, pairwise relatively prime integers, not all positive or all negative.

Example A.1.3. Consider the equation $(bx^2+cy^2)(b^2x^2+c^2y^2) = dz^2$ for $b = 3$, $c = 5$, and $d = 11 \cdot 13$. So $(3x^2+5y^2)(9x^2+25y^2) = 11 \cdot 13z^2$. We may assume that $\gcd(x, y) = 1$. Then by Lemma A.1.1, the gcd of the two terms divides $2 \cdot 3 \cdot 5$.

We can partition the equation by the two factors on the left-hand side and write $(3x^2 + 5y^2) = d_1 \cdot f \cdot z_1^2$, $(9x^2+25y^2) = d_2 \cdot f \cdot z_2^2$, where $d = d_1 d_2$ is the square-free term on the right-hand side of the original equation, $z = f z_1 z_2$, and f is the square-free portion of the gcd. A first observation is that in the second equation, the sum of two squares must equal something that can be written as the sum of two squares, so the square-free term d_2 must be congruent to 1 modulo 4. Thus in this example, $d_2 = 1$ or $d_2 = 13$, so $d_1 = 11 \cdot 13$ or $d_1 = 11$. Now we have several cases, depending on which divisor of $2 \cdot 3 \cdot$ f is. There are 8 cases altogether.

Case 1: The gcd of the two terms is 1. Then $f = 1$ and

case 1.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot z_1^2$. Applying Legendre's equation and using Maple, we see that $3 \cdot 5 \mod 143$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

case 1.2: $3x^2 + 5y^2 = 11 \cdot z_1^2$. Applying Legendre's equation, we see that $3 \cdot 11 \mod 5 = 3$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Case 2: The gcd of the two terms is 2. Then $f = 2$ and

case 2.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 2 \cdot z_1^2$. Applying Legendre's equation, we see that $3 \cdot 2 \cdot 11 \cdot 13 \mod 5 = 3$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

case 2.2: $3x^2 + 5y^2 = 11 \cdot 2 \cdot z_1^2$. Applying Legendre's equation, we see that $2 \cdot 5 \cdot 11 \mod 3 = 2$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Case 3: The gcd of the two terms is 3. Then $f = 3$ and

case 3.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 3 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 3, then 3 must divide y^2, and so 9 must divide y^2, so we can let $y = 3y'$ and rewrite this equation as

$$\begin{aligned} 3x^2 + 5 \cdot 9 \cdot y'^2 &= 11 \cdot 13 \cdot 3 \cdot z_1^2 \\ x^2 + 15 \cdot y'^2 &= 11 \cdot 13 \cdot z_1^2. \end{aligned}$$

Now we can apply Legendre's equation, and we see that $11 \cdot 13 \mod 15 = 8$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

case 3.2: $3x^2 + 5y^2 = 11 \cdot 3 \cdot z_1^2$. Rewriting the equation as above to be $x^2 + 15y'^2 = 11 \cdot z_1^2$ and applying Legendre's equation, we see that 11 is not a quadratic residue modulo 15. Thus there are no nontrivial integer solutions to this equation.

Case 4: The gcd of the two terms is 5. Then $f = 5$ and

case 4.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 5 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 5, then 5 must divide

x^2, and so 25 must divide x^2, so we can let $x = 5x'$ and rewrite this equation as

$$\begin{aligned} 3 \cdot 25x'^2 + 5y^2 &= 11 \cdot 13 \cdot 5 \cdot z_1^2 \\ 15x'^2 + y^2 &= 11 \cdot 13 \cdot z_1^2. \end{aligned}$$

Now we can apply Legendre's equation, and we see that $11 \cdot 13 \mod 15 = 8$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

case 4.2: $3x^2 + 5y^2 = 11 \cdot 5 \cdot z_1^2$. Rewriting the equation as above to be $15x'^2 + y^2 = 11 \cdot z_1^2$ and applying Legendre's equation, we see that 11 is not a quadratic residue modulo 15. Thus there are no nontrivial integer solutions to this equation.

Case 5: The gcd of the two terms is 6. Then $f = 6$ and

case 5.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 6 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 6, then 3 must divide y^2, and so 9 must divide y^2, so we can let $y = 3y'$ and rewrite this equation as

$$\begin{aligned} 3x^2 + 5 \cdot 9y'^2 &= 11 \cdot 13 \cdot 6 \cdot z_1^2 \\ x^2 + 15y'^2 &= 11 \cdot 13 \cdot 2 \cdot z_1^2. \end{aligned}$$

Now we can apply Legendre's equation, and we see that 15 is not a quadratic residue modulo $11 \cdot 13 \cdot 2$. Thus there are no nontrivial integer solutions to this equation.

case 5.2: $3x^2 + 5y^2 = 11 \cdot 6 \cdot z_1^2$. Rewriting the equation as above to be $x^2 + 15y'^2 = 11 \cdot 2z_1^2$ and applying Legendre's equation, we see that $22 \mod 15 = 7$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Case 6: The gcd of the two terms is 10. Then $f = 10$ and

case 6.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 10 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 10, then 5 must divide x^2, and so 25 must divide x^2, so we can let $x = 5x'$ and rewrite this equation as

$$\begin{aligned} 3 \cdot 25x'^2 + 5y^2 &= 11 \cdot 13 \cdot 10 \cdot z_1^2 \\ 15x'^2 + y^2 &= 11 \cdot 13 \cdot 2 \cdot z_1^2. \end{aligned}$$

Now we can apply Legendre's equation, and we see that 15 is not a quadratic residue modulo $11 \cdot 13 \cdot 2$. Thus there are no nontrivial integer solutions to this equation.

case 6.2: $3x^2+5y^2 = 11 \cdot 10 \cdot z_1^2$. Rewriting the equation as above to be $15x'^2+y^2 = 11 \cdot 2 \cdot z_1^2$ and applying Legendre's equation, we see that $22 \mod 15 = 7$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Case 7: The gcd of the two terms is 15. Then $f = 15$ and

case 7.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 15 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 15, then 5 must divide x^2, 3 must divide y^2 and so 25 must divide x^2 and 9 must divide y^2, so we can let $x = 5x'$ and $y = 3y'$ and rewrite this equation as

$$\begin{aligned} 3 \cdot 25x'^2 + 5y^2 &= 11 \cdot 13 \cdot 15 \cdot z_1^2 \\ 15x'^2 + y^2 &= 11 \cdot 13 \cdot 3 \cdot z_1^2 \\ 15x'^2 + 9y'^2 &= 11 \cdot 13 \cdot 3 \cdot z_1^2 \\ 5x'^2 + 3y'^2 &= 11 \cdot 13 \cdot z_1^2. \end{aligned}$$

Now we can apply Legendre's equation, and we see that 15 is not a quadratic residue modulo $11 \cdot 13$. Thus there are no nontrivial integer solutions to this equation.

case 7.2: $3x^2+5y^2 = 11 \cdot 15 \cdot z_1^2$. Rewriting the equation as above to be $5x'^2+3y'^2 = 11 \cdot z_1^2$ and applying Legendre's equation, we see that $3 \cdot 11 \mod 5 = 3$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Case 8: The gcd of the two terms is 30. Then $f = 30$ and case 8.1: $3x^2 + 5y^2 = 11 \cdot 13 \cdot 30 \cdot z_1^2$. We cannot apply Legendre's equation in this case, because the coefficients are not relatively prime. However, if the gcd is 30, then 5 must divide x^2, 3 must divide y^2 and so 25 must divide x^2 and 9 must divide y^2, so we can let $x = 5x'$ and $y = 3y'$ and rewrite this equation as

$$\begin{aligned}
3 \cdot 25x'^2 + 5y^2 &= 11 \cdot 13 \cdot 30 \cdot z_1^2 \\
15x'^2 + y^2 &= 11 \cdot 13 \cdot 6 \cdot z_1^2 \\
15x'^2 + 9y'^2 &= 11 \cdot 13 \cdot 6 \cdot z_1^2 \\
5x'^2 + 3y'^2 &= 11 \cdot 13 \cdot 2 \cdot z_1^2.
\end{aligned}$$

Now we can apply Legendre's equation, and we see that $3 \cdot 2 \cdot 11 \cdot 13 \mod 5 = 3$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

case 8.2: $3x^2+5y^2 = 11 \cdot 30 \cdot z_1^2$. Rewriting the equation as above to be $5x'^2+3y'^2 = 11 \cdot 2 \cdot z_1^2$ and applying Legendre's equation, we see that $2 \cdot 5 \cdot 11 \mod 3 = 2$ is not a quadratic residue. Thus there are no nontrivial integer solutions to this equation.

Therefore there are no nontrivial integer solutions to $(3x^2 + 5y^2)(9x^2 + 25y^2) = 11 \cdot 13z^2$.

In this way one is able to tackle each case of the equation $(bx^2 + cy^2)(b^2x^2 + c^2y^2) = dz^2$, although the method is tedious.

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Curriculum Vitae

Jill Bigley Dunham graduated from Logan View High School, Hooper, Nebraska, in 1996. She received her Bachelor of Science in Computer Engineering from Iowa State University in 2001. She was employed as an engineer by Sprint for two years and for two years as a researcher in the Geography Department at the University of Maryland. She received her Master of Science in Computational Science from George Mason University in 2005.

In the fall of 2009, Jill will be joining the faculty at Hood College in Frederick, Maryland, as an Assistant Professor in the Department of Mathematics.