PROPERTIES OF RINGS AND OF RING EXTENSIONS THAT ARE INVARIANT UNDER GROUP ACTION

by

Amy D. Schmidt A Dissertation Submitted to the Graduate Faculty of George Mason University In Partial fulfillment of The Requirements for the Degree of Doctor of Philosophy Mathematics



Dr. Jay Shapiro, Dissertation Director

Dr. Neil Epstein, Committee Member

Dr. Geir Agnarsson, Committee Member

Dr. John Miller, Committee Member

Dr. David Walnut, Department Chair

Dr. Donna M. Fox, Associate Dean, Office of Student Affairs & Special Programs, College of Science

Dr. Peggy Agouris, Dean, College of Science

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Amy D. Schmidt Master of Science University of Minnesota Duluth, 2011 Bachelor of Science University of Minnesota Duluth, 2009

Director: Dr. Jay Shapiro, Professor Department of Mathematical Sciences

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Abstract

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Amy D. Schmidt, PhD

George Mason University, 2015

Dissertation Director: Dr. Jay Shapiro

We expand the work in invariant theory inspired by Hilbert's Fourteenth Problem [24]. Given a commutative ring with identity R and a subgroup G of the automorphism group of R, the fixed ring is $R^G := \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$. That is, R^G is the collection of elements of R that are fixed by all automorphisms in G. Properties of R inherited by R^G and properties of the extension $R^G \subseteq R$ have been studied extensively. We call properties of R that are inherited by R^G invariant (under the group action by G).

In Chapter 1, we review the necessary basic definitions and literature on invariant theory. We also describe the research questions addressed in this work. In Chapter 2, we continue the study of invariant properties of rings.

In Chapter 2, Section 1, we determine that many properties of domains are invariant. We consider generalizations of UFDs and generalizations of PIDs. We show that many of these properties are invariant even though R^G need not be a UFD or PID whenever R is a UFD or PID, respectively.

In Section 2 of Chapter 2, we consider properties of rings with zero-divisors. Jøndrup and Mouanis have determined that the properties of being a PP-ring [25] and a PF-ring [34], respectively, are invariant. We show that being a pseudo-PF-ring (as defined by Kourki[29]) is not an invariant property.

The major component of this work is Chapter 3, in which we determine invariant properties of ring extensions. Given an extension of commutative rings with identity $R \subset T$ and a subgroup G of the automorphism group of T, we have $R^G \subseteq T^G$, where these fixed rings are defined as above. We often assume that R is G-invariant, i.e., $\sigma(R) \subseteq R$ for all $\sigma \in G$. We call properties of the ring extension $R \subset T$ inherited by $R^G \subseteq T^G$ invariant.

In Chapter 3, Section 1, we show that various significant properties of ring extensions are invariant, e.g., integrality. In Sections 2 and 3, we show that being an integral minimal ring extension and being an integrally closed minimal ring extension, respectively, are invariant properties under certain assumptions. In Section 4, we consider properties related to minimal ring extensions. Lastly, in Section 4 we determine that the FIP and FCP properties of ring extensions are invariant under various hypotheses.

Chapter 1: Introduction

1.1 Preliminaries

Herein all rings are commutative with identity, and all homomorphisms are unital. Given a ring R and a subgroup G of the automorphism group of R, we say that G acts on R and denote the *fixed ring* of this action by $R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$. For $r \in R$ we set $\mathcal{O}_r := \{\sigma(r) \mid \sigma \in G\}$, and we say that the group action is *locally finite* (on R) if \mathcal{O}_r is finite for all $r \in R$. When there is no chance for confusion, we say that G is locally finite. If G is locally finite, we define

$$n_r := |\mathcal{O}_r|, \quad \hat{r} := \sum_{r_i \in \mathcal{O}_r} r_i \quad \text{and} \quad \tilde{r} := \prod_{r_i \in \mathcal{O}_r} r_i.$$

If G is finite, instead we denote by \hat{r} the sum $\sum_{\sigma \in G} \sigma(r)$ and by \tilde{r} the product $\prod_{\sigma \in G} \sigma(r)$ (allowing for duplicates). Given an ideal I in R we denote the orbit of I under G by $\mathcal{O}_I := \{\sigma(I) \mid \sigma \in G\}$. By the First Isomorphism Theorem, $R/I \cong R/\sigma(I)$ (cf. [14, Lemma 2.1 (c)]). It follows that R/I is a domain (field) if and only if $R/\sigma(I)$ is a domain (field). Hence, I is a prime (maximal) ideal if and only if $\sigma(I)$ is a prime (maximal) ideal. As defined in [16], G is strongly locally finite if G is locally finite and \mathcal{O}_P is finite for all prime ideals P in R. If G is finite, then G is strongly locally finite; and if G is strongly locally finite, then G is locally finite.

The total quotient ring of R is denoted tq(R), and if R is a (integral) domain, then the total quotient ring is the field of fractions, denoted qf(R). We denote by Aut(R) the automorphism group of R; Z(R) the set of zero-divisors of R; U(R) the set of units of R; Prin(R) the set of principal ideals; Spec(R) the set of prime ideals; Min(R) the set of minimal prime ideals; Max(R) the set of maximal ideals; $Ann_R(a)$ the collection of elements of Rwhose product with a is 0; and $Rad_R(I) := \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$ the radical in R of an ideal I of R. A non-Noetherian ring with a unique maximal ideal is called *quasilocal*. A ring is *Noetherian* if it has no infinite ascending chains of ideals - the ascending chain condition (ACC). A Noetherian ring with a unique maximal ideal is called *local*.

We say that a ring extension $R \subseteq T$ satisfies the property

- 1. lying over if for every $P \in \text{Spec}(R)$ there exists $Q \in \text{Spec}(T)$ such that $P = Q \cap R$;
- 2. going up if for $P, P' \in \operatorname{Spec}(R)$ and $Q \in \operatorname{Spec}(T)$, where $P \subseteq P'$ and $P = Q \cap R$, there exists $Q' \in \operatorname{Spec}(T)$ such that $Q \subseteq Q'$ and $P' = Q' \cap R$;
- 3. going down if for $P, P' \in \operatorname{Spec}(R)$ and $Q' \in \operatorname{Spec}(T)$, where $P \subseteq P'$ and $P' = Q' \cap R$, there exists $Q \in \operatorname{Spec}(T)$ such that $Q \subseteq Q'$ and $P = Q \cap R$; and
- 4. *incomparable* if for $Q, Q' \in \text{Spec}(T), Q \nsubseteq Q'$ and $Q' \nsubseteq Q$ whenever $Q \cap R = Q' \cap R$.

As in [26, p. 28], we refer to the lying over, going up, going down, and incomparable properties of ring extensions as LO, GU, GD, and INC, respectively.

Given a ring extension $R \subseteq T$, an element $t \in T$ is called *integral (over R)* if it satisfies a monic polynomial over R. If every element of T is integral, then we say $R \subseteq T$ is an *integral extension*. We denote by \overline{R} the collection of integral elements of T, which is subring of Tcalled the *integral closure of* R *in* T [26, Theorem 14]. (We only consider \overline{R} in the context of a ring extension $R \subset T$, so there is no chance for confusion.) If $R = \overline{R}$, then we say that the extension $R \subseteq T$ is *integrally closed*. If R is a domain and T = qf(R), then we use R'to denote the integral closure (of R in T), and if R = R', then we refer to R as *integrally closed*.

We now introduce some constructions used in this dissertation.

Given a field K and a totally ordered group H, a map $v: K \to H \cup \{\infty\}$ satisfying

1.
$$v(xy) = v(x) + v(y)$$
,

- 2. $v(x+y) \ge \min\{v(x), v(y)\}$, and
- 3. $v(x) = \infty$ if and only if x = 0

is called a *valuation*, and $v: K \setminus \{0\} \to H$ is a group homomorphism. By R_v we denote the subset of K with nonnegative valuation under v. This is a subring of K, and its quotient field is K. Any such ring is called a *valuation domain*.

A subset $S \subseteq R$ is called *multiplicatively closed* if $st \in S$ whenever $s \in S$ and $t \in S$. Given such a set S, the *localization of* R *at* S, denoted R_S , is a ring consisting of equivalence classes (r, s), where $r \in R$ and $s \in S$. The equivalence relation is given by $(r, s) \sim (r', s')$ if there exists $t \in S$ such that t(s'r - sr') = 0. (We might as well assume that $0 \notin S$; otherwise $R_S = \{0\}$.) As is common, we use $\frac{r}{s}$ to denote the equivalence class of (r, s), and if $P \in \operatorname{Spec}(R)$, we set $R_P := R_{R \setminus P}$.

1.2 Background & Literature Review

The study of invariant theory, particularly under a group's action on a ring, has been of interest since the late 1800s. David Hilbert's Fourteenth Problem [24] asks if $R = F[x_1, \ldots, x_n]$ and G is a subgroup of the general linear group of R, then is R^G of the form $F[y_1, \ldots, y_m]$, where $y_i \in R$, $n, m \in \mathbb{N}$, and F is a field? Over 50 years after the problem was proposed, Masayoshi Nagata produced a counterexample [35]. This led to the following questions: Given an arbitrary ring R and $G \leq \operatorname{Aut}(R)$, (1) what ring-theoretic properties of R are also satisfied by R^G , and (2) what properties of ring extensions does $R^G \subseteq R$ satisfy? Various assumptions about R and G are often considered. Properties described question (1) are called *invariant*. Much research has been devoted to these questions, including the following results, which will be useful in our work. In particular, Lemma 1.1 is a well-known, elementary result and is a vital tool. We include some proofs for the sake of completeness and to illustrate the group action by G.

Lemma 1.1. If G is locally finite, then R is integral over R^G .

Proof. For each $r \in R$ consider the polynomial

$$p_r(x) := \prod_{s \in \mathcal{O}_r} (x - s).$$

Clearly $p_r(r) = 0$. Since the lead coefficient is 1, and the coefficients are the elementary symmetric polynomials in \mathcal{O}_r , this is a monic polynomial over \mathbb{R}^G . Hence \mathbb{R} is integral over \mathbb{R}^G .

- **Lemma 1.2.** (a) The set of zero-divisors of R is G-invariant, i.e., $\sigma(\mathbb{Z}(R)) = \mathbb{Z}(R)$ for all $\sigma \in G$. It follows that the G-action on R extends uniquely to $\operatorname{tq}(R)$ via $\sigma\left(\frac{a}{b}\right) = \frac{\sigma(a)}{\sigma(b)}$ for any $\sigma \in G$.
- (b) [34, Lemma 2.2 (1)] If G is locally finite and R is reduced, then $r \in R \setminus Z(R)$ whenever $r \in R^G \setminus Z(R^G)$.
- (c) [34, Lemma 2.2 (2)] If G is locally finite and R is reduced, then $tq(R) = R_S$, where $S := R^G \setminus Z(R^G)$. It follows that $(tq(R))^G = tq(R^G)$.
- (d) [14, Lemma 2.1 (a)] The set of units of R is G-invariant, i.e., $\sigma(U(R)) = U(R)$ for all $\sigma \in G$. Moreover, $(U(R))^G = U(R^G)$.
- (e) [14, Lemma 2.1 (b)] If (R, M) is quasilocal, then $(R^G, M \cap R^G)$ is quasilocal.

Proof. (a) Let $\sigma \in G$. If $0 \neq a, b \in R$ such that ab = 0, then $0 \neq \sigma(a), \sigma(b) \in R$ and $\sigma(a)\sigma(b) = \sigma(ab) = 0$. Hence $\sigma(Z(R)) = Z(R)$. Next we show that $\sigma\left(\frac{a}{b}\right) = \frac{\sigma(a)}{\sigma(b)}$ is welldefined. For $a, c \in R$ and $b, d \in R \setminus Z(R)$, suppose that $\frac{a}{b} = \frac{c}{d}$ in tq(R), whence ad = bc. It follows that $\sigma(a)\sigma(d) = \sigma(b)\sigma(c)$, so we have $\frac{\sigma(a)}{\sigma(b)} = \frac{\sigma(c)}{\sigma(d)}$. (Note that we can divide by $\sigma(b)$ and $\sigma(d)$, since $b, d \notin Z(R)$ implies $\sigma(b), \sigma(d) \notin \sigma(Z(R)) = Z(R)$.) Lastly this G-action is unique since

$$\sigma\left(\frac{a}{b}\right) = \sigma(a)\sigma(b^{-1}) = \sigma(a)\sigma(b)^{-1} = \frac{\sigma(a)}{\sigma(b)}.$$

(b) Let $r \in \mathbb{R}^G \setminus \mathbb{Z}(\mathbb{R}^G)$. For the sake of contradiction, suppose that $r \in \mathbb{Z}(\mathbb{R})$. Then there exists $0 \neq s \in \mathbb{R}$ such that rs = 0. It follows that $r\sigma(s) = 0$ for every $\sigma \in G$ and so rf(s) = 0, where f(s) is any symmetric polynomial in \mathcal{O}_s . Since $f(s) \in \mathbb{R}^G$ and r is regular in \mathbb{R}^G , every such f(s) must be 0. Hence

$$s^{n_s} = \prod_{s_i \in \mathcal{O}_s} (s - s_i) = 0.$$

but R is reduced, so s = 0 – contradiction. Thus $r \in R \setminus Z(R)$.

(c) From (b) it follows that $R_S \subseteq \operatorname{tq}(R)$. For the converse, let $\frac{a}{b} \in \operatorname{tq}(R)$. From (a) it follows that $\sigma(b) \in R \setminus \mathbb{Z}(R)$ for every $\sigma \in G$, whence $\tilde{b} \in R^G \setminus \mathbb{Z}(R^G)$. Hence

$$\frac{a}{\bar{b}} = \frac{a}{\bar{b}} \prod_{b_i \in \mathcal{O}_b \setminus \{b\}} b_i \in R_S.$$

(d) The asserted invariance follows from the facts that $\sigma(1) = 1$ and $\sigma(r^{-1}) = \sigma(r)^{-1}$ for any $r \in U(R)$ and any $\sigma \in G$. For the second assertion, it only remains to show that $U(R)^G \subseteq U(R^G)$, which also follows from the fact that $\sigma(r^{-1}) = \sigma(r)^{-1}$.

(e) It is well-known and straightforward that $U(R) = R \setminus M$ if and only if (R, M) is quasilocal. Note that $R^G \setminus (M \cap R^G) = (R \setminus M) \cap R^G = (U(R))^G$. It now follows from (d) that $U(R^G) = R^G \setminus (M \cap R^G)$. Hence $(R^G, R^G \cap M)$ is quasilocal.

Lemma 1.3. Assume that G is locally finite and that R is a domain.

(a) [14, Lemma 2.3 (a)] Each element of qf(R) can be expressed as a fraction of the form $\frac{a}{b}$, where $a \in R$ and $0 \neq b \in R^G$, i.e., if $S := R^G \setminus \{0\}$, then $qf(R) = R_S$.

- (b) [14, Lemma 2.3 (b)] The quotient field of \mathbb{R}^G is $(\operatorname{qf}(\mathbb{R}))^G$, i.e., $\operatorname{qf}(\mathbb{R}^G) = (\operatorname{qf}(\mathbb{R}))^G$.
- (c) [14, Theorem 2.4] The G-action extends uniquely to R'. Moreover, if G is locally finite, then $(R')^G = (R^G)'$.

Proof. We include only the proof of (c), since (a) and (b) follow from Lemma 1.2 (c). For the first assertion, suppose that $r \in R'$. Then r satisfies the polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_i \in R$. It follows that $\sigma(r)$ satisfies $x^n + \sigma(a_{n-1})x^{n-1} + \cdots + \sigma(a_1)x + \sigma(a_0)$, for any $\sigma \in G$. Hence $\sigma(R') \subseteq R'$.

Next we show that $(R^G)' = (R')^G$. If $r \in (R^G)'$, then r is integral over R^G and $r \in qf(R^G)$. Hence, r is integral over R and, by (b), $r \in (qf(R))^G$. Thus $r \in (R')^G$. Conversely, if $r \in (R')^G$, then r is integral over R and $r \in (qf(R))^G$. By the transitivity of integrality [26, Theorem 40] and by Lemma 1.1, r is integral over R^G , and, again by (b), $r \in qf(R^G)$. Thus $r \in (R^G)'$.

Corollary 1.4. [14, Corollary 2.5] Assume that G is locally finite. If R is integrally closed, then R^G is integrally closed.

Proposition 1.5. [17, Theorem 2.2] If G is locally finite, then $R^G \subseteq R$ satisfies GD.

There are many ring theoretic properties that are either known to be invariant or for which there are counterexamples to invariance. A *principal ideal domain (PID)* (respectively, *Bézout domain*) is a domain in which all ideals (respectively, finitely generated ideals) are principal, i.e., generated by a single element. Example 1.6 (c.f. [6, p. 72]) shows that both being a PID and being a Bézout domain are not invariant properties.

Example 1.6. [6, p. 72] Let x be a real variable. Then $R := \mathbb{C}[e^{ix}, e^{-ix}]$ is a PID, hence Bézout domain. If $G := \{1, \sigma\}$ where σ is the complex conjugation map, then $R^G = \mathbb{R}[\sin(x), \cos(x)]$ is neither a PID nor a Bézout domain.

A Dedekind domain (respectively, Prüfer domain) is a domain in which all nonzero ideals (respectively, nonzero finitely generated ideals) are invertible. An ideal I of a domain R is called *invertible* if $II^{-1} = R$, where I^{-1} is the conductor $(R :_{qf(R)} I) = \{x \in qf(R) | xI \subseteq R\}$. Since principal ideals are invertible [26], PIDs are both Bézout domains and Dedekind domains, and these are a subclass of Prüfer domains. The following result is due to Bergman [6].

Proposition 1.7. [6, Proposition 4.1] Assume that G is finite and that R is a domain. If I is an ideal of R^G and IR is an invertible ideal of R, then I is an invertible ideal of R^G . In particular, if R is a Dedekind domain (respectively, Prüfer domain), then R^G is Dedekind domain (respectively, Prüfer domain).

Krull domains comprise another important class of domains, which includes PIDs and Dedekind domains. A domain R is a *Krull domain* if

- (i) $R = \bigcap \{ R_P \mid P \in \operatorname{Spec}(R) \text{ has height } 1 \},$
- (ii) each such R_P is a discrete valuation ring (DVR), and
- (iii) each nonzero $x \in R$ is contained in only finitely many height one prime ideals.
- A DVR is a valuation domain with value group \mathbb{Z} .
- **Proposition 1.8.** (a) [14, Proposition 2.6 (a)] For locally finite G, if R is a Prüfer domain, then R^G is a Prüfer domain.
- (b) [14, Proposition 2.6 (b)] For arbitrary G, if R is a Krull domain, then R^G is a Krull domain.

Unique factorization domains (UFDs) are a subclass of Krull domains. A *UFD* is a domain in which every nonzero non-unit has a unique factorization into irreducibles. Krull domains are contained in the class of integrally closed domains. Whereas Krull domains and integrally closed domains are invariant, if R is a UFD, R^G need not be. This is illustrated in Example 1.9, which is a well-known example. In Chapter 2, Section 1, we use this construction to show some generalizations of UFDs are also not invariant under even a finite group action. **Example 1.9.** Let F be a field of characteristic other than 2. Set R := F[x, y] and $G := \{1, \sigma\}$, where $\sigma(x) := -x$ and $\sigma(y) := -y$. Then R is a UFD, and $R^G = F[x^2, xy, y^2]$ is not, since $x^2y^2 = (xy)(xy)$.

With UFDs now defined, we note that PIDs are UFDs. Also, PIDs can be characterized as Noetherian Bézout domains. From these observations one can see that Example 1.6 also shows UFDs are not invariant: $\sin^2 \theta = (1 + \cos \theta)(1 - \cos \theta)$ in $R^G = \mathbb{R}[\sin \theta, \cos \theta]$. Moreover, R^G is not Bézout, since it is not a UFD, hence not a PID, but it is Noetherian. In Chapter 2, Section 1, we continue to determine if related properties of domains are invariant.

Turning to rings with zero-divisors, we consider PP-rings and PF-rings. A *PP-ring* (also known as a weak Baer ring [28] or a Rickart ring [30]) is a ring in which each principal ideal is a projective module. A *PF-ring* is a ring in which each principal ideal is a flat module [33]. Equivalently, a ring R is a PF-ring if Ann(a)+Ann(b)=R whenever ab = 0 for any $a, b \in R$ [4, Lemma β (ii)] (cf. [22], [33], [28]), where $Ann(a)=\{r \in R \mid ra=0\}$. Jøndrup [25] and Mouanis [34] show these classes of rings are invariant under arbitrary and locally finite group actions, respectively. We investigate a generalization of PF-rings in Chapter 2, Section 2.

Proposition 1.10. [25, Lemma 3] For arbitrary G, if R is a PP-ring, then R^G is a PP-ring.

Proposition 1.11. [34, Theorem 2.7 (1)] Assume that G is locally finite. If R is a PF-ring, then R^G is a PF-ring.

Related to PP-rings are complemented rings; in fact, all PP-rings are complemented. A ring R is said to be *complemented* if for all $a \in R$ there exists a $b \in R$ such that ab = 0and a + b is a regular element. It follows easily that complemented rings are reduced. It is part of the folklore that a ring is complemented if and only if its total quotient ring is von Neumann regular (cf. [18, Proposition 2.4]). A ring R is von Neumann regular if for all $a \in R$ there exists $x \in R$ such that axa = a. Another well-known equivalence is that a ring is von Neumann regular if and only if every finitely generated ideal is generated by an idempotent. Hence, von Neumann regular rings are Bézout rings. Interestingly, the von Neumann regular property is invariant even though the Bézout property is not. The property of being complemented is also invariant. These results, respectively, are also due to Jøndrup [25] (cf. [17, Proposition 3.6 (a)]) and Mouanis [34]. In fact, Mouanis uses Jøndrup's result to establish Proposition 1.13.

Proposition 1.12. [25, Corollary 4] For arbitrary G, if R is von Neumann regular, then R^G is von Neumann regular.

Proposition 1.13. [34, Theorem 2.1] Assume G is locally finite. If tq(R) is von Neumann regular, then $tq(R^G)$ is von Neumann regular.

1.3 Research Questions

Given a ring R and a subgroup G of $\operatorname{Aut}(R)$, we continue to determine what ring-theoretic properties of R are invariant in Chapter 2. We often assume that G is locally finite, and for some results we make stronger assumptions. In Chapter 3, we consider a ring extension $R \subset T$ and $G \leq \operatorname{Aut}(T)$. Since G is not assumed to be a subgroup of $\operatorname{Aut}(R)$, we define $R^G := R \cap T^G$. Upon inspection, one can see that this definition agrees with our original definition of $R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$ given in Section 1. We determine properties of the extension $R \subset T$ that are also satisfied by $R^G \subseteq T^G$. At times we assume that R is G-invariant, i.e., $\sigma(R) \subseteq R$ for all $\sigma \in G$, whence $G \leq \operatorname{Aut}(R)$.

Chapter 2: Properties of Rings

2.1 Domains

We continue the work of determining which classes of domains are invariant under group action. Noetherianness is an important property known to be invariant if G is finite and |G| is a unit of R. As defined in Chapter 1, a ring is Noetherian if its ideals satisfy ACC (all ascending chains terminate). Equivalently, a ring is Noetherian if all ideals are finitely generated. Noetherian domains are a subclass of ACCP domains. The ACCP property is the ascending chain condition for principal ideals. Another subclass of ACCP domains consists of bounded factorization domains (BFDs) [2]. Within the BFD class we have finite factorization domains (FFDs) [2], UFDs, and PIDs. In a BFD each nonzero non-unit is a product of finitely many irreducible elements (*atomic* property), and the length of these factorizations is bounded. In an FFD there are only finitely many such factorizations. We have the following hierarchy of these domains: A UFD is an FFD; an FFD is a BFD; and a BFD is an ACCP domain.

Lemma 2.1. For arbitrary G, if R is a domain, then elements of R^G are associates in R^G if and only if they are associates in R. Hence, the map $\phi : \operatorname{Prin}(R^G) \to \operatorname{Prin}(R)$ given by $xR^G \mapsto xR$ is injective.

Proof. Let $0 \neq x, y \in \mathbb{R}^{G}$. Clearly $\phi(0\mathbb{R}^{G}) = 0\mathbb{R}$. If x and y are associates in \mathbb{R}^{G} , then x = ry, for some $r \in U(\mathbb{R}^{G})$. By [14, Lemma 2.1(a)], $U(\mathbb{R}^{G}) = U(\mathbb{R})^{G}$. Hence $r \in U(\mathbb{R})$. Thus x and y are associates in \mathbb{R} .

Conversely, if x and y are associates in R, then x = ry, for some $r \in U(R)$. For any $\sigma \in G$ we have $x = \sigma(r)y$, whence $0 = (r - \sigma(r))y$. Since R is a domain and $y \neq 0$, it follows that $r = \sigma(r)$, i.e., $r \in U(R)^G = U(R^G)$. Hence, x and y are associates in R^G . \Box

Proposition 2.2. For arbitrary G, if R is a BFD (respectively, FFD, ACCP domain), then R^G is a BFD (respectively, FFD, ACCP domain).

Proof. As pointed out by Anderson et. al in [2], R is a FFD if, and only if, every nonzero principal ideal is contained in only finitely many principal ideals. Similarly, R is a BFD if for each nonzero $x \in R$ the lengths of chains of principal ideals ascending from xR is bounded. Thus, the result follows immediately from Lemma 2.1.

Corollary 2.3. Let G be locally finite. Then if R is a UFD, R^G need not be a UFD but is at least a FFD.

Some non-Noetherian generalizations of UFDs are greatest common divisor (GCD) domains and Schreier domains [8], which are subclasses of the class of integrally closed domains. In a *GCD domain* every two elements have a greatest common divisor. A *Schreier domain* is an integrally closed domain in which every element is primal. An element x in a ring R is *primal* if, for $a, b \in R$, x|ab implies $x = x_1x_2$, $x_1|a$ and $x_2|b$ for some elements x_1 and x_2 in R. UFDs can be characterized as atomic Schreier domains [8, Theorem 2.3]. Cohn [8] also observes UFDs can be characterized as atomic GCD domains.

Proposition 2.4. If R is a GCD domain or a Schreier domain, R^G may not be, even under finite group action where the order of the group is a unit in R.

Proof. As in Example 1.9, consider R = K[x, y], where K is a field of characteristic other than 2 and $G = \{1, \sigma\}$, where $\sigma(x) = -x$ and $\sigma(y) = -y$. In this case $R^G = K[x^2, xy, y^2]$. Note that R and R^G are atomic. Since R^G is not a UFD, it follows that R^G is neither a GCD domain nor a Schreier domain.

As mentioned in Chapter 1, PIDs and Bézout domains are not invariant [6], and Dedekind and Prüfer domains are invariant (cf. [6], [14]) under certain group action. We consider generalizations of these domains introduced by Anderson and Zafrullah [3]. They call a domain R an almost Bézout domain (AB-domain) (respectively, almost Prüfer domain (AP-domain) if given any nonzero $a, b \in R$ there exists $n \in \mathbb{N}$ such that (a^n, b^n) is principal (respectively, invertible). Similarly, R is an *almost principal ideal domain* (API-domain) (respectively, AD-domain) if for any nonempty, nonzero $\{a_\alpha\}_{\alpha \in A} \subseteq R$ there exists $n \in \mathbb{N}$ such that the ideal generated by $\{a^n_\alpha\}_{\alpha \in A}$ is principal (respectively, invertible). Note that AD-domains are *not* referred to as "almost Dedekind" domains as this term already has a different meaning.

Proposition 2.5. Assume that G is finite. If R is an AP-domain (respectively, AD-domain), then R^G is an AP-domain (respectively, AD-domain).

Proof. Let $a, b \in R^G$ (respectively, $\{a_\alpha\}_{\alpha \in A} \subset R^G$). Then there exists $n \in \mathbb{N}$ such that $(a^n, b^n)R$ (respectively, $(a^n_\alpha)_{\alpha \in A}R$) is an invertible ideal in R. By Proposition 1.7, (a^n, b^n) (respectively, $(a^n_\alpha)_{\alpha \in A}$) is an invertible ideal in R^G . Hence R^G is an AP-domain (respectively, AD-domain).

We will show in Proposition 2.7 that being a AB- or API-domain is an invariant property of a ring. First we introduce a few tools that we will use. As in [19], we define the following:

- 1. A fractional ideal of a domain R is an R-submodule A of qf(R) such that $dA \subseteq R$ for some nonzero $d \in R$.
- 2. An *invertible ideal* of R is a fractional ideal A such that there exists a fractional ideal B such that AB = R. In this case, B is the *inverse* of A, which is denoted A^{-1} . We denote by F(R) the collection of invertible ideals of R.
- 3. The *(ideal) class group*, denoted C(R), is the group of invertible ideals modulo principal ideals.

Under multiplication and with R as the identity element, F(R) is an Abelian group, and Prin(R) is a (normal) subgroup. Hence the group C(R) described above is a group under multiplication. By [3, Lemma 4.4] AB-domains (respectively, API-domains) can be characterized as AP-domains (respectively, AD-domains) with torsion class group. Hence, Proposition 2.7 will follow from Proposition 2.5 and the following lemma.

Lemma 2.6. Assume that G is finite and that R is domain.

- (a) There is a group monomorphism $F(\mathbb{R}^G) \to F(\mathbb{R})$ given by $I \mapsto I\mathbb{R}$.
- (b) If C(R) is torsion, then $C(R^G)$ is torsion.

Proof. (a) Let $R^G \neq I \in F(R^G)$. It follows from Lemma 1.3 (b) that $I^{-1}R$ is a fractional ideal of R. Clearly $(IR)(I^{-1}R) = (II^{-1})R = R^G R = R$. Hence, IR is an invertible ideal in R. Clearly (IJ)R = (IR)(JR), where $J \in F(R^G)$. Thus, the asserted map exists and is a homomorphism. It remains to show that it is injective. For the sake of contradiction, suppose that IR = R. Since an invertible ideal is finitely generated [26, Theorem 58], $I = (a_1, \ldots, a_n)$ for some $a_i \in R^G$, and there exist $r_i \in R$ such that $r_1a_1 + \cdots + r_na_n = 1$. It follows that

$$\prod_{\sigma \in G} \left(\sigma(r_1)a_1 + \dots + \sigma(r_n)a_n \right) = 1.$$

As in the proof of [6, Proposition 4.1 (b)], if we expand this product and collect the coefficients of the monomials in the set $\{a_i\}$, we see that the coefficients of these terms are elements of R^G . Hence $1 \in I$ – contradiction. Thus the map $I \mapsto IR$ is injective.

(b) As above, let $I = (a_1, \ldots, a_n) \in F(\mathbb{R}^G)$, whence $I\mathbb{R} \in F(\mathbb{R})$. Then there exists $m \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $I^m \mathbb{R} = a\mathbb{R}$. It follows that

$$I^{m|G|}R = \prod_{\sigma \in G} I^m R = \prod_{\sigma \in G} \sigma(I^m R) = \prod_{\sigma \in G} \sigma(aR) = \tilde{a}R.$$

We will show that $I^{m|G|^2} = \tilde{a}^{|G|} R^G$. Contracting to the fixed ring we have

$$I^{m|G|} \subseteq I^{m|G|} R \cap R^G = \tilde{a}R \cap R^G = \tilde{a}R^G,$$

where the last equality follows from Lemma 2.1. Hence $I^{m|G|^2} \subseteq \tilde{a}^{|G|}R^G$.

For the reverse containment, first note that $\tilde{a}R^G \subseteq \tilde{a}R = I^{m|G|}R$, whence

$$\tilde{a} = \sum_{k_1 + \dots + k_n = m|G|} r_{k_1} a_1^{k_1} \cdots a_n^{k_n},$$

for some $r_{k_1} \in R$. It follows that

$$\tilde{a}^{|G|} = \prod_{\sigma \in G} \sigma(\tilde{a}) = \prod_{\sigma \in G} \left(\sum_{k_1 + \dots + k_n = m|G|} \sigma(r_{k_1}) a_1^{k_1} \cdots a_n^{k_n} \right).$$

As in (a), if we expand this product and collect the terms involving the same powers of the a_i 's, we see that the coefficients of these terms are elements of R^G . Hence $\tilde{a}^{|G|} \in I^{m|G|^2}$; that is $\tilde{a}^{|G|}R^G \subseteq I^{m|G|^2}$. Thus $I^{m|G|^2} = \tilde{a}^{|G|}R^G$, so $C(R^G)$ is torsion.

Proposition 2.7. Assume that G is finite. If R is an AB-domain (respectively, API-domain), then R^G is an AB-domain (respectively, API-domain).

Proof. Apply Lemma 2.6 (b) and Proposition 2.5. \Box

2.2 Rings with Zero-Divisors

We continue the study of annihilator conditions on rings with zero-divisors as described in Chapter 1, Section 2. Generalizing PF-rings, Kourki [29] defines a *pseudo-PF-ring* as a ring R in which Ann(a)+Ann(b)=R whenever $aR \cap bR = \{0\}$ for any $a, b \in R$. Mouanis shows the pseudo-PF property is invariant under restrictive conditions [34, Theorem 2.9]. We can conclude the pseudo-PF property is not invariant in general. To see this we will observe in Lemma 2.8 that locally the pseudo-PF property is equivalent to the ring being uniform as a module over itself. A module is *uniform* if the intersection of any two nonzero submodules is nonzero. **Lemma 2.8.** Let R be a quasilocal ring. Then R is pseudo-PF if and only if it is uniform. *Proof.* Let M be the unique maximal ideal of R. Assume that R is pseudo-PF and not uniform. Then there exist nonzero $a, b \in R$ such that $(a) \cap (b) = \{0\}$. Since a and b are nonzero, both Ann(a) and Ann(b) are proper ideals of R, whence they are both contained in M. It follows that $R = Ann(a) + Ann(b) \subseteq M \subsetneq R$ – contradiction. Thus R is uniform.

Conversely, suppose that R is uniform, and suppose there exist $a, b \in R$ such that $(a) \cap (b) = \{0\}$. Then a = 0 or b = 0. Hence Ann(a) + Ann(b) = R. Thus R is pseudo-PF.

By [1, Lemma 4.1], the idealization R := T(+)M is a uniform *R*-module if and only if *M* is faithful and uniform, where *T* is a ring, and *M* is a *T*-module. An *R*-module *M* is faithful if $Ann_R(M) = \{0\}$. The *idealization* of *M* over *T* is a ring $R = \{(t,m) \mid t \in T, m \in M\}$ containing *T* with component-wise addition and multiplication given by (t,m)(s,n) = (ts,tn+sm) (so (1,0) is the identity). With this construction and Proposition 2.9 we conclude Proposition 2.10.

Proposition 2.9. A ring is a domain if and only if it is reduced and uniform.

Proof. Suppose that R is a domain. Then, of course, R is reduced. Let I and J be nonzero ideals of R, and let $0 \neq a \in I$ and $0 \neq b \in J$. Then $0 \neq ab \in IJ \subseteq I \cap J$. Hence R is uniform.

Conversely, assume that R is reduced and uniform, and assume that R is not a domain. Let $0 \neq a, b \in R$ such that ab = 0. Since R is uniform, there exists nonzero $x \in (a) \cap (b)$, whence x = ra = r'b for some $r, r' \in R$. It follows that $x^2 = rr'ab = 0$ – contradiction.

Proposition 2.10. Let T be a reduced ring of characteristic other than 2 that is not a domain, and let M be a faithful, uniform T-module. Set R := T(+)M, and define $\sigma(t,m) = (t,-m)$. Then for $G = \{1,\sigma\}$, R is uniform but $R^G = T$ is not.

Proof. By [1, Lemma 4.1], R is uniform. Since $R^G = T$ is not a domain, R^G is not uniform by Lemma 2.9.

Corollary 2.11. Let (T, \mathfrak{m}) be a local Noetherian ring that is reduced but not a domain with char $(T) \neq 2$. Let $E := E(T/\mathfrak{m})$ be the injective hull of T/\mathfrak{m} . Set R := T(+)E, and define $\sigma(t, m) = (t, -m)$. Then for $G = \{1, \sigma\}$, R is uniform but $R^G = T$ is not.

Proof. We must show that E is a uniform and faithful T-module. Let $N, N' \subset E$ be nonzero T-modules. Clearly $N \cap T/\mathfrak{m} \subseteq T/\mathfrak{m}$, but, in fact, $N \cap T/\mathfrak{m} = T/\mathfrak{m}$, since T/\mathfrak{m} is a simple module. By the same reasoning, $N' \cap T/\mathfrak{m} = T/\mathfrak{m}$. It follows that $(N \cap N') \cap T/\mathfrak{m} = T/\mathfrak{m} \neq 0$, whence $N \cap N' \neq 0$. Thus E is uniform.

Now we show that E is faithful. Set $A_i := \{x \in E \mid \mathfrak{m}^i x = 0\}$. Then $E = \bigcup A_i$ by [32, Theorem 3.4 (1)]. It follows that

$$\operatorname{Ann}_T(E) = \bigcap \operatorname{Ann}_T(A_i) = \bigcap \left(\bigcap_{x \in A_i} \operatorname{Ann}_T(x)\right).$$

By [32, Theorem 3.4 (2)], $\bigcap (\bigcap_{x \in A_i} \operatorname{Ann}_T(x)) = \mathfrak{m}^{(i)}$, where $\mathfrak{m}^{(i)} := (\mathfrak{m}T_{\mathfrak{m}})^i \cap T$ is the i^{th} symbolic power of \mathfrak{m} . Since (T, \mathfrak{m}) is quasilocal, $\mathfrak{m}^{(i)} = \mathfrak{m}^i$. Hence $\operatorname{Ann}_T(E) = \bigcap \mathfrak{m}^i$, but $\bigcap \mathfrak{m}^i = 0$, by [26, Theorem 79], so $\operatorname{Ann}_T(E) = 0$. Thus E is faithful. By Proposition 2.10, R is uniform but R^G is not.

In particular, we have the following example.

Example 2.12. Set T := K[x, y]/(xy) where K is a field with $\operatorname{char}(K) \neq 2$ and $N := T_m/(mT_m)$ where m is the maximal ideal generated by the images of x and y. Note T_m is not uniform, since it is reduced but not a domain, and N is a simple T_m -module. Since T_m is local and Noetherian, the injective hull M := E(N) is faithful and uniform. Thus, T_m and M satisfy the above proposition.

Remark 2.13. It follows from Lemma 1.2 (e) and Corollary 2.11 that the pseudo-PF property is not invariant.

Chapter 3: Properties of Ring Extensions

In this chapter R is assumed to be a (proper) subring of T, and G is a subgroup of $\operatorname{Aut}(T)$. In comparison to determining invariant properties of rings in Chapter 2, in this chapter we investigate which properties of a ring extension $R \subset T$ are inherited by $R^G \subseteq T^G$. Recall from Chapter 1, Section 3 that $R^G := R \cap T^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$. At times we will assume that $\sigma(R) \subseteq R$ for all $\sigma \in G$, in which case we say that R is G-invariant. It is easy to see that if R is G-invariant, then G is a subgroup of $\operatorname{Aut}(R)$, and if G is locally finite on T, then G is locally finite on R. But if G is strongly locally finite on T, G may not be strongly locally finite on R. To see this define R as in [16, Example 2.3], and set $T := \operatorname{qf}(R)$.

As in [20], we say that $R \subset T$ is a minimal ring extension if there is no ring S such that $R \subset S \subset T$. Clearly, this is true if and only if T = R[u] for all $u \in T \setminus R$. Since $R \subseteq \overline{R} \subseteq T$, where \overline{R} is the integral closure of R in T, if $R \subset T$ is minimal, then either R is integrally closed in T, or T is integral over R (equivalently, T is module finite over R). In the first case we call $R \subset T$ an integrally closed minimal ring extension, and in the second case, we call it an integral minimal ring extension. By [20, Théorème 2.2], if $R \subset T$ is a minimal ring extension, there exists a unique maximal ideal M of R such that $R_P \cong T_P$ for all $P \in \operatorname{Spec}(T) \setminus \{M\}$. This maximal ideal is commonly referred to as the crucial maximal ideal of the extension. In the integral case, $(R :_R T)$ is the crucial maximal ideal, while in the integrally closed case, $(R :_R T)$ is a prime ideal of R adjacent to the crucial maximal ideal.

In 1970, Ferrand and Olivier contributed to the groundbreaking work of classifying minimal ring extensions by determining the minimal ring extensions of a field [20]. More recently, Ayache extended this work to integrally closed domains [5]. Shortly thereafter, Dobbs and Shapiro generalized these results further to arbitrary domains in [13] and then later to certain rings with zero-divisors in [15]. In their second paper, they completely classify the integral minimal ring extensions of an arbitrary ring, as well as the integrally closed minimal ring extensions of a ring with von Neumann regular total quotient ring [15]. In [37] (cf. [11]), Picavet and Picavet-L'Hermitte give another characterization of integral minimal ring extensions. In [7], Cahen et al. characterize integrally closed minimal ring extensions of an arbitrary ring. We will use the characterizations in [37] and [7] to show that being minimal is an invariant property of ring extensions under certain group action. To do so, we treat the integral and integrally closed cases separately in Sections 3.2 and 3.3, respectively.

In Section 3.1, we establish the invariance of several properties related to integral and integrally closed extensions. In Section 3.4, we consider properties of ring extensions related to minimal ring extensions. Lastly, we use our results from Sections 3.2 and 3.3 to show that the FIP and FCP are invariant properties. These properties will be defined in Section 3.5.

3.1 Integrality and Related Properties

Recall Lemma 1.1: If G is locally finite, then $T^G \subseteq T$ is an integral extension. This result is fundamental in this work and in much of the work by Dobbs and Shapiro [14], [16], [17], which inspired this research. As noted in Chapter 1, integrality is a fundamental property of ring extensions. Naturally this is our first result, which follows immediately from Lemma 1.1.

Proposition 3.1. Assume that R is G-invariant and G is locally finite. If $R \subset T$ is an integral extension, then $R^G \subseteq T^G$ is an integral extension.

Proof. This follows from Lemma 1.1 and by the transitivity of integrality [26, Theorem 40].

Proposition 3.2. (a) If R is integrally closed in T, then R^G is integrally closed in T^G .

- (b) Assume that R is G-invariant and G is locally finite. If R is a domain and R' = T, then $(R^G)' = T^G$.
- *Proof.* (a) Let $u \in T^G$ be integral over R^G . Then $u \in T$ is integral over R. Hence $u \in T^G \cap R = R^G$.
- (b) This follows directly from 1.3 (c).

By [26, Theorem 44], an integral extension satisfies LO, GU, and INC. Dobbs and Shapiro [17, Corollary 2.3] show that if $R \subseteq T$ satisfies GD (respectively, GU), then $R^G \subseteq T^G$ satisfies GD (respectively, GU) when R is G-invariant and G is locally finite. We show LO and INC also pass from $R \subset T$ to $R^G \subseteq T^G$ under the same hypothesis.

Proposition 3.3. Assume that R is G-invariant and G is locally finite. If $R \subset T$ satisfies LO, (respectively, INC), then $R^G \subseteq T^G$ satisfies LO (respectively, INC).

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R^G)$. Then there exists $P \in \operatorname{Spec}(R)$ where $\mathfrak{p} = R^G \cap P$. Let $Q \in \operatorname{Spec}(T)$ such that $P = R \cap Q$, and set $\mathfrak{q} := T^G \cap Q$. Then

$$\mathfrak{q} \cap R^G = T^G \cap Q \cap R^G = Q \cap R^G = Q \cap R \cap R^G = P \cap R^G = \mathfrak{p}_{\mathcal{A}}$$

Hence $R^G \subseteq T^G$ satisfies LO.

Suppose $R \subset T$ satisfies INC and $R^G \subseteq T^G$ does not. Then there exist $\mathfrak{q}, \mathfrak{\hat{q}} \in \operatorname{Spec}(T^G)$ such that $\mathfrak{q} \subset \mathfrak{\hat{q}}$ and $\mathfrak{q} \cap R^G = \mathfrak{\hat{q}} \cap R^G =: \mathfrak{p}$. Let Q and \hat{Q} be prime ideals of T lying over \mathfrak{q} and $\mathfrak{\hat{q}}$, respectively, where $Q \subset \hat{Q}$. If $P := Q \cap R$ and $\hat{P} := \hat{Q} \cap R$, then $P \subsetneq \hat{P}$ $(P \neq \hat{P}$ since $R \subset T$ satisfies INC). Moreover,

$$P \cap R^G = (Q \cap R) \cap R^G = Q \cap R^G = Q \cap T^G \cap R^G = \mathfrak{q} \cap R^G = \mathfrak{p}$$

and

$$\hat{P} \cap R^G = \hat{Q} \cap R \cap R^G = \hat{Q} \cap R^G = \hat{Q} \cap T^G \cap R^G = \hat{\mathfrak{g}} \cap R^G = \mathfrak{p}.$$

Since $R^G \subseteq R$ satisfies INC and $\hat{P} \subsetneq P$ in Spec(R), the above conclusion that $\hat{P} \cap R^G = P \cap R^G = \mathfrak{p}$ is a contradiction. Thus $R^G \subseteq T^G$ satisfies INC.

3.2 Integral Minimal Ring Extensions

Recall our riding assumptions in this section are that $R \subset T$ and that G acts on T via automorphisms. We do no necessarily assume that R is G-invariant. In the following lemma we establish several technical results needed for the main result of this section. Proposition 3.5 is another tool for the main result and is also of independent interest.

Lemma 3.4. Assume that G is locally finite (on T) and that $M := (R :_R T)$ is a maximal ideal of R. Set $\mathfrak{m} := M \cap R^G = M \cap T^G$.

(a) If R is integral over R^G and $R^G \neq T^G$, then the conductor $(R^G:_{R^G}T^G)$ equals \mathfrak{m} .

(b) If there exist $N \in \text{Spec}(T)$ containing M, then $M = N \cap R$.

Proof. (a) Let $x \in \mathfrak{m}$. Then $x \in R^G$ and $xt \in R$, for all $t \in T$. If $t \in T^G$, then $xt \in R$, from which it follows that $xt \in T^G \cap R = R^G$. Hence $x \in (R^G :_{R^G} T^G)$. Thus $\mathfrak{m} \subseteq (R^G :_{R^G} T^G)$. Since R is integral over R^G , we have that $\mathfrak{m} \in \operatorname{Max}(R^G)$. Thus $\mathfrak{m} = (R^G :_{R^G} T^G)$.

(b) Clearly $M = N \cap R$ whenever N is a prime ideal of T containing M, since $M \in Max(R)$.

Recall the following definitions from Chapter 1: If G is locally finite on T, then for $t \in T$ we set $\mathcal{O}_t := \{\sigma(t) \mid \sigma \in G\}$, and we define

$$n_t := |\mathcal{O}_t|, \quad \hat{t} := \sum_{t_i \in \mathcal{O}_t} t_i \quad \text{and} \quad \tilde{t} := \prod_{t_i \in \mathcal{O}_t} t_i.$$

Proposition 3.5. Let $M \in Max(R)$ and $\mathfrak{m} := M \cap R^G$. Assume that G is locally finite (on R) such that $char(R^G/\mathfrak{m}) \nmid n_r$ for all $r \in R$ (e.g. if $char(R^G/\mathfrak{m}) = 0$). If the orbit of M in R is a singleton set, i.e., $\mathcal{O}_M = \{M\}$, then the G-action extends to R/M via $\sigma(r+M) = \sigma(r)+M$, for $\sigma \in G$. Moreover, if R is integral over R^G , then $R^G/\mathfrak{m} \cong (R/M)^G$.

Proof. The given action of G on R/M is well-defined: if r + M = s + M, then $\sigma(r) - \sigma(s) \in \sigma(M) = M$. Hence $\sigma(r) + M = \sigma(s) + M$.

As for the moreover, first note that $\mathfrak{m} \in \operatorname{Max}(R^G)$ by integrality. Define $\phi : R^G/\mathfrak{m} \to (R/M)^G$ by $r + \mathfrak{m} \mapsto r + M$. Since R^G/\mathfrak{m} is a field, by showing that ϕ is a (unital) ring homomorphism we can assert that it is an injection. Clearly ϕ preserves ring structure. If $\phi(1 + \mathfrak{m}) = 0 + M$, then $1 \in M$ – contradiction. Hence ϕ is a (unital) ring homomorphism.

Now let $r + M \in (R/M)^G$. Then $r + M = \sigma(r) + M$ for all $\sigma \in G$. Summing the elements of \mathcal{O}_r we have that $n_r r + M = \hat{r} + M$. Since R/M is a field, we have that $r + M = (n_r + M)^{-1}(\hat{r} + M)$. Similarly, since $n_r + \mathfrak{m} \in R^G/\mathfrak{m}$, we have that $y + \mathfrak{m} := (n_r + \mathfrak{m})^{-1} \in R^G/\mathfrak{m}$. It follows that $y + M = (n_r + M)^{-1}$, whence $\phi(y\hat{r} + \mathfrak{m}) = y\hat{r} + M = (n_r + M)^{-1}(\hat{r} + M) = r + M$. Thus ϕ is surjective. Hence $R^G/\mathfrak{m} \cong (R/M)^G$.

Remark 3.6. Proposition 3.5 is true without the assumptions of this chapter that $R \subset T$ and $G \leq \operatorname{Aut}(T)$. That is, it is true if R is any ring and G is a locally finite group acting on R via automorphisms such that R and G satisfy the hypotheses of the proposition.

The technique of averaging the orbit of an element used in Proposition 3.5 to produce $r+M = (n_r+M)^{-1}(\hat{r}+M)$ is a well-known method (see [6, Proposition 1.1]). We generalize this method in the following lemma.

Lemma 3.7. Assume that G is locally finite (on T). Let $t \in T^G$. We show that if $t = r_1u_1 + r_2u_2 + \cdots + r_ku_k$ for some $r_i \in R$ and $u_i \in T^G$, then there exist $m, m_i \in \mathbb{N}$ and $r'_i \in R^G$ such that $0 \neq mt = m_1r'_1u_1 + m_2r'_2u_2 + \cdots + m_kr'_ku_k$ whenever

(a) T is a domain and $char(T) \nmid n_t$ for all $t \in T$, or

(b) |G| is finite and a unit in T.

Proof. For all $t \in T$, fix a subset \mathcal{N}_t of G such that for each $a \in \mathcal{O}_t$ there exists a unique $\sigma \in \mathcal{N}_t$ with $a = \sigma(t)$ (and so $|\mathcal{N}_t| = |\mathcal{O}_t| = n_t$).

First we show that if

$$0 \neq t = q_1 u_1 \dots + q_i u_i + r_{i+1} u_{i+1} + \dots + r_k u_k, \tag{3.1}$$

where $t \in T^G$, $q_i \in R^G$, and $r_j \in R$, then there exists $m \in \mathbb{N}$, $r'_{i+1} \in R^G$, and $s_j \in R$ such that

$$0 \neq mt = m(q_1u_+\dots+q_iu_i) + r'_{i+1}u_{i+1} + s_{i+2}u_{i+2} + \dots + s_ku_k.$$
(3.2)

Applying each $\sigma \in \mathcal{N}_{r_{i+1}}$ to (3.1) and summing establishes (3.2). In particular,

$$m = n_{r_{i+1}}, \quad r'_{i+1} = \widehat{r}_{i+1}, \quad \text{and} \quad s_j = \sum_{\sigma \in \mathcal{N}_{r_{i+1}}} \sigma(r_j) u_j,$$

for $i + 2 \leq j \leq k$. Note that $n_{r_{i+1}}t \neq 0$ under assumption (a). Since i = 1 establishes the base case, the assertion of the lemma now follows from induction. Under assumption (b), the same argument holds replacing $\mathcal{N}_{r_{i+1}}$ with G and $n_{r_{i+1}}$ with |G|.

We have established the machinery needed to prove the main result of this section. We use the characterization provided below for reference.

Theorem 3.8. [37, Theorem 3.3] (cf. [11, Corollary II.2]) The ring extension $R \subset T$ is minimal and integral if and only if $(R :_R T) \in Max(R)$ and one of the following three conditions holds:

- (a) Inert case: $(R :_R T) \in Max(T)$ and $R/(R :_R T) \to T/(R :_R T)$ is a minimal field extension.
- (b) **Decomposed case:** There exist $N_1, N_2 \in Max(T)$ such that $(R:_R T) = N_1 \cap N_2$ and the natural maps $R/(R:_R T) \to T/N_1$ and $R/(R:_R T) \to T/N_2$ are each isomorphisms.

(c) Ramified case: There exists $N \in Max(T)$ such that $N^2 \subseteq (R:_R T) \subset N$, $[T/(R:_R T): R/(R:_R T)] = 2$ and the natural map $R/(R:_R T) \to T/N$ is an isomorphism.

We now present our main result on the invariance of integral minimal extensions.

Theorem 3.9. Let $R \subset T$ be an integral minimal extension with crucial maximal ideal $M = (R :_R T)$. Assume that G is locally finite (on T) such that $R^G \neq T^G$ and $\operatorname{char}(R^G/(M \cap T^G)) \nmid n_r$, for all $r \in R$. Also assume that R is integral over R^G and $\mathcal{O}_M = \{M\}$ (e.g., if R is G-invariant). Then $R^G \subset T^G$ is a minimal extension of the same type as $R \subset T$. Moreover, the crucial maximal ideal of $R^G \subset T^G$ is $(R^G :_{R^G} T^G)$.

Proof. Throughout the argument, set $\mathfrak{m} := (R^G :_{R^G} T^G)$, whence $\mathfrak{m} = M \cap R^G = M \cap T^G$, by Lemma 3.4(a).

Inert case: By Theorem 3.8(a), $M \in Max(T)$ and $R/M \to T/M$ is a minimal field extension. By Proposition 3.5, we may pass to $R/M \subset T/M$ (since $\mathcal{O}_M = \{M\}$). Replacing $R/M \subset T/M$ with $R \subset T$, we show that $T^G = R^G[u]$ for all $u \in T^G \setminus R^G$, i.e., $R^G \subset T^G$ is a minimal field extension. If $u \in T^G \setminus R^G$, then $u \in T \setminus R$, so T = R[u]. Let $t \in T^G$. Then $t = r_k u^k + \cdots + r_1 u + r_0$, for some $k \in \mathbb{N}$ and $r_i \in R$. By Lemma 3.7, there exist $m, m_i \in \mathbb{N}$ and $r'_i \in R^G$ such that $0 \neq mt = m_k r'_k u^k + \cdots + m_1 r'_1 u + m_0 r'_0$. Since R^G is a field, we have that $t = m^{-1}(m_k r'_k u^k + \cdots + m_1 r'_1 u + m_0 r'_0) \in R^G[u]$. Hence, $R^G \subset T^G$ is a minimal field extension. By Theorem 3.8(a), the original fixed ring extension (before passing to the quotient ring extension) $R^G \subset T^G$ is an inert integral minimal extension with crucial maximal ideal $\mathfrak{m} = (R^G :_{R^G} T^G)$.

Decomposed case: By Theorem 3.8(b), there exist $N_1, N_2 \in \text{Max}(T)$ such that $M = N_1 \cap N_2$ and the natural maps $R/M \to T/N_1$ and $R/M \to T/N_2$ are isomorphisms. Set $\mathfrak{n}_1 := N_1 \cap T^G$ and $\mathfrak{n}_2 := N_2 \cap T^G$. By Lemma 1.1, T is integral over T^G , whence $\mathfrak{n}_1, \mathfrak{n}_2 \in \text{Max}(T^G)$. Clearly

$$\mathfrak{m} = M \cap T^G = (N_1 \cap N_2) \cap T^G = \mathfrak{n}_1 \cap \mathfrak{n}_2.$$

Define $\phi : R^G/\mathfrak{m} \to T^G/\mathfrak{n}_1$ via the natural map $r + \mathfrak{m} \mapsto r + \mathfrak{n}_1$. Suppose that $\phi(r + \mathfrak{m}) = 0 + \mathfrak{n}_1$ for some $r \in R^G$. Then $r \in \mathfrak{n}_1 \cap R^G$, but, by Lemma 3.4(b), $\mathfrak{n}_1 \cap R^G = \mathfrak{m}$. Hence $r + \mathfrak{m} = 0 + \mathfrak{m}$. Thus ϕ is injective.

To show that ϕ is surjective, we first note that the *G*-action extends to T/N_1 , since it extends to R/M and $R/M \cong T/N_1$. From Proposition 3.5, we have that $R^G/\mathfrak{m} \cong (R/M)^G \cong (T/N_1)^G$. Let $t + \mathfrak{n}_1 \in T^G/\mathfrak{n}_1$ be nonzero. Then $t + N_1 \in (T/N_1)^G$ is nonzero. (Clearly it is fixed, and if $t \in N_1$, then $t \in N_1 \cap T^G = \mathfrak{n}_1$ – contradiction.) Since $R^G/\mathfrak{m} \cong (T/N_1)^G$ (via composition of the natural maps), there exists $r + \mathfrak{m} \in R^G/\mathfrak{m}$ such that $r + \mathfrak{m} \mapsto r + M \mapsto$ $r + N_1 = t + N_1$. It follows that $(r - t) \in N_1 \cap T^G = \mathfrak{n}_1$. Hence $\phi(r + \mathfrak{m}) = r + \mathfrak{n}_1 = t + \mathfrak{n}_1$, so ϕ is surjective. Thus $R^G/\mathfrak{m} \cong T^G/\mathfrak{n}_1$. The same argument applies to show $R^G/\mathfrak{m} \cong T^G/\mathfrak{n}_2$. By Theorem 3.8(b), $R^G \subset T^G$ is a decomposed integral minimal extension with crucial maximal ideal $\mathfrak{m} = (R^G :_{R^G} T^G)$.

Ramified case: By Theorem 3.8(c), there exists $N \in Max(T)$ such that $N^2 \subseteq M \subset N$, [T/M: R/M] = 2 and the natural map $R/M \to T/N$ is an isomorphism. Set $\mathfrak{n} := N \cap T^G$, and recall that $\mathfrak{m} = M \cap T^G$. Clearly, $\mathfrak{n} \in Max(T^G)$ and $\mathfrak{m} \subsetneq \mathfrak{n}$, since $\mathfrak{m} \notin Max(T^G)$ (as $M \notin Max(T), N \in Max(T)$, and T is integral over T^G). For the other containment, let $x \in \mathfrak{n}^2$. Then $x \in N^2$, so $x \in M$. Hence $x \in M \cap T^G = \mathfrak{m}$. Thus $\mathfrak{n}^2 \subseteq \mathfrak{m}$.

We show that the natural map $\phi : R^G/\mathfrak{m} \to T^G/\mathfrak{n}$ given by $r + \mathfrak{m} \mapsto r + \mathfrak{n}$ is an isomorphism. Suppose that $\phi(r + \mathfrak{m}) = 0 + \mathfrak{n}$ for some $r \in R^G$. Then $r \in \mathfrak{n}$, so $r^2 \in \mathfrak{n}^2$. Since $\mathfrak{n}^2 \subseteq \mathfrak{m}$ and \mathfrak{m} is prime (in fact maximal) in R^G , we have that $r \in \mathfrak{m}$. (Alternatively, $r \in \mathfrak{n} \cap R^G = \mathfrak{m}$, by Lemma 3.4(b).) Hence $r + \mathfrak{m} = 0 + \mathfrak{m}$. Thus ϕ is injective.

Next we show that ϕ is surjective. If $t + \mathfrak{n} \in T^G/\mathfrak{n}$, then $t + N \in (T/N)^G$. Note that, as in the decomposed case, since $R/M \cong T/N$, the G-action extends to T/N. From this and from Proposition 3.5, it follows that $R^G/\mathfrak{m} \cong (R/M)^G \cong (T/N)^G$ via $r + \mathfrak{m} \mapsto r + M \mapsto r + N$. Hence, there exists $r + \mathfrak{m} \in R^G/\mathfrak{m}$ such that $r + \mathfrak{m} \mapsto r + M \mapsto r + N = t + N$, from which it follows that $(r - t) \in N \cap T^G = \mathfrak{n}$. Hence $\phi(r + \mathfrak{m}) = t + \mathfrak{n}$. Thus ϕ is surjective. It remains to show that $[T^G/\mathfrak{m} : R^G/\mathfrak{m}] = 2$. Note that T^G/\mathfrak{m} is not a domain, since $\mathfrak{n}^2 \subseteq \mathfrak{m} \subset n$ implies $\mathfrak{m} = \mathfrak{n}$, if \mathfrak{m} is prime. Hence $T^G/\mathfrak{m} \neq R^G/\mathfrak{m}$, i.e., $[T^G/\mathfrak{m} : R^G/\mathfrak{m}] \geq 2$.

Suppose that $[T^G/\mathfrak{m} : \mathbb{R}^G/\mathfrak{m}] > 2$, and let $\{e_1 + \mathfrak{m}, e_2 + \mathfrak{m}, e_3 + \mathfrak{m}\}$ be an $\mathbb{R}^G/\mathfrak{m}$ -linearly independent set in T^G/\mathfrak{m} . Then each $e_i \notin M$; otherwise, $e_i \in M \cap T^G = \mathfrak{m}$. Hence each $e_i + M$ is nonzero in T/M. Since $[T/M : \mathbb{R}/M] = 2$, without loss of generality we may assume that there exist $t_1 + M, t_2 + M \in T/M$ such that

$$e_3 + M = (t_1 + M)(e_1 + M) + (t_2 + M)(e_2 + M) = t_1e_1 + t_2e_2 + M.$$

As in Lemma 3.7, using $\sigma \in \mathcal{N}_{t_1}$ and summing \mathcal{O}_{t_1} we have that

$$n_{t_1}e_3 + M = \hat{t}_1e_1 + \left(\sum_{\sigma \in \mathcal{N}_{t_1}} \sigma(t_2)\right)e_2 + M.$$

Defining t_3 to be the coefficient of e_2 above and repeating the above technique with respect to t_3 we have that

$$n_{t_3}n_{t_1}e_3 + M = n_{t_3}\hat{t}_1e_1 + \hat{t}_3e_2 + M$$

It follows that $n_{t_3}n_{t_1}e_3 - (n_{t_3}\hat{t}_1e_1 + \hat{t}_3e_2) \in M \cap T^G = \mathfrak{m}$, so

$$n_{t_3}n_{t_1}e_3 + \mathfrak{m} = n_{t_3}\hat{t}_1e_1 + \hat{t}_3e_2 + \mathfrak{m}.$$

Equivalently,

$$(n_{t_3}n_{t_1} + \mathfrak{m})(e_3 + \mathfrak{m}) = (n_{t_3}\widehat{t}_1 + \mathfrak{m})(e_1 + \mathfrak{m}) + (\widehat{t}_3 + \mathfrak{m})(e_2 + \mathfrak{m})$$

is an R^G/\mathfrak{m} -linear combination of $e_1 + \mathfrak{m}, e_2 + \mathfrak{m}, e_3 + \mathfrak{m}$ in T^G/\mathfrak{m} – contradiction. Hence, there cannot exist in T^G/\mathfrak{m} any more than two R^G/\mathfrak{m} -linearly independent elements. Thus $[T^G/\mathfrak{m}: R^G/\mathfrak{m}] \leq 2$. Hence $[T^G/\mathfrak{m}: R^G/\mathfrak{m}] = 2$. By Theorem 3.8(c), $R^G \subset T^G$ is a ramified integral minimal extension with crucial maximal ideal $\mathfrak{m} = (\mathbb{R}^G :_{\mathbb{R}^G} \mathbb{T}^G).$

Remark 3.10. If we were to assume that R is G-invariant in this section (instead of waiting until Section 3.3), then the conditions that R is integral over R^G and $\mathcal{O}_M = \{M\}$ would automatically be satisfied. Integrality follows from Lemma 1.1. To see that $\mathcal{O}_M = \{M\}$, note that $\sigma(M)T = \sigma(MT) \subseteq \sigma(R) = R$, for any $\sigma \in G$. Hence $\sigma(M) \subseteq M$. Since $\sigma(M) \in Max(R)$ (by the First Isomorphism Theorem), we have that $\sigma(M) = M$.

Remark 3.11. It is necessary to assume that $R^G \neq T^G$ in Theorem 3.9, as illustrated in the following.

Example 3.12. The fixed rings are equal, even under finite group action, in the following cases:

Inert case: Set $R := \mathbb{R}$, $T := \mathbb{C}$, and $G := \{1, \sigma\}$, where σ is the conjugacy map. Then $R^G = R = T^G$.

Decomposed case: Let F be a field such that $\operatorname{char}(F) \neq 2$, and set $R := \{(x, x) \mid x \in F\}$ and $T := F \times F$. By [20, Lemme 1.2(b)], $R \subset T$ is a minimal extension. Define $G := \{1, \sigma\}$, where $\sigma((x, x) = (x, -x)$. Then $R^G = T^G$.

Ramified case: Let F and R be as above, and set T := F(+)F. Then by [20, Lemme 1.2(c)], $R \subset T$ is a minimal extension. Define G as above. Then $R^G = T^G$.

3.3 Integrally Closed Minimal Extension

Our riding assumptions in this section are that R is a (proper) subring of T, G is a subgroup of $\operatorname{Aut}(T)$, and R is G-invariant. We will show that the integrally closed minimal property of an extension $R \subset T$ is invariant when G is locally finite. This generalizes Dobbs' and Shapiro's result that the property is invariant if R is a domain and if |G| is finite and a unit in R [16, Theorem 3.6]. They use Ayache's characterization of minimal extensions (overrings) of an integrally closed domain [5, Theorem 2.4]. This result has since been generalized by Cahen et al. [7, Theorem 3.5]. With this more recent characterization we establish Theorem 3.18.

Whereas crucial maximal ideals are historically essential to the study of minimal extensions, Cahen et al. introduce critical ideals and use them extensively in classifying integrally closed minimal extensions of an arbitrary ring [7]. They define a critical ideal for $R \subset T$ as an ideal $I \subset R$ such that $I = \text{Rad}_R((R:_R t))$ for all $t \in T \setminus R$. That is, $\text{Rad}_R((R:_R t))$ is the same ideal for all $t \in T \setminus R$. They show in [7, Lemma 2.11] that if an extension has a critical ideal, then the ideal is prime. Moreover, they show that if $R \subset T$ is a minimal extension, then the critical ideal exists [7, Proposition 2.14(2)] and is maximal [7, Theorem 3.5]. If $R \subset T$ has a critical ideal, we show that $R^G \subset T^G$ has a critical ideal under any group action such that $R^G \neq T^G$.

Lemma 3.13. Let P be the critical ideal of $R \subset T$. If $R^G \neq T^G$, then $\mathfrak{p} := P \cap R^G$ is the critical ideal of $R^G \subset T^G$.

Proof. Let $t \in T^G \setminus R^G$. Then $t \in T \setminus R$. Hence $P = \operatorname{Rad}_R((R :_R t))$, from which it follows that

$$\mathfrak{p} = \operatorname{Rad}_R((R:_R t)) \cap R^G = \operatorname{Rad}_{R^G}((R:_R t) \cap R^G) = \operatorname{Rad}_{R^G}((R^G:_{R^G} t)).$$

Thus \mathfrak{p} is the critical ideal of $R^G \subset T^G$.

We next show that if a critical ideal is maximal, then its orbit (under G) is a singleton set.

Lemma 3.14. Suppose that $M = \operatorname{Rad}_R((R:_R t))$, for all $t \in T \setminus R$, i.e., M is the critical ideal for $R \subset T$. If M is a maximal ideal of R, then $\sigma(M) = M$ for all $\sigma \in G$, i.e. $\mathcal{O}_M = \{M\}.$

Proof. Let $\sigma \in G$ and $t \in T \setminus R$. Note that $\sigma^{-1}(t) \in T \setminus R$, for otherwise, if $\sigma^{-1}(t) \in R$, then $t = \sigma(\sigma^{-1}(t)) \in \sigma(R) = R$ – contradiction. Since M is the critical ideal for $R \subset T$, $M = \operatorname{Rad}_R((R :_R \sigma^{-1}(t)))$. Let $x \in M$ and set $y := \sigma^{-1}(x)$. Then there exists $n \in \mathbb{N}$ such that $x^n t \in R$, from which it follows that $(\sigma^{-1}(x))^n \sigma^{-1}(t) \in \sigma^{-1}(R) = R$. Hence $y = \sigma^{-1}(x) \in \operatorname{Rad}_R((R :_R \sigma^{-1}(t))) = M$. Thus $x = \sigma(y) \in \sigma(M)$, which shows that $M \subseteq \sigma(M)$. Since M is maximal, $M = \sigma(M)$, as desired.

Remark 3.15. In fact, it is not necessary to assume that M is maximal in the preceding lemma. However, it is maximal when it is the critical ideal of an integrally closed minimal ring extension (c.f. [7, Theorem 3.5]), and this is the case in our application of the lemma.

Related to critical ideals are valuation pairs for an extension $R \subset T$. As in [31], for $P \in \operatorname{Spec}(R)$, (R, P) is a valuation pair of T if there is a valuation v on T with $R = \{t \in T \mid v(t) \ge 0\}$ and $P = \{t \in T \mid v(t) > 0\}$. Equivalently, (R, P) is a valuation pair of T if R = S whenever S is an intermediate ring containing a prime ideal lying over P [31]. Rank 1 valuation pairs are one of several equivalences of integrally closed minimal extensions given by Cahen et al [7]. As previously mentioned, the rank of a valuation pair (R, P) of T is the rank of the valuation group. The following lemma describes the relationship between critical ideals and valuation pairs.

Lemma 3.16. [7, Lemma 2.12] Let (R, P) be a valuation pair of T. Then $R \subset T$ has a critical ideal if and only if (R, P) has rank 1. Moreover, under these conditions, P is the critical ideal of $R \subset T$.

Our next result is fundamental to the invariance of integrally closed minimal extensions to be established in Theorem 3.18.

Proposition 3.17. Assume that G is locally finite (on T) such that $R^G \neq T^G$. Let $M \in \text{Max}(R)$ and set $\mathfrak{m} := M \cap R^G$. If $\mathcal{O}_M = \{M\}$, then (R^G, \mathfrak{m}) is a valuation pair of T^G whenever (R, M) is a valuation pair of T.

Proof. Let A be a ring such that $R^G \subseteq A \subseteq T^G$. Then $R \subseteq AR \subseteq T$. First note that AR is integral over A, since R is integral over R^G , hence over A. Let $\mathfrak{q} \in \operatorname{Spec}(A)$ such that

 $\mathfrak{q}\cap R^G=\mathfrak{m},$ and let $Q\in \operatorname{Spec}(AR)$ lie over $\mathfrak{q}.$ From

$$\mathfrak{m}=\mathfrak{q}\cap R^G=(Q\cap A)\cap R^G=Q\cap R^G=(Q\cap R)\cap R^G$$

it follows that $Q \cap R$ is maximal in R, by integrality. We claim $Q \cap R = M$. Suppose not. Then there exists $x \in (Q \cap R) \setminus M$, since $Q \cap R$ and M are incomparable (as maximal ideals). It follows that $\tilde{x} \in Q \cap R^G = \mathfrak{m} = M \cap R^G$. Hence $\sigma(x) \in M$ for some $\sigma \in G$. Since $\mathcal{O}_M = \{M\}$, we have that $x \in \sigma^{-1}(M) = M$ – contradiction. Hence $Q \cap R = M$. Since (R, M) is a valuation pair of T, we have that AR = R, whence $A = R^G$. Thus (R^G, \mathfrak{m}) is a valuation pair of T^G .

Of the several integrally closed minimal extension equivalences in [7, Theorem 3.5], we use the condition that there exists a maximal ideal M such that (R, M) is a rank 1 valuation pair of T where $R \subset T$. With this equivalence, it follows directly from the preceding results that integrally closed minimal extensions are invariant under locally finite group action.

Theorem 3.18. Assume that G is locally finite (on T). If $R \subset T$ is an integrally closed minimal extension, then $R^G \subset T^G$ is an integrally closed minimal extension.

Proof. First we show that $R^G \neq T^G$. Let $t \in T \setminus R$. Then $\tilde{t} \in T^G$. Suppose that $\tilde{t} \in R^G$. Then $\tilde{t} \in R$. By [20, Proposition 3.1], $\sigma(t) \in R$ for some $\sigma \in G$, whence $t = \sigma^{-1}(\sigma(t)) \in \sigma^{-1}(R) = R$ – contradiction. Hence, $\tilde{t} \in T^G \setminus R^G$. Thus, $R^G \subsetneq T^G$.

Let M be the critical ideal for $R \subset T$. By Lemma 3.13, $m := M \cap R^G$ is the critical ideal for $R^G \subset T^G$. Since $R \subset T$ is a minimal extension, the critical ideal M is maximal. By Lemma 3.14 $\mathcal{O}_M = \{M\}$. By Lemma 3.17 (R^G, m) is a valuation pair of T^G . Since mis the critical ideal of $R^G \subset T^G$, this valuation pair has rank 1 by Lemma 3.16. Hence, $R^G \subset T^G$ is an integrally closed minimal extension by [7, Proposition 3.5].

3.4 Minimal Extensions, Flat Epimorphisms, and Normal Pairs

As in the previous section, our riding assumptions in this section are that R is a subring of T, G acts on T via automorphisms, and R is G-invariant. In this section, we generalize the results of Sections 3.2 and 3.3.

In Proposition 3.19 and Corollary 3.20, we show that integral minimal extensions are invariant under stronger assumptions on G but without the restriction of characteristic used in Theorem 3.9. In doing so, we simultaneously re-establish Theorem 3.18.

In Theorem 3.23, we exchange a stronger assumption for a more general result. In particular, we assume that G is strongly locally finite in order to show that flat epimorphic extensions are invariant.

Lastly in Corollary 3.26, we show that normal pairs are invariant. As in [9], we say that (R, T) is a normal pair if S is integrally closed in T whenever $R \subseteq S \subseteq T$. Clearly, if $R \subset T$ integrally closed minimal extension, then (R, T) is a normal pair.

As in Theorems 3.9 and 3.18, certain integral minimal extensions and all integrally closed minimal extensions are invariant under locally finite G-action. In the former, however, we require a certain restriction of characteristic. Assuming that |G| is finite and a unit in the base ring, we can remove this restriction. Of course, if G is finite, then it is locally finite. Hence, the following result and corollary re-establish Theorem 3.18.

Proposition 3.19. Let $R \subset T$ be a minimal extension. Assume that G is finite such that |G| is a unit in R and $R^G \neq T^G$. Then $R^G \subset T^G$ is a minimal extension.

Proof. Let $u \in T^G \setminus \mathbb{R}^G$. Clearly, $u \in T \setminus \mathbb{R}$. Hence, $T = \mathbb{R}[u]$. Let $t \in T^G$. Then $t = r_n u^n + \cdots + r_1 u + r_0$ for some $r_i \in \mathbb{R}$. Applying the averaging technique introduced in Section 3.2 we have that

$$t = |G|^{-1} \sum_{\sigma \in G} \sigma(r_n) u^n + \dots + \sigma(r_1) u + \sigma(r_0).$$

Thus $T^G = R^G[u]$, i.e. $R^G \subset T^G$ is a minimal extension.

Combining Propositions 3.1, 3.2(a), and 3.19, we have the following corollary.

Corollary 3.20. Under the hypotheses of Proposition 3.19, if $R \subset T$ is an integral or integrally closed minimal extension, then $R^G \subset T^G$ is an integral or integrally closed minimal extension, respectively.

Integrally closed minimal extensions are flat epimorphic extensions, by [20, Théormè 2.2]. An extension $R \subset T$ is called *flat epimorphic extension* if T is a flat R-module and if the inclusion map is an epimorphism (in the category of commutative rings). We will introduce an equivalent characterization of flat epimorphisms, but first we introduce the construction of localization at a filter (versus at a multiplicative set).

A collection of ideals \mathcal{F} of a ring R is called a *Gabriel filter* (or *localizing filter*) if it satisfies:

- (i) If $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$.
- (ii) If $I, J \in \mathcal{F}$, then $I \cap J \in \mathcal{F}$.
- (iii) If for an ideal I there exists $J \in \mathcal{F}$ such that $(I:j) \in \mathcal{F}$ for every $j \in J$, then $I \in \mathcal{F}$.

By $R_{\mathcal{F}}$ we denote the localization of R at \mathcal{F} (or ring of quotients with respect to \mathcal{F}). One way to define $R_{\mathcal{F}}$ is as follows: Assume R is \mathcal{F} -torsion free, i.e., $\tau_{\mathcal{F}}(R) = \{0\}$. (If R is not torsion-free, then set $R := R/\tau_{\mathcal{F}}(R)$.) Then $R_{\mathcal{F}} = \pi^{-1}(\tau_{\mathcal{F}}(E(R)/R))$, where $\pi : E(R) \to E(R)/R$ is the canonical projection and E(R) denotes the injective hull of R. For more information on Gabriel filters and localizations, see [21, 38, 39].

Theorem 3.21. [39, Theorem 2.1, Ch. XI] Let $\phi : R \to T$ be a ring homomorphism. Then ϕ is a *flat epimorphism* if and only if the collection $\mathcal{F} = \{I \subset R \mid \phi(I)T = T\}$ where I is an ideal in R is a Gabriel filter, and there exists an isomorphism $\psi : T \to R_{\mathcal{F}}$ such that $\psi \circ \phi : R \to R_{\mathcal{F}}$ is the canonical homomorphism. Such a filter is called *perfect*. Equivalently, flat epimorphisms are *perfect localizations*, so-called because of the above correspondence. By [39, Exercise 8, p. 242], T is a perfect localization of R if and only if for all $t \in T$, $(R :_R t)T = T$. With this definition and Lemma 3.22 we show that perfect localizations (equivalently, flat epimorphic extensions) are invariant in Proposition 3.23.

Lemma 3.22. Assume that G is strongly locally finite (on T) and R is G-invariant. Define $\mathcal{F} := \{I \subset R \mid IT = T\}$ and $\mathcal{F}' := \{J \subset R^G \mid JT^G = T^G\}$. If $I \in \mathcal{F}$, then $I \cap R^G \in \mathcal{F}'$.

Proof. Note that $I \in \mathcal{F}$ if and only if every $P \in \operatorname{Spec}(R)$ containing I is not lain over in T. Also note that $\mathcal{F}' = \{J \subset R^G \mid JR \in \mathcal{F}\}$. Let $I \in \mathcal{F}$ and let $P \in \operatorname{Spec}(R)$ contain $(I \cap R^G)R$. We claim $I \subseteq \sigma(P)$ for some $\sigma \in G$, whence $PT = \sigma^{-1}(\sigma(P)T) = \sigma^{-1}(\sigma(PT)) = T$ (since IT = T). Let $x \in I$. Then $\tilde{x} \in I \cap R^G$, so $\tilde{x} \in P$. It follows that $\sigma(x) \in P$ for some $\sigma \in G$; equivalently, $x \in \sigma^{-1}(P)$. Hence $I \subseteq \bigcup_{Q \in \mathcal{O}_P} Q$. Since G is strongly locally finite, \mathcal{O}_P is finite. It follows that $I \subseteq Q$ for some $Q \in \mathcal{O}_P$ by the Prime Avoidance Lemma [26, Theorem 81]. Hence the claim is satisfied by $\sigma \in G$, where $Q = \sigma(P)$, so PT = T. Thus, every prime containing $(I \cap R^G)R$ is not lain over in T. That is, $(I \cap R^G)R \in \mathcal{F}$, whence $I \cap R^G \in \mathcal{F}'$, as desired.

We are now ready to show that perfect localizations (flat epimorphic extensions) are invariant under strongly locally finite group action using Lemma 3.22.

Theorem 3.23. Assume that G is strongly locally finite (on T) and R is G-invariant. Let \mathcal{F} and \mathcal{F}' be as in Lemma 3.22. Then

(a) \mathcal{F}' is a Gabriel filter whenever \mathcal{F} is a Gabriel filter, and

(b) if $R \subseteq T$ is a flat epimorphic extension, then so is $R^G \subseteq T^G$.

In particular, $T^G = (R^G)_{\mathcal{F}'}$ whenever $T = R_{\mathcal{F}}$.

Proof. (a) Suppose that \mathcal{F} is a Gabriel filter. We check that \mathcal{F}' satisfies the defining conditions (i) through (iii) of a Gabriel filter given above. Let $I \in \mathcal{F}'$, and let J be an

ideal of R^G containing I. Then $IR \in \mathcal{F}$ and $IR \subseteq JR$, so $JR \in \mathcal{F}$. It follows that JT = T, so $JT^G = T^G$, since T is integral over T^G . Hence $J \in \mathcal{F}'$, which establishes condition (i). Now let $I, J \in \mathcal{F}'$. Then IT = T and JT = T. Suppose that $I \cap J \notin \mathcal{F}'$, i.e. $(I \cap J)T^G \neq T^G$. Again by integrality, $(I \cap J)T \neq T$. Let $P \in \operatorname{Spec}(T)$ contain $(I \cap J)T$. Then $I \cap J \subseteq P \cap T^G =: \mathfrak{p}$. It follows that $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$, but then $IT \subseteq P$ or $JT \subseteq P - \operatorname{contradiction}$. Hence $I \cap J \in \mathcal{F}'$, which establishes condition (ii).

It remains to show that \mathcal{F}' satisfies condition (iii). Let J be an ideal of \mathbb{R}^G , and suppose that there exists $I \in \mathcal{F}'$ such that $(J:_{\mathbb{R}^G} a) \in \mathcal{F}'$ for all $a \in I$. We claim $(JR:_{\mathbb{R}} a) \in \mathcal{F}$ for all $a \in IR$, whence $JR \in \mathcal{F}$, i.e., $J \in \mathcal{F}'$. Let $a := a_1r_1 + \cdots + a_nr_n \in IR$, where $a_i \in I$ and $r_i \in R$. For each a_i , clearly $(J:_{\mathbb{R}^G} a_i)\mathbb{R} \subseteq (JR:_{\mathbb{R}} a_i)$. Since $(J:_{\mathbb{R}^G} a_i) \in \mathcal{F}'$, we have that $(J:_{\mathbb{R}^G} a_i)\mathbb{R} \in \mathcal{F}$. Hence $(JR:_{\mathbb{R}} a_i) \in \mathcal{F}$. From $(JR:_{\mathbb{R}} a_i) \subseteq (JR:_{\mathbb{R}} a_ir_i)$ it follows that $(JR:_{\mathbb{R}} a_ir_i) \in \mathcal{F}$. Since $\bigcap_{i=1}^n (JR:_{\mathbb{R}} a_ir_i) \in \mathcal{F}$ and $\bigcap_{i=1}^n (JR:_{\mathbb{R}} a_ir_i) \subseteq (JR:_{\mathbb{R}} a)$, we have that $(JR:_{\mathbb{R}} a) \in \mathcal{F}$, proving the claim. Hence $JR \in \mathcal{F}$, i.e. $J \in \mathcal{F}'$. Thus \mathcal{F}' is a Gabriel filter.

(b) Now we show that $R^G \subseteq T^G$ is a flat epimorphic extension by showing that T^G is a perfect localization of R^G . Let $x \in T^G$. Then $(R :_R x)T = T$, since T is a perfect localization of R. It follows that $(R :_R x) \in \mathcal{F}$, and $(R :_R x) \cap R^G \in \mathcal{F}'$, by Lemma 3.22. We claim $(R :_R x) \cap R^G \subseteq (R^G :_{R^G} x)$, whence $(R^G :_{R^G} x) \in \mathcal{F}'$, since \mathcal{F}' is a Gabriel filter. Let $y \in (R :_R x) \cap R^G$. Then $xy \in R$, but $x \in T^G$ and $y \in T^G$, so $xy \in R^G$. Hence $(R :_R x) \cap R^G \subseteq (R^G :_{R^G} x)$, so $(R^G :_{R^G} x) \in \mathcal{F}'$ as claimed. (In fact, as the reverse containment clearly holds, $(R :_R x) \cap R^G = (R^G :_{R^G} x)$.) Thus $(R^G :_{R^G} x)T^G = T^G$, i.e. T^G is a perfect localization of R^G . In particular, $T^G \cong (R^G)_{\mathcal{F}}$.

Remark 3.24. It would be interesting to know if epimorphic extensions or flat extensions are invariant under any group action.

Normal pairs are another generalization of integrally closed minimal extensions. By [27, Theorem 5.2], (R,T) is a normal pair if and only if R is integrally closed in T and

 $R \subseteq S$ satisfies INC for any intermediate ring S. A pair of rings (R, T) satisfying the latter property is called an *INC-pair* and note that it is equivalent to the definition of an INC-pair given in [10].

We have already seen that integrally closed extensions are invariant in Proposition 3.2 (a). To assert that normal pairs are invariant, it remains to show that INC-pairs are invariant.

Proposition 3.25. Assume that G is locally finite (on T). If (R, T) is an INC-pair, then (R^G, T^G) is an INC-pair.

Proof. Let $R^G \subseteq A \subseteq T^G$, and let $\mathfrak{q} \subseteq \mathfrak{q}'$ be prime ideals of A with the same contraction in R^G . Set $\mathfrak{p} := \mathfrak{q} \cap R^G = \mathfrak{q}' \cap R^G$. Since R is integral over R^G (whence over A), AR is integral over A. Hence, $A \subseteq AR$ satisfies LO and GU. Let $Q \subseteq Q'$ be prime ideals in AR such that $\mathfrak{q} = Q \cap A$ and $\mathfrak{q}' = Q \cap A$. Setting $P := Q \cap R$ and $P' := Q' \cap R$, we have that $P \subseteq P'$ and

$$P \cap R^G = Q \cap R^G = (Q \cap A) \cap R^G = \mathfrak{q} \cap R^G = \mathfrak{p},$$

and $P' \cap R^G = \mathfrak{p}$, by the same reasoning. As an integral extension, $R^G \subseteq R$ satisfies INC, whence P = P'. Since $R \subseteq AR$ satisfies INC, Q = Q'. Hence q = q'. Thus (R^G, T^G) is an INC-pair.

The corollary below now follows easily from Propositions 3.2 (a) and 3.25.

Corollary 3.26. If G is locally finite (on T), then (R^G, T^G) is a normal pair whenever (R, T) is a normal pair.

3.5 FIP and FCP Extensions

As in the previous section, our riding assumptions in this section are that R is a subring of T, G acts on T via automorphisms, and R is G-invariant. We denote the collection of intermediate rings by [R, T], and the collection of proper intermediate rings by (R, T). We say that $R \subset T$ satisfies the finitely many intermediate algebras property (FIP) if [R, T] is finite, and we say that the extension satisfies the finite chain property (FCP) if every chain in [R, T] is finite.

We utilize several significant results in [12] to show that these are invariant properties of ring extensions. By Theorem 3.27 below, while studying extensions satisfying FIP or FCP we need only consider two cases: integral and integrally closed extensions. By [12, Theorem 6.3], in the integrally closed case, the FIP and FCP are equivalent. We begin with this case in Theorem 3.28.

Theorem 3.27. [12, Theorem 3.13] The extension $R \subset T$ satisfies FCP (respectively, FIP) if an only if $R \subseteq \overline{R}$ and $\overline{R} \subseteq T$ satisfy FCP (respectively, FIP).

Theorem 3.28. Assume G is locally finite, and $R \subset T$ is integrally closed. If $R \subset T$ satisfies FCP (FIP), then so does $R^G \subseteq T^G$.

Proof. Let $R = S_0 \subset S_1 \subset \cdots \subset S_n \subset T$ be a maximal chain, i.e., each subextension is minimal. If n = 0, then $R \subset T$ is a minimal extension, in which case $R^G \subset T^G$ is a minimal extension by Proposition 3.18. If n = 1, then $R \subset S_1$ and $S_1 \subset T$ are both integrally closed minimal ring extensions (the latter by [12, Theorem 6.3 (b)]). We will show that S_1 is G-invariant, i.e., G acts on S_1 , whence $R^G \subset S_1^G$ and $S_1^G \subset T^G$ are integrally closed minimal ring extensions by Theorem 3.18.

Let $s \in S_1 \setminus R$. Then $\tilde{s} \in S_1^G$. Note that $R \subset S_1$ is an integrally closed minimal ring extension and so satisfies Samuel's condition, meaning $a \in R$ or $b \in R$ whenever $ab \in R$, for $a, b \in S_1$ [20, Proposition 3.1]. In particular, if $\tilde{s} \in R$, then $\sigma(s) \in R$ for some $\sigma \in G$, whence $s \in \sigma^{-1}(R) = R$ – contradiction. Hence, $\tilde{s} \in S_1^G \setminus R^G$, i.e., $R^G \subsetneq S_1^G$. Let $u \in S_1^G \setminus R^G$. Then $u \in S_1 \setminus R$. Since $R \subset S_1$ is minimal, we have $S_1 = R[u]$. Hence, $\sigma(S_1) = \sigma(R)[\sigma(u)] = R[u] = S_1$ for all $\sigma \in G$. Thus S_1 is G-invariant.

Now assume that if $R = \hat{S}_0 \subset \hat{S}_1 \subset \cdots \subset \hat{S}_{n-1} \subset T$ is a maximal chain, then $R^G = \hat{S}_0 \cap T^G \subseteq \hat{S}_1 \cap T^G \subseteq \cdots \subseteq \hat{S}_{n-1} \cap T^G \subseteq T^G$ is a maximal chain. Consider the aforementioned

maximal chain $R = S_0 \subset S_1 \subset \cdots \subset S_n \subset T$. By the reasoning in the case where $n = 1, S_1$ is integrally closed in T, and S_1 is G-invariant. Hence $R^G \subseteq S_1^G$ is a minimal ring extension (again as in the case where n = 1), and $S_1 \cap T^G \subseteq \cdots S_n \cap T^G \subseteq T^G$ is a maximal chain by the inductive hypothesis. Thus $R^G \subset S_1^G \subseteq \cdots \subseteq S_n^G \subseteq T^G$ is a maximal chain. By Proposition 3.2 (a), $R^G \subset T^G$ is integrally closed. By [12, Theorem 6.3 (a)], $R^G \subseteq T^G$ satisfies FCP.

Before establishing that the integral FCP property is invariant, we introduce two constructions that we will use. As in [40], we say that a ring extension $A \subseteq B$ is seminormal if $b \in B$ and $b^2, b^3 \in A$ imply $b \in A$. Given $R \subset T$, the seminormalization of R in T is the smallest ring ${}_{T}^{+}R \in [R, T]$ such that ${}_{T}^{+}R \subseteq T$ is seminormal. As in [36], we say that $A \subset B$ is t-closed if $b \in B$, $a \in A$, $b^2 - ab \in A$, and $b^3 - ab^2 \in A$ imply $b \in A$. Given $R \subset T$, the t-closure of R in T is the smallest ring ${}_{T}^{t}R \in [R, T]$ such that ${}_{T}^{t}R \subseteq T$ is t-closed. The rings ${}_{T}^{+}R$ and ${}_{T}^{t}R$ can be constructed via elementary subintegral and infra-integral extensions. Again as in [40] (respectively, [36]), a ring extension $A \subset B$ is an elementary subintegral (respectively, infra-integral) extension if B = A[b], where $b \in B$ and $b^2, b^3 \in A$ (respectively, $b^2 - ab, b^3 - ab^2 \in A$, where $a \in A$).

Lemma 3.29. For $R \subset T$, ${}^+_TR$ and t_TR are *G*-invariant.

Proof. By [40, Theorem 2.8], ${}^+_T R$ is the union of all subrings of T obtained from R by a finite number of elementary subintegral extensions. Analogously, by [36, Théorème 2.5], ${}^t_T R$ is the union of all subrings of T obtained from R by a finite number of elementary infra-integral extensions. Let $S \in (R, T)$ and $\sigma \in G$. If S = R[s], where $s \in S$ and $s^2, s^3 \in R$ (respectively, $s^2 - rs, s^3 - rs^2 \in R$, where $r \in R$), then, since R is G-invariant, $\sigma(S) = R[\sigma(s)], \sigma(s) \in \sigma(S)$, and $\sigma(s)^2, \sigma(s)^3 \in R$ (respectively, $\sigma(s)^2 - \sigma(r)\sigma(s), \sigma(s)^3 - \sigma(r)\sigma(s)^2 \in R$, where $\sigma(r) \in R$. Hence, the image of an elementary subintegral (respectively, infra-integral) extension under any element of G is an elementary subintegral (respectively, infra-integral) extension. It follows that ${}^{+}_{T}R$ (respectively, ${}^{t}_{T}R$) is G-invariant.

With this lemma and the following theorem, we show that, under certain assumptions, the integral case of the FCP is invariant in Theorem 3.31.

Theorem 3.30. [12, Theorem 4.6] Let $R \subseteq T$ ben an integral extension. Then $R \subseteq T$ satisfies FCP if and only if $R \subseteq {}^{+}_{T}R$, ${}^{+}_{T}R \subseteq {}^{t}_{T}R$, and ${}^{t}_{T}R \subseteq T$ each satisfy FCP.

Theorem 3.31. Assume G is locally finite such that $\operatorname{char}(R^G/(R^G:_{R^G}T^G)) \nmid n_r$ for all $r \in R$. Also assume at least one of $_T^+R$ or $_T^tR$ is distinct from both R and T, i.e., $_T^+R \in (R,T)$ or $_T^tR \in (R,T)$. If $R \subset T$ is integral and satisfies FCP, then so does $R^G \subseteq T^G$.

Proof. By [12, Theorem 4.2], there exists a maximal chain $R \subset S_1 \subset \cdots \subset S_n \subset T$, where each subextension is integral. If n = 0, then $R \subset T$ is minimal, whence $R^G \subset T^G$ is minimal or $R^G = T^G$ by Theorem 3.9. If n = 1, then let $S := S_1$ and there are three cases: $R^G = S^G = T^G$, $R^G \subset S^G = T^G$, and $R^G = S^G \subset T^G$. Clearly, $R^G = S^G = T^G$ is trivial. If $R^G \subset S^G = T^G$, let $u \in S^G \setminus R^G$. Then $u \in S \setminus R$, whence S = R[u]. For any $\sigma \in G$ we have $\sigma(S) = \sigma(R)[\sigma(u)] = R[u] = S$. Hence S is G-invariant. By Theorem 3.9, $S^G \subset T^G$ is a minimal ring extension.

The case where $R^G = S^G \subset T^G$ is more complicated. Set $M := (R :_R S), N := (S :_S T)$, and $P := N \cap R$, and note that M and N are the crucial maximal ideals of $R \subset S$ and $S \subset T$, respectively. If $P \not\subseteq M$, then $[R,T] = \{R,S,S',T\}$, where $R \subset S'$ and $S' \subset T$ are minimal, by [12, Lemma 2.7]. Without loss of generality assume that ${}^+_TR \in (R,T)$. Then again without loss of generality we may assume ${}^+_TR = S$, whence S is G-invariant, by Lemma 3.29. By Theorem 3.9, $S^G \subset T^G$ is a minimal ring extension.

Now consider $R^G = S^G \subset T^G$ and $P \subseteq M$. Since $R \subset S$ is integral and $N \in Max(S)$, we have $P \in Max(R)$, whence P = M. As in the previous cases, we will show that $S^G \subset T^G$ is a minimal ring extension by Theorem 3.9, except we establish that $S^G \subseteq S$ is an integral extension instead of determining that S is G-invariant. To do so we argue $M \in Max(S)$ and $\mathcal{O}_M = \{M\}$.

Since $R^G \subseteq R$ and $R \subset S$ are integral and $S^G = R^G$, we have that $S^G \subseteq S$ is integral. Since $M \in Max(R)$, we have $M \cap S^G = M \cap R^G$ is maximal in $S^G = R^G$. Hence $M \in Max(S)$. Moreover, since $M = N \cap R$, we have $M \subseteq N$, whence M = N, i.e., $(R :_R S) = (S :_S T)$.

We now show that $\mathcal{O}_M = \{M\}$. Note that $(R:_R T) \subseteq M$. For the sake of contradiction, assume that there exists $r \in M \setminus (R:_R T)$. Since $M = N \cap R$, we have $r \in N$, whence $rT \subseteq S$. It follows that $r^2T \subseteq rS \subset R$, i.e. $r^2 \in (R:_R T)$. Since $(R:_R T) \in \text{Spec}(R)$, we have $r \in (R:_R T)$ – contradiction. Thus $(R:_R T) = M$. By Remark 3.10 $\mathcal{O}_M = \{M\}$.

Having established that $S^G \subseteq S$ is an integral extension, $M \in Max(S)$, and $\mathcal{O}_M = \{M\}$, it now follows from Theorem 3.9 that $S^G \subset T^G$ is a minimal integral ring extension. Thus the base case is complete.

We now proceed with the inductive argument which is similar to that in Theorem 3.28, except we use strong induction. Let 1 < k < n and assume that if $R = \hat{S}_0 \subset \hat{S}_1 \subset \cdots \subset \hat{S}_k \subset T$ is a maximal chain, where each subextension is integral, then $R^G = \hat{S}_0 \cap T^G \subseteq \hat{S}_1 \cap T^G \subseteq \cdots \subseteq \hat{S}_k \cap T^G$ is also a maximal chain with integral subextensions. Consider the aforementioned maximal chain $R \subset S_1 \subset \cdots \subset S_n \subset T$. Without loss of generality we may assume ${}_T^+R \in (R,T)$. Since $R \subset T$ satisfies FCP, clearly $R \subset {}_T^+R$ and ${}_T^+R \subset T$ satisfy FCP. Without loss of generality we may assume ${}_T^+R = S_k$. Hence,

$$R \subset S_1 \subset \cdots \subset S_{k-1} \subset S_k = {}^+_T R$$

and

$$^+_T R = S_k \subset S_{k+1} \subset \dots \subset S_n \subset T$$

are maximal chains, where each subextension is integral. By Lemma 3.29, the inductive hypothesis applies to the above chains. Hence, $R^G \subseteq S_1 \cap T^G \subseteq \cdots \subseteq S_{k-1} \cap T^G \subseteq ({}^+_T R)^G$

and $({}^{+}_{T}R)^{G} \subseteq S_{k+1} \cap T^{G} \subseteq \cdots \subseteq S_{n} \cap T^{G} \subseteq T^{G}$ are maximal chains where each subextension is integral. Thus, $R^{G} \subseteq S_{1}^{G} \subseteq \cdots \subseteq S_{n}^{G} \subseteq T^{G}$ is a maximal chain, where each subextension is integral. By Proposition 3.1, $R^{G} \subseteq T^{G}$ is an integral extension. By [12, Theorem 4.2] $R^{G} \subseteq T^{G}$ satisfies FCP.

With stronger assumptions we can show that being an integral extension satisfying FIP is an invariant property. In fact a similar argument applies to the integrally closed FCP (FIP) and integral FCP cases.

Lemma 3.32. Assume |G| is finite and a unit in R. If $A \in [R^G, T^G]$, then $AR \cap T^G = A$.

Proof. Note that $R \subseteq AR \subseteq T$. Clearly, $A \subseteq AR \cap T^G$. For the reverse containment, let $x \in AR \cap T^G$. Then there exists $a_i \in A$ and $r_i \in R$ such that $x = a_1r_1 + \cdots + a_nr_n$. As in Lemma 3.7, we have $|G|x = a_1\hat{r}_1 + \cdots + a_n\hat{r}_n$, whence $x = |G|^{-1}(a_1\hat{r}_1 + \cdots + a_n\hat{r}_n)$. Hence $x \in AR^G = A$. Thus $AR \cap T^G \subseteq A$

Proposition 3.33. Assume |G| is finite and a unit in R, and $R \subset T$ is integral. If $R \subset T$ satisfies FIP, then so does $R^G \subseteq T^G$.

Proof. Let |[R,T]| = n, and let $A \in [R^G, T^G]$. By the above lemma, $R \subsetneq AR \subsetneq T$, i.e. $AR \in [R,T]$, and if $A \neq B \in [R^G, T^G]$, then $AR \neq BR$. Hence $|[R^G, T^G]| \leq n$. \Box

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Biography

Amy D. Schmidt grew up in Minnesota. She attended the University of Minnesota Duluth, where she received her Bachelor of Science in Mathematics in 2009. There she also received her Master of Science in Applied and Computational Mathematics in 2011. She then received her Doctorate in Mathematics from George Mason University in 2015.