#### A THREE-FACTOR MORTGAGE DEFAULT OPTION PRICING MODEL WITH APPLICATIONS TO THE LOAN MODIFICATIONS

by

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By

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# DEDICATION

to my wife, our unborn daughter, and my parents

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### ABSTRACT

#### A THREE-FACTOR MORTGAGE DEFAULT OPTION PRICING MODEL WITH APPLI-CATIONS TO THE LOAN MODIFICATIONS

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The classic contingent-claims pricing model views the borrower's right to default on a mortgage as a put option. By defaulting on a mortgage the borrower effectively sells the property to the lender with the current value of the mortgage. The primary goal of this dissertation is to develop a three-factor structural default option pricing model to explain and evaluate the default options in the residential mortgage contracts. Home price, interest rate and net transaction cost are the three underlying factors of this model. Because a borrower can default at any time when a mortgage payment is due, the mortgage default option is by nature a path dependent Bermudan-American type option. Similar to the American type equity options, there is no analytical solution to the mortgage default option price. By applying the least-squares Monte Carlo (LSM) method to numerically evaluate the mortgage default option prices under different economic scenarios, this dissertation attempts to explain the borrowers' behaviors of strategic defaulting on their mortgages. In addition, this dissertation applies the mortgage default pricing model to an important mortgage research area - loan modifications. The effectiveness of the strategic default prevention of the payment reduction modification method and the equity sharing modification method are quantitatively compared. This dissertation also proposes a flexible parametrized loan modification framework by generalizing and extending the existing modification methods.

## Chapter 1 Introduction

As of the end of 2009, 67% of American households owned about 80 million residential houses. According to a 2010 National Housing Survey<sup>1</sup> of Federal National Mortgage Association (Fannie Mae), even during the unprecedented housing market meltdown, nearly 2/3 of Americans still prefer owning a house to renting. Among the 80 million residential houses, about 53 million (or two-thirds) households have some forms of mortgage as part of their underlying financing packages. As a consequence, the residential mortgage debt tops American personal debt list in terms of the debt amount outstanding. A residential mortgage is a loan secured by an underlying residential property and is scheduled to amortize overtime. When the borrower is delinquent for a few consecutive mortgage payments, the lender starts to repossess the property by going through the foreclosure process. Mortgage default and associated foreclosure process cause significant loss to the total wealth of the society. The massive mortgage defaults in the subprime mortgage sector is considered as one of the most important triggers of the "Great Recession" between 2008 and 2010. To understand, explain and predict the mortgage default has become one of the central focuses of the mortgage finance researchers and practitioners.

<sup>&</sup>lt;sup>1</sup> Fannie Mae's 2010 National Housing Survey (http://www.fanniemae.com/media/pdf/2010/Housing-Survey-Fact-Sheet-040610.pdf)

### 1.1 Objectives

The classic contingent-claims or option pricing theory views the mortgage borrower's right to default as a put option. By defaulting on a mortgage the borrower effectively sells the property to the lender with the current value of the mortgage. The financial benefit obtained from exercising a default option is the difference between the current value of the mortgage and the underlying house price under the assumption of frictionless transaction. The theory assumes that rational borrowers default only when their default options are in-the-money. This dissertation utilizes the option pricing theory to understand and model the borrowers' behaviors of strategic defaulting on their mortgages under a three-factor pricing framework. This dissertation also quantitatively compares the effectiveness of the strategic default prevention of the payment reduction modification method and the equity sharing modification method. In addition, a unified parametrized loan modification framework is introduced to extend the existing modification methods.

The first objective of this dissertation is to introduce an additional stochastic factor – net transaction cost to the two-factor model used in the conventional mortgage default option research. The conventional mortgage default option research uses the house price index (HPI) and the risk-free interest rate as the fundamental underlying factors of the default option price. Exercising of a default option, however, is usually associated with a significant transaction cost. Modeling the transaction cost as a stochastic factor helps to explain the behaviors of the borrowers that do not exercise the default option ruthlessly when it is in-the-money. The most significant transaction costs of the default option are the moral cost and social stigma cost. This dissertation applies a few individual component models to explain the role of the transaction cost in the mortgage default, and these models are compared for

effectiveness.

The second objective is to accurately model the house price process in the default option valuation framework. The conventional mortgage default option valuation research uses the geometric Brownian motion process to explain the dynamics of the house price. However, the unprecedented U.S. national house price crash between 2007 - 2010 invalidates the fundamental assumption that the house price appreciation follows a log-normal distribution. This dissertation applies a jump-diffusion model to better capture and explain the jump nature of the aggregate house price. For the individual house price, a top-down hedonic approach is used to explain the additional volatility at the individual house level.

The third objective is to apply a no-arbitrage interest rate model to default option pricing research. The equilibrium mean-reversion interest rate model used in the conventional two-factor mortgage default pricing models has long been criticized for not being able to capture the empirical term structure of the interest rate. As a consequence, the equilibrium type of interest rate model is rarely used by market practitioners. This dissertation applies a no-arbitrage interest rate term structure model which takes the empirical term structure as an input and precisely replicate the input term structure as part of its output, which can be directly adopted by the mortgage finance market practitioners.

The fourth objective is to apply the least-square Monte Carlo (LSM) method to the valuation of the default option. Unlike the equity options whose strike prices are fixed, the strike price of the default option is a function of time and the underlying risk-free interest rate. And another mortgage option – prepayment option impacts the default option implicitly. This research incorporates the changes of the strike price as an additional equation to be solved by LSM and set up the cancellation

conditions to determine the default option and prepayment option prices simultaneously.

The fifth objective is to introduce a unified parametrized modification framework which generalizes and extends the major loan modification methods. The loan modification has become one of the critical components of the mortgage finance industry after the housing market meltdown between 2007 - 2010. There are a couple of major loan modification methods that focus on modifying different attributes of the underlying mortgages. Payment reduction modification method focuses on reducing borrowers' monthly mortgage payments while equity sharing modification method focuses on the reduction of the negative equity. By introducing a generalized framework, different loan modification methods are compared for the effectiveness of strategic default reduction. This framework is also used to design optimal loan modification methods for individual borrowers to improve the participation rate of the loan modification program.

### 1.2 Content Guide

The dissertation proceeds as follows. In Chapter 2, I provide an overview of the mortgage finance industry and a survey of the previous mortgage default research. In Chapter 3, I discuss the unique features of the mortgage default option and the factors that impact the option value. In Chapter 4, I introduce the three-factor model. The stochastic factors in this model include the house price, the interest rate and the net transaction cost. In addition, the historical correlations between these three factors are discussed. In Chapter 5, I first set up the principles for pricing the mortgage default option. Then the least square Monte Carlo method is described

and two improvements to the computational performance of the LSM method are described. In Chapter 6, I apply the LSM method and the three-factor model to study the impacts of different factors to the default option value. In Chapter 7, I first quantitatively compare the two major existing loan modification methods. Then I create a parametrized loan modification framework to generalize and extend the existing loan modification methods. In Chapter 8, the contributions of this study are summarized and potential future studies are listed.

## **Chapter 2 Background**

This chapter provides a summary of background information about U.S. mortgage market and previous mortgage default research. I start with an overview of the U.S. mortgage market. The market structure, the participants and the products are discussed. In addition, the investment risks in mortgage related products are summarized. Then a brief history of the U.S. mortgage finance market is presented. This chapter ends with a review of the previous mortgage default research and the differences between this dissertation and the previous work.

### 2.1 An Overview of the U.S. Mortgage Market

#### 2.1.1 Mortgage Market Structure

A mortgage is defined as a loan that is secured by an underlying real estate property. Under the mortgage contract, if a borrower defaults on the mortgage, the underlying property will be repossessed by the lender. Mortgages play a vital role in the U.S. housing market. Approximately two-thirds of the 80 million residential properties, that is 53 million units in the United States have some form of mortgage. The residential mortgage debt accounts for the largest share of the overall consumer credit market. According to Federal Reserve, in the third quarter of 2010, the face value of the outstanding mortgage debt in the U.S. is \$14.0 trillion, which accounts for 58% of the \$24.2 trillion total outstanding household debt. Mortgage market consists of a primary market and a secondary market. The role of the primary market is to originate new mortgages. There are a number of different financial institutions involved in the primary mortgage market. The mortgage industry chain starts with a borrower's new mortgage application to the mortgage originator. An originator is a financial institute that initially takes a borrower's application and makes necessary steps to process the application. If the borrower qualifies for the originator's underwriting standard, the application will be approved and the loan contract will be signed and closed. The originator is compensated by a one-time origination fee. Once the loan is closed, a servicer is required to handle the daily operations, which include collecting and remitting periodic interest and principal payments, remitting property tax and home owner insurance payments, informing delinquent borrowers, and managing foreclosures. The servicer is compensated by a periodic service fee which usually is a fraction of the loan's unpaid balance. The originator may or may not keep the servicing right. In most cases, the originator will sell the servicing right to a third party company that specializes in servicing and is optimized by the economy of scales. Another important player in the primary market is the private mortgage insurance (PMI) company. If a borrower is considered more risky according to the underwriting standard (for example, having a loan value to house value ratio greater than 80%), the originator may require the borrower to purchase PMI. The PMI company will cover a fraction of the losses should the borrower default. In return, the PMI company is compensated by a periodic insurance premium from the borrower based on the unpaid loan balance.

The secondary mortgage market is the place where mortgage loans and mortgage backed securities (MBS) are traded. It plays a major role in providing funding to the primary market. In the secondary market, the originators sell the mortgages to the MBS issuers and the mortgage whole loan investors to obtain the liquidity to make more originations. MBS issuers like Fannie Mae and Federal Home Loan Mortgage Corporation (Freddie Mac) purchase mortgages and pool them into MBS using a technique called securitization. Then the MBS issuers sell MBS products to the global investors. This chain in the mortgage industry enables the local borrowers to obtain low cost funding for their mortgages through the global investment community.

Based on the underlying real estate property types, mortgages can be classified into residential mortgages and commercial mortgages. The property underlying residential mortgages includes single family dwelling, multifamily dwelling, and farm house. The property underlying commercial mortgages includes office property, retail property, and health care property. According the a publication of Census Bureau, the residential mortgages account for 83% of the total mortgage market by 2009. <sup>2</sup> Figure 2.1 shows the detail market shares of different mortgages based on property types.

Residential mortgages in the United States can be further classified by the loan underwriting process which generally is based on the borrower's risk factors like credit scores and documentation. Prime mortgages are defined as loans made to the borrowers with sterling credit scores and sufficient documentation of their income and assets. Alternative approval (alt-A) mortgages are those loans made to the borrowers with good credit scores but without enough documentation to support their income and asset claims. Subprime mortgages are those loans made to the borrowers with blemished credit history. Among these three mortgage types, a subprime mortgage is considered the most risky product and usually bears the

<sup>&</sup>lt;sup>2</sup>Data source: http://www.census.gov/compendia/statab/2011/tables/11s1191.pdf

highest mortgage rate.

There are many mortgage products in the U.S. residential mortgage market. Based on the interest rate type, mortgage can be classified into fixed rate mortgage (FRM) and adjustable rate mortgage (ARM). FRM has a constant rate for the term of the loan while ARM has an adjustable rate that is scheduled to reset periodically during the term of the loan. Another important mortgage product is hybrid ARM. As its name implies, it has a fixed rate for the initial 3 to 10 years of the term and an adjustable rate after the initial fixed period. Mortgage loans can also be classified by the loan term. Most mortgages have 30 or 15 years as the original terms. A small portion of mortgages have 10 years, 20 years or 40 years as their original terms tailored for the individual borrower's circumstances.





Figure 2.1: Mortgage Market Share

#### 2.1.2 Risks in Mortgage Investment

A mortgage investment, like other fixed income investments, bears interest rate risk. The changes of the market interest rate will cause fluctuations in the value of the mortgage investment. The value of the mortgage investment is negatively correlated with the market interest rate.

Mortgage investors also face unique risks like the prepayment risk and the default risk. The prepayment risk is defined as the risk that the mortgage borrower terminates the mortgage contract early by selling the underlying property or refinancing it into a new loan. The existence of the prepayment risk reduces the expected return of the mortgage investment and increases the uncertainty in the cash flow timing.

Another risk in the mortgage investment is the default risk, which is defined as the risk that the mortgage borrower fails to make the mortgage payment to the lender. In practice, the lending institutes generally consider a residential mortgage in default when the borrower missed three consecutive payments. Usually, a mortgage default is followed by the foreclosure process which is a legal procedure for the lender to claim the underlying property. The duration and cost of the foreclosure process varies from state to state due to different foreclosure laws at the state level. A default usually leads to a significant loss to the principal of the investment as the lender usually can recover only a portion of the unpaid loan balance through foreclosure and the disposition of the underlying property. In addition to the foreclosure and disposition cost, lenders suffer from the extra depreciation of the foreclosed properties. Cambell, Giglio, and Pathak (2009) indicate that due to lack of maintenance, a foreclosure process decreases the value of the underlying house by around 28%. The foreclosed property also depresses the neighboring proprieties

and thus reduces the overall wealth of the society.

There are many triggers of the mortgage default. The first class of triggers by nature is involuntary. Life events, like becoming temporarily disabled, incurring unexpected medical expenses or loss of job, or simply the payment shock of an ARM after the initial teaser period could make the borrowers unable to continue mortgage payments and thus default. The second class is voluntary default or strategic default – the borrowers simply stop paying the mortgage and walk away from the properties even if they could afford their mortgage payments. The voluntary or strategic default accounts for a significant portion of the current defaults. Guiso, Sapienza and Zingales (2009) estimate that the strategic default accounts for 26% of overall defaults in 2009.

For agency MBS investors, there is little default risk because the government agencies, such as Fannie Mae, Freddie Mac, and Government National Mortgage Association (Ginnie Mae), guarantee the timely payments of the principals and interests of their MBS. And these government agencies are explicitly backed by the full faith of the U.S. government.

### 2.2 A Brief History of the U.S Mortgage Finance Market

The modern mortgage finance market started in the year of 1938 when Fannie Mae was created by Congress with the mission to raise the housing affordability and provide liquidity into the housing finance market after the Great Depression. By purchasing mortgage loans from banks, Fannie Mae alone created a secondary mortgage market that allows the banks to free up their capital and originate more loans. Another important creation of the modern mortgage financing industry was the introduction of the long term level-pay fully amortizing mortgage, or the 30 year fixed rate mortgage. Before Fannie Mae was created, all mortgages were like short term corporate debt that pays interest only periodically and pays the principal in a lump sum at the end of the contract. The long term level-pay fully amortizing mortgage enables the borrowers to pay off their principals over the terms (e.g. 30 year) of the mortgages with fixed monthly payments. With the help of a liquid secondary market, origination banks could easily sell these long term mortgages to Fannie Mae and obtain the capital they needed for new business.

Fannie Mae was rechartered into a private company in 1968. The company maintained a hybrid mode to serve the interest of its share holders as well as public missions of the federal government. A company operating under this hybrid operation mode is also referred to as a Government Sponsored Enterprise (GSE). A division of Fannie Mae was spun off to continue the government lending programs, and was named Ginnie Mae, which has retained the government agency status. Fannie Mae maintained 32 years monopoly in the secondary mortgage market until 1970 when another GSE, Freddie Mac, was chartered by Congress with the purpose of competing with Fannie Mae.

In the same year, Ginnie Mae issued the first MBS by securitizing a pool of similar mortgages together and selling the securities to the traditional mortgage investors. The MBS holders receive principal and interest payments of the underlying pool on a pro rata basis. Freddie Mac quickly followed. One year after its creation, Freddie Mac issued its first MBS as "participation certificates". Fannie Mae waited until 1981 to issue its first MBS. In addition to the agency backed securities, the first private labeled securities (PLS), or private MBS was issued by Bank of America in 1977. MBS soon became the main trading vehicle of the secondary mortgage market and provided great liquidity into this market. By 2006, at the peak of the recent housing market booming, 56% of the total outstanding \$10.5 trillion<sup>3</sup> "one-to-four unit" residential mortgages were issued as MBS by the GSEs and the private trusts.

The collateralized mortgage obligation (CMO) was another invention of the mortgage financing industry. The first CMO was issued by Freddie Mac in 1983. CMO is a class of specially designed investment vehicles to divide the principal and the interest payments from the underlying pools into different tranches according to a defined set of rules. Based on these rules, the overall risks of the underlying mortgage pools are re-allocated among different tranches. This feature of CMO enabled the mortgage financing industry to attract additional liquidity from investors that were not traditionally involved with mortgage investment.

During the 1980s and 1990s, Fannie Mae and Freddie Mac experienced rapid growth in the secondary mortgage market. By 2000, 36% of the \$5.1 trillion of outstanding "one-to-four unit" mortgages were guaranteed and securitized by these two GSEs. During this time period, the private sector also expanded its mortgage financing business and created an active PLS market. By 2000, \$0.43 trillion outstanding "one-to-four unit" PLS were issued. However, GSEs still dwarfed all their private competitors in the secondary mortgage market at the beginning of this century.

The PLS market started to grab a larger share of the secondary mortgage market in 2004. Figure 2.2 compares the market shares of the agency MBS and PLS in new MBS issuance volume from 1996 to 2009<sup>4</sup>. The PLS market share was relatively

<sup>&</sup>lt;sup>3</sup>Data source: Federal Reserve Bulletin

<sup>&</sup>lt;sup>4</sup>Data Source: Securities Industry and Financial Markets Association



MBS Issuance Market Share

Figure 2.2: Market Shares of New MBS Issuance

small before 2004. In 2003, the new MBS issuance in the PLS market was about \$0.58 trillion and accounted for only 18% of the market. However, PLS issuance reached \$1.2 trillion and about 51% of the total market share by 2006.

Two reasons caused this astonishing growth in the PLS market. The first and probably the most widely cited reason is the lower standard of PLS securitization. In order to grab a larger share of the highly profitable mortgage securitization market, PLS issuers lowered their securitization standard by purchasing mortgages that were traditionally not qualified for GSE MBS. These loans were originated to the borrowers with blemished credit (subprime) or no documents to support their income and asset claims (Alt-A). The PLS securitization standards were further relaxed to include mortgages that were characterized with high risk factors such as high loanto-value (LTV) ratios and highly risky products like option adjustable rate mortgage (option ARMs) and hybrid ARMs. These relaxed securitization standards not only allowed originators to maximize their profits by eliciting borrowers from non-prime sector but implicitly encouraged existing borrowers in the prime sector to cash out equity from their properties like credit cards through re-financing.

The second reason is the increased popularity of PLS CMOs backed by subprime mortgages. The PLS market would not be profitable if the issuers could not sell the securitized PLS to the investors. The PLS issuers innovated the CMO product further by creating various internal and external credit enhancements to increase the credit ratings of the PLS backed by the risky mortgages. These credit enhancements include extra tranches, over-collateralization, bond insurance and credit default swaps (CDS). With the help of these credit enhancements, the senior tranches of CMOs obtained investment grades from the rating agencies and started becoming popular in mutual fund and pension fund investors because these instruments bear extra spreads over the corporate bonds with the the same ratings. The subprime market reached \$600 billion in 2006 accounting for 20% of the U.S. mortgage market comparing to 9% in 1996.

It is widely believed that the historically low interest rates also played a partial role in the U.S. housing bubble. Federal Reserve cut the Fed Fund Rate from 5.5% to 1% during 2001-2003 (see Figure 2.3). With the help of the low interest rate and the expanded subprime market, the U.S. mortgage financing industry was flooded with liquidity, and the outstanding volume of "one-to-four unit" mortgages topped \$10 trillion in 2006, more than double the volume in 2000. The pressure of sustaining profit drove the mortgage originators and mortgage lenders to aggressively lower their underwriting and securitization standards further to keep up the high volume of mortgage originations and PLS issuance. In order to prevent further slip of their market shares, the GSEs loosened their standards to purchase and securitize riskier



HPI and Fed Fund Rate

Figure 2.3: Quarterly National Home Price Index and Fed Fund Rate

loans as well. With the excessive liquidity and loosened underwriting standards in the mortgage financing market, not surprisingly, the home ownership rate hit a record high at 69% in 2004 (and maintained at the same level during 2005 and 2006)<sup>5</sup> and house price index reached new historical high at 189.9<sup>6</sup> in 2006 Q2 (see Figure 2.3).

As shown in Figure 2.3, the U.S. national house price reached the turning point in the summer of 2006. After the house prices reached historical highs, many regions of the United States saw house prices decline. However, the full impact of the depressed house prices was not felt until 2007, when the subprime sectors faced higher than expected default and foreclosure rate. During 2007, as the subprime lenders were no longer able to sell their loans to the secondary market, most of them had already failed or put themselves for sale. These lenders include the nation's

<sup>&</sup>lt;sup>5</sup>Data source: U.S Census Bureau

<sup>&</sup>lt;sup>6</sup>Data source: S&P/Case-Schiller U.S. National Home Price Index

largest mortgage lenders like Countrywide Financial and New Century Financial. The severe losses of the mortgage investment in the subprime sector effectively shut down the PLS market. By 2008, PLS issuance was \$36 billion and just about 3% of market share (see Figure 2.2) and the agency MBS again dominated the mortgage finance market.

The housing market continued to slump during 2007-2009. By the first quarter of 2009, the average national house price dropped 32% from its peak at 2006 Q2. At the same time, the serious delinquency rate<sup>7</sup> of subprime loans rose rapidly. During the period of 2007-2009, serious delinquency rate of subprime loans increased from 14.4% to 30.6%. The subprime mortgage crisis quickly spread into the prime sector, serious delinquency rate of prime mortgages increased from a moderate 1.7% in 2007 to a devastating level of 7.0% at the end of 2009 (see Figure 2.4). The U.S national average serious delinquency rate was about 9.7% by the end of 2009. This implies that approximately 4.6 million borrowers<sup>8</sup> were in serious trouble of keeping up their mortgage payments and would lose their houses to foreclosure.

The consequence of the bursting of the housing bubble and associated foreclosure crisis was devastating. The crisis in the mortgage industry eventually triggered the greatest global financial crisis since the Great Depression. During the first half of 2008, the heavy losses that had occurred in the mortgage investment divisions had put some of the largest investment banks, commercial banks and financial institutions in the near insolvent status. On September 07, 2008, the Federal Housing Finance Agency (FHFA), the regulator of the GSEs, placed Fannie Mae and Freddie Mac under government conservatorship. This was an attempt to rebuild the capital

 $<sup>^7\</sup>mathrm{A}$  serious delinquent loan is defined with 90+ days past due status or already in the foreclosure stage

<sup>&</sup>lt;sup>8</sup>Based on 47 million mortgages outstanding by 2009 Q2 reported by First American CoreLogic



Figure 2.4: Serious Delinquency Rate of Prime and Subprime Markets (2004-2010) Data source: Mortgage Bankers Association's National Delinquency Survey

market confidence with a pledge of \$200 billion capital injection from U.S. Treasury to make the GSEs solvent.<sup>9</sup> The capital market hailed the decision initially before quickly turned into turmoil again. During the same month, Lehman Brothers, one of the largest investment banks, filed for bankruptcy. Another top investment bank, Merrill Lynch, sold itself to Bank of America due to heavy losses in its mortgage related investments. American Insurance Group (AIG), the largest insurance in the world, suffered an astonishing loss from its CDS trading and almost collapsed. The credit market completely dried up in the fear of a total collapse of the financial market.

In October of 2008, the U.S. government mounted one of the largest financial rescues in history by creating the Troubled Asset Relief Program (TARP) with a commitment of up to \$700 billion to purchase mortgage related securities from financial

<sup>&</sup>lt;sup>9</sup>The pledge was increased to "unlimited support" in 2009

institutions. The purpose of TARP is to bring liquidity and confidence back to the credit market. TARP, combined with the other bail-out measures such as loosened monetary policies and the home buying incentives, gradually restored the order to the financial market and stabilized the house prices. By 2009 Q1, the national house price increased for the first time in 3 years.

Two years after being placed under the government conservatorship, Fannie Mae, Freddie Mac together with Federal Housing Agency provided more than 90% of liquidity for housing finance by the end of 2010. It is clear that legislation and policies are needed to reduce the government's role in housing finance market and encourage a robust and healthy private mortgage finance sector. At the same time, the market expects a clear exit strategy from the government to end the conservatorship of the two GSEs. In February of 2011, in a White Paper from the Obama Administration, three options for reforming the housing finance market were suggested to Congress. This marked the beginning of the housing finance market reform, although a lengthy political debate is expected and any concrete reform action will have to wait for years. Nonetheless, this White Paper delivered a few clear messages that will help reshape the future housing finance market. It is hoped that it will have a robust private mortgage sector with greatly reduced government involvement. Fannie Mae and Freddie Mae would be wound down at least in their current form, and have an increased level of mortgage borrowing cost.

Another notable development since 2009 in the mortgage finance industry is the government sponsored loan modification program. With the unprecedented home price decline and skyrocketing foreclosures since 2007, in an effort to stabilize the national housing market, the U.S Treasury in March of 2009 initiated Home Afford-able Modification Program (HAMP) to provide affordable loan modifications to the

borrowers with delinquent payments or with imminent risk of keeping up the mortgage payments. As of November of 2009, more than 728,000 loans have been modified under this program. The main approach of the government sponsored loan modification program is to reduce the mortgage payment by lowering the coupon, increasing the loan term or forbearing a certain percentage of the principal in order to prevent default and foreclosure. Meanwhile, the program provides pecuniary incentives to the borrowers for staying current after the loan modifications. It is the key to understand the default rate of the modified loans to evaluate the effectiveness of the HAMP programs as well as to design other methods to improve the effectiveness of the overall loan modification effort.

#### 2.3 Previous Work

According to Quercia and Stegman (1992) and Elul (2006) mortgage default has been regarded as one of the most important areas in the mortgage finance research.

The mortgage default research has been focused on developing theoretical frameworks from a few distinct perspectives. Quercia and Stegman (1992) classify the mortgage default research into three types. The first type of research is from the financial institution's perspective, which considers the loss severity as the main risk of the mortgage default. This type of research focuses on explaining and predicting the loss severity rate of the large mortgage investment portfolios held by the financial institutions. Crawford and Rosenblatt (1995) use empirical loss severity data to support the empirical effect of frictions on the default option price. Calem and LaCour-Little (2004) use simulated variables that affect the mortgage default and the conditional loss probability to calculate the risk based capital for the mortgage portfolios of the financial institutions. In a recent study, Qi and Yang (2009) find that the loss severity rate can largely be explained by various characteristics of the loan, the underlying property, and the default related processes.

The second type of research is from the mortgage lender's perspective. This type of study utilizes multivariate regression techniques to identify the correlations between the default rates and the specific characteristics of the loans, the borrowers, and the underlying properties. In an important paper, Deng, Quigley, and Order (2000) introduce a competing risks model of mortgage termination that simultaneously estimates the proportional hazard rates of prepayment and default by regressing on the important loan and property attributes such as the loan-to-value ratios (LTV). Calhoun and Deng (2002) find that the impact of the competing risks of default and prepayment are similar for FRM and ARM by applying a multinomial-logit model to 1.3 million mortgages originated over the period of 1979-1993. Hayre et al. (2008) present a default model by regressing the subprime default rates between 2002 and 2006 with the underlying collateral characteristics such as the loan-to-value ratios (LTV), the credit scores, and the loan purposes. Pennington-Cross and Ho (2010) find the default rate of subprime hybrid loans are significantly correlated with the teaser rates and and the current LTVs. In a recent study, Bhutta, Dokko, and Shan (2011) find that the strategic default decision is correlated with the level of negative equity by regressing the empirical strategic default rates with the LTVs, the credit scores, and other borrower and property characteristics through a two-step estimation technique.

The third type of research focuses on the borrower's decision at the time of default. The fundamental assumption is that when a mortgage payment is due the borrower will maximize his utility by choosing one of the following options: continuing the scheduled payment, prepayment, or default. This type of study provides a structural framework to explain the borrower's behaviors at the time of default. One of the borrower's utility functions is the option-based model. In this model, the borrower's right to default on a mortgage loan is considered as a put option because by defaulting the borrower effectively sells the property to the lender for the current value of the mortgage. Kau et al. (1995) apply the option pricing theory and the implicit difference numerical method to evaluate the mortgage at the origination under different simulations from a two-factor model. Kau and Slawson (2002) incorporate two simple variable transaction cost into the valuation of default option under a two-factor framework. Downing, Stanton, and Wallance (2005) develop a two-factor mortgage pricing model by estimating the model parameters from the termination rates of Freddie Mac MBS issued between 1991 and 2002.

As an American type option, mortgage default option does not have an analytical solution of its price. There are many studies in the research of the numerical methods for default option valuation. Kau et al. (1995) apply the implicit difference numerical method in the default option valuation under a two-factor model. In an important paper, Hilliard, Kau, and Slawson (1998) present a bivariate binomial technique to evaluate the mortgage prepayment and default options under a two-factor risk-neutral framework. Sharp and Newton (2008) introduce a new singular perturbation approach to approximate the finite difference numerical solution of the default option value under a two-factor risk-neutral framework. There are other advancements in the research of the numerical method for generic American option valuation. In an influential paper, Longstaff and Schwartz (2001) introduce the least squares Monte Carlo (LSM) method to solve the American option value through a set of conditional expectations estimated by the least square regressions
of the future financial benefits. Clement, Lamberton, and Protter (2002) prove that the LSM converges to an approximation of the true price under general conditions and the normalized error of this method is asymptotically Gaussian. Stentoft (2004) proves that the LSM converges to the true price in a two-period situation and converges to an approximation of the true price in a multi-period setting. Chaudhary (2005) improves the LSM method by applying the Brownian bridge approximation and quasi-random sequences to the LSM to solve equity option values with up to 64 dimensions.

The recent housing market meltdown between 2006 and 2010 has stimulated a wealth of research in explaining and modeling the mortgage default. Souissi (2007) utilizes the option pricing theory to stress-testing the Canadian mortgage portfolio by applying the bivariate-binomial lattice numerical method under the conventional two-factor framework. Foote, Gerardi, and Willen (2008) use an intuitive duration model to explain the proportional hazard rate of default in the Massachusetts housing market over the period of 1991-2007. Krainer, LeRoy, and O (2009) use a simple one-factor model to explain the prepayment and default rates of California mortgages under both the costless transaction environment and the costly transaction environment. Bajari, Chu, and Park (2010) use an utility maximization econometric model to explain borrowers' behaviors in the optimal default and sub-optimal default decisions. Ambrose and Buttimer (2011) apply the same bivariate-binomial lattice numerical method to solve the default option value for a new adjustable balance mortgage under the conventional two-factor framework.

This dissertation differs from the previous literature of default option valuation in at least four respects. First, the three-factor model explicitly incorporates the net transaction cost as an additional stochastic factor to the conventional two-factor model. Most of the previous research do not consider the transaction cost in the underlying costless environment. Some literature such as Foote, Gerardi, and Willen (2008) and Krainer, LeRoy, and O (2009) take a fixed transaction cost into the default option valuation. The stochastic transaction cost model in this dissertation provides additional flexibility in modeling and explaining borrowers' default behaviors. Second, this dissertation introduces LSM method into the mortgage default option research. Compared to the conventional numerical methods such as bivariate-binomial lattice method and finite different method used in the previous literature such as Sharp and Newton (2008) and Ambrose and Buttimer (2011), the LSM method expands the numerical solution of the default option value to the multi-factor general stochastic processes, rather than one-factor or two-factor geometric Brownian processes required by the numerical methods used in the previous research. Third, this dissertation expands the underlying stochastic models of the house price and the interest rate used in the conventional two-factor model. Compared to the geometric Brownian motion model and the equilibrium CIR type model used in the previous literature such as Downing, Stanton, and Wallance (2005) and Ambrose and Buttimer (2011), a jump-diffusion house price model and a noarbitrage interest rate model are used by this dissertation to better capture the complex dynamics of the house price and the interest rate featured with the price jumps and the empirical market term structure. Finally, this dissertation improves the computational performance of LSM by using a new parameter setting in selecting Sobol quasi-random sequence. Based on the unique feature of the default option, a simple adaptive LSM with specific simulation paths and error tolerance settings is introduced to reduce the simulation horizon of the LSM simulations and improve the computational performance of LSM in the mortgage default option valuation.

An interesting development in the mortgage finance industry since 2009 is the government-sponsored loan modification programs. The purpose of these government loan modification programs is to stabilize the national housing market by modifying the borrowers' monthly mortgage payments to reduce mortgage default and foreclosure. The loan modification program has stimulated strong research interests in the comparison and optimization of the loan modification methods. Foote, Gerardi, and Willen (2008), by using Massachusetts housing market data over the period of 1991-2007, conclude that the borrower's negative equity need not, and probably cannot be addressed effectively and the loan modification program needs to focus on the mortgage payment reduction only. Guiso, Sapienza, and Zingales (2009), however, argue that the model of Foote et al. which is calibrated with limited empirical data set does not have sufficient predictive power when the magnitude of the negative equity approaches 40%. Posner and Zingales (2009) propose an alternative equity sharing loan modification method to the government sponsored payment reduction modification method. This proposal explicitly addresses the borrower's negative equity issue by reducing the mortgage principal while giving the mortgage lender an equity appreciation interest. Goodman (2010) suggests to adopt a similar equity sharing method to address the mounting foreclosure crisis of different mortgage market sectors. Ambrose and Buttimer (2011) propose an adjustable balance mortgage (ABM) product whose balance is reset periodically to eliminate borrower's negative equity and reduce the borrower's default option value.

This dissertation builds a unified parametric loan modification framework by considering each of the arguments of the above literature. However, this unified parametric loan modification framework differs from the previous literature in at least three respects. First, this unified parametric loan modification framework generalizes the existing loan modification methods of the previous literature such as Foote, Gerardi, and Willen (2008) and Posner and Zingales (2009). The payment reduction modification and the equity sharing loan modification, two distinct loan modification methods in the previous literature, become two instances of this unified loan modification framework. Second, this parametric loan modification framework extends the existing loan modification methods by allowing modifications of additional mortgage terms, rather than being limited to the mortgage payment and the loan balance in the previous literature such as Posner and Zingales (2009) and Ambrose and Buttimer (2011). Last, unlike the previous literature such as Posner and Zingales (2009) and Goodman (2010), this loan modification framework sets up a clear borrower's utility maximization function to evaluate different loan modification methods quantitatively. This function can also be used to define the terms of the optimal modification method under the unified framework.

# Chapter 3 Features of the Mortgage Default Option

The mortgage default option shares a similar fundamental payoff structure with the standard equity put option. However, due to the unique features of the mortgage finance market and the mortgage contract, a few important aspects of the default option are substantially different from the ones in the equity option. Furthermore, unlike the equity option, the value of a default option is impacted by the perception of the borrower (the option holder). Thus the values of the default options need to be adjusted to reflect the heterogeneity of the individual borrowers.

## 3.1 Mortgage Default Option - An American Style Put Option

A home owner's right to default on an underlying mortgage can be considered as a put option which is implicitly specified in the original mortgage contract. When a borrower defaults, he stops paying the mortgage and effectively sells the underlying property to the lender with the current value of the mortgage. This implicit default option gives the borrower the right to gain financially when the house price is less than the current value of the mortgage, When the house price is greater than the value of the mortgage, the borrower will continue the mortgage payment and defer the default decision to the next month. The payoff of the default option is zero or the difference between the current value of the mortgage and the house price, whichever is bigger. For example, a borrower has a mortgage worth \$300,000 and the underlying house worth \$200,000. The borrower could immediately gain a financial benefit of \$100,000 by defaulting on the current mortgage and relocating to

a similar house assuming there is no transaction cost. One simple explanation of the financial benefit of the default is that the borrower saves the monthly mortgage payment difference between the new mortgage and the old mortgage and the present value of these future monthly savings worth \$100,000. If the default option's strike price is approximated with the remaining balance assuming no transaction cost, the payoff structure of the immediate exercise is shown in Figure 3.1. The payoff of the immediate exercise of the default option increases as the underlying house price decreases.

The financial option style is defined by when the option can be exercised. The most frequently traded equity options are either European style or American style. The European style option can only be exercised at the expiration date while American option can be exercised at any time before the expiration date. Another popular option style is Bermudan which can be exercised at a set of times before expiration. Theoretically, the mortgage borrower can default at any time and thus is considered as an American style option. However, in practice, the borrower only needs to make a decision when a mortgage payment is due, the default option should be classified as Bermudan. Nevertheless, the mortgage default option as an American option or a Bermudan option is valued through the similar pricing frameworks.

A mortgage default option has two unique features that distinguish it from the standard equity option. These two features include the stochastic nature of its strike price and the co-existence with the prepayment option.

### 3.1.1 Stochastic Strike Price

The first unique feature of the mortgage default option is that its strike price follows



Figure 3.1: Default Option Payout as a Function of the house price

a stochastic process. Instead of being a fixed number, the strike price of a mortgage default option is a function of time and the discount interest rate. More specifically, the strike price at a particular month is the present value of all the future mortgage payments discounted by the interest rates; thus the strike price is a function of a deterministic decreasing component and a stochastic component.

The strike price of the default option is a decreasing function of time *t*. Assuming there is neither prepayment nor default, the principal of the mortgage is gradually paid down through a pre-defined contractual amortization schedule. For an FRM, the fixed monthly payment is calculated at the origination of the mortgage with the following formula:

$$Payment_{t} = \frac{C_{\theta}/12}{1 - (1 + \frac{C_{\theta}}{12})^{-Term}} * Balance$$
(3.1)

Where  $C_0$  is the fixed coupon rate at origination and  $C_0 = C_1 = \ldots = C_{\text{Term}}$ , Term is the term of the mortgage in the unit of month, Balance is the original borrowed amount.  $K_t$ , the strike price of the mortgage default option at month t is the present value of all the future mortgage payments:

$$K_t = \sum_{\tau=t}^{T} \text{Payment}_{\tau} / \text{Dfactor}_{\tau}$$
(3.2)

where  $Dfactor_{\tau}$  is the discount factor associated with the time period  $\tau$  and is calculated as

Dfactor<sub>$$au$$</sub> =  $\prod_{i=t}^{\tau} (1 + r_t/12)$  (3.3)

 $r_t$  is the discount interest rate of month t. Assuming the discount interest rate is always equal to the borrower's coupon rate, the strike price of the mortgage is the same as the remaining balance. Figure 3.2 is an example of the yearly remaining balance of a 30 year fixed rate mortgage assuming there is no prepayment and no default. This mortgage in the example has an original balance of \$100,000, an original term of 360 months and a coupon rate at 6%. Assuming the discount interest rate is fixed at 6%, the strike price of the default option equals to the remaining balance which reduces over time.

The strike price of the default option is also a decreasing function of the discount interest rate, r. Under the risk-neutral valuation principle, the discount interest rate is the risk-free interest rate. If the risk-free interest rates are lower than the coupon rate, the value of the mortgage is greater than the remaining balance. This makes the default option more attractive because the borrower has a higher strike price of the default option. If the risk-free interest rates are lower than the coupon



Figure 3.2: Strike Prices of a FRM default option

rate, the value of the mortgage is less than the remaining balance. This will reduce the value of the default option and discourage the borrower from defaulting. The difference between the risk-free interest rate and the borrower's coupon can be used as a proxy to determine the relative differences between the strike price and the remaining balance. Figure 3.3 shows the mortgage values with different riskfree interest rates.

The risk-free interest rate follows a stochastic process. According to Equations (3.2)-(3.3), the strike price of the default option follows a stochastic process. Please refer to Chapter 4 for a detailed discussion of the stochastic model of the risk-free interest rate.

The determination of the strike price for the default option of an ARM is more complicated. For an ARM, since the amortization schedule is dependent on the



#### Mortgage Values with Different Discount Rates

Figure 3.3: Mortgage values (strike prices) with different discount rates

underlying reset rate, the monthly payment follows a similar stochastic process as the underlying reset rate. The first step of determining the monthly payment of an ARM is to determine the coupon rate. The coupon rate of an ARM resets periodically to a new rate that depends on the underlying reset index plus a fixed margin. The coupon rate is capped by the lifetime cap and the periodic caps. And it is floored by the life time floor and the periodic floors. The coupon rate can be specified as:

1

$$\begin{cases}
C_t = \max\{\min(\operatorname{Index}_t + \operatorname{Margin}, C_{t-1} + \operatorname{Cap}_{\operatorname{sub}}, C_0 + \operatorname{Cap}_{\operatorname{life}}), \\
C_{t-1} - \operatorname{Floor}_{\operatorname{sub}}, C_0 - \operatorname{Floor}_{\operatorname{life}}\}, when \ t = n * \operatorname{Reset} + 1 \\
C_t = C_{t-1}, \ when \ t \neq n * \operatorname{Reset} + 1
\end{cases}$$
(3.4)

where  $Index_t$  is the rate of the underlying reset index,  $Cap_{sub}$  is the periodic cap,  $Cap_{life}$  is the life time cap,  $Floor_{sub}$  is the periodic floor,  $Floor_{life}$  is the life time floor, Reset is the reset period (usually 12 months in the United States), n is an integer and  $0 \le n < \frac{\text{Term}}{\text{Reset}}$ . For an ARM, the mortgage payment needs to be recalculated on each of the reset dates. The mortgage payment for each of the reset period can be calculated by applying the  $C_t$  derived in Equation (3.4) to the C used in Equation (3.1).

The dynamics of the strike price of an ARM's default option are substantially more complicated than a FRM's default option. Because the ARM's reset index is usually different with the risk-free interest rate, the present value of the future mortgage payments are subject to two underlying stochastic processes.

#### 3.1.2 Coexistence with the Prepayment Option

The second unique feature of the default option is that its value is significantly impacted by another option – the prepayment option. The prepayment option gives the borrower the right to terminate the mortgage contract earlier by repaying the remaining balance to the lender. The prepayment clause is usually explicitly specified in the residential mortgage contract. In the United States, most borrowers have the right to prepay their loans at the book values without prepayment penalty, although some mortgages are originated with prepayment penalty clauses. These penalties, however, usually decrease over time and expire after 5 years since origination.

The prepayment option can be considered as a call option since the borrower effectively purchases the underlying property with the remaining mortgage balance. Assuming that a rational borrower will not prepay unless it is to his financial benefit to do so. The prepayment option is in-the-money when the mortgage value is greater than the remaining balance. The prepayment option is out-of-the-money when the mortgage value is less than the remaining balance. In the case of the prepayment option, the mortgage plays the role of the underlying instrument and the option's strike price is the remaining balance of the mortgage. For the default option, the house price plays the role of the underlying instrument and the strike price is the current mortgage value.

Downing, Stanton, and Wallace (2005) find that the existence of the prepayment option affects the value of the default option. When the borrower chooses to prepay, the borrower purchases the house from the lender with the current mortgage balance and the mortgage contract is terminated. Thus the default option is voided and its value is set to zero. When the borrower chooses to default, the borrower sells the house back to the lender with the current mortgage balance and the mortgage contract is terminated. The prepayment option is voided and its value is set to zero. Because the exercise of the prepayment option and the exercise of the default option are mutually exclusive, the existence of the prepayment option reduces the value of the default option, and vice versa. Intuitively, the most complex scenario for option valuation is when both of the options are in-the-money and the less complex scenario of option valuation is when at least one of the options is out-of-the-money.

Prepayment is generally classified into two major types. The first category is the housing turnover. Housing turnover occurs when the borrower sells the house and uses the sale proceedings to repay the mortgage back to the lender. Housing turnover is caused by reasons like relocation and housing upgrade or downgrade. It is widely believed that the housing turnover rate is negatively correlated with the market mortgage rate — a lower mortgage rate increases the housing affordability and home sales. Intuitively, due to life events, some degree of housing turnover persists regardless of the interest rate level. Lowell and Corsi (2006) find signifi-



Figure 3.4: Refinance Share of Mortgage Activities

cant housing turnovers in the high mortgage rate environment between 1999 and 2000. Seasonality is another important factor that impacts the housing turnover rate. Usually, home sales are higher in the summer and become slow during the winter.

The second type of prepayment is refinancing. Mortgage refinancing is to replace the current mortgage with a new mortgage under different terms. Historically, refinance represented the majority of the prepayment activities. Figure 3.4 shows the refinance share of the weekly mortgage activities between Aug, 2010 and Jan, 2011.<sup>10</sup> During this period, refinance accounted for an average of 77% of the mortgage activities with a range of 69% to 83%.

Mortgage borrowers refinance for two major reasons: to lower their mortgage in-

<sup>&</sup>lt;sup>10</sup> Data is obtained from the weekly surveys of the Mortgage Banker Association between Aug, 2010 and Jan, 2011

terest rates or to take out equities from the increased house values. The refinancing activity driven by the first reason is usually referred as the rate refinance. A rational borrower will initiate a rate refinance if the difference in value between the new mortgage and the old mortgage is great enough to recover the fees, points and all other costs associated with the refinance. The most important determinant of the rate refinance is the level of the mortgage rate. Intuitively, rate refinance is negatively correlated with the mortgage rate – a higher mortgage rate will lead to less incentive for borrowers to initiate a rate refinance. Lowell and Corsi (2006) identify three periods: 1990-1993, 1997-1998, and 2000-2004 with progressively strong refinance responses to the decreases of the mortgage rate by using data from Mortgage Banker Association (MBA) and Freddie Mac.

The other type of refinance is the cash-out refinance. The borrower repays the old mortgage by entering a new mortgage with a higher balance. The borrower subsequently takes out the difference in the balance between the new mortgage and the old mortgage for other expenditures like home improvement and debt consolidation. Because the equity from the cash-out refinance has the borrower's house as the underlying collateral, the borrower usually enjoys a significantly lower interest rate than the rate on an unsecured personal loan. A sufficient condition for a cash-out finance to occur is that the appreciation of the underlying house value is great enough to recover all refinancing related costs. Lowell and Corsi (2006) point out the cash-out refinance is less sensitive to the interest rate environment because many causes of cash-out refinance like home improvement, debt consolidation and medical expenses are not interest rate sensitive. In fact, when the interest rate is high, the refinance market is dominated by cash-out activities because the rate refinance is not in the borrower's financial benefit.

The homeowners are usually charged with certain types of costs for refinancing. Loan origination fees and points are the two variable costs associated with the loan principal amount. A loan origination fee is the compensation of the lender's work associated with the eligibility evaluation and the documentation preparation. Points are charges to either reduce the borrower's coupon rate or to provide additional compensation to the lender. There are many other fixed costs associated with refinancing such as application fees, appraisal fees, legal fees, closing costs, inspection fees, and so on. These fixed costs are usually not associated with the size of the loan amount. According to a publication of Federal Reserve,<sup>11</sup> the total variable costs of the loan origination fee and points could range from 0 to 4.5% of the loan principal amount and the total fixed costs could range from \$2,000 - \$4,000. For a rate refinance to occur, the difference in the coupon rate between the new mortgage and the old mortgage should be great enough to recover the fixed and variable costs. Lowell and Corsi (2006) claim that 35 basis points (0.35%) of spread on interest rates is sufficient enough for a borrower to initiate a rate refinance. Follian and Tzang (1988) find that the spread of 60 basis points is needed to trigger a rate refinance. The gap between the mortgage rate spreads from these two research may be explained by the recent advancement of technology that significantly reduced the cost of the lender for processing the loan applications.

There are other forms of prepayment. Curtailment happens when the borrower chooses to prepay a portion of the principal in addition to the scheduled monthly payment. Curtailment does not result in a smaller monthly payment. Instead, the lender deducts the prepaid amount from the unpaid balance and makes the effective term of the mortgage shorter. In practice, curtailment is hard to measure and is

<sup>&</sup>lt;sup>11</sup> A consumer's guide to mortgage refinancings, Federal Reserve. (http://www.federalreserve.gov/pubs/refinancings/default.htm)

usually treated as other prepayment types. In the agency MBS analysis, default is also treated as a source of prepayment because the GSEs guarantee the timely payments of the principals and the interests. This classification is not applicable to this dissertation because its focus is to explain the borrower's behavior of defaulting at the individual mortgage loan level.

Prepayments due to housing turnover and cash-out refinance have little impact on the default option value. The sufficient condition for a normal housing turnover or cash-out refinance is that the underlying house value is greater than or equal to the remaining loan balance. Under this condition, since there is positive equity in the underlying house, the borrower will not choose to default. However, negative equity on the underlying house will not prevent the borrower from a rate refinance. In a news release of Federal Housing Finance Agency<sup>12</sup>, Home Affordable Refinance Program (HARP) was extended to June 30, 2011 in order to help eligible borrowers refinance their mortgage loans with negative equity up to 25%. If the borrower commits other collateral, the refinance can be processed even at a higher negative equity level. As the impact of the other prepayments is trivial to the default option price, without loss of generality, this dissertation focuses on explaining the impact of the rate refinancing prepayment to the default option value.

# 3.2 Factors Affecting the Default Option Prices

Due to the complexity of the mortgage finance market and the mortgage contracts, there are many factors that directly affect the default option value. These factors can be classified into three categories: the macro economic environment, the mortgage attributes, and the borrower's characteristics.

<sup>&</sup>lt;sup>12</sup> http://www.fhfa.gov/webfiles/15466/HARPEXTENDED3110.pdf

#### 3.2.1 Macro-economic Environment

The macro-economic factors are properties of the external market environment of the mortgage contracts. These factors are independent with the mortgage contract and the individual borrower's perception. The house price index and the risk-free interest rate are the two major economic factors that directly affect the value of the mortgage default option. The house price has been proved to be the most important economic factor that impacts the borrower's strategic default decision. The role of the house price to the default option is the same as the stock price to its put option. The level of the house price will directly determine the value of the default option. Assuming no transaction cost, if the house price is higher than the value of the mortgage, the default option is out of money. If the house price is lower than the value of the mortgage, the default option is in-the-money. A rational borrower will not exercise the default option when it is out of money. However, even the default option is in-the-money, a rational borrower may defer the exercise to the next period if he expects a higher financial benefit by exercising the option in a later month. A few other leading economic indicators like unemployment rate and demographic composition are also used in the mortgage finance research. However, these factors are also the explaining variables to the fluctuations of the house prices.

The risk-free interest rate is used in this dissertation. Under the risk-neutral valuation principle, the risk-free interest rate is the expected rate of return for all investment including mortgage finance investment. Thus the risk-free interest rate is not only to the discount rate of the future cash flows but also the prevailing refinancing rate for prepayment. The dynamics of the house price and the risk-free interest rate are discussed and modeled in Chapter 4.

#### 3.2.2 Mortgage Attributes

The mortgage attributes significantly affect the default option values. Different mortgages on similar houses may have very different default option values. The mortgages can be classified into different mortgage product types based on the method of determining the coupon rate. The dominant mortgage types in the United States are the fixed rate mortgage (FRM) and the adjustable rate mortgage (ARM). An FRM has a fixed coupon rate through the entire loan term while an ARM has an adjustable coupon rate which resets periodically based on an underlying reset index. The mortgage type can be further classified by the loan term e.g. FRM with 30 year term and FRM with 15 year term. ARM is believed to be more risky than FRM due to the possible monthly payment shock caused by the fluctuations of the underlying reset index. Figure 3.5 compares the yearly payments of an FRM and an ARM. Both mortgages have 15 years term. The ARM index rate starts at 6% and follows a Wiener process with 1% as the yearly volatility. The ARM product has a payment spike at the year 2 which could possibly cause a payment shock to the borrower and increase the chance of default.

The mortgage coupon rate is the price tag that the lender places on the mortgage at initiation. If the lender determines that a mortgage has a higher default risk, the coupon rate will be set higher to compensate the extra risk. At initiation, the coupon rate is a good proxy for the riskiness of the mortgage. However, with seasoning of the loan and fluctuations of the house price, the coupon rate can no longer be used as a risk proxy. Instead, it indirectly impacts the default option value through the calculation of the strike price – current value of the mortgage.

Another important loan attribute is the original loan to house value ratio (LTV).



Figure 3.5: Yearly Payments of FRM and ARM

The original LTV measures the down payment that the borrower commits to the mortgage. For example, if a borrower pays down \$10,000 and borrows \$90,000 to purchase a house worth \$100,000, the LTV of this mortgage is 90%. Original LTV is an important attribute used by lenders to assess the risks of mortgages. Traditionally, a mortgage with LTV equal to or less than 80% is considered less risky. With a 80% original LTV, the borrower's default option is always out-of-the-money unless the house price drops more than 20%.

The loan age is another attribute widely used in the mortgage default research. With the seasoning of the loan, the default option generally becomes less valuable because the monthly payments reduce the remaining loan balance and increase the borrower's equity in the house. Previous studies found evidence that the default rate of a new mortgage increases in the first few years and then decreases over time. The reason is that for the first few years little principal is paid down because the interest portion consists most of the monthly payment. And when the house price decreases, the default rate increases. However, after a few years, the borrower builds up some equity in the house and a normal house price depreciation is unlikely to make the default option in-the-money.

#### 3.2.3 Borrower and Property Characteristics

The transaction cost of exercising a default option is substantially higher than the transaction cost of exercising an equity option. This transaction cost is subjective to the borrowers' perceptions. In the standard option pricing theory, the option price is not affected by the perception of the option holder. However, in the case of the default option, borrowers' perceptions affect their decision making processes. There is a great amount of heterogeneity in the transaction costs of the default options which are mainly caused by the diversification of the individual borrowers' characteristics.

Crawford and Rosenblatt (1995) indicate that the net transaction cost of default should be calculated by considering both pecuniary and non-pecuniary values or penalties of default. The transaction penalty consists of relocation costs, moral costs, social stigma costs, deficiency judgment costs, and so on. The transaction benefits include the free rents that the borrower enjoys between the default date and the foreclosure date. In general, the transaction penalty is greater than the transaction benefit and the net transaction cost is positive. This helps to explain the borrower's behavior to delay the default decision even when the house price is below the current value of the mortgage.

Among the penalty components of the net transaction costs, the social stigma costs

and the moral constraint costs are considered the most important cost factors. However, unlike other pecuniary transaction costs, the economic value of these nonpecuniary penalties cannot be easily quantified and modeled. In addition, the value of these penalties is perceived very differently among different borrowers. However, it is important to realize that the moral level barriers could be penetrated by the increase of the economic incentives. Guiso, Sapienza, Zingales (2009) find evidence that even in the higher level moral group the percentage of borrowers to declare strategic default increases significantly as the economic incentive increases. The important implication of this finding is that the moral constraint costs at different moral levels could be measured by economic value.

There is evidence that some characteristics may impact the borrowers perceptions toward moral constraint cost more significantly than the other characteristics. The borrower's age, education, and income are a few notable characteristics of this category. Guiso, Sapienza, Zingales (2009) claim that the younger people and the older people have lower moral views on strategic default than the middle-aged group. It may be explained by the reason that the dominant roles of the middle-aged group in the social and economic activities of the society. Another notable characteristic is the borrower's education level. Guiso, Sapienza, Zingales (2009) conclude that more educated people have lower moral views on strategic default than less educated people. One explanation is that due to their advanced education in economics and finance the more educated people may view the strategic default more from economic perspective than from moral perspective.

The borrowers with higher credit score may tend to view the social stigma cost more significantly than the borrowers with lower credit scores. Bhutta, Dokko and Shan (2010) find the median borrower in the group with credit scores between 620 and 680 will walk away from their houses if the negative equity reaches 50% while the median borrower in the group with credit scores higher than 720 will walk away from their houses only when the negative equity reaches 68%. The possible explanation for the difference is that compared with the lower credit score group, the higher credit score group will face a significantly higher future credit cost after the default.

One may question if the borrower's total wealth may change his perception toward the moral constraint cost. Guiso, Sapienza, Zingales (2009) find no evidence to support the hypothesis that there are significant correlations between the probability of the strategic default and the borrower's total wealth. In their survey, the borrowers are classified into two groups: one group with financial asset less than \$50,000 and one group with financial asset greater than \$50,000. When the negative equity of the underlying house is \$50,000, the wealthier group tends to declare default more often. But when the negative equity of the underlying house reaches \$100,000, the less wealthy group has a higher probability to default strategically. Their classification of the wealthy group by using \$50,000 financial asset as threshold seems arbitrary. There may be other patterns between the wealth and the perception toward moral constraint if there are more categories in the wealth dimension. This claim is not supported by available data. However, a similar characteristic, the income level, does affect the borrower's moral view. In the same research, Guiso, Sapienza, Zingales (2009) find the borrower group with higher income tends to have a higher moral view.

The borrower's characteristics may affect the other components of the net transaction cost as well. The cost of future deficiency judgment is impacted by the state residency of the borrower. The deficiency judgment is a legal process in which a lender takes an unsecured money judgment against the borrower in order to recapture the difference between the mortgage value and the sale proceedings of the house. The availability of the deficiency judgment is determined by the state law. Eleven states are non-recource states. A non-recourse state is defined that when the borrowers of this state default the lenders can only recover the underlying properties and can not recover from the borrowers' other personal assets. In contrast, if a borrower in the recourse states defaults, the lender can pursue the borrower's personal asset if the lender can not recover the mortgage balance from the sale proceedings of the underlying property. Ghent and Kudlyak (2009) find that borrowers from non-recourse states have a higher chance to strategically default. Bhutta, Dokko and Shan (2010) also find that the median borrowers from the recourse states have a lower negative equity threshold to strategically default than the borrowers from the non-recourse states.

Another component of the net transaction cost that may be affected by the borrower's characteristics is the relocation cost. Guiso, Sapienza, Zingales (2009) use factors like the tenure of the residency in the property and number of children in the household to explain the variations of the relocation cost. They find that borrowers who lived more than 5 years in the underlying properties are 7% less likely to default than the borrowers who lived less than 5 years in the properties. They explain this behavior by the personal attachment and possible re-modelings. In addition to these non-economic reasons, because of the continuous monthly payments, the significant reduction of the mortgage balance may also be used to explain the differences in the strategic default rates.

Another notable finding is that the borrower's view on strategic default is affected by the decisions of other borrowers. According to the survey of Guiso, Sapienza, Zingales (2009), the borrower who knows somebody defaulted strategically is 8% more likely to declare default. However, their research also shows that there is no evidence that the increase of the default probability comes from the weaker moral views. Most likely, the increase is achieved through the relief of the social pressure.

In this chapter, I have discussed the unique features and underlying factors of the default option compared to the standard equity option. These features of the default option provide the fundamentals for the mortgage default option modeling and valuation. In Chapter 4, I build the underlying three-factor model based on the underlying factors introduced in this chapter. In Chapter 6, I quantitatively study the impact of these factors to the prices of the default options.

# Chapter 4 The Model

The fundamental state variable underlying the contingent claims of the mortgage default option is the house price. To value a mortgage default option, in the conventional one-factor and two-factor models, the underlying house price process is modeled by adapting the equity price model of Black and Scholes (1973) and Merton (1973). In this model, the house price is considered to follow an Ito process. In a risk-neutral world without prepayments and transaction costs, the default option price can be derived from the Black-Scholes formula or by applying the backward induction method on a binomial tree of the house price.

The geometric Brownian motion model, however, fails to explain the event of the national level housing market crash between 2006 and 2009. During this three year period, the average U.S. house prices dropped 30% from peak to trough. This indicates that similar to all other financial quantities the house price does not strictly follow the log-normal random walk process and it follows a process with unexpected negative and positive jumps instead. Merton (1976) introduces a jump-diffusion model with a mixture of both continuous and jump processes. In this dissertation, I use Merton's jump-diffusion process to model the house price in order to provide insights into the borrowers' default decisions should a housing market crash happen again. In addition, a volatility model dependent on the individual house price is introduced to provide additional house price volatility at the individual house level.

Another important state variable that affects the value of the default option is the interest rate. When it takes the form of the discount rate, the underlying inter-

est rate directly determines the present value of the future payments or the strike price of the default option. When it takes the form of the reset rate, the underlying interest rate determines the coupon rates and payments for the adjustable rate mortgages. In this dissertation, a non-arbitrage Hull-White interest rate model is used to capture both the initial empirical term structure of the interest rate and its stochastic volatility.

The third state variable affecting the default decision is the net default transaction cost. The existence of the transaction cost significantly reduces the value of the default option. The net transaction cost consists of transaction costs and pecuniary benefits. Transaction costs include the relocation costs, deficiency judgment payments, moral constraint costs, and social stigma costs. Pecuniary benefits include the free rents during the foreclosure period. In this dissertation, relocation costs, deficiency judgment payments and free rental benefits are modeled as fractions of the underlying house price. And a mean-reversion model is used to explain the stochastic process of the social stigma cost, which is approximated by the additional future financing costs caused by the impaired credits. In addition, I use a standard normal distribution to model the individual borrower's moral constraint cost, which is believed to be heterogeneous among borrowers and is resilient to the changes of the social environment.

By considering the three underlying state variables that affect the borrower's decision to default, I develop a three-factor model to explain the economic scenarios underlying the valuation of the mortgage default option.

### 4.1 Factor One - the House Price

In the mortgage finance research, the house price has been proved to be the most important driver of the borrowers' default decisions because the level of the house price will determine the moneyness of the default option. However, the borrower will not know the exact house price before the house is sold. The individual borrower will have to estimate the house price through the house price index measured at the aggregated level. The accuracy of the aggregated house price is impacted by issues like limited house sales in a certain period and the heterogeneity of the houses in the sample. Different measurements such as the repeat-sales method and the hedonic method are developed to serve different usages of the house price index. In addition, there is strong evidence that the aggregated house price follows a jump-diffusion process with the possibility of both positive and negative jumps. In this research, I use a jump-diffusion process to model the repeat-sales type of aggregated house price index.

According to the option pricing theory, it is critical to provide an accurate description of the volatility of the individual underlying house price to avoid underestimation of the default option on the individual house. In this dissertation, I develop a top-down approach to provide additional volatility to the individual house prices.

### 4.1.1 Measurement of the Aggregated House Price Index

The accurate measurement of the house price is critical to understand and model the process of house price and its influence on the mortgage default option. However, it is difficult to develop the house price measurement at both individual and aggregated level because the housing market exhibits great magnitude of heterogeneity and infrequency of sales. A few distinct aggregated measurements have been developed and used in today's housing finance research and applications. Unlike normal consumer products, houses exhibit great magnitude of heterogeneity. Most of the attributes of a house can differ from another house: location, community, school district, maintenance status, bedrooms, building styles, and so on. Quigley (1995) refers to the combination of all the attribute variations as the "quality" of the house which is not directly observable. Intuitively, the house price is positively correlated with the house quality. As the house quality changes overtime, it is difficult to measure the house price changes caused by other factors. In addition, the houses are infrequently traded which makes the measurement sample not representative because of the sample is either too small or biased. Due to the existence of these two measurement problems, it is difficult to define a uniformly represented and unbiased measurement for the house prices.

Rappaport (2007) generalized three distinct approaches to measure the aggregated house price index. The first approach is the simple average of all the observed house prices for a given time period. The assumption is that the sample size in this approach is big enough to minimize the impact of the heterogeneity and sales infrequency. However, the average house price index can not be isolated from the impact of the continuous changes of the house quality. The National Association of Realtors (NAR) existing home median value is an example of the average house index approach.

The second approach is the repeat sales price measurement. This approach focuses on the houses that have been traded more than once or the underlying mortgages have been refinanced at least once. The aggregated house price changes are measured through a regression on the price changes of the underlying properties from different time periods. The fundamental assumption of this approach is that the quality of the same houses does not change over time and the heterogeneity of the sample house sales is thus controlled. However, this measurement can significantly deviate from the real aggregated house prices because of the limited number of house sales during the measurement period. The widely followed S&P Case Shiller Home Price Index is a repeat sales type price index.

The third approach is the hedonic measurement. This method assumes that the quality or the service of the house consists of different attributes such as number of bedrooms, number of bathrooms, kitchen, location, lot size, and so on. The correlations between the observed house prices and the house attributes are estimated through regressions. Based on the estimated correlations, the values of the house attributes are derived. The aggregated house price is then estimated by applying the estimated house attribute prices to a set of appropriate attributes representative of the overall housing market. This method controls heterogeneity of the houses and the constant change of the house quality. However, due to the large amount of data needed for the house attributes, the only well known hedonic index is the Census Constant Quality Index of New One-Family Homes Sold.

It is believed that the choice of the best housing price measurement depends on the purpose of its intended usage. For assessing the overall housing affordability, the average measurement like NAR series suits the purpose best as it captures both the price appreciation and the housing quality changes. For estimating the changes of the house price appreciation or depreciation, the repeat sales measurement is the best as it controls the house quality changes and only estimates the house price changes. Since one of the main interests of this research is to price the mortgage default option based on the changes of the underlying house price, the repeat sales type of index like S&P Case/Shiller index will be used for modeling house price changes.

#### 4.1.2 The Aggregated House Price Model

The fundamental model of the repeat sales type of home price index can be derived from the Black-Scholes-Merton's equity model. Case and Shiller (1987) and Abraham and Schauman (1991) specifically add a random walk error term to the dynamics of the aggregated house prices for a repeat sales type of measurement. The variance of the aggregated house prices increases with the elapsed time between the time 0 and the time t. I model the house price similar to the underlying process of a stock with dividends:

$$\frac{\mathrm{d}H}{H} = (\alpha - s)\mathrm{d}t + \sigma_H \mathrm{d}z_H.$$
(4.1)

Equation (4.1) is an Ito process where H is the aggregated house price,  $\alpha$  is the expected return of the underlying property, s is the service flow that the default option holder can not benefit from,  $\sigma_H$  is the instantaneous standard deviation of the house price, and  $dz_H$  is a Wiener process. By applying risk-neutral principle, the house price process can be re-written as

$$\frac{\mathrm{d}H}{H} = (r-s)\mathrm{d}t + \sigma_H \mathrm{d}z_H,\tag{4.2}$$

where *r* is the risk-free interest rate.

For a mortgage default option f which is dependent on the house price process in

Equation (4.2), by applying Ito's Lemma, its price must satisfy the following partial differential equation:

$$\frac{\partial f}{\partial t} + (r-s)H\frac{\partial f}{\partial H} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 f}{\partial H^2} = rf.$$
(4.3)

Because the mortgage default option is an American option, there is no analytical solution to its price f in Equation (4.3). The conventional approach to derive the option price in this equation is to use numerical methods such as binomial tree or finite difference method. The approach is to evaluate the payoff function of the mortgage default option at each exercise point and solve the default option price f backward from the last exercise point. This kind of method is sometimes called backward induction method. However, the accuracy of the numerical solution depends on the ideal house price process specified in Equation (4.1), which assumes the house price follows a geometric Brownian motion through time which produces a log-normal distribution for the house price between any two points in time.

Historical data suggests that the aggregated house prices tend to have more outliers than a simple geometric Brownian motion. Figure 4.1 is the histogram of the quarterly log rate of returns of the U.S. national HPI from 1987 to 2011. The distribution has a fat tail to the left which suggests the house price dynamics include a possibility of negative price jumps. The sample kurtosis of the rate of return is 4.92 which suggests a fatter tail than normal distribution. Figure 4.2 is the QQ-norm plot of the same house price index rate of returns which further supports a possibility of jumps.

Merton (1976) presents a jump-diffusion model which follows a mixture of both continuous and jump processes. In this model, the total change of the equity prices



Figure 4.1: The Histogram of U.S. Quarterly HPI Rate of Returns Data Source: S&P/Case-Shiller U.S. National HPI 1987Q1 - 2011Q1



Figure 4.2: Q-Q Norm Plot of the U.S HPI Returns

has two components:

- The regular movement of the equity prices, which is mainly decided by factors like demand and supply, changes in the overall economic outlook, and other information that causes marginal changes in the equity price. This diffusion component is modeled by the regular geometric Brownian motion.
- The "jump" price changes caused by abnormal important new information of the underlying equity that has more than a marginal effect on the price. This jump component is modeled by a Poisson process.

The house price can be interpreted by a similar two component jump-diffusion model. The regular diffusion process of the house price changes is determined by the normal factors like temporary imbalance between housing supply and demand, demographic changes and outlook of the housing industry. The jump component of the house price changes is caused by abnormal new information on the housing sector and the overall economic outlook. For example, the sudden decline of the average U.S. national house prices from 2006 to 2009 was triggered by the unexpected subprime mortgage crisis.

I apply Merton's jump-diffusion model to the dynamics of the house price process. Poisson distribution is used to model the jump component of the house price process caused by an abnormal event. I assume the events are independently and identically distributed. The probability of the occurrence of such an event for a time interval of length  $\tau$  can be described as

$$\Pr(no \ event \ occurs \ from \ t \ to \ t + \tau) = 1 - \lambda \tau + O(\tau)$$
(4.4)

$$Pr(an \ event \ occurs \ from \ t \ to \ t + \tau) = \lambda \tau + O(\tau)$$
(4.5)

$$Pr(the event occurs more than once from t to t + \tau) = O(\tau),$$
(4.6)

where  $O(\tau)$  is defined as the asymptotic order symbol and  $\lambda$  is the expected number of events per unit time.

I define *J* as an independently and identically distributed random variable to describe the impact of the event on the aggregated house price. By neglecting the diffusion part, the house price  $H_{t+\tau}$  will become  $JH_t$ , given that one event occurs between *t* and  $t+\tau$ . Similar to the diffusion stochastic processes in Equation (3.1), the aggregated house price follows a jump-diffusion process can be written as

$$\frac{\mathrm{d}H_t}{H_t} = (\alpha - s - \lambda k)\mathrm{d}t + \sigma_H \mathrm{d}z_H + \mathrm{d}q_H, \tag{4.7}$$

where  $q_H$  is the independent Poisson event described in Equation (4.4) to (4.6),  $dq_H$ and  $dz_H$  are independent,  $\lambda$  is the expected number of events per unit time, k is the expected value of the random variable of *J*-1, where *J*-1 is the percentage change of the house price if the Poisson event occurs. The  $\sigma_H dz_H$  specifies the unanticipated house price change due to the normal information, and the  $dq_H$  specifies the unexpected house price change due to the abnormal information. If  $\lambda = 0$  (which implies  $dq_H \equiv 0$ ), the house price dynamics would be the same as in Equation (3.1). By applying the risk-neutral principle, Equation (4.7) can be written in a more intuitive form

$$\frac{\mathrm{d}H}{H} = \begin{cases} (r-s)\mathrm{d}t + \sigma_H \mathrm{d}z_H & if the Poisson event does not occur\\ (r-s)\mathrm{d}t + \sigma_H \mathrm{d}z_H + (J-1) & if the Poisson event does occur \end{cases}$$
(4.8)

where only one abnormal event can occur in dt. And when an event occurs, the variable *J*-1 produces a finite jump in the current house price *H* to *J*H. The sample paths of the house price *H* are continuous and smooth for most of the time and with finite jumps of different signs and sizes at discrete points in time. To model a 20% house price jump, the value of *J* needs to be set to 0.8.

I can further generalize the jump process by considering J as a random variable drawn from a distribution with probability density function P(J). Lai and Van Order (2010) observe that the housing bubble from 2003 -2005 consists of a number of positive shocks. Their observation is consistent with the house price growth patterns in Figures 4.1 and 4.2 that the house price jumps include jumps to both directions. One condition of the distribution of P(J) is that it needs to ensure that the jump level J is floored at 0 since house price can not be negative and the probability of a large positive jump is di minis. Weibull distribution can be used to model these features of the house price jumps. Equation (4.9) specifies the probability density function (pdf) of a Weibull random variable J.

$$f(J;\psi,\varphi) = \begin{cases} \frac{\varphi}{\psi} \left(\frac{J}{\psi}\right)^{\varphi-1} e^{-(J/\psi)^{\varphi}} & J \ge 0\\ 0 & J < 0 \end{cases}$$
(4.9)

where  $\varphi > 0$  is the shape parameter and  $\psi > 0$  is the scale parameter. The mean of this distribution can be calculated by using  $\psi\Gamma(1 + 1/\varphi)$ , where  $\Gamma$  is the gamma function and  $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ . By setting  $\varphi$  and  $\psi$  properly, I can obtain a Weibull distribution with high kurtosis. Figure 4.3 is a probability density function of jump levels following a Weibull distribution with  $\varphi = 12$  and  $\psi = 0.9$ . The mean can be calculated as  $0.9\Gamma(1 + 1/12) = 0.86$  by using the gamma function. This value indicates that the expected jump is -14%. In this dissertation, I use the Weibull



Figure 4.3: *J* following a Weibull distribution

distribution to draw the random jump quantity.

In summary, there are three random variables in our aggregated house price model: the diffusion component, the timing of the jump, and the value of the jump. These three quantities are assumed to be independent to each other. Figure 4.4 compares five sample aggregated house price paths simulated from Equation (4.8) and (4.9). It shows clear house price jumps with different sizes and directions.

#### 4.1.3 The Individual House Price Model

By the option pricing theory, it is critical to develop an accurate description of the volatility of the individual house price to evaluate the default option of an individual house or a cohort of similar houses. However, due to the heterogeneity, the aggregate house price model can not sufficiently capture the volatility of the individual


Figure 4.4: The Jump-Diffusion Process of the Aggregated HPI

house price.

Quigley (1995) proposes a simple hybrid method to measure the house price index. This method explicitly combines samples of single sales with samples of repeat sales to calculate the aggregated house price index. This method estimates the systematic component of aggregate housing price changes through the repeat sales. At the same time, this method leverages the single sales data and the hedonic method to correct the estimation bias caused by the small sample size of the repeat sales. The equation of this hybrid method is specified below:

$$h_t = \beta X_t + H_t + \delta_t \tag{4.10}$$

where  $h_t$  is the individual house price index at time t,  $H_t$  is the aggregated house price index at time t,  $X_t$  is the characteristic vector of the underlying house at time t,  $\beta$  is the constant parameter vector to translate the house characteristics into house quality, and  $\delta_t$  is the random errors associated the individual house price following  $N(0, \sigma_h^2)$  distribution. Assuming the constant quality of the individual house such that  $X_t = X_{t-1}... = X_0$ , the expected house price appreciation for the individual house equals to the expected price appreciation of the aggregated house price.

$$E(h_t - h_{t-1}) = E(H_t - H_{t-1})$$
(4.11)

By normalizing both the individual house price index and the aggregated house price index, the expected individual house price is equal to the expected aggregated house price index at time *t*.

The individual house price volatility  $\delta_t$  can be thought as the price fluctuations caused by the supply-demand of the overall hedonic characteristics of the underlying dwelling. Quigley (1995) regresses Los Angeles condominium sales data between 1980 and 1991 to estimate the error terms in the hybrid model and concludes that the hybrid model improves the precision of the index volatility estimation.

One implicit but important feature of this hybrid method is that it can be used to measure and predict the house prices at more granular level of the housing market. By adding a set of sector specific hedonic components to the aggregate repeat sales index, it becomes a sector specific house price index. This is a top-down modeling approach - the variance of a particular sector is measured and modeled by introducing the sector specific volatility in addition to the overall housing market volatility based on the hedonic sector information.

There are many hedonic attributes of a house: the house size, the location, the house price, and so on. Li and Rosenblatt (1997), based on the historical house

price appreciations of three Primary Metropolitan Statistical Areas (PMSA) of California and 1990 Census information, point out that the house price appreciations are correlated with certain neighborhood indicators such as house prices. The directions of these correlations, however, are not clear in their research. In addition, their research indicates that there is a positive correlation between the house price appreciation volatility and the house price. Zhou (2009) provides evidence to support this hypothesis that the house price volatility is a U-shaped function of the house price. Due to the elastic supply and the consistent demand of the middle level priced house, the middle level house price is less volatile than the low-end and high-end house prices. Figure 4.5 shows a synthetic U-Shaped relationship between the house price and the house price volatility within a PMSA. In this figure, the house price volatility decreases as the individual house price increases from the low end to the middle level, then the volatility bottoms up when the individual house price increases from the middle level to the high end.

Intuitively, other notable hedonic house attributes such as location, square footage, number of bedrooms are positively correlated with the house price within an area for a given time period. The individual house price can be considered as a good unbiased proxy to explain the correlations between the house price volatility and the overall hedonic attributes. I specify the volatility of the individual house price relative to the aggregated house price as a cosine function,

$$dh_{t+1} = H_t \left( 2 + \cos\left(\frac{h_t}{H_t}\pi\right) \right) \sigma_h dz_h$$
(4.12)

where  $h_t$  is the individual house value at time t and for simplicity reason is capped at two times of the aggregated house price  $H_t$ ,  $\sigma_h$  is the additional volatility of



Figure 4.5: House Price and Volatility

the individual house,  $dz_h$  is an independent Wiener process. The cosine function ensures the individual house price volatility is a U-shaped function of the individual house price. The house price volatility increases as the individual house price moves away from the average level to both directions and then plateau when the individual house price reaches 0 or two times of the average house price. The Wiener process  $dz_h$  ensures the expected value of  $dh_t$  equal to zero.

I specify the individual house price process below:

$$h_t = H_t + \mathrm{d}h_t \tag{4.13}$$

by applying Equation (4.12), the formula becomes

$$h_t = H_t \left[ 1 + \left( 2 + \cos\left(\frac{h_t}{H_t}\pi\right) \right) \sigma_h dz_h \right]$$
(4.14)





Figure 4.6: Simulated Aggregate HPI and Individual HPIs

Figure 4.6 compares the individual house prices simulated from Equation (4.14) with the aggregated house price simulated from Equation (4.8) for a period of 40 months. The median house price at the beginning of the simulation is \$300k. The dotted red line represents a simulated house price path with a starting house price at \$600k. As expected from Equation (4.14), this house price path exhibits the largest volatility among the three simulated paths. The dotted blue line represents a simulated house price at \$150k. As expected, this house price path exhibits larger volatility than the aggregated house price but has smaller volatility than the house price path with the initial house price \$600k.

## 4.2 Factor Two - the Interest Rate

Another state variable that impacts the value of the default option is the risk-free interest rate is the expected rate of return for all investments including mortgage finance investment. The risk-free interest rate is not only the discount rate of the future cash flows but also the prevailing refinancing rate. As the discount rate, the risk-free interest rate is directly used to discount the future mortgage payments to derive the strike price of the future mortgage payment is lower, which in turn reduces the strike price and the value of the default option. When the risk-free interest rate is lower, the present value of the future mortgage payment is higher, which in turn increases the strike price and the value of the default option. In the risk-neutral world, the risk-free interest rate is also the prevailing refinance rate for prepayment where the lenders expect the risk-free interest rate as the rate of return investment.

Another type of interest rate impacting the mortgage default option value is the reset interest rate of ARM. The London Interbank Offered Rate (LIBOR) and the U.S. Treasury rate are among the most commonly used reset indices. The reset index directly determines the coupon rate and the monthly payment of an ARM product. Although the coupon rate is subject to a few rate caps and floors, a higher index rate usually leads to a higher coupon rate and thus a higher monthly payment. For an adjustable mortgage, both the risk-free interest rate and the reset index impact its default option value. In this dissertation, I focus on explaining the dynamics of the risk-free interest rate and its impact to the value of the default option of the fixed rate mortgage.

A two-factor model with the house price and the risk-free interest rate as the state variables can help understand the interactions between the prepayment option and the default option of a fixed rate mortgage. The instantaneous spot rate of this model is assumed to contain all the information about the future risk-free interest rates, and thus can be used to derive the entire term structure. For simplicity reason, the equilibrium model of the short term risk-free interest rate is traditionally used in a multi-variable framework which takes the functional form proposed by Cox, Ingersoll and Ross (1985).

$$dr = \gamma(\Theta - r)dt + \sigma_r \sqrt{r} dz_r.$$
(4.15)

This model form is known as the CIR model which is a mean-reverting process, where  $\Theta$  is the long-term level of the interest rate,  $\gamma$  is the speed of the interest rate reversion, and  $\sigma_r \sqrt{r}$  is the volatility of the interest rate. Equation (4.15) implies that as the short term interest rate increases, its volatility increases. This model captures the behavior of the interest rate reverting toward the long-term level at a certain speed with a stochastic noise component.

If I take the simple versions of the house price model in Equation (4.1) and the interest rate model in Equation (4.15), by applying risk-neutral principle and Ito's lemma, any derivative f that is dependent on the house price h and instantaneous interest rate r must satisfy the following partial differential equation:

$$\frac{\partial f}{\partial t} + (r-s)h\frac{\partial f}{\partial h} + \gamma(\Theta - r)\frac{\partial f}{\partial r} + \frac{1}{2}h^2\sigma_h^2\frac{\partial^2 f}{\partial h^2} + \frac{1}{2}r\sigma\frac{\partial^2 f}{\partial r^2} + \rho h\sqrt{r}\sigma_h\sigma_r\frac{\partial^2 f}{\partial h\partial r} = rf \quad (4.16)$$

where  $\rho$  is the correlation of  $dz_r$  and  $dz_H$ . The value of the derivative in Equation (4.16) can be numerically solved by applying the finite difference method with

two set of boundary conditions. Another way of solving this equation is to apply the bi-variate binomial technique using backward induction through the nodes of the binomial tree.

The disadvantage of the equilibrium model is that it does not capture the term structure of the risk-free interest rate that is empirically observed in the current market environment. If the model cannot be calibrated to fit the empirical interest rate term structure, the underlying debt and derivatives cannot be valued correctly and thus provides arbitrage opportunities.

A no-arbitrage model is designed to be exactly consistent with the empirical interest rate term structure. The fundamental difference between an equilibrium model and no-arbitrage model is the current market's interest rate term structure. In an equilibrium model like CIR model, the current interest rate term structure is an output of the model instead of an input. In a no-arbitrage model, the current interest rate term structure is an input to the model.

Ho and Lee (1986) propose the first no-arbitrage model for the interest rate term structure,

$$\mathrm{d}r = \theta(t)\mathrm{d}t + \sigma_r\mathrm{d}z \tag{4.17}$$

where  $\sigma_r$  is the constant instantaneous volatility of the short term rate and  $\theta(t)$  is a function of *t* calibrated to ensure the interest rate term structure of this model fit the empirical market structure.  $\theta(t)$  also defines the expected direction that dr at time *t* and can be written analytically.

$$\theta(t) = \frac{\partial F(0,t)}{\partial t} + \sigma^2 t$$
(4.18)

where F(0,t) is the instantaneous forward rate for a maturity *t* directly observed at time t=0 from the market.

Hull and White (1990) propose a modified one-factor no-arbitrage model with an additional mean-reversion feature to the Ho-Lee model.

$$dr = [\theta(t) - ar] dt + \sigma_r dz$$
(4.19)

where *a* is the mean reversion rate. This model can also be viewed as a Vasicek model with an exact fit to the empirical market term structure. Equation (4.19) implies that at time *t*, the short rate *r* reverts to the level of  $\frac{\theta(t)}{a}$  at rate *a*. And Ho-Lee model is equivalent to Hull-White model when a = 0. The function  $\theta(t)$  can be derived from the current term structure:

$$\theta(t) = \frac{\partial F(0,t)}{\partial t} + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$
(4.20)

By ignoring the last term of this equation, the drift process of *r* is expected to be determined by the slope of the initial instantaneous forward rate curve,  $\frac{\partial F(0,t)}{\partial t}$ . When *r* deviates from the initial curve, it reverts back to the curve at rate *a*.

The Hull-White model provides a more accurate and flexible interest rate term structure framework than the traditionally used CIR model in the mortgage default research. As long as there are available numeric methodologies to help derive the option price, the no-arbitrage model like Hull-White model is almost always better to explain the dynamics of the risk-free interest rate which impacts the borrower's mortgage default decision. The detail calibration method of the Hull-White model can be found in many literature and is out of scope of this dissertation. The focus of this dissertation is to apply Hull-White no-arbitrage model to explain and evaluate the impact of the interest rate dynamics to the default option valuation. The disadvantage of applying Hull-White model in this study, however, is that no PDE function like Equation (4.16) can be derived.

## 4.3 Factor Three - the Net Transaction Cost

The role of the transaction cost in the contingent-claims default option model has been debated extensively. Kau, Keenan, and Kim (1991) find that the frictionless model is sufficient to explain the empirical observations of the delayed default option exercises. They argue that by delaying the default decision to the future, rational borrowers default only when their house prices fall substantially below the mortgage value, thus optimizing their financial benefits. Quigley and Van Order (1995), on the other hand, argue that the transaction cost is an important factor explaining why borrowers do not exercise the default option ruthlessly when the house price falls below the mortgage value. They tested their claim using two rich bodies of data from Freddie Mac: one with micro default information of all mortgages purchased by Freddie Mac and another one with data of all mortgages underlying the repeat sales of houses purchased by Freddie Mac. In their research, the frictionless model is not supported. Since then, other researchers have explicitly tested and rejected the frictionless model. However, this type of research failed to show that the transaction cost alone can explain why the borrowers default only when the house price falls substantially below the mortgage value.

In the contingent-claims default framework, borrowers' actions can only be explained by the economic reasons. When the borrowers face different choices, they will choose the action to maximize their financial benefits. In this dissertation, I propose to use a combination of the explanations from both a frictionless model and a transaction cost model to explain the delayed exercises of the default option.

#### 4.3.1 The Net Transaction Cost Formula

Neither a frictionless model nor a transaction cost model is sufficient to explain the actions of the borrowers to delay the exercises of their default options. I make an attempt to explain this behavior by a combination effect of the penalty of the net transaction cost and delaying for better financial benefits in the future. In my model, for any given month when the default option is in-the-money, the borrower will defer the default decision to a later month if the following two conditions are met: if the immediate financial benefit cannot recover the net transaction cost or if the expected financial benefit from a delayed exercise is bigger.

Under this framework, a borrower makes the default decision only based on three state variables: the house price, the interest rate, and the transaction cost. The transaction cost is determined by many factors and follows a much more complicated process than just a constant pecuniary amount or a constant percentage of the underlying mortgage value that is used in the traditional default research. Intuitively, when the default option is deeply in-the-money and the future transaction cost is expected to be significantly higher than the current level, the borrower will tend to default immediately rather than to default in a later month. When the expected future transaction cost is in par with or slightly below the current level, it is still in the borrower's interest to default sooner because the mortgage value will gradually decrease over time and thus reduces the financial benefit of exercising the default option. Only when the expected future transaction cost is significantly below the current level, the borrowers will tend to delay their default decisions to the future.

Crawford and Rosenblatt (1995) argue that a rational borrower defaults only when the house price is below the mortgage value by an amount of the net transaction cost. The net transaction cost consists of transaction costs and pecuniary benefits. Transaction costs include the relocation costs, deficiency judgment payments, moral constraint costs, and social stigma costs. Pecuniary benefits include the free rents during the foreclosure period. The net transaction cost of the default option is specified in the following equation:

$$\text{Transaction}_{t} = \text{Relo}_{t} + \delta(F_{t} - h_{t}) + \text{MC}_{t} + g_{t} - \sum_{k=t}^{t+\tau} \text{Rent}_{k}$$
(4.21)

In this formula, Relot is the relocation cost,  $h_t$  represents the house price at default month t,  $F_t$  is the mortgage value at month t,  $\delta$  is the expected recovery rate function that the lender can recover from a deficiency judgment, MC<sub>t</sub> represents the moral constraint cost,  $g_t$  represents the social stigma cost, Rent<sub>k</sub> represents the free rent of month k during the foreclosure period which is regarded as the second most significant financial benefit to the defaulted borrower other than the elimination of the negative equity, and  $\tau$  is the time lag between the default month and the foreclosure month.

The transaction cost of the default option should only be measured as the direct cost to the defaulted borrowers. Additional costs such as foreclosure costs and disposition costs have little, if any, impact to the default decision. Unlike the lenders of the refinance transactions, the lenders of the default transactions can't transfer their costs to the borrowers who stopped paying the mortgage at the first place.

In the following sections, details of the four cost and benefit components in the net transaction cost formula are discussed.

#### 4.3.2 The Relocation Cost Model

Under the context of the mortgage default, the relocation cost to a defaulted borrower mainly consists of the moving costs and the opportunity costs for his time spent on locating a similar house. There is a great heterogeneity among defaulted borrowers in the moving costs and the opportunity costs and it is very difficult to develop a formula to precisely measure these two components of the relocation costs. Guiso, Sapienza and Zingales (2009) find that the default probability is negatively correlated to the loan age. Their explanation is that the relocation cost increases as the borrower's attachment to the underlying house increases. However, as I discussed in Chapter 3, the correlation may also be explained by the increased level of equity in the house due to the scheduled amortization over time. In addition, it is unnecessary to develop an over complex model for the relocation cost component which has been considered as a secondary impact factor to the default decision. Intuitively, the moving cost and the opportunity cost should be positively correlated with the underlying house price. It is not unreasonable to assume that the relocation cost can be represented in a linear equation as a fraction of the house price

$$Relo_t = \rho h_t \tag{4.22}$$

where  $\rho$  is relocation percentage to the underlying house price ratio.

#### 4.3.3 The Deficiency Judgment Cost Model

The deficiency judgment is a legal process that a lender takes an unsecured money judgment to against the borrower in order to recapture the difference between the mortgage value and the sale proceedings of the house. As discussed in Chapter 3, lenders from eleven non-recourse states can only recover losses from the sale proceedings of the underlying property should a default happens. On the other hand, lenders from other 39 recourse states are entitled to pursue the borrower's personal asset for the difference between the mortgage balance and the sale proceedings of the house.

The deficiency judgment cost to a rational borrower is the expected value of the deficiency judgment payments and is determined by the expected recovery rate  $\delta$  and the negative equity  $F_t - h_t$  at the default month t. In the non-recourse state, there is no deficiency judgment cost and  $\delta = 0$ . Even in the recourse state, it is not in the lender's interest to initiate a deficiency judgment process if the expected deficiency payment cannot recover the expenses of the legal process. Ghent and Kudlyak (2010) argue that the lender's expected recovery rate is positively correlated with the value of the house. I propose a simple recovery model below:

$$\delta_{i,t}(F_t - h_t) = \Phi h_t(F_t - h_t)$$
(4.23)

where  $\delta_{i,t}$  is the expected recovery rate at month t for state i,  $\Phi$  is a marginal constant recovery rate associated with one unit of the house price. For non-recourse states,  $\Phi = 0$ . And for recourse states,  $\Phi = e^{(-D/h_i)}$  where D is a lender targeted average house value that determines the probability of the recourse action.

If the detailed historical deficiency judgment data is made available, a more accu-

rate deficiency recovery model might be established at the state level by regressing the recovery rate with the house price and other key factors.

#### 4.3.4 The Social Stigma Cost Model

The economic value of the social stigma cost has been gradually recognized by both practitioners and researchers in the mortgage finance industry. On June 20, 2010, Fannie Mae announced a 7 year lock-out policy to penalize the strategically defaulted borrowers. According to this policy, borrowers who strategically defaulted on their mortgages will be penalized by automatically disqualifying future Fannie Mae backed mortgage loans for a 7 year period. The implication of this policy is to increase social stigma cost of the strategically defaulted borrowers by denying them the access to the future financing opportunities. This is one of the first attempts of the mortgage finance industry to increase the transaction cost of the default option in order to prevent strategic defaults and foreclosures.

Similar to other social behaviors, the average cost of social stigma follows a mean reversion process. The historical spread between the prime mortgage origination rate and the subprime mortgage origination rate can be used as a proxy to study the economic cost of one of the most important components of the social stigma cost – credit impairment. The subprime mortgages are generally defined as those mortgage loans made to the borrowers with credit scores less than 620. The lower credit core is usually caused by certain credit events including delinquency and default. Bhutta, Dokko and Shan (2010) estimate on average a mortgage default event reduces the borrower's credit score by 21% which is a reduction that may lead to a downgrade of the borrower's credit rating from prime to subprime. The

subprime borrower is penalized by an additional risk premium or a rate spread to the prime rate which the prime borrowers pay for their mortgages. This subprimeprime mortgage rate spread is essentially an average price tag for the additional risks of the borrowers with impaired credits.

Chomsisengphet and Pennigton-Cross (2006) calculate the subprime-prime rate spread of FRM30 for the period of 1995-2004 based on Freddie Mac Market Survey and LoanPerformance data. Their subprime-prime rate spread exhibits a tendency of reverting to the 2% long term equilibrium level. Demyanyk and Van Hemert (2007) apply similar techniques to derive the subprime-prime mortgage rate spread for the period of 2001 - 2006. Their results are consistent with Chomsisengphet and Pennigton-Cross (2006). Figure 4.7 shows the subprime-prime spread between 2001 and 2007. It shows that the spread declined rapidly during the time period of 2001-2004 and started to revert back to the long term average after 2005.

From the fundamental social and public policy perspective, the average cost of social stigma has a long-run equilibrium and tends to be pulled back to this long-run average over time. More specifically, when the cost of social stigma is too high, the public policy will have to be adjusted to lower this cost to accommodate the involuntary events such as illness and loss of job. On the other hand, if the cost of social stigma is too low, the policy also must be adjusted to increase the cost in order to avoid a large scale of moral hazards. In this dissertation, I use a CIR type model to approximate the process of the average social stigma cost perceived by the society,

$$dg = \gamma(b - g)dt + \sigma_g \sqrt{g} dz_g$$
(4.24)

where b is the long term level of the social stigma cost,  $\gamma$  is the speed of the cost



Figure 4.7: Historical Prime and Subprime Origination Rate Spread Data source: Demyanyk and Van Hemert (2007)

reversion, and  $\sigma_g$  is the volatility of the social stigma cost.

A more precise social stigma cost model might be derived by first setting up a subprime-prime rate spread model by fitting the historical spread volatility data. Then applying the expected future credit activities with the forecasted spread rates to derive the social stigma cost.

#### 4.3.5 The Moral Cost Models

Guiso, Sapienza, Zingales (2009) indicate that the most significant transaction costs of default option is the moral constraint cost. Their survey shows that no household would choose strategic default if the incentive from the negative equity of their house is less than 10% of the house value. And as the negative equity in the house increases so does the percentage of the households choosing to default strategically. However, the same survey also shows that a significant portion (55%) of households would not default even when the negative equities of their houses equal to \$300k. Their findings provide three fundamental arguments for modeling the moral constraint cost: first, the cost of moral constraints can be measured pecuniarily; second, the cost of the moral constraints is perceived differently among the borrowers; third, the moral barriers can be penetrated if the economic incentives are sufficient.

In order to capture the heterogeneity of the borrowers' perceptions toward the moral constraint cost, I introduce an individual moral cost model to classify borrowers into different moral groups. I define the level of the morality as  $L_{i,t}$  for morality group i at time t, where  $L_{0,t}$  is the lowest morality level and  $L_{n,t}$  is the highest morality level. Each morality level  $L_{i,t}$  is associated with the perceived moral cost MC<sub>i,t</sub> and the percentage of the borrowers  $P_{i,t}$  in this group. As the number of the morality groups  $n \to \infty$ ,  $P_{i,t}$  becomes a continuous probability density function of MC<sub>i,t</sub>. In this dissertation, I assume the asymptotic distribution of MC<sub>i,t</sub> follows a normal distribution.

$$MC_{i,t} \sim \mathcal{N}(MC_{m,t}, \sigma_{i,t}^2)$$
 (4.25)

Where  $MC_{m,t}$  is the average moral cost of all the borrowers at time *t* and  $\sigma_{i,t}$  is the associated standard deviation.

The model can be used to study the impact of the moral views to the default option collectively hold by a group of borrowers. This normal distribution implies that the percentage of borrowers with moral constraint cost below a certain level x can be derived through the cdf function  $\Phi(x)$  of this normal distribution.

There is evidence to support that the borrower's moral view is resilient to the changes of the social environment. Guiso, Sapienza, Zingales (2009) find that the borrower who knows somebody defaulted strategically is 8% more likely to declare default. However, the same research also shows that there is no evidence that the increase of the default probability comes from the weaker moral views. This research implies that  $MC_{m,t}$  and  $\sigma_{m,t}$  in Equation (4.25) can be approximated by their values at time 0.

The usefulness of this model depends on the validity of the assumption of the distribution. There is little data to support the assumption that the borrower moral constraint cost follows a normal distribution. Figure 4.8, based on the high level survey data of Guiso, Sapienza and Zingales (2009), shows that the distribution of the moral cost group seems different with the normal distribution assumption. However, as shown in the same graph, their survey does not distinguish between "high" and "higher" moral cost categories. Additional empirical data is needed to test if the borrower's moral constraint cost follows a normal distribution.

It is possible to model the moral constraint cost more precisely. As I discussed in Chapter 3, the borrower's moral view toward the default is correlated with his age, education level, income and credit score. Similar to the concept of the credit score, a "moral score" can be calculated by regressing the moral levels with the borrower's characteristics and the historical events. Due to the limitation in the data availability, for the purpose of this dissertation, the volatility of the moral constraint cost for individual borrowers is limited to the normal distribution in Equation (4.25).



Distribution of Survey Moral Cost Values

Figure 4.8: Moral Cost Category and Associated Borrower Percentage

#### 4.3.6 The Rental Benefit Model

Ambrose, Buttimer and Capone (1997) indicate that the borrower's default decision is made at the foreclosure date instead of the last payment due date. Under this theory, in addition to the relief from negative equity the benefit to the borrower also includes the ability to consume the house during the default period. This pecuniary benefit can also be thought as the free rents enjoyed by the borrower during the period between default date and foreclosure date.

In the net transaction cost formula,  $\sum_{k=t}^{t+\tau} \text{Rent}_k$  represents the free rents enjoyed by the borrower between the default month t and the foreclosure month  $t + \tau$ . This benefit is regarded as the second most significant financial benefit to the defaulted borrower other than the elimination of the negative equity. During the foreclosure period, the borrower is still legally entitled to consume the underlying dwelling.

The borrower may choose to continue live in the property or rent the property out after relocation. It is true that in reality not every single defaulted borrower will take advantage of this financial benefit of free rents. However, in this dissertation, I assume the borrowers are rational and they will take this financial benefit into their default decision processes.

There is empirical evidence that the rents are positively correlated with the house price. Figure 4.9 shows that the normalized United States national level house price index<sup>13</sup> and the rental index<sup>14</sup> during the period of Jan 2001 to Dec 2010 are positively correlated with a coefficient 0.35. In the existing literature, the rent-house price ratio has been used to forecast the future house price level and the rent is considered as the fundamental determinant to the value of the underlying property with a similar role of dividends to the equity valuation. In this dissertation, I utilize the rent-house price ratio to determine the rental level with a given house price level.

$$Rent_t = h_t u_t \tag{4.26}$$

where  $u_t$  is the rent-house price ratio for the month t.

It is widely believed that the rent-house price ratio follows a mean reversion process. Gallin (2004) shows the rents and house prices ratio reverts back to the equilibrium level over the three-year horizon. Figure 4.9 also shows that after 2006 the United States national level rent-house price ratio reverted back to its Jan 2000 level proceeded with a 5-6 years rapidly decreasing period. For the purpose of this dissertation, I use the long-term equilibrium as the rent-house price ratio to

 $<sup>^{13}</sup>$  S&P Case and Shiller Home Price Index for Composite 20 Cities and is normalized to 100 for Jan 2001

<sup>&</sup>lt;sup>14</sup> Rent of primary residence index from Consumer Price Index of Bureau of Labor Statistics and is normalized to 100 for Jan 2001



House Price Index and Rent Index

Figure 4.9: United States National Level Rent Index and Home Price Index

determine the rental benefit. A more accurate rent model could be developed by modeling the rent-price ratio with a mean reversion process.

Based on the individual cost and benefit component models, I re-write the net transaction cost formula in Equation (4.21)

Transaction<sub>t</sub> = 
$$h_t \rho + \Phi h_t (F_t - h_t) + \mathrm{mc}_t + g_t - \sum_{k=t}^{t+\tau} h_k \mu_k$$
 (4.27)

If I assume the house price and the rent-house price ratio are approximately the same during the foreclosure period, Equation (4.27) becomes

$$Transaction_t = h_t [\rho + \Phi(F_t - h_t) - \tau \mu] + mc_t + g_t$$
(4.28)

# 4.4 The Three-Factor Model

Based on the specific models of the house price, the interest rate and the net transaction cost I introduced through Equations (4.1)-(4.28), the final three-factor model is summarized below:

$$\begin{cases} h_t = H_t \left[ 1 + (2 + \cos(\frac{h_t}{H_t}\pi))\sigma_h dz_h \right] \\ r_t = r_{t-1} + \left[\theta(t) - ar_{t-1}\right] dt + \sigma_r dz_r \\ \text{Transaction}_t = h_t [\rho + \Phi(F_t - h_t) - \tau \mu_t] + \text{mc}_t + g_t \end{cases}$$
(4.29)

where the aggregated house price  $H_t$  follows the jump-diffusion process defined in

$$\frac{\mathrm{d}H_t}{H_t} = (r_t - s - \lambda k)\mathrm{d}t + \sigma_H\mathrm{d}z_H + \mathrm{d}q_H$$

and *k* is the expected value of an independent random variable of *J* which follows a Weibull distribution, and  $dq_H$  is an independent Poisson process, and the social stigma cost  $g_t$  follows the CIR type mean reversion process defined in

$$\mathrm{d}g = \gamma(b-g)\mathrm{d}t + \sigma_g\sqrt{g}\mathrm{d}z_g$$

The three Wiener processes,  $dz_H$ ,  $dz_r$ , and  $dz_m$  in the above equations follow a correlation matrix:

$$R = \begin{bmatrix} 1 & \rho_{H,r} & \rho_{H,g} \\ \rho_{H,r} & 1 & \rho_{r,g} \\ \rho_{H,g} & \rho_{r,g} & 1 \end{bmatrix}$$
(4.30)

where  $\rho_{H,r}$  is the correlation between the aggregated house price drift and interest rate drift;  $\rho_{H,g}$  is the correlation between the aggregated house price drift and social



Figure 4.10: A Simulated Economic Scenario with Three Underlying Factors

stigma cost drift;  $\rho_{r,g}$  is the correlation between interest rate drift and social stigma cost drift.

Figure 4.10 depicts an economic scenario which consists the three underlying factors simulated from the three-factor model.

# 4.5 The Empirical Correlations of the Factors

Equation (4.30) represents the correlation matrix of the three stochastic factors: the house price, the risk-free interest rate, and the social stigma cost of the underlying model I introduced in this chapter. It is important to understand the empirical correlations among these factors for modeling and simulation studies.

There are plenty of studies that examine the correlations between house prices and interest rates. However, there is no conclusive answer to this question. Kau, Keenan and Kim (1994) present different reasons explaining why house prices could be either positively or negatively correlated with interest rates. Capozza, Kazarian and Thomson (1998) find no evidence that there is significant correlations between house prices and interest rates based on a large panel of data of 64 metropolitan areas for 25 years. Mayer and Hubbard (2009) find that the global house prices are negatively correlated with real mortgage rates by using house price data of several countries including United States between 1997-2007. They argue that most of the house price variances can be explained by the real mortgage rate changes. The remaining house price variance can be explained by other factors such as economy growth and over supply of the housing stock. In this dissertation, I follow the conclusion of Mayer and Hubbard (2009) that the house prices and mortgage rates are negatively correlated.

A stochastic mean reversion model is used to explain the variations of the average social stigma cost that the defaulted borrowers incurred in the long term. Under this context, the social stigma cost should be positively correlated with the house price as the higher the social stigma cost the lower the default probability. In the longer term, with lower default rate and less foreclosed properties, the house price is expected to be higher. In the short term, it is widely believed that the subprime expansion of credit is one of the most important triggers of the recent United States housing bubble. Mian and Sufi (2009) find that the credit expansion could be responsible of driving up house prices in the subprime areas. Although in short term social stigma cost is negatively correlated with the house price, this correlation is largely reflected through the negative correlations between the interest rates and the house prices. The direct consequence of the short term credit expansion is the rapid drop of the interest rate. The lowered interest rate drives up the house

price. The recent burst of the housing bubble demonstrates again that the long run correlation between the social stigma cost and the house price stays positive.

Intuitively, the average social stigma cost is positively correlated with the interest rate. By definition, the social stigma cost is the sum of the defaulted borrowers' future disadvantages in terms of financing related activities compared to the borrowers without credit impairment. The social stigma cost is measured approximately through the refinance rate spread between borrowers with impaired credit and borrowers without credit impairment. It is widely believed that the subprime-prime spread is positively correlated with the level of the prime rate. I conclude that the social stigma cost is positively correlated with the interest rate.

In this chapter, I set up a three-factor model based on the stochastic processes of the house price, the risk-free interest rate and the net transaction cost. This model captures the volatility of the house price at both the aggregated level and the individual level. The model also captures the heterogeneity among borrowers in terms of net transaction cost by considering both the characteristics of the underlying properties and the individual borrower's moral view. In the next chapter, I introduce the numerical methods to evaluate the default option price under this three-factor pricing framework.

# **Chapter 5 Numerical Methods**

As an American type option, a mortgage default option does not have an analytical solution to its price. The classic Monte Carlo simulation method used in the European option pricing cannot be used either, because the default option price is path dependent and the expected optimal payoff cannot be simply derived along a single path. The default option price, however, can be approximated through some specialized numerical algorithms. In this chapter, I first describe the valuation principles that any numerical method needs to follow to derive the correct default option price. Then I describe the finite difference method which is a widely used numerical method to derive the American option price in the one-factor or twofactor models. After that, I describe the least-square Monte Carlo (LSM) simulation method that I will adapt to solve the option price underlying the three-factor model.

## 5.1 Valuation Principles of the Mortgage Default Option

A borrower faces three choices when a mortgage payment is due. He can terminate the mortgage contract by choosing to exercise either the prepayment option or the default option. He can also defer the decision to the next period by making the current payment. I assume that a borrower will always choose the action that maximizes his financial position. Even when the default option or the prepayment option is clearly in-the-money, it may not be optimal to exercise the option immediately as the present value of the future exercise may represent a better financial payoff. The question of calculating the option price becomes an optimization problem. The following principles are to be followed to solve the optimization problem and derive the option values numerically.

- At month t, the borrower may prepay the mortgage immediately when the present value of the future payments K<sub>t</sub> is higher than the sum of the unpaid mortgage balance UPB<sub>t</sub> and the prepayment cost PCost<sub>t</sub>. The immediate exercise value of the prepayment option is determined by P<sub>t</sub> =max{K<sub>t</sub>-UPB<sub>t</sub> PCost<sub>t</sub>, 0}. When the borrower chooses to prepay, the borrower purchases the house with the current mortgage balance and the mortgage contract is terminated. The values of the default option and the continuation are set to 0.
- At month t, the borrower may default the mortgage immediately when the present value of the remaining payments K<sub>t</sub> is higher than the sum of the current house price h<sub>t</sub> and the default transaction cost DCost<sub>t</sub>. The immediate exercise value of the default option is determined by D<sub>t</sub> = max{K<sub>t</sub> h<sub>t</sub> DCost<sub>t</sub>, 0}. When the borrower chooses to default, the borrower effectively sells the house to the lender at the current house value and the mortgage contract is terminated. The values of the prepayment option and the continuation are set to 0.
- Since only one mortgage option can be exercised at a given time, I define *M<sub>t</sub>* = max{*P<sub>t</sub>*, *D<sub>t</sub>*} as the mortgage option value at time *t*. At month *T*, the expiration month of the mortgage, like other American option investors, the borrower will exercise the mortgage option if it is in-the-money or passively let it expire if it is out of money. For any month t before T, two scenarios need

to be examined further.

- If  $M_t = 0$ , which means both default option and prepayment option are out-of-the-money at time *t*, a rational borrower will make the mortgage payment and defer the decision to the next month.
- If  $M_t > 0$ , one of the options is in-the-money. However, immediate exercise without considering the future values of the options may not be the optimal decision. Let  $m_t$  be the present value of the future exercise of the mortgage option. If  $M_t \ge m_t$ , the immediate exercise of the mortgage option is optimal. If  $M_t < m_t$ , the immediate exercise of the mortgage option is not optimal and the decision should be deferred to the next month. Since the future is unknown to the borrower,  $m_t$  can only be estimated through the expectation functions of the future cash flows generated by the options conditional on the probability space of the economic environment.
- By applying the dynamic programming and the backward induction, for the optimal option values of month *T*-1, I only need to compare the value of the immediate exercise with the expectation function of the future cash flows from month *T* and decide the optimal strategy for each of the simulation paths. After the optimal strategy is defined for month *T*-1, the optimal exercise strategy for month *T*-2 is derived similarly by comparing the financial benefit of immediate exercise of month *T*-2 with the present value of the optimal strategy at month *T*-1. The same procedure is applied to all the months until month 0 is reached. The optimal exercise strategy for each of the paths can be derived.
- The values of the options are then determined by discounting the exercise

benefit cash flows along each simulated path. Based on the optimal strategy along each path, the paths can be classified into default paths, prepayment paths and maturity paths. The default path and the prepayment path are defined as the default option or the prepayment option is exercised during the term of the loan along this path. The maturity path is defined that no option is exercised during the term of the loan. The default option value is calculated as the sum of the discounted cash flows from the default paths divided by the total number of the simulated paths. And the prepayment value is calculated as the sum of the discounted cash flows from the prepayment value is calculated by the total number of the simulated paths.

## 5.1 The Finite Difference Method

For the simple one-factor model in Equation (4.3), one of the most widely used numerical methods in financial derivative pricing – the finite difference method can be used to solve the underlying stochastic equation. The finite difference method converts the differential equation into a set of difference equations and solve these equations iteratively.

In the case of the mortgage default option, the *X*-axis in the grid of the finite difference method is time t, where *T* is used to denote the term of the mortgage default option or the maturity of the underlying mortgage loan which is also the right boundary of the *x* axis in the grid. I divide *T* into *N* equally spaced intervals of length  $\Delta t$ . Since the borrower only needs to make the default decision when a mortgage payment is due, I set  $\Delta t = 1$  Month (or  $\frac{1}{12}$ Year) and N = 12T. The *Y*-axis in the grid is the house price. I use  $H_{\text{max}}$  to represent the upper boundary of this axis in the grid.  $H_{\text{max}}$  represents a sufficient high house price which makes the underlying default option's value equal to zero. I divide  $H_{\text{max}}$  into M equally spaced intervals of house price  $\Delta h$ . The option price at any node of the grid can be denoted as  $f_{i,j}$ . I can derive the default option price on all the nodes in the grid by applying the following three conditions to the set of difference equations derived from Equation (4.3): f = 0 when  $H_t = H_{\text{max}}$ , f = K when  $H_t = 0$ , and  $f_{N,j} = \max[K - j\Delta s, 0]$ , where K is the strike price. If K is a fixed value, the equation can be solved iteratively using the difference equations. For a mortgage default option,  $K_t$  can be calculated by solving the Equation (3.2). After  $K_t$  is calculated, the last boundary condition of the the finite difference equation can be written as  $f_{N,j} = \max[K - j\Delta s, 0]$ . By iteratively solving the difference equations for each node, the option price can be calculated by discounting the future optimal cash flows.

### 5.2 The Least Square Monte Carlo Method

The finite difference method becomes computationally impractical to solve differential equations with more than two stochastic variables. Longstaff and Schwartz (2001) propose a Monte Carlo simulation based approach to approximate the value of the American-type option. This method is easily applicable to the multi-factor models and provides accurate approximation to the value of the option whose underlying asset follows a general stochastic process such as a jump diffusion model. This method is based on the assumption that the optimal exercise strategy is a conditional expectation function of the option's payoffs from continuation. In this method, the expectation function is estimated from the cross-sectional information in all simulated paths using least squares. This method is also known as the least square Monte Carlo method (LSM).

I use the same notation as in Longstaff and Schwartz (2001) to define the LSM framework that will be used to approximate the values of the mortgage default options. I assume an underlying complete probability space denoted by  $(\Omega, \mathcal{F}, P)$ , where  $\Omega$  is the set of all possible realizations of the economy between time 0 and T,  $\mathcal{F}_t$  is a sigma field at time t such that  $\mathcal{F}_t \subset \mathcal{F}_s$  for  $t < s \leq T$  and  $\mathcal{F}_T \subset \mathcal{F}$ , and P is a probability measure of  $\mathcal{F}$ . Let Q be a martingale measure on the filtration  $\{\mathcal{F}_t; t \in [0,T]\}$  defined under a risk-neutral assumption. I use  $C(\omega, s; t, T)$  to denote the cash flows of the option in a particular path  $\omega$ , with the conditions that the option is not exercised before t and the the investor is a rational person strictly following the optimal exercise strategy for all  $s, t < s \leq T$ .

I assume the default option can only be exercised at *K* discrete times  $0 < t_1 \le t_2 \le \cdots \le t_k = T$ , corresponding to the times when a payment is due; *K* is the total number of payments remaining in the mortgage contract. On the other hand, I consider that the default exercise can happen at anytime during a payment cycle. In that case, I can take *K* to be a sufficiently large number to achieve an approximation to the true value within a given threshold. As an American-type option, the default option's value is maximized if the borrower defaults on the property immediately once the financial benefit is greater than the value of continuation. At month *t*, the financial benefit of continuation is unknown to the borrower. By applying the no-arbitrage principle, the financial benefit of continuation after month *t* is the expected function of the future cash flows with respect to the martingale measure

Q. Then at time  $t_k$ , the value of the default option  $V(\omega; t_k)$  can be written as

$$V(\omega; t_k) = E_Q \left[ \sum_{j=k+1}^{K} \exp\left( -\int_{t_k}^{t_j} r(\omega, s) \mathrm{d}s \right) C(\omega, s; t, T) \middle| \mathcal{F}_{t_k} \right]$$
(5.1)

where  $r(\omega, s)$  is the risk-free interest rate at time *s* of path  $\omega$ . Based on Equation (5.1), the value of the default option is calculated by comparing the value of the immediate exercise of the option with the conditional expected value. And the optimal option value is then determined by the maximum of these two values.

Similar to the finite difference method and other numerical methods to solve the value of the American option, the numerical algorithm of the LSM method also sets up a backward recursive mechanism to approximate the conditional expectation function of the Equation (5.1) at  $t_{K-1}, t_{K-2}, \ldots, t_1$ . Considering a special case, at time  $t_{K-1}$ , one month before the maturity of the default option, the underlying assumption is that the value of the continuation  $V(\omega; t_{k-1})$  in the Equation (5.1) can be represented as a series of linear basis functions. This assumption can be justified by considering  $V(\omega; t_{k-1})$  to be an element of the Hilbert space of the square-integrable functions and the conditional expectation of this element can be represented by a series of countable orthonormal basis functions. There are many types of orthonormal basis functions including Hermite, Legendre, Laguerre, Chebyshev, Gegenbauer, and Jacobi polynomials. I will use the Hermite polynomials:

$$H_0(x) = 1$$
 (5.2)

$$H_1(x) = x \tag{5.3}$$

$$H_2(x) = x^2 - 1 \tag{5.4}$$

$$H_3(x) = x^3 - 3x \tag{5.5}$$

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$
(5.6)

The conditional expectation function  $V(\omega; t_{k-1})$  in Equation (5.1) can be represented as

$$V(\omega; t_{k-1}) = \sum_{j=0}^{\infty} a_j H_j(x)$$
(5.7)

In order to solve the Equation (5.7), I use the first  $M < \infty$  basis functions to approximate the value of  $V(\omega; t_{k-1})$  as  $V_M(\omega; t_{k-1})$ . The value of  $V_M(\omega; t_{k-1})$  is estimated by conducting a regression of the discounted cash flow  $C(\omega, s; t_{k-1}, T)$  with the selected basis functions on the selected paths at time  $t_{k-1}$ . As suggested by Longstaff and Schwartz (2001), it is more computationally efficient to choose the in-themoney paths in the regression function to approximate the conditional expectation function using fewer basis functions. Longstaff and Schwartz (2001) prove that the fitted value of the regression function  $\hat{V}_M(\omega; t_{k-1})$  converges in probability to the true value of  $V_M(\omega; t_{k-1})$  as the number of the simulation paths approaches infinity.

After  $\hat{V}_M(\omega; t_{k-1})$  is calculated by the fitted regression function, the value of the default option at time  $t_{k-1}$  of path  $\omega$  is determined by the maximum value of the immediate exercise value and the conditional expectation  $\hat{V}_M(\omega; t_{k-1})$ . This calculation of  $\hat{V}_M(\omega; t_{k-1})$  is repeated for all in-the-money paths. After the values of the default option of all paths at time  $t_{k-1}$  are calculated, the default option cash flows  $C(\omega, s; t_{k-2}, T)$  at time  $t_{k-2}$  can then be determined. A similar regression of  $C(\omega, s; t_{k-2}, T)$  and the basis functions will lead to the values of  $\hat{V}_M(\omega; t_{k-2})$  and

thus the exercise decisions for all paths at time  $t_{k-2}$ . The same procedure will be repeated recursively until the exercise decisions at each exercise time for all paths have been made. At last the default option value at time zero is determined by discounting optimal exercise prices back to time zero and taking the average of all paths.

When there are multiple state variables underlying the option price, the basis functions should include the terms in each of the state variables as well as the crossproducts of these terms. It will become computationally impractical if the number of basis functions needed grows exponentially with the number of the state variables. However, Longstaff and Schwartz (2001) point out the number of basis functions used to approximate the conditional expectation function are not necessarily linked exponentially to the number of the state variables. Instead, the number of the basis functions in the case of three or higher dimensional case may be a small set of the total available basis functions and the computation can be very manageable. In this research, I will use a small set of the available basis functions of the three-factor model to conduct the least square estimation.

I use a simple numerical example to illustrate the LSM method. I assume there is no prepayment, a synthetic interest-only balloon mortgage product presents the simplest amortization schedule. By definition, the borrower of the interest-only balloon mortgage does not pay down the principal and pays the interest only during the entire loan term. At the end of the loan term, the borrower pays back the whole original borrowed amount to the lender. Since there is no amortization, the strike price of the default option remains constant as the original borrowed loan amount. I use an example of the interest-only balloon mortgage to help understand the application of the LSM to value the default option. The basic characteristics of

Path	month $= 0$	month = $1$	month $= 2$	month $= 3$
1	1.00	0.91	0.90	0.90
2	1.00	1.00	1.01	1.03
3	1.00	1.00	0.99	0.99
4	1.00	1.00	1.00	1.01
5	1.00	0.99	0.98	0.97
6	1.00	1.01	1.00	1.00
7	1.00	0.98	1.00	1.00
8	1.00	1.00	1.00	0.90
9	1.00	0.98	0.98	0.97
10	1.00	1.00	1.00	1.01

Table 5.1: house price Index Matrix

this loan are: the original borrowed amount is \$100,000, the original house price is \$100,000, and the borrower coupon rate is 4%. The mortgage matures at month 3. And the default option is exercisable at month 1, 2, and 3. The value of the default option only depends on the underlying house price. For simplicity, I use ten sample paths for the underlying house price. By general definition, the house price index (HPI) starts from month 0 with the initial value 1. The HPI values in the subsequent months are used to derive the house prices of those months. For example, the HPI value at month 1 of path 1 is 0.91, the house price at month 1 of path 1 is equal to 0.91/1 \* \$100,000 = \$91,000.

The sample HPI paths are simulated under a jump-difussion process. For the jump part, I set the values of *J* to 0.9 and $\lambda$  to 0.05. For the diffusion part, I set  $\sigma_H$ , the instantaneous standard deviation of house price, to 0.02. These sample paths are shown in Table 5.1. There are two jumps - the first one happens at the month 1 of path 1 and the second one occurs on month 3 of path 8.
# Table 5.2: Financial Benefits at Month 3

Path	month $= 3$
1	\$10,000
3	\$1,000
5	\$3,000
8	\$10,000
9	\$3,000

The first step to solve the maximum value of the option for each path is to determine the terminal value of the default option at the expiration date assuming there is no exercise before the expiration date at month 3. After that, I apply the LSM algorithm recursively to derive the conditional expected value of the option at each time point along each simulation path. The financial benefit of the default option is obtained by deducting the value of the underlying house price from the total loan amount if the house price index is less than 1. The optimal financial gain realized by the borrower at month 3 are shown in the Table 5.2.

The second step is to determine the value of the default option at month 2. At month 2, the borrower will decide if to default immediately or wait until the final expiration date at month 3. I use the paths where the default option is in-the-money at month 2 since the conditional expectation function over these paths reflects the exercise-continuation scenarios that the borrower faces. In addition, the computation over the selected paths are much more efficient than computation over the full paths. According to the house price index in Table 5.1, there are 4 paths where the default option is in-the-money. I use *X* to denote the house price index at month 2 and *Y* denote the corresponding present value of the financial benefit at month 3. The regression table at month 2 is given below.

Table 5.3: Regression at Month 2

Path	Y	Χ
1	\$10,000/(1+0.06/12)	0.90
3	\$1,000/(1+0.06/12)	0.99
5	\$3,000/(1+0.06/12)	0.98
9	\$3,000/(1+0.06/12)	0.98

Table 5.4: Exercise v.s. Continuation at Month 2

Path	Exercise	Continuation
1	\$10,000	\$9,950
3	\$1,000	\$995
5	\$2,000	\$2,985
9	\$2,000	\$2,985

I apply the first 3 Hermite polynomials: a constant, *X*, and  $X^2$  in the regression function to determine the conditional expected value of the continuation at month 2. The values of path 1, 3, 5 and 9 are used in setting up the regression. The resulting regression function is  $E[Y|X] = -1.01 + 2251244X - 1243781X^2$ . The comparison of the immediate exercise versus the continuation for the 4 paths is shown in Table 5.4.

For path 1 and 3, it is optimal to exercise the default option immediately at month 2. For path 5 and 9, the optimal choice is continuation since the expected value of continuation is greater than the value of the immediate exercise. This leads to the

Path	month $= 1$	month $= 2$	month $= 3$
1	-	\$10,000	0
3	-	\$1,000	0
5	-	0	\$3,000
8	-	0	\$10,000
9	-	0	\$3,000

Table 5.5: Financial Benefit Matrix at Month 2

optimal financial benefits of the borrower for each path if there is no exercise on month 1. Please note the exercise of the default option is a one-time benefit for each path - if the option is exercised at month 2, there is no additional financial benefit at month 3.

The third step is to determine the value of the default option at month 1. I again regress the present values of the subsequent financial benefits of the default option Y with the house price index X at month 1. I use the realized financial benefit along each path to determine Y. The realized financial benefit is either the value realized by exercise or 0 if there is no terminal value at expiration date. The financial benefit realized at month 2 will be discounted back one period and the financial benefit realized at month 3 will be discounted back two periods. From the house price index matrix, there are four paths where the default option is in-the-money at month 1. The vectors of Y and X that will be used to set up the conditional expectation function are given below.

The conditional expected function is estimated again by regressing *Y* on a constant, *X* and  $X^2$ . The resulting function is  $E[Y|X] = 3123598 - 6486491X + 3368021X^2$ . I

Table 5.6: Regression at month 1

Path	Y	Χ
1	\$10,000/(1+0.06/12)	0.91
5	\$3,000/(1+0.06/12) <sup>2</sup>	0.99
7	\$0/(1+0.06/12) <sup>2</sup>	0.98
9	\$3,000/(1+0.06/12) <sup>2</sup>	0.98

Table 5.7: Exercise Versus Continuation at Month 1

Path	ath Exercise Continuation	
1	\$9,000	\$9,950
5	\$1,000	\$2,970
7	\$2,000	\$1,485
9	\$2,000	\$1,485

obtain the expected values of continuation as the fitted values of the model. Then I compare the value of immediate exercise with the value of continuation for each path in Table 5.7

By comparing the exercise value and the continuation value, the optimal strategy for the path 1 and 5 is continuation. For path 7 and 9, the optimal strategy is to exercise immediately. And the final option cash flow matrix is given in Table 5.8.

As the optimal financial benefits for all paths have been identified, the default option value can be calculated by discounting these benefits back to time zero and

Path	month $= 1$	month $= 2$	month $= 3$
1	0	\$10,000	0
3	0	\$1,000	0
5	0	0	\$3,000
7	\$2,000	0	0
8	0	0	\$10,000
9	\$2,000	0	0

Table 5.8: Financial Benefit Matrix

averaging over all paths. The resulting default option value is \$2,768. This value is around \$100 higher than the value of \$2,659 for a synthetic European default option if I assume the default option can only be exercised at month 3.

In addition, this financial benefit matrix and associated stop rule matrix can be used to calculate the default probabilities, an important analytics for the mortgage finance industry to measure the default risk of a pool of similar mortgage products. The mortgage default rate is usually defined as the single monthly mortality (SMM) rate, which indicates, for any given month, the fraction of the mortgage balance that had defaulted during the month. In this example, the month 1 SMM is 20%, SMM of month 2 is 25% (2 out of 8 remaining paths), SMM of month 3 is 33% (2 out of 6 remaining paths), the cumulative default rate for the first 3 months is 60%. This is an important advantage of the LSM method compared to other numerical methods for practice purposes and may be used for future studies.

Path	month $= 1$	month $= 2$	month $= 3$
1	0	1	0
2	0	0	0
3	0	1	0
4	0	0	0
5	0	0	1
6	0	0	0
7	1	0	0
8	0	0	1
9	1	0	0
10	0	0	0

Table 5.9: Stopping Rule

# 5.3 Improvements to the Computational Performance of LSM

The evaluation of the default option with LSM method requires K (the number of remaining mortgage payment) least square regressions on N simulated paths with M orthonormal basis functions. The computational performance of LSM decreases when N and M increase. Under the context of the mortgage default option valuation, I use two methods to improve the computational performance of LSM.

## 5.3.1 Quasi-Random Sequence in LSM

A quasi-random sequence is also referred to as a low-discrepancy sequence. In this sequence, all values of N and its subsequence  $x_1, x_2, \ldots, x_N$  has a low discrepancy. There are many ways of constructing a quasi-random sequence. Niederreiter(1992)

provides a detailed discussion on the definition and generation of the quasi-random sequence.

Due to its low discrepancy, quasi-random sequence has been used in Monte Carlo method for a faster convergence in the low-dimensional numerical integration. In the best scenario, the rate of convergence of quasi-random sequence is close to  $O(N^{-1})$  compared with  $O(N^{-1/2})$  of the traditional pseudo random sequence. However, it is known that the upper bound of the convergence rate of the quasi-random sequence is  $O\left(\frac{(log N)^d}{N}\right)$ , where *d* is the dimension of the sequence and in the case of mortgage default valuation d = K. The quasi-random sequence performance decreases with the dimension *d*. Another issue with the quasi-random sequence is the multi-dimensional clustering caused by the correlations between the dimensions.

In this dissertation, the quasi-random sequence is used in LSM to solve the default option value. Chaudhary (2005) applies quasi-random sequences in LSM to solve equity option values with up to 64 dimensions. He also uses Brownian bridge approximation to reduce the effective dimensions of the quasi-random sequences. My approach differs from Chaudhary (2005) in two aspects. First, I drop the Brownian bridge approximation and use a modified Sobol quasi-random sequence by discarding the first n = d + 100 points to reduce the high-dimensional clustering. Second, I apply the Sobol sequence in the LSM to solve default option value with up to 360 dimensions.

Figure 5.1 compares the two dimensional spaces covered by the modified Sobol quasi-random sequences and the pseudo random sequences of 1000 simulation paths. In this graph, the modified Sobol quasi-random sequences are used for dimensions (months) 359 and 360 and are generated by discarding the first 459 and 460 points of the original Sobol sequences. Figure 5.1 shows that the mod-



Figure 5.1: Space Coverage of Sobol and Pseudo Random Sequences

ified Sobol quasi-random sequences cover more spaces than the pseudo random sequences. In addition, the modified Sobol quasi-random sequences are more symmetric and less clustering, which are also the features of better convergence.

Figure 5.2 compares the convergence rates of the modified Sobol quasi-random sequence and the pseudo random sequence. It is clear that the default option value with the modified Sobol quasi-random sequence converges faster than the one with the pseudo random sequence. In addition, by using a sample of default option valuations, the estimated convergence rate of Sobol sequence is  $O(N^{-0.62})$  compared to the convergence rate of the pseudo random sequence at  $O(N^{-1/2})$ .



Figure 5.2: Convergence of Sobol and Pseudo Random Sequences

#### 5.3.2 An Adaptive LSM

The mortgage default option value decreases overtime because the monthly scheduled amortization gradually reduces the loan balance. A mortgage loan has little default value if it is deeply seasoned. This feature can be utilized to design an adaptive algorithm to reduce the number of remaining mortgage payment K required in the backward induction of the LSM.

The purpose of the adaptive LSM is to discard the months with little value in default option and essentially reduce the simulation horizon. In this method, the number of remaining mortgage payment K is reduced to t when  $t \leq K$  and the resulted default option value is close enough to the original default option value. In LSM the default option value is estimated through a series of least squares of the future exercise benefits. Thus I use the sum of the exercise benefits as a proxy to determine

	LSM		Adaptive	LSM		
Paths	Default	Speed	Default	Speed	Error	Efficiency
400	21576	167	21596	125	0.09%	25%
500	21235	215	21447	156	0.99%	27%
600	21916	263	22006	190	0.41%	28%
1000	21250	488	21328	353	0.37%	28%

Table 5.10: Performance of the Adaptive LSM

the stopping rule of the adaptive algorithm for searching the new K

$$K = t \text{ when } t \le K \text{ and } |(E_t - E_K)| < \delta,$$
(5.8)

where  $E_t = \sum_{i=1}^{t} \sum_{n=1}^{N} e_{i,n}$  is the sum of the immediate exercise benefits from month 1 to month *t* for all paths *N*,  $e_{i,n}$  is the immediate exercise value of month *i* for path *n*,  $\delta$  is the an error tolerance level.

Table 5.10 summarizes the performance of the adaptive LSM in the default option pricing. I find that when the number of paths  $N \ge 400$  and  $\delta = 0.01$ , the adaptive LSM achieves significant performance improvement in reducing the computational time by about 25% while keeping the relative error within  $\delta < 0.01$ .

By applying both the modified Sobol quasi-random sequence and an adaptive LSM algorithm, the computational performance of LSM in the mortgage default option evaluation is significantly improved. The numerical results of the applications of these computational methods to the default option valuation are discussed in the next chapter.

# **Chapter 6 Numerical Results**

In this chapter, the relationships of the parameters of the three-factor model and the mortgage default option value are studied through simulations. As discussed in the previous chapters, there are three groups of parameters that impact the default option value. The first group is a set of economic environment variables which includes the house price, the interest rate and the social stigma costs. The second group is a set of individual borrower's characteristics which determines the borrower's perception toward the moral costs. The third group is a set of mortgage and property specific attributes. The impact of these variables will be studied separately by controlling other variables.

# 6.1 Impact of the Economic Environment Variables

# 6.1.1 Impact of the Aggregate House Price

The aggregate house price is modeled through a jump-diffusion process. The impact of the aggregate house price to the default option value is studied through the parameters of this process. Table 6.1 shows the detail information of the base case used in this chapter.

Table 6.2 shows the impact of the service flow rate s and the volatility of the aggregate house price  $\sigma_H$ . As expected, the default option value increases when sincreases or  $\sigma_H$  increases. Figure 6.1 shows that the relationship between the default option value and s is a convex curve and for different  $\sigma_H$  the default value

Economic Environment	Borrower Information	Mortgage and Property
$r_0 = 0.035$	$\rho = 0.04$	Coupon = 0.035
s = 0.015	$g_0 = 15K$	Term = 360
$\sigma_H = 0.1$	$\gamma_g = 0.1$	LTV = 1.1
$\lambda = 0.05$	$b_g = 15K$	$h_0 = 300K$
$\varphi = 20$	$\sigma_g = 50$	$\Phi = 0$
$\phi = 0.96$	$MC_0 = 10K$	$\tau = 15$
$\sigma_h = 0.01$		$\mu = 0.002$
$\sigma_r = 0.005$		
$\theta(t) = 0.06$		

Table 6.1: Base Case

converges as the *s* decreases. It shows that the relationship between the default option value and  $\sigma_H$  is also a convex curve.

Table 6.3 shows that the impact of *s* in the diffusion process and the jump  $\lambda$  in the jump process. As expected,  $\lambda$  is positively correlated with the default option price.

Table 6.4 shows the impact of the shape parameter  $\varphi$  and the scale parameter  $\phi$  of the Weibull distribution in the jump model. As expected, the default option value is negatively correlated with both  $\phi$  and  $\varphi$ . However, the correlations between these parameters and the default option value are relatively low and the impact of these parameters to the default option value is low.

		$\sigma_H$				
s	Option	0.01	0.025	0.05	0.075	0.1
0	Default	16849	16899	17362	18076	19487
0	Prepay	127	144	351	743	915
0.015	Default	17489	17362	17862	19125	20819
	Prepay	55	134	248	503	918
0.00	Default	17951	17947	18777	20195	22645
0.03	Prepay	77	83	136	414	712
0.05	Default	18835	19238	20711	23073	25927
0.05	Prepay	15	52	78	383	920
0.08	Default	24590	25598	28034	30730	35361
	Prepay	18	95	205	630	908

Table 6.2: Impact of *s* and  $\sigma_H$ 





Figure 6.1: Impact of s and  $\sigma_H$ 

		$\lambda$					
s	Option	0	0.05	0.1	0.15	0.2	0.5
	Default	18545	19903	24099	24550	24647	25758
0.005	Prepay	968	908	1341	1049	1263	412
0.015	Default	19521	20819	24984	25019	25823	28029
	Prepay	904	918	1334	513	1401	493
0.03	Default	21340	22645	27199	27430	28316	30308
	Prepay	718	712	1452	642	1339	984
0.04	Default	22807	24126	28833	30603	31196	34580
0.04	Prepay	613	828	1305	505	1221	1182
0.05	Default	25181	25927	31272	33546	34188	36720
0.05	Prepay	555	920	1571	548	1370	1573

Table 6.3: Impact of s and  $\lambda$ 

Table 6.4: Impact of  $\varphi$  and  $\phi$ 

		$\phi$					
$\varphi$	Option	0.85	0.9	0.95	1	1.05	1.10
20	Default	18261	17955	17577	17164	17003	16940
20	Prepay	129	69	63	64	62	54
20	Default	18110	17596	17370	17067	16894	16899
30	Prepay	91	94	45	62	62	39
FO	Default	17946	17400	17109	16977	16793	16846
50	Prepay	89	87	45	62	69	54
76	Default	17851	17363	17065	16964	16793	16803
75	Prepay	90	72	46	54	69	54
100	Default	17805	17259	17043	16944	16793	16795
100	Prepay	90	71	46	54	69	54

### 6.1.2 Impact of the Individual House Value

The individual house value is modeled through an additional volatility factor to the aggregated house price as shown in Equation (4.14). Table 6.5 shows the impact of the initial house value  $h_0$  and the individual house price volatility  $\sigma_h$  to the default option value. As expected, the default option value increases as the volatility  $\sigma_h$  increases with a given  $h_0$ .

Figure 6.2 shows the relationship between  $h_0$  and the default option value. In the Yaxis, I use the default option value to  $h_0$  ratio to measure the relative default option values across different properties. Based on Equation (4.14), U-shaped curves are expected. In Figure 6.2, although the right hand side of these curves are similar to the right part of a U-shaped curve, the left hand side of the curves are increasing instead of decreasing. The explanation is that the significant social stigma cost and moral cost in this example reduced the default option value and the impact is more significant when both  $h_0$  and the default option value are low.

#### 6.1.3 Impact of the Risk-Free Interest Rate

The risk-free interest rate is modeled through a Hull-White no-arbitrage model as shown in Equation (4.19). There are three parameters: the initial shape of the forward rate curve  $\theta(t)$ , the reversion speed a, and the volatility of the short term rate  $\sigma_r$  in this model. In this dissertation, I limit the term structure of the initial forward curve to linear, which is determined by a slope parameter. Table 6.6 summarizes the impact of the slope of the initial forward rate curve and a. As expected, the slope of the initial forward curve is negatively correlated with the default option value. With a higher future risk-free interest rate, the present value

		$\sigma_h$					
$h_0$	Option	0.01	0.02	0.03	0.04	0.05	0.06
1001	Default	367	863	1779	3139	4827	7818
100K	Prepay	2517	2270	1941	1640	1589	1333
2001-	Default	7371	8642	10418	12103	13321	16187
200k	Prepay	1409	1101	855	874	1339	1470
300k	Default	17506	18001	19062	20813	22718	24797
	Prepay	63	47	152	171	377	450
4001-	Default	27164	29365	33959	38718	45382	54517
400K	Prepay	86	29	96	823	1465	2161
500k	Default	38178	46393	57593	75234	98671	122462
book	Prepay	28	145	1796	2739	1649	1070
6001	Default	49646	60932	80628	103117	133710	163259
600k	Prepay	37	510	1962	1769	1310	1292

Table 6.5: Impact of  $h_0$  and  $\sigma_h$ 

Default Option Values v.s. Idividual House Value



Figure 6.2: Impact of  $h_0$ 

		a				
Slope	Option	0.01	0.02	0.05	0.1	0.2
0.10/	default	4084	3497	2230	1149	372
0.1%	prepay	4	3	1	0	0
	default	9207	8420	6525	4309	2195
0.05%	prepay	18	16	9	3	0
0.0050/	default	12733	11896	9929	7636	5014
0.025%	prepay	56	55	55	18	2
0	default	17506	16702	14625	12126	9142
0	prepay	63	65	80	69	39
0.0050/	default	22236	21687	19415	17359	14999
-0.025%	prepay	188	155	184	237	334
-0.05%	default	27714	27111	26078	24044	22510
	prepay	468	442	471	689	1123
0.106	default	40153	40158	40147	40264	40905
-0.1%	prepay	1239	1483	1905	2313	2494

Table 6.6: Impact of Forward Curve Slope and a

of the mortgage payments will be lower, which reduces the value of the default option. The reversion speed *a* is also negatively correlated with the default option value. The correlation is more significant when the slope of the initial interest rate curve is positive. This is caused by a floor of zero of the risk-free interest rate. When the risk-free interest rate is lower, there is a higher chance that the interest rate will be floored at zero and the higher reversion speed will have less impact under this scenario.

Figure 6.3 shows the impact of the volatility  $\sigma_r$  to the default option value. As expected, the default option price is positively correlated to the volatility  $\sigma_r$ .



Figure 6.3: Impact of  $\sigma_r$ 

# 6.2 Impact of the Individual Borrower and Property Characteristics

# 6.2.1 Impact of the Social Stigma Cost

The social stigma cost is modeled through a mean-reversion CIR model as shown in Equation (4.24). There are three parameters in this model: the long term level of the social stigma cost b, the speed of the cost reversion  $\gamma$ , and the volatility of the social stigma cost  $\sigma_g$ . Table 6.7 shows the impact of b and  $\gamma$ . As expected, the default option value has a negative correlation with both b and  $\gamma$ . The correlation between default option value and  $\gamma$ , however, is less significant. Figure 6.4 shows the impact of b to the default option value is a convex function. For b less than \$40k, an increase of \$1 in the stigma cost approximately translates into a reduction of \$0.65

		$\gamma$					
b	Option	0.01	0.02	0.05	0.075	0.1	0.2
15K	Default	17482	17487	17489	17508	17506	17422
1010	Prepay	63	63	63	63	63	63
2017	Default	14844	14842	14839	14836	14834	14809
20K	Prepay	491	491	492	492	492	498
30K	Default	5782	5781	5676	5673	5669	5460
	Prepay	7251	7256	7380	7382	7382	7576
FOV	Default	733	730	450	446	426	425
JUK	Prepay	11669	11671	11905	11913	11921	11940
1002	Default	0	0	0	0	0	0
100K	Prepay	12532	12532	12532	12532	12532	12533

Table 6.7: Impact of b and  $\gamma$ 

in the default option value. For *b* bigger than \$40k, an increase of \$1 in the social stigma cost approximately translates into a reduction of \$0.03 in the default option value. This finding is very useful in determining the appropriate social stigma cost for policy makers to balance social stigma costs for different sectors of the society. Figure 6.4 also shows that  $\gamma$  has little impact to the default option value.

## 6.2.2 Impact of the Average Moral Cost

The borrower's view toward moral cost is modeled through a simple normal distribution as shown in Equation (4.25). Table 6.8 shows the impact of this model to the default option value. As expected, the initial moral cost  $MC_0$  is negatively correlated with default option value and the volatility of the moral cost  $\sigma_m$  is positively correlated with the default option value. The reduction of the default option value



Figure 6.4: Impact of *b* 

caused by increased initial moral cost  $MC_0$  is similar to the effect of increased social stigma cost.

# 6.2.3 Impact of the Deficiency Judgment Cost

The deficiency judgment cost is modeled through Equation (4.23). The most important parameter in the deficiency judgment model is the house value D that is used by lender to determine if to pursue a deficiency judgment. Figure 6.5 shows that the impact of D to the default option value is a concave function. It is clear that the lender has little benefit by reducing the deficiency judgment threshold from \$1 million to \$400k. And if the lender uses a house value less than \$300k (the average house price used in this example) as the deficiency judgment threshold, the default option value will be significantly reduced.

		$0_m$					
$\mathrm{MC}_{0}$	Option	0	100	500	1000	2000	3000
10k	Default	17506	17441	17400	17397	17419	17504
	Prepay	63	63	58	58	65	65
25K	Default	4371	4383	4607	4634	5712	6712
	Prepay	8387	8387	8233	8237	7375	6605
50K	Default	62	63	64	65	378	390
	Prepay	12293	12293	12290	12296	12023	12049
75K	Default	0	0	0	0	0	0
	Prepay	12497	12497	12497	12493	12496	12499
100K	Default	0	0	0	0	0	0
	Prepay	12537	12537	12537	12537	12537	12535

Table 6.8: Impact of  $MC_0$  and  $\sigma_m$ 

Default Option Value v.s. Average Price for Deficiency Judgment



Figure 6.5: Impact of Deficiency Judgment Factor to the Default Option Value

τ	Option	0.05%	0.1%	0.2%	0.3%	0.5%	1%
6	Default	12587	13504	14577	15601	17506	21645
	Prepay	1735	1099	520	265	63	34
12	Default	13504	14577	16524	18328	21645	31772
	Prepay	1099	520	108	55	34	4
18	Default	14224	15601	18328	20799	26426	44046
	Prepay	640	265	55	40	7	3
24	Default	14577	16524	19975	23462	31772	58135
	Prepay	520	108	40	13	4	0
36	Default	15601	18328	23462	29476	44046	89170
	Prepay	265	55	13	3	3	0

Table 6.9: Impact of Foreclosure Lag and Rent-House Price Ratio

 $\mu$ 

# 6.2.4 Impact of the Rental Benefit

The rental benefit of a mortgage default option is a pecuniary benefit that the borrower enjoys during the foreclosure period. In this dissertation, the rental benefit is modeled through Equation (4.26). There are two parameters in this model: the number of months between the default month and the foreclosure month  $\tau$  and the rent-house value ratio  $\mu$ . As expected, the default option value is positively correlated with both  $\tau$  and  $\mu$ . And the default option value increases significantly with  $\tau$ . This finding suggests that by reducing the average time lag between the default month and the foreclosure month, the mortgage lender not only reduces the foreclosure loss in the traditional loss mitigation view but reduces the financial incentives of the strategic default.

# 6.3 Impact of the Mortgage Loan Attributes

# 6.3.1 Impact of the Loan to Value (LTV) Ratio

Loan to value ratio (LTV), is defined as the ratio of the mortgage unpaid balance to the underlying house price. This ratio is widely used in the mortgage finance industry to measure the borrower's equity and assess the mortgage default risk. In this section, I study the impact of the LTV ratios to the default option values.

Table 6.10 shows that the default option price is positively correlated with LTV or negatively correlated with the negative equity. It also shows the cancellation effect of the two options – the higher default option value the lower the value of the prepayment option, and vice versa. Another notable finding is that the default option value is significant (\$13,440) even when the negative equity is only at 7.5% level (e.g. LTV = 1.075), which indicates that the borrower may choose to default at a much lower threshold than previously believed. Figure 6.6 shows the cancellation effects between default option value and prepayment option value.

# 6.3.2 Impact of the Coupon Rate

The borrower's coupon rate is another important loan attribute that impacts the option values. For a fixed rate mortgage, it directly determines the level of the periodic payment and the values of the prepayment option and the default option. A mortgage loan with a higher coupon rate will have a higher present value of the future mortgage payments. This leads to a higher strike price that is the present value of these future payments, thus a higher value in the default option.

$UPB_0$	$LTV_0$	Prepay Value	Default Value
270k	0.9	10241	0
285k	0.95	10513	334
300k	1	10594	895
315k	1.05	8263	4152
322.5k	1.075	997	13440
330k	1.1	134	17471
345k	1.15	46	25392
360k	1.2	15	34754
450k	1.5	2	104892

Table 6.10: Default Option Values with Different LTVs

Prepay and Default Values of Different LTVs



Figure 6.6: Cancellation Effect of Default Option and Prepayment Option

In this section, I study the impact of mortgage coupon rate to the values of the mortgage options. The coupon ranges from 1% to 6%. Since the level of equity (LTV) will determine if default option or prepayment option dominates the overall mortgage option value, I control LTV to study the impact of coupon to the values of the options within each equity level.

Table 6.11 shows the detailed comparison of the default option values with different coupon rates for each LTV bucket. When the borrower has a non-negative equity (LTV = 0.9 or 1.0), the prepayment option has a higher value than the default option. As expected, the value of prepayment option increases as the coupon rate increases. Another finding under this scenario is that the changes of coupon rate have very limited impact to the value of the default option because of the dominance of the prepayment option.

When the borrower has negative equity (LTV>1.0), the default option becomes more valuable than the prepayment option because the default benefit dominates the prepayment benefit for most of the simulated economic scenarios. Even when the negative equity is as low as -10%, the default option value increases significantly with the coupon rate. Figure 6.7 shows a clear transition of the prepayment option value and the default option value when LTV increases from 0.9 to 1.2.

# 6.3.3 Impact of the Loan Age

The loan age is another loan attribute that impacts the default option value. Schultz, Flanagan and Muth (2005) find that the default rate peaks at month 40. I find a similar pattern through simulations. Figure 6.8 shows the distribution of the default option value associated with each of the loan age buckets. It is clear that most of the

		Coupon						
$LTV_0$	Option	0.01	0.02	0.03	0.035	0.04	0.05	0.06
0.0	Default	0	0	0	0	0	0	0
0.9	Prepay	1	341	4396	9839	17951	40247	67626
10	Default	0	0	263	455	723	1015	2500
1.0	Prepay	1	411	4901	10637	18953	41490	68263
1.05	Default	0	257	1971	4267	6784	16719	32378
	Prepay	1	364	3716	7943	14498	30104	47133
1 1	Default	30	913	8539	17506	28824	60332	99504
1.1	Prepay	1	32	80	63	216	1227	3442
1.2	Default	201	4523	21530	34918	50818	90540	133474
	Prepay	0	5	3	5	45	213	1764

Table 6.11: Default and Prepay Values with Different Coupons



Figure 6.7: Default and Prepayment Values with Different LTVs and Coupons



Figure 6.8: Marginal Effect of the Loan Age

default option value is concentrated in the age buckets of the first 4-5 years. There is little opportunity to gain financially by exercising the default option beyond the time bucket of 7 years.

# **Chapter 7 Applications to the Loan Modifications**

This chapter provides a special application of the proposed three-factor default option valuation framework to the loan modifications. I start with an overview of the existing loan modification methods and the development of the government sponsored loan modification programs. Then two major existing loan modification methods are compared quantitatively. At last, I make an attempt to create an unified and parametrized modification framework for different modification methods.

# 7.1 An Overview of the Loan Modification Methods

A loan modification is "a process where the terms of a mortgage are modified outside the original terms of the contract agreed to by the lender and borrower".<sup>15</sup> Any modification made to the original mortgage contract can be defined as a loan modification. In this dissertation, a loan modification is referred to a modification that is used to prevent borrowers from defaulting. The loan modification has long been used by the mortgage lenders to modify the original mortgage contracts in order to keep the borrowers from defaulting and foreclosure.

Lenders are motivated by the expectation that the value of a performing loan with new terms is higher than the sale proceedings from a foreclosed property with a distressed value due to lack of maintenance. The purpose of the loan modification is to reduce the default option value and keep the borrowers from defaulting.

<sup>&</sup>lt;sup>15</sup>http://en.wikipedia.org/wiki/Mortgage\_modification

An interesting development since 2009 in the mortgage finance industry is the involvement of the government in loan modifications, which brought lots of highlights into this otherwise little known sector. In an effort to recover the housing market from the unprecedented house price crash and skyrocketing foreclosure since 2007, United States Department of Treasury initiated the Making Home Affordable Program (MHAP) in March 2009 by providing financial assistance to the borrowers with difficulties of keeping up their mortgage payments. MHAP contains a few initiatives: Home Affordable Refinance Program (HARP), Home Affordable Modification Program (HAMP) and Home Affordable Foreclosure Alternatives Program (HAFAP).

The current literature and practice classify the loan modification methods into two main categories: the payment reduction loan modification and the equity sharing loan modification.

# 7.1.1 The Payment Reduction Loan Modification

The focus of the payment reduction modification is to reduce the borrower's monthly mortgage payment by changing one or more terms of the original mortgage contract. The supporters of this modification method believe that the fundamental driver of the mortgage default is the solvency problem of the borrower and by addressing the borrower's cash flow problem the default risk will be eliminated. The most notable payment reduction modification program is HAMP.

HAMP is regarded as the most important initiative among the MHAP programs as it is designed to directly help most of the distressed borrowers. HAMP modifies the eligible borrowers' mortgage contracts in order to make their monthly payments to an affordable level, which is determined by a target 31% of debt to income (DTI) ratio. The DTI ratio is calculated by dividing the borrower's monthly mortgage payment (including all required tax and fees) with the borrower's monthly income. In order to reduce the monthly mortgage payment, HAMP modifies a few key terms of the original contract, which includes modifying an ARM into a FRM with a lower rate, extending the mortgage to a longer term product i.e. 40 years, and forbearing a portion of the principal of the mortgage to the maturity.

I use an example to show how HAMP works. Consider a borrower who pays 6% coupon for a fixed rate mortgage of \$500,000 on a \$400,000 house. The mortgage has 25 years as the remaining term. Based on the amortization formula of Equation (3.4), the current monthly payment is about \$3,222. And I assume that the borrower can no longer afford this monthly payment and turns to HAMP for help. In order to keep this borrower from defaulting, HAMP first uses the borrower's monthly income to determine the affordable monthly payment, in this case, say \$2,400. HAMP then modifies the loan into a lower rate mortgage with an extended term while keeps the balance intact at \$500,000. The modified loan has a new fixed rate of 4.05% and a new loan term of 30 years. Under this contract, the borrower's new monthly fixed payment is reduced to \$2,400, a 25% relief from the current monthly payment.

Since its initiation in 2009, HAMP has modified more than 1.5 million mortgages and about 590k modifications are active and performing.<sup>16</sup> On average, the median payment reduction for a HMAMP modification is 37% or about \$500 per month.<sup>17</sup>

<sup>&</sup>lt;sup>16</sup>http://www.treasury.gov/initiatives/financial-stability/results/MHA-

Reports/Documents/March%202011%20MHA%20Report%20FINAL.PDF

<sup>&</sup>lt;sup>17</sup>http://www.makinghomeaffordable.gov/news/latest/Pages/Obama-Administration-Releases-April-Housing-Scorecard.aspx

## 7.1.2 The Equity Sharing Loan Modification Program

Posner and Zingales (2009) and Goodman (2010) separately propose an equity sharing loan modification method as an alternative to the payment reduction modification method. Their research claims that because the payment reduction modifications such as HAMP only focus on the reduction of the borrower's monthly payment, the borrower's negative equity, which is regarded as the most important driver of the strategic default, is ignored. In their proposal, if a borrower's mortgage is under water, the principal will be reduced to the estimated market house value to eliminate the negative equity. Meanwhile, the lender will be compensated by a portion of the future appreciation of the underlying house.

I use the same example in Section 7.1.1 to illustrate how the equity sharing modification works. Under the equity sharing modification method, the borrower's mortgage balance is reduced from \$500,000 to the estimated market house value at \$400,000. Meanwhile, the new mortgage contract gives the lender a portion, say 50%, of the future appreciation of the house value. The modified mortgage rate will stay at 6%. Based on the amortization formula of Equation (3.4), the modified monthly payment is \$2,400, the same monthly payment as the payment reduction modification in the previous section.

In addition, Posner and Zingales (2009) propose six pragmatic criteria to evaluate a loan modification program:

- 1. The program should only help borrowers that can afford the modified mortgage payments.
- 2. The program should be easy to borrowers to understand and easy to lenders to initiate.

- 3. The program should make both lenders and borrowers better off.
- 4. The program should "minimize the negative long term effects on the credit market" and should not decrease future mortgage credit availability.
- 5. The program should "minimize the burden to taxpayers".
- 6. The program should be fair from both moral and economic perspective.

They evaluate the HAMP program based on these criteria and determine that HAMP does not meet criteria 1, 4, and 5. In the same research, they examine the equity sharing method as well and claim that the equity sharing method meets all the six criteria above and is a better method than the payment reduction modification. They argue that the equity sharing modification generates the best results by keeping both the borrowers and the lenders better off financially and preserving the overall economic wealth of the society.

# 7.1.3 Comparison of the loan Modification Methods

The claims of Posner and Zingales (2009) are mainly supported by qualitative arguments from the public policy perspective. And there is little quantitative result to support the claims in their research. In this section, I quantitatively compare the equity sharing modification with the payment reduction modification under the three-factor default option valuation framework.

The borrower's utility function includes two utility inflow components: the default option value and the present value of the expected future house price. It also contains a utility outflow component: the present value of the future monthly mortgage payments. The utility functions  $U_e$  and  $U_p$  of the equity sharing modification and payment reduction modification can be defined as:

$$U_e = F_e - K_e - \max(0, \frac{h_T - S(h_T - h_0)}{d_T})$$
(7.1)

$$U_p = F_p - K_p + \frac{h_T}{d_T} \tag{7.2}$$

where *S* is the lender's share of the equity appreciation under the equity sharing program,  $d_t = \exp\left(-\int_0^t r(t) dt\right)$  is the discount factor of the month *t* and  $d_T$  is the discount factor of the loan expiration month *T*,  $h_0$  is the current house value and  $h_T$  is the expected house value at maturity month *T*,  $F_e$  is the default option value under the equity sharing modification,  $F_p$  is the default option value under the payment reduction modification,  $K_e = \sum_{t=0}^{T} d_t P_e$  is the present value of the future monthly payments of the equity sharing modification,  $K_p = \sum_{t=0}^{T} d_t P_p$  is the present value of the payment reduction modification respectively. The max $(0, \frac{h_T - S(h_T - h_0)}{d_T})$  function reflects the fact that under the equity sharing modification lenders do not share the depreciation of the house value with borrowers.

In order to fairly compare these two modifications, I make the following assumptions.

- Assumption 1: the borrower can choose one of the two modifications to maximize his utility function.
- Assumption 2: the borrower and the lender have the same expectation of the service flow rate.

- Assumption 3: the lender does not get financial support from the taxpayers for loan modifications.
- Assumption 4: no additional loan modification will be allowed after the first modification and the borrower is not allowed to prepay his modified mort-gage.
- Assumption 5: if the loan is modified under the equity sharing modification, the value of the house will be evaluated at the end of the loan term and the lender will claim his share of the house value appreciation.

If I assume  $P_e = P_p$  for all the future months of the loan term, it is obvious that the present value of the two payments are the same  $K_e = K_p$ . And according to Section 5.1, the determinants of the default option value F are the house price h and the present value of the future mortgage payments K (the cost factors are the same as the borrower of both modifications programs is the same person), the default option values of the two modified loans are the same  $F_e = F_p$ .

Because the difference between the utility functions of the two modification programs:  $U_e - U_p = \min(0, \frac{-S(h_T - h_0)}{d_T})$ , if the borrower expects a future house price appreciation (service flow is less than the risk-free interest rate),  $\min(0, \frac{-S(H_T - H_0)}{d_T}) < 0$ and  $U_e < U_p$ . And if the borrower expects the house price does not appreciate,  $\min(0, \frac{-S(H_T - H_0)}{d_T}) = 0$  and  $U_e = U_p$ . Thus for any future house appreciation scenario,  $U_e \leq U_p$  always holds when monthly payments of the two modifications are equal.

This conclusion indicates that if the monthly payments of the two programs are the same, from the borrower's perspective, the equity sharing program will provide less

utility and the borrower will not choose the equity sharing modification program. And from the lender's perspective, although the equity sharing program provides a better expected payoff, it does not attract the borrowers to participate at all nor it reduces the borrower's incentive to default. In this case, contrary to the claims of Posner and Zingales (2009), the equity sharing modification is less effective than the payment reduction modification from the perspectives of both the borrower and the lender.

The equity sharing program, however, becomes attractive when its monthly payment is less than the monthly payment of the payment reduction method. When the monthly payments are not the same, the utility difference between  $U_e$  and  $U_p$ becomes  $(F_e - F_p) - (K_e - K_p) - \max(0, \frac{S(h_T - h_0)}{d_T})$ . I set the utility difference to 0, it becomes an equation

$$\max(0, \frac{S(h_T - h_0)}{d_T}) = (F_e - K_e) - (F_p - K_p)$$
(7.3)

Since the lender will not initiate the equity sharing modification if the house value is expected to depreciate, I assume  $h_T > h_0$ , the equation becomes

$$\frac{S(h_T - h_0)}{d_T} = (K_p - F_p) - (K_e - F_e)$$
(7.4)

Equation (7.3) indicates that the additional utility required by the borrower to choose the equity sharing program can be achieved by reducing the payments or increasing the default option value.

I use the same example in Section 7.1.1 to show how the borrower's utility function can be equalized between the two loan modification programs. The basic information of the original loan term and the economic environment information are sum-

Economic Environment	Borrower Information	Mortgage and Property
$r_0 = 0.04$	$\mu = 0.002$	Coupon = 0.06
s = 0.0166	$\rho = 0.04$	Term = 360
$\sigma_H = 0.1$	$g_0 = 15K$	LTV = 1.25
$\lambda = 0$	$\gamma_g = 0.1$	$h_0 = 400K$
$\sigma_h = 0.01$	$b_g = 15K$	$\tau = 15$
$\sigma_r = 0$	$\sigma_g = 50$	$\Phi = 0$
$\theta(t) \equiv 0.04$	$\mathrm{MC}_0 = 10K$	

Table 7.1: Base Case of the Modified Loan

marized in Table 7.1. I define  $D_u = (K_p - F_p) - (K_e - F_e)$ . When  $P_e = P_p = 2400$ ,  $(K_e - F_e) = (K_p - F_p)$  and  $D_u = 0$ . If both the borrower and the lender expect \$400,000 or 100% of house value appreciation ( $\exp(r - s) = 0.0234$ ), the expected payoff to the lender from the house appreciation  $\frac{S(h_T - h_0)}{d_T}$  is \$99,463. Table 7.2 summarizes the changes of the borrower's utility under the equity sharing modification when the monthly payment is decreased from \$2,400 (Coupon = 6\%) to \$1,554 (Coupon = 2.36\%).

When the monthly payment of equity sharing modification is reduced to \$1,864 (Coupon = 3.8%), the borrower's utility of the equity sharing modification is equal to the utility of the payment reduction modification. From the lender's perspective, the equity sharing modification significantly reduces the default option value from \$74,778 to \$15,182 comparing to the payment reduction modification. Although the lender's expected utility paid to the borrower is the same for the two modifications, the equity sharing modification reduces the borrower's incentive to default and will significantly mitigate the loss associated with default and foreclosure. Under this condition, when the borrower's utility function is equalized by lowering the
Coupon	$P_e$	$K_e$	$F_e$	$K_e - F_e$	$P_p$	$K_p - F_p$	$D_u$	S
6.00%	2400	492312	74213	418099	2400	417207	0	0.0%
5.70%	2322	476587	64119	412468	2400	417207	4739	4.8%
5.50%	2271	466232	57804	408428	2400	417207	8779	8.8%
5.00%	2147	440803	42455	398349	2400	417207	18859	19.0%
4.50%	2027	416057	29830	386228	2400	417207	30979	31.1%
4.00%	1910	392023	18855	373168	2400	417207	44039	44.3%
3.80%	1864	382614	15182	367432	2400	417207	49775	50.0%
3.50%	1796	368727	10553	358174	2400	417207	59033	59.4%
3.00%	1686	346194	4738	341456	2400	417207	75751	76.2%
2.75%	1633	335221	2725	332496	2400	417207	84711	85.2%
2.50%	1580	324448	1380	323068	2400	417207	94139	94.6%
2.36%	1552	318672	927	317744	2400	417207	99463	100.0%

Table 7.2: Borrower's Utility of Equity Sharing Modification

monthly payment of the equity sharing modification, the equity sharing modification is a better choice to both the lender and the overall wealth of the society.

If I allow the lender's equity share S in Equation (7.3) to vary, the equity modification provides more flexibilities to the borrower. Table 7.2 shows how the changes of S can provide the same utility to the borrower. If the borrower would like to take a larger share of the future house price appreciation (or the lender takes a small S), for example 81%, the lender will charge a higher rate (5.00%) in order to keep the borrower's total utility the same. If the borrower would like to take a lower rate, for example 3.00%, the lender will ask for 76.2% of the future appreciation for return.

### 7.2 A Parametrized Modification Framework

The process of attracting borrowers to participate into the loan modification program is similar to the fundamental idea of the hypothesis testing. I define the Type A error of a loan modification program as the percentage of the eligible borrowers that don't participate into the program and the Type B error as the percentage of the non-eligible borrowers actually participate into the program. If I control the Type A error, a good loan modification program should have a relatively small Type B error.

The purpose of reducing the Type B error is to reduce the overall cost of the modification program by lowering the default rate of the participants of the loan modifications. These defaulted participants should be considered as non-eligible borrowers. In practice, however, it is difficult to distinguish the non-eligible borrowers from the eligible borrowers. If the income level (as used in HAMP) is used to control the Type B error, in addition to the six criteria proposed by Posner and Zingales (2009), another important criteria for a successful modification program is to provide different solutions to different borrowers who may have very different financial circumstances and different expectations of the future economic environment to reduce the Type A error and improve the efficiency of the modification program.

I propose a unified parametrized modification framework. In this framework, the coupon rate, the loan term, and the lender's share of equity are all configurable parameters. The traditional payment reduction modification method and the equity sharing modification method become instances of this framework. The modification parameters are configured such that the borrower's utility for each of the configuration is the same under a given expected future house value expectation. I use an

equation to generalize this framework:

$$U(C_0, T_0, S_0) \mid \mathcal{E}(H_T) = U(C_i, T_i, S_i) \mid \mathcal{E}(H_T) = \ldots = U(C_n, T_n, S_n) \mid \mathcal{E}(H_T) ,$$
(7.5)

where U is the expected borrower's utility function,  $C_i$ ,  $T_i$ , and  $S_i$  are the modified coupon rate, loan term, and lender's share of appreciation of the  $i^{th}$  modification configuration respectively.

The equation is conditional on a given expected future house value  $E(H_T)$ . In the example of Section 7.1.3, the utility function of the equity sharing modification is defined as U(3.80%, 360, 50%) | 800K. The utility function of the payment reduction modification is defined as U(4.05%, 360, 0%) | 800K. When the house price appreciates to exact \$800,000 or the house price actually depreciates, U(3.80%, 360, 50%) | 800K = U(4.05%, 360, 0%) | 800K. Under this framework, the borrower may choose any of the parameter configurations. Figure 7.1 shows the curve of equivalent borrower's utility under future house value at \$800,000. On this curve, every point represents a configuration of lender's equity share *s* and borrower's coupon rate, which has the same expected borrower's utility with other points on this curve.

This flexible equity sharing framework expands the base population of the eligible borrowers. Under this program, the monthly payment can be reduced to an affordable level which is lower than the lowest monthly payment that the payment reduction modification method can modify into. In the same example above, assuming a 70% recovery rate from the foreclosure and disposition, the lender expects to recover \$280,000 from the foreclosure process and has to record a \$220,000 loss – \$100,000 due to the market house value depreciation and \$120,000 due to the



Figure 7.1: Equivalent Borrower's Utility Curve

additional depreciation from foreclosure.

Under the payment reduction modification, the lender will not modify the loan with a monthly payment less than \$1,665 since otherwise the present value of the payments will be less than the proceedings from the disposition of the house. If a borrower is determined that cannot afford the \$1,665 monthly payment, he will be excluded from the payment reduction modification. However, under this improved equity sharing modification framework, the borrower's mortgage can be modified with a new monthly payment less than \$1,665. For example, under the equity sharing program, a borrower can pay a monthly payment \$1,633 (Coupon=2.75%) and compensates the lender with a 85.2% share of the future house appreciation.

The modification framework, however, has its own limitations. Figure 7.1 shows

that even the lender claims 100% of the future house value appreciation, the modified coupon rate can only be reduced to 2.36% and thus the eligibility of the modification program is also limited.

In addition, this modification framework may improve the participation rate of the eligible borrowers. Under this framework, the borrowers have extra incentives to participate into the modification if he has an expectation of the future house value that is different with the market expectation. In the risk-neutral framework with a constant risk-free interest rate as in the base case, the expectation of the service flow *s* determines expected house value. For example, when s = 0.0027 and  $r_0 - s = 0.0373$ ,  $\exp(h_t) = 1.2$  million. In the base case, if the borrower expects the house value appreciation is more than 50%, he will choose the modification which has the lowest lender's equity share and highest monthly payment that he can afford. And if the borrower expects the house value appreciation is less than 50%, he will choose the modification which has the lowest payment and highest lender's equity shares. In both cases, the borrower's expected utility function is maximized.

Table 7.1 shows the borrower's utilities under different coupon and *S* combinations when the borrower's expectation is different with the market expectations. It is clear that the best strategy for a borrower to achieve better utility is to reduce the lender's share of equity when he expects the house value appreciation is more than market expectation; and increase the lender's share of equity when he expects the house the house value appreciation is less than the market expectation.

In summary, this flexible modification framework may expand the population of the modification eligible borrowers and improve the participation rate of the eligible borrowers. In addition, it inherits the benefit of the traditional equity sharing

Coupon	S	$U \mid 600K$	$U \mid 800K$	$U \mid 1.2M$
6.00%	0.0%	-268905	-219173	-119711
5.70%	4.8%	-265639	-218276	-123549
5.50%	8.8%	-263623	-218282	-127598
5.00%	19.0%	-258584	-218282	-137678
4.50%	31.1%	-252523	-218282	-149798
4.00%	44.3%	-245994	-218282	-162858
3.80%	50.0%	-243108	-218258	-168558
3.50%	59.4%	-238496	-218282	-177852
3.00%	76.2%	-230137	-218282	-194570
2.75%	85.2%	-225646	-218266	-203506
2.50%	94.6%	-220943	-218282	-212958
2.36%	100.0%	-218268	-218264	-218255

 Table 7.3: Borrower's Utility Under Different House Value Expectations

method in reducing the borrower's incentive to default while keeping the borrower's utility the same.

# **Chapter 8 Conclusions**

### 8.1 Contributions

This dissertation developed a three-factor mortgage default option valuation framework that extended the existing two-factor mortgage default models by introducing the net transaction cost as a new underlying stochastic factor. This dissertation also introduced the LSM numerical method into the mortgage default option research and made specific improvements to the computational performance of LSM for evaluating mortgage default options. In particular, there are four major contributions of this dissertation to the mortgage default option research.

- The three-factor model explicitly incorporates the net transaction cost as an additional stochastic factor to the conventional two-factor model. Most of the previous research does not consider the transaction cost, and when it is included in the analysis, it is considered as a fixed cost (for example, Foote, Gerardi, and Willen, 2008, and Krainer, LeRoy, and O, 2009). The stochastic transaction cost model in this dissertation provides additional flexibility in modeling and explaining borrowers' default behaviors.
- This dissertation introduced the LSM method into the mortgage default option research. Compared to the conventional numerical methods such as bivariate-binomial lattice method and finite different method which are currently used in the mortgage default research (for example, Sharp and Newton, 2008, and

Ambrose and Buttimer, 2011), the LSM method expands the numerical solution of the default option value to the multi-factor general stochastic processes, rather than one-factor or two-factor geometric Brownian processes required by the conventional numerical methods.

- This dissertation expanded the underlying stochastic models of the house price and the interest rate used in the conventional two-factor model. Compared to the geometric Brownian motion model and the equilibrium CIR type model used in the previous literature such as Downing, Stanton, and Wallance (2005) and Ambrose and Buttimer (2011), a jump-diffusion house price model and a no-arbitrage interest rate model are used by this dissertation to better capture the complex dynamics of the house price and the interest rate such as price jumps and empirical market term structure.
- This dissertation introduced and studied two modifications of the LSM method to its computational performance. One modification was the use of a quasirandom number generator instead of a pseudo-random generator. The simulated paths using quasi-random number are symmetric and cover more spaces, which have a better convergence rate in the LSM method. A second modification was to make the method adaptive. Based on the unique feature of the default option, this adaptive LSM with specific simulation paths and error tolerance settings was introduced to reduce the simulation horizon of the LSM method and improve the computational performance of the LSM in the mortgage default option valuation.

In addition, this dissertation built a unified parametrized loan modification framework and made contributions to the loan modification research in at least three respects:

- The unified parametric loan modification framework generalizes the existing loan modification methods in the previous literature such as Foote, Gerardi, and Willen (2008) and Posner and Zingales (2009). The payment reduction modification and the equity sharing loan modification, two distinct loan modification methods in the previous literature, become two instances of this unified loan modification framework.
- This parametric loan modification framework extends the existing loan modification methods by allowing modifications of additional mortgage terms, rather than modifications limited to the mortgage payment and the loan balance in the previous literature such as Posner and Zingales (2009) and Ambrose and Buttimer (2011).
- Unlike the previous literature such as Posner and Zingales (2009) and Goodman (2010), this loan modification framework sets up a clear borrower's utility maximization function to evaluate different modifications. This function is then used to define the modification terms of the optimal modification method under the unified framework.

The model and the methodology developed in this dissertation were used in a numerical study of various factors related to mortgage defaults. This research concluded with the following useful findings:

• The shape and scale parameters of the Weibull distribution of the house price jumps have little impact to the default option value.

- The reduction of default option value by increasing the social stigma cost and moral cost is less effective when the social stigma cost and moral constraint cost are already high.
- By lowering the threshold of deficiency judgment, the default option value is significantly reduced.
- By reducing the average time lag between default month and foreclosure month, mortgage lenders not only could reduce foreclosure expenditures, but could also reduce the default option value and prevent strategic defaults.
- A small amount of negative equity could generate a significant value in the default option and motivate the borrower to default.
- Contrary to the belief that the equity sharing modification is more effective than the payment reduction modification, the equity sharing modification is actually less effective when the borrower's modified monthly payments of both modifications are the same.
- The equity sharing modification becomes more effective only when it offers an equivalent borrower's utility and a lower monthly payment.

## 8.2 Future Work

Like all other research, there are more topics that could be covered in this dissertation. The work presented in this dissertation may be extended in a few directions.

There are a few improvements that can be made in the three-factor model. The individual house price is modeled through the ratio of the individual house price to

the aggregate house price. It can be modeled more accurately if additional information of the individual house hedonic properties is made available. The moral cost is currently approximated through a normal distribution, an empirical study of the distribution of the moral cost may be considered. The relationship between moral level and the wealth of the individual could be explored. It will be most helpful if a moral score system is established with the individual person's credit and behavior history to provide a more accurate assessment of the moral level. As mentioned in Chapter 4, the mortgage default is impacted by a contagious effect if there was a default in the neighborhood. Further extension of the model to the contagious effect could be explored as well.

The efficiency of the LSM method can be further improved by comparing different orthonormal basis functions with different number of basis functions.

This default option pricing framework can also be applied to the default option valuation of different mortgage product types including ARMs, Hybrid ARMs, Option ARMs, and step rate mortgages. A numerical application of the flexible modification framework to other modification attributes like the forbearance percentage and the loan term may also be considered. In addition, the share of the lender's equity in the underlying house may worth further research to achieve the optimal house value appreciation. Bibliography

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# CURRICULUM VITAE

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