VARIATIONAL AND QUASI-VARIATIONAL PROBLEMS WITH GRADIENT CONSTRAINTS AND APPLICATIONS TO SANDPILE GROWTH

by

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Mathematics

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with Gradient Constraints

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Dedication

To Hodaja, Rebekka, and Tabea

Acknowledgments

I am very grateful for the guidance and support of my advisor Dr. Carlos Rautenberg, who made it possible for me to accomplish this academic project. He is an inspiring and patient teacher, naturally conveying the excitement for concepts, theorems, and proofs. I have enjoyed his kind and motivating encouragement in areas directly and indirectly connected to this scientific endavour.

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Abstract

VARIATIONAL AND QUASI-VARIATIONAL PROBLEMS
WITH GRADIENT CONSTRAINTS

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In this dissertation, we study variational inequalities (VIs) and quasi-variational inequalities (QVIs) with gradient constraints in diffusive and non-diffusive settings together with several related problems. Specifically, we consider evolutionary, as well as stationary versions of the aforementioned problems, and address existence and uniqueness of solutions, differentiability properties in the context of optimal control, and rigorous Fenchel dualization approaches under low regularity of the data.

Initially, we address features of the prototypical problem under study: The Prigozhin model of sandpile growth. In particular, we establish an illustrating example and show closed forms for its multiple solutions and prove that the elementary regularization of constraints leads to uniqueness.

On the class of problems arising from the semi-discretization of the evolutionary version, we study existence of solutions under low regularity assumptions; we analyze the cases where the bounds of the gradient constraints are non-negative integrable functions, and also Borel measures. In the latter, we identify new mathematical tools for the application of the direct method. A complete characterization of the Fenchel pre-dual problem leads to the study of minimization problems in a non-standard state space given by vectorial Borel measures with

square integrable divergences. The duality description is then exploited for the development of a primal-dual solution algorithm and numerical tests are shown.

For a stationary problem formulation which includes a diffusive operator, we provide novel results on the Newton differentiability of the control-to-state map. This is of interest in the investigation of sensitivity features and optimal control. In this framework, the control is the material source term and the state corresponds to the stationary shape of the material pile. The mathematical enabling tool here is a new implicit function theorem for Newton differentiable maps.

In the evolutionary sandpile growth setting, an optimal control problem with a QVI as constraint, is considered. The main goal in this problem is to keep a part of the domain free of material accumulation by controlling the initial supporting surface. We consider fully discrete and semi-discrete approaches for this problem and provide an existence of solutions result.

Chapter 1: Introduction

An increasing number of significant problems in applied sciences involve partial differential operators as well as constraints on the first order derivatives of the state variable, thus leading to nonsmooth distributed parameter systems. A large class of problems of this type relate variational principles (or energy minimization) together with a known or unknown bound on the norm of the gradient of the state variable. In this setting, when the constraint is known a priori, the resulting mathematical problem is in general a *variational inequality* (VI). However, and in contrast, in many of the gradient constrained problems, the upper bound of the gradient constraint depends also on the state variable itself. This additional complexity in the form of an *implicit* constraint results in what is called a *quasi-variational inequality* (QVI).

The simplest situation where gradient constraints are found is within the elastoplastic torsion of a cylindrical body. Here, the description of the stress variable u in the cross section of the body divides the material domain into an elastic $\{x : |\nabla u| < \alpha\}$ and a plastic $\{x : |\nabla u| = \alpha\}$ region where α corresponds to the limit plasticity threshold. In the elastic region, the variable u further requires a constitutive law associated to elasticity, and this leads to a variational inequality. However more complex situations may arise that may lead to a quasi-variational formulation: In the elastoplastic setting thus we may consider the constraint $\{x : |\nabla u| \leq \alpha(u)\}$, where the plasticity threshold is also dependent on the state variable: This is common when considering that the temperature of the material is not uniform, it is further dependent on the deformation of the material, and that α is temperature dependent.

Variational and quasi-variational inequalities with gradient constraints are not only found in elastoplasticity, but in friction mechanics, superconductivity, and also arise as the result of competition of a finite resource in generalized Nash games. Structurally speaking, QVIs are

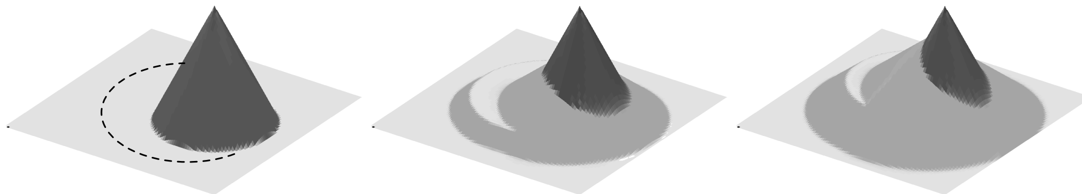
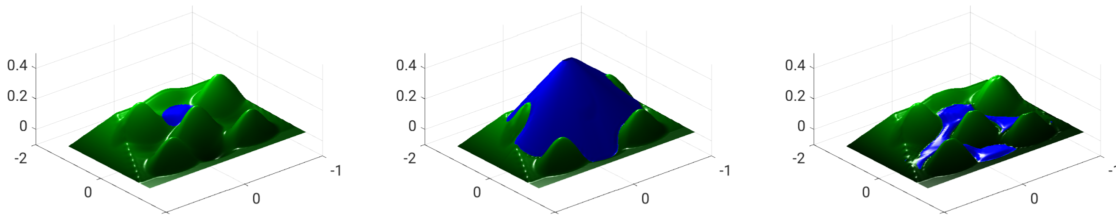


Figure 1.1: Evolution of sand poured over a steep structure.

significantly more complex than VIs, in fact QVIs in addition to being nonsmooth problems are nonconvex and in general possess multiple solutions. Derivation of solution algorithms, and the study of differentiability properties (of the control-to-state map), are among the features hindered by these highly complex nonlinearities.

This dissertation concerns the study of variational and quasi-variational inequalities with gradient constraints that include diffusive and non-diffusive operators. The prototypical model that is extended and studied within this dissertation is the Prigozhin model of sandpile accumulation. Mathematically, the model corresponds to a variational or quasi-variational inequality (according to its setting) with a gradient constraint that may be determined by a discontinuous operator; it was developed and studied by Leonid Prigozhin, see [1–7], and provides a solid description of the behavior of piles of granular cohensionless materials. See Figure 1.1 for an example of the capabilities of the model, where sand is poured over a steep supporting structure and the evolution of the pile exhibits a fully non-trivial shape. The steepness of the pile is mainly dependent on a material parameter called *angle of repose* which is determined as the angle established by the growth cone of material when being poured from a point source. Different granular cohensionless materials (sand, gravel, couscous,...) possess different angles of repose. In fact, the model is versatile enough to be able to describe the water accumulation on a topographical map; such a feature is observed when considering water as a material with a zero angle of repose. See Figure 1.2.

The focus of this work is mostly on stationary problems but some features involving evolutionary ones are also discussed. Each chapter is self-contained, and corresponds to a



time= 0, source location (blue)   time= T, high angle of repose   time= T, low angle of repose

Figure 1.2: Accumulation of sand and water over certain topography

completed paper or to work under development. The organization of the overall dissertation is given as follows:

Chapter 2 describes initial properties of the main model under study: The Prigozhin accumulation model for cohensionless and granular materials. We deal with the derivation of the model, properties of the semi-discretization approach, regularization of the constraint and how the latter affects multiplicity of solutions. Specifically, it is shown that regularization of the upper bound operator in the QVI setting leads to uniqueness of solutions.

Chapter 3 concerns the stationary non-diffusive problem and where the upper bound of the gradient constraint can be a highly irregular function, i.e., an element in $L^1(\Omega)$, or a Borel measure. We deal with existence theory in this highly irregular setting by providing novel mathematical tools for the application of the direct method of calculus of variations. In addition, we conduct a rigorous identification of the Fenchel pre-dual problem. The latter leads to a study of variational problems on a non-standard state space of vectorial Borel measures with square integrable divergences. In addition, a primal-dual solution algorithm is established, and numerical tests are provided.

Chapter 4 is devoted to the study of differentiability properties of the control-to-state map for the stationary model. In this case the control corresponds to the forcing term, and a novel result of Newton type differentiability is obtained. Such a result is concluded by a new kind of implicit function theorem involving Newton differentiability.

Chapter 5 studies the semi-discretization as well as the full discretization of the evolutionary Prigozhin model and optimal control thereof. In this chapter, the consider that the control variable is the initial supporting structure. The motivation for this problem is flood prevention by minimal topographical changes.

The material in this dissertation partially corresponds to the following publications that I have co-authored:

(1) with A. N. Ceretani, and C. N. Rautenberg, "The stationary Boussinesq equations with do-nothing boundary conditions," Proceedings of VII MACI 2019, Río Cuarto, Argentina, vol. 7, 2019.

(2) with A. N. Ceretani, and C. N. Rautenberg, "On existence and uniqueness of solutions to a Boussinesq system with nonlinear and mixed boundary conditions," Journal of Mathematical Analysis and Applications, vol. 490, no. 1, p. 124201, 2020.

(3) with H. Antil, C. N. Rautenberg, and D. Verma, "Non-diffusive variational problems with distributional and weak gradient constraints," arXiv preprint arXiv:2106.12680, 2021.

(4) with C. N. Rautenberg, "Differentiability and control of a model for granular material accumulation," arXiv preprint arXiv:2106.12653, 2021.

Chapter 2: QVIs and VIs with gradient constraints

We start by introducing the variational inequality (VI) with gradient constraints, which in different forms is of interest in this work. The modeling capabilities of this problem formulation are then demonstrated in the context of an application of sandpile growth: The derivation of the VI from the physical properties of this application is presented. Furthermore we then regard a more general setting of a quasi-variational inequality (QVI) in presence of supporting structures with steep slopes. Semi-discretization in time, uniqueness, and existence are also discussed.

We begin by considering the following evolutionary problem: Suppose that $f : (0,T) \times \Omega \to \mathbb{R}$ together with the initial state $u_0 : \Omega \to \mathbb{R}$ are given, where $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is a bounded domain with a Lipschitz boundary. We assume that the boundary $\partial\Omega$ is partitioned into a connected Dirichlet boundary part Γ_D and a boundary part Γ_N that is related to non-permeability. Furthermore, let $\alpha : \Omega \to \mathbb{R}^+$ be a given nonnegative function. In the first part of this chapter, we additionally impose on the initial state the feasibility assumption

$$|\nabla u_0| \leq \alpha \text{ a.e. (almost everywhere) in } \Omega. \tag{2.0.1}$$

In the physical motivation that we introduce below, this condition reflects a flatness assumption on the underlying surface, and it hinges on this condition if the sandpile models motivates a VI or more generally a QVI.

The accumulation dynamics are driven by a diffusive or non-difussive operator A; the two most common cases are determined by $A = -c\Delta$ for a sufficiently small $c > 0$ and $A = 0$ for the diffusive and non-diffusive cases respectively. The overall dynamics are described as follows: Suppose that $u : (0,T) \times \Omega \to \mathbb{R}$ satisfying $u(0,x) = u_0(x)$ for all $x \in \Omega$, is a solution to the following problem:

Find $u \in \mathcal{K}$ such that

$$\int_0^T \langle \partial_t u(t) + A(u(t)) - f(t), v(t) - u(t) \rangle_{U_{\Gamma_D}(\Omega)', U_{\Gamma_D}(\Omega)} \, \mathrm{d}t \geq 0, \quad \text{for all } v \in \mathcal{K}, \tag{2.0.2}$$

with the set $\mathcal{K}$ is given by

$$\mathcal{K} := W(0,T) \cap \{w \;:\; w(t) \in K \text{ a.e. in } (0,T)\}, \tag{2.0.3}$$

and where

$$W(0,T) := \{u \in L^2(0,T; H^1_{\Gamma_D}(\Omega)) : \partial_t u \in L^2(0,T; L^2(\Omega))\},$$

and $U_{\Gamma_D}(\Omega) = H^1_{\Gamma_D}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$.

The set K is convex and it arises by a nonlinear law with a bound on the first order derivative terms. In the standard case K is given by

$$K := \{v \in U_{\Gamma_D}(\Omega) : |\nabla v| \leq \alpha \text{ a.e. in } \Omega\}, \tag{2.0.4}$$

where ∇ is the weak gradient, and $|\cdot|$ denotes the Euclidian norm.

2.1 Modeling of the sandpile growth

A possible motivation for the above class of problems is based on the study of accumulation of granular cohensionless materials. This approach was pioneered by Prigozhin [4, 6, 7] limited to homogeneous materials and a continuous support structure. The derivation of the model is included for the sake of completeness. In such a model, $f : (0,T) \times \Omega \to \mathbb{R}$ represents the (density) rate of a granular material being deposited on a supporting structure $u_0 : \Omega \to \mathbb{R}$. In this setting, assuming enough regularity on f, the quantity $\int_0^T \int_\Omega f \, \mathrm{d}x \, \mathrm{d}t$ is the total

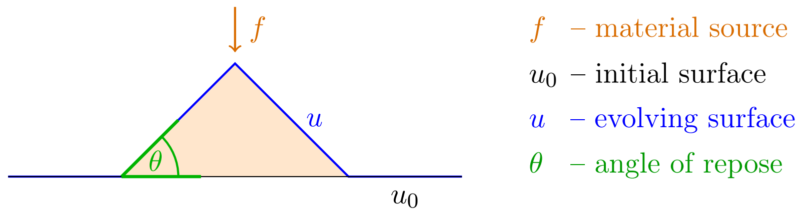


Figure 2.1: Angle of repose in one dimension

amount of material deposited on Ω over the time interval $[0, T]$.

The accumulation of granular cohesionless materials, such as sand, exhibit a material specific critical angle, denoted by θ and called *angle of repose*. On slopes of this angle, further added material flows in the direction of steepest descent. This angle can be observed on the resulting surface when such a material is disposed onto a flat surface from a point source.

Letting $\alpha = \tan(\theta)$, the angle of repose condition is given by

$$|\nabla u| \leq \alpha \quad \text{a.e. in } \Omega. \tag{2.1.1}$$

In the case that $\alpha > 0$ is constant on Ω, this corresponds to the classical case of a granular homogeneous cohesionless material, if $\alpha : \Omega \to \mathbb{R}$ is not constant, the value of α at a point determines the local angle of repose. In the second case, heterogenous sandpiles can be formed [8].

Slopes which are less steep than the angle of repose are considered stable, in such regions material accumulates, and no flow occurs. Denoting the flux of material by $\vec{\phi} : \Omega \to \mathbb{R}^{\mathrm{d}}$, this property is expressed by

$$|\nabla u| < \alpha \Rightarrow \vec{\phi} = 0. \tag{2.1.2}$$

This and the following functional equalities and inequalities are understood in the almost everywhere sense, which we omit to explicate for readability. Due to gravity, if positive flux occurs, material moves only in the direction of steepest descent, i.e.,

$$-\nabla u \cdot \vec{\phi} = |\nabla u||\vec{\phi}|. \tag{2.1.3}$$

Furthermore the following law of mass conservation is in place:

$$u_t + \operatorname{div} \vec{\phi} = f. \tag{2.1.4}$$

We assume that on Γ_D material is allowed to freely leave the domain Ω and that on Γ_N material can not; Γ_N can be interpreted as an impermeable wall, as no flux can occur across this boundary part. This leads to the following boundary conditions: On Γ_D we observe that $u = 0$ and on Γ_N we observe that $\vec{n} \cdot \vec{\phi} = 0$ where $\vec{n}$ is the unit outer normal vector.

From (2.1.1)–(2.1.3), it follows that either $\vec{\phi} = 0$, which is the case if $|\nabla u| < \alpha$, or $-\nabla u \cdot \vec{\phi} = |\nabla u||\vec{\phi}| = \alpha|\vec{\phi}|$ in the case of $|\nabla u| = \alpha$. Hence

$$-\nabla u \cdot \vec{\phi} = \alpha|\vec{\phi}|. \tag{2.1.5}$$

For every $v \in K$, as defined in (2.0.4), it holds that

$$\nabla v \cdot \vec{\phi} \geq -|\nabla v||\vec{\phi}| \geq -\alpha|\vec{\phi}|,$$

and thus

$$\alpha|\vec{\phi}| + \nabla v \cdot \vec{\phi} \geq 0.$$

Adding (2.1.5), we get

$$\nabla(v - u) \cdot \vec{\phi} \geq 0,$$

and integration over Ω yields

$$0 \leq \int_\Omega \nabla(v-u)\cdot\vec{\phi}\,\mathrm{d}x = -\int_\Omega (v-u)\cdot \operatorname{div}\vec{\phi}\,\mathrm{d}x + \int_{\partial\Omega}(v-u)\,\vec{n}\cdot\vec{\phi}\,\mathrm{d}S,$$

where the second term on the right hand side vanishes: Note that on Γ_D we have that $v = u = 0$, and on Γ_N we observe that $\vec{n}\cdot\vec{\phi} = 0$. Here, $\mathrm{d}S$ is the boundary measure and $\vec{n}$ is the unit outer normal vector. The inequality

$$(\operatorname{div}\vec{\phi}, u - v) \geq 0 \tag{2.1.6}$$

follows directly. Finally, applying the mass conservation law (2.1.4), this is equivalent to

$$(\partial_t u - f, v - u) \geq 0, \quad \forall\, v \in K, \tag{2.1.7}$$

for a.e. $t \in (0,T)$.

One interesting case is worth describing: Provided that $v = u \pm 1$ is feasible, e.g., if $\Gamma_D = \emptyset$, a conservation law of material is in place, specifically, it can be inferred from (2.0.2) that $\int_\Omega (u(T) - u_0)\,\mathrm{d}x = \int_0^T \int_\Omega f\,\mathrm{d}x\mathrm{d}t$.

Realistically, shapes of sandpiles do not resemble perfect cones, but some smoothing occurs, rounding the top and the tails of accumulations. This is due to the stochastic nature of the interactions of particles. A Deterministic way to capture this effect is by introducing a diffusivity term $Au = -c\Delta u$ with a small coefficient $c > 0$. Including this term, the variable u satisfies

$$(\partial_t u + Au - f, v - u) \geq 0, \quad \forall\, v \in K. \tag{2.1.8}$$

2.1.1 A simple example

A simple example to illustrate the behavior of the model above is the following. We assume that $u_0 = 0$, and that Ω is a circle for $\mathrm{d} = 2$ or on an interval for $\mathrm{d} = 1$, respectively, and

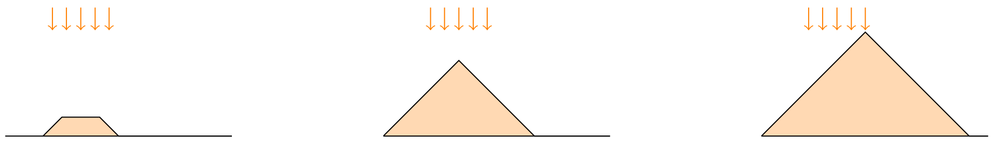


Figure 2.2: Illustration of a basic example of sandpile growth for d = 1

that $\Gamma_D = \partial\Omega$. Suppose that the source term f is the characteristic function of a circle (in two dimensions) or an interval (in one dimension). In this setting, first a truncated, and then a full cone evolves. As its support reaches the boundary, some added material leaves the domain, letting the free surface further grow only in other areas.

2.2 Semi-discretization and the stationary problem

The study of solutions to (2.0.2) usually makes use of the semi-discretization (in time) of the problem via an implicit Euler method. In particular, we approximate the partial time derivative $\partial_t u$ by $(u_n - u_{n-1})/k$ for some time-step $k > 0$. For sake of simplicity we consider the non-diffusive setting. The arising class of problems is then given by

$$\begin{aligned} &\text{Find } u_n \in K \text{ such that} \\ &\int_\Omega (u - u_{n-1})(v - u)\,\mathrm{d}x \geq \langle f_n, v - u\rangle_{U_{\Gamma_D}(\Omega)', U_{\Gamma_D}(\Omega)}, \quad \text{for all } v \in K, \end{aligned} \tag{VI(u_{n-1}, f_n)}$$

where $f_n = \int_{(n-1)k}^{nk} f(\tau)\,\mathrm{d}\tau$.

Letting $g := f_n + u_{n-1}$, this variational problem (assuming enough regularity for g) corresponds to the first order condition of the minimization problem

$$\begin{aligned} &\text{Minimize (min)} \quad \frac{1}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x - \int_\Omega g(x)u(x)\,\mathrm{d}x \quad \text{over } u \in U_{\Gamma_D}(\Omega), \\ &\text{subject to (s.t.)} \quad u \in K. \end{aligned} \tag{2.2.1}$$

Uniqueness and existence can be shown using standard tools (e.g., see [9]). In Chapter 3 we focus on problems of this type with generalized gradient bounds. Therein, proofs of existence and uniqueness for $g \in L^2(\Omega)$ are also presented in the cases where the gradient constraint is a function with low regularity, namely that $\alpha \in L^1(\Omega)$, or further in the case where α is a non-negative Borel measure.

2.3 The evolutionary QVI

In this section, we drop the assumption of a relatively flat underlying surface (2.0.1) and regard the more general case where the slopes of the underlying surface can potentially be steeper than the angle of repose. In this case, the upper bound of $|\nabla u|$ in the constraint set is generalized to

$$M(u, u_0) := \begin{cases} \alpha, & \text{if } u > u_0; \\ \max(\alpha, |\nabla u_0|), & \text{if } u \le u_0, \end{cases} \tag{2.3.1}$$

and the gradient constraint (2.1.1) is replaced by

$$|\nabla u| \le M(u, u_0) \quad \text{a.e. in } \Omega. \tag{2.3.2}$$

This accounts for the fact that the solution is not bound by the angle of repose at places where no material accumulates, i.e., where the solution coincides with the underlying surface. If the underlying surface u_0 does not exhibit steep slopes, then $M(u, u_0) = \alpha$ and the general case reduces to the one previously discussed.

The generalization of the derivation in Section 2.1 is direct: In the same way as in (2.3.2) compared to (2.1.1), the upper bound α is replaced by $M(u, u_0)$ throughout. Merely a short argument for the generalization of (2.1.5), i.e.,

$$-\nabla u \cdot \vec{\phi} = M(u, u_0)|\vec{\phi}|, \tag{2.3.3}$$

needs to be appended: If $M(u, u_0) \neq \alpha$, then by (2.3.2), it holds that $u = u_0$, from which it follows that $\nabla u = \nabla u_0$, and hence, $|\nabla u| = M(u, u_0)$. Together with (2.1.3), this shows that (2.3.3) holds true.

Next, we introduce the evolutionary QVI which arises from the general formulation: We assume that $f \in W(0,T)'$, hence $f \in U_{\Gamma_D}(\Omega)'$ a.e. in $(0,T)$. The QVI in the Bochner space setting is given as follows:

$$\begin{aligned}
&\text{Find } u \in \mathcal{K}(M(u, u_0)) \text{ such that} \\
&\int_0^T \langle \partial_t u(t) - f(t), v(t) - u(t) \rangle_{U_{\Gamma_D}(\Omega)', U_{\Gamma_D}(\Omega)} \mathrm{d}t \geq 0, \quad \text{for all } v \in \mathcal{K}(M(u, u_0))
\end{aligned} \tag{QVI(u_0)}$$

with the initial condition $u(0) = u_0$, and where

$$\mathcal{K}(\varphi) := \{v \in W(0,T) : v(t) \in K(\varphi) \text{ a.e. in } (0,T)\},$$

and

$$K(\varphi) := \{v \in U_{\Gamma_D}(\Omega) : |\nabla v| \leq \varphi \text{ a.e. in } \Omega\}.$$

2.3.1 Regularization of the gradient bound

The pointwise gradient bound (2.3.1) is discontinuous in the u argument. While this makes sense in modeling, from the viewpoint of wellposedness this presents an unsurmountable obstacle. This motivates that and for $\varepsilon > 0$, we introduce the regularization

$$M^\epsilon(u, u_0) := \begin{cases} \alpha, & u > u_0 + \epsilon, \\ \max(\alpha, |\nabla u_0|) + \frac{u - u_0}{\epsilon} \left(\alpha - \max(\alpha, |\nabla u_0|)\right), & u_0 < u \leq u_0 + \epsilon, \\ \max(\alpha, |\nabla u_0|), & u \leq u_0. \end{cases} \tag{2.3.4}$$

2.3.2 Semi-discretization of the QVI

In the same vein as for the VI above, approximating $\partial_t u$ via an implicit Euler method, the generalization of (VI(u_{n-1}, f_n)) is given by the general time discrete QVI that is given in its general form next:

$$\text{Find } u \in K(\Phi(u)) \text{ such that} \qquad \qquad (\text{QVI}(u_{n-1}, f_n, \Phi))$$
$$\int_\Omega (u - u_{n-1})(v - u)\,\mathrm{d}x \geq \langle f_n, v - u\rangle_{U_{\Gamma_D}(\Omega)', U_{\Gamma_D}(\Omega)}, \quad \forall v \in K(\Phi(u)).$$

where $u_{n-1} \in U_{\Gamma_D}(\Omega)$ denotes a previous timestep solution (or the initial surface), $f_n \in U_{\Gamma_D}(\Omega)'$, and we use either $\Phi(u) = M(u, u_0)$ or $\Phi(u) = M^\varepsilon(u, u_0)$; note that we observe in both cases that $\Phi(w) \geq \alpha$ for all w. We rely heavily on the following non-increasing property of the map Φ in both the previous cases:

$$w_1 \leq w_2 \quad \text{a.e.} \qquad \Longrightarrow \qquad \Phi(w_2) \leq \Phi(w_1) \quad \text{a.e.}$$

In contrast to the VI case, an equivalent minimization problem in general is not possible to be obtained, and (QVI(u_{n-1}, f_n, Φ)) is not necessarily uniquely solvable as we show by means of a 1-dimensional example in Section 2.3.3. First, we establish some increasing properties of solutions.

Lemma 1. Let $f \geq 0$ and $\Phi(u) = M(u, u_0)$ or $\Phi(u) = M^\varepsilon(u, u_0)$. If u solves QVI(u_0, f, Φ), and $u_0 \in K(\Phi(u_0))$, then it holds true that $u \geq u_0$.

Proof. Let u solve QVI(u_0, f, Φ) and suppose the opposite, namely that the set

$$S := \{x \in \Omega \,:\, u_0(x) > u(x)\}$$

has positive measure. Define $v = \max(u_0, u)$, and note that because $w \mapsto \Phi(w)$ is a non-increasing map, on S it holds that

$$|\nabla v| = |\nabla u_0| \leq \Phi(u_0) \leq \Phi(u),$$

and on $\Omega \setminus S$

$$|\nabla v| = |\nabla u| \leq \Phi(u).$$

Hence it holds that $v \in K(\Phi(u))$, and thus

$$\int_\Omega (u - u_0)(v - u)\,\mathrm{d}x = -\|(u - u_0)\chi_S\|_2^2 < 0 \leq \langle f, v - u\rangle_{U_{\Gamma_D}(\Omega)', U_{\Gamma_D}(\Omega)};$$

which violates that u is a solution to $\mathrm{QVI}(u_0, f, \Phi)$. □

2.3.3 A 1-D example of multiple solutions

In this section, we show how the regularization of M resulting in M^ϵ leads to the uniqueness of solutions. In particular, we provide an example where $\mathrm{QVI}(u_0, f, M(\cdot, u_0))$ has multiple solutions, however the respective regularized problem $\mathrm{QVI}(u_0, f, M^\varepsilon(\cdot, u_0))$ has a unique solution.

We assume here that $\mathrm{d} = 1$, $\Omega = (-2, 2)$, and $\Gamma_D = \{-2, 2\} = \partial\Omega$, i.e., we consider $U_{\Gamma_D}(\Omega) = H_0^1(\Omega)$. Let the initial surface be given by

$$u_0 = \max(0, 1 - |x|), \tag{2.3.5}$$

the gradient bound be given by $\alpha = 0.5$, and let $f = c\delta_0$ for a $0 \leq c < 0.5$, where δ_0 is Dirac's delta centered at zero.

First, we give necessary and sufficient conditions for solutions of $\mathrm{QVI}(u_0, f, \Phi)$. Based on these conditions, we can show that $\mathrm{QVI}(u_0, f, M(\cdot, u_0))$ has multiple solutions while for any $\epsilon > 0$ there exists a unique solution to $\mathrm{QVI}(u_0, f, M^\epsilon(\cdot, u_0))$.

Proposition 1. For the given example, it holds that $u \in H_0^1(\Omega)$ is a solution of $\mathrm{QVI}(u_0, f, \Phi)$ if and only if the following two conditions are satisfied: There exist $a, b \in \mathbb{R}$ with $-2 \leq a \leq -1$ and $1 \leq b \leq 2$, such that

$$u'(x) = -\mathrm{sgn}(x)\Phi(u)(x) \quad \text{on} \quad (a, b), \qquad \text{and} \qquad u = 0 \text{ on } \Omega \setminus (a, b), \tag{C1}$$

a.e., and it holds that

$$\int_\Omega u - u_0 \,\mathrm{d}x = \langle f, 1 \rangle. \tag{C2}$$

Proof. *i) (C1) is a necessary condition.* Let u be a solution of $\mathrm{QVI}(u_0, f, \Phi)$. Noting that $\operatorname{supp} u_0 \subset \operatorname{supp} u$ by Lemma 1, let $[a, b]$ denote the largest interval which contains $\operatorname{supp} u_0 = [-1, 1]$ and is contained in $\operatorname{supp} u$.

Consider that the negation of (C1) holds true. Initially, we assume the violation of the first equality in (C1) and without loss of generality we assume that this violation occurs on a set of positive measure $S \subset (0, b)$, i.e.,

$$u'(x) \neq -\Phi(u)(x), \qquad \text{a.e. in } S.$$

Since $-u' \leq |u'| \leq \Phi(u)$ holds for any solution of $\mathrm{QVI}(u_0, f, \Phi)$, it immediately follows that

$$u'(x) > -\Phi(u)(x), \qquad \text{a.e. in } S.$$

As test function in $\mathrm{QVI}(u_0, f, \Phi)$, regard

$$v(x) = \begin{cases} \left(u(0) - \int_0^x \Phi(u)(\xi)\,\mathrm{d}\xi\right)^+, & \text{if } x \in (0, b), \\ u(x), & \text{otherwise.} \end{cases}$$

By $|u'| \leq \Phi(u)$, it holds that $v \leq u$ on Ω: We only need to prove this in $(0, b)$, and note that either $v'(x) = -\Phi(u)(x)$ which leads to $v \leq u$, or $v(x) = 0$ which also implies the same inequality given that all solutions u in this case are non-negative by Lemma 1. It further holds on $(0, b)$ that by definition of u_0 we have

$$u_0(x) = \left(u_0(0) - \int_0^x \Phi(u_0)(\xi)\,\mathrm{d}\xi\right)^+. \tag{2.3.6}$$

By Lemma 1 we have $u \geq u_0$, and then by definition of v it holds that $v(x) \geq u_0(x)$ for $x \notin (0, b)$. Further, since $u \geq u_0$, the non-increasing property of Φ, implies that $\Phi(u) \leq \Phi(u_0)$. Thus, by definition of v and (2.3.6), we also have that $v(x) \geq u_0(x)$ for $x \in (0, b)$. Thus, $v(x) \geq u_0(x)$ for every x.

Because $|u'| \leq \Phi(u)$ a.e. on $(0, b)$, and $u' > -\Phi(u)$ on S, a set of positive measure in $(0, b)$, there exists a $0 < \sigma < b$ such that

$$u_0(x) \leq v(x) = \left(u(0) - \int_0^x \Phi(u)(\xi)\,\mathrm{d}\xi\right)^+ < u(0) + \int_0^x u'(\xi)\,\mathrm{d}\xi = u(x) \quad \text{for every } x \in (\sigma, b).$$

Thus

$$\int_\Omega (u - u_0)(v - u)\,\mathrm{d}x > 0 = c \cdot (u(0) - u(0)) = \langle f, v - u\rangle,$$

which implies that u does not solve the QVI, a contradiction.

Secondly, we assume the violation of the second equality in (C1). By construction of the interval $[a, b]$, it holds that $u(a) = 0 = u(b)$ and thus testing with

$$v(x) = \begin{cases} u(x), & \text{if } x \in [a, b], \\ 0, & \text{otherwise}, \end{cases}$$

yields $u = 0$ on $\Omega \setminus (a, b)$, a contradiction. Hence (C1) is a necessary condition.

ii) (C2) *is a necessary condition.* Let u be a solution of $\mathrm{QVI}(u_0, f, \Phi)$. It follows from the first part of the proof that (C1) holds true. Since $\Phi(u) > 0$ by definition, and because $\operatorname{supp} u_0 \subset \operatorname{supp} u$ by Lemma 1, it holds that a and b are uniquely defined satisfying $-2 \leq a \leq -1$ and $1 \leq b \leq 2$.

First, we regard the case that $-2 < a$ and $b < 2$. We define $\overline{\mu} = \min(a+2, 2-b)$. For any $\mu \in (-\overline{\mu}, \overline{\mu})$, define

$$m(\mu) = \begin{cases} \alpha \cdot \operatorname{sgn}(\mu) \cdot (x+2) & \text{if } -2 \leq x \leq -2 + |\mu|, \\ \alpha \cdot \operatorname{sgn}(\mu) \cdot (2-x) & \text{if } 2 - |\mu| \leq x \leq 2, \\ \alpha \cdot \mu & \text{otherwise,} \end{cases}$$

and let $v_\mu := u + m(\mu)$, which satisfies $v_\mu \in K(\Phi(u))$. Since u is a solution of the QVI and $f = c\delta_0$, for every $\mu \in (-\overline{\mu}, \overline{\mu})$, it holds that

$$\alpha\mu \int_\Omega (u - u_0)\,\mathrm{d}x + \mathcal{O}(\mu^2) = \int_\Omega (u-u_0)(v_\mu - u)\,\mathrm{d}x \geq \langle f, v_\mu - u\rangle = \alpha\mu\langle f, 1\rangle. \tag{2.3.7}$$

Multiplying both sides with $(\alpha|\mu|)^{-1}$, and letting $\mu \to 0^+$ and $\mu \to 0^-$, we get the equality (C2) due to $\mathcal{O}(\mu^2) = o(\mu)$.

We complete this part of the proof by showing that $a = -2$ can not hold. The same follows for $b = 2$ by symmetry. Suppose that $a = -2$. Then, by the first equation of (C1), $u' = \Phi(u) \geq \alpha = 0.5$ on $(-2, 0)$. Due to the boundary condition $u(-2) = 0$, and because $u \geq u_0$ by Lemma 1, it thus follows that

$$u(x) \geq \begin{cases} 0.5 \cdot x + 1, & \text{if } -2 \leq x \leq 0, \\ u_0(x), & \text{if } 0 < x \leq 2. \end{cases}$$

Hence by direct computation we have

$$\int_\Omega u - u_0 \, \mathrm{d}x \geq 0.5. \tag{2.3.8}$$

However for $-2 < \mu < 0$, v_μ is feasible and (2.3.7) holds true, hence we can let $\mu \to 0^-$, and therefore conclude that

$$\int_\Omega u - u_0 \, \mathrm{d}x \leq \langle f, 1\rangle = c < 0.5.$$

in contradiction to (2.3.8).

iii) (C1) *together with* (C2) *is a sufficient condition.*

Let u be a function which satisfies (C1) and (C2). Because $\Phi(u) \geq 0$ holds by definition, (C1) implies that $|u'(x)| \leq \Phi(u)(x)$ for every $x \in \Omega$ and hence $u \in K(\Phi(u))$. We need to show that for every $v \in K(\Phi(u))$

$$\int_\Omega (u - u_0)(v - u) \, \mathrm{d}x \geq \langle f, v - u\rangle \tag{2.3.9}$$

holds true.

For $v \in K(\Phi(u))$ it holds that $|v'(x)| \leq \Phi(u)(x)$. Then by (C1), for a.e. $x \in (a, b)$ we have

$$\operatorname{sgn}(x)u'(x) = -\Phi(u)(x) \leq -|v'(x)| \leq v'(x) \leq |v'(x)| \leq \Phi(u)(x) = -\operatorname{sgn}(x)u'(x),$$

and hence $v'(x) - u'(x) \geq 0$ if $x \in (a, 0)$ and $v'(x) - u'(x) \leq 0$ if $x \in (0, b)$. Defining the constant $l := (v-u)(0)$, then

$$v(y) - u(y) = l - \int_y^0 v'(x) - u'(x)\,\mathrm{d}x \geq l, \quad \text{for } y \in (a,0), \tag{2.3.10}$$

$$v(y) - u(y) = l + \int_0^y v'(x) - u'(x)\,\mathrm{d}x \geq l, \quad \text{for } y \in (0,b). \tag{2.3.11}$$

Because $u \geq u_0$ and $u - u_0 = 0$ on $\Omega \setminus (a,b)$, from (C2) we can conclude

$$\int_\Omega (u-u_0)(v-u)\,\mathrm{d}x \geq \int_\Omega (u-u_0)\cdot l\,\mathrm{d}x = \langle f, l\rangle = \langle f, v-u\rangle.$$

□

We are now able to show that in the unregularized case multiple solutions exist, while in the regularized case a unique solution exists.

Proposition 2. Let $\varepsilon > 0$, and $\Phi(u) = M^\varepsilon(u, u_0)$. In the example under study, the problem $\mathrm{QVI}(u_0, f, M^\varepsilon(u,u_0))$ admits a unique solution.

Proof. We use that by Proposition 1, solutions of $\mathrm{QVI}(u_0, f, M^\varepsilon(u,u_0))$ can be equivalently characterized by (C1) in conjunction with (C2).

First we show that for any fixed function value $u(0) \geq 0$ of a solution u, (C1) fully characterizes u everywhere on Ω, and subsequently, we show that by (C2) only one such solution exists.

In the first step, we determine the solution for $x \in (0,2]$, as for $[-2,0)$ the same construction can be applied symetrically. Due to the discontinuity of $|u_0'|$ and therefore of $M^\varepsilon(u,u_0)$ at 1, we regard the intervals $(0,1]$ and $[1,2]$ separately: On the first interval, condition (C1) yields $u'(x) = -M^\varepsilon(u, u_0)$, hence

$$-u'(x) = \begin{cases} 0.5 & u(x) > -x + \varepsilon \\ 1 - \frac{u(x)+x}{2\varepsilon} & -x < u \leq -x + \varepsilon \\ 1 & u(x) \leq -x \end{cases} \tag{2.3.12}$$

or equivalently

$$u'(x) = -1 + \max(0, 0.5 \min(1, (u(x) + x) \cdot \varepsilon^{-1})), \quad \text{for } x \in [0, 1].$$

Together with the given initial value $u(0)$ is uniquely solvable by [10, Theorem I.2.3] since the right hand side is Lipschitz continuous in u for any fixed $x \in [0, 1]$. On the subsequent interval $[1, 2]$ it holds that $M^\varepsilon(u, u_0) = \alpha$, and therefore by (C1), the solution for $x \in (1, 2]$ is given by

$$u(x) = \max(u(1) - \alpha(x - 1), 0), \quad \text{for } x \in [1, 2]. \tag{2.3.13}$$

Finally, to show uniqueness, assume that there exist two distinct solutions u^1 and u^2. Because the function value at 0 defines the function on Ω, as we have shown above, without loss of generality, suppose that $u^1(0) < u^2(0)$. Since both functions satisfy (2.3.12), by [10, theorem I.2.2] and by continuity of u^1 and u^2, there can not be a point $\xi \in (0, 1)$ such that $u^1(\xi) = u^2(\xi)$, and thus

$$u^1(x) < u^2(x) \quad \text{for every } x \in (0, 1). \tag{2.3.14}$$

By continuity of both functions it follows that $u^1(1) \leq u^2(1)$ holds true and therefore (2.3.13) implies that

$$u^1(x) \leq u^2(x) \quad \text{for every } x \in [1, 2]. \tag{2.3.15}$$

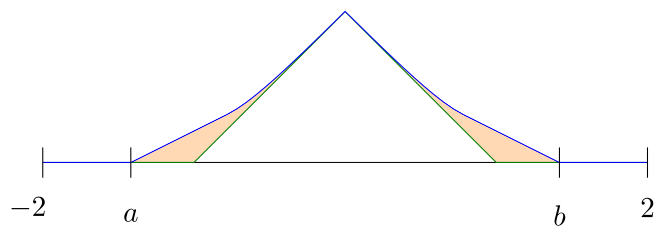


Figure 2.3: Unique solution of the regularized problem

Because the same inequalities as in (2.3.14) and (2.3.15) hold symmetrically for $x \in [-2, 0)$, by (C2) we obtain the contradiction

$$\langle f, 1\rangle = \int_\Omega u^1 \,\mathrm{d}x < \int_\Omega u^2 \,\mathrm{d}x = \langle f, 1\rangle.$$

□

Remark 2.1. The closed form solution to the example problem can be obtained by resolution of (2.3.12) and (2.3.13), and can by basic calculus be shown to be given by

$$u(x) = \begin{cases} (u(0) - u_0(0))\, e^{\frac{|x|}{2\epsilon}} + 1 - |x|, & \text{if } 0 \le |x| \le b, \\ \max(0, u(b) - \alpha(|x| - b)), & \text{if } b < |x| < 2, \end{cases} \tag{2.3.16}$$

where $b = \min\left(1, 2\varepsilon \log\left(\varepsilon/(u(0) - u_0(0))\right)\right)$. Furthermore, $u(0)$ is uniquely defined by (C2), i.e., $\int_\Omega u - u_0 \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x$, due to the monotonous dependence of $u(x)$ on $u(0)$, as seen above in (2.3.12) and (2.3.13).

Proposition 3. The problem $\mathrm{QVI}(u_0, f, M(u, u_0))$ has uncountably many solutions.

Proof. We provide a family of functions for which (C1) and (C2) are satisfied, and which by Proposition 1 are solutions of the QVI.

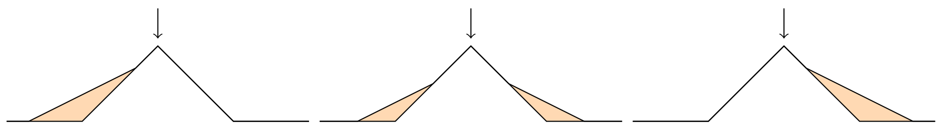


Figure 2.4: 1-D example for non-uniqueness: solutions u^0, $u^{1/2}$, u^1.

For every $\lambda \in [0,1]$, define $u^\lambda \in H_0^1(\Omega)$ by

$$u^\lambda(x) = \begin{cases} \max(u_0(x), (x - a(\lambda))\alpha), & x \leq 0, \\ \max(u_0(x), (b(\lambda) - x)\alpha), & x \geq 0, \end{cases}$$

where $a(\lambda) := -1 - \sqrt{(1-\lambda)/2}$ and $b(\lambda) := 1 + \sqrt{\lambda/2}$. Further define $j(\lambda) := -1 + \sqrt{(1-\lambda)/2}$ and $k(\lambda) := 1 - \sqrt{\lambda/2}$.

To show that (C1) is satisfied, due to symmetry, it suffices to consider $x \in (0,2)$: On $(0, k(\lambda))$, where $k(\lambda) \leq 1$, it holds that $u^\lambda(x) = u_0(x)$ and because $|u_0'(x)| \geq \alpha$ it follows that $(u^\lambda)'(x) = u_0'(x) = -|u_0'(x)| = -M^\epsilon(u^\lambda, u_0)(x)$. On the interval $(k(\lambda), b(\lambda))$, we have $u^\lambda(x) > u_0(x)$ and hence $(u^\lambda)'(x) = -\alpha = -M^\epsilon(u^\lambda, u_0)(x)$, and finally on $(b(\lambda), 2)$, it holds that $u^\lambda(x) = u_0(x) = 0$ because $b(\lambda) \geq 1$. In the same way we can show that $u^\lambda(x) = M^\epsilon(u^\lambda, u_0)(x)$ for $x \in (-2, 0)$, and hence condition (C1) is satisfied.

Since $\operatorname{supp}(u^\lambda - u_0) = (a(\lambda), j(\lambda)) \cup (k(\lambda), b(\lambda))$, condition (C2) is satisfied as well:

$$\int_{-2}^{2} u^\lambda(x) - u_0(x)\,\mathrm{d}x = \int_{a(\lambda)}^{j(\lambda)} (x - a(\lambda))\frac{1}{2}\,\mathrm{d}x + \int_{k(\lambda)}^{b(\lambda)} (b(\lambda) - x)\frac{1}{2}\,\mathrm{d}x = \frac{1-\lambda}{4} + \frac{\lambda}{4} = \langle f, 1\rangle.$$

By Proposition 1, we can conclude that u^λ is a solution of $\mathrm{QVI}(u_0, f, M(u, u_0))$ for every $\lambda \in [0,1]$. $\square$

A sample of the family of solutions in the previous proposition can be seen in Figure 2.4. There we can see that the multiplicity of solutions is not an artifact of the quasi-variational model but actually a feature. Given that intensity is concentrated on the peak of the supporting surface u_0, there is no information of where material should go to, and hence all possibilities are valid. Interestingly, this seems possible to be embedded into a random variable approach; this is, of course, beyond the scope of this dissertation.

In the next chapter we will derive a dual problem which can for example be used for an efficient solution scheme. In fact the condition under which a Fenchel dual can be derived is more general then the setting above, as the gradient bound α is only required to be Lebesgue measurable.

Chapter 3: The stationary problem with measure constraint

The semi-discretization of the evolutionary sandpile problem in the VI case, (2.0.2), can be posed as a minimization problem, (2.2.1), as we have seen in the previous chapter. In the present chapter we concentrate on this minimization problem in a more general setting which allows for a less regular gradient constraint:

$$\begin{aligned} &\min \quad \frac{1}{2}\int_\Omega |u(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x \quad \text{ over } u \in U_{\Gamma_D}(\Omega), \\ &\text{s.t.} \quad u \in K, \end{aligned} \tag{$\mathbb{P}$}$$

where the constraint set is given by

$$K = \{v \in U_{\Gamma_D}(\Omega) : |Gv|_p \leq \alpha\}. \tag{3.0.1}$$

with $1 \leq p \leq +\infty$; a full explanation on the sense in which the constraint is taken is given briefly. In order to consider less regular gradient bounds α than in the introductory chapter, the space U_{Γ_D} is chosen accordingly as a BV-space or a Sobolev space, which is described in detail below, and G denotes an appropriate gradient operator. This generalized setting allows for solutions with jumps at certain locations. Note that the underlying evolutionary problem can be posed in such a generalized setting analogously.

In this section, we assume that the boundary $\partial\Omega$ is partitioned into a Dirichlet boundary part Γ_D and a non-Dirichlet boundary part Γ_N, both composed of a finite number of connected parts, such that

$$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega, \qquad \text{and} \qquad \Gamma_D \cap \Gamma_N = \emptyset.$$

The restriction of u to the Γ_D part of the boundary is assumed to be zero, and no restrictions are assumed on Γ_N. Notice that on Γ_N, Neumann boundary conditions may not arise (due to non-diffusive nature of the variational problem).

In this chapter, we rigorously identify the Fenchel pre-dual of ($\mathbb{P}$), and we address questions of uniqueness and existence for ($\mathbb{P}$) and its Fenchel pre-dual problem. In fact we consider the aforementioned questions under low regularilty assumptions on α, i.e., when α is a measure or an integrable function.

We briefly discuss the two possible scenarios that we consider:

(i) If α is a **nonnegative integrable function**, then $U_{\Gamma_D}(\Omega)$ is a Sobolev-type space and $G = \nabla$ is the weak gradient, so that $|\nabla v|_p$ is the ℓ_p-norm of the weak gradient of v. Hence, $|\nabla v|_p \leq \alpha$ in (3.0.1) is considered in the almost everywhere (a.e.) in Ω sense.

(ii) If α is a **nonnegative Borel measure**, then $U_{\Gamma_D}(\Omega)$ is a subset of functions of bounded variation $\mathrm{BV}(\Omega)$. In this case, $G = \mathrm{D}$ is the distributional gradient, and $|\mathrm{D}v|_p$ the total variation measure of $\mathrm{D}v$ associated to the ℓ_p-norm, and the constraint $|\mathrm{D}v|_p \leq \alpha$ is understood in the measure sense.

Both instances, (i) and (ii), are related, in fact (i) may be considered as a special case of (ii): Letting $\alpha \in \mathrm{M}^+(\Omega)$ in case (ii), where $\mathrm{M}^+(\Omega)$ denotes the set of nonnegative Borel measures, enables us to handle the delicate case $\alpha \in L^1(\Omega)^+$ in (i) by assuming that the measure is absolutely continuous with respect to the Lebesgue measure. Next we shall provide a brief description of modeling capabilities of (i) and (ii) in the context of a particular application.

A description of the qualitative behavior of Problem (2.0.2) is displayed in Figure 3.1. We assume two materials with different angles of repose α_1 and α_2 with $\alpha_1 > \alpha_2$ are poured on the discontinuous structure $u_0(x) := \chi_{(x_1,x_2)}(x)$ for $x \in \Omega := (0,1)$ and $0 < x_1 < x_2 < 1$. The intensity of the material being deposited is given by $f(t,x) = f_1\chi_{(x_0,x_2)}(x) + f_2\chi_{(x_2,1)}(x)$ for some points x_0, and x_2, and some $f_1, f_2 > 0$, i.e., the first and second materials are poured

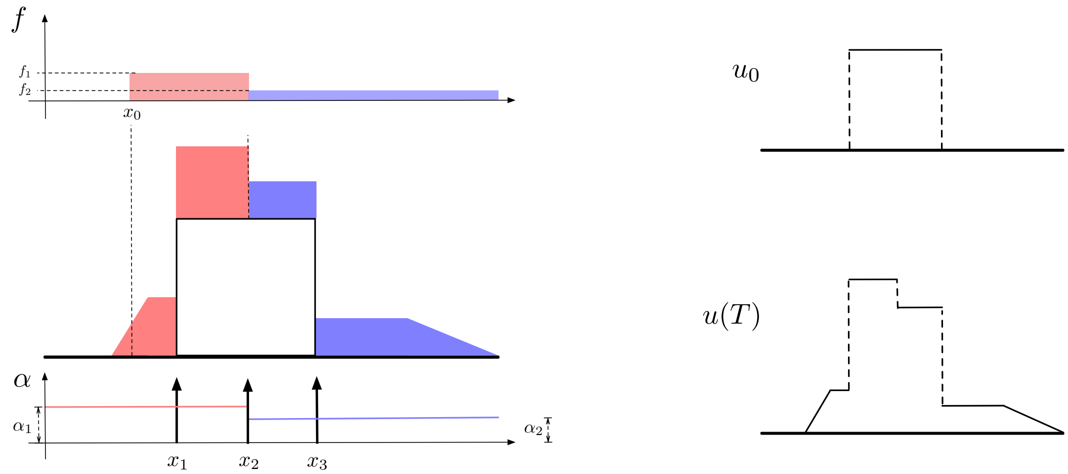


Figure 3.1: Accumulation of two kinds (magenta and blue) of granular materials on discontinuous surface. (**LEFT**) Depiction of $f(t,x) = f_1\chi_{(x_0,x_2)}(x) + f_2\chi_{(x_2,1)}(x)$, the accumulation of both materials, and $\mathrm{d}\alpha = \alpha_1\chi_{(x_1,x_2)}(x)\mathrm{d}x + \alpha_2\chi_{(x_2,1)}(x)\mathrm{d}x + \sum_{i=1}^{3}\delta(x - x_i)$. (**RIGHT**) The value of the initial supporting structure u_0 and the final distribution $u(T)$.

with density rates f_1 and f_2, respectively, during the entire time interval $(0,T)$. We further assume that a sharp edge can form at x_2 with maximum height of 1, and in addition discontinuities of maximum size 1 can be preserved at the locations of the discontinuities of u_0. Finally, the gradient constraint α is then given by $\mathrm{d}\alpha = \alpha_1\chi_{(0,x_2)}(x)\,\mathrm{d}x + \alpha_2\chi_{(x_2,1)}(x)\,\mathrm{d}x + \sum_{i=1}^{3}\delta(x - x_i)$, and the material is assumed to escape freely at the boundary points of Ω. On the right side of Figure 3.1, we see the comparison between u_0 and $u(T)$, the solution at time $T > 0$; on the left we see the depiction of f, α, and the accumulation regions of both materials.

Closely related to problem ($\mathbb{P}$), we consider the following class of problems

$$\min \frac{1}{2}\int_\Omega |\mathrm{div}\,\boldsymbol{p}(x) - f(x)|^2\,\mathrm{d}x + J(\boldsymbol{p}) \qquad \text{over } \boldsymbol{p} \in V_{\Gamma_N}(\Omega). \tag{$\mathbb{P}^*$}$$

We prove that ($\mathbb{P}^*$) is the *Fenchel pre-dual* of problem ($\mathbb{P}$), i.e., the Fenchel dual [11] of ($\mathbb{P}^*$) under certain conditions is ($\mathbb{P}$). Several choices for $V_{\Gamma_N}(\Omega)$ and J are explored which are directly related to the nature of α. In all cases considered, $V_{\Gamma_N}(\Omega)$ contains d-dimensional vector fields with divergences in $L^2(\Omega)$. In particular, we consider the following settings:

(i) If α is a **nonnegative measurable function** (additional assumptions are later explained but continuity is enough to guarantee what follows), then we explore two options for J:

$$J(\boldsymbol{p}) = \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x, \qquad \text{and} \qquad J(\boldsymbol{p}) = \int_\Omega \alpha \mathrm{d}|\boldsymbol{p}|_q.$$

In the first case $V_{\Gamma_N}(\Omega)$ is a subspace of $L^1(\Omega)^{\mathrm{d}}$. In the second case $V_{\Gamma_N}(\Omega)$ is contained in the space of $\mathbb{R}^{\mathrm{d}}$-valued Borel measures, so that the second functional denotes the integral of α with respect to the total variation measure of $\boldsymbol{p}$ induced by the ℓ^q-norm. The two functionals are closely related, and the first can be seen as a restriction of the second one to measures that are absolutely continuous with respect to the Lebesgue measure.

(ii) If α is a **nonnegative Borel measure**, then $V_{\Gamma_D}(\Omega)$ is contained in the space of maps that are α measurable, with J given by

$$J(\boldsymbol{p}) = \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha.$$

A few words are in order concerning ($\mathbb{P}$) and ($\mathbb{P}^*$). Although the objective functional in ($\mathbb{P}$) is smooth and amenable, the constraint set K makes the entire problem highly nonlinear and nonsmooth. The latter also holds for ($\mathbb{P}^*$) given the nature of the functional J. The development of solution algorithms for both problems is a rather delicate issue that requires appropriate regularization methods that can handle the nonsmoothness in an asymptotic fashion.

Here we focus on functional analytic properties of ($\mathbb{P}$) and ($\mathbb{P}^*$) together with duality relationship properties. Additionally, we develop a mixed finite type method to solve the optimality conditions corresponding to ($\mathbb{P}$) and ($\mathbb{P}^*$).

Some Bibliography

The structure of Problems ($\mathbb{P}^*$) and ($\mathbb{P}$) and their inherent difficulties are analogous to the ones that appear in the context of plasticity; see [12,13] and references therein. In particular, the first class of applications for diffusive variational problems with gradient constraints is the elasto-plastic torsion problem. Such a problem has been thoroughly analyzed by Brézis, Caffarelli, Evans, Friedman, Gerhardt, and others; see [14–21]. Further, a complete account of the literature can be found in [22]. A significant amount of the aforementioned works focuses on regularity of solutions, the free boundary, and the equivalence of the gradient constrained problem to a double obstacle one.

The modeling of the evolution of the magnetic field in critical-state models of type-II superconductors also leads to a problem like (2.0.2) with the addition of a diffusive operator and a state-dependent constraint in some cases; see [7, 23–28].

Analogous problems are found in mathematical imaging involving total variation regularization [29–31] and more specifically in the weighted total variation version [32]. There, in contrast to the work here, the L^∞-norm on the gradient is replaced by the L^1-norm, leading to a pre-dual problem with a pointwise bound in its state variable.

3.1 Organization of the chapter

Elementary results about the generalized gradient constraint are given in Section 3.2.1. In Section 3.3, we prove existence and uniqueness of the solution to problem ($\mathbb{P}$) for the cases when α is either a nonnegative Lebesgue measurable function or a nonnegative Borel measure. Existence of solutions to problem ($\mathbb{P}^*$) is addressed in Section 3.4, while for the case when $\boldsymbol{p}$ is a function we require $\mathrm{d} = 1$, when $\boldsymbol{p}$ is a measure the dimension restriction is dropped. The relation between problems ($\mathbb{P}$) and ($\mathbb{P}^*$) are considered in Section 3.5, where a rigorous Fenchel duality result establishes a link between these two problems. In particular, in Section 3.5.1, we address the case where α is a function and the variable $\boldsymbol{p}$ is either a function or a measure. The case when α is a measure and an extension of the duality result of the previous section is given in Section 3.5.2. Finally in Section 3.6, we introduce a mixed finite element method to solve the underlying problems and present a range of numerical tests.

3.2 Notation and Preliminaries

The purpose of this section is to introduce notation involving spaces, and convergence notions that are used throughout the chapter; in particular, we address the well-known notions of Sobolev spaces and the space of functions of bounded variation. We refer the reader to Attouch et al. [33] that we follow closely for this introduction together with the book of Adams and Fournier [34].

For a Banach space X, we denote its corresponding norm as $\|\cdot\|_X$. For an element F in the topological dual X' of X, the duality pairing of F and an arbitrary element $x \in X$ is written as $\langle F, x\rangle_{X',X}$. Throughout the chapter, all Banach spaces are assumed to be real vector spaces.

The inner product on the Lebesgue space $L^2(\Omega)$ of (equivalence classes of) functions that are square integrable on Ω is denoted as $(\cdot,\cdot)$, so that $(f,g) := \int_\Omega f(x)g(x)\,\mathrm{d}x$ for $f,g \in L^2(\Omega)$ where $\mathrm{d}x$ refers to integration with respect to the Lebesgue measure.

The Sobolev space of functions in $L^r(\Omega)$ for $1 \leq r < +\infty$ with weak gradients in $L^r(\Omega)^{\mathrm{d}}$ is denoted by $W^{1,r}(\Omega)$, and it is endowed with the norm

$$\|v\|_{W^{1,r}(\Omega)} := \|v\|_{L^r(\Omega)} + \|\nabla v\|_{L^r(\Omega)^{\mathrm{d}}},$$

where ∇v denotes the weak gradient of v. In the case $r = 2$, we use the notation $H^1(\Omega) := W^{1,2}(\Omega)$. Given that Ω is assumed Lipschitz, restriction of a function $v \in W^{1,r}(\Omega)$ to the boundary $\partial\Omega$ is well-defined via the continuous trace map $\gamma_0 : W^{1,r}(\Omega) \to L^r(\partial\Omega)$. Furthermore, the closed subspace of functions in $W^{1,r}(\Omega)$ that are zero on Γ_D is denoted by

$W^{1,r}_{\Gamma_D}(\Omega)$, i.e.,

$$W^{1,r}_{\Gamma_D}(\Omega) := \{v \in W^{1,r}(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_D\}.$$

Similarly, we define $H^1_{\Gamma_D}(\Omega) := W^{1,2}_{\Gamma_D}(\Omega)$. The space of real-valued Borel measures $\mathrm{M}(\Omega)$ is endowed with the norm $\|\mu\|_{\mathrm{M}(\Omega)} := |\mu|(\Omega)$, where $|\mu|$ is defined for an arbitrary open set O as

$$|\mu|(O) = \sup\left\{\langle \mu, z\rangle_{\mathrm{M}(\Omega),C_0(\Omega)} \;:\; z \in C_0(\Omega),\ \mathrm{supp}(z) \subset O,\ |z(x)| \leq 1,\ \text{for every } x \in O\right\}.$$

Note that $\langle \mu, z\rangle_{\mathrm{M}(\Omega),C_0(\Omega)} = \int_\Omega z \,\mathrm{d}\mu$, and that $|\mu|$ defines a Borel measure in $\mathrm{M}^+(\Omega)$, the subset of nonnegative elements of $\mathrm{M}(\Omega)$, i.e., $\sigma \in \mathrm{M}^+(\Omega)$ if $\sigma(B) \geq 0$ for every Borel set $B \subset \Omega$.

We denote by $\mathrm{BV}(\Omega)$, the space of functions v in $L^1(\Omega)$, for which the total variation semi-norm

$$\int_\Omega |\mathrm{D}v|_p = \sup\left\{\int_\Omega v \,\mathrm{div}\,\boldsymbol{p}\ \mathrm{d}x \;:\; \boldsymbol{p} \in C^1_0(\Omega)^{\mathrm{d}},\ |\boldsymbol{p}(x)|_q \leq 1,\ \text{for every } x \in \Omega\right\}$$

is finite and where q is the Hölder conjugate of p, i.e., $1/p + 1/q = 1$; see [33, Section 10.1]. The space $\mathrm{BV}(\Omega)$ is a Banach space endowed with the norm

$$\|v\|_{\mathrm{BV}(\Omega)} := \|v\|_{L^1(\Omega)} + \int_\Omega |\mathrm{D}v|_p.$$

The operator D represents the *distributional gradient*, and for a $v \in \mathrm{BV}(\Omega)$, $\mathrm{D}v$ is a $\mathbb{R}^d$-valued Borel measure. We use $|\mathrm{D}v|_p$ to denote the total variation measure (associated to the ℓ^p-norm) of $\mathrm{D}v$, and the total mass $|\mathrm{D}v|_p(\Omega)$ is by definition

$$|\mathrm{D}v|_p(\Omega) = \int_\Omega |\mathrm{D}v|_p.$$

Furthermore, the Lebesgue decomposition result applied to $\mathrm{D}v$ implies that there exist measures $\mathrm{D}_a v$ and $\mathrm{D}_s v$ such that

$$\mathrm{D}v = \mathrm{D}_a v + \mathrm{D}_s v,$$

with $\mathrm{D}_a v$ and $\mathrm{D}_s v$ respectively being absolutely continuous and singular with respect to the d-dimensional Lebesgue measure.

We define now the notions of *weak* and *intermediate convergence* of sequences in $\mathrm{BV}(\Omega)$ which provide different topologies on the space $\mathrm{BV}(\Omega)$. The former is obtained by a subsequence of a bounded sequence in $\mathrm{BV}(\Omega)$. Moreover, the latter is sufficient to preserve boundary conditions in the sense of the trace as stated in Theorem 3.1 below.

Definition 3.1 (Weak convergence for $\mathrm{BV}(\Omega)$)**.** Let $\{u_n\}$ be a sequence in $\mathrm{BV}(\Omega)$ and $u^* \in \mathrm{BV}(\Omega)$. We say that u_n converges to u^* weakly, denoted as $u_n \rightharpoonup u^*$ in $\mathrm{BV}(\Omega)$, if

$$u_n \to u^* \text{ in } L^1(\Omega), \qquad \text{and} \qquad |\mathrm{D}u_n|_p \rightharpoonup |\mathrm{D}u^*|_p \text{ in } \mathrm{M}(\Omega).$$

Recall that if $\{\mu_n\}$ is a sequence of measures in $\mathrm{M}(\Omega)$ then $\mu_n \rightharpoonup \mu$ in $\mathrm{M}(\Omega)$ for some $\mu \in \mathrm{M}(\Omega)$, that is, μ_n weakly converges to μ, if

$$\int_\Omega g \, \mathrm{d}\mu_n \to \int_\Omega g \, \mathrm{d}\mu,$$

for all $g \in C_0(\Omega)$.

The definition 3.1 is understood in light of the following fact: If $\{u_n\}$ is a bounded sequence in $\mathrm{BV}(\Omega)$, there exists $u^* \in \mathrm{BV}(\Omega)$ such that along a subsequence $u_n \rightharpoonup u^*$ in $\mathrm{BV}(\Omega)$. The latter follows since the embedding $\mathrm{BV}(\Omega) \hookrightarrow L^1(\Omega)$ is compact (see Attouch et al. [33, Theorem 10.1.4.]) for Lipschitz domains, and since a bounded sequence of measures admits a weakly convergent subsequence.

We shall use the direct method of calculus of variations to establish existence of solutions to problems in $\mathrm{BV}(\Omega)$ and with Dirichlet homogeneous boundary conditions on Γ_D. The space of interest is $\mathrm{BV}_{\Gamma_D}(\Omega)$ defined as

$$\mathrm{BV}_{\Gamma_D}(\Omega) := \{v \in \mathrm{BV}(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_D\},$$

where γ_0 is a trace operator; see [33, section 10.2]. Notice that we use the same notation for the trace operator in Sobolev spaces $W^{1,p}(\Omega)$. There is a fundamental issue with the trace in $\mathrm{BV}(\Omega)$ and the application of the direct method as we show next with an standard example adapted from [33].

Consider a bounded sequence $\{u_n\}$ in $\mathrm{BV}_{\Gamma_D}(\Omega)$. Then, we can extract a subsequence (not relabeled) of $\{u_n\}$ such that $u_n \rightharpoonup u^*$ in $\mathrm{BV}(\Omega)$. The problem is that in general it is not possible to say that $u^* \in \mathrm{BV}_{\Gamma_D}(\Omega)$: Let $\Omega = (0,1)$ with $\Gamma_D = \{0\}$, and consider $\{v_n\}$ defined as

$$v_n(x) = \begin{cases} nx, & \text{if } 0 < x < 1/n, \\ 1, & \text{if } 1/n \leq x < 1. \end{cases}$$

Then, $v_n \in \mathrm{BV}_{\Gamma_D}(\Omega)$, and $v_n \rightharpoonup v^* \in \mathrm{BV}(\Omega) \backslash \mathrm{BV}_{\Gamma_D}(\Omega)$, with $v^* = 1$. The underlying reason is that the trace operator in $\mathrm{BV}(\Omega)$ is not continuous with respect to *weak convergence*, but it is with respect to the *intermediate convergence* subsequently defined. We further notice that $|\mathrm{D}v_n|(0,1) = 1$ and $|\mathrm{D}v^*|(0,1) = 0$, this discrepancy is central to the issue we are considering.

Definition 3.2 (Intermediate convergence)**.** Let $\{u_n\}$ be a sequence in $\mathrm{BV}(\Omega)$ and $u^* \in \mathrm{BV}(\Omega)$. We say that u_n converges to u^* in the sense of intermediate convergence if

$$u_n \to u^* \text{ in } L^1(\Omega), \qquad \text{and} \qquad \int_\Omega |\mathrm{D}u_n|_p \to \int_\Omega |\mathrm{D}u^*|_p.$$

The name *intermediate convergence* arises since it describes a stronger topology than the one of weak convergence, but not as strong as the norm one. The importance of the intermediate convergence can be seen in the following result which holds in our case since $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is a Lipschitz bounded domain. We refer to Attouch et al. [33, Theorem 10.2.2] for its proof.

Theorem 3.1. The trace operator $\gamma_0 : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ is continuous when $\mathrm{BV}(\Omega)$ is equipped with the intermediate convergence and when $L^1(\partial\Omega)$ is equipped with the strong convergence.

We also note that $C^\infty(\overline{\Omega})$ is dense in $\mathrm{BV}(\Omega)$ in the intermediate convergence topology, for a proof see [33, Theorem 10.1.2].

3.2.1 The gradient constraint

A few words are in order concerning the gradient constraint given in the set K defined in (3.0.1). Although in the case when $G = \nabla$ the situation is somewhat standard, if $G = \mathrm{D}$, the distributional gradient for BV functions, several nontrivial explanations are required. In the cases where α is a Borel measure and $v \in \mathrm{BV}(\Omega)$, the inequality

$$|\mathrm{D}v|_p \leq \alpha \tag{3.2.1}$$

in (3.0.1) is understood in the sense of measures, i.e., (3.2.1) holds true if

$$\int_\Omega w|\mathrm{D}v|_p \leq \int_\Omega w \,\mathrm{d}\alpha \;\text{ for all } w \in C_0^\infty(\Omega) \text{ with } w \geq 0 \text{ in } \Omega, \tag{3.2.2}$$

and equivalently, for every Borel measurable set $S \subset \Omega$, it holds that

$$\int_S |\mathrm{D}v|_p \leq \int_S \mathrm{d}\alpha. \tag{3.2.3}$$

Given that nonnegative Borel measures are inner and outer regular ([33, Proposition 4.2.1]) the condition (3.2.2) is equivalent to

$$\int_O |\mathrm{D}v|_p \leq \int_O \mathrm{d}\alpha \tag{3.2.4}$$

for all open sets $O \subset \Omega$.

It is possible to replace $C_0^\infty(\Omega)$ in (3.2.2) by $C^\infty(\overline{\Omega})$, which we discuss next.

Proposition 4. The condition in (3.2.2) is equivalent to

$$\int_\Omega w|\mathrm{D}v|_p \leq \int_\Omega w\,\mathrm{d}\alpha \quad \text{for every } w \in C^\infty(\overline{\Omega}) \text{ with } w \geq 0 \text{ in } \Omega. \tag{3.2.5}$$

Proof. Suppose that (3.2.2) holds true and let K_n be a sequence of closed sets such that

$$\int_{\Omega\setminus K_n} |\mathrm{D}v|_p \to 0 \quad \text{and} \quad \int_{\Omega\setminus K_n} \mathrm{d}\alpha \to 0. \tag{3.2.6}$$

The sequence $\{K_n\}$ exists given that measures in $\mathrm{M}^+(\Omega)$ are inner regular; see [33, Proposition 4.2.1]. Let $\tilde{w} \in C^\infty(\overline{\Omega})$ be nonnegative and arbitrary.

Accordingly, let $\{w_n\}$ in $C_0^\infty(\Omega)$ be nonnegative, uniformly bounded in Ω, and such that $w_n = \tilde{w}$ in K_n. Hence $|\tilde{w}| + |w_n|$ can be uniformly estimated by a constant, and by (3.2.6) it holds that

$$\int_\Omega (\tilde{w}-w_n)|\mathrm{D}v| = \int_{\Omega\setminus K_n} (\tilde{w}-w_n)|\mathrm{D}v| \to 0 \quad \text{and} \quad \int_\Omega (\tilde{w}-w_n)\,\mathrm{d}\alpha = \int_{\Omega\setminus K_n} (\tilde{w}-w_n)\,\mathrm{d}\alpha \to 0.$$

Since the inequality in (3.2.5) holds for every w_n by initial assumption, it also holds in the limit for $\tilde{w}$. Furthermore, (3.2.5) immediately implies (3.2.2), so the result is proven. $\square$

3.3 Existence Theory for $(\mathbb{P})$

In this section, we discuss the existence and uniqueness of solution to the problem $(\mathbb{P})$. We start with the case when α is a measure, and the case when α is a function follows as a special one. In particular, existence of solutions is studied in the function spaces $U_{\Gamma_D}(\Omega) = \mathrm{BV}_{\Gamma_D}(\Omega)$ and $U_{\Gamma_D}(\Omega) = W^{1,1}_{\Gamma_D}(\Omega)$. Both of these spaces share the same difficulty: Bounded sequences do not necessarily admit convergent (in some sense) subsequences that preserve the zero boundary condition on Γ_D in the limit. The main purpose of this section is to overcome this obstacle.

3.3.1 The case when α is a nonnegative Borel measure

We consider in this section that $\alpha \in \mathrm{M}^+(\Omega)$ and hence the state space is given by

$$U_{\Gamma_D}(\Omega) = \mathrm{BV}_{\Gamma_D}(\Omega).$$

We start by proving the following lemma which gives sequential precompactness of some classes of bounded sets in $\mathrm{BV}_{\Gamma_D}(\Omega)$. These bounded sets are subsets of K which in this case is defined as

$$K = \{v \in \mathrm{BV}_{\Gamma_D}(\Omega) : |\mathrm{D}v|_p \leq \alpha\}.$$

Lemma 2. Let $\alpha \in \mathrm{M}^+(\Omega)$ and $M > 0$, then the set

$$K^* = K \cap \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \leq M\}$$

is sequentially precompact in the sense of the intermediate convergence of $\mathrm{BV}(\Omega)$.

Proof. Let $\{v_n\}$ be a sequence in K^*, then it is bounded in $\mathrm{BV}(\Omega)$, and thus $v_n \rightharpoonup v^*$ in $\mathrm{BV}(\Omega)$ for some $v^* \in \mathrm{BV}(\Omega)$ along a subsequence (not relabelled). Since $|\mathrm{D}v_n|_p \rightharpoonup |\mathrm{D}v^*|_p$

in $\mathrm{M}(\Omega)$, and $|\mathrm{D}v_n|_p \leq \alpha$ it follows that for every open set $O \subset \Omega$ that

$$|\mathrm{D}v^*|_p(O) \leq \liminf_{n\to\infty} |\mathrm{D}v_n|_p(O) \leq \alpha(O), \tag{3.3.1}$$

where we have used the lower-semicontinuity property for open sets of weak convergence of measures; see [33, Proposition 4.2.3]. Additionally, since elements in $\mathrm{M}(\Omega)$ are outer (and inner) regular, we have that for a Borel set B it holds that $\mu(B) = \inf \mu(O)$ where the infimum is taken over all open sets such that $B \subset O$; see [33, Proposition 4.2.1]. Thus,

$$|\mathrm{D}v^*|_p(B) \leq \alpha(B) \tag{3.3.2}$$

follows from (3.3.1) by taking the infimum over $\{O \text{ open } : B \subset O\}$.

In order to prove that v_n converges to v^* in the sense of intermediate convergence, we are only left to prove that $|\mathrm{D}v_n| \rightharpoonup |\mathrm{D}v^*|$ narrowly in $\mathrm{M}^+(\Omega)$ (see [33, Proposition 10.1.2]). The latter meaning that $\int_\Omega \varphi|\mathrm{D}v_n| \to \int_\Omega \varphi|\mathrm{D}v^*|$ for each continuous and bounded φ on Ω. Given that $\alpha \in \mathrm{M}^+(\Omega)$ we have that for each $\epsilon > 0$ there exists a compact set $\Lambda_\epsilon \subset \Omega$ such that

$$\alpha(\Omega \setminus \Lambda_\epsilon) \leq \epsilon.$$

Since $v_n \in K$, then $|\mathrm{D}v_n| \leq \alpha$, and hence for each $\epsilon > 0$ the compact set $\Lambda_\epsilon \subset \Omega$, is such that

$$|\mathrm{D}v_n|(\Omega \setminus \Lambda_\epsilon) \leq \epsilon, \qquad \text{for all } n \in \mathbb{N}.$$

Then, by Prokhorov Theorem (see [33, Theorem 4.2.3]), there is a subsequence of $\{|\mathrm{D}v_n|\}$ (not relabelled) that $|\mathrm{D}v_n| \rightharpoonup |\mathrm{D}v^*|$ narrowly in $\mathrm{M}^+(\Omega)$. That is, along a subsequence, v_n converges to v^* in the sense of intermediate convergence. This implies that

$$v^* \in \mathrm{BV}_{\Gamma_D}(\Omega),$$

by virtue of Theorem 3.1 and the fact that $v_n \in \mathrm{BV}_{\Gamma_D}(\Omega)$ for all $n \in \mathbb{N}$. $\square$

The above results particularly means that for a sequence $\{v_n\}$ in K that is bounded in $\mathrm{BV}(\Omega)$, there exists a subsequence that converges to some $u^* \in \mathrm{BV}(\Omega)$ in the sense of intermediate convergence. Further, $u^* \in \mathrm{BV}_{\Gamma_D}(\Omega)$ and also $u^* \in K$. A direct consequence of the above lemma is the following result.

Theorem 3.2. If $\alpha \in \mathrm{M}^+(\Omega)$, then there exists a unique solution to ($\mathbb{P}$) in $\mathrm{BV}_{\Gamma_D}(\Omega)$.

Proof. Consider an infimizing sequence $\{u_n\}$ for ($\mathbb{P}$). It follows that $\{u_n\}$ is bounded in $L^2(\Omega)$ and hence Lemma 2 is applicable. That is, there is a subsequence of $\{u_n\}$ (not relabelled) such that $u_n \rightharpoonup u^*$ in $L^2(\Omega)$, and $u_n \to u^*$ in the sense of the intermediate convergence for $\mathrm{BV}(\Omega)$, and further $u^* \in K$. Finally, by exploiting the weakly lower semicontinuity property of the objective functional in ($\mathbb{P}$), we have that $u^* \in K$ is a minimizer. $\square$

Next we discuss the case when α is a function.

3.3.2 The case when α is an integrable function

In this section, we let $\alpha : \Omega \to \mathbb{R}$ be a nonnegative and integrable function, leading to

$$U_{\Gamma_D}(\Omega) = W^{1,1}_{\Gamma_D}(\Omega).$$

This case can be interpreted (to some extent) as a special case of the one in the previous subsection under the assumption that α is a measure absolutely continuous with respect to the Lebesgue measure. However, we proceed in a slightly different fashion by considering α as a function and the state space contained in $W^{1,1}(\Omega)$; this provides further insight on bounded sequences in K and in Sobolev spaces. In this case, we have K given by

$$K = \{v \in W^{1,1}_{\Gamma_D}(\Omega) : |\nabla v|_p \leq \alpha \ \text{ a.e.}\}.$$

Next we state a version of Lemma 2 adapted to the current setting which can be used to prove existence of solutions to ($\mathbb{P}$).

Lemma 3. Let $\alpha \in L^1(\Omega)^+$ and $M > 0$, then every sequence $\{v_n\}$ in the set

$$K^* = K \cap \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \leq M\}$$

admits a subsequence satisfying

$$v_n \to v^* \text{ in } L^1(\Omega), \qquad \text{and} \qquad \int_\Omega |\nabla v_n(x)|_p \,\mathrm{d}x \to \int_\Omega |\nabla v^*(x)|_p \,\mathrm{d}x,$$

for some $v^* \in K^*$, which is also the weak limit in $W^{1,1}_{\Gamma_D}(\Omega)$ of the same subsequence.

The above can be seen as a consequence of equi-integrability of the set K. Recall that a family of functions $\mathcal{F} \subset L^1(\Omega)$ is *equi-integrable* provided that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every set $A \subset \Omega$ with $|A| < \delta$ we have that $\int_A |u| \,\mathrm{d}x < \epsilon$ for all $u \in \mathcal{F}$. Further, the Dunford-Pettis theorem states that if $\{u_n\}$ is a bounded sequence in $L^1(\Omega)$ and is equi-integrable, then $u_n \rightharpoonup u$ along a subsequence for some $u \in L^1(\Omega)$. Hence, since K is bounded in $W^{1,1}(\Omega)$, and the gradients are equi-integrable, it is simple to infer strong convergence in $L^1(\Omega)$ together with weak convergence of the gradients in $L^1(\Omega)$. The improvement of the latter convergence is done again via Prokhorov's result as in the proof of Lemma 2 leading to an equivalent of the intermediate convergence in $\mathrm{BV}(\Omega)$. The trace preservation follows directly from the same proof. Further note that the convergence determined does not imply strong convergence in $W^{1,1}(\Omega)$ since this space is not uniformly convex. Another formulation of the above lemma is that bounded sets with equi-integrable gradients are compact in $W^{1,1}_{\Gamma_D}(\Omega)$ when endowed with the metric

$$d(v,u) := \|u - v\|_{L^1(\Omega)} + \left| \int_\Omega |\nabla u(x)|_p \,\mathrm{d}x - \int_\Omega |\nabla v(x)|_p \,\mathrm{d}x \right|.$$

Using Lemma 3 and following the same argument as before for Theorem 3.2, we have

Theorem 3.3. If $\alpha \in L^1(\Omega)^+$, then there exists a unique solution to ($\mathbb{P}$) in $W^{1,1}_{\Gamma_D}(\Omega)$.

3.4 Existence Theory for the pre-dual problem ($\mathbb{P}^*$)

The focus of this section is on existence and uniqueness of solutions of problem ($\mathbb{P}^*$) under different functional analytic settings. In particular, we focus on two cases where $\boldsymbol{p}$ is either (i) a function or (ii) a Borel measure. In the first case, we let α be either a function or a measure; here, existence results are limited to $\mathrm{d} = 1$. On the other hand, in the second case we establish an existence and uniqueness result for $\boldsymbol{p}$ with arbitrary $\mathrm{d} \in \mathbb{N}$, for a specific class of α's (to be specified later). Furthermore, this second case requires a nonstandard space of vector measures with divergences in $L^2(\Omega)$. Remarkably, a version of the integration-by-parts formula still holds in this general setting; such a construct is rather recent [35]. We start with the case when $\boldsymbol{p}$ is a function.

3.4.1 The case when $\boldsymbol{p}$ is a function and α is either a function or a measure

We begin this section by considering that $\alpha \in L^1(\Omega)^+$ and J is defined as

$$J(\boldsymbol{p}) = \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x. \tag{3.4.1}$$

Moreover, we define

$$\|\boldsymbol{p}\|_{\alpha,2} := \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x + \|\mathrm{div}\,\boldsymbol{p}\|_{L^2(\Omega)},$$

for $\boldsymbol{p} \in C^\infty(\overline{\Omega})^{\mathrm{d}}$. We assume that if $\mathrm{d}=1$ and $\Gamma_N = \emptyset$ then α is not identically zero, and if $\mathrm{d} > 1$ then $\alpha > 0$ a.e. in Ω. The space $V_{\Gamma_N}(\Omega)$ is defined by

$$V_{\Gamma_N}(\Omega) = \overline{E(\Omega)}^{\|\cdot\|_{\alpha,2}}, \tag{3.4.2}$$

where

$$E(\Omega) := \{\boldsymbol{p} \in C^\infty(\overline{\Omega})^{\mathrm{d}} \;:\; \overline{\operatorname{supp}(\boldsymbol{p})} \cap \Gamma_N = \emptyset\}.$$

It follows that $V_{\Gamma_N}(\Omega)$ is a Banach space: If $\mathrm{d} > 1$, this is clear given that $\alpha > 0$ a.e. in Ω. If $\mathrm{d} = 1$, then $V_{\Gamma_N}(\Omega) = H^1_{\Gamma_N}(\Omega)$ which follows from the fact that $J(\boldsymbol{p}) + \frac{1}{2}\int_\Omega |\boldsymbol{p}'(x)|^2 \,\mathrm{d}x$ is an equivalent norm (to the usual one) on $H^1_{\Gamma_N}(\Omega)$. The latter is due to $J(\boldsymbol{p}) = \int_\Omega \alpha(x)|\boldsymbol{p}(x)| \,\mathrm{d}x$ being a seminorm in $H^1_{\Gamma_N}(\Omega)$ and norm on the constants, i.e. for $a \in \mathbb{R}$, $J(a) = |a|\alpha(\Omega) = 0$ iff $a = 0$; see [36, Chapter 1.4]. We can now establish existence of a solution to problem ($\mathbb{P}^*$).

Theorem 3.4. Let $\mathrm{d} = 1$, $\alpha \in L^1(\Omega)^+$, and if $\Gamma_N = \emptyset$ then suppose that α is not identically zero. Consider J as defined in (3.4.1) on $V_{\Gamma_N}(\Omega)$ as in (3.4.2). Then, there exists a unique solution to ($\mathbb{P}^*$).

Proof. The proof is based on the direct method. Let $\mathcal{J} \colon V_{\Gamma_N}(\Omega) \to \mathbb{R}$ be the objective function in ($\mathbb{P}^*$), that is,

$$\mathcal{J}(\boldsymbol{p}) := \frac{1}{2}\int_\Omega |\boldsymbol{p}'(x) - f(x)|^2 \,\mathrm{d}x + J(\boldsymbol{p}),$$

and let $\{\boldsymbol{p}_n\}_{n=1}^\infty$ in $V_{\Gamma_N}(\Omega)$ be an infimizing sequence of $\mathcal{J}$. Note that $\frac{1}{2}\int_\Omega |\boldsymbol{p}'(x)|^2 \,\mathrm{d}x + \int_\Omega \alpha|\boldsymbol{p}(x)| \,\mathrm{d}x$ is a norm in $H^1_{\Gamma_N}(\Omega)$; see [36, Chapter 1.4]. Hence, $\{\boldsymbol{p}_n\}_{n=1}^\infty$ is bounded in $V_{\Gamma_N}(\Omega)$, and there exists a weakly convergent (not relabeled) subsequence $\{\boldsymbol{p}_n\}_{n=1}^\infty$ such that $\boldsymbol{p}_n \rightharpoonup \bar{\boldsymbol{p}}$ in $H^1_{\Gamma_N}(\Omega)$. By the compact embedding of $H^1_{\Gamma_N}(\Omega) \hookrightarrow C(\overline{\Omega})$ (see [34, Chapter

6]) we have existence of a subsequence (not relabeled) $\boldsymbol{p}_n \to \bar{\boldsymbol{p}}$ in $C(\overline{\Omega})$. Finally, weak lower semicontinuity of $\mathcal{J}(\boldsymbol{p})$ yields that $\bar{\boldsymbol{p}} \in V_{\Gamma_N}(\Omega)$ is a solution to ($\mathbb{P}^*$). The strict convexity of the objective functional provides uniqueness to the solution. $\square$

An analogous approach can be considered when α is a nonnegative Borel measure (and not identically zero), that is, when $\alpha \in \mathrm{M}^+(\Omega)$. In particular, we set

$$J(\boldsymbol{p}) = \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha, \tag{3.4.3}$$

and we construct the space $V_{\Gamma_N}(\Omega)$ in the same way as in (3.4.2), but with the norm $\|\cdot\|_{\alpha,2}$ defined as

$$\|\boldsymbol{p}\|_{\alpha,2} := \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha + \|\mathrm{div}\, \boldsymbol{p}\|_{L^2(\Omega)},$$

and assuming that if $\mathrm{d} = 1$ and $\Gamma_N = \emptyset$ then α is not identically zero, and if $\mathrm{d} > 1$ then $\alpha(B) > 0$ if $|B| > 0$ and $B \subset \Omega$ is a Borel set.

The existence result of Theorem 3.4 follows mutatis mutandis: Since $\frac{1}{2}\int_\Omega |\boldsymbol{p}'(x)|^2 \,\mathrm{d}x + \int_\Omega |\boldsymbol{p}|\mathrm{d}\alpha$ is again a norm in $H^1_{\Gamma_N}(\Omega)$, see [36, Chapter 1.4], the exact argument is applicable in this case.

We can now focus on the case when $\boldsymbol{p}$ is a measure which provides a general setting for the problem of interest in terms of existence, uniqueness, and duality results.

3.4.2 The case when $\boldsymbol{p}$ is a measure and α is a function

We focus now on problem ($\mathbb{P}^*$) when J is defined as

$$J(\boldsymbol{p}) = \int_\Omega \alpha \,\mathrm{d}|\boldsymbol{p}|_q, \tag{3.4.4}$$

and $\boldsymbol{p}$ is a Borel measure. Notice that the above functional can be seen as a generalization of the functional in (3.4.1). The latter corresponds to the case when $\boldsymbol{p}$ is absolutely continuous with respect to the Lebesgue measure.

The functional analytic setting in this section, requires $\boldsymbol{p}$ to be a measure with divergence in $L^2(\Omega)$, and α to be measurable with respect to $|\boldsymbol{p}|_q$. We start with a proper definition of such spaces and their properties.

We disregard the possible "boundary conditions" for the variable $\boldsymbol{p}$, so that $\Gamma_N = \emptyset$, and we define $V_{\Gamma_N}(\Omega)$ as follows:

$$V_{\Gamma_N}(\Omega) := W(\Omega) = \{\boldsymbol{p} \in \mathrm{M}(\Omega)^{\mathrm{d}} : \operatorname{div} \boldsymbol{p} \in L^2(\Omega)\}, \tag{3.4.5}$$

where $\mathrm{M}(\Omega)^{\mathrm{d}}$ corresponds to the $\mathbb{R}^{\mathrm{d}}$-valued Borel measures in $\Omega \subset \mathbb{R}^{\mathrm{d}}$. Specifically, for $\boldsymbol{p} \in \mathrm{M}(\Omega)^{\mathrm{d}}$ it follows that $\boldsymbol{p} \in W(\Omega)$ if there exists $h \in L^2(\Omega)$ such that

$$\int_\Omega \nabla\varphi \cdot \mathrm{d}\boldsymbol{p} = -\int_\Omega \varphi h \,\mathrm{d}x, \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{3.4.6}$$

and we define $\operatorname{div} \boldsymbol{p} := h$. The space $W(\Omega)$ is a Banach space when endowed with the norm

$$\|\boldsymbol{w}\|_{W(\Omega)} := |\boldsymbol{w}|_q(\Omega) + \|\operatorname{div} \boldsymbol{w}\|_{L^2(\Omega)}, \tag{3.4.7}$$

where $q \in [1, +\infty]$ and

$$|\boldsymbol{w}|_q(\Omega) := \sup\left\{\langle \boldsymbol{w}, \boldsymbol{v}\rangle : \boldsymbol{v} \in C_c(\Omega)^{\mathrm{d}} \ \text{ with } \ |\boldsymbol{v}(x)|_p \le 1, \quad \forall x \in \Omega\right\}.$$

Note that above $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathrm{M}(\Omega)^{\mathrm{d}}$ and $C_c(\Omega)^{\mathrm{d}}$, and hence

$$\langle \boldsymbol{w}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{w} = \sum_{i=1}^{d} \int_\Omega v_i \,\mathrm{d}w_i.$$

Similarly to the definition of $|\boldsymbol{w}|_q(\Omega)$, we can define $|\boldsymbol{w}|_q(A)$ for any open set A, and subsequently for an arbitrary Borel set A. Hence, $|\boldsymbol{w}|_q$ induces a nonnegative measure (the *total variation measure of* $\boldsymbol{w}$); in addition $|\boldsymbol{w}|_q(\Omega) = \int_\Omega \mathrm{d}|\boldsymbol{w}|_q$. Note that the space $W(\Omega)$ contains regular maps, clearly if $\boldsymbol{p} \in C_c^1(\Omega)^{\mathrm{d}}$ then $\boldsymbol{p} \in W(\Omega)$, in this case "$\mathrm{d}|\boldsymbol{p}|_q = |\boldsymbol{p}|_q \, \mathrm{d}x$" where $\mathrm{d}x$ is the Lebesgue measure.

A note on the space $W(\Omega)$ is in order. Although one may be inclined to think that vector fields whose divergences are in $L^2(\Omega)$ would always have better regularity than just the measure type, this is not true. We consider an example developed by Šilhavý [35] to show otherwise. Let $u \in \mathrm{BV}(\Omega)$ with $\Omega \subset \mathbb{R}^2$, and define $\boldsymbol{p} = (\mathrm{D}u)^\perp$ with $(a_1, a_2)^\perp = (a_2, -a_1)$ with $\mathrm{D}u$ the distributional (measure valued) gradient of u; it follows that $\operatorname{div} \boldsymbol{p} = 0$. This can be seen as follows: $C^\infty(\overline{\Omega})$ is dense (in the sense of the intermediate convergence) in $\mathrm{BV}(\Omega)$, this means in particular that $\lim \int_\Omega \nabla\varphi \cdot \boldsymbol{p}_n \, \mathrm{d}x = \int_\Omega \nabla\varphi \cdot \mathrm{d}\boldsymbol{p}$ for such a smooth sequence defined as $\boldsymbol{p}_n = (\mathrm{D}u_n)^\perp$ with $u_n \in C^\infty(\overline{\Omega})$. Since also $\int_\Omega \nabla\varphi \cdot \boldsymbol{p}_n \, \mathrm{d}x = 0$, the result follows by taking the limit and from (3.4.6).

Following Šilhavý [35], we have a form of integration-by-parts formula together with a trace result. We denote by $\mathrm{Lip}^B(\Lambda)$ the space of Lipschitz maps $h : \Lambda \to \mathbb{R}$ for $\Lambda \subset \mathbb{R}^k$ and endow it with the norm

$$\|h\|_{\mathrm{Lip}^B(\Lambda)} := \mathrm{Lip}(h) + \sup_{x \in \Lambda} |h(x)|,$$

where $\mathrm{Lip}(h)$ is the Lipschitz constant of h on Λ. It follows that for each $\boldsymbol{p} \in W$ there exists a linear functional $\mathrm{N}_{\boldsymbol{p}} : \mathrm{Lip}^B(\partial\Omega) \to \mathbb{R}$ such that for all $v \in \mathrm{Lip}^B(\overline{\Omega})$ we have

$$\mathrm{N}_{\boldsymbol{p}}(v|_{\partial\Omega}) = \int_\Omega \nabla v \cdot \mathrm{d}\boldsymbol{p} + \int_\Omega v \operatorname{div} \boldsymbol{p} \, \mathrm{d}x. \tag{3.4.8}$$

Further, $\mathrm{N}_{\boldsymbol{p}}$ is bounded in the following sense

$$|\mathrm{N}_{\boldsymbol{p}}(g)| \leq (|\boldsymbol{p}|_q(\Omega) + |\operatorname{div} \boldsymbol{p}|(\Omega)) \, \|g\|_{\mathrm{Lip}^B(\partial\Omega)} \leq C \|\boldsymbol{p}\|_V \|g\|_{\mathrm{Lip}^B(\partial\Omega)},$$

for some $C > 0$, and all $\boldsymbol{p} \in W$ and all $g \in \mathrm{Lip}^B(\partial\Omega)$. Provided that $\boldsymbol{p}$ and v have enough differential regularity, we observe

$$\mathrm{N}_{\boldsymbol{p}}(v|_{\partial\Omega}) = \int_{\partial\Omega} v\, \boldsymbol{p} \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}$$

as expected. Thus, (3.4.8) is an extension of the usual integration-by-parts formula.

We are now ready to state and prove the existence and uniqueness result for problem ($\mathbb{P}^*$) under the setting above.

Theorem 3.5. Let $\alpha \in C(\overline{\Omega})$ be such that $\alpha(x) > 0$ for all $x \in \overline{\Omega}$, and consider J defined by (3.4.4) on $V_{\Gamma_N}(\Omega) = W(\Omega)$ as given in (3.4.5). Then, problem ($\mathbb{P}^*$) admits a unique solution.

Proof. Note first that J is well-defined given that α is measurable with respect to all Borel measures. Consider an infimizing sequence $\{\boldsymbol{p}_n\}$. Since $\min_{x\in\overline{\Omega}} \alpha(x) > 0$, then $\{\boldsymbol{p}_n\}$ is bounded in $V_{\Gamma_N}(\Omega)$. Hence, we can extract a subsequence (not relabelled) such that $\boldsymbol{p}_n \rightharpoonup \boldsymbol{p}^*$ in $\mathrm{M}(\Omega)^{\mathrm{d}}$ for some $\boldsymbol{p}^* \in \mathrm{M}(\Omega)^{\mathrm{d}}$ and $\operatorname{div} \boldsymbol{p}_n \rightharpoonup h$ in $L^2(\Omega)$ for some $h \in L^2(\Omega)$. Furthermore, for $\varphi \in C_c^\infty(\Omega)$ arbitrary

$$(\varphi, \operatorname{div} \boldsymbol{p}^*)_{L^2(\Omega)} = -\int_\Omega \nabla\varphi \cdot \mathrm{d}\boldsymbol{p}^* = -\lim_{n\to\infty} \int_\Omega \nabla\varphi \cdot \mathrm{d}\boldsymbol{p}_n = \lim_{n\to\infty} (\varphi, \operatorname{div} \boldsymbol{p}_n)_{L^2(\Omega)} = (\varphi, h)_{L^2(\Omega)},$$

so that $h = \operatorname{div} \boldsymbol{p}^*$, i.e., $\boldsymbol{p}^* \in W(\Omega)$.

Since the map $\boldsymbol{p} \mapsto |\boldsymbol{p}|_q$ is weakly lower semicontinuous, $\alpha\boldsymbol{p}_n \rightharpoonup \alpha\boldsymbol{p}^*$ in $\mathrm{M}(\Omega)^{\mathrm{d}}$, and $|\boldsymbol{q}|_q = \alpha|\boldsymbol{p}|_q$ for $\boldsymbol{q} = \alpha\boldsymbol{p}$, we have that $\boldsymbol{p}^*$ is a minimizer by a weakly lower semicontinuity argument. Uniqueness follows from the strict convexity of the objective functional. $\square$

At this point, one would be tempted to extend the result to the case where $\Gamma_N \neq \emptyset$, for example, by defining

$$V_{\Gamma_N}(\Omega) = \{\boldsymbol{p} \in W(\Omega) : \mathrm{N}_{\boldsymbol{p}}(v|_{\partial\Omega}) = 0 \quad \forall v \in \mathrm{Lip}^B_{\Gamma_D}(\overline{\Omega})\}. \tag{3.4.9}$$

While the space above is well-defined, it is not clear if the weak limits of sequences in the space also belong to it. In fact, if $\boldsymbol{p}_n \in V_{\Gamma_N}(\Omega)$ is bounded, then

$$\int_\Omega \nabla v \cdot \mathrm{d}\boldsymbol{p}_n = -\int_\Omega v \operatorname{div} \boldsymbol{p}_n \, \mathrm{d}x,$$

for each $v \in \mathrm{Lip}^B_{\Gamma_D}(\overline{\Omega})$. However, the weak limit along a subsequence argument is not enough to pass to the limit on the left hand side given that ∇v is not necessarily of compact support. This remains an open problem.

3.5 Duality relation between ($\mathbb{P}$) and ($\mathbb{P}^*$)

In this section, we discuss the dual problem corresponding to ($\mathbb{P}^*$). We start with the case when α is a Lebesgue measurable function and further subdivide it into two subsections. In Section 3.5.1 we first discuss the case when the pre-dual variable $\boldsymbol{p}$ is a function, and subsequently we assume that the variable $\boldsymbol{p}$ is a measure. Next in Section 3.5.2, we consider the case where α is a measure and the pre-dual variable $\boldsymbol{p}$ is a function. In general, we prove that

Problem ($\mathbb{P}$) is the Fenchel dual of Problem ($\mathbb{P}^*$).

In order to keep the discussion self-contained, we introduce the following notation and terminology. For an extended real valued function $\psi : X \to \mathbb{R} \cup \{\infty\}$ over a Banach space X, by ψ^* we denote its convex conjugate, which is defined by (e.g. see [11, p. 16])

$$\psi^* : X^* \to \mathbb{R} \cup \{\infty\}, \qquad \psi^*(x^*) = \sup_{x \in X} \left\{ \langle x^*, x \rangle_{X^*,X} - \psi(x) \right\}. \tag{3.5.1}$$

Provided that the operator div : $V \to L^2(\Omega)$ is defined for a Banach space V, and it is bounded, its adjoint $(\mathrm{div})^* : L^2(\Omega) \to V^*$ is well-defined and is given by $\langle (\mathrm{div})^* v, \boldsymbol{p}\rangle_{V^*,V} = (v, \mathrm{div}\, \boldsymbol{p})$ for all $v \in L^2(\Omega)$ and all $\boldsymbol{p} \in V$.

3.5.1 The case when α is a function

We first consider the case where α is a nonnegative Lebesgue measurable function and we accordingly set

$$J(\boldsymbol{p}) = \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \qquad \text{or} \qquad J(\boldsymbol{p}) = \int_\Omega \alpha \mathrm{d}|\boldsymbol{p}|_q,$$

in ($\mathbb{P}^*$) for the cases when $\boldsymbol{p}$ is a function or a measure, respectively. For each of the choices of J above, we will also establish the strong duality to ($\mathbb{P}$). We assume throughout this section (and for the sake of simplicity) that

$$\alpha \in C(\overline{\Omega}), \quad \text{and} \quad \alpha(x) > 0,$$

for all $x \in \overline{\Omega}$ as discussed in the introduction of this chapter, together with

$$U_{\Gamma_D}(\Omega) = W^{1,1}_{\Gamma_D}(\Omega), \qquad \text{and} \qquad G = \nabla\,,$$

and hence,

$$K = \{v \in W^{1,1}_{\Gamma_D}(\Omega) : |\nabla v|_p \leq \alpha \ \text{ a.e. in } \Omega\}.$$

Note that in Section 3.3 we proved the existence and uniqueness of solution to ($\mathbb{P}$).

We compute the dual problem to ($\mathbb{P}^*$) and show that it is given by problem ($\mathbb{P}$). Defining $F : L^2(\Omega) \to \mathbb{R}$ by

$$F(v) := \frac{1}{2}\int_\Omega |v(x) - f(x)|^2 \,\mathrm{d}x, \tag{3.5.2}$$

the problem ($\mathbb{P}^*$) can be written as

$$\inf_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)} J(\boldsymbol{p}) + F(\operatorname{div}\boldsymbol{p}), \tag{3.5.3}$$

for $\operatorname{div} : V_{\Gamma_N}(\Omega) \to L^2(\Omega)$, where the space $V_{\Gamma_N}(\Omega)$ is chosen based on whether $\boldsymbol{p}$ is a function or a measure.

By [11, p. 61], the Fenchel dual of ($\mathbb{P}^*$) with respect to the perturbation function

$$\phi : V_{\Gamma_N}(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\}, \qquad \phi(\boldsymbol{p}, u) = J(\boldsymbol{p}) + F(\operatorname{div}\boldsymbol{p} - u)$$

is given by

$$\inf_{u\in L^2(\Omega)} J^*(\operatorname{div}^* u) + F^*(-u), \tag{3.5.4}$$

where the convex conjugates $J^* : (V_{\Gamma_N}(\Omega))^* \to \mathbb{R} \cup \{\infty\}$, $F^* : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ of J and F are defined according to (3.5.1), see also [11, p. 17] for more details.

Duality when $\boldsymbol{p}$ is a function

Now we show that the problem ($\mathbb{P}$) is the dual to problem ($\mathbb{P}^*$). In this section, we assume that $V_{\Gamma_N}(\Omega)$ is given by (3.4.2), and that

$$J(\boldsymbol{p}) = \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x.$$

We start by proving the following result:

Theorem 3.6. For every $u \in L^2(\Omega)$, it holds that $J^*(\operatorname{div}^* u) = I_K(u)$.

We break the proof of the above theorem into the Lemmas 4 and 5, which we state after the following observation.

Remark 3.1. Observe that $J^*(\operatorname{div}^* u)$ only takes the value 0 or $+\infty$: By the definition of the convex conjugate J^*, for any $u \in L^2(\Omega)$ it holds that

$$J^*(\operatorname{div}^* u) \geq (u, \operatorname{div} \mathbf{0}) - \int_\Omega \alpha(x)|\mathbf{0}|_q \,\mathrm{d}x = 0. \tag{3.5.5}$$

If $J^*(\operatorname{div}^* u) > 0$, i.e. there exists a $\boldsymbol{p} \in V_{\Gamma_N}(\Omega)$ such that

$$\langle \operatorname{div}^* u, \boldsymbol{p} \rangle_{V_{\Gamma_N}(\Omega)^*, V_{\Gamma_N}(\Omega)} - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x > 0,$$

we can scale $\boldsymbol{p}$ by an arbitrarily large $\lambda \in \mathbb{R}^+$ leading to $J^*(\operatorname{div}^* u) = +\infty$.

Lemma 4. Let $u \in L^2(\Omega)$ with $J^*(\operatorname{div}^* u) = 0$. Then the following hold true:

(i) $u \in \mathrm{BV}(\Omega)$;

(ii) $|\mathrm{D}u|_p \leq \alpha$;

(iii) $\mathrm{D}u = \nabla u$ and $u \in W^{1,1}(\Omega)$;

(iv) $\gamma_0(u) = 0$ on Γ_D

and therefore $u \in K$.

Proof. (i) First we show that $J^*(\operatorname{div}^* u) = 0$ implies $u \in \mathrm{BV}(\Omega)$.

Suppose $u \notin \mathrm{BV}(\Omega)$. Then, since $C_0^1(\Omega)^{\mathrm{d}} \subset V_{\Gamma_N}(\Omega)$, we have that

$$\begin{aligned} J^*(\operatorname{div}^* u) &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ \langle \operatorname{div}^* u, \boldsymbol{p} \rangle_{V_{\Gamma_N}(\Omega)^*, V_{\Gamma_N}(\Omega)} - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \right\} \\ &\geq \sup_{\substack{\boldsymbol{p} \in C_0^1(\Omega)^{\mathrm{d}} \\ |\boldsymbol{p}|_q \leq 1}} \left\{ (u, \operatorname{div} \boldsymbol{p}) - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \right\} \\ &\geq \sup_{\substack{\boldsymbol{p} \in C_0^1(\Omega)^{\mathrm{d}} \\ |\boldsymbol{p}|_q \leq 1}} \{(u, \operatorname{div} \boldsymbol{p})\} - \int_\Omega \alpha(x) \,\mathrm{d}x \end{aligned} \tag{3.5.6}$$

Then, by using definition of a function of bounded variation, see [33, Definition 10.1.1], we have that the supremum on the right hand side of the above inequality is $+\infty$ if $u \notin \mathrm{BV}(\Omega)$ and hence, $u \in \mathrm{BV}(\Omega)$ if $J^*(\operatorname{div}^* u) < +\infty$.

(ii) As $u \in \mathrm{BV}(\Omega)$, we have that $\mathrm{D}u \in \mathrm{M}(\Omega)^{\mathrm{d}}$ and the inequality $|\mathrm{D}u|_p \leq \alpha$ is understood in the sense of (3.2.4). Hence, if

$$\int_O |\mathrm{D}u|_p - \int_O \alpha(x)\,\mathrm{d}x \leq 0, \tag{3.5.7}$$

for an arbitrary open set $O \subset \Omega$, then the required condition $|\mathrm{D}u|_p \leq \alpha$ immediately follows.

By the assumption $J^*(\operatorname{div}^* u) = 0$, and using integration by parts, we observe that

$$\begin{aligned}
0 = J^*\left(\operatorname{div}^* u\right) &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \langle \operatorname{div}^* u, \boldsymbol{p}\rangle - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x \\
&\geq \sup_{\boldsymbol{p} \in C_0^1(\Omega)^{\mathrm{d}}} \left\{\int_\Omega \boldsymbol{p}\mathrm{D}u - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x\right\} \\
&\geq \sup_{\substack{\boldsymbol{p} \in C_0^1(O)^{\mathrm{d}} \\ |\boldsymbol{p}|_q \leq 1}} \left\{\int_O \boldsymbol{p}\mathrm{D}u - \int_O \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x\right\} \\
&\geq \sup_{\substack{\boldsymbol{p} \in C_0^1(O)^{\mathrm{d}} \\ |\boldsymbol{p}|_q \leq 1}} \left\{\int_O \boldsymbol{p}\mathrm{D}u\right\} - \int_O \alpha(x)\,\mathrm{d}x \\
&= \int_O |\mathrm{D}u|_p - \int_O \alpha(x)\,\mathrm{d}x,
\end{aligned}$$

where the last inequality follows using the definition of $\int_O |\mathrm{D}u|_p$ and (3.5.7).

(iii) By (i) and (ii), it holds that

$$\int_S |\mathrm{D}u|_p \le \int_S \alpha(x)\,\mathrm{d}x, \tag{3.5.8}$$

for every Borel set S (see (3.2.3)), and especially for every Borel set of Lebesgue measure zero, it follows that $|\mathrm{D}u|_p$ vanishes on every set of measure zero, and hence $\mathrm{D}u$ is absolutely continuous w.r.t. the d-dimensional Lebesgue measure, and therefore $\mathrm{D}u = \nabla u$, i.e., the distributional gradient is a weak gradient. Thus, $u \in W^{1,1}(\Omega)$.

(iv) To obtain the boundary conditions on u, we will show that if $J^*(\mathrm{div}^* u) = 0$, then $\gamma_0(u) = 0$. Since $u \in \mathrm{BV}(\Omega)$, then using [33, Theorem 10.2.2] we have that $\gamma_0(u) \in L^1(\partial\Omega)$ and

$$\begin{aligned}
0 = J^*(\mathrm{div}^* u) &= \sup_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)} \left\{ \langle \mathrm{div}^* u, \boldsymbol{p}\rangle_{V^*,V} - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x \right\} \\
&= \sup_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)\cap C^1(\overline{\Omega})} \left\{ (u, \mathrm{div}\,\boldsymbol{p}) - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x \right\} \\
&= \sup_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)\cap C^1(\overline{\Omega})} \left\{ -\int_\Omega \boldsymbol{p}(x)\cdot\nabla u(x)\,\mathrm{d}x + \int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}\cdot\vec{n}\,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} \right. \\
&\qquad\qquad \left. - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x \right\}.
\end{aligned}$$

Whence for all $\boldsymbol{p} \in V_{\Gamma_N}(\Omega) \cap C^1(\overline{\Omega})$, we have

$$-\int_\Omega \boldsymbol{p}(x)\cdot\nabla u(x)\,\mathrm{d}x + \int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}\cdot\vec{n}\,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x \le 0.$$

Subsequently for all $\boldsymbol{p} \in V_{\Gamma_N}(\Omega) \cap C^1(\overline{\Omega})$, we arrive at

$$\left|\int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}\cdot\vec{n}\,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| \le \int_\Omega |\boldsymbol{p}(x)\ \nabla u(x)|\,\mathrm{d}x + \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q\,\mathrm{d}x. \tag{3.5.9}$$

To get (3.5.9), for a $\boldsymbol{p} \in V_{\Gamma_N}(\Omega) \cap C^1(\overline{\Omega})$, choose $s \in \{-1, 1\}$ such that

$$\int_{\Gamma_D} \gamma_0(u) s\boldsymbol{p} \cdot \vec{n} \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} \geq 0,$$

then for $\boldsymbol{w} = s\boldsymbol{p} \in V_{\Gamma_N}(\Omega) \cap C^1(\overline{\Omega})$, we obtain that

$$\begin{aligned}
\left| \int_{\Gamma_D} \gamma_0(u) \boldsymbol{p} \cdot \vec{n} \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} \right| &= \int_{\Gamma_D} \gamma_0(u) s\boldsymbol{p} \cdot \vec{n} \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} = \int_{\Gamma_D} \gamma_0(u) \boldsymbol{w} \cdot \vec{n} \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} \\
&\leq \int_{\Omega} \boldsymbol{w}(x) \, \nabla u(x) \, \mathrm{d}x + \int_{\Omega} \alpha(x) |\boldsymbol{w}(x)|_q \, \mathrm{d}x \\
&\leq \left| \int_{\Omega} s\boldsymbol{p}(x) \, \nabla u(x) \, \mathrm{d}x + \int_{\Omega} \alpha(x) |s\boldsymbol{p}(x)|_q \, \mathrm{d}x \right| \\
&\leq \int_{\Omega} |\boldsymbol{p}(x) \, \nabla u(x)| \, \mathrm{d}x + \int_{\Omega} \alpha(x) |\boldsymbol{p}(x)|_q \, \mathrm{d}x.
\end{aligned}$$

Now for $\varepsilon > 0$, by inner regularity [37, p. 95, Proposition 15.1], there exist closed subsets $\Gamma_D^\varepsilon \subset \Gamma_D$ and $\Omega^\varepsilon \subset \Omega$ such that

$$\left| \int_{\Omega} |\nabla u(x)|_p \, \mathrm{d}x - \int_{\Omega^\varepsilon} |\nabla u(x)|_p \, \mathrm{d}x \right| < \varepsilon, \quad \left| \int_{\Omega} \alpha(x) \, \mathrm{d}x - \int_{\Omega^\varepsilon} \alpha(x) \, \mathrm{d}x \right| < \varepsilon \quad \text{and}$$

$$\left| \int_{\Gamma_D} |\gamma_0(u)| \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} - \int_{\Gamma_D^\varepsilon} |\gamma_0(u)| \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} \right| < \varepsilon.$$

Then, by Urysohn's lemma there exists $\phi_\varepsilon \in C^\infty(\overline{\Omega})$ satisfying, $0 \leq \phi_\varepsilon \leq 1$, such that

$$\phi_\varepsilon = 1 \text{ on } \Gamma_D^\varepsilon \quad \text{and} \quad \phi_\varepsilon = 0 \text{ on } \Omega^\varepsilon \cup \overline{\Gamma}_N.$$

Then for any $\boldsymbol{q} \in C^1(\overline{\Omega})$, applying (3.5.9) to $\boldsymbol{p} = \boldsymbol{p}_\varepsilon := \phi_\varepsilon \, \boldsymbol{q} \in V_{\Gamma_N}(\Omega) \cap C^1(\overline{\Omega})$, we obtain that

$$\left|\int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}_\varepsilon \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| \leq \int_\Omega |\boldsymbol{p}_\varepsilon(x) \cdot \nabla u(x)| \,\mathrm{d}x + \int_\Omega \alpha(x)|\boldsymbol{p}_\varepsilon(x)|_q \,\mathrm{d}x$$

$$\leq \int_{\Omega\setminus\Omega^\varepsilon} |\boldsymbol{p}_\varepsilon(x) \cdot \nabla u(x)| \,\mathrm{d}x + \int_{\Omega\setminus\Omega^\varepsilon} \alpha(x)|\boldsymbol{p}_\varepsilon(x)|_q \,\mathrm{d}x.$$

Further, from

$$\int_{\Gamma_D} \gamma_0(u)\boldsymbol{q} \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} = \int_{\Gamma_D\setminus\Gamma_D^\varepsilon} \gamma_0(u)(\boldsymbol{q} - \boldsymbol{p}_\varepsilon) \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} + \int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}_\varepsilon \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}$$

we infer that

$$\left|\int_{\Gamma_D} \gamma_0(u)\boldsymbol{q} \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| - \left|\int_{\Gamma\setminus\Gamma_D^\varepsilon} \gamma_0(u)(\boldsymbol{p}_\varepsilon - \boldsymbol{q}) \cdot \vec{n}\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| \leq \left|\int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}_\varepsilon \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right|.$$

Next, using the two inequalities above in conjunction with

$$\left|\int_{\Omega\setminus\Omega^\varepsilon} |\boldsymbol{p}_\varepsilon(x) \cdot \nabla u(x)| + \alpha(x)|\boldsymbol{p}_\varepsilon(x)|_q \,\mathrm{d}x\right| \leq 2\varepsilon\|\boldsymbol{q}\|_{L^\infty(\Omega)},$$

and

$$\left|\int_{\Gamma\setminus\Gamma_D^\varepsilon} \gamma_0(u)(\boldsymbol{p}_\varepsilon - q) \cdot \vec{n}\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| \leq 2\varepsilon\|\boldsymbol{q}\|_{L^\infty(\Omega)},$$

we obtain that

$$\left|\int_{\Gamma_D} \gamma_0(u)\boldsymbol{q} \cdot \vec{n} \,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1}\right| \leq 4\varepsilon\|\boldsymbol{q}\|_{L^\infty(\Omega)}.$$

Now since $\boldsymbol{q} \in C^1(\overline{\Omega})$ and $\varepsilon > 0$ have been chosen arbitrarily, it follows that

$$\left| \int_{\Gamma_D} \gamma_0(u) \boldsymbol{q} \cdot \vec{n} \, \mathrm{d}\, \mathcal{H}^{\mathrm{d}-1} \right| = 0, \quad \text{for all } \boldsymbol{q} \in C^1(\overline{\Omega}).$$

This immediately leads to the required result, $\gamma_0(u) = 0$ a.e. on Γ_D, and the proof is complete.

□

Finally, the converse result remains to be shown, i.e., if $u \in K$, then $J^*(\mathrm{div}^* u) = 0$; we prove this next.

Lemma 5. If $u \in K$, then $J^*(\mathrm{div}^* u) = 0$.

Proof. Since $u \in K$, therefore by the definition of K, it holds that $u \in W^{1,1}_{\Gamma_D}(\Omega)$ and $|\nabla u|_p \leq \alpha$ a.e. in Ω. Next, using the definition of the convex conjugate J^* of J, we obtain that

$$\begin{aligned} J^*(\mathrm{div}^*\, u) &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ \langle \mathrm{div}^*\, u, \boldsymbol{p} \rangle_{V_{\Gamma_N}(\Omega)^*, V_{\Gamma_N}(\Omega)} - \int_\Omega \alpha(x) |\boldsymbol{p}(x)|_q \, \mathrm{d}x \right\} \\ &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ (u, \mathrm{div}\, \boldsymbol{p}) - \int_\Omega \alpha(x) |\boldsymbol{p}(x)|_q \, \mathrm{d}x \right\}. \end{aligned} \tag{3.5.10}$$

Next, by using the density of $C^1(\overline{\Omega})^{\mathrm{d}} \cap V_{\Gamma_N}(\Omega)$ in $V_{\Gamma_N}(\Omega)$, from (3.5.10), we obtain that

$$
\begin{aligned}
J^*(\operatorname{div}^* u) &= \sup_{\boldsymbol{p}\in C^1(\overline{\Omega})^{\mathrm{d}}\cap V_{\Gamma_N}(\Omega)} \left\{ \int_\Omega u(x)\ \operatorname{div} \boldsymbol{p}(x)\ \mathrm{d}x - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \right\} \\
&= \sup_{\boldsymbol{p}\in C^1(\overline{\Omega})^{\mathrm{d}}\cap V_{\Gamma_N}(\Omega)} \left\{ -\int_\Omega \boldsymbol{p}(x)\cdot\nabla u(x)\,\mathrm{d}x - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \right. \\
&\qquad\qquad \left. + \int_{\Gamma_D} \gamma_0(u)\boldsymbol{p}\cdot\vec{n}\,\mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} \right\} \\
&\leq \sup_{\boldsymbol{p}\in C^1(\overline{\Omega})^{\mathrm{d}}\cap V_{\Gamma_N}(\Omega)} \left\{ \int_\Omega |\nabla u(x)|_p|\boldsymbol{p}(x)|_q \,\mathrm{d}x - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_q \,\mathrm{d}x \right\} \\
&\leq 0.
\end{aligned}
$$

Thus, since $J^*(\operatorname{div}^* u)$ is nonnegative (we can set $\boldsymbol{p} \equiv 0$ in the definition of J^*), it follows that $J^*(\operatorname{div}^* u) = 0$ and the proof is complete. □

Next we compute the conjugate function of the function F.

Proposition 5. The conjugate function of F defined in (3.5.2) is given by

$$
F^*(u) = \frac{1}{2}\|u\|^2_{L^2(\Omega)} + (f, u) \tag{3.5.11}
$$

Proof. The proof is an immediate consequence of the definition of F^*. Recalling the definition of F^* and rearranging the terms, we obtain that

$$
\begin{aligned}
F^*(u) = \sup_{v\in L^2(\Omega)} \{(u,v) - F(v)\} &= \sup_{v\in L^2(\Omega)} \left\{ (u,v) - \frac{1}{2}\|v-f\|^2_{L^2(\Omega)} \right\} \\
&= \sup_{v\in L^2(\Omega)} \left\{ (u+f,v) - \frac{1}{2}\|v\|^2_{L^2(\Omega)} \right\} - \frac{1}{2}\|f\|^2_{L^2(\Omega)}.
\end{aligned}
$$

The result then follows from elementary calculus. □

Proposition 6 (Strong duality)**.** The problem ($\mathbb{P}$) is the Fenchel dual to problem ($\mathbb{P}^*$), and for these problems the equality strong duality, i.e.,

$$\inf_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)} J(\boldsymbol{p}) + F(\operatorname{div}\boldsymbol{p}) = -\inf_{u\in L^2(\Omega)} J^*(\operatorname{div}^* u) + F^*(-u) \tag{3.5.12}$$

holds. Further, $\overline{\boldsymbol{p}}$ solves problem ($\mathbb{P}^*$) if and only if the following extremality relation holds:

$$\overline{u} = f - \operatorname{div}\overline{\boldsymbol{p}} \quad \text{in } \Omega, \qquad \text{and} \qquad \nabla\overline{u} \in \partial J(\overline{\boldsymbol{p}}), \tag{3.5.13}$$

where $\overline{u}$ denotes the solution of ($\mathbb{P}$), and $\partial J(\boldsymbol{q})$ denotes the subdifferential of J at a point $\boldsymbol{q}$.

Proof. As a corollary to Theorem 3.6 and Proposition 5, it immediately follows that the dual of problem ($\mathbb{P}^*$) which is given in (3.5.4) is identical to problem ($\mathbb{P}$). Using that J and F are convex and continuous proper functions and bounded from below, equality (3.5.12) and the extremality relation (3.5.13) follow from the application of Theorem III.4.1 and Proposition III.4.1 in [11, p. 59] in its decomposed form, which is described in Remark III.4.2 therein, where condition (4.20) is satisfied by any $\boldsymbol{p} \in V_{\Gamma_N}(\Omega)$. □

Remark 3.2. The duality between ($\mathbb{P}$) and ($\mathbb{P}^*$) holds symmetrically, i.e. ($\mathbb{P}^*$) is the dual to problem ($\mathbb{P}$) as well. Defining the perturbation function $\phi : V_{\Gamma_N}(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ by $\phi(\boldsymbol{p}, u) = J(\boldsymbol{p}) + F(\operatorname{div}\boldsymbol{p} - u)$, following the framework in [11, pp. 58–60], ($\mathbb{P}^*$) can be written as

$$\inf_{\boldsymbol{p}\in V_{\Gamma_N}(\Omega)} \phi(\boldsymbol{p}, 0)$$

and the application of [11, (4.20) in p. 61] yields that ϕ is convex, l.s.c., proper, and bounded from below, given that the same holds true for J and F. Thus, it follows from [11, p. 49] that $\phi^{**} = \phi$ and that ($\mathbb{P}^*$) is identical to its bidual problem

$$\inf_{p \in V_{\Gamma_N}(\Omega)} \phi^{**}(\boldsymbol{p}, 0),$$

i.e., to the dual problem to ($\mathbb{P}$) with respect to the perturbation function ϕ^*.

Note that though we assume $\alpha \in C(\overline{\Omega})$, results within this section hold for $\alpha \in L^1(\Omega)$. However recall that the existence result for this case (c.f. Section 3.4.1) only stands in the case $\mathrm{d} = 1$.

Duality when $\boldsymbol{p}$ is a measure

We consider now the duality result in the framework of the variable $\boldsymbol{p}$ in the space of Borel measures with $L^2(\Omega)$ divergences. Surprisingly, the dual problem remains the same. We recall in this framework that

$$\Gamma_N = \emptyset \qquad \text{and} \qquad V_{\Gamma_N}(\Omega) = W(\Omega),$$

as in (3.4.5). Since we already assumed that $\alpha \in C(\overline{\Omega})$ is positive, existence of a unique solution follows from Theorem 3.5.

We again propose to follow the Fenchel dual approach and let $J : V_{\Gamma_N}(\Omega) \to \mathbb{R}$ and $F : L^2(\Omega) \to \mathbb{R}$ be

$$J(\boldsymbol{p}) := \int_\Omega \alpha \,\mathrm{d}|\boldsymbol{p}|_q \qquad \text{and} \qquad F(v) := \frac{1}{2}\int_\Omega |v(x) - f(x)|^2 \,\mathrm{d}x.$$

In this setting, it also holds that for every $u \in L^2(\Omega)$ we have $J^*(\operatorname{div}^* u) = I_K(u)$, i.e., Theorem 3.6. In fact, we show that the Lemmas 4 and 5 remain true under the functional analytic setting of this section.

Proof of Lemma 4. (i) By choosing $\mathrm{d}\boldsymbol{p} = \hat{\boldsymbol{p}}(x)\,\mathrm{d}x$ with $\hat{\boldsymbol{p}} \in C_0^1(\Omega)^{\mathrm{d}}$ and $|\hat{\boldsymbol{p}}|_q \leq 1$, we obtain the same inequality as (3.5.6). Moreover, by following similar steps as before, we can show that $u \in \mathrm{BV}(\Omega)$.

(ii) The proof follows identically as before by considering $\mathrm{d}\boldsymbol{p}_\epsilon = \hat{\boldsymbol{p}}_\epsilon(x)\,\mathrm{d}x$ with $\hat{\boldsymbol{p}}_\epsilon \in C_0^1(\Omega)^{\mathrm{d}}$.

(iii) The same proof applies.

(iv) Note that $|\nabla u| \leq \alpha$ a.e. implies that $u \in W^{1,\infty}(\Omega)$, given that $\alpha \in C(\overline{\Omega})$. As shown before, in Remark 3.1, $J^*(\mathrm{div}^* u) < +\infty$ implies $J^*(\mathrm{div}^* u) = 0$ which yields for $\mathrm{d}\boldsymbol{p} = \hat{\boldsymbol{p}}(x)\,\mathrm{d}x$ with $\hat{\boldsymbol{p}} \in C^1(\overline{\Omega})$ the following:

$$-\int_\Omega \hat{\boldsymbol{p}}(x) \cdot \nabla u(x)\,\mathrm{d}x + \int_{\partial\Omega} u\hat{\boldsymbol{p}} \cdot \vec{n}\,\mathrm{d}\mathcal{H}^{\mathrm{d}-1} - \int_\Omega \alpha(x)|\hat{\boldsymbol{p}}(x)|_q\,\mathrm{d}x \leq 0, \text{ for all } \hat{\boldsymbol{p}} \in C^1(\overline{\Omega})^N,$$

and the proof follows identically leading to $u|_{\Gamma_D} = 0$.

□

Proof of Lemma 5. Let $u \in K$, then from the definition of K, it follows that $u \in W_0^{1,1}(\Omega)$ and $|\nabla u|_p \leq \alpha$. Furthermore,

$$\begin{aligned}
J^*(\mathrm{div}^* u) &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ -\int_\Omega \nabla u \cdot \mathrm{d}\boldsymbol{p} + \mathrm{N}_{\boldsymbol{p}}(u|_{\partial\Omega}) - \int_\Omega \alpha\,\mathrm{d}|\boldsymbol{p}|_q \right\} \\
&= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ -\int_\Omega \nabla u \cdot \mathrm{d}\boldsymbol{p} - \int_\Omega \alpha\,\mathrm{d}|\boldsymbol{p}|_q \right\} \\
&\leq \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ \int_\Omega |\nabla u|_p\,\mathrm{d}|\boldsymbol{p}|_q - \int_\Omega \alpha\,\mathrm{d}|\boldsymbol{p}|_q \right\} \\
&= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ \int_\Omega (|\nabla u|_p - \alpha)\,\mathrm{d}|\boldsymbol{p}|_q \right\} \\
&\leq 0
\end{aligned}$$

i.e., it follows that $J^*(\mathrm{div}^* u) = 0$. The proof is complete. □

From Theorem 3.6, it follows that the duality result of Proposition 6 also holds in this setting; the proof is straightforward.

3.5.2 The case when α is a measure

In this section, we will extend the duality result of Proposition 6 by letting α be a nonnegative Borel measure, that is, $\alpha \in \mathrm{M}^+(\Omega)$. However, $\boldsymbol{p}$ is a function in this setting. In its more general form, in problem ($\mathbb{P}^*$), we set

$$J(\boldsymbol{p}) = \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha\,. \tag{3.5.14}$$

The results in this subsection are a generalization of the case of the Lebesgue integrable constraint α, that was presented in Section 3.5.1. We shall assume that $\boldsymbol{p} \in V_{\Gamma_N}(\Omega)$, see (3.4.2) for the definition of $V_{\Gamma_N}(\Omega)$.

Since $\alpha \in \mathrm{M}^+(\Omega)$, as we discussed at the beginning of this chapter, we let

$$U_{\Gamma_D}(\Omega) = \mathrm{BV}_{\Gamma_D}(\Omega) \qquad \text{and} \qquad G = \mathrm{D}, \tag{3.5.15}$$

the distributional gradient, and hence

$$K = \{v \in \mathrm{BV}_{\Gamma_D}(\Omega) : |\mathrm{D}v|_p \leq \alpha\}.$$

We prove that the dual problem to ($\mathbb{P}^*$) is given by ($\mathbb{P}$) with inequality constraint $|\mathrm{D}u|_q \leq \alpha$ being understood in the sense of (3.2.5).

Recall that in Section 3.3 we have shown existence and uniqueness of solution to ($\mathbb{P}$). Here, we show that dual of problem ($\mathbb{P}^*$) is given by ($\mathbb{P}$). We start by writing ($\mathbb{P}^*$) as

$$\inf_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} F(\operatorname{div} \boldsymbol{p}) + J(\boldsymbol{p}),$$

with $J : V_{\Gamma_N}(\Omega) \to \mathbb{R}$ as in (3.5.14) and $F : L^2(\Omega) \to \mathbb{R}$, as before, given by

$$F(v) := \frac{1}{2} \int_\Omega |v(x) - f(x)|^2 \, \mathrm{d}x.$$

We prove now that Theorem 3.6 holds also true in the current setting. For brevity, we only discuss the essential modifications needed in Lemmas 4 and 5.

Proof of Theorem 3.6. This proof follows along the same lines as the proof to Theorem 3.6. We start by observing that the discussion in Remark 3.1 holds in the current setting as well, i.e., $J^*(\operatorname{div}^* u)$ only takes the values 0 and $+\infty$. We now prove the result.

The proof that $J^*(\operatorname{div}^* u) = 0$ implies that $u \in K$ follows along the lines of Lemma 4. Indeed (i) and (ii) in Lemma 4 apply directly, and for (iv) everything follows in the same way, when $\mathrm{D}u$ and $\mathrm{d}\alpha$ are measures instead of the functions ∇u and $\alpha(x)$.

On the other hand, the converse (Lemma 5), i.e., $u \in K$ implies that $J^*(\operatorname{div}^* u) = 0$ follows from the calculations below. Recall that if $u \in K$, then $u \in \mathrm{BV}_{\Gamma_D}(\Omega)$ and $|\mathrm{D}u|_p \le \alpha$ in the sense of (3.2.5). Therefore,

$$\begin{aligned}
0 \le J^*(\operatorname{div}^* u) &= \sup_{\boldsymbol{p} \in V_{\Gamma_N}(\Omega)} \left\{ \langle \operatorname{div}^* u, \boldsymbol{p} \rangle_{(V_{\Gamma_N}(\Omega))^*, V_{\Gamma_N}(\Omega)} - \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha \right\} \\
&= \sup_{\boldsymbol{p} \in C^1(\overline{\Omega})^{\mathrm{d}} \cap V_{\Gamma_N}(\Omega)} \left\{ - \int_\Omega \boldsymbol{p} \, \mathrm{D}u - \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha + \int_{\Gamma_D} \gamma_0(u) \boldsymbol{p} \cdot \vec{n} \, \mathrm{d}\,\mathcal{H}^{\mathrm{d}-1} \right\} \\
&= \sup_{\boldsymbol{p} \in C^1(\overline{\Omega})^{\mathrm{d}} \cap V_{\Gamma_N}(\Omega)} \left\{ - \int_\Omega \boldsymbol{p} \, \mathrm{D}u - \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha \right\} \\
&\le \sup_{\boldsymbol{p} \in C^1(\overline{\Omega})^{\mathrm{d}} \cap V_{\Gamma_N}(\Omega)} \left\{ \int_\Omega |\boldsymbol{p}|_q |\mathrm{D}u|_p - \int_\Omega |\boldsymbol{p}|_q \mathrm{d}\alpha \right\} \le 0,
\end{aligned}$$

and the proof is complete. □

Finally, note that it follows identically as before that the polar function of F is given by

$$F^*(u) := \frac{1}{2}\|u\|^2_{L^2(\Omega)} - (f,u). \tag{3.5.16}$$

Hence, the duality result of Proposition 6 also holds in the case where α is a measure, with ∇ replaced by D.

3.6 A Finite Element Method with Applications

The purpose of this section is to illustrate the applicability of the proposed primal-dual approach to solve Problems ($\mathbb{P}$) and ($\mathbb{P}^*$). We assume throughout this section that $p = q = 2$.

Recall that Problem ($\mathbb{P}$) in the case that $\alpha \in L^\infty(\Omega)^+$ is given by

$$\min \quad \frac{1}{2}\int_\Omega |u(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x \quad \text{over } u \in W^{1,\infty}_{\Gamma_D}(\Omega), \quad \text{s.t. } |\nabla u|_2 \le \alpha \text{ a.e.} \tag{3.6.1}$$

and that the pre-dual problem ($\mathbb{P}^*$) is given by

$$\min \quad \frac{1}{2}\|\operatorname{div} \boldsymbol{p} - f\|^2_{L^2(\Omega)} + \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_2\,\mathrm{d}x \quad \text{over } \boldsymbol{p} \in V_{\Gamma_N}(\Omega)\,. \tag{3.6.2}$$

Now the first order (necessary and sufficient) optimality condition corresponding to (3.6.2) in the strong form is given by: Find $\boldsymbol{p} : \Omega \to \mathbb{R}^{\mathrm{d}}$ satisfying

$$\begin{aligned} -\nabla\left(\operatorname{div} \boldsymbol{p} - f\right) + \partial\left(\|\alpha|\boldsymbol{p}|_2\|_{L^1(\Omega)}\right) \ni 0 \quad &\text{in } \Omega,\\ \boldsymbol{p}\cdot\vec{n} = 0 \quad &\text{on } \Gamma_N, \end{aligned} \tag{3.6.3}$$

where ∂ denotes the subdifferential operator. In order to solve (3.6.3), recall from the extremality conditions (3.5.13), that if $\overline{u}$ and $\overline{\boldsymbol{p}}$ are solutions to (3.6.1) and (3.6.2), respectively, they satisfy

$$\overline{u} := -\mathrm{div}\,\overline{\boldsymbol{p}} + f \quad \text{a.e. in } \Omega\,. \tag{3.6.4}$$

Then, a primal-dual system arises from (3.6.3) and (3.6.4), which in the weak form becomes the following variational inequality of second kind: Find $(\boldsymbol{p}, u) \in V_{\Gamma_N}(\Omega) \times L^2(\Omega)$ such that

$$(u, -\mathrm{div}\,(\mathbf{v} - \boldsymbol{p})) + \int_\Omega \alpha(x)|\mathbf{v}(x)|_2\,\mathrm{d}x - \int_\Omega \alpha(x)|\boldsymbol{p}(x)|_2\,\mathrm{d}x \geq 0 \qquad \text{for all } \mathbf{v} \in V_{\Gamma_N}(\Omega), \tag{3.6.5}$$

$$(u, w) + (\mathrm{div}\,\boldsymbol{p}, w) = (f, w) \quad \text{for all } w \in L^2(\Omega)\,. \tag{3.6.6}$$

Due to their nonlinear and nonsmooth nature, it is challenging to solve (3.6.5)–(3.6.6).

We shall proceed by introducing the Huber-regularization for $\phi(\boldsymbol{p}) := |\boldsymbol{p}|_2$ in the last term under the integral in (3.6.2). This regularization is C^1 with piecewise differentiable first order derivative. Therefore one can use Newton type methods to solve the resulting regularized system. For a given parameter $\tau > 0$, the Huber regularization of ϕ is given by

$$\phi_\tau(\boldsymbol{p}) := \begin{cases} |\boldsymbol{p}|_2 - \frac{1}{2}\tau, & |\boldsymbol{p}|_2 > \tau, \\ \frac{1}{2\tau}|\boldsymbol{p}|_2^2, & |\boldsymbol{p}|_2 \leq \tau\,. \end{cases}$$

As $\tau \to 0$, $\phi_\tau(\boldsymbol{p}) \to \phi(\boldsymbol{p})$. Moreover, $\phi_\tau(\cdot)$ is continuously differentiable with derivative given by

$$\phi'_\tau(\boldsymbol{p}) := \begin{cases} \frac{\boldsymbol{p}}{|\boldsymbol{p}|_2}, & |\boldsymbol{p}|_2 > \tau \\ \frac{1}{\tau}\boldsymbol{p}, & |\boldsymbol{p}|_2 \leq \tau\,. \end{cases}$$

Replacing $\phi(\cdot) = |\cdot|_2$ in (3.6.2) by $\phi_\tau(\cdot)$, the regularized primal-dual system corresponding to (3.6.5)–(3.6.6) is given by

$$(u, -\operatorname{div} \mathbf{v}) + \int_\Omega \alpha \phi_\tau'(\boldsymbol{p}) \cdot \mathbf{v} = 0, \qquad \text{for all } \mathbf{v} \in V_{\Gamma_N}(\Omega), \tag{3.6.7}$$

$$(u, w) + (\operatorname{div} \boldsymbol{p}, w) = (f, w), \qquad \text{for all } w \in L^2(\Omega). \tag{3.6.8}$$

Notice, that $\phi_\tau'(\cdot)$ is piecewise differentiable and the second order derivative is given by

$$\phi_\tau''(\boldsymbol{p}) =:= \begin{cases} \frac{1}{|\boldsymbol{p}|_2}\left(I_{d\times d} - \frac{\boldsymbol{p}\boldsymbol{p}^\top}{|\boldsymbol{p}|_2^2}\right), & |\boldsymbol{p}|_2 > \tau\,, \\ \frac{1}{\tau} I_{d\times d}, & |\boldsymbol{p}|_2 \le \tau\,, \end{cases}$$

where $I_{d\times d}$ is the $d \times d$ identity matrix.

3.6.1 Finite Element Discretization

We discretize $\boldsymbol{p}$ and u using the lowest order Raviart-Thomas ($\mathbb{RT}_0$) and piecewise constant ($\mathbb{P}_0$) finite elements, respectively. Whenever needed, the integrals are computed using Gauss-quadrature which is exact for polynomials of degree less than equal to 4. For each fixed τ, we solve the discrete saddle point system (3.6.7)–(3.6.8) using Newton's method with backtracking line-search strategy. We stop the Newton iteration when each residual in $L^2(\Omega)$-norm is smaller than 10^{-8}. Each linear solve during the Newton iteration is done using direct solve. Starting from $\tau = 10$, a continuation strategy is applied where in each step we reduce τ by a factor of 1.30 until τ is less than equal to 10^{-6}. We initialize the Newton's method by zero. To compute the solution for the next τ, we use the solution corresponding to the previous τ as the initial iterate for the Newton's method.

3.6.2 Numerical Examples

Next, we report results from various numerical experiments. In all examples we consider $\Omega = (0,1)\times(0,1)$ and we assume that $\Gamma_N = \emptyset$, $\Gamma_D = \partial\Omega$, and hence pure Dirichlet boundary conditions on u on the entire boundary are set. In the first example, we construct exact solutions $(\boldsymbol{p}, u)$ when f and α are constants. We compare these exact solutions with our finite element approximation. These experiments validate our finite element implementation for constant α and f and provide optimal rate of convergence. Additionally, we solve ($\mathbb{P}^*$) and ($\mathbb{P}$) first for a fixed α and vary f and next we fix f and vary α. In our second experiment, we consider a more generic f with different features such as cone, valley and flat regions. In our final experiment, we consider α to be a measure.

Example 1. Note initially that if α and f are constants, it is possible to calculate an exact solution. By setting

$$m(x) := \min\left\{f, \alpha(x-1), \alpha(x-0)\right\},$$

the exact u and $\boldsymbol{p}$ are given as:

$$u(x, y) = \min\{m(x), m(y)\},$$

and $\boldsymbol{p}(x, y) =$

$$\begin{cases} \left(\frac{1}{\alpha}(m(y)-m(x))\operatorname{sgn}(0.5-x)\left(f-\frac{1}{2}\left(m(x)+m(y)\right)\right), 0\right)^\top, & \text{if } |x-0.5| > |y-0.5|, \\ \left(0, \frac{1}{\alpha}(m(x)-m(y))\operatorname{sgn}(0.5-y)\left(f-\frac{1}{2}\left(m(x)+m(y)\right)\right)\right)^\top, & \text{otherwise.} \end{cases}$$

Notice that in this example, u is again Lipschitz continuous. In Figure 3.2 (top panel), we have shown the $\|\boldsymbol{p}-\boldsymbol{p}_h\|_{L^2(\Omega)}$ and $\|u-u_h\|_{L^2(\Omega)}$ when $\Omega = (0,1)^2$, $f = 1$, and $\alpha = 1$. We observe optimal rate of convergence in both cases. In the bottom row, the left panel shows u_h, the middle panel shows $|\nabla u_h|_2$, and the right panel shows $\boldsymbol{p}_h$. We observe that, in this example, the gradient constraints are active in the entire region. Notice, that at the corners (which are sets of measure zero), the gradient is undefined.

Next, we fix the number of $\boldsymbol{p}_h$ and u_h unknowns to be 197,120 and 131,072, respectively. Figure 3.3 shows our results for 3 different experiments. In all cases, we have used a fixed $\alpha = 1$. The rows correspond to $u_h, |\nabla u_h|_2$, and $\boldsymbol{p}_h$. The columns correspond to $f = 1$,

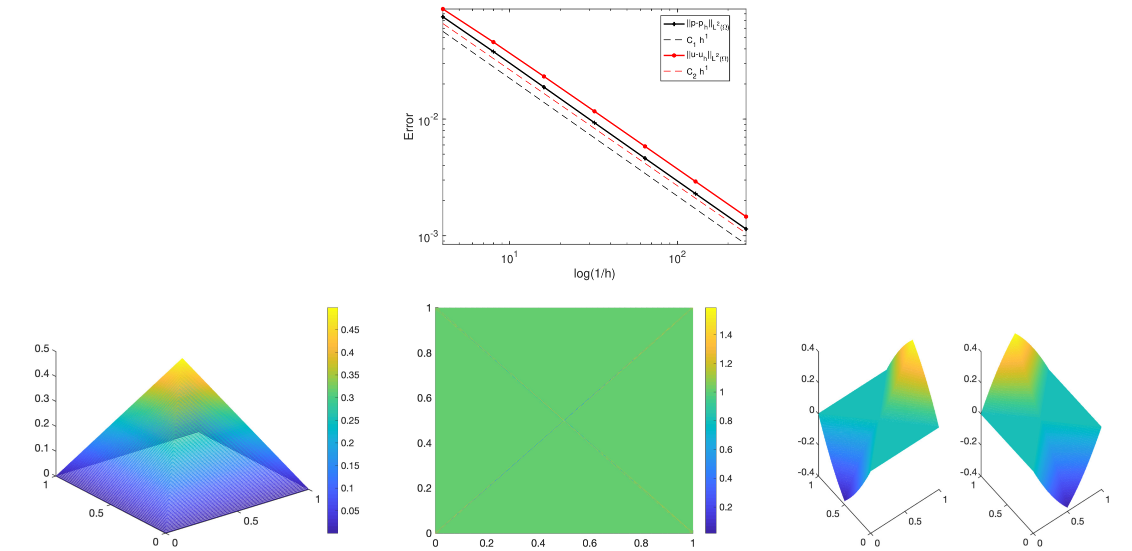


Figure 3.2: **Example 1:** *Top panel* - We have shown the $L^2(\Omega)$-error between the computed solution $(u_h, \boldsymbol{p}_h)$ and the exact solution $(u, \boldsymbol{p})$. The optimal linear rate of convergence is observed. Bottom panel: Computed u_h (left), $|\nabla u_h|_2$ (middle), $\boldsymbol{p}_h$ (right). Notice that we are touching the constraints in the entire region, except where the gradient is undefined. We have omitted the plots of the exact $(u, \boldsymbol{p})$ as they look exactly same as $(u_h, \boldsymbol{p}_h)$.

$f = 0.25$, and $f = 0.1$. As expected, for a large value of f, we observe steep slope, but for smaller values of f, plateau regions appear. We also observe that the active region shrinks as f decreases since the gradient is zero at the top of the plateau. The dual variable $\boldsymbol{p}_h$ also changes significantly with f.

In Figure 3.4 we again show results from three different experiments. In all cases, we have used a fixed $f = 1$. The rows correspond to $u_h, |\nabla u_h|_2$, and $\boldsymbol{p}_h$. The columns corresponds to

$$\alpha = 1, \;\; \alpha = 0.5, \;\; \text{and } \alpha = \begin{cases} 0.75, & y \leq 1 - x, \\ 1.0, & \text{otherwise}, \end{cases}$$

respectively. In all cases, we observe that the gradient constraints are active in the entire domain (except on a set of measure zero). For the case of piecewise constant α, nonsmoothness in $\boldsymbol{p}_h$ is clearly visible.

Example 2. In this example, we set

$$f = 10^{-3} + u_0,$$

where

$$u_0 := \begin{cases} \min\{0.2, 0.5(x^2+y^2)\}, & y \leq 1-x, \\ \max\left\{1 - 5\sqrt{(x-0.7)^2+(y-0.7)^2}, \min\left\{0.2, 0.5(x^2+y^2)\right\}\right\}, & 1-x < y, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, we set $\alpha = 2.5$. Figure 3.5 (left panel) shows a plot of f. Figure 3.5 (right panel) shows the computed solution u_h. In Figure 3.6, we have shown $|\nabla u_h|_2$ (left panel), and $\boldsymbol{p}_h$ (right panel). Notice that the gradient constraints are active. Moreover, we also observe significant flat regions, where the gradient is zero.

In Figure 3.7, we have displayed u_h, $|\nabla u_h|_2$ and $\boldsymbol{p}_h$ when $\alpha = 1.5$.

Example 3. In this example, we consider f given by

$$f(x,y) = \begin{cases} 0.25, & \text{if } (x,y) \in \Omega,\ 0.5 \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

The main novelty and challenge in this example is the fact that we let α to be a measure. Specifically

$$\int_\Omega v \mathrm{d}\alpha = \int_\Omega v \, \mathrm{d}x + 10^2 \int_\omega v \mathrm{d}\mathcal{H}^1,$$

for all $v \in C_c^\infty(\Omega)$ and where $\omega := \{(x,y) \in \Omega : y = 0.5\}$, i.e., α consists of the Lebesgue measure $\mathrm{d}x$ and a weighted line measure on ω.

Let h denotes the mesh size, then α^h is approximated as

$$\mathrm{d}\alpha^h = \left(1 + \frac{\chi_{\omega^h}(x,y)}{h}\right) \mathrm{d}x,$$

where

$$\omega^h := \{(x,y) \in \Omega \,:\, 0.5 - 10^2 h \leq y \leq 0.5,\ x \in (0,1)\}.$$

As $h \downarrow 0$, we approximate the measure in the sense that $\int_\Omega v \mathrm{d}\alpha^h \to \int_\Omega v \mathrm{d}\alpha$ for all $v \in C_c^\infty(\Omega)$.

When $h = 8.4984 \times 10^{-5}$, the results are shown in Figure 3.8 (top row). Finally, when $h = 2.1412 \times 10^{-5}$ the results are provided in Figure 3.8 (bottom row). We notice that as $h \downarrow 0$, we indeed approximate the measure: In fact, we observe a clear discontinuity on the solution u, the size of the jump is below 100 which is the upper bound on the distributional gradient on ω.

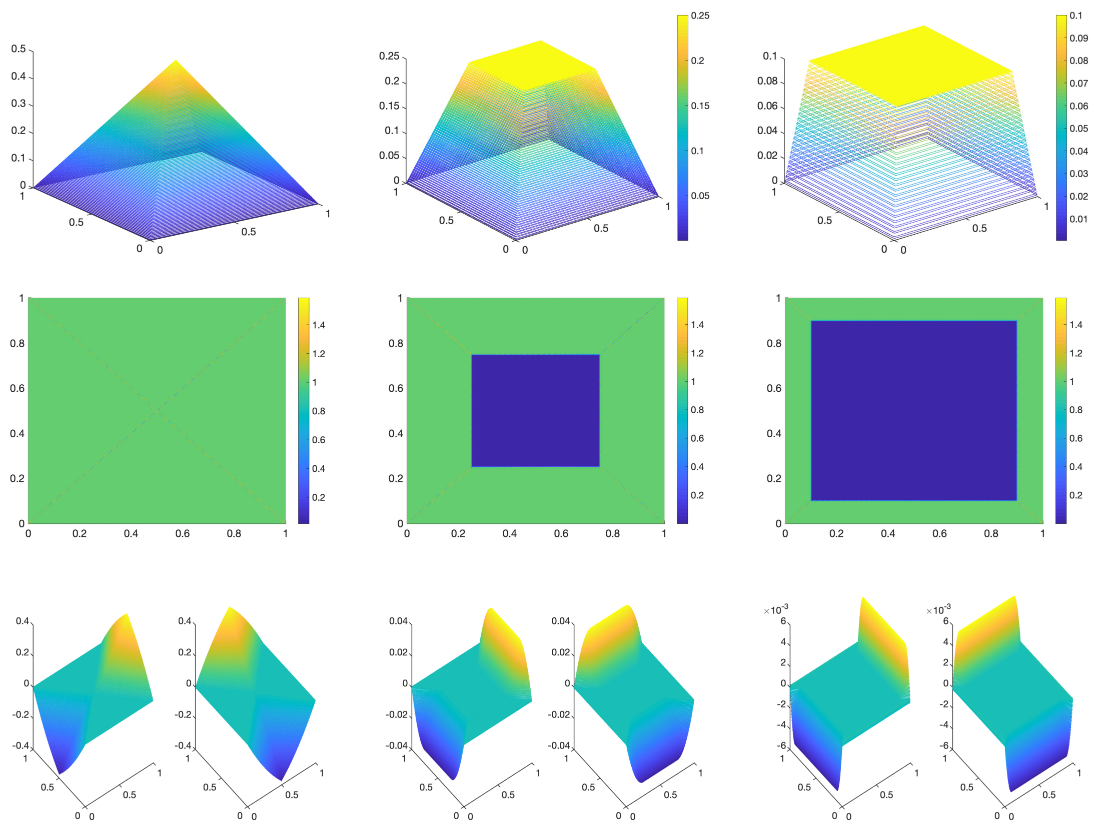


Figure 3.3: **Example 1 (fixed α, varying f).** The rows correspond to u_h, $|\nabla u_h|_2$, and $\boldsymbol{p}_h$. The columns represent $f = 1$, 0.25, and 0.1. In all cases, we observe that the gradient constraints are active but the activity region shrinks as f decreases, this is expected since the gradient on the plateau region is zero. The behavior of $\boldsymbol{p}$ also changes considerably with f.

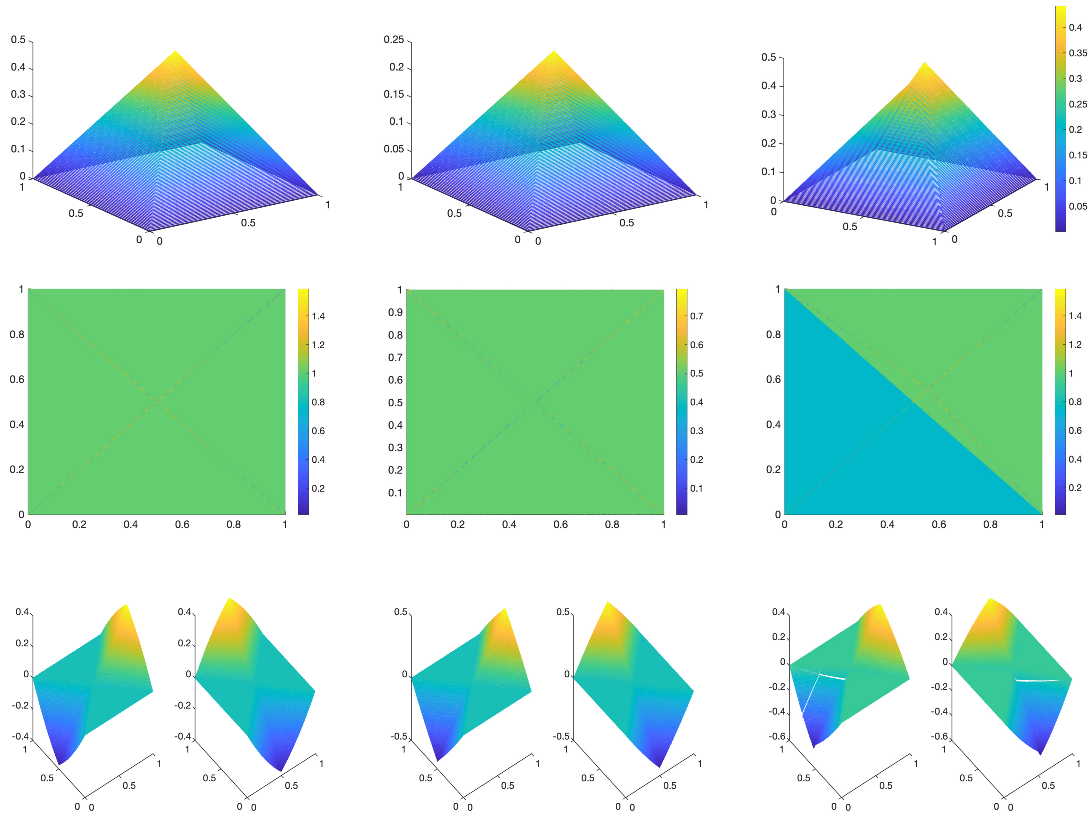


Figure 3.4: **Example 1 (fixed f, varying α).** The rows correspond to u_h, $|\nabla u_h|_2$, and $\boldsymbol{p}_h$. The first two columns represent constant $\alpha = 1$ and 0.5. The third column corresponds to α with jump discontinuity. In all cases, we observe that the gradient constraints are active in the entire region, except on a set of measure zero. Moreover, discontinuity in $\boldsymbol{p}$ in the last column is clearly visible.

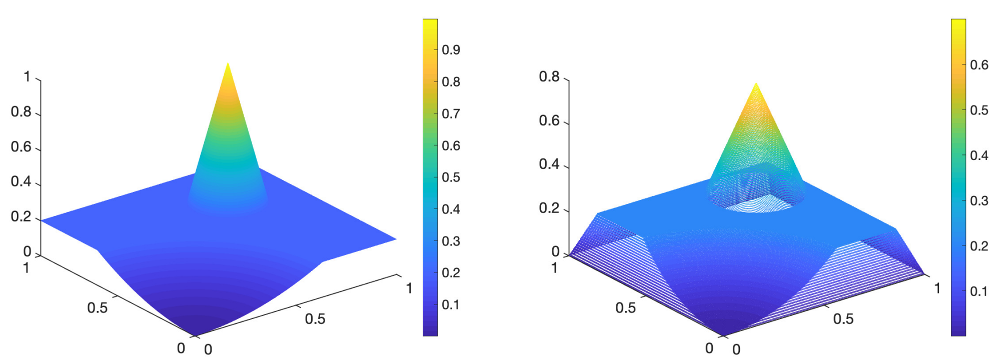


Figure 3.5: **Example 2 ($\alpha = 2.5$).** Left panel: f. Right panel: the computed solution u_h.

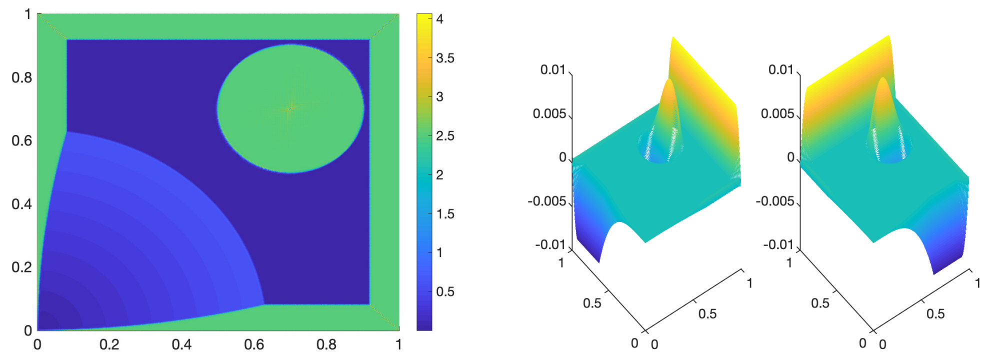


Figure 3.6: **Example 2** ($\alpha = 2.5$)**.** Left panel: $|\nabla u_h|_2$. Right panel: the computed solution $\boldsymbol{p}_h$.

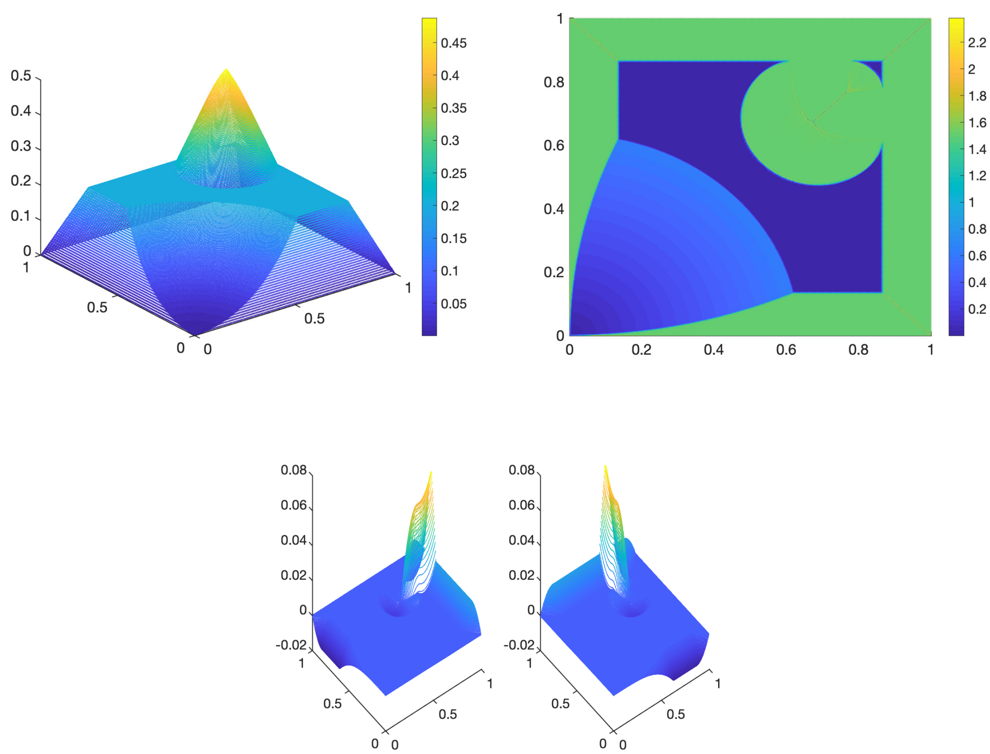


Figure 3.7: **Example 2** ($\alpha = 1.5$)**.** Top row: u_h and $|\nabla u_h|_2$. Bottom row: $\boldsymbol{p}_h$.

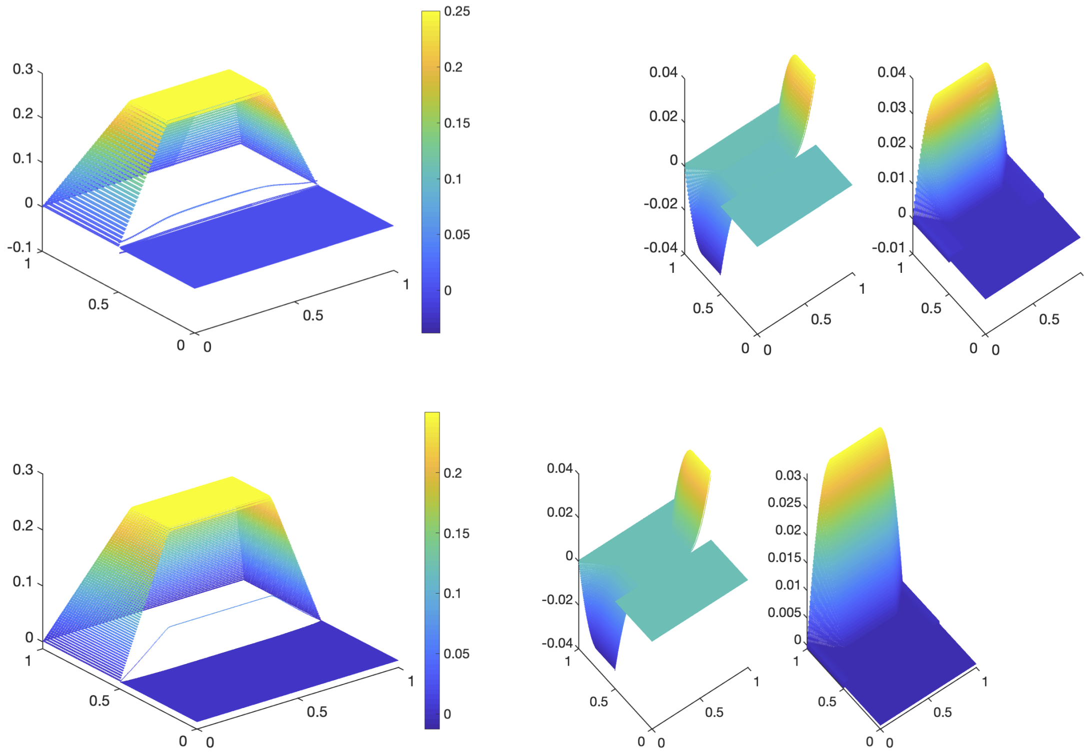


Figure 3.8: **Example 3 (α measure).** Top row: u_h and $\boldsymbol{p}_h$ when the mesh size $h = 8.4984 \times 10^{-5}$. Bottom row: u_h and $\boldsymbol{p}_h$ when the mesh size $h = 2.1412 \times 10^{-5}$. We notice that as $h \downarrow 0$, we accurately approximate the action of the measure α as a distributional gradient constraint.

Chapter 4: Differentiability and Control of a Stationary Problem

In the setting of the sandpile growth problem, a particular task of interest is the sensitivity of the resulting (final) shape of the free surface by controlling the location and intensity of the material source. We consider in this chapter the stationary problem of the diffusive evolutionary VI (2.1.8) with a Laplacian diffusivity operator. In particular, we address a regularization approach for this problem via a family of nonlinear partial differential equations, and provide a novel result of Newton differentiability of the control-to-state map. Further, we discuss solution algorithms for the state equation as well as for the optimization problem.

The reader should be aware that the issue of directional differentiability is further more complicated if instead of the VI problem the QVI is considered, see [38–44], where such a problem is considered in the obstacle setting.

We assume in this chapter that $\Omega \subset \mathbb{R}^{\mathrm{d}}$ with $\mathrm{d} = 1, 2$. The stationary source term $\tilde{f} : \Omega \to \mathbb{R}$ represents the (density) rate of a granular material being deposited on the smooth supporting structure $u_0 : \Omega \to \mathbb{R}$ with $u_0|_{\partial\Omega} = 0$, i.e., here we regard the case where material can escape the domain freely at the boundary $\partial\Omega$. In the limit time $\to \infty$, the function $u : \Omega \to \mathbb{R}$ describing the surface of the outmost layer of material is approximated by the solution to the stationary variational inequality which arises when regarding (2.1.8) with $Au = -\varepsilon\Delta u$ and assuming $\partial_t u = 0$. In more rigorous terms, u is the solution to the problem: Find $u \in K$ such that

$$\langle -\epsilon\Delta u - f, v - u\rangle_{H^{-1},H_0^1} \geq 0, \tag{4.0.1}$$

for all $v \in K$ with

$$K := \{v \in H_0^1(\Omega) : |Dv| \le \alpha \text{ a.e.}\}$$

and where $f = \tilde{f} + \epsilon \Delta u_0$, and $H_0^1(\Omega)$ is the space of $L^2(\Omega)$ functions such that their weak gradients belong to $L^2(\Omega)^{\mathrm{d}}$, together with function values vanishing at the boundary $\partial\Omega$ in the sense of the trace; see [34].

We assume that D is either the weak gradient or an approximation thereof, and $0 < \epsilon \ll 1$. The function $\alpha : \Omega \to \mathbb{R}$ is strictly and uniformly above zero, i.e.,

$$\alpha(x) \ge \nu > 0, \tag{4.0.2}$$

for almost all $x \in \Omega$. If the pile is homogeneous and $|\nabla u_0| \le \tan(\theta)$, then $\alpha \equiv \tan(\theta)$. In the case of an inhomogeneous pile (more than one material present), θ is not longer a constant and neither is α. There is a further more complex case (not treated within this chapter but addressed in several places within this dissertation) when $|\nabla u_0| > \tan(\theta)$ on a positive measure set within Ω; in this case α is actually dependent on u and the problem is a quasi-variational inequality. This approach was pioneered by Prigozhin [4, 6, 7].

A control problem of interest associated to problem (4.0.1) corresponds to the selection of a source of material f so that the accumulation of material $u = u(f)$ is close to some desired structure u^d while f also is minimized in some sense. This leads to the minimization of the functional

$$J(u(f), f) := \frac{1}{2} \int_\Omega |u(f) - u^d|^2 \, \mathrm{d}x + \lambda \|f\|_{Y'}^2, \tag{4.0.3}$$

for some space $Y' \subset H^{-1}(\Omega)$ where $H^{-1}(\Omega)$ is the topological dual to $H_0^1(\Omega)$. In the same vein, the design of algorithms for the minimization of (4.0.3) and the study of optimality conditions require information on the differentiability of the map $f \mapsto u(f)$. This is an extremely complex task due to the constraint K. In light of this, problem (4.0.1) is replaced by the regularized version

$$-\epsilon\Delta u + \gamma\mathcal{P}(u) = f \quad \text{in } H^{-1}(\Omega), \tag{4.0.4}$$

for some $\mathcal{P} : H_0^1(\Omega) \to H^{-1}(\Omega)$ monotone, nonsmooth, and vanishing at K, whose specific form is given later in this chapter. Further, $\gamma > 0$ and we recover (in some sense explained later) the original problem when $\gamma \to \infty$.

The abstract version of the problem above can be formulated as the following minimization problem with a non-smooth equation as constraint:

$$\begin{aligned} &\min \;\; J(u, f) \\ &\text{s.t.} \;\; A(u) + F(u) = f \end{aligned} \tag{P}$$

for some objective function $J : X \times Y' \to \mathbb{R}$, with $Y' \subset X'$, and where $A : X \to X'$ is a strongly monotone, and continuously Fréchet differentiable operator. Additionally, $F : X \to X'$ is monotone, and Newton differentiable (see the next section for a definition). Existence of solutions to (P) is available under mild conditions.

In this chapter we study the Newton differentiability properties of the solution map $f \mapsto u(f)$ associated to the constraint in (P) and how this permeates to $f \mapsto J(u(f), f)$. In particular, this would lead to (P) being suitable for semismooth Newton approaches.

The rest of the chapter is organized as follows. In Section 4.1 we study results involving differentiability and monotonicity, and further establish an abstract result, Theorem 4.1, for the characterization of the sensitivity of the control-to-state map. Subsequently, in Section 4.2, we apply the abstract results to our specific application. There we improve known results for the Newton differentiability of convex regularization of gradient type constraints, and also consider analogous results for operators approximating the gradient. Several lemmas are proven that culminate in Theorem 4.2 which establishes the control-to-state differentiability result for the application example. This chapter ends with a discussion on solution algorithms and future research directions.

4.1 Preliminaries and abstract results

We assume throughout this section that X is a reflexive and real Banach space, and additionally assume that the control space Y' is such that $Y' \subset X'$ with Y a reflexive and real Banach space. The typical example of application here is $X = H_0^1(\Omega)$ and $Y' \simeq Y = L^2(\Omega)$.

We start with monotonicity definitions used throughout the chapter. We say that $A : X \to X'$ is *strongly monotone* if there exist $c > 0$ and $q > 1$ such that

$$\langle A(u+h) - A(u), h\rangle \geq c\|h\|^q, \tag{4.1.1}$$

for all $u, h \in X$. Further, we say that it is *monotone* if (4.1.1) holds for $c = 0$; see for example [45].

Two differentiability concepts are used in this work: Fréchet, and Newton (or slant) differentiability. For the definition of the Fréchet one we refer the reader to [46] or any nonlinear functional analysis book. We introduce now the Newton differentiability concept; see [47] for a solid introduction on the subject. Let X, Y be real Banach spaces and $\mathcal{D} \subset X$ be an open set. Then $F : \mathcal{D} \subset X \to Y$ is called *Newton differentiable* at u if there exists an open neighborhood $\mathcal{N}(u) \subset \mathcal{D}$ and mappings

$$G_F : \mathcal{N}(u) \to \mathcal{L}(X, Y)$$

such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(u+h) - F(u) - G_F(u+h)h\|_Y}{\|h\|_X} = 0.$$

Note that Newton derivatives are in general not unique, and it is direct to prove that if $F : \mathcal{D} \subset X \to Y$ is continuously Fréchet differentiable, then it is Newton differentiable.

In order to simplify notation, we use the Landau o notation: We denote $\|r(h)\|_Y =$

$o(\|h\|_X)$ for a map $r : X \to Y$ if the following holds true,

$$\lim_{\|h\|_X \to 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0,$$

i.e., $\|r(h)\|_Y = o(\|h\|_X)$ implies that r vanishes faster than h as $h \to 0$.

We start now with a result that establishes that for a strongly monotone differentiable map, its derivative is also strongly monotone under relatively mild conditions.

Lemma 6. Let $A : \mathcal{D} \subset X \to X'$ with $\mathcal{D}$ open be Fréchet differentiable and satisfy for some fixed $c > 0$ and $2 \leq q < 3$

$$\langle A(u+h) - A(u), h \rangle \geq c\|h\|_X^q, \tag{4.1.2}$$

for all $u, u+h \in \mathcal{D}$.

In addition, suppose that

$$\langle A(u+h) - A(u), h \rangle = \langle A'(u)h, h \rangle + \langle w(u,h), h \rangle \tag{4.1.3}$$

with $\|w(u,h)\|_{X^*} \leq M\|h\|_X^2$, for all $u \in \mathcal{D}$, and all $h \in B_r(0, X)$ for some sufficiently small $r = r(u) > 0$.

Then for each $u \in \mathcal{D}$, $A'(u)$ is strongly monotone with $q = 2$, i.e.,

$$\langle A'(u)h, h \rangle \geq \tilde{c}\|h\|_X^2,$$

for some $\tilde{c} > 0$ and all $h \in X$.

Proof. Let $u \in \mathcal{D}$ be fixed. From (4.1.2) and (4.1.3) we obtain for all $h \in B_s(0, X)$ with $s > 0$ sufficiently small that

$$c\|h\|_X^q \leq \left\langle A'(u)h, h \right\rangle + M\|h\|_X^3.$$

Let $h = t\tilde{h}$ for some $\tilde{h} \in B_s(0, X)$ with $\|\tilde{h}\|_X = s$, and $t \in [0, 1]$, then

$$t^{q-2} c s^q \leq \langle A'(u)\tilde{h}, \tilde{h}\rangle + M t s^3.$$

If $q = 2$, simply take $t = 0$ and the result follows. If $q \in (2, 3)$, then define $g(t) = t^{q-2} c s^q - M t s^3$, and choose s sufficiently small so that the positive maximum of g is achieved in $(0, 1)$; this can be done since $q < 3$. In fact, since $q \in (2, 3)$, the maximum value is given by $g(t^*) = \tilde{c} s^2$ for some $\tilde{c} > 0$ independent of s. Thus, $\tilde{c} s^2 \leq \langle A'(u)\tilde{h}, \tilde{h}\rangle$ and hence

$$\tilde{c}\|\tilde{h}\|_X^2 \leq \langle A'(u)\tilde{h}, \tilde{h}\rangle.$$

Scaling by $\gamma > 0$ and using that $A'(u)$ is linear, completes the proof. □

A few words are in order on the assumption in (4.1.3). This condition is satisfied if A is twice Fréchet differentiable on the open set $\mathcal{D}$ and its second order derivative A'' is uniformly bounded in $\mathcal{D}$; the result follows from the Taylor remainder theorem, see [48].

We show next that in some cases the Newton derivative also inherits monotonicity properties of the original map. In particular, this requires some continuity assumption on the Newton derivative with respect to the base point.

Lemma 7. Let $F : \mathcal{D} \subset X \to X'$ be Newton differentiable with Newton derivative G_F, with $\mathcal{D}$ open, and suppose that F is monotone, i.e.,

$$\langle F(u+h) - F(u), h\rangle \geq 0, \tag{4.1.4}$$

for all $u, u+h \in \mathcal{D}$. If $w \mapsto \langle G_F(w)h, h\rangle$ is continuous at $w = u \in \mathcal{D}$ for each h, then $G_F(u)$ is monotone, i.e.,

$$\langle G_F(u)h, h\rangle \geq 0$$

for all $h \in X$.

Proof. By (4.1.4) and the definition of Newton derivative, we observe that

$$\langle G_F(u+h)h, h\rangle \geq \langle r(h), h\rangle \,,$$

for some $r(h) = o(\|h\|_X)$. Let $h = t\tilde{h}$ with $\tilde{h} \in X$ fixed, then

$$\langle G_F(u+t\tilde{h})\tilde{h}, \tilde{h}\rangle \geq \frac{\langle r(t\tilde{h}), t\tilde{h}\rangle}{\|t\tilde{h}\|_X^2}\|\tilde{h}\|_X^2,$$

and the result follows by taking the limit of $t \to 0$. $\square$

We now establish the main theorem of the section and the tool which later allows us to determine differentiability properties of the control-to-state map in the introduction of this chapter.

Theorem 4.1. Let $A : X \to X'$ satisfy the assumptions of Lemma 6 for $q = 2$ and $\mathcal{D} = X$, and $F : X \to X'$ be monotone, continuous, and Newton differentiable, and suppose that either F satisfies the assumptions of Lemma 7 or that its Newton derivative $G_F(u) : X \to X'$ is monotone.

Then, for $f \in Y'$, $u(f)$ is well-defined as the unique solution $u \in X$ to the equation

$$A(u) + F(u) = f \quad \text{ in } \; X'. \tag{4.1.5}$$

In addition,

$$Y' \ni f \mapsto u(f) \in X$$

is Newton differentiable with Newton derivative $Y' \ni f \mapsto G_u(f) \in \mathcal{L}(Y', X)$ defined as follows: For $h \in Y'$, $w(f) = G_u(f)h$ is the unique solution $w \in X$ to

$$A'(u(f))w + G_F(u(f))w = h \quad \text{ in } X' \tag{4.1.6}$$

where A' is the Fréchet derivative of A.

Proof. Note first that $u(f)$ is well-defined for any $f \in Y'$ given that there exists a unique solution to (4.1.5); the result follows by standard methods in monotone operator theory, see [45, Theorem II.2.1]. Further, by Lemma 6, we have that for any u, $A'(u)$ is strongly monotone, and $G_F(u)$ is monotone by initial assumption or Lemma 7, so that w, the unique solution to (4.1.6), is also well-defined; again by [45, Theorem II.2.1].

Since A is continuously Fréchet differentiable, it also is Newton differentiable. Then $E := A + F$ is Newton differentiable with derivative $G_E := A' + G_F$ satisfying

$$\langle G_E(u)v, v\rangle \geq \langle A'(u)v, v\rangle \geq \tilde{c}\|v\|_X^2,$$

for all $u, v \in X$.

For any $f, h \in Y'$, define $d(h) = u(f + h) - u(f)$. Considering (4.1.5) with f and $f + h$ and subtracting the results, we obtain

$$E(u(f) + d(h)) - E(u(f)) = h.$$

Since A satisfies (4.1.2) with $q = 2$, and F is monotone, it follows that $Y' \ni f \mapsto u(f) \in X$ is Lipschitz continuous. Then, for a map $r : Y' \to X'$ satisfying $\|r(h)\|_{X'} = o(\|d(h)\|_X)$, we have $\|r(h)\|_{X'} = o(\|h\|_{Y'})$. Subsequently, from the definition of the Newton derivative,

$$G_E(u(f) + d(h))d(h) = h + r(h), \tag{4.1.7}$$

where

$$\|r(h)\|_{X'} = o(\|h\|_{Y'}).$$

In contrast, from (4.1.6), for some $w = w(f + h)$ we have that

$$G_E(u(f + h))w(f + h) = h,$$

and substracting this from (4.1.7), we obtain

$$G_E(u(f+h))R(h) = r(h), \tag{4.1.8}$$

where

$$R(h) := u(f+h) - u(f) - w(f+h).$$

By testing in (4.1.8) with $R(h)$, we observe due to the strong monotonicity of $G_E(u(f+h))$ that

$$\tilde{c}\|R(h)\|_X^2 \le \langle G(u(f+h))R(h), R(h)\rangle = \langle r(h), R(h)\rangle,$$

and since

$$\langle r(h), R(h)\rangle \le \|r(h)\|_{X'}\|R(h)\|_X,$$

we observe

$$\|R(h)\|_X = o(\|h\|_{Y'}),$$

and the proof is finished. □

We now aim at applying the results in this section to our sandpile control problem.

4.2 Application to the sandpile control problem

In the framework of (P), we consider the problem in the introduction of this chapter associated to the control of the stationary accumulation of granular material. In this section, we fix

$$X = H_0^1(\Omega),$$

and $X \subset Y \subset L^2(\Omega)$ with Y a real Banach space, e.g., $Y = L^2(\Omega)$. Note that this implies that $L^2(\Omega) \subset Y' \subset X'$ since we identify $L^2(\Omega)$ with its topological dual.

In (4.0.4), we define the constraint regularization operator $\mathcal{P} : X \to X'$ as

$$\langle \mathcal{P}(u), w\rangle_{X',X} = \int_{\Omega^+(u)} P(Du)\cdot Dw \,\mathrm{d}x = \int_{\Omega^+(u)} (|Du|-\alpha)^{\pm}\frac{(Du\cdot Dw)}{|Du|}\,\mathrm{d}x, \tag{4.2.1}$$

where

$$\Omega^+(u) := \{x\in\Omega : |Du(x)|>0 \quad \text{a.e.}\}.$$

Additionally, $P(u) := q(u)b(u)$ for

$$q(u) := \frac{u}{|u|}, \qquad \text{and} \qquad b(u) := (|u|-\alpha)^{\pm},$$

and $(\cdot)^{\pm} := \min(1, \max(0,\cdot))$ in the pointwise sense: for $g:\Omega\to\mathbb{R}$, then

$$(g(x))^{\pm} = \begin{cases} 0, & \text{if } g(x)\le 0,\\ g(x), & \text{if } 0\le g(x)\le 1,\\ 1, & \text{if } 1\le g(x). \end{cases}$$

Two possible choices for D are considered: $D=\nabla$, the weak gradient, and $D=D_\mu$, where $D_\mu : L^p(\Omega)\to L^p(\Omega)^{\mathrm{d}}$ is a bounded linear operator for all $1\le p\le +\infty$, approximating the gradient (e.g., by means of incremental quotients). The parameter $\mu>0$ can be considered to obtain $D_\mu\to\nabla$ in some sense, as $\mu\downarrow 0$. In addition, note that (formally) we can write $\mathcal{P}(u) = D'P(Du)$.

For both cases, we have that $\mathcal{P}$ corresponds to the derivative of the convex functional

$$J_{\mathcal{P}}(u) = \int_\Omega K(|Du(x)|-\alpha(x))\mathrm{d}x,$$

where

$$K(t) = \begin{cases} \int_0^t (y)^{\pm} \mathrm{d}y, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

It follows that $\mathcal{P}$ is monotone given that $J_{\mathcal{P}}$ is convex, and further from its definition we observe that $\mathcal{P}$ is continuous. Thus, in both the cases $D \in \{\nabla, D_\mu\}$, we obtain that for every $f \in Y'$, since $-\Delta$ is strongly monotone, equation (4.0.4) has a unique solution by the same argument as in the beginning of the proof of Theorem 4.1. Hence, for each $\gamma > 0$, there exists $u_\gamma \in X$ solution to (4.0.4), and standard arguments exploiting the monotonicity of $J_{\mathcal{P}}$ determine that $u_\gamma \to u^*$ in X as $\gamma \to \infty$, where u^* is the solution to (4.0.1); see for example [26].

Next we show that $P(u) = q(u)b(u)$ is Newton differentiable between appropriate spaces, and later we use the result to obtain a Newton differentiability result for $\mathcal{P}(u) = D'P(Du)$. The following lemma is an improvement of the result in [26].

Lemma 8. The operator $P : L^p(\Omega)^{\mathrm{d}} \to L^q(\Omega)^{\mathrm{d}}$ with $2 \leq 2q \leq p$ is Newton differentiable. A Newton derivative G_P can be defined as

$$G_P(u) = q(u)G_b(u) + b(u)Q(u), \tag{4.2.2}$$

where

$$Q(u)(x) := \frac{1}{|u(x)|}\left(\mathrm{id} - \frac{u(x)u^T(x)}{|u(x)|^2}\right),$$

and

$$G_b(u)(x) := G_{\max}^{\min}(|u(x)| - \alpha(x))\frac{u^T(x)}{|u(x)|}$$

is a Newton derivative of $b : L^p(\Omega)^{\mathrm{d}} \to L^q(\Omega)$, and

$$G_{\max}^{\min}(w)(x) := \chi_{(0,1)}(w(x))$$

is a Newton derivative of $(\cdot)^{\pm} : L^p(\Omega) \to L^q(\Omega)$.

Proof. The fact that $G_b(u)$ is a Newton derivative of $b : L^p(\Omega)^{\mathrm{d}} \to L^q(\Omega)$ follows from $G_{\max}^{\min}(u)(x) = \chi_{(0,1)}(u(x))$ being a Newton derivative of $(\cdot)^{\pm} : L^p(\Omega) \to L^q(\Omega)$, and this is similarly obtained as the Newton derivative of the max function alone, see [47] and compare to [26].

Initially, we observe that

$$\begin{aligned} P(u+h) - P(u) - G_P(u+h)h =& b(u+h)\left(q(u+h) - q(u) - Q(u+h)h\right) \\ &+ \left(q(u) - q(u+h)\right)\left(b(u+h) - b(u)\right) \\ &+ q(u+h)\left(b(u+h) - b(u) - G_b(u+h)h\right) \\ =& I + II + III, \end{aligned}$$

and in what follows we show that

$$I + II + III = o(\|h\|_p).$$

Consider initially I. Note that we have

$$I = b(u+h)\left(-\frac{u}{|u+h||u|}\left(|u+h| - |u| - \frac{(u+h)^T h}{|u+h|}\right) + \frac{(u+h)^T h}{|u+h|^2}\left(\frac{u+h}{|u+h|} - \frac{u}{|u|}\right)\right).$$

Since

$$\left|b(u+h)\frac{u}{|u+h||u|}\right| \leq \frac{1}{\nu},$$

due to the fact that $\alpha \geq \nu$ a.e. in Ω, and because

$$G_{|\cdot|}(u+h)h = \frac{(u+h)^T h}{|u+h|},$$

when $u+h \geq \nu$, where $G_{|\cdot|}$ is a Newton derivative of $|\cdot| : L^p(\Omega) \to L^q(\Omega)$, we observe that the first part of the sum in I is $o(\|h\|_p)$.

Regarding the second term, if $u+h \leq \nu$, then $b(u+h) = 0$, and if $u+h \geq \nu$, then

$$\left| \frac{(u+h)^T h}{|u+h|^2} \right| \leq \frac{|h|}{\nu} \quad \text{and} \quad \left(\frac{u+h}{|u+h|} - \frac{u}{|u|} \right) \leq 2\frac{|h|}{\nu},$$

and since $b(u+h) \leq 1$, we get a bound of $2\frac{|h|^2}{\nu^2}$. An application of Hölder's inequality implies that the second term of I is also $o(\|h\|_p)$ as we see next. Suppose that $z : \Omega \to \mathbb{R}$ satisfies $|z| \leq |h|^2$. Thus, by applying Hölder's inqualitiy to $|h|^{2q}$ and 1 for the exponents $\frac{p}{2q} \geq 1$ and $\left(\frac{p}{2q}\right)'$, we get

$$\int_\Omega |z|^q \mathrm{d}x \leq \int_\Omega |h|^{2q} \mathrm{d}x \leq C_1(\Omega) \left(\int_\Omega |h|^p \mathrm{d}x \right)^{2q/p}.$$

Therefore (by taking the q-th root), we get

$$\|z\|_{L^q(\Omega)} \leq C_2(\Omega) \left(\int_\Omega |h|^p \mathrm{d}x \right)^{2/p} = C_2(\Omega)\|h\|^2_{L^p(\Omega)},$$

which proves the statement.

We turn our attention now to II. For a.e. $x \in \Omega$ it holds that

$$|b(u(x)+h(x)) - b(u(x))| \leq |h(x)|\chi_{\Omega_\nu}(x), \tag{4.2.3}$$

where

$$\Omega_\nu := \{x \in \Omega : |u(x)| < \nu \wedge |u(x) + h(x)| < \nu \ \text{ a.e.}\}.$$

Thus for $x \in \Omega \setminus \Omega_\nu$ we observe that

$$|q(u) - q(v)| \leq 2 \min\left(\frac{|h|}{|u|}, \frac{|h|}{|u+h|}\right) \leq 2\frac{|h|}{\nu},$$

thus II is bounded by $|h|^2/\nu$. As above, this implies $II = o(\|h\|_p)$.

Finally, we consider III. Since G_b is the Newton derivative of $b : L^p(\Omega)^{\mathrm{d}} \to L^q(\Omega)$, and $|q|$ is bounded by 1, we have that $III = o(\|h\|_p)$.

□

Let $\mathcal{P}^\nabla : X \to X'$ and $\mathcal{P}^{D_\mu} : X \to X'$ be defined as $\mathcal{P}$ in (4.2.1) for $D = \nabla$ and $D = D_\mu$, respectively. Although these operators are well-defined as maps from X to X', in order to obtain differentiability properties in the case of $\mathcal{P}^\nabla$, the operator needs to be defined in slightly different spaces as we see next.

Lemma 9. The maps $\mathcal{P}^\nabla$ and $\mathcal{P}^{D_\mu}$ are Newton differentiable when defined as $\mathcal{P}^\nabla : X \to (W_0^{1,\infty}(\Omega))'$ and $\mathcal{P}^{D_\mu} : X \to X'$. The general expression of a Newton derivative $G_{\mathcal{P}}$ in these cases is given by

$$\langle G_{\mathcal{P}}(u)v, w\rangle = \int_{\Omega^+(u)} (G_P(Du)Dv) \cdot Dw \,\mathrm{d}x, \tag{4.2.4}$$

for

(i) all $u, v \in X$, $w \in W_0^{1,\infty}(\Omega)$, and the duality pairing considered between $(W_0^{1,\infty}(\Omega))'$ and $W_0^{1,\infty}(\Omega)$ in the case $D = \nabla$ and $\mathcal{P} = \mathcal{P}^\nabla$;

(ii) all $u, v, w \in X$, and the duality pairing considered between X' and X in the case of $D = D_\mu$ and $\mathcal{P} = \mathcal{P}^{D_\mu}$.

Proof. Consider (i) first. The map $\nabla : X \to L^2(\Omega)^{\mathrm{d}}$ is Fréchet differentiable with derivative ∇, and by Lemma 8 the map $P : L^2(\Omega)^{\mathrm{d}} \to L^1(\Omega)^{\mathrm{d}}$ is Newton differentiable. Then $u \mapsto P(\nabla u)$ is Newton differentiable (since it is the composition of a Newton and a Fréchet differentiable mapping [47]) as map from $X \to L^1(\Omega)^{\mathrm{d}}$ with Newton derivative $u \mapsto G_P(\nabla u)\nabla$. From here, and application of Hölder's inequality can be used to show that $\mathcal{P}^{\nabla} : X \to (W_0^{1,\infty}(\Omega))'$ is Newton differentiable (analogously as in [26, Corollary A.3]), with Newton derivative given by (4.2.4).

Next consider (ii). In the case of D_μ, we use that for $\mathrm{d} = 1, 2$, by the Sobolev embedding theorem (e.g. see [34]), X is embedded in $L^p(\Omega)$ for any $2 \le p < \infty$, and by the same result we have that $L^q(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$ for $q > 1$. In addition, $D_\mu : X \to L^p(\Omega)^{\mathrm{d}}$ is Fréchet differentiable with derivative D_μ for $2 \le p < \infty$, and $P : L^p(\Omega)^d \to L^q(\Omega)$ with $2 \le 2q \le p$ is Newton differentiable. Then, as in the previous item, $u \mapsto P(D_\mu u)$ is Newton differentiable as map from $X \to L^q(\Omega)$ with Newton derivative $u \mapsto G_P(D_\mu u)D_\mu$. By choosing a $q > 1$, an application of Hölder's inequality shows that $\mathcal{P}^{D_\mu} : X \to X'$ is Newton differentiable with Newton derivative given by (4.2.4). $\square$

Next we prove the existence of a solution to the sensitivity equation: Note that this requires to show the monotonicity of $G_{\mathcal{P}}(u)$ directly. The reason for this is that $\mathcal{P}$ does not satisfy the continuity assumption of Lemma 7, i.e., continuity of $w \mapsto \langle G_{\mathcal{P}}(w)h, h\rangle$.

Lemma 10. There exists a unique $w \in X$ such that

$$\langle -\epsilon\Delta w, z\rangle_{X',X} + \gamma\langle G_{\mathcal{P}}(u)w, z\rangle_{X',X} = \langle h, z\rangle_{X',X} \tag{4.2.5}$$

for all $z \in X$, for $\mathcal{P} = \mathcal{P}^{\nabla}$ and for $\mathcal{P} = \mathcal{P}^{D_\mu}$.

Proof. The map $-\Delta$ is strongly monotone and linear, then we only need to show that $G_{\mathcal{P}}(u) \in \mathcal{L}(X, X')$ is monotone, i.e.,

$$\langle G_{\mathcal{P}}(u)z, z\rangle_{X',X} \ge 0,$$

for all $z \in X$.

Note that $|G_P(Du)| \in L^\infty(\Omega)$ for each $u \in X$, then we are only left to prove that $D' G_P(Du) D \geq 0$. Exploiting the structure of $G_P(Du)$ of (4.2.2) we have

$$\langle G_{\mathcal{P}}(u)z, z\rangle_{X',X} = (q(Du)G_b(Du)Dz, Dz) + (b(Du)Q(Du)Dz, Dz).$$

For the first term we have

$$G_{\min}^{\max}(|Du| - \alpha)\frac{(Du)^T Dz}{|Du|^2} \cdot \frac{(Du)^T Dz}{|Du|^2} \geq 0,$$

and since $b(Du) \geq 0$ and

$$Q(Du)Dz \cdot Dz = \frac{1}{|Du|}\left(|Dz|^2 - \frac{\left((Du)^T Dz\right)^2}{|Du|^2}\right) \geq 0,$$

the second term is also monotone and the proof is complete. □

We are now in shape to apply the results from Section 4.1 to this specific control problem.

Theorem 4.2. For $\mathcal{P} = \mathcal{P}^{D_\mu}$, the solution map $Y' \ni f \mapsto u(f) \in X$ of (4.0.4) is Newton differentiable.

Proof. The result follows as direct application of the abstract result in Theorem 4.1 where the hypotheses are covered by Lemma 9 and 10. □

A significant obstacle of the "non-ϵ differentiability gap" in Lemma 8 results in the lack of an analogous result for the case $\mathcal{P} = \mathcal{P}^{\nabla}$. However, some of these issues can be resolved in the algorithmic area with the introduction of a "lifting operator", see [49].

A direct corollary of the previous result concerns the Newton differentiability of the reduced functional for the control problem under study.

Corollary 1. Provided that $Y' \ni f \mapsto \|f\|_{Y'}^2$ is Newton differentiable, the functional $Y' \ni f \mapsto J(u(f), f)$ where J is defined in (4.0.3) is Newton differentiable.

Proof. Note that $u \mapsto \frac{1}{2}\int_\Omega |u - u^d|^2 \, dx$ is Fréchet differentiable. Then, the proof is an application of the previous theorem and the composition result of Newton differentiable maps in [50]. □

4.3 Solution algorithms

The idea of this section is to provide a solid idea of the applicability of the results of the previous section for the development of solution algorithms for (4.0.4) and (P).

As defined previously, the map $E : X \to X'$ is given by

$$E(u) = -\epsilon\Delta u + \gamma\mathcal{P}(u) - f,$$

so that the state equation (4.0.4) can be written as

$$E(u) = 0.$$

It is known that for every $f \in H^{-1}(\Omega)$, this equation is uniquely solvable in all cases contemplated in this chapter, namely for $D = \nabla$ and $D = D_\mu$.

In the case $D = D_\mu$, and given that E is Newton differentiable as a map $X \to X'$, a Newton derivative of E is given by

$$G_E(u)v = -\epsilon\Delta v + \gamma G_{\mathcal{P}}(u)v,$$

where $G_{\mathcal{P}}$ is made explicit in (4.2.4). Further, since the constant in the strong monotonicity of $-\epsilon\Delta$ is identical to ϵ, and we have proven that $G_{\mathcal{P}}(u)$ is monotone, it follows that $G_E(u)$

is nonsingular and its inverse is uniformly bounded by ϵ^{-1}, i.e.,

$$\|G(u)^{-1}\| \leq \frac{1}{\epsilon}.$$

Hence, solutions to $E(u) = 0$ are suitable to be approximated by a function space version of a *semismooth Newton method*; see [47, Theorem 8.16].

The semismooth Newton iteration is then given by

$$u^{n+1} := u^n + v^*,$$

where the Newton step v^* is defined as the solution of

$$\gamma G_E(u^n)v = -E(u^n). \tag{4.3.1}$$

This yields (see [47, Theorem 8.16]) that the sequence $\{u^n\}_{n\in\mathbb{N}}$ defined by the iteration (4.3.1) converges superlinearly to the unique solution $u^* \in X$ of the state equation $E(u) = 0$, provided that the initial iterate u_0 is sufficiently close to u^*. The fact that $D = D_\mu$ is considered, leads to the idea to further apply a path-following method simultaneously on γ and μ.

Under mild conditions we have proven that $Y' \ni f \mapsto J(u(f), f)$ is Newton differentiable. In the case where Y' does not have a large number of degrees of freedom, a descent approach for the overall problem is possible. However the computation of the entire derivative is prohibitive if the dimension of Y' is not small. On the other hand, if the approach is such that the entire derivative is not needed (only a small number of components are), a descent method is directly suitable.

4.4 Future research

We have provided several abstract results that link notions of differentiability to monotonicity ones. In fact, in Theorem 4.1, the hypotheses contemplate the possibility of heavily nonlinear operators, e.g., the $p-$Laplacian defined as $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p \in [2,3)$ is within the scope of the theorem. The application of the abstract results to the specific example of control of the pile of granular material is then tackled in Theorem 4.2 and its corollary. There, a function space approach is suitable provided that the gradient is considered in its approximated version. The result of Newton differentiability in Lemma 9 is of interest in its own right; it provides a better result than the one standing in the literature for the case $D = \nabla$. The short discussion on the algorithmic development is a current area of active research.

Chapter 5: Initial State Control for the Evolutionary QVI

We regard a control problem for the evolutionary QVI (2.0.2) without diffusion. The objective is to reshape the initial surface in a minimal way with the aim of keeping a particular region free of material accumulation. In mathematical terms, the control variable within this chapter corresponds to the initial state u_0 and the state variable u, the shape of the poured pile, is aimed to be as close as possible to u_0 on a certain prescribed region Ω_0. In contrast to the previous chapters, here we take the approach of first discretizing in space.

Throughout this chapter, we assume for simplicity that $\Gamma_D = \partial\Omega$, hence that the supporting structure $u_0 : \Omega \to \mathbb{R}$ satisfies $u_0|_{\partial\Omega} = 0$ where $\Omega \subset \mathbb{R}^{\mathrm{d}}$. By $u : (0,T) \times \Omega \to \mathbb{R}$ for $t \in (0,T)$ we denote the height of the accumulation of a granular cohensionless material that is disposed. We assume that $u(t)|_{\partial\Omega} = 0$ which implies that material is allowed to abandon the domain freely. The material is characterized by its angle of repose $\theta > 0$ which corresponds to the steepest angle at which a sloping surface formed from a point source of material is stable. The (density) rate of a granular material being deposited at each point of the domain Ω is given by $f : (0,T) \times \Omega \to \mathbb{R}$.

As in Chapter 3, we generalize the gradient constraint to include any ℓ_p-norm, (2.0.2) is expressed by the following problem.

Problem ($\mathbf{QVI}(u_0)$). Find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ and

$$u \in \mathcal{K}^p(u,u_0) := \{z \in H_0^1(\Omega) : |\nabla z|_p \leq M_p(u,u_0)\},$$

a.e. in $(0,T)$ and for which

$$(\partial_t u - f, v - u) \geq 0,$$

for all $v \in \mathcal{K}^p(u, u_0)$ a.e. in $(0, T)$.

The operator $M_p(w, u_0) : \Omega \to \mathbb{R}$ is given by

$$M_p(w, u_0) := \begin{cases} \alpha, & \text{if } w > u_0; \\ \max(\alpha, |\nabla u_0|_p), & \text{if } w = u_0; \end{cases}$$

where $\alpha = \tan(\theta)$ (cf. (2.3.1)). In particular, this means that if material has accumulated then the gradient constraint is the material dependent one, and if it has not, we may allow higher gradients on the supporting surface. This actually allows material to slide off high slopes into other regions.

The choice of p determines possible shapes of u and hence the possible structures of the piles. In particular (and formally), if we consider a point source f, in the case $p = 2$, the structure of u (for a flat u_0) corresponds to a growing cone. Other cases like $p = \infty$ would imply that a point source of sand would generate a pyramid structure with sides aligned with the horizontal and vertical axis instead. In the latter case, note that $v \in \mathcal{K}^\infty(u, u_0)$ implies that

$$-M_\infty(u, u_0) \leq \partial_{x_i} v \leq M_\infty(u, u_0) \quad \text{a.e. in } \Omega \text{ and for } i = 1, 2, \ldots, \mathrm{d}.$$

The control/optimization problem can be stated in words as: How to modify u_0 (slightly) with respect to some reference structure u_0^{ref}, in order to maintain a certain region $\Omega_0 \subset \Omega$ relatively free of material. This can be stated mathematically as follows.

Problem ($\mathbb{P}$). Let $\sigma > 0$, $f \in L^2(\Omega)^+$ and $u_0^{\mathrm{ref}} \in L^2(\Omega)$ be given, and consider

$$\min \qquad \int_0^T \int_{\Omega_0} (y - u_0)\,\mathrm{d}x\,\mathrm{d}t + \frac{\sigma}{2}\int_\Omega (u_0 - u_0^{\mathrm{ref}})^2 \mathrm{d}x, \quad \text{over} \quad u_0 \in H_0^1(\Omega);$$

$$\text{s.t.} \quad y \text{ solves } \mathbf{QVI}(u_0),$$

$$u_0 \in \mathcal{A},$$

where

$$\mathcal{A} := \{z \in H_0^1(\Omega) : u_0^{\mathrm{ref}} + \lambda_0 \leq z \leq u_0^{\mathrm{ref}} + \lambda_1 \ \text{ a.e. } \},$$

with $\lambda_i \in H_0^1(\Omega)$ for $i = 1, 2$ and $\lambda_0 \leq \lambda_1$ a.e. in Ω.

Smoothing of M

Since M_p is discontinuous and an unsurmountable obstacle (theoretically and numerically), in the same vein as in Section 2.3.1, in general we re-define as

$$M_p(w, u_0) := \begin{cases} \alpha, & \text{if } w > u_0 + \epsilon; \\ \max(\alpha, |\nabla u_0|_p)\frac{(u_0+\epsilon-w)}{\epsilon}\epsilon + \alpha\frac{(w-u_0)}{\epsilon}, & \text{if } u_0 + \epsilon \geq w > u_0; \\ \max(\alpha, |\nabla u_0|_p), & \text{if } w = u_0. \end{cases}$$

(omitting an ϵ–superscript for readability). Smoother approximations $\tilde{M}_p$ can be obtained using higher order interpolants, a regularization of the max function, and of the $\mathbb{R}^{\mathrm{d}}$ norm. Hence, we assume throughout the work that

$$\tilde{M}_p \in C^k(\mathbb{R}, \mathbb{R}),$$

with $k \geq 2$.

5.1 The semi-discrete problem

We consider now the discretized in space problem for $\Omega \subset \mathbb{R}^2$. Given the nonnegative $\mathbf{f} : (0,T) \to \mathbb{R}^N$, the semi-discretized QVI problem can then be formulated as: Find $\mathbf{u} : (0,T) \to \mathbb{R}^N$ such that

$$\mathbf{u}(t) \in \mathcal{K}^p(\mathbf{u}(t), \mathbf{u}_0) \ : \quad \big(\mathbf{u}'(t) - \mathbf{f}(t), \mathbf{v} - \mathbf{u}(t)\big)_{\mathbb{R}^N} \geq 0, \quad \forall\, \mathbf{v} \in \mathcal{K}^p(\mathbf{u}(t), \mathbf{u}_0), \quad (\mathrm{QVI}_N(\mathbf{u}_0))$$

for almost all $t \in (0,T)$.

The two most popular choices for $\mathcal{K}^p(\mathbf{u}(t), \mathbf{u}_0)$ are given by $p = 2$ and $p = \infty$, where

$$\mathcal{K}^2(\mathbf{w}, \mathbf{z}) := \{\mathbf{v} \in \mathbb{R}^N : \quad \sqrt{|(\mathbf{D}_1\mathbf{v})_i|^2 + |(\mathbf{D}_2\mathbf{v})_i|^2} \leq (M_2(\mathbf{w}, \mathbf{z}))_i \quad i = 1, \ldots, N\},$$

and

$$\begin{aligned}\mathcal{K}^\infty(\mathbf{w}, \mathbf{z}) := \{\mathbf{v} \in \mathbb{R}^N : -(M_\infty(\mathbf{w}, \mathbf{z}))_i \leq (\mathbf{D}_j\mathbf{v})_i \leq (M_\infty(\mathbf{w}, \mathbf{z}))_i, \\ j = 1, 2, \text{ and } i = 1, 2, \ldots, N\},\end{aligned} \tag{5.1.1}$$

with

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{N \times N}, \quad \text{and} \quad M_2, M_\infty : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N.$$

A few words are in order concerning $\mathbf{D}_1$, and $\mathbf{D}_2$. The former represent discrete approximations of the partial derivatives $\partial/\partial x$ and $\partial/\partial y$, respectively. In this vein, $\mathbf{D} := (\mathbf{D}_1, \mathbf{D}_2)$ is the approximation of the gradient, such that $\mathbf{D} : \mathbb{R}^N \to \mathbb{R}^{2N}$.

For $1 \leq p < +\infty$

$$\mathcal{K}^p(\mathbf{w}, \mathbf{z}) := \{\mathbf{v} \in \mathbb{R}^N\!: \ |(\mathbf{D}\mathbf{v})_i|_p := (|(\mathbf{D}_1\mathbf{v})_i|^p + |(\mathbf{D}_2\mathbf{v})_i|^p)^{\frac{1}{p}} \leq (M_p(\mathbf{w}, \mathbf{z}))_i \quad i = 1, \ldots, N\}, \tag{5.1.2}$$

where the map M_p defined as

$$(M_p(\mathbf{w},\mathbf{z}))_i := \begin{cases} \alpha, & \text{if } \mathbf{w}_i > \mathbf{z}_i + \epsilon; \\ \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p)\frac{(\mathbf{z}_i+\epsilon-\mathbf{w}_i)}{\epsilon}\epsilon + \alpha\frac{(\mathbf{w}_i-\mathbf{z}_i)}{\epsilon}, & \text{if } \mathbf{z}_i + \epsilon \geq \mathbf{w}_i > \mathbf{z}_i; \\ \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p), & \text{if } \mathbf{w}_i = \mathbf{z}_i, \end{cases} \tag{5.1.3}$$

where $\mathbf{D}\mathbf{z} := (\mathbf{D}_1\mathbf{z}, \mathbf{D}_2\mathbf{z})$, and $(\mathbf{D}\mathbf{z})_i := ((\mathbf{D}_1\mathbf{z})_i, (\mathbf{D}_2\mathbf{z})_i)$. Although M_p is only continuous, we can consider differentiable approximations $\tilde{M}_p$ of M_p, of the form

$$(\tilde{M}_p(\mathbf{w},\mathbf{z}))_i := \begin{cases} \alpha, & \text{if } \mathbf{w}_i > \mathbf{z}_i + \epsilon; \\ \frac{a}{\epsilon^3}\mathbf{w}_i^3 + \frac{b}{\epsilon^3}\mathbf{w}_i^2 + \frac{c}{\epsilon^3}\mathbf{w}_i + \frac{d}{\epsilon^3}, & \text{if } \mathbf{z}_i + \epsilon \geq \mathbf{w}_i > \mathbf{z}_i; \\ \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p), & \text{if } \mathbf{w}_i = \mathbf{z}_i, \end{cases}$$

where

$$a = 2(\max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p) - \alpha), \qquad b = -3(\max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p) - \alpha)(\epsilon + 2\mathbf{z}_i)$$

$$c = 6(\max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p) - \alpha)\mathbf{z}_i(\epsilon + \mathbf{z}_i),$$

$$d = \epsilon^3 \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p) + 3\epsilon(\alpha - \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p))(\mathbf{z})_i^2 + 2(\alpha - \max(\alpha, |(\mathbf{D}\mathbf{z})_i|_p))(\mathbf{z})_i^3$$

We prove now that the quasi-variational inequality ($\mathrm{QVI}_N(\mathbf{u}_0)$) has at least one solution.

Theorem 1. *Let* $\mathbf{u}_0 \in \mathbb{R}^N$ *and* $\mathbf{f} : (0,T) \to \mathbb{R}^N$ *such that* $\mathbf{f} \in L^2(0,T)$. *Then, there exists a solution* $\mathbf{u} : (0,T) \to \mathbb{R}^N$ *to* ($\mathrm{QVI}_N(\mathbf{u}_0)$) *such that*

$$\mathbf{u} \in C([0,T]) \quad \textit{and} \quad \mathbf{u}' \in L^2(0,T). \tag{5.1.4}$$

Proof. *Step 1: Existence of solutions to the regularized variational inequality*

We consider the case where $p = 2$ without loss of generality, as the general case for $1 \leq p \leq +\infty$ can be done analogously. Given $\gamma > 0$, consider the nonlinear ordinary differential equation

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{f}(t) - \gamma G(t, \mathbf{u}(t)), \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned} \tag{5.1.5}$$

with

$$G(t, \mathbf{u}(t)) := \mathbf{D}^{\mathrm{T}}(|\mathbf{D}\mathbf{u}(t)|_2^2 - M(t)^2)^+ \mathbf{D}\mathbf{u}(t), \tag{5.1.6}$$

where $M(t) := M_p(\mathbf{z}(t), \mathbf{u}_0)$ for some arbitrary $\mathbf{z} \in C(\mathbb{R})$, $\mathbf{u}_0 \in \mathbb{R}^N$, and

$$(\mathbf{h}(t))^+ := (\max(h_1(t), 0), \max(h_2(t), 0), \ldots, \max(\tau_M(t), 0)),$$

for $\mathbf{h} : (0, T) \to \mathbb{R}^N$. Note that $\mathbb{R}^N \ni \mathbf{h} \mapsto G(t, \mathbf{h}) \in \mathbb{R}^N$ is monotone for each t in $\mathbb{R}^N$, i.e.,

$$(G(t, \mathbf{h}_1) - G(t, \mathbf{h}_2), \mathbf{h}_1 - \mathbf{h}_2) \geq 0, \qquad \forall\, \mathbf{h}_1, \mathbf{h}_2 \in \mathbf{R}^N. \tag{5.1.7}$$

The latter can be inferred as the derivative of the convex function $J(\mathbf{h}) = ((|\mathbf{D}\mathbf{h}|_2^2 - M(t)^2)^+ \mathbf{D}\mathbf{h}, \mathbf{D}\mathbf{h})$ is given by $J(\mathbf{h})'\mathbf{d} = G(t, \mathbf{h})\mathbf{d}$.

The integral formulation of (5.1.5) is then given by

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{f}(s)\mathrm{d}s - \gamma \int_0^t G(s, \mathbf{u}(s))\mathrm{d}s =: \Lambda(\mathbf{u})(s). \tag{5.1.8}$$

It follows that $\Lambda : C([0, T]) \to C([0, T])$, and further $\Lambda(\mathbf{u})'(s) \in L^2(0, T)$ for $\mathbf{u} \in C([0, T])$.

We show existence of a solution to (5.1.8) for each $\gamma > 0$ via the application of the Leray-Schauder theorem (e.g. see [51]). First, note that $\Lambda : C([0, T]) \to C([0, T])$ is *continuous*.

Additionally, let $\{\mathbf{u}_n\}$ be a bounded sequence in $C([0,T])$, hence it follows that $\{\Lambda(\mathbf{u}_n)\}$ is also bounded in $C([0,T])$, and further $\{\Lambda(\mathbf{u}_n)'\}$ is bounded in $L^2(0,T)$: If $C>0$ is such that

$$\sup_n \|\mathbf{u}_n\|_{C([0,T])} \leq C,$$

then

$$\|\Lambda(\mathbf{u}_n)'\|_{L^2(0,T)} \leq \|\mathbf{f}\|_{L^2(0,T)} + \gamma\|\mathbf{D}^{\mathrm{T}}\|\|\mathbf{D}\|CT^{1/2}(\|\mathbf{D}\|^2C^2 + \|M\|_{C([0,T])}).$$

It follows, by the compact embedding of $V = \{\mathbf{v} \in L^2(0,T) : \mathbf{v}' \in L^2(0,T)\}$ into $C([0,T])$, that $\Lambda(\mathbf{u}_n) \to \mathbf{g}$ for some $\mathbf{g} \in C([0,T])$ along a subsequence. Thus, $\Lambda : C([0,T]) \to C([0,T])$ is *completely continuous*. Finally, we prove that the set

$$Y := \{\mathbf{u} \in C([0,T]) : \mathbf{u} = \lambda\Lambda(\mathbf{u}) \text{ for some } \lambda \in (0,1)\},$$

is bounded. First, if $\mathbf{u} \in Y$ then $\mathbf{u}(0) = \lambda\mathbf{u}_0$ and

$$\mathbf{u}'(t) = \lambda\mathbf{f}(t) - \gamma\lambda G(t,\mathbf{u}(t)).$$

Hence, taking the inner product above with $\mathbf{u}$ and the integral from 0 to $s<T$, we obtain

$$\begin{aligned}\|\mathbf{u}(s)\|_2^2 - \|\lambda\mathbf{u}_0\|_2^2 &= 2\lambda\int_0^s \mathbf{f}(t)\cdot\mathbf{u}(t)\mathrm{d}t - 2\lambda\gamma\int_0^s G(t,\mathbf{u}(t))\mathbf{u}(t)\mathrm{d}t\\ &\leq 2\lambda\left(\sup_{t\in[0,T]}\|\mathbf{u}(t)\|_2\right)\int_0^T \|\mathbf{f}(t)\|_2\mathrm{d}t,\end{aligned}$$

where we have used $G(s,\mathbf{h})\mathbf{h} \geq 0$ for all $\mathbf{h} \in \mathbb{R}^n$, and all $s \in (0,T)$. It follows that

$$\sup_{s\in[0,T]}\|\mathbf{u}(s)\|_2 \leq C_1(\mathbf{u}_0,\mathbf{f}) < +\infty, \tag{5.1.9}$$

i.e., all elements from Y are bounded. Therefore, by the Leray-Schauder theorem, we have a solution $\mathbf{u}^\gamma$ for each γ.

Step 2: Uniqueness of solutions to the regularized variational inequality. In order to prove uniqueness, suppose we have two solutions $\mathbf{u}_i^\gamma$ for $i = 1, 2$. Then, since both functions satisfy (5.1.5), we subtract term by term, and test the equation with $\mathbf{u}_1^\gamma - \mathbf{u}_2^\gamma$ to then integrate from 0 to $s < T$. Then, we observe

$$\|(\mathbf{u}_1^\gamma - \mathbf{u}_2^\gamma)(s)\|_2^2 = -\gamma \int_0^t (G(s, \mathbf{u}_1^\gamma(s)) - G(s, \mathbf{u}_2^\gamma(s)), \mathbf{u}_1^\gamma(t) - \mathbf{u}_2^\gamma(t))\mathrm{d}t \leq 0,$$

where we use (5.1.7), and hence solutions are unique to (5.1.5).

Step 3: Existence and uniqueness of solutions to the variational inequality problem. An analogous argument to the one used above shows the uniform boundedness of $\mathbf{u}_\gamma$ for $\gamma > 0$: From (5.1.5), we consider the inner product with $\mathbf{u}_\gamma$ and integrate from 0 to t, so that

$$\|\mathbf{u}^\gamma(t)\|_2^2 - \|\mathbf{u}_0\|_2^2 \leq \left(\sup_{t\in(0,T)} \|\mathbf{u}^\gamma(s)\|_2\right) \int_0^T \|\mathbf{f}(s)\|_2 \mathrm{d}s,$$

where we have used $G(s, \mathbf{h})\mathbf{h} \geq 0$ for all $\mathbf{h} \in \mathbb{R}^n$, and all $s \in (0, T)$. This implies that

$$\sup_{\gamma>0} \sup_{s\in[0,T]} \|\mathbf{u}^\gamma(s)\|_2 \leq C_1(\mathbf{u}_0, \mathbf{f}) < +\infty. \tag{5.1.10}$$

By testing in (5.1.5) with an arbitrary $\mathbf{v} \in L^2(0, T)$ such that $\mathbf{v}' \in L^2(0, T)$, we have

$$\begin{aligned}\gamma \int_0^T G(s, \mathbf{u}^\gamma(s))\mathbf{v}(s)\mathrm{d}s &= \int_0^T \mathbf{f}(s)\mathbf{v}(s)\mathrm{d}s - \int_0^T (\mathbf{u}^\gamma)'(s)\mathbf{v}(s)\mathrm{d}s \qquad (5.1.11)\\ &\leq \left(\int_0^T \|\mathbf{f}(s)\|_2^2 \mathrm{d}s\right)^{1/2} \left(\int_0^T \|\mathbf{v}(s)\|_2^2 \mathrm{d}s\right)^{1/2} + \int_0^T \mathbf{u}^\gamma(s)\mathbf{v}'(s)\mathrm{d}s + \mathbf{u}^\gamma(T)\mathbf{v}(T) - \mathbf{u}_0\mathbf{v}(0)\\ &\leq \left(\int_0^T \|\mathbf{f}(s)\|_2^2 \mathrm{d}s\right)^{1/2} \left(\int_0^T \|\mathbf{v}(s)\|_2^2 \mathrm{d}s\right)^{1/2} + C_1(\mathbf{u}_0, \mathbf{f})T^{1/2} \left(\int_0^T \|\mathbf{v}'(s)\|_2^2 \mathrm{d}s\right)^{1/2}.\end{aligned}$$

Since $V = \{\mathbf{v} \in L^2(0,T) : \mathbf{v}' \in L^2(0,T)\}$ is continuously and compactly embedded in $C([0,T])$, we have that

$$\gamma \int_0^T G(s,\mathbf{u}(s))\mathbf{v}(s) \leq C_2(\mathbf{u}_0,\mathbf{f},T)\left(\left(\int_0^T \|\mathbf{v}(s)\|_2^2 \mathrm{d}s\right)^{1/2} + \left(\int_0^T \|\mathbf{v}'(s)\|_2^2 \mathrm{d}s\right)^{1/2}\right)$$

for some $C_2(\mathbf{u}_0,\mathbf{f},T)$. Thus, we observe

$$\sup_{\gamma>0} \|\gamma G(s,\mathbf{u}(s))\|_{V^*} \leq C_2(\mathbf{u}_0,\mathbf{f},T),$$

and hence $\sup_{\gamma>0} \|(\mathbf{u}^\gamma)'\|_{V^*} \leq C_3(\mathbf{u}_0,\mathbf{f},T)$. In particular, this means that

$$\sup_{\gamma>0} \|(\mathbf{u}^\gamma)'\|_{L^2(0,T)} \leq C_3(\mathbf{u}_0,\mathbf{f},T). \tag{5.1.12}$$

Note that $\{\mathbf{u}^\gamma\}_{\gamma>0}$ is bounded in V, so we can choose a a sequence $\mathbf{u}^n := \mathbf{u}^{\gamma_n}$ with $\gamma_n \to \infty$ such that $\mathbf{u}^n \rightharpoonup \mathbf{u}^*$ for some $\mathbf{u}^* \in V$. Further, since V is continuously and compactly embedded in $C([0,T])$, we have

$$\mathbf{u}^n \to \mathbf{u}^* \text{ in } C([0,T]) \qquad \text{and} \qquad (\mathbf{u}^n)' \rightharpoonup (\mathbf{u}^*)' \text{ in } L^2(0,T).$$

Further, from (5.1.11), we observe that

$$\lim_{n\to\infty} \int_0^T G(s,\mathbf{u}^n(s))\mathbf{v}(s)\mathrm{d}s = \int_0^T ((|\mathbf{D}\mathbf{u}^*(t)|_2^2 - M(t)^2)^+ \mathbf{D}\mathbf{u}^*(t), \mathbf{D}\mathbf{v}(s))\mathrm{d}s = 0, \tag{5.1.13}$$

from which we infer that $|\mathbf{D}\mathbf{u}^*(t)|_2 \leq M(t)$, i.e., $\mathbf{u}^* \in \mathcal{K}^2(\mathbf{z}(t),\mathbf{u}_0)$.

Testing (5.1.5) with $\mathbf{w} = \mathbf{v} - \mathbf{u}^n$, where $\mathbf{v} \in \mathcal{K}^2(\mathbf{z}(t),\mathbf{u}_0)$ we observe

$$\int_0^T ((\mathbf{u}^n)'(t) - \mathbf{f}(t), \mathbf{v}(t) - \mathbf{u}^n(t))\mathrm{d}t = \gamma \int_0^T (G(t,\mathbf{v}(t)) - G(t,\mathbf{u}(t)), \mathbf{v}(t) - \mathbf{u}(t)^n), \tag{5.1.14}$$

where we have used that $G(t, \mathbf{v}) = 0$. Using the fact that $\mathbf{h} \mapsto G(t, \mathbf{h})$ is monotone, we have that the right hand side of the above is nonnegative, and then taking the limit as $n \to \infty$ leads to

$$\int_0^T ((\mathbf{u}^*)'(t) - \mathbf{f}(t), \mathbf{v}(t) - \mathbf{u}^*(t))\mathrm{d}t \geq 0. \tag{5.1.15}$$

Since $\mathbf{v}$ was arbitrary, a simple density argument shows that $\mathbf{u}^*$ solves

$$\mathbf{u}(t) \in \mathcal{K}^2(\mathbf{z}(t), \mathbf{u}_0) \quad : \quad \big(\mathbf{u}'(t) - \mathbf{f}(t), \mathbf{v} - \mathbf{u}(t)\big)_{\mathbb{R}^N} \geq 0, \quad \forall\, \mathbf{v} \in \mathcal{K}^2(\mathbf{z}(t), \mathbf{u}_0), \tag{5.1.16}$$

and uniqueness follows by monotonicity arguments.

Step 4: Existence of solutions to the quasi-variational inequality problem. Denote the solution map of (5.1.16) by $\mathbf{u} = S(\mathbf{z})$. By identical arguments as above, the map $S : V \to V$ is compact, further since $M_2(\mathbf{z}(t), \mathbf{u}_0) \leq \max(\alpha, |\mathbf{D}\mathbf{u}_0|_\infty) =: \beta$, we observe that

$$S : \mathcal{K}^2_\beta \to \mathcal{K}^2_\beta,$$

where $\mathcal{K}^2_\beta := \{\mathbf{v} \in C([0,T]) : |\mathbf{D}\mathbf{u}(t)|_2 \leq \beta \text{ a.e}\}$. Thus by Schauder's fixed point theorem, there exists a fixed point to $\mathbf{u} = S(\mathbf{u})$, i.e., ($\mathrm{QVI}_N(\mathbf{u}_0)$) has a solution satisfying (5.1.4). $\square$

The previous theorem allows us to pose the problem in a slightly more regular space than the initially given. In particular, the entire effort of the proof is given within

Problem ($\mathbb{P}_N$). Let $\sigma > 0$, $\mathbf{f} : (0,T) \to \mathbb{R}^N$ nonnegative and $\mathbf{a}, \mathbf{u}_0^{\mathrm{ref}} \in \mathbb{R}^N$ be given.

Consider

$$\min \qquad J(\mathbf{u},\mathbf{u}_0) := \int_0^T \mathbf{a}^\intercal(\mathbf{u}(t)-\mathbf{u}_0)\mathrm{d}t + \frac{\sigma}{2}(\mathbf{u}_0-\mathbf{u}_0^{\mathrm{ref}})^\intercal(\mathbf{u}_0-\mathbf{u}_0^{\mathrm{ref}}) \quad \text{over} \quad \mathbf{u}_0 \in \mathbb{R}^N;$$

$$\text{s.t.} \quad \mathbf{u} \text{ solves } \mathrm{QVI}_N(\mathbf{u}_0),$$

$$\mathbf{u} \in V := \{\mathbf{v} \in L^2(0,T) : \mathbf{v}' \in L^2(0,T)\}$$

$$\mathbf{u}_0 \in \mathcal{A},$$

where

$$\mathcal{A} := \{\mathbf{z} \in \mathbb{R}^N : \mathbf{u}_0^{\mathrm{ref}} + \boldsymbol{\lambda}_0 \le \mathbf{z} \le \mathbf{u}_0^{\mathrm{ref}} + \boldsymbol{\lambda}_1\},$$

where $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \in \mathbb{R}^N$ with $0 \le \boldsymbol{\lambda}_0 \le \boldsymbol{\lambda}_1$.

Before we prove existence of solutions to the above control problem, we consider the following definition:

Definition 1 (Mosco convergence). *Let $\mathbb{K}$ and $\mathbb{K}_n$, for each $n \in \mathbb{N}$, be non-empty, closed and convex subsets of a reflexive Banach space V. Then the sequence $\{\mathbb{K}_n\}$ is said to converge to $\mathbb{K}$ in the sense of Mosco as $n \to \infty$, denoted by*

$$\mathbb{K}_n \xrightarrow{\mathrm{M}} \mathbb{K},$$

if the following two conditions are satisfied:

(I) *For each $w \in \mathbb{K}$, there exists $\{w_{n'}\}$ such that $w_{n'} \in \mathbb{K}_{n'}$ for $n' \in \mathbb{N}' \subset \mathbb{N}$ and $w_{n'} \to w$ in V.*

(II) *If $w_n \in \mathbb{K}_n$ and $w_n \rightharpoonup w$ in V along a subsequence, then $w \in \mathbb{K}$.*

Theorem 2. *There exists a solution to Problem* $(\mathbb{P}_N)$.

Proof. *Step 1: Properties of the infimizing sequence.* Note first that due to Theorem 1 for each $\mathbf{u}_0 \in \mathcal{A}$ there exists a $\mathbf{u} \in V$ that solves $\mathrm{QVI}_N(\mathbf{u}_0)$. Then, there exists an infimizing sequence $\{(\mathbf{u}_n, \mathbf{u}_0^n)\}$ for problem $(\mathbb{P}^h)$, so that for each $n \in \mathbb{N}$

$$(\mathbf{u}_n, \mathbf{u}_0^n) \in V \times \mathcal{A}, \quad \mathbf{u}_n \text{ solves } \mathrm{QVI}_N(\mathbf{u}_0^n), \quad \text{ and } J(\mathbf{u}_n, \mathbf{u}_0^n) \to \inf J.$$

Since $\mathbf{u}_0^n \in \mathcal{A}$ for every $n \in \mathbb{N}$, it follows that there exists a subsequence (not relabelled) such that

$$\mathbf{u}_0^n \to \mathbf{u}_0^*$$

for some $\mathbf{u}_0^* \in \mathcal{A}$.

Further, for each n, $\mathbf{u}_n$ solves $\mathrm{QVI}_N(\mathbf{u}_0^n)$ in V. In particular, we shown next that the same bounds in (5.1.10) and (5.1.12) hold true, that is

$$\sup_{s\in[0,T]} \|\mathbf{u}_n(s)\|_2 \leq \sup_{\mathbf{u}_0\in\mathcal{A}} C_1(\mathbf{u}_0, \mathbf{f}) < +\infty \qquad \text{and} \qquad \|\mathbf{u}_n'\|_{L^2(0,T)} \leq \sup_{\mathbf{u}_0\in\mathcal{A}} C_3(\mathbf{u}_0, \mathbf{f}, T) < +\infty, \tag{5.1.17}$$

where $C_1(\mathbf{u}_0, \mathbf{f})$ and $C_3(\mathbf{u}_0, \mathbf{f}, T)$ are independent of n, so the bounds hold uniformly in $n \in \mathbb{N}$. In order to prove this fix n, note that since $\mathbf{u}_n \in V$, then $\mathbf{u}_n \in C([0,T])$, and by the proof in Theorem 1 hence there exists $\mathbf{z}_k \in V$ such that

$$\mathbf{z}_k(t) = \mathbf{u}_0^n + \int_0^t \mathbf{f}(s)\mathrm{d}s - k\int_0^t G^n(s, \mathbf{z}_k(s))\mathrm{d}s \tag{5.1.18}$$

where

$$G^n(s, \mathbf{z}(s)) := \mathbf{D}^{\mathrm{T}}(|\mathbf{D}\mathbf{z}(t)|_2^2 - M^n(t)^2)^+\mathbf{D}\mathbf{z}(t),$$

with M^n continuous and defined by

$$M^n(t) = M_2(\mathbf{u}_n(t), \mathbf{u}_0^n). \tag{5.1.19}$$

We further observe by the proof of Theorem 1 that

$$\mathbf{z}_k \to \mathbf{u}_n \text{ in } C([0,T]) \qquad \text{and} \qquad (\mathbf{z}_k)' \rightharpoonup (\mathbf{u}_n)' \text{ in } L^2(0,T),$$

and that the following bounds holds true

$$\sup_{k\in\mathbb{N}} \sup_{s\in[0,T]} \|\mathbf{z}_k(s)\|_2 \le C_1(\mathbf{u}_0, \mathbf{f}) \qquad \text{and} \qquad \sup_{k\in\mathbb{N}} \|\mathbf{z}_k'\|_{L^2(0,T)} \le C_3(\mathbf{u}_0, \mathbf{f}, T),$$

so that the bounds in (5.1.17) follow by noting that $\sup_{\mathbf{u}\in\mathcal{A}} C_1(\mathbf{u}_0, \mathbf{f})$ and $\sup_{\mathbf{u}\in\mathcal{A}} C_3(\mathbf{u}_0, \mathbf{f}, T)$ are finite.

From (5.1.17) we obtain that along a subsequence (not relabeled)

$$\mathbf{u}_n \to \mathbf{u}^* \text{ in } C([0,T]) \qquad \text{and} \qquad (\mathbf{u}_n)' \rightharpoonup (\mathbf{u}^*)' \text{ in } L^2(0,T), \tag{5.1.20}$$

for some $\mathbf{u}^* \in V$.

Step 2: $\mathbf{u}^*$ is a solution to $\mathrm{QVI}_N(\mathbf{u}_0^*)$. Note that by (5.1.20), then M^n defined in (5.1.19) satisfies

$$M^n \to M^* \text{ in } C([0,T]) \quad \text{for} \quad M^*(t) := M_2(\mathbf{u}_0^*, \mathbf{u}^*(t)),$$

and by definition $M^n(t) \ge \alpha > 0$. Thus, we have that

$$\mathscr{K}^2(\mathbf{u}_n, \mathbf{u}_0) \xrightarrow{\mathrm{M}} \mathscr{K}^2(\mathbf{u}^*, \mathbf{u}_0), \tag{5.1.21}$$

in the sense of Mosco using the V topology where

$$\mathscr{K}^2(\mathbf{z},\mathbf{u}_0) := \{\mathbf{w}\in V : \mathbf{w}(t)\in \mathcal{K}^2(\mathbf{z}(t),\mathbf{u}_0) \text{ for all } t\in[0,T]\}.$$

We prove initially (II) in Definition 1. If $\mathbf{w}_n \in \mathscr{K}^2(\mathbf{u}_n,\mathbf{u}_0^n)$ and $\mathbf{w}_n \rightharpoonup \mathbf{w}^*$ in V for some $\mathbf{w}^*$, then $\mathbf{w}^* \in \mathscr{K}^2(\mathbf{u}^*,\mathbf{u}_0^*)$: Since V is continuously and compactly embedded in $C([0,T])$, we observe that $\mathbf{w}_n \to \mathbf{w}^*$ in $C([0,T])$, and since

$$\sqrt{|(\mathbf{D}_1\mathbf{w}_n(t))_i|^2 + |(\mathbf{D}_2\mathbf{w}_n(t))_i|^2} \le (M_2(\mathbf{u}_n(t),\mathbf{u}_0^n))_i,$$

for $t\in[0,t]$ and $i=1,\dots,N$, thus

$$\sqrt{|(\mathbf{D}_1\mathbf{w}^*)_i|^2 + |(\mathbf{D}_2\mathbf{w}^*)_i|^2} \le (M_2(\mathbf{u}^*,\mathbf{u}_0^*))_i,$$

i.e., $\mathbf{w}(t)\in\mathcal{K}^2(\mathbf{u}^*(t),\mathbf{u}_0^*)$ for all $t\in[0,T]$ which proves the statement.

Secondly we turn the attention to (I) in Definition 1. Note that $M^n \ge \alpha > 0$ and $M^n \to M^*$ in $C([0,T])$ so that

$$\beta_n := \left(1 + \frac{\|M^n - M^*\|_{C([0,T])}}{\alpha}\right)^{-1},$$

is such that $\beta_n \uparrow 1$ and if $\mathbf{w}^* \in \mathscr{K}^2(\mathbf{u}^*,\mathbf{u}_0^*)$, then $\beta_n\mathbf{w}^* \in \mathscr{K}^2(\mathbf{u}_n,\mathbf{u}_0^n)$ and $\beta_n\mathbf{w}^* \to \mathbf{w}^*$ in V. That is, we have prove (5.1.21).

Hence, the set convergence in (5.1.21) implies that $\mathbf{u}^* \in \mathscr{K}^2(\mathbf{u}^*,\mathbf{u}_0^*)$ satisfies

$$\int_0^T ((\mathbf{u}^*)'(t) - \mathbf{f}(t), \mathbf{v}(t) - \mathbf{u}^*(t))\mathrm{d}t \ge 0, \qquad \text{for all } \mathbf{v}\in\mathscr{K}^2(\mathbf{u}^*,\mathbf{u}_0^*).$$

Further, a density argument shows that $\mathbf{u}^*$ is actually a solution to $\mathrm{QVI}_N(\mathbf{u}_0^*)$.

Finally, exploiting the lower semicontinuity of the objective functional, we observe that

$$J(\mathbf{u}^*, \mathbf{u}_0^*) \leq \liminf_{n\to\infty} J(\mathbf{u}^n, \mathbf{u}_0^n) = \lim_{n\to\infty} J(\mathbf{u}^n, \mathbf{u}_0^n) = \inf J.$$

$\square$

A few words are in order concerning the previous result. Although, the previous theorem provides the basic existence result for our problem of interest, we still require a result that is plausible to be computationally implemented. In particular, we require a result that also involves time-discretization of the implicit Euler type.

5.2 Discrete approximation method

Taking any natural number $M \in \mathbb{N}$, we consider the discrete grid/mesh on $(0, T)$ defined by

$$T_M := \{0, \tau_M, ..., T - \tau_M, T\}, \quad \tau_M := \frac{T}{M},$$

with the *stepsize of discretization* τ and the *mesh points* $t_j^M := j\tau_M$ as $j = 0, ..., M$. Then the quasi-variational inequality in ($\mathrm{QVI}_N(\mathbf{u}_0)$) is replaced by the problem

$$\mathbf{u}_j^M \in \mathcal{K}^p(\mathbf{u}_0, \mathbf{u}_j^M) \quad : \quad \left(\frac{\mathbf{u}_j^M - \mathbf{u}_{j-1}^M}{\tau_M} - \mathbf{f}_j^M, \mathbf{v} - \mathbf{u}_j^M\right)_{\mathbb{R}^N} \geq 0, \qquad \forall\, \mathbf{v} \in \mathcal{K}^p(\mathbf{u}_0, \mathbf{u}_j^M), \quad j = 1, 2, \ldots, M, \tag{$\mathrm{QVI}_N^M(\mathbf{u}_0)$}$$

where

$$\mathbf{f}_j^M = \int_{(j-1)\tau_M}^{j\tau_M} \mathbf{f}(t)\mathrm{d}t \qquad j = 1, 2, \ldots, M.$$

Given $\{\mathbf{u}_j^M\}$ satisfying ($\mathrm{QVI}_N^M(\mathbf{u}_0)$), we consider its *piecewise linear extension* $\mathbf{u}^M(t)$ to the continuous-time interval $(0,T)$, i.e.,

$$\mathbf{u}^M(t) := \sum_{j=1}^{M} \mathbf{u}_j^M \chi_{I_j}(t), \qquad \text{where} \quad I_j = [(j-1)\tau_M, j\tau_M) \qquad j = 1,2,\ldots,M.$$

For a given $\mathbf{u}_0 \in \mathbb{R}^N$, existence of solutions $\mathbf{u} = \{\mathbf{u}_j^M\}_{j=1}^M$ to ($\mathrm{QVI}_N^M(\mathbf{u}_0)$) follows analogously as in Theorem 1. Further, the discrete version of the optimal control problem ($\mathbb{P}^h$) is given by

Problem ($\mathbb{P}_N^M$). Let $\sigma > 0$, $\mathbf{f} : (0,T) \to \mathbb{R}^N$ nonnegative and $\mathbf{a}, \mathbf{u}_0^{\mathrm{ref}} \in \mathbb{R}^N$ be given. Consider

$$\begin{aligned}
&\min \qquad J^M(\mathbf{u},\mathbf{u}_0) := \sum_{j=1}^{M} \mathbf{a}^\intercal(\mathbf{u}_j^M - \mathbf{u}_0)\tau_M + \frac{\sigma}{2}(\mathbf{u}_0 - \mathbf{u}_0^{\mathrm{ref}})^\intercal(\mathbf{u}_0 - \mathbf{u}_0^{\mathrm{ref}}),\\
&\text{over} \qquad\qquad \mathbf{u}_0^M, \mathbf{u}_1^M, \mathbf{u}_2^M, \ldots, \mathbf{u}_M^M \in \mathbb{R}^N;\\
&\text{s.t.} \quad \mathbf{u} = \{\mathbf{u}_j^M\}_{j=1}^M \text{ solves } \mathrm{QVI}_N^M(\mathbf{u}_0),\\
&\qquad\ \ \mathbf{u}_0 \in \mathcal{A}.
\end{aligned}$$

In the same vein of Theorem 2, and analogously we have.

Theorem 3. *There exists a solution to Problem* ($\mathbb{P}_N^M$)*.*

Chapter 6: Conclusion and Future Research Directions

In this dissertation, we have studied variational and quasi-variational inequalities with gradient constraints and related problems. The novel contributions of this work are severalfold:

i. For a specific one dimensional QVI stationary problem we have determined necessary and sufficient conditions for a function to a be a solution, and proven that regularization of the upper bound of the gradient constraint leads to uniqueness of solutions. In words, conditions can be stated as follows. A function is a solution provided that (a) it satisfies the gradient equality constraint or it is identical to the supporting structure and (b) material is conserved. The extension of these conditions to general problems is under study.

ii. We have developed new theory to tackle VIs with Borel measures as gradient bounds and with the state space given by a subspace of the functions of bounded variation. The results are further applicable to problems posed in standard Sobolev spaces and when the gradient bounds are given by integrable functions. The identification of the Fenchel pre-dual problem was performed, and leads to the study of variational problems on the space of vectorial Borel measures with square integrable divergences. The primal-dual structure was then used to provide a solution algorithm. The extension of the entire work into an evolutionary setting is ongoing.

iii. A version of the implicit function theorem that deals with Newton differentiable maps was established and it is directly applicable to the regularized stationary diffusive QVI problem. In addition, an improvement on the state of the art Newton differentiability result of the regularized gradient constrained penalty was provided. The conjunction of both results leads to a Newton differentiability result of the control to state map for the QVI problem.

iv. The semi-discretized (in time) and fully discretized (in time and space) QVI evolutionary problem was considered as constraint of an overall optimization problem. The design variable was taken to be the initial supporting structure and results concerning existence were determined. The extension to the fully continuous setting is envisaged.

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Curriculum Vitae

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