A New Valuation On Lattice Polytopes

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Dedication

I dedicate this dissertation to my wife April, my daughter Valentina and my son Hamilton.

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Abstract

A NEW VALUATION ON LATTICE POLYTOPES

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The present thesis investigates a new family of valuations on convex polytopes. The work done can be regarded in two perspectives. On one hand, it can be considered as a generalization of the well-studied theory of Ehrhart polynomials. On the other hand, it can be seen as a special case of translation-invariant valuations on convex bodies.

We start by defining the valuation on convex polytopes in general, then we restrict our attention to lattice polytopes, that is, polytopes with integer vertices. The h^*-vectors of this valuation are studied and some partial classification results are made. Next, this valuation is extended to a more general function related to the discrete mixed volume. Finally, we mention an application to the study of sparse resultants.

Chapter 1: Introduction

The present thesis introduces a new family of valuations on convex polytopes, at the crossroads of enumerative and geometric combinatorics. The "classical" setup is to model an enumerative problem as counting integer points in a polyhedral object. Ehrhart Theory provides suitable geometric tools when the parameter in question is geometric dilation of a given polytope.

The core of this thesis is divided into five chapters. The background will be discussed in Chapter 2, where we will provide rigorous definitions of all the concepts needed in later chapters.

A map φ from convex polytopes in $\mathbb{R}^d$ into an abelian group is a valuation if for polytopes P and Q such that $P \cup Q$ is convex, we obtain the value on $P \cup Q$ by adding the values on P and Q and subtracting the value on their intersection $P \cap Q$: $\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$.

The origin of the notion of valuation can be traced back to Dehn's solution of Hilbert's Third Problem. However, the starting point for a systematic investigation of general valuations was Hadwiger's fundamental characterization of the linear combinations of intrinsic volumes as the continuous valuations that are rigid motion invariant [13]. McMullen's deep result on the polynomial expansion of translation invariant valuations is among the seminal contributions to the structure theory of the space of translation-invariant valuations [18].

The first example of a valuation that usually comes to mind is the volume. Another one is the Euler characteristic, which is constant equal to 1 on non-empty polytopes. There are far more valuations, but we will be interested in those related to counting integer points in convex bodies.

A classical result [7] from the 1960's shows that if P is a full-dimensional convex polytope in $\mathbb{R}^d$ with integer vertices, then the number of integer points in dilates of P is a polynomial of degree d in the factor of dilation. Such a polynomial is known as the Ehrhart polynomial of P [2]. Ehrhart polynomials appear all over enumerative and algebraic combinatorics. For some of their coefficients an interpretation can be given: the leading coefficient corresponds to the volume, the second highest coefficient is related to half the surface area, and the constant coefficient is the Euler characteristic of P. Still, a full understanding of Ehrhart polynomials is far out of sight.

In Chapter 3, we will define the new valuation φ_x to be the function that counts the number of lattice points in dilates of a polytope, when the lattice is translated by a real vector x. Note that when x is an integer vector, φ_x reduces to the standard lattice point enumerator. We will show that φ_x is indeed a valuation, and that it is invariant under translation by integer vectors.

We will define a relation $\sim$ on the points in $\mathbb{R}^d$, induced by φ_x. It will turn out to be an equivalence relation as we will prove so. By making use of this relation and Minkowski's existence and uniqueness theorem, we will prove the main theorem in this chapter which states that φ_x together with $\sim$ determine the polytope P uniquely up to translation.

In Chapter 4, we will consider the expansion of $\varphi_x(nP)$ in the polynomial basis $\binom{n+d}{d}, \binom{n+d-1}{d}, \ldots, \binom{n}{d}$, where the coefficients form the so called $h*$-vector of P. We will then study these $h*$-vectors, comparing them with the counterparts of Ehrhart polynomials. In particular, we will derive similar results concerning positivity and monotonicity while we obtain some differences in their corresponding values.

In the same spirit, we will define the cone, the lattice and the semigroup generated by those polynomials and we will investigate some of their properties.

Chapter 4 will end with a partial classification of two objects: the h^*-vectors and the groups generated by the lattice of polynomials. We will provide plenty of examples to illustrate our results and to give an idea on the difficulty of a full classification. Also, counterexamples to some intuitive questions will be mentioned as well.

In Chapter 5, we will extend our valuation to the mixed setting. We will define the mixed valuation induced by φ_x as well as a new notion of shifted mixed volume that extends the classical one. We will elaborate on some definitions and properties therein, making analogy with the well-known ones about mixed valuations in general.

Finally, Chapter 6 has a purely applied nature and deals with problems of sparse resultants. The problem of computing all common zeros of a system of polynomials is of fundamental importance in a wide variety of scientific and engineering applications. Sparse elimination exploits the structure of polynomials by measuring their complexity in terms of Newton polytopes instead of total degree. The sparse, or Newton, resultant generalizes the classical homogeneous resultant and its degree is a function of the mixed volume of the Newton polytopes. We will apply φ_x to discuss some special cases of these resultants, based on the work in [5] and [28].

Most of the work in the present thesis will eventually appear in [1].

Chapter 2: Basics

In this chapter we introduce the main objects together with their basic properties. Our main sources which we recommend for further reading are Ziegler's *Lectures on Polytopes* [30], the book by Beck and Robins on integer point enumeration [4], the survey paper on the Smith Normal Form by Morris Newman [19], and the book *Convex Bodies*: *The Brunn-Minkowski Theory* by Rolf Schneider [21]. Basic knowledge of linear algebra and combinatorics is assumed.

2.1 Polytopes

Let $\mathbb{N}$ be the set of natural numbers $\{1, 2, 3, \dots\}$. For $n \geq 1$, let $[n] := \{1, \dots, n\}$. A set $C \subseteq \mathbb{R}^d$ is said to be convex if the segment $[x, y] = \{(1-\lambda)x + \lambda y \colon 0 \leqslant \lambda \leqslant 1\}$ is contained in C whenever $x, y \in C$. If a set is not itself convex, its convex hull is the smallest convex set containing it

$$\operatorname{conv}(C) := \{\lambda_1 c_1 + \cdots + \lambda_m c_m \colon c_i \in C, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}. \tag{2.1}$$

More explicitly, all the points in $\operatorname{conv}(C)$ may be obtained by forming a particular set of linear combinations of the elements in C. Linear combinations of the form (2.1) are called convex combinations. The affine hull of a set $C \subseteq \mathbb{R}^d$ is the smallest affine space containing it

$$\text{aff}(C) := \{\lambda_1 c_1 + \cdots + \lambda_m c_m \colon c_i \in C, \lambda_i \in \mathbb{R}, \sum_{i=1}^{m} \lambda_i = 1\}. \tag{2.2}$$

A polytope is the convex hull of a finite set in $\mathbb{R}^d$. If the finite set of points is $C = \{c_1, \ldots, c_m\} \subseteq \mathbb{R}^d$, then the corresponding polytope can be expressed as in (2.1). In low dimensions, polytopes are familiar figures from geometry:

- A polytope in $\mathbb{R}$ is the empty set, a point or a line segment.
- A polytope in $\mathbb{R}^2$ is the empty set, a point, a line segment or a convex polygon.
- A polytope in $\mathbb{R}^3$ is the empty set, a point, a line segment, a convex polygon lying in a plane, or a three-dimensional polyhedron.
- etc. . .

The dimension of a polytope P is defined to be the dimension of its affine hull, $\dim(P) := \dim(\text{aff}(P))$.

There is another equivalent way to define polytopes. A hyperplane $H \subseteq \mathbb{R}^d$ is an affine space of dimension $d-1$, that is, there exist $a \in \mathbb{R}^d$, $a \neq 0$, and $b \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^d \colon a^t x = b\}$. H divides $\mathbb{R}^d$ into two halfspaces $H^+ = \{x \in \mathbb{R}^d \colon a^t x \geq b\}$ and $H^- = \{x \in \mathbb{R}^d \colon a^t x \leq b\}$. H is called a supporting hyperplane for a polytope P if P is fully contained in either H^+ or H^- and the intersection of P with H is nonempty. A subset of $\mathbb{R}^d$ is called a polyhedron if it is the intersection of finitely many halfspaces. With this setting in mind, a polytope is simply a bounded polyhedron in $\mathbb{R}^d$.

For us, the most important polytopes will be convex hulls of sets of points with integer coordinates. Such polytopes are often called lattice polytopes in the literature. Thus, a lattice polytope is a set of the form $\operatorname{conv}(S)$, where $S \subseteq \mathbb{Z}^d$ is a finite set.

$F \subseteq P$ is called a face of P if there is a supporting hyperplane H of P such that $F = H \cap P$. By definition, the empty set $\emptyset$ and P itself are faces as well. Faces of dimension 0, 1, $\dim(P) - 1$ are called vertices, edges and facets respectively. The dimension of $\emptyset$ is -1. A face is proper if $\dim(F) \leq \dim(P) - 1$ and P is a full-dimensional polytope in $\mathbb{R}^d$ if $\dim(P) = d$.

The boundary ∂P of a polytope P is the set of points contained in a proper face. The relative interior of P is defined by $\operatorname{relint}(P) = P \setminus \partial P$. We collect some simple but basic facts about polytopes:

Theorem 2.1 [30]. Let $P \subseteq \mathbb{R}^d$ be a polytope and $V := \operatorname{vert}(P)$ be the set of the vertices of P. Let F be a face of P.

1. Every polytope is the convex hull of its vertices: $P = \operatorname{conv}(\operatorname{vert}(P))$.

2. The face F is a polytope, with $\operatorname{vert}(F) = F \cap V$.

3. Every intersection of faces of P is a face of P.

4. The faces of F are exactly the faces of P that are contained in F.

$\square$

A polytope Δ with vertices $v_0, v_1, \ldots, v_r$ is an r-dimensional simplex if $v_0, v_1, \ldots, v_r$ are affinely independent. The convex hull of any subset of vertices is a simplex itself and a face of Δ. We will denote by Δ_d the standard d-simplex, i.e. the convex hull of the origin and the d unit vectors in $\mathbb{R}^d$.

For $n \in \mathbb{N}$ and P a polytope, we define the n-th dilation of P as

$$nP = \{nx \colon x \in P\}. \tag{2.3}$$

The Minkowski sum of two polytopes P and Q is defined as

$$P + Q = \{x + y \colon x \in P, y \in Q\}. \tag{2.4}$$

Both operations, taking the dilation or the Minkowski sum, yield again a polytope.

2.2 Valuations

Throughout we will denote by $\wp(\mathbb{R}^d)$ the set of polytopes in $\mathbb{R}^d$. As a subset of $\wp(\mathbb{R}^d)$, we will denote the set of polytopes with integer vertices, or lattice polytopes, by $\wp(\mathbb{Z}^d)$.

Let G be an abelian group. A valuation on polytopes is a map $\varphi \colon \wp(\mathbb{R}^d) \to G$ such that $\varphi(\emptyset) = 0$ and

$$\varphi(P \cap Q) = \varphi(P) + \varphi(Q) - \varphi(P \cup Q) \tag{2.5}$$

for all P, $Q \in \wp(\mathbb{R}^d)$ with $P \cup Q \in \wp(\mathbb{R}^d)$. φ is called simple if $\varphi(P) = 0$ for all polytopes of dimension strictly less than d, and φ is called homogeneous of degree r if for all $n \in \mathbb{N}$ and all $P \in \wp(\mathbb{R}^d)$ we have $\varphi(nP) = n^r \varphi(P)$. Moreover, φ is called translation-invariant if $\varphi(P + t) = \varphi(P)$ for all $P \in \wp(\mathbb{R}^d)$ and for all $t \in \mathbb{R}^d$.

An example of a simple and homogeneous valuation of degree d is the d-dimensional volume Vol_d. Another fundamental valuation on polytopes is the Euler characteristic

$\chi\colon \wp(\mathbb{R}^d) \to \mathbb{Z}$ with $\chi(P) = 1$ for every non-empty polytope P.

Theorem 2.2 [18]. Let $\varphi\colon \wp(\mathbb{R}^d) \to G$ be a valuation and let $P_1, \ldots, P_m \in \wp(\mathbb{R}^d)$ such that $P = P_1 \cup P_2 \cup \ldots P_m \in \wp(\mathbb{R}^d)$. Then,

$$\varphi(P) = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \varphi\left(\bigcap_{i \in I} P_i\right) \tag{2.6}$$

Equation (2.6) is known as the inclusion-exclusion property for valuations. □

The characteristic function of a polytope P is defined as

$$\mathbf{1}_P(x) = \begin{cases} 1 & \text{if } x \in P, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Theorem 2.3 [18]. Let $P \in \wp(\mathbb{R}^d)$ be an r-dimensional polytope.
Let $\varphi\colon \wp(\mathbb{R}^d) \to G$ be a translation-invariant valuation. Then,

1. (Polynomiality) $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ in n of degree at most r, for all $n \in \mathbb{N}$.

2. (Reciprocity) $\varphi_P(-n) = (-1)^r \varphi(\mathrm{relint}(-nP))$.

□

In particular, the second part of the theorem gives an interpretation for the evaluation of φ_P at negative integers. Here $-P$ is the image of P under the symmetry through the origin, i.e. $-P = \{-x\colon x \in P\}$.

2.3 Ehrhart Polynomials

Lattice point enumeration in polytopes is a classical topic in geometric combinatorics. A classical theorem by Ehrhart [7] states that the function counting lattice points in the n-th dilate of an r-dimensional lattice polytope P in $\mathbb{R}^d$ agrees with a polynomial $\mathrm{Ehr}_P(n)$ of degree r for $n \geq 1$, the Ehrhart polynomial of P. It follows that the generating function — the Ehrhart series — is rational of the form

$$\overline{\mathrm{Ehr}}_P(t) := 1 + \sum_{n\geq 1} \mathrm{Ehr}_P(n)t^n = \frac{h_0^*(P) + h_1^*(P)t + \cdots + h_r^*(P)t^r}{(1-t)^{r+1}} := \frac{\delta_P(t)}{(1-t)^{r+1}}. \tag{2.8}$$

The vector $h^* = h^*(P) := (h_0^*(P), h_1^*(P), \ldots, h_d^*(P))$ is called the h^*-vector of P, where $h_i^*(P) = 0$ for all $i > r$. The polynomial $\delta_P(t)$ is called the δ-polynomial, also known as the h^*-polynomial of P. The importance of the h^*-vector stems from the fact that it encodes the Ehrhart polynomial of P in a different polynomial basis:

$$\mathrm{Ehr}_P(n) = h_0^*(P)\binom{n+r}{r} + h_1^*(P)\binom{n+r-1}{r} + \cdots + h_r^*(P)\binom{n}{r}. \tag{2.9}$$

Although a complete classification of Ehrhart polynomials seems out of sight, there are non-trivial constraints on the set of Ehrhart polynomials. We mention a couple of them below.

Theorem 2.4 [24]. Let P be a lattice polytope in $\mathbb{R}^d$ and let $h^*(P) = (h_0^*(P), h_1^*(P), \ldots, h_d^*(P))$ be the h^*-vector of its Ehrhart polynomial. Then the following statements hold:

1. (Nonnegativity) $h_i^* \geq 0$ for all $i \geq 0$.

2. (Monotonicity) For two lattice polytopes P and Q in $\mathbb{R}^d$ such that $P \subseteq Q$, $h_i^*(P) \leq h_i^*(Q)$ for all $0 \leq i \leq d$.

$\square$

Theorem 2.5 [2]. (Ehrhart-Macdonald Reciprocity)

Let P be a lattice polytope in $\mathbb{R}^d$. Then, $\mathrm{Ehr}_P(-n) = (-1)^{dim(P)}\mathrm{Ehr}_{relint(nP)}$.

In other words, evaluating the Ehrhart polynomial at negative integers has a combinatorial meaning, namely it equals the number of integer points in the relative interior of nP (up to sign). $\square$

Theorem 2.6 [7]. Let P be a d-dimensional lattice polytope in $\mathbb{R}^d$ and let $h^*(P) = (h_0^*(P), h_1^*(P), \ldots, h_d^*(P))$ be the h^*-vector of its Ehrhart polynomial. Then:

1. $h_0^*(P) = 1$

2. $h_1^*(P) =\mid P \cap \mathbb{Z}^d \mid -d - 1$

3. $h_d^*(P) =\mid \mathrm{relint(P)} \cap \mathbb{Z}^d \mid$

4. $h_0^*(P) + h_1^*(P) + \cdots + h_d^*(P) = d!\ \mathrm{Vol}_d(P)$.

$\square$

2.4 Smith Normal Form

Let R be a commutative ring with an identity 1. An element a of R is a unit if an element b of R exists such that $ab = ba = 1$. Let n be a positive integer and let $R_{n\times n}$ stand for the ring of $n \times n$ matrices over the ring R. An element A of $R_{n\times n}$ is unimodular if an element B of $R_{n\times n}$ exists such that $AB = BA = I_n$, where I_n is the identity matrix of order n. The unimodular matrices of $R_{n\times n}$ form a multiplicative group, denoted by $GL_n(R)$. Here we are working with square matrices, but the same concepts work for rectangular matrices.

Two matrices A and B in $R_{n\times n}$ are said to be equivalent if there exist matrices $U, V \in GL_n(R)$ such that $B = UAV$. If the elementary row (column) operations on an integer matrix are:

1. multiply a row (column) by -1,
2. interchange two rows (columns),
3. add an integer multiple of one row (column) to another,

then multiplication on the left by a unimodular integer matrix corresponds to a sequence of elementary row operations, while multiplication on the right corresponds to a sequence of elementary column operations. It is straightforward that equivalent matrices must have the same rank.

Theorem 2.7 [23]. Every matrix $A \in R_{n\times n}$ of rank r is equivalent to a diagonal matrix $D = \mathrm{diag}\,(s_1, s_2, \ldots, s_r, 0, \ldots, 0)$, where $s_i \neq 0$ for $1 \leq i \leq r$ and $s_1 \mid s_2 \mid \cdots \mid s_r$. Furthermore, the s_i's are unique up to multiplication by a unit. □

The elements $s_i := s_i(A)$ are known as the invariant factors of A, and are basic to the problem of determining when two matrices of $R_{n\times n}$ are equivalent. The matrix D is called the Smith Normal Form of A, and is denoted by $\mathrm{SNF}(A)$. It follows from Smith's theorem that two matrices of $R_{n\times n}$ are equivalent if and only if they have the same rank and the same invariant factors.

For our purposes in this thesis, we will assume $R = \mathbb{Z}$, the ring of integers. The units here are ± 1, and the $n \times n$ unimodular matrices over $\mathbb{Z}$ are those of determinant ± 1.

Let A be a matrix in $\mathbb{Z}_{n\times n}$ and let k be an integer such that $1 \leq k \leq n$. Choose k rows and k columns, and compute all the determinants of the submatrices constructed from these choices. There are $\binom{n}{k}^2$ such choices. Finally, take the greatest common divisor (gcd) of all of these determinants. This number will be denoted by $d_k(A)$, the k-th determinantal divisor of A. Notice that if A is of rank r, then only the first r such numbers will be different from zero. For completeness, we define $d_0(A) = 1$.

The relationship between the determinantal divisors and the invariant factors is quite simple:

$$d_k(A) = s_1(A)s_2(A)\ldots s_k(A) \quad \text{for} \quad 1 \leq k \leq n, \tag{2.10}$$

or equivalently:

$$s_k(A) = \frac{d_k(A)}{d_{k-1}(A)} \quad \text{for} \quad 1 \leq k \leq n. \tag{2.11}$$

Historically, the original purpose behind the invention of this concept by Smith was to solve systems of linear diophantine equations. Suppose we want to find all integral solutions of the diophantine system $Ax = b$, where $A \in \mathbb{Z}_{n\times n}$ and b is an $n \times 1$ integral vector. We first find the Smith Normal Form $S = UAV$ of A, and replace the system by the equivalent system $Sy = c$, where $x = Vy$, and $c = Ub$. If A is of rank r, then

$$S = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.12}$$

where D is a nonsingular diagonal $r \times r$ matrix.

Put $c = (c', c'')^t$, $y = (y', y'')^t$ (t denoting the transpose), where c' and y' are $r \times 1$ and c'' and y'' are $(m-r) \times 1$. Then, $Sy = c$ if and only if $Dy' = c'$. Thus, the system has integral solutions if and only if $D^{-1}c'$ is an integral vector. Consequently, a particular solution in this case is given by $x = V(D^{-1}c', 0)^t$.

Let A be an $n \times n$ integral matrix of full rank n. We define the lattice generated by A as

$$\mathcal{L}(A) = \{Ax \colon x \in \mathbb{Z}^n\} \tag{2.13}$$

For example, take the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, $\mathcal{L}(A) = \mathbb{Z}^2$.

Another example is $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. In this case, $\mathcal{L}(A)$ is the lattice of all integer points whose coordinates sum to an even number.

We define $G := \mathbb{Z}^n/\mathcal{L}(A)$ for some matrix $A \in \mathbb{Z}_{n\times n}$. For the sake of simplicity, we may assume that A has full rank n. Then, G is a finitely generated abelian group. Suppose that $\mathrm{SNF}(A) = \mathrm{diag}\,(s_1, s_2, \ldots, s_n)$, where $s_i \neq 0$ for $1 \leq i \leq n$. A classical theorem from Group Theory states that G can be uniquely written in the "canonical form" as a direct product

$$G = C(s_1) \times C(s_2) \times \cdots \times C(s_n), \tag{2.14}$$

of cyclic groups $C(s_i)$ of order s_i, respectively. If $s_i = 1$ for some i, then we may regard $C(s_i)$ as the trivial group.

2.5 Mixed Volume

Recall that Vol_d denotes the d-dimensional Euclidean volume. One significant property of the volume is that for convex polytopes, and more generally convex bodies, $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$, the function $\mathrm{Vol}_d(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_r P_r)$ agrees with a multivariate polynomial of degree at most d for all $\lambda_1, \ldots, \lambda_r \geq 0$.

When $r = d$, we can single out one particular term in the polynomial expression $\mathrm{Vol}_d(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_d P_d)$ that has special meaning. We define the d-dimensional mixed volume of a collection of polytopes $P_1, P_2, \ldots, P_d$ in $\mathbb{R}^d$, denoted by $\mathrm{MV}_d(P_1, P_2, \ldots, P_d)$, to be the coefficient of $\lambda_1 \lambda_2 \ldots \lambda_d$ in the above polynomial, normalized by $\dfrac{1}{d!}$.

Mixed volumes arise in a lot of mathematical disciplines and give rise to the deep theory of geometric inequalities. We will mention, without proofs, some of their main properties.

Theorem 2.8 [21]. Let $P_1, P_2, \ldots, P_d$ be polytopes in $\mathbb{R}^d$. Then, the following properties hold:

1. $\mathrm{MV}_d(P_1, P_2, \ldots, P_d)$ is invariant if the P_i's are replaced by their images under a volume-preserving affine linear transformation of $\mathbb{R}^d$.

2. $\mathrm{MV}_d(P_1, P_2, \ldots, P_d)$ is symmetric and linear in each variable.

3. $\mathrm{MV}_d(P_1, P_2, \ldots, P_d) \geq 0$.

4. $\mathrm{MV}_d(P_1, P_2, \ldots, P_d) = 0$ if one of the P_i's has dimension zero, i.e. P_i consists of a point, and $\mathrm{MV}_d(P_1, P_2, \ldots, P_d) > 0$ if every P_i has dimension d.

5. The mixed volume of any collection of polytopes can be computed as

$$\mathrm{MV}_d(P_1, P_2, \ldots, P_d) = \frac{1}{d!} \sum_{k=1}^{d} (-1)^{d-k} \sum_{\substack{I \subset [d] \\ |I|=k}} \mathrm{Vol}_d \left(\sum_{i \in I} P_i \right), \qquad (2.15)$$

   where $\sum_{i \in I} P_i$ is the Minkowski sum of polytopes indexed by I

6. $\mathrm{MV}_d(P_1, P_2, \ldots, P_d) \leq \mathrm{MV}_d(Q_1, Q_2, \ldots, Q_d)$ for all polytopes $P_i \subseteq Q_i$ and for $1 \leq i \leq d$.

$\square$

In the discrete setting, the counterpart to the Euclidean volume $\mathrm{Vol}_d(P)$ of a polytope P is the discrete volume $\mathrm{Ehr}_P(1) =| P \cap \mathbb{Z}^d |$. Also, the counterpart to the mixed volume is the discrete mixed volume which we define next. Let $P_1, P_2, \ldots, P_k$ be lattice polytopes in $\mathbb{R}^d$. The discrete mixed volume of $P_1, P_2, \ldots, P_k$ is defined as

$$\mathrm{DMV}(P_1, P_2, \ldots, P_k) := \sum_{J \subseteq [k]} (-1)^{k-|J|} \mid P_J \cap \mathbb{Z}^d \mid . \qquad (2.16)$$

where $P_J := \sum_{j \in J} P_j$ is the Minkowski sum of polytopes for $\emptyset \neq J \subseteq [k]$ and $P_\emptyset = \{0\}$.

With the same notation as above, this furnishes the definition of a mixed Ehrhart polynomial as follows

$$\mathrm{MEhr}_{P_1,P_2,\dots,P_k}(n) := \mathrm{DMV}(nP_1, nP_2, \dots, nP_k) = \sum_{J\subseteq[k]} \mathrm{Ehr}_{P_J}(n) \in \mathbb{Q}[n]. \tag{2.17}$$

Mixed Ehrhart polynomials and their coefficients with respect to various bases of the vector space of polynomials of degree at most d were studied in [12]. In Chapter 5, we will extend some of their results to the shifted scenario.

Khovanskii [16] relates the evaluation $\mathrm{MEhr}_{P_1,P_2,\dots,P_k}(-1)$ to the arithmetic genus of a compactified complete intersection with Newton polytopes $P_1, P_2, \dots, P_k$. A slight variant of equation (2.17) has been employed in [27] as a specific means to study higher dimensional mixed versions of Pick's formula in connection with the combinatorics of intersections of tropical hypersurfaces.

Finally, we recall a result independently due to Bernstein and McMullen. For a reference, see for example theorem 19.4 in the book *Convex and Discrete Geometry* by Peter Gruber [11].

Theorem 2.9. For lattice polytopes $P_1, P_2, \dots, P_k$ in $\mathbb{R}^d$, the function

$$\mathrm{Ehr}_{P_1,P_2,\dots,P_k}(n_1, n_2, \dots, n_k) :=\mid (n_1P_1 + n_2P_2 + \cdots + n_kP_k) \cap \mathbb{Z}^d \mid \tag{2.18}$$

agrees with a multivariate polynomial for all $n_1, n_2, \dots, n_k \in \mathbb{N}$. The degree of such polynomial in n_i is $\dim(P_i)$ for $1 \le i \le k$. $\square$

Chapter 3: Shifted Lattice Point Enumeration

3.1 Introduction

Valuations, as defined in section 2.2, are maps into an abelian group that satisfy the inclusion-exclusion property. We recall the following notation:

$\wp(\mathbb{R}^d)$ The set of polytopes in $\mathbb{R}^d$.

$\wp(\mathbb{Z}^d)$ The set of polytopes in $\mathbb{R}^d$ with integer vertices.

One valuation of particular importance is the lattice point enumerator. It is the function that counts the number of integer points in dilates of lattice polytopes. In section 3.2, we define the main object of study: the shifted lattice point enumerator φ_x. This chapter is concerned with the investigation of geometric properties of this function.

We start by defining φ_x as a generalization of the Ehrhart function and we prove it is indeed a valuation. We then discuss some trivial and non-trivial implications on the class of lattice polytopes. We mention an extension to general convex polyhedra as well.

Section 3.3 deals with an equivalence relation induced by φ_x. We define such a relation and study the equivalence classes determined by it. The setting is switched from the Euclidean space $\mathbb{R}^d$ to the d-dimensional torus $\mathbb{T}^d$. We prove that these classes are sufficient to determine a polytope P up to integer translation. This constitutes the main theorem of Chapter 3. The final section 3.4 contains a variety of examples that illustrate the main properties of φ_x.

The starting point of this chapter is the following fundamental result. For more details, we refer to the paper *The Minkowski Problem for Polytopes* by Daniel Klain.

Theorem 3.1 [17]. Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^d$ are unit vectors that span $\mathbb{R}^d$ and suppose that $a_1, a_2, \ldots, a_k > 0$. Then there exists a polytope $P \in \wp(\mathbb{R}^d)$ having facet unit normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $a_1, a_2, \ldots, a_k$ if and only if

$$a_1 u_1 + a_2 u_2 + \cdots + a_k u_k = 0. \tag{3.1}$$

Moreover, this polytope is unique up to translation. $\square$

Proofs of this theorem and its many generalizations abound in the literature. Once the surface data are suitably defined, the Minkowski theorem can also be generalized to the context of compact convex sets [21]. Minkowski's original proof involves two steps. First, the existence of a polytope satisfying the given facet data is demonstrated by a linear optimization argument. In the second step, the uniqueness of that polytope (up to translation) is then shown to follow from the equality conditions of Minkowski's inequality, a generalized isoperimetric inequality for mixed volumes.

Next we state an elegant theorem due to Georg Alexander Pick (1859-1942), dating back to 1899. For the sake of simplicity, we sketch one of its many proofs.

Theorem 3.2 (Pick's Formula) [2]. Let P be a convex lattice polygon. Let b be the number of lattice points on the boundary of P and i the number of lattice points in the interior of P. Then, the area of P can be expressed as

$$A(P) = \frac{b}{2} + i - 1 \tag{3.2}$$

Proof (sketch). We start by proving that Pick's formula is additive in the following sense. Let $P = P_1 \cup P_2$ where P_1 and P_2 are lattice polygons intersecting in a common edge. For $k = 1, 2$, let b_k be the number of integer points on the boundary of P_k and i_k be the number of integer points in the interior of P_k. We show that

$$A(P_1) + A(P_2) = \frac{b_1}{2} + i_1 - 1 + \frac{b_2}{2} + i_2 - 1.$$

Clearly, $A(P) = A(P_1) + A(P_2)$.

If we denote by L the number of integer points on the common edge, then:

$i = i_1 + i_2 + L - 2$ and $b = b_1 + b_2 - 2L + 2$.

Substituting these identities in equation (3.2) proves the claim.

Now, any convex polygon can be decomposed into triangles that share a common vertex. Hence, it suffices to prove Pick's formula for triangles. Furthermore, any integral triangle can be embedded into an integral rectangle as shown in the figure below. Finally, it remains to show the theorem for integral rectangles whose edges are parallel to the coordinates axes and for rectangular triangles two of whose edges are parallel to the coordinate axes. $\square$

3.2 The valuation φ_x

Let $x \in \mathbb{R}^d$ and P a polytope in $\mathbb{R}^d$. Define the function

$$\varphi_x \colon \wp(\mathbb{R}^d) \longrightarrow \mathbb{Z}_{\geq 0}$$

$$P \longmapsto \mid P \cap (x + \mathbb{Z}^d) \mid$$

Note that when $x \in \mathbb{Z}^d$, $\varphi_x(P) =\mid P \cap \mathbb{Z}^d \mid$ is just the usual lattice point enumerator.

Clearly, φ_x is not translation-invariant in general. However, φ_x is integer translation invariant. For $t \in \mathbb{Z}^d$, we have:

$$\begin{aligned}
\varphi_x(P+t) &=\mid (P+t) \cap (x+\mathbb{Z}^d) \mid \\
&=\mid P \cap (x-t+\mathbb{Z}^d) \mid \\
&=\mid P \cap (x+\mathbb{Z}^d) \mid \\
&= \varphi_x(P).
\end{aligned} \tag{3.3}$$

Recall that a valuation is a map that satisfies the inclusion-exclusion principle.

Theorem 3.3. The map φ_x is a valuation on the class of polytopes in $\mathbb{R}^d$.

Proof. Let P and Q be two polytopes in $\mathbb{R}^d$ such that $P \cup Q \in \wp(\mathbb{R}^d)$. Then:

$$\begin{aligned}
\varphi_x(P \cup Q) &=\mid (P \cup Q) \cap (x+\mathbb{Z}^d) \mid \\
&=\mid (P \cap (x+\mathbb{Z}^d)) \cup (Q \cap (x+\mathbb{Z}^d)) \mid \\
&=\mid P \cap (x+\mathbb{Z}^d) \mid + \mid Q \cap (x+\mathbb{Z}^d) \mid - \mid (P \cap (x+\mathbb{Z}^d)) \cap (Q \cap (x+\mathbb{Z}^d)) \mid \\
&=\mid P \cap (x+\mathbb{Z}^d) \mid + \mid Q \cap (x+\mathbb{Z}^d) \mid - \mid (P \cap Q) \cap (x+\mathbb{Z}^d) \mid \\
&= \varphi_x(P) + \varphi_x(Q) - \varphi_x(P \cap Q).
\end{aligned} \tag{3.4}$$

Hence, φ_x is a valuation on $\wp(\mathbb{R}^d)$. $\square$

Remark 3.4. As x ranges through the Euclidean space $\mathbb{R}^d$, $\varphi_x(P)$ can only assume a finite set of values in $\mathbb{Z}_{\geq 0}$ for any given polytope P. This is due to two facts: on one hand, polytopes are closed and bounded sets in $\mathbb{R}^d$. On the other hand, the lattice $\mathbb{Z}^d$ is a discrete subset of $\mathbb{R}^d$. $\square$

Next, we prove one type of invariance satisfied by φ_x. We denote by $SL_d(\mathbb{Z})$ the set of $d \times d$ integer matrices with determinant ± 1.

Theorem 3.5. Let $P \in \wp(\mathbb{R}^d)$ and let $x \in \mathbb{R}^d$. For any $\psi \in SL_d(\mathbb{Z})$, we have $\varphi_{\psi(x)}(\psi(P)) = \varphi_x(P)$.

Proof. Let P, x and ψ be as given above.

$$\begin{aligned}
\varphi_{\psi(x)}(\psi(P)) &= \mid \psi(P) \cap (\psi(x) + \mathbb{Z}^d) \mid \\
&= \mid (\psi(P) - \psi(x)) \cap \mathbb{Z}^d \mid \\
&= \mid \psi(P - x) \cap \mathbb{Z}^d \mid \\
&= \mid (P - x) \cap \mathbb{Z}^d \mid \\
&= \mid P \cap (x + \mathbb{Z}^d) \mid \\
&= \varphi_x(P).
\end{aligned} \tag{3.5}$$

$\square$

Note that the two sets $(P - x) \cap \mathbb{Z}^d$ and $\psi(P - x) \cap \mathbb{Z}^d$ do not have to be equal, but they always have the same cardinality.

Corollary 3.6. $\varphi_{-x}(-P) = \varphi_x(P)$.

Proof. Let $\psi = -I_d$ where I_d is the $d \times d$ identity matrix. Then, $\psi(x) = -x$ and $\psi(P) = -P$. The result now follows from the previous theorem. $\square$

We will provide illustrative examples in section 3.4, mainly in dimensions 2 and 3 for the sake of visualization.

3.3 Equivalence Classes Induced by φ_x

Definition 3.7. Let $P \in \wp(\mathbb{R}^d)$ and let $x, y \in \mathbb{R}^d$. Define a relation $\sim_P$ on $\mathbb{R}^d$ as follows

$$x \sim_P y \text{ if and only if } \varphi_x(P) = \varphi_y(P). \tag{3.6}$$

It turns out that $\sim_P$ is an equivalence relation, as is shown below.

Proposition 3.8. The relation $\sim_P$ is an equivalence relation.

Proof. We show that $\sim_P$ is reflexive, symmetric and transitive.

1. Clearly, $\varphi_x(P) = \varphi_x(P)$. Hence, $x \sim_P x$.

2. Let $x \sim_P y$. Then, $\varphi_x(P) = \varphi_y(P)$. This is the same as $\varphi_y(P) = \varphi_x(P)$. Hence, $y \sim_P x$. The converse is obvious too.

3. Let $x, y, z \in \mathbb{R}^d$ such that $x \sim_P y$ and $y \sim_P z$. Then, $\varphi_x(P) = \varphi_y(P)$ and $\varphi_y(P) = \varphi_z(P)$. This implies that $\varphi_x(P) = \varphi_z(P)$. Hence, $x \sim_P z$. $\square$

An equivalence relation provides a partition of a set into equivalence classes. We denote by $\mathcal{C}(P)$ the collection of equivalence classes induced by $\sim_P$. Elements of $\mathcal{C}(P)$ are called regions. They are of the form

$$R_i = \{x \in \mathbb{R}^d \colon \varphi_x(P) = a_i\}, \tag{3.7}$$

where a_i is a fixed element in $\mathbb{Z}_{\geq 0}$, called the multiplicity of R_i. Note that $\dim(R_i) \in \{0, 1, \ldots, d\}$, that is, the regions can be subsets of any dimension.

Now, we change the setting from the Euclidean space $\mathbb{R}^d$ to the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d \setminus \mathbb{Z}^d$. For our purposes, we identify the torus with the semi-open d-dimensional unit hypercube in $\mathbb{R}^d$: $\mathbb{T}^d \simeq [0,1)^d$. We then get a partition of $\mathbb{T}^d$ into disjoint regions R_i, each of multiplicity a_i. We call such a partition the Shabby partition. Loosely speaking, one can reconstruct the polytope P by performing a 'gluing' of a_i copies of R_i. We will show later that this construction is unique up to integer translation.

As mentioned in remark 3.4, for a polytope $P \in \wp(\mathbb{R}^d)$, the function $\varphi_x(P)$ assumes a finite set of values in $\mathbb{Z}_{\geq 0}$ as x ranges through $\mathbb{R}^d$. This implies that $\mathcal{C}(P)$ has finite cardinality, i.e., there is a finite number of regions for a given polytope. We call this number the Shabby number of P and we denote it by $\alpha(P)$.

Theorem 3.9. Let $P \in \wp(\mathbb{R}^d)$. Then, $\mathcal{C}(P)$ determines P uniquely up to integer translation.

Proof. Consider the natural projection

$$\pi\colon \mathbb{R}^d \longrightarrow \mathbb{T}^d$$

$$x \longmapsto x + \mathbb{Z}^d = \{x\}$$

Here, $\{x\} = (\{x_1\}, \{x_2\}, \ldots, \{x_d\})$, where $\{\}$ denotes the fractional part of a real number. Let P be a d-dimensional polytope in $\mathbb{R}^d$. For each facet F of P, $\pi(F) := F^*$ is a $(d-1)$-dimensional subset of $\mathbb{T}^d$. Note that F and F^* have the same unit normal vector.

By looking at the Shabby partition induced by P, each F^* determines the normal vector corresponding to its preimage F. Moreover, each F^* belongs to some region in $\mathcal{C}(P)$. The multiplicity of such region determines the $(d-1)$-volume of the corresponding facet F. By Minkowski's existence and uniqueness theorem (Theorem 3.1), the polytope P is uniquely determined up to translation.

It remains to show that it is indeed integer translation. In other words, we need to show that if P and Q are two polytopes having the same Shabby diagram, then $P = Q + t$ for some $t \in \mathbb{Z}^d$.

Assume P and Q have the same Shabby diagram. By the above result, we know that $P = Q + t$ for some $t \in \mathbb{R}^d$. Suppose t is not an integer vector. Let F be a facet of Q, then $F + t$ is a facet of P. Since $t \notin \mathbb{Z}^d$, then $\pi(F) \neq \pi(F + t)$. However, since P and Q have the same Shabby partition, $\pi(F)$ and $\pi(F + t)$ both have to be facets of $\pi(Q)$ in $\mathbb{T}^d$. The same holds for every facet of Q and for every vector t. This is impossible for any given polytope unless $t \in \mathbb{Z}^d$. This completes the proof. $\square$

3.4 Examples

In this section, we will provide several examples to illustrate the main ideas behind the valuation φ_x. The figures herein will help visualize what happens in the torus (in low dimensions of course). The figures on the left represent a polytope P and the ones on the right represent the Shabby partition $\mathcal{C}(P)$.

Example 3.10. Let $P = [0, 3]$ be a line segment in dimension 1. Then :

$$\varphi_x(P) = \begin{cases} 4 & \text{if } x \in \mathbb{Z}, \\ 3 & \text{otherwise.} \end{cases} \tag{3.8}$$

In this case, $\mathcal{C}(P) = \{R_1, R_2\}$ where $R_1 := \{0\}$ and $R_2 := (0, 1)$.

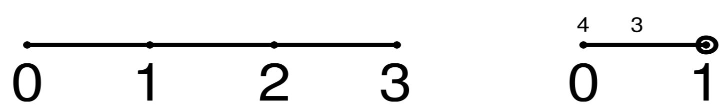


Figure 3.1: Line segment

Note that $R_1 \cup R_2 = \mathbb{T}_1$ and $R_1 \cap R_2 = \emptyset$. □

Example 3.11. Let $P = \Delta_2$, the standard 2-simplex in dimension 2. Then :

$$\varphi_x(P) = \begin{cases} 3 & \text{if } x \in \mathbb{Z}^2, \\ 1 & \text{if } x \in P \setminus \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

In this case, $\mathcal{C}(P) = \{R_1, R_2, R_3\}$ where $R_1 := \{(0,0)\}$, $R_2 := P \setminus \mathbb{Z}^2$ and $R_3 := \mathbb{T}^2 \setminus (R_1 \cup R_2)$. $\square$

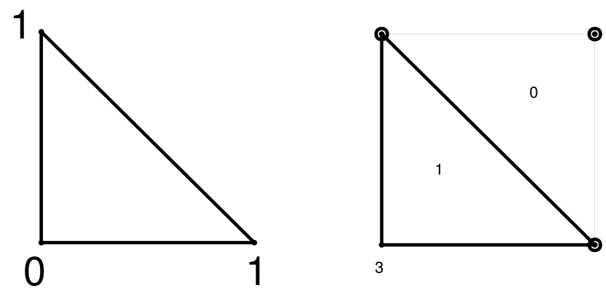


Figure 3.2: Standard 2-simplex

Example 3.12. Let $P = \operatorname{conv}\{(0,0),(1,0),(1,1),(0,3)\}$. Then :

$$\varphi_x(P) = \begin{cases} 6 & \text{if } x \in R_1, \\ 4 & \text{if } x \in R_2, \\ 3 & \text{if } x \in R_3, \\ 2 & \text{if } x \in R_4, \\ 1 & \text{if } x \in R_5. \end{cases} \tag{3.10}$$

In this case, $\mathcal{C}(P) = \{R_1, R_2, R_3, R_4, R_5\}$ where :

$R_1 := \{(0,0)\}$,

$R_2 :=\operatorname{conv}\{(0,0),(0,1)\} \setminus \{(0,1),(0,0)\}$,

$R_3 :=\operatorname{conv}\{(0,0),(0,1),(\frac{1}{2},0)\} \setminus \{\operatorname{conv}\{(0,0),(0,1)\}$,

$R_4 :=$conv$\{(\frac{1}{2},0),(1,0),(0,1),(\frac{1}{2},1)\} \setminus (R_3 \cup$ conv$\{(0,1),(\frac{1}{2},1)\} \cup \{(1,0)\})$,

$R_5 = \mathbb{T}^2 \setminus (R_1 \cup R_2 \cup R_3 \cup R_4)$. $\square$

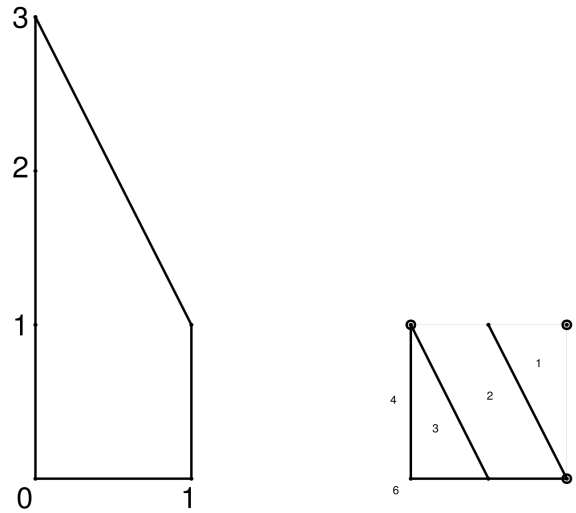


Figure 3.3: The polygon P

Example 3.13. Let P be the unit cube. Then :

$$\varphi_x(P) = \begin{cases} 8 & \text{if } x \in R_1, \\ 4 & \text{if } x \in R_2, \\ 2 & \text{if } x \in R_3, \\ 1 & \text{if } x \in R_4. \end{cases} \tag{3.11}$$

In this case, $\mathcal{C}(P) = \{R_1, R_2, R_3, R_4\}$ where :

$R_1 := \{(0,0)\}$,

R_2 is the union of the relative interior of the twelve edges of the cube,

R_3 is the union of the relative interior of the six facets of the cube,

R_4 is the relative interior of the cube. $\square$

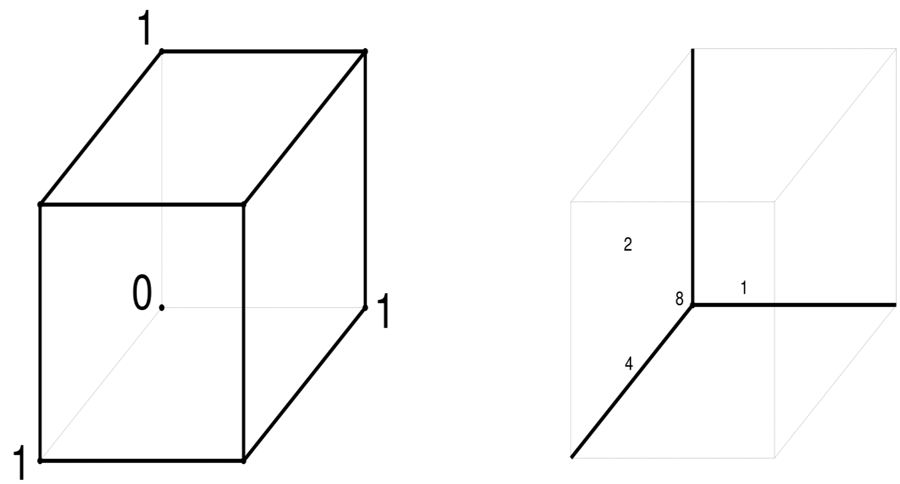


Figure 3.4: The unit cube

Chapter 4: The Study of the h^*-vectors

4.1 Introduction

For an abelian group G, we denote by $G[t]$ the set of formal sums and $G[[t]]$ the set of formal power series, in the variable t with coefficients in G. Both $G[t]$ and $G[[t]]$ are considered as $\mathbb{Z}[t]$-modules. An element $F(t) = \sum\limits_{n\geq 0} a_n t^n \in G[[t]]$ is rational if there are $h(t) \in G[t]$ and $q(t) \in \mathbb{Z}[t] \setminus \{0\}$ such that $q(t).F(t) = h(t)$ and we write

$$\sum_{n\geq 0} a_n t^n = \frac{h(t)}{q(t)}. \tag{4.1}$$

The following theorem is a characterization of polynomiality.

Theorem 4.1 [26]. Let $f\colon \mathbb{Z} \longrightarrow G$ and let $d \in \mathbb{N}$. Then the following statements are equivalent:

1. $\sum\limits_{n\geq 0} f(n)t^n = \dfrac{h(t)}{(1-t)^{d+1}}$, where $h(t) \in G[t]$ and $\deg(h) \leq d$,

2. f is a polynomial in n of degree at most d.

□

Let φ be a $\mathbb{Z}^d$-valuation, that is, a valuation invariant under integer translation:

$\varphi(P+t) = \varphi(P)$ whenever $t \in \mathbb{Z}^d$ and $P \in \wp(\mathbb{Z}^d)$. Let P be a polytope in $\mathbb{R}^d$ with integer vertices. We denote by $\varphi(nP)$ the value of φ at the n-th dilate of P. It is due to McMullen that $\varphi(nP)$ agrees with a polynomial for $n \geq 0$. The next theorem is another version of Theorem 2.3 written in terms of generating functions.

Theorem 4.2 [18]. Let φ and P be as above. Then :

$$\sum_{n\geq 0} \varphi(nP)t^n = \frac{h_0^*(\varphi, P) + h_1^*(\varphi, P)t + \cdots + h_r^*(\varphi, P)t^r}{(1-t)^{r+1}}, \tag{4.2}$$

where $h_r^*(\varphi, P) = \varphi(\text{relint}(-P))$. In particular, $\varphi(nP)$ agrees with a polynomial of degree at most $r := \dim(P)$ for $n \geq 0$. $\square$

As $\varphi(nP)$ agrees with a polynomial for $n \geq 0$, it is natural to ask if there is an interpretation for the evaluation of this polynomial at negative integers. An answer to this question was also given by McMullen (see the second part of Theorem 2.3).

Theorem 4.2 allows us to write $\varphi(nP)$ as

$$\varphi(nP) = h_0^*(\varphi, P)\binom{n+r}{r} + h_1^*(\varphi, P)\binom{n+r-1}{r} + \cdots + h_r^*(\varphi, P)\binom{n}{r}, \tag{4.3}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Hence, every translation-invariant valuation φ comes with the notion of an h^*-vector $h^*(\varphi, P) = (h_0^*, h_1^*, \ldots, h_d^*)$ with $h_i^* = 0$ for $i > r$.

In [14], the authors introduced the notion of combinatorial positivity of translation-invariant valuations on convex polytopes that extends the nonnegativity of Ehrhart

h^*-vectors. They gave a surprisingly simple characterization of combinatorially positive valuations that implies Stanley's nonnegativity and monotonicity of h^*-vectors. We will mention, without proofs, some of the results in [14] that will be used later in this chapter.

Let φ be a $\mathbb{Z}^d$-valuation with h^*-vector $(h_0^*, h_1^*, \ldots, h_d^*)$. Then, φ is called combinatorially positive if $h_i^* \geq 0$ for all i, and combinatorially monotone if $h_i^*(\varphi, P) \leq h_i^*(\varphi, Q)$ for all i whenever $P \subseteq Q$. The main result in [14] is the following simple complete characterization.

Theorem 4.3 [14]. For a translation-invariant valuation $\varphi\colon \wp(\mathbb{Z}^d) \longrightarrow \mathbb{R}$, the following statements are equivalent :

1. φ is combinatorially monotone,
2. φ is combinatorially positive,
3. $\varphi(\mathrm{relint}(\Delta)) \geq 0$ for all simplices $\Delta \in \wp(\mathbb{Z}^d)$.

□

Note that the condition (3) in the above theorem is clearly equivalent to the condition that $\varphi(\mathrm{relint}(P)) \geq 0$ for all polytopes $P \in \wp(\mathbb{Z}^d)$. This is due to the fact that every such polytope admits a triangulation $\mathcal{C}$ into simplices in $\wp(\mathbb{Z}^d)$, and thus

$$\varphi(\mathrm{relint}(P)) = \sum_{\substack{\Delta \in \mathcal{C} \\ \Delta \nsubseteq \delta P}} \varphi(\mathrm{relint}(\Delta)) \geq 0. \tag{4.4}$$

In the remainder of this thesis, we restrict the valuation φ_x to $\wp(\mathbb{Z}^d)$, the set of polytopes in $\mathbb{R}^d$ with integer vertices. In section 4.2, we state and prove several properties of the h^*-vector of φ_x, in analogy to their counterparts in Ehrhart polynomials. Section 4.3 contains a partial classification of such vectors in dimensions 1 and 2, in the spirit of the classification given by Scott in [22].

In section 4.4, we use the h^*-vectors as generators of three objects: the semigroup, the cone and the lattice of polynomials induced by φ_x. This in turn gives another partial classification of the abelian groups formed by the lattices. Throughout the chapter, several examples will be given to illustrate the main results therein.

4.2 Properties of the h^*-vectors

Let $P \in \wp(\mathbb{Z}^d)$ and let $x \in \mathbb{R}^d$. By definition, φ_x is a $\mathbb{Z}^d$-valuation. Theorem 4.2 asserts that $\varphi_x(nP)$ agrees with a polynomial in n of degree at most $r = \dim(P)$, for $n \geq 0$. We can then write $\varphi_x(nP)$ as

$$\varphi_x(nP) = h_0^* \binom{n+r}{r} + h_1^* \binom{n+r-1}{r} + \cdots + h_r^* \binom{n}{r}, \tag{4.5}$$

and we can define the h^*-vector of φ_x as $h^*(\varphi_x, P) = (h_0^*, h_1^*, \ldots, h_d^*)$ with $h_i^* = 0$ for all $i > r$.

Note that if $x \in \mathbb{Z}^d$, then $\varphi_x(nP)$ is just the Ehrhart polynomial of P and $h^*(\varphi_x, P)$ is just the h^*-vector of P, sometimes called the δ-vector of P.

Proposition 4.4. Let P and Q be two polytopes in $\wp(\mathbb{Z}^d)$ such that $P \subseteq Q$. Then, $h_i^*(\varphi_x, P) \leq h_i^*(\varphi_x, Q)$ for all i.

Proof. We have : $\varphi_x(\text{relint}(P)) = |\ \text{relint}(P) \cap (x+\mathbb{Z}^d)\ | \geq 0$ since φ_x is a counting function and such a function is always nonnegative. Moreover, φ_x is a $\mathbb{Z}^d$-valuation. By Theorem 4.3, a translation-invariant valuation is combinatorially monotone if and only if it is nonnegative on the relative interior of polytopes. This completes the proof. $\square$

Corollary 4.5. $h_i^* \geq 0$ for all i.

Proof. Clearly, $\emptyset$ is a polytope and $\emptyset \subseteq P$ for all $P \in \wp(\mathbb{Z}^d)$.
Hence, $h_i^*(\varphi_x, P) \geq h_i^*(\varphi_x, \emptyset) = 0$. $\square$

Next we characterize the values of $h_i^*(\varphi_x, P)$ for $i = 0, 1, r$.

Proposition 4.6. Let $h_i^*(\varphi_x, P)$ be the h^*-vector of φ_x and P. Then :

1. $h_0^* = 0$ if $x \notin \mathbb{Z}^d$,

2. $h_1^* = \varphi_x(P)$ if $x \notin \mathbb{Z}^d$,

3. $h_r^* = \varphi_x(\text{relint}(-P)) = (-1)^r \varphi_x(nP)|_{n=-1}$.

Proof. Recall that when $x \in \mathbb{Z}^d$, $h_0^* = 1$ and $h_1^* =|\ P \cap \mathbb{Z}^d\ | -r-1$.

1. For $n = 0$, equation (4.5) becomes $\varphi_x(0.P) = h_0^*\binom{r}{r} + h_1^*\binom{r-1}{r} + \cdots + h_r^*\binom{0}{r}$. Now, $\binom{a}{b} = 0$ whenever $a < b$ and $\binom{a}{b} = 1$ whenever $a = b$. Hence, $\varphi_x(0.P) = h_0^*$. On the other hand, $\varphi_x(0.P) =|\ 0.P \cap (x+\mathbb{Z}^d)\ |=|\ \mathbf{0} \cap (x+\mathbb{Z}^d)\ |=|\ \emptyset\ |= 0$ if $x \notin \mathbb{Z}^d$. Thus, $h_0^* = 0$.

2. For $n = 1$, equation (4.5) becomes $\varphi_x(P) = h_0^*\binom{r+1}{r} + h_1^*\binom{r}{r} + \cdots + h_r^*\binom{1}{r}$. Substituting $h_0^* = 0$ and $\binom{r}{r} = 1$, we get $\varphi_x(P) = h_1^*$.

3. This follows from Theorem 4.2 and Theorem 2.3, using the fact that φ_x is a $\mathbb{Z}^d$-valuation. $\square$

The final result of this section states that the sum of the coordinates of the $h*$-vector of a full-dimensional polytope P is a multiple of the Euclidean volume of P. This is similar to Ehrhart polynomials (Theorem 2.6).

Given a geometric object $S \subset \mathbb{R}^d$, its Euclidean volume is one of the fundamental data of S. It is defined by the integral

$$\mathrm{Vol}_d(S) = \int_S dx. \tag{4.6}$$

By the definition of the integral in the Riemannian sense, we can think of computing $\mathrm{Vol}_d(S)$ by approximating S with d-dimensional boxes that get smaller and smaller. To be precise, if we take the boxes with side length $\frac{1}{n}$ then they each have volume $\frac{1}{n^d}$. Equivalently, we can think of the boxes as filling out the space between grid points in the lattice $\left(\frac{1}{n}\mathbb{Z}^d\right)$. This means that volume computation can be approximated by counting boxes, or equivalently, counting lattice points in $\left(\frac{1}{n}\mathbb{Z}^d\right)$:

$$\mathrm{Vol}_d(S) = \lim_{n\to\infty} \frac{1}{n^d} \left| S \cap \left(\frac{1}{n}\mathbb{Z}^d\right) \right|. \tag{4.7}$$

This is the same as

$$\mathrm{Vol}_d(S) = \lim_{n\to\infty} \frac{1}{n^d} \ \left| \, nS \cap \mathbb{Z}^d \, \right| . \tag{4.8}$$

Theorem 4.7. Let $P \in \wp(\mathbb{Z}^d)$ be a polytope of dimension d.

Let $h^*(\varphi_x, P) = (h_0^*, h_1^*, \ldots, h_d^*)$ be the corresponding h^*-vector. Then :

$$h_0^* + h_1^* + \cdots + h_d^* = d! \ \mathrm{Vol}_d(P) \tag{4.9}$$

Proof. If the lattice $\mathbb{Z}^d$ is shifted by a vector in $\mathbb{R}^d$, the limit at ∞ in equation (4.8) does not change. Hence,

$$\mathrm{Vol}_d(P) = \lim_{n\to\infty} \frac{1}{n^d} \ \mid nP \cap (x + \mathbb{Z}^d) \mid = \lim_{n\to\infty} \frac{1}{n^d} \ \varphi_x(nP). \tag{4.10}$$

Using equation (4.5), we can write

$$\varphi_x(nP) = \left(\frac{h_0^* + h_1^* + \cdots + h_d^*}{d!} \right) n^d + \text{lower terms.} \tag{4.11}$$

Finally, combining Equations (4.10) and (4.11), we get the desired result. $\square$

We will mention two corollaries that follow from the results above. But first, we recall that the valuation φ_x takes only a finite set of values in $\mathbb{Z}_{\geq 0}$ as x ranges through $\mathbb{R}^d$ (Remark 3.4). This means that for a polytope $P \in \wp(\mathbb{Z}^d)$, $\varphi_x(nP)$ consists of a finite set of polynomials in n, each of degree at most d. We denote these polynomials by $p_1(n), p_2(n), \ldots, p_\alpha(n)$ where $\alpha := \alpha(P)$ is the Shabby number of P, the number of regions in $\mathcal{C}(P)$.

Said differently, each region R_i in the Shabby partition of P is assigned a polynomial $p_i(n)$. For consistency, we always refer to $p_1(n)$ as the usual Ehrhart polynomial of P. As seen before, the leading coefficient of $p_1(n)$ is the volume of P and the constant term is 1.

Corollary 4.8. For $i = 1, 2, \ldots, \alpha(P)$, the leading coefficient of $p_i(n)$ is $\mathrm{Vol}_d(P)$.

Proof. By Equation (4.11), the leading coefficient of $\varphi_x(nP)$ is $\dfrac{h_0^* + h_1^* + \cdots + h_d^*}{d!}$, which is in turns equal to $\mathrm{Vol}_d(P)$ by Equation (4.9). $\square$

Corollary 4.9. If $x \notin \mathbb{Z}^d$, then the constant term in $\varphi_x(nP)$ is always 0.

Proof. In general, the constant term of a polynomial is equal to its value at 0. For $n = 0$, $\varphi_x(0.P) = h_0^* = 0$ whenever $x \notin \mathbb{Z}^d$ (Proposition 4.6). $\square$

Ehrhart polynomials have been extensively studied in the literature (see [2] and the references therein). They are considered as the discrete counterpart to the Euclidean volume of polytopes. Moreover, they are invariant under the action of $SL_d(\mathbb{Z})$.

Definition 4.10. Two lattice polytopes P and Q in $\mathbb{R}^d$ are called unimodularly equivalent if

$$Q = \psi(P) + m, \tag{4.12}$$

for some $\psi \in SL_d(\mathbb{Z})$ and $m \in \mathbb{Z}^d$.

In [25], Stanley constructed two families of polytopes whose Ehrhart polynomials coincide, but yet they are not unimodularly equivalent. Those families are called the order polytope and the chain polytope associated with a finite poset. We refer to the paper [25] for the definitions and results. To our knowledge, it remains an open problem to find additional conditions on two given polytopes to ensure that they are unimodularly equivalent.

Observation 4.11. We mention two interesting remarks:

1. The set of polynomials corresponding to φ_x is not determined by the Ehrhart polynomial. In other words, there exist lattice polytopes in $\mathbb{R}^d$ having the same Ehrhart polynomial but different sets of polynomials.

   We give a simple example in dimension 2.

   Consider the unit square S and the triangle $P = \text{conv}\{(0,0),(1,0),(0,2)\}$. They both have the same Ehrhart polynomial $p_1(n) = n^2 + 2n + 1$.

   However, as x ranges through $\mathbb{R}^2$, $\varphi_x(nS)$ gives rise to two other polynomials: $n^2 + n$ and n^2, while $\varphi_x(nP)$ gives rise to three other polynomials: $n^2 + n$, n^2 and $n^2 - n$.

2. The set of polynomials corresponding to φ_x and a polytope P does not determine P up to unimodular equivalence. In other words, there exist two polytopes P and Q that have the same set of polynomials induced by φ_x, without being unimodularly equivalent to each other. As an example in dimension 2, let

   $P = \text{conv}\,\{(0,0),(3,0),(2,9),(5,9)\}$ and $Q = \text{conv}\,\{(0,0),(1,0),(3,27),(4,27)\}$.

   Both polytopes yield five polynomials under dilation via the valuation φ_x:

   $27n^2 + 4n + 1, 27n^2 + 4n, 27n^2 + 3n, 27n^2 + n, 27n^2$. However, there do not exist $\psi \in SL_d(\mathbb{Z})$ and $m \in \mathbb{Z}^d$ that satisfy Definition 4.10.

$\square$

We end this section by providing a couple of examples to illustrate the properties of the h^*-vectors of φ_x. This is done by going back and forth between the representation of polynomials in monomial basis and in binomial basis. In general, a polynomial p of degree d is determined by $d+1$ points $(x, p(x)) \in \mathbb{R}^2$. Namely, let $p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$. Evaluating p at distinct values $x_1, x_2, \ldots, x_{d+1}$

gives

$$\begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{bmatrix} = \begin{bmatrix} x_1^d & x_1^{d-1} & \dots & x_1 & 1 \\ x_2^d & x_2^{d-1} & \dots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d+1}^d & x_{d+1}^{d-1} & \dots & x_{d+1} & 1 \end{bmatrix} \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_0 \end{bmatrix}, \tag{4.13}$$

so that

$$\begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} x_1^d & x_1^{d-1} & \dots & x_1 & 1 \\ x_2^d & x_2^{d-1} & \dots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d+1}^d & x_{d+1}^{d-1} & \dots & x_{d+1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{bmatrix}. \tag{4.14}$$

Equation (4.14) is the famous identity known as Lagrange interpolation formula.

Example 4.12. Let $P = \text{conv}\,\{(1,0),(0,1),(-1,0),(0,-1)\}$ in dimension 2. Then, $\mathcal{C}(P) = \{R_1, R_2, R_3, R_4\}$ and $\varphi_x(nP) = \{p_1(n), p_2(n), p_3(n), p_4(n)\}$, where:

$R_1 := \{(0,0)\}$, $p_1(n) = 2n^2 + 2n + 1$;

$R_2 := \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$, $p_2(n) = 2n^2 + 2n$;

$R_3 := \text{conv}\{(0,0),(1,1)\} \cup \text{conv}\{(1,0),(0,1)\} \setminus \left(\mathbb{Z}^2 \cup \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \right\}\right)$, $p_3(n) = 2n^2 + n$;

$R_4 := \mathbb{T}^2 \setminus (R_1 \cup R_2 \cup R_3)$, $p_4(n) = 2n^2$.

The area of P is 2. For $x \in R_4$, we get $h_0^* = 0$ and $h_1^* = p_4(1) = 2$ by Proposition 4.6. By Theorem 4.7, $h_0^* + h_1^* + h_2^* = 2!(2) = 4$. Hence, $h_2^* = 2$.

Thus, $h^*(\varphi_x, P) = (0, 2, 2)$ for $x \in R_4$. $\square$

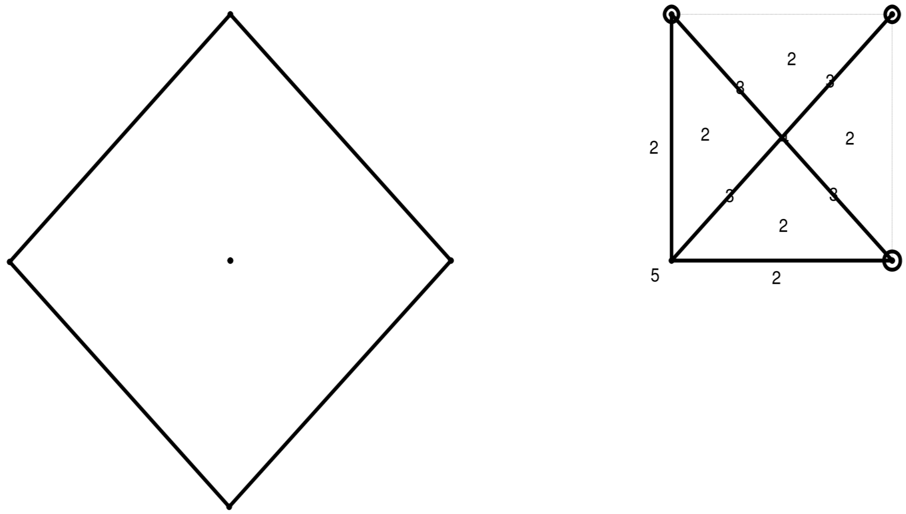


Figure 4.1: The polygon P

Example 4.13. Let $P = \text{conv}\{(0,0,0),(1,0,0),(0,2,0),(1,2,0),(0,2,1),(1,2,1),(1,0,1),(0,0,1)\}$ in dimension 3. Then, $\mathcal{C}(P) = \{R_1, R_2, R_3, R_4, R_5, R_6\}$ and $\varphi_x(nP) = \{p_1(n), p_2(n), p_3(n), p_4(n), p_5(n), p_6(n)\}$, where:

$R_1 := \{(0,0,0)\}$, $p_1(n) = 2n^3 + 5n^2 + 4n + 1$;

$R_2 := \text{conv}\{(0,0,0),(0,1,0)\} \setminus \mathbb{Z}^3$, $p_2(n) = 2n^3 + 4n^2 + 2n$;

$R_3 := \text{conv}\{(0,0,0),(1,0,0)\} \setminus \mathbb{Z}^3$, $p_3(n) = 2n^3 + 3n^2 + n$;

$R_4 := \text{relint}(\text{conv}\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}) \cup \text{relint}(\text{conv}\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\})$, $p_4(n) = 2n^3 + 2n^2$;

$R_5 := \text{relint}(\text{conv}\{(0,0,0),(1,0,0),(1,0,1),(0,0,1)\})$, $p_5(n) = 2n^3 + n^2$;

$R_6 := \mathbb{T}^3 \setminus (R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5)$, $p_6(n) = 2n^3$.

The volume of P is 2. For $x \in R_3$, we get $h_0^* = 0$, $h_1^* = p_3(1) = 6$ and $h_3^* = -p_3(-1) = 0$ by Proposition 4.6.

By Theorem 4.7, $h_0^* + h_1^* + h_2^* + h_3^* = 3!(2) = 12$. Hence, $h_2^* = 6$.

Thus, $h^*(\varphi_x, P) = (0,6,6,0)$ for $x \in R_3$. $\square$

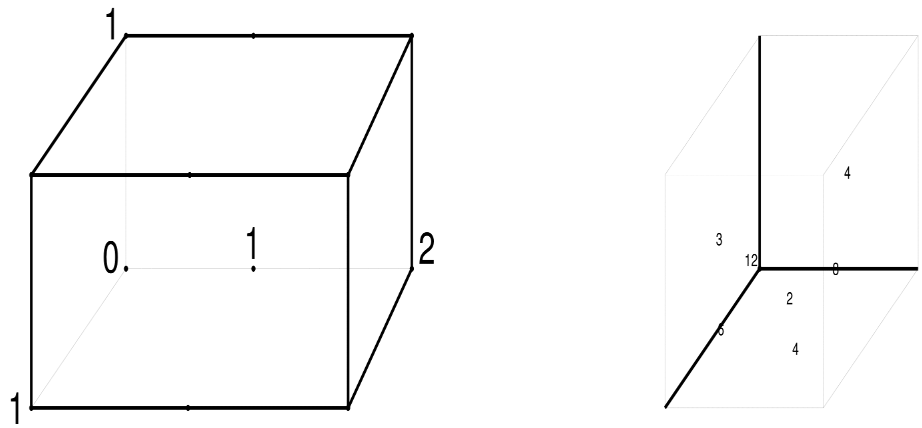


Figure 4.2: The polytope P

4.3 Partial Classification of the h^*-vectors

We start by discussing what happens in dimension 1. This is quite simple as follows. Let $P \in \wp(\mathbb{Z})$ be a one-dimensional lattice polytope. Without loss of generality, we may assume $P = [0, l]$ for some $l \in \mathbb{N}$. Let $h^*(\varphi_x, P) = (h_0^*, h_1^*)$, then $h_0^* + h_1^* = l$ by Theorem 4.7. We have two cases:

Case 1: $x \in \mathbb{Z}$

In this case, $h_0^* = 1$ (Theorem 2.6) and $h_1^* = l - 1$.

Hence, $h^*(\varphi_x, P) = (1, l - 1)$.

Case 2: $x \notin \mathbb{Z}$

In this case, $h_0^* = 0$ (Proposition 4.6) and $h_1^* = l$.

Hence, $h^*(\varphi_x, P) = (0, l)$.

Before we move to dimension 2, we mention the following result.

Theorem 4.14 [22]. Let $P \subset \mathbb{R}^2$ be a lattice polygon. Denote by A and i the area of P and the number of integer points in its relative interior, respectively. If $i \geq 1$, then

$$|\ P \cap \mathbb{Z}^2\ | \quad \leq 3i + 6, \tag{4.15}$$

with only one exception when $P = 3\Delta_2$. Using Pick's formula (Theorem 3.2), this is equivalent to

$$A \leq 2i + 2. \tag{4.16}$$

$\square$

We will distinguish between three cases: $i = 0$, $i = 1$ and $i > 1$. Polygons with no integer points and with only one integer point in their relative interior were classified by Rabinowitz [20]. He showed that, up to unimodular transformation, there are three families of polygons with $i = 0$:

1. Triangles with vertices $(0,0), (0,1)$ and $(p,0)$ for some $p \in \mathbb{N}$,
2. $2\Delta_2$, i.e. $\text{conv}\{(0,0),(2,0),(0,2)\}$,
3. Trapezoids with vertices $(0,0),(0,1),(p,0)$ and $(q,1)$ for some $p, q \in \mathbb{N}$.

On the other hand, there are 16 polygons with one integer point in their relative interior, up to unimodular transformation: 5 triangles, 7 quadrilaterals, 3 pentagons and 1 hexagon. Instead of enumerating them, we illustrate them in the figure at the end of this section.

Now we discuss each of the three cases separately. The first two cases ($i = 0, 1$) are based on the classification provided in [20].

$\underline{i = 0}$: As mentioned above, there are three families satisfying this condition. We investigate each of them to find the corresponding h^*-vectors of φ_x when $x \in \mathbb{Z}^2$ and when $x \notin \mathbb{Z}^2$.

1. Let P be a triangle with vertices $(0,0),(0,1)$ and $(p,0)$ for some $p \in \mathbb{N}$. Then, $\mathcal{C}(P)$ consists of $p+2$ regions in the torus $\mathbb{T}^2$. Let $x \in \mathbb{Z}^2$. Then, $h_0^* = 1$ and $h_1^* =\mid P\cap\mathbb{Z}^2 \mid -3 = p+2-3 = p-1$. Since $i = 0$, then $h_2^* =\mid \text{relint}(P)\cap\mathbb{Z}^2 \mid= 0$. Hence, $h^*(\varphi_x, P) = (1, p-1, 0)$.

   Assume $x \notin \mathbb{Z}^2$. Then, a polynomial is associated to each of the remaining $p+1$ regions. The corresponding h^*-vectors are $h^*(\varphi_x, P) = (0, a, p-a)$ for $a = 0, 1, \ldots, p$.

   Example 3.11 illustrates this idea for $p = 1$.

2. Let $P = 2\Delta_2$. For $x \in \mathbb{Z}^2$, $h^*(\varphi_x, P) = (1, 3, 0)$ since $\mid P \cap \mathbb{Z}^2 \mid= 6$.

   For $x \notin \mathbb{Z}^2$, there are two regions in the torus with corresponding vectors $(0, 3, 1)$ and $(0, 1, 3)$ since the area of P is 2.

3. Let P be a lattice trapezoid with vertices $(0,0),(0,1),(p,0)$ and $(q,1)$ for some $p, q \in \mathbb{N}$. Without loss of generality, we may assume $q \leq p$. Let $x \in \mathbb{Z}^2$. Since $\mid P \cap \mathbb{Z}^2 \mid= p+q+2$, then $h^*(\varphi_x, P) = (1, p+q-1, 0)$.

   Now let $x \notin \mathbb{Z}^2$. We split this case into two subcases: $q = p$ and $q < p$.

   If $q = p$, we get three regions with corresponding vectors $(0, 2p, 0)$, $(0, p+1, p-1)$ and $(0, p, p)$, or two regions if $p = 1$ since the first two would coincide.

   If $q < p$, we get $p-q+2$ regions with corresponding vectors $(0, p+q, 0)$ and $(0, p-a, q+a)$ for $a = 0, 1, \ldots, p-q$.

   Example 3.12 illustrates this idea for $p = 3$ and $q = 1$.

<u>$i = 1$</u>: We compute the h^*-vectors corresponding to the 16 polygons having only one integer point in their relative interior. In doing so, we make repetitive use of Theorem

2.6, Proposition 4.6 and Theorem 4.7. The figure at the end of this section lists these computations.

It remains to discuss the case when the number of integer points in the relative interior of lattice polygons P is bigger than 1, i.e., $i > 1$. When $x \in \mathbb{Z}^2$, the h^*-vector has the form $(1, h_1^*, i)$ where $h_1^* =\mid P \cap \mathbb{Z}^2 \mid -3 \leq 3i + 6 - 3 = 3i + 3$ by Theorem 4.14. A trivial lower bound for h_1^* is $h_1^* \geq i$ since the number of integer points on the boundary of any lattice polygon is at least 3.

Assume $x \notin \mathbb{Z}^2$. We show that $\varphi_x(P) < \mid P \cap \mathbb{Z}^2 \mid$ for all $P \subseteq \wp(\mathbb{Z}^2)$. This implies that $h_1^* = \mid P \cap (x + \mathbb{Z}^2) \mid < \mid P \cap \mathbb{Z}^2 \mid \leq 3i + 6$ by Theorem 4.14. Likewise, since $h_2^* = \varphi_x(-\text{relint}(P)) = \varphi_{-x}(\text{relint}(P))$, then $h_2^* < \mid P \cap \mathbb{Z}^2 \mid \leq 3i + 6$. Hence, an h^*-vector $(0, h_1^*, h_2^*)$ must satisfy these two conditions.

Proposition 4.15. Let P be a lattice polygon in $\mathbb{R}^2$ and let x be a vector in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Then, $\varphi_x(P) < \mid P \cap \mathbb{Z}^2 \mid$.

Proof. Since P is a 2-dimensional lattice polygon, then it admits a unimodular triangulation Δ, that is, a triangulation using 2-simplices unimodularly equivalent to the standard 2-simplex Δ_2. The number of triangles in Δ is $b + 2i - 2$, where b and i are the number of integer points on the boundary and in the relative interior of P, respectively. Hence,

$$P = \bigcup_{\delta_i \in \Delta} \delta_i. \tag{4.17}$$

From Example 3.11, $\varphi_x(\Delta_2) = 0$ or 1 whenever $x \notin \mathbb{Z}^2$. Thus, $\varphi_x(\delta_i) \leq 1$ for all $i = 1, 2, \ldots, b + 2i - 2$. Applying the inclusion-exclusion principle to the valuation φ_x

(Theorem 2.2), we get:

$$\varphi_x(P) = \sum_{\emptyset \neq I \subseteq [b+2i-2]} (-1)^{|I|-1} \varphi_x \left(\bigcap_{i \in I} \delta_i \right). \tag{4.18}$$

Since three or more triangles can intersect at most in one point and $\varphi_x(\{\text{point}\}) = 0$ when $x \notin \mathbb{Z}^2$, Equation (4.18) becomes

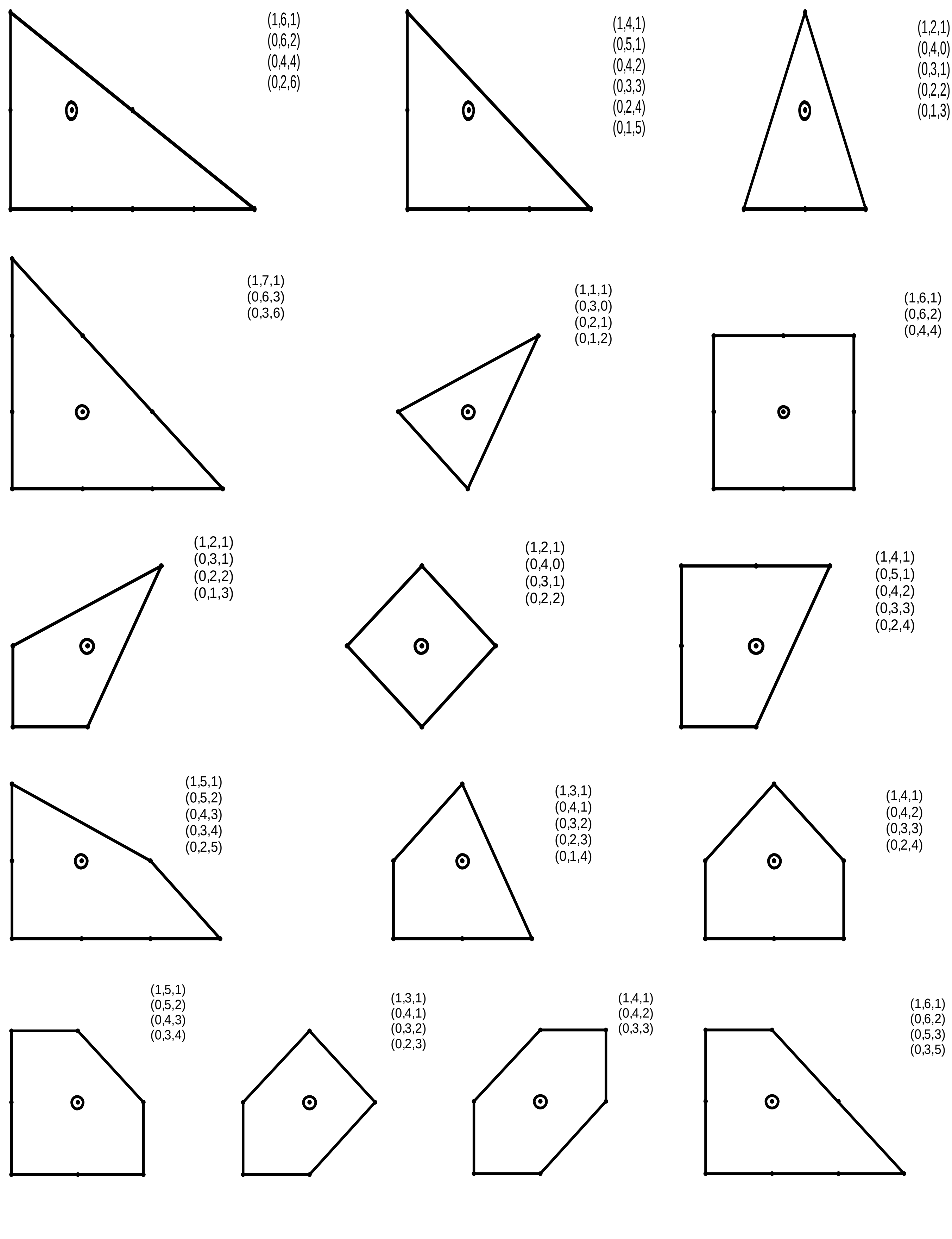
$$\begin{aligned}
\varphi_x(P) &= \sum_{i \in [b+2i-2]} \varphi_x(\delta_i) - \sum_{i \neq j} \varphi_x(\delta_i \cap \delta_j) \\
&\leq b + 2i - 2 - \sum_{i \neq j} \varphi_x(\delta_i \cap \delta_j) \qquad (4.19) \\
&\leq b + 2i - 2.
\end{aligned}$$

However, for every lattice interior point p, the triangles with common vertex p cannot all have one lattice point in their relative interior when shifted in any direction. Hence, Equation (4.19) becomes

$$\begin{aligned}
\varphi_x(P) &\leq b + 2i - 2 - i + 1 \\
&= b + i - 1 \\
&= \mid P \cap \mathbb{Z}^2 \mid -1 \\
&< \mid P \cap \mathbb{Z}^2 \mid .
\end{aligned} \tag{4.20}$$

This completes the proof. $\square$

We conclude this section with with an illustration of the 16 exceptional lattice polygons, together with their corresponding h^*-vectors (see figure on next page). It is based on the classification that appeared in [20].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4.4 The h^*-vectors as generators

Let $x \in \mathbb{R}^d$. For a polytope $P \in \wp(\mathbb{Z}^d)$ and $n \in \mathbb{Z}_{\geq 0}$, $\varphi_x(nP)$ consists of a finite set of polynomials in n, each of degree at most d. As before, we denote these polynomials by $p_1(n), p_2(n), \ldots, p_\alpha(n)$ where $\alpha := \alpha(P)$ is the Shabby number of P and $p_1(n)$ is the Ehrhart polynomial of P.

We define the following:

1. Let $S(P)$ be the semigroup generated by the p_i's

$$S(P) := \{\sum_{i=1}^{\alpha} a_i p_i \colon a_i \in \mathbb{Z}_{\geq 0}\}. \tag{4.21}$$

2. Let $L(P)$ be the lattice generated by the p_i's

$$L(P) := \{\sum_{i=1}^{\alpha} a_i p_i \colon a_i \in \mathbb{Z}\}. \tag{4.22}$$

3. Let $C(P)$ be the cone generated by the p_i's

$$C(P) := \{\sum_{i=1}^{\alpha} a_i p_i \colon a_i \in \mathbb{R}_{\geq 0}\}. \tag{4.23}$$

Observation 4.16. In general, for any polytope $P \in \wp(\mathbb{Z}^d)$, $S(P) \subseteq C(P)$ and $S(P) \subseteq L(P)$. This is clear from the definition. $\square$

Proposition 4.17. Let $P, Q \in \wp(\mathbb{Z}^d)$ such that $S(P) \subseteq S(Q)$. Then, $C(P) \subseteq C(Q)$ and $L(P) \subseteq L(Q)$.

Proof. We show the first inclusion. The second one is quite similar.

Assume $S(P) = \{\sum_{i=1}^{\alpha(P)} a_i p_i \colon a_i \in \mathbb{Z}_{\geq 0}\}$ and $S(Q) = \{\sum_{j=1}^{\alpha(Q)} b_j q_j \colon b_j \in \mathbb{Z}_{\geq 0}\}$, where the p_i's and the q_j's are the components of $\varphi_x(nP)$ and $\varphi_x(nQ)$, respectively, for $n \in \mathbb{Z}$.

Let p be an element of $C(P)$, say $p = \sum_{i=1}^{\alpha(P)} c_i p_i$ where $c_i \in \mathbb{R}_{\geq 0}$.

Since $S(P) \subseteq S(Q)$, then there exist nonnegative integers $b_{i,1}, b_{i,2}, \ldots, b_{i,\alpha(Q)}$ such that $p_i = \sum_{j=1}^{\alpha(Q)} b_{i,j} q_j$ for $i = 1, 2, \ldots, \alpha(P)$. Thus,

$$\begin{aligned} p &= \sum_{i=1}^{\alpha(P)} c_i p_i \\ &= \sum_{i=1}^{\alpha(P)} c_i \sum_{j=1}^{\alpha(Q)} b_{i,j} q_j \\ &= \sum_{j=1}^{\alpha(Q)} \sum_{i=1}^{\alpha(P)} c_i b_{i,j} q_j. \end{aligned} \tag{4.24}$$

Since $c_i \in \mathbb{R}_{\geq 0}$ and $b_{i,j} \in \mathbb{Z}_{\geq 0}$ for all i, j, then $\sum_{i=1}^{\alpha(P)} c_i b_{i,j} \in \mathbb{R}_{\geq 0}$.

This implies that $p \in C(Q)$ and hence $C(P) \subseteq C(Q)$. $\square$

Instead of dealing with polynomials in the monomial basis, we regard them as vectors in the binomial basis. In other words, S, L and C are generated by the h^*-vectors associated with the valuation φ_x.

By identifying the d-dimensional torus with $[0, 1)^d$, it becomes clear that the valuation φ_x induces $d + 1$ regions $R_1, R_2, \ldots, R_{d+1}$ when applied to the d-dimensional unit hypercube C_d. More precisely, R_i is the union of the relative interior of the

$(i-1)$-dimensional faces of the hypercube and $\varphi_x(C_d) = 2^{d-i+1}$ when $x \in R_i$, for $i = 1, 2, \ldots, d+1$. Hence, the Shabby number of C_d is $\alpha(C_d) = d+1$.

Next we show that $\alpha(\Delta_d) = d+1$ as well. This will allow us to characterize the generators of $S(\Delta_d)$, $L(\Delta_d)$ and $C(\Delta_d)$.

Lemma 4.18. Let Δ_d be the standard d-simplex in $\mathbb{R}^d$. Then, $\alpha(\Delta_d) = d+1$.

Proof. Recall that Δ_d is the convex hull of the origin and the unit vectors e_i in $\mathbb{R}^d$, for $i = 1, 2, \ldots, d$. Equivalently, Δ_d can be written as

$$\Delta_d = \{x_1 \geq 0\} \cap \{x_2 \geq 0\} \cap \ldots \{x_d \geq 0\} \cap \{x_1 + x_2 + \cdots + x_d \leq 1\}. \tag{4.25}$$

Denote by R_j the regions of $\mathcal{C}(\Delta_d)$, for $j = 1, 2, \ldots, \alpha(\Delta_d)$ and set R_1 to be the origin in $\mathbb{R}^d$. Then, $\varphi_x(\Delta_d) = d+1$ when $x \in R_1$.

Let $\mathcal{H}$ be the hyperplane in $\mathbb{R}^d$ defined by the equation $x_1 + x_2 + \cdots + x_d = 1$ and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$. The number of remaining regions in $\mathcal{C}(\Delta_d)$ is equal to the minimum integer k such that $\mathbf{1} \in k\Delta_d$. But $k\mathcal{H} = \{x_1 + x_2 + \cdots + x_d = k\}$ and $x_i \geq 0$ for $i = 1, 2, \ldots, d$. Hence, $k = d$ and therefore $\alpha(\Delta_d) = d+1$. $\square$

Theorem 4.19. $S(\Delta_d)$, $L(\Delta_d)$ and $C(\Delta_d)$ are generated by the unit vectors e_i in $\mathbb{R}^{d+1}$, for $i = 1, 2, \ldots, d+1$.

Proof. By the previous lemma, $\alpha(\Delta_d) = d+1$. This gives rise to $d+1$ polynomials and hence $d+1$ corresponding h^*-vectors. For each such vector,

$$h_0^* + h_1^* + \cdots + h_d^* = d!\ \mathrm{Vol}_d(\Delta_d) \tag{4.26}$$

by Theorem 4.7. Now, $\mathrm{Vol}_d(\Delta_d) = \dfrac{1}{d!}$. Hence, Equation (4.26) becomes

$$h_0^* + h_1^* + \cdots + h_d^* = 1 \tag{4.27}$$

Moreover, $h_i \geq 0$ for all i by Corollary 4.5.

Thus, for each vector, there exists $i \in \{0, 1, \ldots, d\}$ such that $h_i = 1$ and $h_j = 0$ for all $j \neq i$. This implies that the generators of $S(\Delta_d)$, $L(\Delta_d)$ and $C(\Delta_d)$ are the unit vectors in $\mathbb{R}^{d+1}$. □

Corollary 4.20. For all $P \in \wp(\mathbb{Z}^d)$, $S(P) \subseteq S(\Delta_d)$.
Hence, $L(P) \subseteq L(\Delta_d)$ and $C(P) \subseteq C(\Delta_d)$.

Proof. This follows immediately from the fact that every nonnegative integer vector in $\mathbb{R}^{d+1}$ can be written as a nonnegative integer combination of the unit vectors e_i for $i = 1, 2, \ldots, d+1$.

The second claim is a consequence of Proposition 4.17. □

In Proposition 4.4, we showed that if P and Q are two polytopes in $\wp(\mathbb{Z}^d)$ such that $P \subseteq Q$, then $h_i^*(\varphi_x, P) \leq h_i^*(\varphi_x, Q)$ for all i. However, inclusion of polytopes does not imply reverse inclusion of semigroups, lattices nor cones as we note in the following example.

Example 4.21. Let $P = 2\Delta_2$, $Q = \mathrm{conv}\{(0,0), (2,0), (0,3)\}$ and R be the unit square. Then, $R \subseteq P \subseteq Q$. The corresponding sets of h^*-vectors are $\{(1,3,0), (0,3,1), (0,1,3)\}$ for P, $\{(1,4,1), (0,5,1), (0,4,2), (0,3,3), (0,2,3), (0,1,5)\}$ for Q and $\{(1,1,0), (0,2,0), (0,1,1)\}$ for R.

It is easy to verify that $S(P) \not\subseteq S(R)$, $C(P) \not\subseteq C(R)$ and $L(Q) \not\subseteq L(P)$. However,

$L(P) \subseteq L(R)$. $\square$

Although the reverse inclusion property does not hold in general, we show that it holds in a special case.

Theorem 4.22. For a polytope $P \in \wp(\mathbb{Z}^d)$, $S(kP) \subseteq S(lP)$ whenever $k, l \in \mathbb{N}$ and k is a positive multiple of l.

Proof. Fix $k, l, a \in \mathbb{N}$ such that $k = al$ and $x \in \mathbb{R}^d$. Let $Q_k := kP$, $Q := lP$ and let $y := \frac{1}{a}x$. Assume $S(Q)$ is generated by $p_1(n), p_2(n), \dots, p_\alpha(n)$ where $\alpha := \alpha(P)$ and let $q(n) := \varphi_x(nQ_k) \in S(Q_k)$. We show that $q(n) \in S(Q)$.

First we have the following:

$$\begin{aligned} q(n) = \varphi_x(nQ_k) &= \mid nQ_k \cap (x + \mathbb{Z}^d) \mid \\ &= \mid nkP \cap (x + \mathbb{Z}^d) \mid \\ &= \mid nalP \cap (x + \mathbb{Z}^d) \mid \\ &= \left| nQ \cap \frac{1}{a}(x + \mathbb{Z}^d) \right| \\ &= \left| nQ \cap \left(\frac{x}{a} + \frac{1}{a}\mathbb{Z}^d \right) \right| \\ &= \left| nQ \cap \left(y + \frac{1}{a}\mathbb{Z}^d \right) \right|. \end{aligned} \tag{4.28}$$

Recall the projection map $\pi \colon \mathbb{R}^d \longrightarrow \mathbb{T}^d$ defined in Theorem 3.9.

Let $S_a := \left\{ y + \frac{1}{a}\mathbb{Z}^d \colon a \in \mathbb{N} \right\}$. Then, $\pi(S_a)$ is a finite set of cardinality a^d since $\frac{1}{a}$ is a rational number. We can write $\pi(S_a) := \{y_1, y_2, \dots, y_{a^d}\}$ where the y_i's are distinct

points in $\mathbb{T}^d$. Moreover, since $\mathcal{C}(P)$ partitions $\mathbb{T}^d$, each y_i belongs to some region R_j for $i = 1, 2, \ldots, a^d$ and $j = 1, 2, \ldots, \alpha$. Hence, there exist nonnegative integers b_j such that b_j points of $\pi(S_a)$ belong to R_j, for $j = 1, 2, \ldots, \alpha$.
Now Equation 4.29 becomes:

$$\begin{aligned} q(n) &= \left| nQ \cap (\pi(S_a) + \mathbb{Z}^d) \right| \\ &= \left| nQ \cap (\{y_1, y_2, \ldots, y_{a^d}\} + \mathbb{Z}^d) \right| \\ &= \left| \bigcup_{i=1}^{a^d} (nQ \cap (y_i + \mathbb{Z}^d)) \right| \\ &= \sum_{i=1}^{a^d} \left| nQ \cap (y_i + \mathbb{Z}^d) \right| \\ &= \sum_{i=1}^{a^d} \varphi_{y_i}(nQ) \\ &= \sum_{j=1}^{\alpha} b_j p_j(n) \end{aligned} \tag{4.29}$$

Note that in Equation (4.29) we used the inclusion-exclusion property of the valuation φ_x and the fact that the y_i's are distinct. Since $b_j \in \mathbb{Z}_{\geq 0}$ for $j = 1, 2, \ldots, \alpha$, then $q(n) \in S(Q)$ and therefore $S(kP) \subseteq S(lP)$. $\square$

We turn our attention now to $L(P)$, the lattice generated by the h^*-vectors of $\varphi_x(nP)$ for $P \in \wp(\mathbb{Z}^d)$ and as x ranges through $\mathbb{R}^d$. From the discussion in section 2.4, one can form an abelian group G as a quotient of $\mathbb{Z}^{d+1}$ by $L(P)$ and then use the Smith Normal Form of the matrix of vectors to write G as a direct product of cyclic groups.

For a polytope P, denote by M the matrix whose columns are the h^*-vectors of $\varphi_x(nP)$. Clearly, M has $d+1$ rows and $\alpha(P)$ columns. As an example, consider the unit square in dimension 2. We saw before that the corresponding h^*-vectors are $(1,1,0)$, $(0,2,0)$ and $(0,1,1)$. Then,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathrm{SNF}(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{4.30}$$

Hence, $G \simeq \mathbb{Z}_2$.

In the last part of this chapter, we attempt to partially classify the groups formed by the lattices $L(P)$ for lattice polytopes in dimensions 1 and 2.

The case when $d = 1$ is straightforward. Let $P \in \wp(\mathbb{Z})$ be a one-dimensional lattice polytope. Without loss of generality, we may assume $P = [0, l]$ for some $l \in \mathbb{N}$. As we saw in Section 4.3, there are two h^*-vectors for every P, namely $(1, l-1)$ and $(0, l)$. This implies that

$$M = \begin{bmatrix} 1 & 0 \\ l-1 & l \end{bmatrix}, \text{ and } \mathrm{SNF}(M) = \begin{bmatrix} 1 & 0 \\ 0 & l \end{bmatrix}. \tag{4.31}$$

Hence, $G \simeq \mathbb{Z}_l$.

We move now to the case $d = 2$. Let P be a two-dimensional lattice polytope and let $\alpha := \alpha(P)$. Then, $L(P)$ is generated by α vectors in $\mathbb{Z}^{d+1}$, say $h^{*,1}, h^{*,2}, \ldots, h^{*,\alpha}$.

Recall that $h^{*,1} = (1, h_1^{*,1}, h_2^{*,1})$ and $h^{*,i} = (0, h_1^{*,i}, h_2^{*,i})$ for $i = 2, 3, \ldots, \alpha$. With this set up, the matrix M becomes:

$$M = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ h_1^{*,1} & h_1^{*,2} & \ldots & h_1^{*,\alpha} \\ h_2^{*,1} & h_2^{*,2} & \ldots & h_2^{*,\alpha} \end{bmatrix}, \text{ and } \mathrm{SNF}(M) = \begin{bmatrix} s_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & s_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & s_3 & 0 & \ldots & 0 \end{bmatrix}, \tag{4.32}$$

where the s_i's are called the invariant factors of M (see Section 2.4). By Equation (2.11), $s_i = \dfrac{d_i}{d_{i-1}}$ for $i = 1, 2, 3$, where the d_i's are the determinantal divisors of M and $d_0 = 1$. Recall that the i-th determinantal divisor of M is the gcd of all $(i \times i)$-minors of M.

For $i = 1$, d_1 is simply the gcd of all entries of M. Since the first entry of M is 1, then $d_1 = 1$. This implies that $s_1 = \dfrac{d_1}{d_0} = 1$.

For $i = 2$, $s_2 = \dfrac{d_2}{d_1} = d_2$ since $d_1 = 1$. Now, d_2 is the gcd of all (2×2)-minors of M. By examining the matrix M, we can write

$$d_2 = \gcd\left(\left(h_1^{*,i}\right)_{i=2}^{\alpha}, \left(h_2^{*,i}\right)_{i=2}^{\alpha}, \left(h_1^{*,i} h_2^{*,j} - h_1^{*,j} h_2^{*,i}\right)_{\substack{i,j=1 \\ i \neq j}}^{\alpha} \right). \tag{4.33}$$

By elementary properties of the greatest common divisor and since the third term of the right hand side of Equation (4.33) is an integer combination of the first two terms, we get

$$d_2 = \gcd\left(\left(h_1^{*,i}\right)_{i=2}^{\alpha}, \left(h_2^{*,i}\right)_{i=2}^{\alpha} \right). \tag{4.34}$$

Finally, for $i = 3$, we have $s_3 = \dfrac{d_3}{d_2}$ where d_3 is the gcd of all the (3×3)-minors of M. Since the first row of M consists of zeros except the first entry, then

$$d_3 = \gcd\left(\left(h_1^{*,i}h_2^{*,j} - h_1^{*,j}h_2^{*,i}\right)_{\substack{i,j=2\\ i\neq j}}^{\alpha}\right). \tag{4.35}$$

Equation (4.35) can be simplified further. By Theorem 4.7, $h_0^* + h_1^* + h_2^* = 2A$ where A denotes the area of P. Hence, for $i \geq 2$, we have $h_1^{*,i} + h_2^{*,i} = 2A$ since $h_0^{*,i} = 0$. This allows us to write

$$\begin{aligned} h_1^{*,i}h_2^{*,j} - h_1^{*,j}h_2^{*,i} &= \left(2A - h_2^{*,i}\right)h_2^{*,j} - \left(2A - h_2^{*,j}\right)h_2^{*,i} \\ &= 2Ah_2^{*,j} - h_2^{*,i}h_2^{*,j} - 2Ah_2^{*,i} + h_2^{*,i}h_2^{*,j} \\ &= 2A\left(h_2^{*,j} - h_2^{*,i}\right). \end{aligned} \tag{4.36}$$

Combining Equations (4.35) and (4.36), we get

$$\begin{aligned} d_3 &= \gcd\left(2A\left(h_2^{*,j} - h_2^{*,i}\right)_{\substack{i,j=2\\ i\neq j}}^{\alpha}\right) \\ &= 2A\gcd\left(\left(h_2^{*,j} - h_2^{*,i}\right)_{\substack{i,j=2\\ i\neq j}}^{\alpha}\right) \end{aligned} \tag{4.37}$$

Equations (4.34) and (4.37) together give the expression

$$d_3 = 2A\frac{\gcd\left(\left(h_2^{*,j} - h_2^{*,i}\right)_{\substack{i,j=2\\ i\neq j}}^{\alpha}\right)}{\gcd\left(\left(h_1^{*,i}\right)_{i=2}^{\alpha}, \left(h_2^{*,i}\right)_{i=2}^{\alpha}\right)}. \tag{4.38}$$

Equivalently, Equation (4.38) can be written as

$$d_3 = 2A \frac{\gcd\left(\left(h_1^{*,i} - h_1^{*,j}\right)_{\substack{i,j=2 \\ i \neq j}}^{\alpha}\right)}{\gcd\left(\left(h_1^{*,i}\right)_{i=2}^{\alpha}, \left(h_2^{*,i}\right)_{i=2}^{\alpha}\right)}. \tag{4.39}$$

Remark 4.23. For $d \geq 3$, the task of classifying the group G becomes more complicated due to the increase in the size of the matrix M. However, one can guarantee that $s_1 = 1$ in any dimension. This is due to two facts: on one hand, $d_0 = 1$ by convention. On the other hand, $d_1 = 1$ since the first row of M consists of zeros except the first entry which is 1. This implies that in any dimension, the group G can be written as a direct product of at most d cyclic groups.

One simple case to note is when $P = \Delta_d$, the standard d-simplex in $\mathbb{R}^d$. By Theorem 4.19, $L(P)$ is generated by the unit vectors e_i in $\mathbb{R}^{d+1}$ for $i = 1, 2, \ldots, d+1$. This means that $M = I_{d+1}$, the $(d+1) \times (d+1)$ identity matrix. Hence, the group associated with Δ_d is simply the trivial group, for all d. $\square$

Chapter 5: Shifted Mixed Valuation

5.1 Introduction

The notion of combinatorial mixed valuations associated to translation-invariant valuations on polytopes was introduced in [15]. Let φ be a $\mathbb{Z}^d$-valuation, that is, a valuation that is invariant under integer translation: $\varphi(P+t) = \varphi(P)$ whenever $t \in \mathbb{Z}^d$ and $P \in \wp(\mathbb{Z}^d)$. For $k \geq 0$, the authors defined the k-th combinatorial mixed valuation associated to φ by

$$\mathrm{CM}_k\varphi(P_1,\ldots,P_k) := \sum_{I\subseteq[k]} (-1)^{k-|I|}\varphi(P_I), \tag{5.1}$$

for $P_1,\ldots,P_k \in \wp(\mathbb{Z}^d)$ and where $P_I := \sum\limits_{i\in I} P_i$ is the Minkowski sum of polytopes and $P_\emptyset := \{0\}$. By convention, $\mathrm{CM}_0\varphi = \varphi(\{0\})$ and for all choices of $k > d$ polytopes, $\mathrm{CM}_k\varphi(P_1,\ldots,P_k) = 0$. We drop the index k and simply write $CM\varphi$ when no confusion arises.

Clearly, $\mathrm{CM}_k\varphi$ is symmetric and a $\mathbb{Z}^d$-valuation in each of its arguments. For $\varphi = \mathrm{Vol}_d$ and $k = d$, the definition recovers the usual mixed volume discussed in Section 2.5: $\mathrm{CMVol}_d(P_1,\ldots,P_d) = d!\mathrm{MV}_d(P_1,\ldots,P_d)$. For $\varphi = \mid P\cap\mathbb{Z}^d \mid$ (the discrete volume), the definition recovers the discrete mixed volume discussed also in Section 2.5.

For $k < d$, the discrete mixed volume was investigated by Bihan [3] in the context of fewnomial bounds and tropical intersection theory. In particular, using irrational mixed decompositions, Bihan showed that the discrete mixed volume is always nonnegative. Following [15], we call a $\mathbb{Z}^d$-valuation φ combinatorially mixed monotone if

$$\mathrm{CM}\varphi(P_1, \ldots, P_k) \leq \mathrm{CM}\varphi(Q_1, \ldots, Q_k) \tag{5.2}$$

for all $k \geq 0$ and all polytopes $P_i \subseteq Q_i$ in $\wp(\mathbb{Z}^d)$ for $i = 1, 2, \ldots, k$.

Recall that a $\mathbb{Z}^d$-valuation φ is combinatorially monotone if $h_i^*(\varphi, P) \leq h_i^*(\varphi, Q)$ for all i whenever $P \subseteq Q$ (see Section 4.1). The next result gives sufficient conditions for combinatorially mixed monotonicity.

Theorem 5.1 [15]. Let $\varphi\colon \wp(\mathbb{Z}^d) \longrightarrow G$ be a $\mathbb{Z}^d$-valuation with values in a partially ordered group G. If φ is combinatorially monotone, then $\mathrm{CM}\varphi$ is combinatorially mixed monotone.

The discrete mixed volume was defined in Equation (2.16), which led to the definition of mixed Ehrhart polynomials as in Equation (2.17). In this chapter, we generalize these notions and define the shifted discrete mixed volume induced by the valuation φ_x.

In section 5.2, we provide the suitable definitions of the shifted discrete mixed volume DMV_x and the mixed valuation $M\varphi_x$. Then, we use the above theorem to prove that $M\varphi_x$ is combinatorially mixed monotone and nonnegative.

The following result is the cornerstone of this chapter. It is due to McMullen [18] and it underlies most of the theory of translation-invariant valuations.

Theorem 5.2 [18]. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice or vector space over a subfield of $\mathbb{R}$ and $\wp(\Lambda)$ the collection of polytopes with vertices in Λ. Let $\varphi\colon \wp(\Lambda) \longrightarrow G$ be a

Λ-valuation with values in an abelian group G, that is, a valuation invariant under translation with vectors in Λ. Then, for any polytopes $P_1, P_2, \ldots, P_k \in \wp(\Lambda)$, the function

$$\varphi_{P_1,P_2,\ldots,P_k}(n_1, n_2, \ldots, n_k) := \varphi(n_1P_1 + n_2P_2 + \cdots + n_kP_k) \tag{5.3}$$

is a polynomial of degree at most $\dim(P_1 + P_2 + \cdots + P_k)$. $\square$

Section 5.2 continues with the study of the coefficients of the polynomial corresponding to $M\varphi_x$. The results are similar in spirit to those in [12] and we use several arguments that appeared therein. We end this section with a characterization of the shifted discrete mixed volume via counting lattice points in $\mathbb{R}^d$.

In section 5.3, we define and investigate the h^{**}-vectors corresponding to $M\varphi_x$. As the mixed valuation $M\varphi_x$ is written in terms of the original valuation φ_x, we relate these vectors to the ones discussed in chapter 4. In doing so, we prove some useful properties they satisfy. We end this section by dealing with the special case when $P_1 = P_2 = \cdots = P_k$ and how it gives an easy way to expand $M\varphi_x$.

Finally, section 5.4 contains some examples illustrating the main concepts and results in this chapter. We build on the examples provided in the previous chapters and we make the connection between the original valuation φ_x and the mixed valuation $M\varphi_x$.

5.2 The Shifted Mixed Valuation $M\varphi_x$

Let $x \in \mathbb{R}^d$ and $P \in \wp(\mathbb{Z}^d)$. Recall the main object of study in this thesis: $\varphi_x(P) =\mid P \cap (x + \mathbb{Z}^d) \mid$. As discussed in chapter 3, φ_x is a valuation on the set of lattice polytopes and it is invariant under integer translation. Various properties for φ_x were established before, such as polynomiality, monotonicity, nonnegativity among others.

Let $P = (P_1, P_2, \ldots, P_k)$ be a collection of lattice polytopes in $\mathbb{R}^d$ such that $P_1 + P_2 + \cdots + P_k$ is of full dimension d. For $n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}$, we denote $\bar{n}P := (n_1P_1, n_2P_2, \ldots, n_kP_k)$ and we define

$$\begin{aligned}\varphi_x(\overline{n}P) &:= \varphi_x(n_1P_1 + n_2P_2 + \cdots + n_kP_k) \\ &= \Big|(n_1P_1 + n_2P_2 + \cdots + n_kP_k) \cap (x + \mathbb{Z}^d)\Big|.\end{aligned} \tag{5.4}$$

By Theorem 5.2, since φ_x is a $\mathbb{Z}^d$-valuation, $\varphi_x(\overline{n}P)$ agrees with a multivariate polynomial of total degree d, for all $n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}$. The degree of $\varphi_x(\overline{n}P)$ in n_i is $\dim(P_i)$ for $i = 1, 2, \ldots, k$. In monomial basis, we can write $\varphi_x(\overline{n}P)$ as

$$\varphi_x(\overline{n}P) = \sum_{\gamma \in \mathbb{Z}^k_{\geq 0}} c_\gamma \overline{n}^\gamma, \tag{5.5}$$

where $\overline{n}^\gamma = n_1^{\gamma_1} n_2^{\gamma_2} \ldots n_k^{\gamma_k}$ for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k_{\geq 0}$.

In analogy with the definition of the discrete mixed volume in Equation (2.16), we define the shifted discrete mixed volume as follows.

Definition 5.3. Let $P_1, P_2, \ldots, P_k \in \wp(\mathbb{Z}^d)$ and let $x \in \mathbb{R}^d$. The shifted discrete mixed volume of $P_1, P_2, \ldots, P_k$ is

$$\mathrm{DMV}_x(P_1, P_2, \ldots, P_k) := \sum_{J \subseteq [k]} (-1)^{k-|J|} \varphi_x(P_J), \tag{5.6}$$

where $P_J := \sum\limits_{j \in J} P_j$ is the Minkowski sum of polytopes indexed by J and $P_\emptyset = \{0\}$.

The discrete mixed volume DMV led to the definition of mixed Ehrhart polynomials in Equation (2.17). Similarly, the shifted discrete mixed volume DMV_x gives rise to the following definition.

Definition 5.4. Let $P_1, P_2, \ldots, P_k \in \wp(\mathbb{Z}^d)$ and let $x \in \mathbb{R}^d$. Define

$$\begin{aligned} M\varphi_x(nP_1, nP_2, \ldots, nP_k) &:= \mathrm{DMV}_x(nP_1, nP_2, \ldots, nP_k) \\ &= \sum_{J \subseteq [k]} (-1)^{k-|J|} \varphi_x(nP_J), \end{aligned} \tag{5.7}$$

where P_J is as above.

In a similar way as in the paper [11], we deal with the shifted mixed volume separately as an analogue to the continuous mixed volume and we study its behavior under simultaneous dilations of the polytopes P_i under the name $M\varphi_x$.

Using the terminology from Section 5.1, since φ_x is a $\mathbb{Z}^d$-valuation, $M\varphi_x$ is a combinatorial mixed valuation. This means it is symmetric and a $\mathbb{Z}^d$-valuation in each of its arguments $P_1, P_2, \ldots, P_k$. Being an alternating sum of polynomials and since $\dim(P_1 + P_2 + \cdots + P_k) = d$, $M\varphi_x(nP_1, nP_2, \ldots, nP_k)$ agrees with a univariate polynomial of degree at most d, say

$$M\varphi_x(nP_1, nP_2, \ldots, nP_k) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0. \tag{5.8}$$

For the sake of abbreviation in this section, we write P to denote the collection of polytopes $P_1, P_2, \ldots, P_k$ and nP to denote $nP_1, nP_2, \ldots, nP_k$. Note that for $n = 1$, $M\varphi_x(P) = \mathrm{DMV}_x(P)$, the shifted discrete mixed volume of $P_1, P_2, \ldots, P_k$.

Our first result concerns the monotonicity and nonnegativity of the mixed valuation $M\varphi_x$.

Theorem 5.5. Let $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_k$ be lattice polytopes in $\mathbb{R}^d$ such that $P_i \subseteq Q_i$ for $i = 1, 2, \ldots, k$. Then, $M\varphi_x(P) \leq M\varphi_x(Q)$. In other words, $M\varphi_x$ is combinatorially mixed monotone.

Proof. In Proposition 4.4, we proved that φ_x is combinatorially monotone, that is, for $P, Q \in \wp(\mathbb{Z}^d)$ such that $P \subseteq Q$, $h_i^*(\varphi_x, P) \leq h_i^*(\varphi_x, Q)$ for all i. Using Theorem 5.1, this implies that $M\varphi_x$ is combinatorially mixed monotone. $\square$

Corollary 5.6. $M\varphi_x(P) \geq 0$ for all collections of lattice polytopes in $\mathbb{R}^d$ and for all $x \in \mathbb{R}^d$.

Proof. Setting $Q_i = \{0\}$ for $i = 1, 2, \ldots, k$, we get:

$$\begin{aligned}
M\varphi_x(Q_1, Q_2, \ldots, Q_k) &= M\varphi_x(\{0\}, \{0\}, \ldots, \{0\}) \\
&= \sum_{J \subseteq [k]} (-1)^{k-|J|} \varphi_x(\{0\}) \\
&= \sum_{J \subseteq [k]} (-1)^{k-|J|} \Big| \{0\} \cap (x + \mathbb{Z}^d) \Big|.
\end{aligned} \tag{5.9}$$

Note that $\varphi_x(\{0\}) = 0$ whenever $x \notin \mathbb{Z}^d$.

If $x \in \mathbb{Z}^d$, then $\varphi_x(\{0\}) = 1$ and Equation (5.9) becomes

$$\begin{aligned} M\varphi_x(Q_1, Q_2, \ldots, Q_k) &= \sum_{J \subseteq [k]} (-1)^{k-|J|}\ 1 \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \\ &= 0. \end{aligned} \tag{5.10}$$

The last equality in Equation (5.10) is a well-known combinatorial identity. Hence, since $\{0\} \subseteq P_i$ for $i = 1, 2, \ldots, k$, $M\varphi_x(P) \geq M\varphi_x(Q) = 0$. $\square$

Next we compare the coefficients c_i appearing in Equation (5.8) with c_γ in Equation (5.5). For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k$, we write $|\ \gamma\ | := \gamma_1 + \gamma_2 + \cdots + \gamma_k$.

Theorem 5.7. Let $P = (P_1, P_2, \ldots, P_k)$ be a collection of lattice polytopes in $\mathbb{R}^d$. Let c_γ and c_i be given as in Equations (5.5) and (5.8). Then, for $i = 0, 1, \ldots, k$,

$$c_i = \sum_{\gamma} c_\gamma, \tag{5.11}$$

where the sum runs over all $\gamma \in \mathbb{Z}^k_{\geq 1}$ such that $|\ \gamma\ |= i$.

Proof. Let $x \in \mathbb{R}^d$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{Z}^k_{\geq 0}$. When $n_1 = \cdots = n_k = n$, Equation (5.5) becomes

$$\begin{aligned}\varphi_x(nP) &= \sum_{\gamma \in \mathbb{Z}^k_{\geq 0}} c_\gamma n^{\gamma_1} n^{\gamma_2} \dots n^{\gamma_k} \\ &= \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|}.\end{aligned} \tag{5.12}$$

Using Definition 5.4 and Equation (5.12), we can write

$$\begin{aligned}M\varphi_x(nP) &= \sum_{J \subseteq [k]} (-1)^{k-|J|} \varphi_x(nP_J) \\ &= \sum_{J \subseteq [k]} (-1)^{k-|J|} \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|},\end{aligned} \tag{5.13}$$

where P_J is the Minkowski sum of polytopes indexed by J.

We claim that the expression in (5.13) vanishes unless $J = [k]$ and $\gamma_i > 0$ for all $i \in [k]$.

This implies that

$$M\varphi_x(nP) = \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 1} \\ |\gamma| = i}} c_\gamma n^{|\gamma|}. \tag{5.14}$$

Comparing the coefficients in Equations (5.8) and (5.14), we get $c_i = \sum\limits_{\substack{\gamma \in \mathbb{Z}^k_{\geq 1} \\ |\gamma| = i}} c_\gamma$, for $i = 0, 1, \dots, k$.

Proof of the claim. We examine the monomial $n^{|\gamma|} := n_1^{\gamma_1} n_2^{\gamma_2} \dots n_k^{\gamma_k}$ and we use the fact that $0^0 = 1$. Without loss of generality, we may assume that $\gamma_k = 0$. Then,

$n^{|\gamma|} = n_1^{\gamma_1} n_2^{\gamma_2} \dots n_{k-1}^{\gamma_{k-1}}$. Note that for any subset $J \subsetneq [k]$, except $J = [k-1]$, the monomial $n^{|\gamma|}$ vanishes since at least one of the polytopes $P_1, P_2, \dots, P_k$ does not appear in the expression of $\varphi_x(nP_J)$, i.e., some coefficient n_i is zero. Hence, Equation (5.13) simplifies to

$$\begin{aligned} M\varphi_x(nP) &= (-1)^{k-(k-1)} \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|} + (-1)^{k-k} \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|} \\ &= (-1) \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|} + \sum_{i=0}^{d} \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = i}} c_\gamma n^{|\gamma|} \qquad (5.15) \\ &= 0. \end{aligned}$$

Thus, $\gamma_i > 0$ for all $i \in [k]$ and $n^{|\gamma|} = n^{\gamma_1 + \gamma_2 + \dots + \gamma_k}$. As mentioned before, for any subset $J \subsetneq [k]$, $n^{|\gamma|} = 0$ whenever $\gamma_i > 0$ for all i. This completes the proof of the claim and therefore the theorem. $\square$

Now that we have a formula that relates the coefficients of $M\varphi_x$ to those of φ_x, we use it to give an interpretation to some of them.

Corollary 5.8. In the expression (5.8), we have $c_i = 0$ for all $0 \leq i < k$.

Proof. By theorem (5.7), $c_i = \sum_\gamma c_\gamma$, where the sum runs over all $\gamma \in \mathbb{Z}^k_{\geq 1}$ such that $\mid \gamma \mid = i$. This implies that $\gamma_1 + \gamma_2 + \dots + \gamma_k \geq k$ since $\gamma_i \geq 1$ for all $i = 1, 2, \dots, k$. Thus, $c_i = 0$ for $i < k$. $\square$

Equation (5.12) gave a polynomial expression of $\varphi_x(nP)$ when P consists of a collection of lattice polytopes $P_1, P_2, \ldots, P_k$ in $\mathbb{R}^d$. By inspection, the leading coefficient of this polynomial is $\sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = d}} c_\gamma$. Using the same reasoning as in Theorem 4.7 and Corollary 4.8, we note that this coefficient is simply $\mathrm{Vol}_d(nP_1 + nP_2 + \cdots + nP_k)$.

A well-known property of mixed volumes (see [21]) is the following

$$\mathrm{Vol}_d(nP_1 + nP_2 + \cdots + nP_k) = \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = d}} \binom{d}{\gamma_1, \gamma_2, \ldots, \gamma_k} \mathrm{MV}_d(P_1^{\gamma_1}, P_2^{\gamma_2}, \ldots, P_k^{\gamma_k}) n^{|\gamma|}, \tag{5.16}$$

where $\binom{d}{\gamma_1, \gamma_2, \ldots, \gamma_k} := \dfrac{d!}{\gamma_1! \gamma_2! \ldots, \gamma_k!}$ is the multinomial coefficient and $P_i^{\gamma_i}$ means that P_i appears γ_i times for $i = 1, 2, \ldots, k$. This implies that

$$\sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = d}} c_\gamma = \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 0} \\ |\gamma| = d}} \binom{d}{\gamma_1, \gamma_2, \ldots, \gamma_k} \mathrm{MV}_d(P_1^{\gamma_1}, P_2^{\gamma_2}, \ldots, P_k^{\gamma_k}). \tag{5.17}$$

By combining Equation (5.17) and Theorem 5.7 for $i = d$, we have thus proved

Corollary 5.9.

$$c_d = \sum_{\substack{\gamma \in \mathbb{Z}^k_{\geq 1} \\ |\gamma| = d}} \binom{d}{\gamma_1, \gamma_2, \ldots, \gamma_k} \mathrm{MV}_d(P_1^{\gamma_1}, P_2^{\gamma_2}, \ldots, P_k^{\gamma_k}). \tag{5.18}$$

$\square$

A special case of the previous corollary occurs when $k = d$. We record it separately here since it gives an expression of $M\varphi_x$ that is independent of x.

Corollary 5.10. For $k = d$ and $x \in \mathbb{R}^d$, $M\varphi_x(nP) = d!MV_d(P_1, P_2, \ldots, P_d)n^d$ for any collection P of d lattice polytopes in $\mathbb{R}^d$.

Proof. The conditions $\gamma_i \geq 1$ and $\mid \gamma \mid = \gamma_1 + \gamma_2 + \cdots + \gamma_k = d$ together with $k = d$ imply that $\gamma_i = 1$ for all $i = 1, 2, \ldots, d$. The multinomial coefficient $\binom{d}{\gamma_1, \gamma_2, \ldots, \gamma_k}$ then simply becomes $d!$. Plugging these identities in the right hand side of Equation (5.18), we get $c_d = d!MV_d(P_1, P_2, \ldots, P_d)$.

Moreover, by Corollary 5.8, $c_i = 0$ for all $i < k = d$.

Replacing all the coefficients in Equation (5.8), we end up with the desired result. □

The last result of this section is a combinatorial interpretation of the shifted discrete mixed volume introduced in Definition 5.3. It identifies DMV_x as a function that counts the number of certain lattice points in the space. The only requirement is that the dimension of the Minkowski sum of the polytopes is equal to the sum of the dimensions of the polytopes.

Theorem 5.11. Let $P_1, P_2, \ldots, P_k$ be a collection of lattice polytopes in $\mathbb{R}^d$ such that $\dim(P_1 + P_2 + \cdots + P_k) = \dim(P_1) + \dim(P_2) + \cdots + \dim(P_k)$ and let $x \in \mathbb{R}^d$. For $J \subseteq [k]$, denote by P_J the Minkowski sum of polytopes indexed by J. Then, the shifted discrete mixed volume $\mathrm{DMV}_x(P_1, P_2, \ldots, P_k)$ counts the number of lattice points in $(P_1 + P_2 + \cdots + P_k) \cap (x + \mathbb{Z}^d)$ that are not contained in any subsum P_J for $J \subsetneq [k]$.

Proof. The hypothesis on the dimension implies that $P_1 + P_2 + \cdots + P_k$ is affinely isomorphic to the cartesian product $P_1 \times P_2 \times \cdots \times P_k$. This implies that for any subset $J \subseteq [k]$, P_J is the intersection of $P_1 + P_2 + \cdots + P_k$ with the linear span of P_J. Moreover, for any subsets $I, J \subseteq [k]$, the intersection $P_I \cap P_J$ is exactly $P_{I \cap J}$. Using Equation (5.6) and the above implication, we get

$$
\begin{aligned}
\mathrm{DMV}_x(P_1, P_2, \ldots, P_k) &= \sum_{J \subseteq [k]} (-1)^{k-|J|} \varphi_x(P_J) \\
&= \varphi_x(P_1 + P_2 + \cdots + P_k) + \sum_{J \subsetneq [k]} (-1)^{k-|J|} \varphi_x(P_J) \\
&= \varphi_x\Big((P_1 + P_2 + \cdots + P_k) \setminus \bigcup_{J \subsetneq [k]} P_J\Big) \\
&= \Big|\Big((P_1 + P_2 + \cdots + P_k) \setminus \bigcup_{J \subsetneq [k]} P_J\Big) \cap (x + \mathbb{Z}^d)\Big|.
\end{aligned}
\tag{5.19}
$$

This completes the proof. $\square$

5.3 The h^{**}-vectors of $M\varphi_x$

For a collection of lattice polytopes $P_1, P_2, \ldots, P_k$ in $\mathbb{R}^d$ and a real vector x, we defined the mixed valuation $M\varphi_x$ to be the shifted discrete mixed volume evaluated at integer dilates of the polytopes P_i. It turned out that it is indeed a polynomial of degree d whenever $\dim(P_1 + P_2 + \cdots + P_k) = d$. In monomial basis, $M\varphi_x(nP_1, nP_2, \ldots, nP_k)$ can be written as $c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0$, as in Equation (5.8).

In this section as well, we use the abbreviation $nP := nP_1, nP_2, \ldots, nP_k$ and we consider every polytope to be full-dimensional, that is, $\dim(P_i) = d$ for all i. When we switch to the binomial basis, the above polynomial becomes

$$M\varphi_x(nP) = h_0^{**}\binom{n+d}{d} + h_1^{**}\binom{n+d-1}{d} + \cdots + h_d^{**}\binom{n}{d}. \tag{5.20}$$

We call $h^{**}(M\varphi_x, P) := (h_0^{**}, h_1^{**}, \ldots, h_d^{**})$ the h^{**}-vector of $M\varphi_x$ and P. We use the notation h^{**} to make a distinction with the h^*-vectors discussed in Chapter 4.

We recall some combinatorial identities that will be used in this section. We omit the proofs and refer to [10] for a reference.

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i}\binom{n+d-i}{d} = 1 \tag{5.21}$$

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i} = 0 \tag{5.22}$$

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\binom{d+i-j}{d} = \binom{d-j}{d-k}. \tag{5.23}$$

By definition 5.4, for $x \in \mathbb{R}^d$, $M\varphi_x(nP) = \sum_{J\subseteq[k]}(-1)^{k-|J|}\varphi_x(nP_J)$, where P_J is the Minkowski sum of polytopes indexed by J. Note that when J is the empty set $\emptyset$, $\varphi_x(nP_J) = \varphi_x(nP_\emptyset) = \varphi_x(n\{0\}) = \varphi_x(\{0\})$. This allows us to write

$$M\varphi_x(nP) = (-1)^k\varphi_x(\{0\}) + \sum_{\emptyset\neq J\subseteq[k]}(-1)^{k-|J|}\varphi_x(nP_J). \tag{5.24}$$

Now we find a relation between the h^{**}-vector of $M\varphi_x$ and the h^*-vectors of φ_x and P_J for all subsets J of $[k]$. We distinguish two cases: when $x \in \mathbb{Z}^d$ and when

$x \notin \mathbb{Z}^d$.

If $x \in \mathbb{Z}^d$, then $\varphi_x(\{0\}) = 1$. Multiplying the identity (5.21) by $(-1)^k$, we get

$$\sum_{i=0}^{d}(-1)^{k+i}\binom{d}{i}\binom{n+d-i}{d} = (-1)^k. \tag{5.25}$$

By combining Equations (5.24) and (5.25) and writing them in terms of the corresponding h^{**}- and h^*-vectors, we get

$$h_i^{**} = (-1)^{k+i}\binom{d}{i} + \sum_{\emptyset \neq J \subseteq [k]} (-1)^{k-|J|} h_i^*(\varphi_x, P_J), \tag{5.26}$$

for $i = 0, 1, \ldots, d$.

If $x \notin \mathbb{Z}^d$, then $\varphi_x(\{0\}) = 0$. Using the same equations as above, we get

$$h_i^{**} = \sum_{\emptyset \neq J \subseteq [k]} (-1)^{k-|J|} h_i^*(\varphi_x, P_J), \tag{5.27}$$

for $i = 0, 1, \ldots, d$.

Next we use some of the properties of the h^*-vectors studied in Chapter 4 to find their counterparts for $M\varphi_x$. Recall that $h_0^* = 1$ if $x \in \mathbb{Z}^d$ and $h_0^* = 0$ if $x \notin \mathbb{Z}^d$, for all lattice polytopes in $\mathbb{R}^d$.

Proposition 5.12. Let $P_1, P_2, \ldots, P_k$ be full-dimensional lattice polytopes in $\mathbb{R}^d$. Let $M\varphi_x$ be as defined in Section 5.2. Then, $h_0^{**} = 0$.

Proof. Due to the distinction made in the previous paragraph, we divide the proof into two cases.

Case 1: $x \in \mathbb{Z}^d$.

When $i = 0$, Equation (5.26) gives

$$
\begin{aligned}
h_0^{**} &= (-1)^k \binom{d}{0} + \sum_{\emptyset \neq J \subseteq [k]} (-1)^{k-|J|}\, 1 \\
&= (-1)^k + \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \\
&= 0,
\end{aligned} \tag{5.28}
$$

where the last equality follows from identity (5.22).

Case 2: $x \notin \mathbb{Z}^d$.

When $i = 0$, Equation (5.27) implies $h_0^{**} = \sum_{\emptyset \neq J \subseteq [k]} (-1)^{k-|J|}\, 0 = 0$. □

When $i = 1$, recall that $h_1^* = \varphi_x(P) - d - 1$ if $x \in \mathbb{Z}^d$ and $h_1^* = \varphi_x(P)$ if $x \notin \mathbb{Z}^d$, for all lattice polytopes in $\mathbb{R}^d$. The first result is a known fact from Ehrhart theory and the second one was proved in Proposition 4.6.

Proposition 5.13. Let $P_1, P_2, \ldots, P_k$ be full-dimensional lattice polytopes in $\mathbb{R}^d$. Let $M\varphi_x$ be as defined in Section 5.2. Then, $h_1^{**} = \mathrm{DMV}_x(P)$.

Proof. Again we divide the proof into two cases.

Case 1: $x \in \mathbb{Z}^d$.

When $i = 1$, Equation (5.26) gives

$$
\begin{aligned}
h_1^{**} &= (-1)^{k+1}\binom{d}{1} + \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\Big(\varphi_x(P_J) - d - 1\Big) \\
&= (-1)^{k+1}d + \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) - \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}(d+1) \\
&= (-1)^{k+1}d + \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) - (d+1) \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\ 1 \\
&= (-1)^{k+1}d + \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) - (d+1) \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} \\
&= (-1)^{k+1}d + \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) + (d+1)(-1)^{k} \\
&= \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) + (-1)^{k} \\
&= \sum_{J\subseteq[k]} (-1)^{k-|J|}\varphi_x(P_J) \\
&= \mathrm{DMV}_x(P).
\end{aligned} \tag{5.29}
$$

Note that we used the identity (5.22) and the definition of the shifted discrete mixed volume as given in Equation (5.6).

Case 2: $x \notin \mathbb{Z}^d$.

When $i = 1$, Equation (5.27) implies $h_1^{**} = \sum_{\emptyset\neq J\subseteq[k]} (-1)^{k-|J|}\ \varphi_x(P_J) = \mathrm{DMV}_x(P)$ since $\varphi_x(\{0\}) = 0$ in this case. $\square$

One crucial property in Chapter 4 was that the sum of the coordinates of the h^*-vector of a full-dimensional polytope P is equal to the Euclidean volume of P

multiplied by a factor of $d!$ (see Theorem 4.7). We will make use of this property to prove a similar result for the h^{**}-vector of $M\varphi_x$, when $k = d$.

Theorem 5.14. Let $P_1, P_2, \ldots, P_d$ be full-dimensional lattice polytopes in $\mathbb{R}^d$. Let $M\varphi_x$ be as defined in Section 5.2. Then, $\sum\limits_{i=0}^{d} h_i^{**} = (d!)^2 MV_d(P)$.

Proof. For the same reasons as in the previous two proofs, we divide our work into two cases and show that both of them lead to the desired result.

Case 1 : $x \in \mathbb{Z}^d$.

By taking the sum over Equation (5.26), we obtain

$$\begin{aligned}
\sum_{i=0}^{d} h_i^{**} &= \sum_{i=0}^{d} \left[(-1)^{d+i} \binom{d}{i} + \sum_{\emptyset \neq J \subseteq [d]} (-1)^{d-|J|} h_i^*(\varphi_x, P_J) \right] \\
&= \sum_{i=0}^{d} (-1)^{d+i} \binom{d}{i} + \sum_{i=0}^{d} \sum_{\emptyset \neq J \subseteq [d]} (-1)^{d-|J|} h_i^*(\varphi_x, P_J) \\
&= \sum_{\emptyset \neq J \subseteq [d]} (-1)^{d-|J|} \sum_{i=0}^{d} h_i^*(\varphi_x, P_J) \\
&= \sum_{\emptyset \neq J \subseteq [d]} (-1)^{d-|J|} d! \operatorname{Vol}_d(P_J) \\
&= d! \sum_{\emptyset \neq J \subseteq [d]} (-1)^{d-|J|} \operatorname{Vol}_d(P_J) \\
&= d!(d! MV_d(P)) \\
&= (d!)^2 MV_d(P).
\end{aligned} \tag{5.30}$$

Here we used the identity (5.22) and the property (2.15) of the mixed volume.

Case 2: $x \notin \mathbb{Z}^d$.

By taking the sum over Equation (5.27) and doing the same algebra work as in Equation (5.30), the same result occurs. $\square$

The last result in this section is a nice expression of $M\varphi_x$ in terms of the h^*-vector of φ_x and P_i when $P_1 = P_2 = \cdots = P_k$. This result is then used to find a simple expression of the shifted discrete mixed volume of P.

Theorem 5.15. Let P be a full-dimensional lattice polytope in $\mathbb{R}^d$ and suppose $\mathbf{P} = (P, \ldots, P)$ is a collection of k copies of P. Then, for $x \in \mathbb{R}^d$,

$$M\varphi_x(n\mathbf{P}) = \sum_{j=0}^{d} \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{in+d-j}{d} \right) h_j^*, \tag{5.31}$$

where $h^*(\varphi_x, P) = (h_0^*, h_1^*, \ldots, h_d^*)$ is the h^*-vector of φ_x and P.

Proof. Recall that in binomial basis, $\varphi_x(nP) = \sum_{j=0}^{d} h_j^* \binom{n+d-j}{d}$ by Equation (4.5).

Using Definition (5.4) and the hypothesis $P_1 = P_2 = \cdots = P_k = P$, we can write

$$\begin{aligned} M\varphi_x(n\mathbf{P}) &= \sum_{J\subseteq[k]} (-1)^{k-|J|}\varphi_x(nP_J) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\varphi_x(niP) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=0}^{d} h_j^*\binom{in+d-j}{d} \\ &= \sum_{j=0}^{d}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\binom{in+d-j}{d}\right)h_j^*. \end{aligned} \tag{5.32}$$

□

Corollary 5.16. $\mathrm{DMV}_x(\mathbf{P}) = \sum\limits_{j=0}^{d}\binom{d-j}{d-k}h_j^*$.

Proof. Note that $\mathrm{DMV}_x = M\varphi_x(n\mathbf{P})$ evaluated at $n=1$. Substituting $n=1$ in Equation (5.32), we get: $\mathrm{DMV}_x(\mathbf{P}) = \sum\limits_{j=0}^{d}\left(\sum\limits_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\binom{i+d-j}{d}\right)h_j^*$.

The inner sum can be replaced by $\binom{d-j}{d-k}$ using identity (5.23) and the result follows immediately. □

5.4 Examples

In this short section, we build on previous examples mentioned in this thesis and we provide new ones to illustrate the main results appearing in Chapter 5. The main theme is the interplay between the h^*- and the h^{**}-vectors via the shifted mixed valuation and the shifted discrete mixed volume.

Example 5.17. In dimension 2, let $P_1 = \Delta_2$ and P_2 be the unit square. Let $P = P_1 + P_2$ and let $x = (\frac{1}{2}, 0)$. By Chapter 4, we know that $h^*(\varphi_x, P_1) = (0,1,0)$ and $h^*(\varphi_x, P_2) = (0,2,0)$. Also, it is easy to verify that $h^*(\varphi_x, P) = (0,5,2)$. This implies that $\varphi_x(nP_1) = \frac{1}{2}n^2 + \frac{1}{2}n$, $\varphi_x(nP_2) = n^2 + n$ and $\varphi_x(nP) = \frac{7}{2}n^2 + \frac{3}{2}n$.
By Definition 5.4, $M\varphi_x(nP_1, nP_2) = -\varphi_x(nP_1) - \varphi_x(nP_2) + \varphi_x(nP_1 + nP_2) = 2n^2$.
This coincides with Corollary 5.10 as $2!\ \mathrm{MV}_d(P_1, P_2) = \frac{7}{2} - \frac{1}{2} - 1 = 2$ and
$M\varphi_x(nP_1, nP_2) = 2!\ \mathrm{MV}_d(P_1, P_2)n^2$. Corollary 5.8 is also satisfied since all coefficients except the leading one are zero.
Moreover, $2n^2$ can be written in binomial basis as $2\binom{n+1}{2} + 2\binom{n}{2}$. This is equivalent to saying that the h^{**}-vector is $(0,2,2)$. Hence, we recover:
$h_0^{**} = 0$,
$h_1^{**} = \mathrm{DMV}_x(P_1, P_2) = -1 - 2 + 5 = 2$ and
$h_2^{**} = (2!)(2!)\ \mathrm{MV}_d(P_1, P_2) - h_0^{**} - h_1^{**} = 4 - 0 - 2 = 2.$ $\square$

Example 5.18. In dimension 3, let P be the unit cube and let $x = (\frac{1}{2}, 0, 0)$. Then, $\varphi_x(nP) = n^3 + 2n^2 + n$ and $\varphi_x(2nP) = 8n^3 + 8n^2 + 2n$. Using the formulas developed in this chapter, we find $\mathrm{DMV}_x(P, P) = -4 - 4 + 18 = 10$ and
$M\varphi_x(nP, nP) = -\varphi_x(nP) - \varphi_x(nP) + \varphi_x(2nP) = -(n^3 + 2n^2 + n) - (n^3 + 2n^2 + n) + 8n^3 + 8n^2 + 2n = 6n^3 + 4n^2$. In binomial basis, this is equivalent to the expression $10\binom{n+2}{3} + 24\binom{n+1}{3} + 2\binom{n}{3}$, which gives rise to $h^{**} = (0, 10, 24, 2)$.
On the other hand, we know that:

$h_0^{**} = 0,$

$h_1^{**} = \mathrm{DMV}_x(P,P) = 10,$

$h_2^{**} = -h_2^*(P) - h_2^*(P) + h_2^*(2P) = -2 - 2 + 28 = 24$ and

$h_3^{**} = -h_3^*(P) - h_3^*(P) + h_3^*(2P) = -0 - 0 + 2 = 2.$

The two results coincide! $\square$

Example 5.19. In dimension 3, Let $P = \Delta_3$ and let $x = (1,0,0)$. Then,

$\varphi_x(nP) = \frac{1}{6}n^3 + n^2 + \frac{11}{6}n + 1$ and $\varphi_x(2nP) = \frac{4}{3}n^3 + 4n^2 + \frac{11}{3}n + 1$. This is equivalent to $h^*(\varphi_x, P) = (1,0,0,0)$ and $h^*(\varphi_x, 2P) = (1,6,1,0)$.

A simple calculation using Equation (5.26) implies

$h_0^{**} = 0,$

$h_1^{**} = \mathrm{DMV}_x(P,P) = -3 - 0 - 0 + 6 = 3,$

$h_2^{**} = 3 - h_2^*(P) - h_2^*(P) + h_2^*(2P) = 3 - 0 - 0 + 1 = 4$ and

$h_3^{**} = -1 - h_3^*(P) - h_3^*(P) + h_3^*(2P) = -1 - 0 - 0 + 0 = -1.$

Note that

$$\begin{aligned} M\varphi_x(nP,nP) &= 1 - \varphi_x(nP) - \varphi_x(nP) + \varphi_x(2nP) \\ &= 1 - (\frac{1}{6}n^3 + n^2 + \frac{11}{6}n + 1) - (\frac{1}{6}n^3 + n^2 + \frac{11}{6}n + 1) + (\frac{4}{3}n^3 + 4n^2 + \frac{11}{3}n + 1) \\ &= n^3 + 2n^2, \end{aligned} \tag{5.33}$$

which is consistent with the vector computed above. $\square$

Remark 5.20. The last example reveals a remarkable observation. Unlike the non-negativity of the h^*-vector discussed in Chapter 4, the h^{**}-vector can have negative components! In particular, we got $h_3^{**} = -1$. $\square$

Chapter 6: Application to Sparse Elimination Theory

6.1 Introduction

This last chapter aims at discussing an application of the valuation φ_x in the field of sparse elimination theory. We provide the necessary background in Section 6.1, where the material is taken mainly from chapter 7 of the book *Using Algebraic Geometry* by Cox, Little and O'Shea [6]. Section 6.2 contains the contribution of our work into this context.

Sparse elimination exploits the structure of a multivariate polynomial by considering its Newton polytope instead of its total degree. The central object in elimination theory is the resultant, which characterizes the solvability of an overconstrained system in a certain field by providing a condition independent of the variables. The sparse resultant considers only affine roots and generalizes the classical resultant of n homogeneous polynomials in n variables in the sense that they coincide when all polynomial coefficients are nonzero. The sparse resultant coincides with the Sylvester resultant if the system is comprised of two univariate polynomials. We refer to [29] for the classical theory.

Unlike its classical counterpart, however, the sparse resultant depends on the nonzero monomials only and therefore it has lower degree for sparse inputs. More precisely, the sparse resultant degree is a function of Bernstein's bound on the number of affine roots, which is at most equal to the classical Bézout bound on the number of projective roots of an $n \times n$ polynomial system.

Bernstein's Theorem (see Theorem 6.3 below) bounds the number of common roots by the mixed volume of the respective Newton polytopes. Thus, mixed volumes determine the degree of the sparse resultant, express the effective degree of the system, and, in short, give a measure of the intrinsic complexity of the problem in the context of sparse elimination.

Sparse elimination theory considers Laurent polynomials in n variables, where the exponents are allowed to be arbitrary integers. The polynomial ring is defined as $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] := K[x, x^{-1}]$, for some base field K. We shall be interested in polynomial roots in $(\bar{K}^*)^n$, where $\bar{K}$ is the algebraic closure of K and $K^* := K \backslash \{0\}$.

Definition 6.1. Let f be an element in $K[x, x^{-1}]$. The finite set $\mathcal{A} \subset \mathbb{Z}^n$ of all monomial exponents corresponding to nonzero coefficients is called the support of f. The Newton polytope of f is the convex hull of the set $\mathcal{A}$ and is denoted by $\mathrm{NP}(f) = \mathrm{conv}(\mathcal{A}) \subset \mathbb{R}^n$.

Newton polytopes model the sparse structure in polynomials. For example, the polynomial $x^2y^2 + x^2 + 2x + y + 5xy$ has $\mathrm{conv}\{(1,0), (0,1), (2,2), (2,0), (1,1)\}$ as its Newton polytope, whereas the Newton polytope of the dense polynomial of the same total degree is $4\Delta_2$. By dense polynomial we mean a polynomial in which every coefficient is nonzero.

Newton polytopes provide a bridge from algebra to geometry since they permit certain algebraic problems to be cast in geometric terms. We refer to Chapter 2 for the definition and properties of polytopes, Minkowski sums and mixed volumes as they play an important role in this chapter.

We mention now, without proofs, two major results on the number of solutions of systems of polynomials. The first one is called Bézout's theorem and the second one is called Bernstein's theorem, sometimes referred to as the BKK bound (Bernstein, Kushnirenko, Khovanskii). For simplicity, we assume $K = \mathbb{C}$.

Theorem 6.2. Given homogeneous polynomials $f_1, f_2, \ldots, f_n \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ of degree $d_1, d_2, \ldots, d_n$, respectively, such that the system $f_1 = f_2 = \cdots = f_n = 0$ has finitely many solutions, the number of such solutions is bounded above by the product of the degrees $d_1 d_2 \ldots d_n$. $\square$

Theorem 6.3. Given Laurent polynomials $f_1, f_2, \ldots, f_n$ over $\mathbb{C}$ with finitely many common zeroes in $(\mathbb{C}^*)^n$, let $Q_i := \mathrm{NP}(f_i)$ be the Newton polytope of f_i in $\mathbb{R}^n$, for $i = 1, 2, \ldots, n$. Then the number of common zeroes in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $\mathrm{MV}_n(Q_1, Q_2, \ldots, Q_n)$.
Moreover, for generic choices of the coefficients of the polynomials, the number of common zeroes is exactly $\mathrm{MV}_n(Q_1, Q_2, \ldots, Q_n)$. $\square$

Note that the two bounds coincide for dense polynomials: the Newton polytope of such a polynomial is the standard n-simplex dilated by the degree of the polynomial. By the properties of the mixed volume from Chapter 2, this implies that the mixed volume of the Newton polytopes is simply the product of the degrees of the corresponding polynomials.

We discuss now the mixed sparse resultant. Fix $n+1$ finite sets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ in $\mathbb{Z}^n$ and consider $n+1$ Laurent polynomials f_i with support $\mathcal{A}_i$ for $i = 0, 1, \ldots, n$. The resultant measures whether or not the $n+1$ equations in n variables

$$f_0(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0 \tag{6.1}$$

have a solution as follows.

Theorem 6.4 [9]. Assume $Q_i := \mathrm{conv}(\mathcal{A}_i)$ is an n-dimensional polytope in $\mathbb{R}^n$ for $i = 0, 1, \ldots, n$. Then there is an irreducible polynomial $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_n}$ in the coefficients of the f_i's such that the system (6.1) has a solution if and only if $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_0, f_1, \ldots, f_n) = 0$.

Moreover, if $\mathbb{Z}^n$ is generated by the differences of elements in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$, then $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_n}$ is homogeneous in the coefficients of the f_i's of degree $\mathrm{MV}_n(Q_0, Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n)$ for $i = 0, 1, \ldots, n$. $\square$

Before computing the sparse resultants, we introduce several notions of subdivisions of polytopes. Let $Q = Q_0 + Q_1 + \cdots + Q_n$ be the Minkowski sum of the polytopes described above and assume $\dim(Q) = n$. A polyhedral subdivision of Q consists of finitely many n-dimensional polytopes $R_1, R_2, \ldots, R_s$ such that $Q = R_1 \cup R_2 \cup \cdots \cup R_s$ and for $i \neq j$, the intersection $R_i \cap R_j$ is a face of both R_i and R_j. We call the R_i's the cells of the subdivision.

A polyhedral subdivision is called a mixed subdivision if each cell R_i can be written as a Minkowski sum

$$R_i = F_0 + F_1 + \cdots + F_n, \tag{6.2}$$

where each F_i is a face of Q_i and $n = \dim(F_0) + \dim(F_1) + \cdots + \dim(F_n)$.
A cell R_i is called a mixed cell if $\dim(F_i) \leq 1$ for all i. This is equivalent to one face being a vertex and all the others being edges.

Obtaining the sparse resultant boils down to the construction of a matrix M in the polynomial coefficients, whose determinant is a nontrivial multiple of the sparse resultant. We sketch the main ideas below. The algorithmic details are adopted from

[8] and the geometric ones are taken from [4].

Pick $n+1$ linear lifting forms $l_i \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ for $i = 0, 1, \ldots, n$. Then define the lifted Newton polytopes

$$\bar{Q}_i := \{(p_i, l_i(p_i)) \colon p_i \in Q_i\} \tag{6.3}$$

and take their Minkowski sum $\bar{Q} := \bar{Q}_0 + \bar{Q}_1 + \cdots + \bar{Q}_n \subset \mathbb{R}^{n+1}$.
The lower envelope of $\bar{Q}$ with respect to the vector $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is defined as the union of all facets whose inner vector has positive last component. The projection of all facets of the lower envelope of $\bar{Q}$ onto Q induces a mixed subdivision of Q.

The rows and columns of M are indexed by the integer points $E := (Q + \delta) \cap \mathbb{Z}^n$, for some small $\delta \in \mathbb{R}^n$ chosen to be sufficiently generic so that every perturbed lattice point lies strictly inside a mixed cell. We partition E into $n+1$ disjoint subsets $E = S_0 \cup S_1 \cup \cdots \cup S_n$, where S_i consists of integer points in E that belong to some shifted mixed cell $R + \delta$ with a decomposition as in Equation (6.2), such that F_i is a vertex. We denote $v(e) := F_i$, for each $e \in E$. Since $Q_i = \mathrm{conv}(\mathcal{A}_i)$, it follows that $v(e) \in \mathcal{A}_i$.

Finally, for each $e \in S_i$, we consider the equations

$$(x^{e - v(e)}) f_i = 0. \tag{6.4}$$

This provides one equation for each e, which means that the total number of equations is $\mid E \mid$. On the other hand, it is not hard to show that the right-hand side of Equation (6.4) can be written as a linear combination of the monomials x^β for $\beta \in E$. Hence, the last expression becomes a system of $\mid E \mid$ equations in $\mid E \mid$ variables.

We wrap this construction with the main result that summarizes the properties of the matrix M.

Theorem 6.5 [5]. The matrix M constructed above is well-defined, square, generically nonsingular and its determinant is a nonzero multiple of the sparse resultant $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_0, f_1, \ldots, f_n)$. $\square$

6.2 Numerical Application

The cardinality of the set E defined in the previous section is reminiscent of the valuation $\varphi_{-\delta}$, the main object of study in this thesis! It is clear that $\mid E \mid$ is a major factor in the complexity of finding the sparse resultant of a system of equations, since it dictates the size of the matrix M in the polynomial coefficients.

In what follows, we adopt the same notation introduced in Section 6.1. The results from Chapter 4 are used to find the minimal size of such set E in the special case where the Minkowski sum Q is a multiple of a given polytope P. In this situation, one can reduce the work from Q to P and use the formulas already established in the previous chapters.

Instead of reiterating the work done throughout the body of the thesis, we provide one numerical example that illustrates this reduction and helps visualize the techniques described in Section 6.1.

Example 6.6. Consider the following system

$$
\begin{aligned}
f_0 &= 2x + 3y + 5 = 0 \\
f_1 &= x + 6y + 1 = 0 \\
f_2 &= 2x^2 + 3y^2 + 5xy - 2 = 0.
\end{aligned}
\tag{6.5}
$$

The corresponding Newton polytopes are:

$Q_1 := \mathrm{NP}(f_1) = \Delta_2$,

$Q_2 := \mathrm{NP}(f_2) = \Delta_2$,

$Q_3 := \mathrm{NP}(f_3) = 2\Delta_2$.

Hence, $Q := Q_1 + Q_2 + Q_3 = \Delta_2 + \Delta_2 + 2\Delta_2 = 4\Delta_2$. A mixed subdivision of Q is shown in the figure below, as well as the shifted polytope $Q + \delta$ for some $\delta \in \mathbb{R}^2$.

By Chapter 5, the h^*-vectors of φ_x and Δ_2 are $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$. This implies that

$$
\varphi_x(n\Delta_2) = \begin{cases} \frac{1}{2}n^2 + \frac{3}{2}n + 1 & \text{if } x \in \mathbb{Z}^2, \\ \frac{1}{2}n^2 + \frac{1}{2}n & \text{if } x \in \Delta_2 \setminus \mathbb{Z}^2, \\ \frac{1}{2}n^2 - \frac{1}{2}n & \text{otherwise.} \end{cases}
\tag{6.6}
$$

For $n = 4$, $\varphi_x(4\Delta_2)$ attains the smallest value for any vector $x \in \mathbb{T}^2 \setminus \Delta_2$, namely $\varphi_x(4\Delta_2) = \frac{1}{2}(4^2) - \frac{1}{2}(4) = 8 - 2 = 6$.

We can now find the set E of integer points and the subsets S_0, S_1 and S_2:

$E = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (1, 3)\}$,

$S_0 = \{(3, 1), (2, 2), (2, 1)\}$,

$S_1 = \{(1, 3), (1, 2)\}$,

$S_2 = \{(1,1)\}$.

The corresponding vertices are $(1,0)$ for S_0, $(0,1)$ for S_1 and $(0,0)$ for S_2.

By applying Equation (6.4), we get the system

$$
\begin{aligned}
&2x^3y + 3x^2y^2 + 5x^2y = 0\\
&2x^2y^2 + 3xy^3 + 5xy^2 = 0\\
&2x^2y + 3xy^2 + 5xy = 0\\
&x^2y^2 + 6xy^3 + xy^2 = 0\\
&x^2y + 6xy^2 + xy = 0\\
&2x^3y + 3xy^3 + 5x^2y^2 - 2xy = 0.
\end{aligned}
\tag{6.7}
$$

The system (6.7) is translated into the matrix of coefficients

$$
M = \begin{bmatrix}
0 & 5 & 2 & 0 & 3 & 0\\
0 & 0 & 0 & 5 & 2 & 3\\
5 & 2 & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 6\\
1 & 1 & 0 & 6 & 0 & 0\\
-2 & 0 & 2 & 0 & 5 & 3
\end{bmatrix},
\tag{6.8}
$$

whose determinant can be easily computed and is equal to -1836. It can be shown that the resultant of the system (6.5) is 918, which is indeed a divisor of $\det(M)$.

□

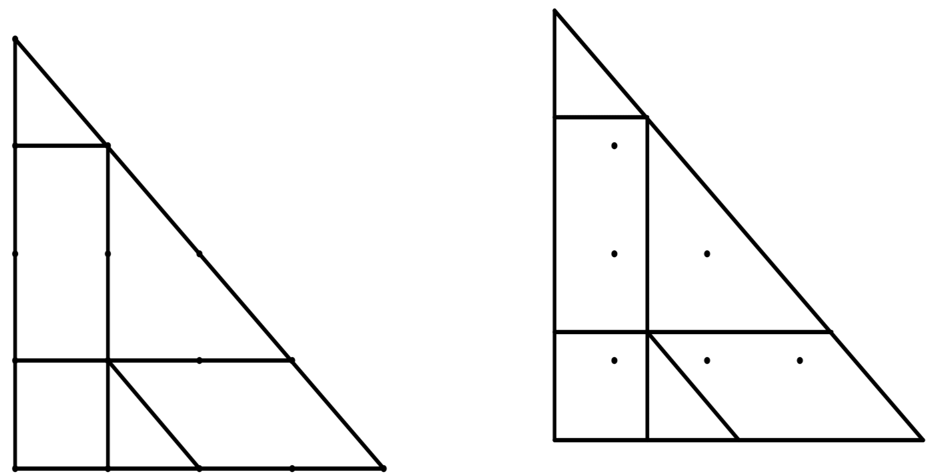

Figure 6.1: The mixed subdivision and the shift

Remark 6.7. There are several ways in which the sparse resultant can be used to solve systems of polynomial equations. We did not mention any of them here since they are out of the scope of the present thesis. We refer to the book [6] for more technical details and to the paper [5] for the study of the complexity of the related algorithms. □

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Biography

Elie Alhajjar grew up in a small town in Beirut, Lebanon. He attended Notre Dame University of Lebanon where he received his Bachelor of Science in Mathematics in 2007 and his Masters of Science in Mathematics in 2010. He then received his Doctorate in Mathematics from George Mason University in 2017, under the supervision of Dr. James F. Lawrence. He will be joining West Point Academy in Fall 2017.