WEIGHTED COMPOSITION OPERATORS FROM ANALYTIC FUNCTION SPACES INTO A CLASS OF WEIGHTED-TYPE BANACH SPACES

by

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Dedication

To my parents, my sister Rabab, and my brother Essam,

their love and support sustained me throughout my life

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Table of Contents

List of Symbols

$\mathbb{D}$	The open unit disk
$H(\mathbb{D})$	The algebra of analytic functions on $\mathbb{D}$
$S(\mathbb{D})$	The subset of $H(\mathbb{D})$ consisting of the functions mapping $\mathbb{D}$ into itself
$Aut(\mathbb{D})$	The group of conformal automorphisms of $\mathbb{D}$
X	Banach space of analytic functions on $\mathbb{D}$
$K(z)$	The point-evaluation functional at z
$\mathcal{H}$	The reproducing kernel Hilbert space
K_z	The reproducing kernel at z
μ	A positive continuous function
H_μ^∞	The weighted-type Banach space
$\mathcal{B}_\mu$	The Bloch-type space
$\mathcal{B}_{\mu,0}$	The little weighted Bloch space
$\mathcal{Z}_\mu$	The weighted Zygmund spce
$\mathcal{Z}_{\mu,0}$	The little Zygmund-type space
$\mathcal{V}_n$	The weighted-type Banach space
$\mathcal{V}_{n,0}$	The little weighted-type Banach space
H^∞	The Hardy space
H^p	The Hardy space
A_α^p	The weighted Bergman space
S^p	The derivative Hardy space
φ	Analytic self-map of $\mathbb{D}$
ψ	Analytic function on $\mathbb{D}$
M_ψ	The multiplication operator with symbol ψ
C_φ	The composition operator with symbol φ
$W_{\psi,\varphi}$	The weighted composition operator with symbols ψ and φ
$\lVert \cdot \rVert$	Norm
$\lVert \cdot \rVert_e$	Essential norm
$A \asymp B$	There exist positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$

Abstract

WEIGHTED COMPOSITION OPERATORS FROM ANALYTIC FUNCTION SPACES INTO A CLASS OF WEIGHTED-TYPE BANACH SPACES

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This dissertation is under the supervision of Dr. Flavia Colonna, professor of Mathematical Sciences at George Mason University. It concerns weighted composition operators from analytic function spaces into a class of weighted-type Banach spaces.

In Chapter 2, we characterize the bounded and compact weighted composition operators from a large class of Banach space X of analytic functions on the open unit disk into Zygmund-type spaces $\mathcal{Z}_\mu$, where μ is a positive continuous function. Under more restrictive conditions, we provide an approximation of the essential norm of such operators. We also show that all bounded weighted composition operators from X to an important subspace $\mathcal{Z}_{\mu,0}$ of $\mathcal{Z}_\mu$, called the little Zygmund-type space, are compact and characterize such operators.

In Chapter 3, we generalize our work in part one to characterize the bounded and compact weighted composition operators from a large class of Banach space X of analytic functions on the open unit disk into the weighted-type Banach spaces $\mathcal{V}_n$ for $n \geq 0$. Such spaces have $\mathcal{Z}_\mu$ as a special case for $n = 2$. The cases when $n = 0, 1$ have been studied by Colonna and Tjani in 2016. Under more restrictive conditions, we provide an approximation of the essential norm of such operators. We also show that all bounded weighted composition

operators from X to the little weighted-type space $\mathcal{V}_{n,0}$ spaces are compact and characterize such operators.

In Chapter 4, we apply our results to the cases when X is the Hardy space H^p for $1 \leq p \leq \infty$ and the weighted Bergman space A^p_α for $\alpha > -1$ and $1 \leq p < \infty$.

Since our general results from Chapters 2 and 3 are not applicable to the case when X is the space S^p of analytic functions whose derivatives are in the Hardy space H^p (for $p \geq 1$), independently, we carry out in Chapter 5 the study of the weighted composition operators mapping S^p into the weighted-type Banach spaces $\mathcal{Z}_\mu$ and $\mathcal{V}_n$.

This work is motivated by recent studies that have been done by Colonna and Tjani. Indeed, in [9], Colonna and Tjani studied the weighted composition operators from a general reproducing kernel Hilbert space of analytic functions $\mathcal{H}$ to the Banach spaces H^∞_μ and $\mathcal{B}_\mu$. They characterized the bounded and the compact weighted composition operators from $\mathcal{H}$ into H^∞_μ and $\mathcal{B}_\mu$. Moreover, they obtained an approximation of the essential norm of such operators. In [10], they extended their results to the weighted composition operators from a large class of Banach space X to the same target spaces H^∞_μ and $\mathcal{B}_\mu$.

In [8], Colonna and Tjani analyzed the weighted composition operators acting on a general reproducing kernel Hilbert space of analytic functions $\mathcal{H}$ but taking as target space the weighted Zygmund space $\mathcal{Z}_\mu$. They gave characterizations of the boundedness and compactness of such operators and provided an approximation of the essential norm.

In [11], they studied the weighted composition operators mapping into Bloch-type spaces when the domain is a large class of Banach spaces X for which the results in the earlier works were not applicable. They also focused on the weighted composition operators acting on the space S^p of the analytic functions on the unit disk whose derivative is in the Hardy space H^p for $p \geq 1$.

Finally, the boundedness and the compactness of the weighted composition operators into a subspace of the main target space has also been discussed in [9], [10], and [8].

Chapter 1: Preliminaries

1.1 General classes of analytic function spaces on the open unit disk $\mathbb{D}$

We shall denote by $H(\mathbb{D})$ the algebra of analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ the subset of $H(\mathbb{D})$ consisting of the functions mapping $\mathbb{D}$ into itself, and $Aut(\mathbb{D})$ the group (under composition) of conformal automorphisms of $\mathbb{D}$, namely the bijective analytic self-maps of $\mathbb{D}$. It is well-known (e.g. [12], p. 132) that the conformal automorphisms of $\mathbb{D}$ are the linear fractional transformations of the form

$$S(z) = \lambda\Big(\frac{a-z}{1-\bar{a}z}\Big), \quad \text{for } z \in \mathbb{D},$$

where λ is a constant of modulus 1 and $a \in \mathbb{D}$.

1.1.1 The reproducing kernel Hilbert space $\mathcal{H}$

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ with inner product $\langle \cdot, \cdot \rangle$. We say that $\mathcal{H}$ a *reproducing kernel Hilbert space* if all point evaluations are bounded linear functionals. By the Riesz Representation Theorem, for each $z \in \mathbb{D}$, there exists a unique element K_z of $\mathcal{H}$, called the *reproducing kernel at* z, such that for each $f \in \mathcal{H}$,

$$f(z) = \langle f, K_z \rangle.$$

1.1.2 The functional Banach space X

Following the terminology used in [14], a Banach space X of complex-valued functions on $\mathbb{D}$ is called a *functional Banach space on* $\mathbb{D}$ if

- the vector operations are the pointwise operations;
- for $f, g \in X$, $f(z) = g(z)$ for each $z \in \mathbb{D}$ implies $f = g$;
- for $z, w \in \mathbb{D}$, $f(z) = f(w)$ for each function $f \in X$ implies $z = w$;
- for each $z \in \mathbb{D}$, the linear functional $f \mapsto f(z)$ is continuous.

A functional Banach space on $\mathbb{D}$ whose functions are analytic on $\mathbb{D}$ is called a *Banach space of analytic functions.*

Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$. For each $z \in \mathbb{D}$, define

$$K(z) = \sup\{|f(z)| : f \in X, \|f\| \leq 1\} = \|\Lambda_z\|, \tag{1.1}$$

where Λ_z denotes the point-evaluation functional at z. Thus, for any function $f \in X$ and $z \in \mathbb{D}$,

$$|f(z)| \leq \|f\| K(z). \tag{1.2}$$

Note that if X is a reproducing kernel Hilbert space of analytic functions, then

$$K(z) = \|K_z\| = \sqrt{\langle K_z, K_z \rangle}.$$

1.2 Weighted-type Banach spaces of analytic functions

The spaces in this section are studied explicitly in [4, 6, 26].

Let μ is a positive continuous function on $\mathbb{D}$, which will be referred to as a *weight.*

1.2.1 The weighted-type Banach space H_μ^∞

The weighted-type Banach space H_μ^∞ is defined as the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H_\mu^\infty} := \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty.$$

Under the norm $\|\cdot\|_{H_\mu^\infty}$, H_μ^∞ is a Banach space. We define $H_{\mu,0}^\infty$ to be the subspace of H_μ^∞ whose elements f satisfy the condition

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

1.2.2 The weighted Bloch-type space $\mathcal{B}_\mu$

For a weight μ, the Bloch-type space $\mathcal{B}_\mu$ is the Banach space of analytic functions f on $\mathbb{D}$ such that

$$b_\mu(f) := \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty,$$

with norm

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f).$$

Clearly $f \in \mathcal{B}_\mu$ if and only if $f' \in H_\mu^\infty$ and

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f'\|_{H_\mu^\infty}.$$

The little weighted Bloch space $\mathcal{B}_{\mu,0}$ is the subspace of $\mathcal{B}_\mu$ of the functions f such that

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0.$$

1.2.3 The weighted Zygmund spce $\mathcal{Z}_\mu$

The weighted Zygmund space $\mathcal{Z}_\mu$ is the Banach space of analytic functions whose elements f satisfy the condition

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Thus, $f \in \mathcal{Z}_\mu$ if and only if $f' \in \mathcal{B}_\mu$. The norm in $\mathcal{Z}_\mu$ is defined by

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| = |f(0)| + \|f'\|_{\mathcal{B}_\mu}.$$

The little weighted Zygmund space $\mathcal{Z}_{\mu,0}$ is the subspace of $\mathcal{Z}_\mu$ of the functions f such that

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0.$$

1.2.4 The weighted-type Banach space $\mathcal{V}_n$

For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the weighted-type Banach space $\mathcal{V}_n$ is the Banach space of analytic functions $f \in H(\mathbb{D})$, whose elements f satisfy the condition

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f^{(n)}(z)| < \infty,$$

where we adopt the convention that $f^{(0)} = f$.

The norm of $\mathcal{V}_n$ is defined by

$$\|f\|_{\mathcal{V}_n} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f^{(n)}(z)|.$$

For the weight $\mu(z) = 1-|z|^2$, we have $\mathcal{V}_0 = H_\mu^\infty$, $\mathcal{V}_1 = \mathcal{B}_\mu$, and $\mathcal{V}_2 = \mathcal{Z}_\mu$.

1.3 Other important spaces of analytic functions

1.3.1 The Hardy space H^∞

The Hardy space H^∞, is the Banach space of bounded analytic functions f on $\mathbb{D}$ with norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

1.3.2 The Hardy space H^p

For $1 \le p < \infty$, the Hardy space H^p is the Banach space of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

with norm

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}. \tag{1.3}$$

Note that the inclusion of the factor $1/(2\pi)$ is to normalize the arc length measure.

1.3.3 The weighted Bergman space A^p_α

For $1 \le p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α is the space consisting of the analytic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where A is the normalized area measure on $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. The norm of A^p_α is defined as

$$\|f\|_{p,\alpha} = \Big((\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z)\Big)^{1/p}.$$

In the special case when $\alpha = 0$, A^p_α is the classical Bergman space A^p.

1.3.4 The space S^p for $p \geq 1$

The space S^p is defined as the set of all analytic functions on $\mathbb{D}$ whose first derivative is in the Hardy space H^p. The norm of S^p, defined as

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}, \tag{1.4}$$

yields a Banach space structure.

1.4 Weighted composition operators

Let φ be an analytic self-map of the open unit disk $\mathbb{D}$ and ψ an analytic function on $\mathbb{D}$. The multiplication operator with symbol ψ is defined by

$$M_\psi f = \psi f \quad \text{for every } f \in H(\mathbb{D}).$$

The composition operator with symbol φ is defined by

$$C_\varphi f = f \circ \varphi \quad \text{for every } f \in H(\mathbb{D}).$$

The weighted composition operator with symbols ψ and φ is defined as

$$W_{\psi,\varphi} f = M_\psi C_\varphi f = \psi(f \circ \varphi)$$

for any f analytic function on $\mathbb{D}$.

The weighted composition operator $W_{\psi,\varphi}$ is linear on a Banach space of analytic functions on $\mathbb{D}$ as a consequence of M_ψ and C_φ both being linear. Indeed, if f and g are two analytic functions and a and b are constants, then

$$\begin{aligned} W_{\psi,\varphi}(af+bg) &= M_\psi C_\varphi(af+bg) \\ &= \psi((af+bg)\circ\varphi) \\ &= \psi((af\circ\varphi)+(bg\circ\varphi)) \\ &= \psi(af\circ\varphi)+\psi(bg\circ\varphi) \\ &= a\psi(f\circ\varphi)+b\psi(g\circ\varphi) \\ &= aW_{\psi,\varphi}f+bW_{\psi,\varphi}g. \end{aligned}$$

1.4.1 Boundedness

The weighted composition operator $W_{\psi,\varphi}$ between two normed spaces X and Y is bounded if there is a positive constant C such that for every $f \in X$

$$\|W_{\psi,\varphi}f\|_Y \leq C\|f\|_X,$$

The norm of the operator $W_{\psi,\varphi}$ is given by

$$\|W_{\psi,\varphi}\| = \sup_{\|f\|_X\leq 1} \|W_{\psi,\varphi}f\|_Y.$$

1.4.2 Compactness

The bounded weighted composition operator $W_{\psi,\varphi}$ between two normed spaces X and Y is *compact* if the image under $W_{\psi,\varphi}$ of any bounded subset of X is a relatively compact subset of Y, i.e., the image has compact closure in Y. Thus, the operator $W_{\psi,\varphi}$ is compact if and only if for any positive constant C

$$W_{\psi,\varphi}(\{f\in X : \|f\|_X \leq C\}) \equiv W_{\psi,\varphi}(\{f\in X : \frac{\|f\|_X}{C} \leq 1\})$$

is relatively compact. Using linearity, it follows that $W_{\psi,\varphi}$ is compact if and only if

$$W_{\psi,\varphi}(\{f \in X : \|f\|_X \leq 1\})$$

is relatively compact.

The following two results due to Colonna and Tjani will be useful to characterize the compactness of the operators under consideration in this work.

Lemma 1.4.1. ([7], Lemma 8) *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose*

(i) *the point evaluation functionals on Y are continuous;*

(ii) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*

(iii) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets. Then T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X converging to 0 uniformly on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y.*

Proposition 1.4.1. ([10], Proposition 2.1) *Let X be a Banach spaces of analytic functions on $\mathbb{D}$. Then the mapping $z \mapsto K(z)$ is bounded on compact subsets of $\mathbb{D}$.*

Remark 1.4.1. The conclusion of Lemma 1.4.1 holds under just the assumptions (i) and (ii) if $T : X \to Y$ is a weighted composition operator, since any bounded sequence in X is uniformly bounded on compact subsets of $\mathbb{D}$ and any sequence converging to 0 in norm also converges uniformly on compact sets.

Furthermore, if X is reflexive, then the unit ball of X is relatively compact with respect to the topology of uniform convergence on compact subset of $\mathbb{D}$.

1.4.3 Essential norm

Definition 1.2. The *essential norm* of a bounded linear operator S between Banach spaces X and Y is defined as

$$\|S\|_e := \inf\{\|S - K\| \mid K : X \to Y \text{ compact}\}.$$

Thus, a bounded linear operator S is compact if and only if $\|S\|_e := 0$.

In [10], the authors restricted to Banach spaces with the following properties, that will be used in this work under appropriate modifications. In particular, they have imposed several conditions that were used in varying degrees to enable them to state and prove their results. These conditions and corresponding modifications are listed below.

(I) There is a constant $C > 0$ such that

$$K(rz) \leq CK(z),$$

for all $z \in \mathbb{D}$ and $0 < r < 1$.

(II) The unit ball of X is relatively compact with respect to the topology of uniform convergence on compact subset of $\mathbb{D}$.

(III) $\lim_{|z|\to 1} K(z) = \infty$.

(IV) $K(z)$ is bounded below by a positive constant on compact subsets of $\mathbb{D}$.

(V) For $0 < r < 1$, the linear operator $\mathcal{T}_r$ mapping an analytic function f on $\mathbb{D}$ to the function $f_r(z) = f(rz), z \in \mathbb{D}$, is compact on X.

(VI) There is a constant $C > 0$ such that

$$\|\mathcal{S}f\| \leq C\|f\|$$

for all $\mathcal{S} \in Aut(\mathbb{D})$, $f \in X$.

(VII) There is a constant $C > 0$ such that

$$(1-|z|^2)^j |f^{(j)}(z)| \leq C\|f\| K(z)$$

for all $f \in X$, $z \in \mathbb{D}$, and $j = 1, \ldots, n$.

(VIII) $\sup_{0<r<1} \|\mathcal{T}_r\| < \infty$.

The following result due to Colonna and Tjani will be useful to provide an approximation of the essential norm of the operators under consideration in this work.

Lemma 1.4.2. ([10], Lemma 3.1) *Let X be a banach space of analytic functions.*

(a) *If X contains functions non-vanishing at 0 and satisfies either (I) or (VIII), then for each $r \in (0,1)$,*

$$\sup_{\|f\|\leq 1} \sup_{z\in\mathbb{D}} \frac{1}{K(z)} |((I-\mathcal{T}_r)f)(z)| < \infty.$$

(b) *If X satisfies (II) and (VIII), or X is reflexive and satisfies (VIII), or X satisfies (VII), then for each $s \in (0,1)$ and each $\varepsilon > 0$, there exists $r \in (0,1)$ such that*

$$\sup_{\|f\|\leq 1} \sup_{|z|\leq s} |((I-\mathcal{T}_r)f)(z)| < \varepsilon.$$

(c) *Assume X satisfies (IV) and one of the following three sets of conditions:*

- *(II) and (VIII);*
- *(VIII) and X is reflexive;*
- *(VII).*

Then for each $s \in (0,1)$ and each $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\sup_{\|f\|\leq 1} \sup_{|z|\leq s} \frac{1}{K(z)} |((I-\mathcal{T}_r)f)'(z)| < \varepsilon.$$

Remark 1.4.2. By part (b), $((I - \mathcal{T}_r)f)^{(n)} \to 0$ uniformly on compact subsets as $r \to 1^-$. Therefore, under the same conditions of part (c), since $K(z)$ is bounded below by a positive constant on compact subsets of $\mathbb{D}$, we also have

$$\sup_{\|f\|\leq 1} \sup_{|z|\leq s} \frac{1}{K(z)} |((I - \mathcal{T}_r)f)^{(n)}(z)| < \varepsilon.$$

In this work we use the notation $A \asymp B$ to mean that there exist positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. We shall use the convention of C as a positive constant which is independent of the functions under consideration and may change at each occurrence.

Chapter 2: Weighted composition operators into Zygmund-type spaces

In this chapter, we characterize the bounded and compact weighted composition operators from a large class of Banach spaces X of analytic functions on $\mathbb{D}$ into Zygmund-type spaces. Under more restrictive conditions, we provide an approximation of the essential norm of such operators. We also show that all bounded weighted composition operators from X to the little Zygmund-type space are compact and characterize such operators.

2.1 Bounded and compact weighted composition operators mapping into $\mathcal{Z}_\mu$

We begin the section with a characterization of boundedness that extends ([8], Theorem 3.1).

Theorem 2.1.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (VI) *and* (VII). *Let μ be a positive continuous function on $\mathbb{D}$, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is bounded if and only if the following quantities are finite:*

$$\begin{aligned}
M_1 &:= \sup_{z\in\mathbb{D}} \mu(z)|\psi''(z)|K(\varphi(z)),\\
M_2 &:= \sup_{z\in\mathbb{D}} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{1-|\varphi(z)|^2}K(\varphi(z)),\\
M_3 &:= \sup_{z\in\mathbb{D}} \mu(z)\frac{|\psi(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}K(\varphi(z)),
\end{aligned}$$

in which case $\|W_{\psi,\varphi}\| \asymp M_1 + M_2 + M_3$.

Proof. Suppose $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is bounded. First, we prove that $M_3 \le C\|W_{\psi,\varphi}\|$ for some $C > 0$.

For $p \in \mathbb{D}$, consider the disk automorphism that interchanges 0 and p:

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

The first and second derivatives evaluated at p are given by

$$\alpha_p'(p) = \frac{1}{|p|^2 - 1}, \text{ and } \alpha_p''(p) = -\frac{2\overline{p}}{(1 - |p|^2)^2}. \tag{2.1}$$

Fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \le 1$. Consider the function defined on $\mathbb{D}$ by $g_w(z) = [\alpha_{\varphi(w)}(z)]^2 f(z)$. Then by direct calculation $g_w(\varphi(w)) = 0 = g_w'(\varphi(w))$, $g_w''(\varphi(w)) = \frac{2f(\varphi(w))}{(1-|\varphi(w)|^2)^2}$, and by (VI) with $j = 2$, we see that $g_w \in X$ and

$$\|g_w\| = \|\alpha^2_{\varphi(w)} f\| \le C\|f\| \le C.$$

Thus,

$$\begin{aligned}
\mu(w)|(W_{\psi,\varphi} g_w)''(w)| &= \mu(w)\big|\psi''(w) g_w(\varphi(w)) + 2\psi'(w)\varphi'(w) g_w'(\varphi(w)) \\
&\qquad + \psi(w)\big(\varphi''(w) g_w'(\varphi(w)) + \varphi'(w)^2 g_w''(\varphi(w))\big)\big| \\
&= \mu(w)\frac{2|\psi(w)||\varphi'(w)|^2|f(\varphi(w))|}{(1 - |\varphi(w)|^2)^2}.
\end{aligned}$$

Thus, since $W_{\psi,\varphi}$ is bounded, we have

$$\mu(w)\frac{|2\psi(w)||\varphi'(w)|^2|f(\varphi(w))|}{(1 - |\varphi(w)|^2)^2} \le \|W_{\psi,\varphi} g_w\|_{\mathcal{Z}_\mu} \le \|g_w\|\|W_{\psi,\varphi}\| \le C\|W_{\psi,\varphi}\|.$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.1) and (1.2) we obtain

$$M_3 = \sup_{w\in\mathbb{D}} \mu(w)\frac{|\psi(w)||\varphi'(w)|^2}{(1-|\varphi(w)|^2)^2}K(\varphi(w)) \leq C\|W_{\psi,\varphi}\|. \tag{2.2}$$

We next prove that $M_2 \leq C\|W_{\psi,\varphi}\|$. Fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \leq 1$, and let $h_w(z) = \alpha_{\varphi(w)}(z)f(z)$. Observe that $h_w(\varphi(w)) = 0$, and by condition (VI) with $j = 1$, $\|h_w\| = \|\alpha_{\varphi(w)}f\| \leq C\|f\| \leq C$. Thus by (2.1) we obtain

$$h'_w(\varphi(w)) \quad = \quad \frac{f(\varphi(w))}{|\varphi(w)|^2-1}. \tag{2.3}$$

It follows that

$$\begin{aligned}\mu(w)|(W_{\psi,\varphi}h_w)''(w)| = \mu(w)\big|h'_w(\varphi(w))\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)\\ +\psi(w)\varphi'(w)^2h''_w(\varphi(w))\big|. \end{aligned}\tag{2.4}$$

By condition (VII) with $j = 2$, we may choose the above constant C large enough so that

$$(1-|\varphi(w)|^2)^2|h''_w(\varphi(w))| \leq C\|h_w\|K(\varphi(w)) \leq CK(\varphi(w)). \tag{2.5}$$

Since $W_{\psi,\varphi}$ is bounded, $\|W_{\psi,\varphi}h_w\|_{\mathcal{Z}_\mu} \leq \|h_w\|\|W_{\psi,\varphi}\| \leq C\|W_{\psi,\varphi}\|$. Hence, by (2.4),

$$\begin{aligned}\mu(w)\big|h'_w(\varphi(w))\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big) + \psi(w)\varphi'(w)^2h''_w(\varphi(w))\big|\\ \leq C\|W_{\psi,\varphi}\|.\end{aligned}$$

Thus, by the triangle inequality

$$\mu(w)\big|h_w'(\varphi(w))\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)\big|$$
$$\leq C\|W_{\psi,\varphi}\| + \mu(w)\big|\psi(w)\varphi'(w)^2 h_w''(\varphi(w))\big|.$$

Using (2.3) and (2.5), we obtain

$$\begin{aligned}
&\mu(w)\big|h_w'(\varphi(w))\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)\big| \\
&= \mu(w)\left|\frac{f(\varphi(w))}{1-|\varphi(w)|^2}\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)\right| \\
&\leq C\|W_{\psi,\varphi}\| + \mu(w)\big|\psi(w)\varphi'(w)^2 h_w''(\varphi(w))\big| \\
&= C\|W_{\psi,\varphi}\| + \mu(w)\frac{(1-|\varphi(w)|^2)^2}{(1-|\varphi(w)|^2)^2}\big|\psi(w)\varphi'(w)^2 h_w''(\varphi(w))\big| \\
&\leq C\|W_{\psi,\varphi}\| + C\mu(w)\frac{|\psi(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^2}K(\varphi(w)). \qquad (2.6)
\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2) and (2.6) we obtain

$$\mu(w)\frac{|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)|}{1-|\varphi(w)|^2}K(\varphi(w))$$
$$\leq C\|W_{\psi,\varphi}\| + C\mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^2}K(\varphi(w)).$$

Taking the supremum over all $w \in \mathbb{D}$, and using (2.2), we have

$$\begin{aligned}
M_2 &= \sup_{w\in\mathbb{D}} \mu(w)\frac{\big|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big|}{1-|\varphi(w)|^2}K(\varphi(w)) \\
&\leq C\|W_{\psi,\varphi}\| + CM_3 \\
&\leq C\|W_{\psi,\varphi}\|. \qquad (2.7)
\end{aligned}$$

Lastly, we prove that $M_1 \le C\|W_{\psi,\varphi}\|$. Fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \le 1$. Then,

$$\begin{aligned}\psi''(w)f(\varphi(w)) &= (W_{\psi,\varphi}f)''(w) - f'(\varphi(w))\big(2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big) \\ &\quad - \psi(w)f''(\varphi(w))\varphi'(w)^2.\end{aligned}$$

so, by the boundedness of the operator, we have

$$\mu(w)|(W_{\psi,\varphi}f)''(w)| \le \|W_{\psi,\varphi}f\|_{\mathcal{Z}_\mu} \le \|W_{\psi,\varphi}\|\|f\| \le \|W_{\psi,\varphi}\|.$$

Thus by (VII) for $j = 1, 2$, we obtain

$$\begin{aligned}\mu(w)\big|\psi''(w)f(\varphi(w))\big| &\le \|W_{\psi,\varphi}\| \\ &\quad + \mu(w)\big|f'(\varphi(w))\big(2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big)\big| \\ &\quad + \mu(w)\big|\psi(w)f''(\varphi(w))\varphi'(w)^2\big| \\ &\le \|W_{\psi,\varphi}\| \\ &\quad + C\mu(w)\frac{\big|2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big|}{1-|\varphi(w)|^2}K(\varphi(w)) \\ &\quad + C\mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^2}K(\varphi(w)). \qquad (2.8)\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \le 1$, by (1.2) and (2.8), we obtain

$$\begin{aligned}\mu(w)|\psi''(w)|K(\varphi(w)) &\le \|W_{\psi,\varphi}\| \\ &\quad + C\mu(w)\frac{\big|2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big|}{1-|\varphi(w)|^2}K(\varphi(w)) \\ &\quad + C\mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^2}K(\varphi(w)).\end{aligned}$$

Hence, taking the supremum over all $w \in \mathbb{D}$, and using (2.7) and (2.2), we obtain

$$M_1 \leq \|W_{\psi,\varphi}\| + CM_2 + CM_3 \leq C\|W_{\psi,\varphi}\|.$$

Therefore, M_1, M_2 and M_3 are finite and for some positive constant C,

$$\|W_{\psi,\varphi}\| \geq C(M_1 + M_2 + M_3).$$

Conversely, suppose M_1, M_2 and M_3 are finite. Let $f \in X$ with $\|f\| \leq 1$, and fix $w \in \mathbb{D}$. Then, by (1.1) and (VII) for $j = 1, 2$, we have

$$\begin{aligned} \mu(w)|(W_{\psi,\varphi}f)''(w)| &= \mu(w)\big|\psi''(w)f(\varphi(w)) \\ &\quad + f'(\varphi(w))\big(2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big) \\ &\quad + \psi(w)f''(\varphi(w))\varphi'(w)^2\big| \\ &\leq \mu(w)|\psi''(w)|K(\varphi(w)) \\ &\quad + C\mu(w)\frac{\big|2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big|}{1 - |\varphi(w)|^2}K(\varphi(w)) \\ &\quad + C\mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1 - |\varphi(w)|^2)^2}K(\varphi(w)) \\ &\leq M_1 + CM_2 + CM_3. \end{aligned} \tag{2.9}$$

Taking the supremum over all $w \in \mathbb{D}$, by (2.9) we obtain

$$\sup_{w \in \mathbb{D}} \mu(w)|(W_{\psi,\varphi}f)''(w)| \leq C(M_1 + M_2 + M_3).$$

Thus by the definition of $\|\cdot\|_{\mathcal{Z}_\mu}$, (1.1), and (VII) for $j=1$, we have

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathcal{Z}_\mu} &\leq \big|\psi(0)f(\varphi(0))\big| + \big|\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))\big| \\ &\quad + C(M_1+M_2+M_3) \\ &\leq \left(\big|\psi(0)\big| + \big|\psi'(0)\big| + C\frac{\big|\psi(0)\varphi'(0)\big|}{1-|\varphi(0)|^2}\right) K(\varphi(0)) \\ &\quad + C(M_1+M_2+M_3),\end{aligned}$$

proving the boundedness of $W_{\psi,\varphi}$ and, choosing C sufficiently large, the upper estimate

$$\|W_{\psi,\varphi}\| \leq C(M_1+M_2+M_3).$$

□

In the following theorem we characterize compact weighted composition operators from a Banach space X of analytic functions into $\mathcal{Z}_\mu$ under certain conditions on X. The following result complements Theorem 3.3 in [8].

Theorem 2.1.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (II), (VI), *and* (VII). *Suppose $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is bounded and suppose*

$$\delta := \inf_{z\in\mathbb{D}} K(\varphi(z)) > 0. \tag{2.10}$$

Then $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is compact if and only if $A_j(\psi,\varphi) = 0$ for $j = 1,2,3$, where

$$\begin{aligned}A_1(\psi,\varphi) &:= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)|\psi''(z)|K(\varphi(z)), \\ A_2(\psi,\varphi) &:= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{1-|\varphi(z)|^2} K(\varphi(z)), \\ A_3(\psi,\varphi) &:= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2} K(\varphi(z)).\end{aligned}$$

Note that condition (2.10) clearly holds if the space X contains the constants since then $K(z) \geq 1/\|1\|$ for all $z \in \mathbb{D}$.

Proof. Suppose $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is compact. If $\|\varphi\|_\infty < 1$, the result is obvious since the supremum over an empty set is zero. So assume $\|\varphi\|_\infty = 1$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $1/2 < |\varphi(a_n)| \to 1$ as $n \to \infty$ and

$$A_3(\psi, \varphi) = \lim_{n\to\infty} \mu(a_n) \frac{|\psi(a_n)\varphi'(a_n)^2|}{(1 - |\varphi(a_n)|^2)^2} K(\varphi(a_n)).$$

Fix $\varepsilon > 0$. By (1.2), for each $n \in \mathbb{N}$ there exists $f_n \in X$ with $\|f_n\| \leq 1$ such that

$$|f_n(\varphi(a_n))| > K(\varphi(a_n)) - \varepsilon. \tag{2.11}$$

Since the sequence $\{\|f_n\|\}$ is bounded and by Proposition 1.4.1, the map $z \mapsto K(z)$ is bounded on compact sets in $\mathbb{D}$. Using (1.2) we see that $\{f_n\}$ is uniformly bounded on compact sets. For each $n \in \mathbb{N}$ and $z \in \mathbb{D}$, define

$$F_n(z) = \frac{\big(1 - |\varphi(a_n)|^2\big)z}{1 - \overline{\varphi(a_n)}z} f_n(z) = \big(\varphi(a_n) - \alpha_{\varphi(a_n)}(z)\big) f_n(z). \tag{2.12}$$

Then

$$F_n(\varphi(a_n)) = \varphi(a_n) f_n(\varphi(a_n)) \tag{2.13}$$

and by (1.2),

$$|F_n(z)| \leq \frac{1 - |\varphi(a_n)|^2}{1 - |z|} K(z).$$

Since $|\varphi(a_n)| \to 1$ and, as noted above, $z \mapsto K(z)$ is bounded on compact subsets of $\mathbb{D}$, the sequence $\{F_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. By (VI), we see that

$F_n \in X$ and

$$\begin{aligned} \|F_n\| &\leq |\varphi(a_n)|\|f_n\| + \|\alpha_{\varphi(a_n)} f_n\| \\ &\leq \|f_n\| + C\|f_n\| \\ &\leq C\|f_n\| \\ &\leq C. \end{aligned} \tag{2.14}$$

For $n \in \mathbb{N}$ and $z \in \mathbb{D}$, define $H_n(z) = F_n(z)(\alpha_{\varphi(a_n)}(z))^2$. By (2.14) and condition (VI) for $j = 2$, we have

$$\|H_n\| = \|F_n \alpha^2_{\varphi(a_n)}\| \leq C.$$

Moreover, H_n converges to zero uniformly on compact subsets of $\mathbb{D}$ since F_n converges to zero uniformly on compact subsets of $\mathbb{D}$ and $\alpha^2_{\varphi(a_n)}$ is bounded on $\mathbb{D}$. In addition, since $\alpha_{\varphi(a_n)}(\varphi(a_n)) = 0$, we see that $H_n(\varphi(a_n)) = H_n'(\varphi(a_n)) = 0$. Therefore, by (2.1), we have

$$\begin{aligned} H_n''(\varphi(a_n)) = F_n''&(\varphi(a_n))\alpha_{\varphi(a_n)}(\varphi(a_n))^2 \\ &+ 4F_n'(\varphi(a_n))\alpha_{\varphi(a_n)}(\varphi(a_n))\alpha'_{\varphi(a_n)}(\varphi(a_n)) \\ &+ 2F_n(\varphi(a_n))\big(\alpha'_{\varphi(a_n)}(\varphi(a_n))^2 + \alpha_{\varphi(a_n)}(\varphi(a_n))\alpha''_{\varphi(a_n)}(\varphi(a_n))\big) \\ &= \frac{2F_n(\varphi(a_n))}{(1 - |\varphi(a_n)|^2)^2}. \end{aligned} \tag{2.15}$$

Since condition (II) holds, we may apply Lemma 1.4.1 and Remark 1.4.1. Thus, $\|W_{\psi,\varphi} H_n\|_{\mathcal{Z}_\mu} \to 0$, hence

$$\lim_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi} H_n)''(a_n)| = 0. \tag{2.16}$$

Using (2.15) and since $H_n(\varphi(a_n)) = H_n'(\varphi(a_n)) = 0$, it follows from (2.16) that

$$\begin{aligned}
\lim_{n\to\infty} \mu(a_n)&\frac{|2\psi(a_n)\varphi'(a_n)^2 F_n(\varphi(a_n))|}{(1-|\varphi(a_n)|^2)^2} \\
&= \lim_{n\to\infty} \mu(a_n)\Big|\psi''(a_n)H_n(\varphi(a_n)) + 2\psi'(a_n)\varphi'(a_n)H_n'(\varphi(a_n)) \\
&\qquad + \psi(a_n)\Big(\varphi''(a_n)H_n'(\varphi(a_n)) + \varphi'(a_n)^2 H_n''(\varphi(a_n))\Big)\Big| \\
&= \lim_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi}H_n)''(a_n)| = 0. \qquad (2.17)
\end{aligned}$$

By (2.11), (2.13), and (2.17), we obtain

$$\begin{aligned}
\mu(a_n)&\frac{|2\psi(a_n)\varphi'(a_n)^2\varphi(a_n)|}{(1-|\varphi(a_n)|^2)^2}\big(K(\varphi(a_n)) - \varepsilon\big) \\
&\le \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|\varphi(a_n)f_n(\varphi(a_n))| \\
&= \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|F_n(\varphi(a_n))| \to 0 \qquad (2.18)
\end{aligned}$$

as $n \to \infty$. Since $W_{\psi,\varphi}$ is bounded, by Theorem 2.1.1, the quantities M_1, M_2 and M_3 are finite.

By condition (2.10), we have

$$\begin{aligned}
\mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2\varphi(a_n)|}{(1-|\varphi(a_n)|^2)^2} &\le \sup_{w\in\mathbb{D}} \mu(w)\frac{|2\psi(w)\varphi'(w)^2\varphi(w)|}{(1-|\varphi(w)|^2)^2} \\
&\le \frac{1}{\delta}\sup_{w\in\mathbb{D}} \mu(w)\frac{|2\psi(w)\varphi'(w)^2\varphi(w)|}{(1-|\varphi(w)|^2)^2}K(\varphi(w)) \\
&= \frac{2}{\delta}M_3. \qquad (2.19)
\end{aligned}$$

Therefore, from (2.18) and (2.19), we obtain

$$\begin{aligned}
&\mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2\varphi(a_n)|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n)) \\
&\le \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2\varphi(a_n)|}{(1-|\varphi(a_n)|^2)^2}\,\varepsilon \\
&\quad + \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|F_n(\varphi(a_n))| \\
&\le \frac{2}{\delta}M_3\,\varepsilon + \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|F_n(\varphi(a_n)) \\
&\to \frac{2}{\delta}M_3\,\varepsilon \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Since ε is arbitrary, it follows that $A_3(\psi,\varphi)=0$.

Let us next show that $A_2(\psi,\varphi)=0$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $\frac{1}{2}<|\varphi(a_n)|\to 1$ as $n\to\infty$ and

$$A_2(\psi,\varphi)=\lim_{n\to\infty}\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)).$$

Fix $\varepsilon>0$ and, as done above, for each $n\in\mathbb{N}$, choose $f_n\in X$ such that $\|f_n\|\le 1$ and (2.11) holds relative to this ε. Define F_n in terms of f_n as done above and set $G_n=F_n\alpha_{\varphi(a_n)}$. By (2.14) and condition (VI) for $j=1$, we have

$$\|G_n\|=\|F_n\alpha_{\varphi(a_n)}\|\le C\|F_n\|\le C. \tag{2.20}$$

The sequence $\{G_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Moreover,

$G_n(\varphi(a_n)) = 0$ and by (2.1),

$$\begin{aligned} G_n'(\varphi(a_n)) &= F'(\varphi(a_n))\alpha_{\varphi(a_n)}(\varphi(a_n)) + \alpha'_{\varphi(a_n)}(\varphi(a_n))F(\varphi(a_n)) \\ &= \frac{F_n(\varphi(a_n))}{|\varphi(a_n)|^2 - 1}. \end{aligned} \tag{2.21}$$

By the compactness of the operator $W_{\psi,\varphi}$ and Lemma 1.4.1, $\|W_{\psi,\varphi}G_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi}G_n)''(a_n)| = 0.$$

Thus, since $G_n(\varphi(a_n)) = 0$, corresponding to the number ε fixed above, there is an $N \in \mathbb{N}$ such that

$$\begin{aligned} \mu(w)|(W_{\psi,\varphi}G_n)''(a_n)| &= \mu(a_n)\big|G_n'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\big) \\ &\qquad + \psi(a_n)\varphi'(a_n)^2 G_n''(\varphi(a_n))\big| \\ &< \frac{\varepsilon}{2}, \end{aligned}$$

whenever $n > N$. Hence, by (2.20) and (VII) for $j = 2$, for every $n > N$, we have

$$\begin{aligned} &\mu(a_n)\big|G_n'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\big)\big| \\ &\qquad < \frac{\varepsilon}{2} + \mu(a_n)\big|\psi(a_n)\varphi'(a_n)^2 G_n''(\varphi(a_n))\big| \\ &\qquad \le \frac{\varepsilon}{2} + C\mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n))|. \end{aligned}$$

By (2.21) and since we have shown above that $A_3(\psi,\varphi)=0$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\mu(a_n)\frac{|\big(2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)\big)F_n(\varphi(a_n))|}{1-|\varphi(a_n)|^2}\\
&\qquad< \varepsilon + C\lim_{n\to\infty}\mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n))\\
&\qquad= \varepsilon + CA_3(\psi,\varphi)\\
&\qquad= \varepsilon. \qquad (2.22)
\end{aligned}$$

By (2.11), recalling (2.13) and since $1/2<|\varphi(a_n)|$, for all n sufficiently large, from (2.22), we obtain

$$\begin{aligned}
&\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}(K(\varphi(a_n))-\varepsilon)\\
&\quad\le\ \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}|f_n(\varphi(a_n))|\\
&\quad=\ \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}\frac{|F_n(\varphi(a_n))|}{|\varphi(a_n)|}\\
&\quad\le\ \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}2|F_n(\varphi(a_n))|\\
&\quad< 2\varepsilon. \qquad (2.23)
\end{aligned}$$

Moreover, recalling (2.10), we have

$$
\begin{aligned}
&\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2} \\
&\quad \le \sup_{w\in\mathbb{D}} \mu(w)\frac{|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)|}{1-|\varphi(w)|^2} \\
&\quad \le \frac{1}{\delta}\sup_{w\in\mathbb{D}} \mu(w)\frac{|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)|}{1-|\varphi(w)|^2}K(\varphi(w)) \\
&\quad = \frac{1}{\delta}M_2. \qquad (2.24)
\end{aligned}
$$

Therefore, by (2.23), (2.22), and (2.24), we have

$$\lim_{n\to\infty}\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)) \le \varepsilon\frac{1}{\delta}\,M_2+2\varepsilon = C\varepsilon.$$

Since ε is arbitrary, we deduce that $A_2(\psi,\varphi)=0$.

Finally, we show that $A_1(\psi,\varphi)=0$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $\frac{1}{2}<|\varphi(a_n)|\to 1$ as $n\to\infty$ and

$$A_1(\psi,\varphi)=\lim_{n\to\infty}\mu(a_n)|\psi''(a_n)|K(\varphi(a_n)).$$

Fix $\varepsilon>0$ and for each $n\in\mathbb{N}$, construct f_n and F_n as above in terms of ε and of the sequence $\{a_n\}$, which we recall, are bounded sequences in X. By Lemma 1.4.1, the uniform convergence of $\{F_n\}$ to zero on compact subsets of $\mathbb{D}$ guarantees that $\|W_{\psi,\varphi}F_n\|_{\mathcal{Z}_\mu}\to 0$ as $n\to\infty$.

By the definition of the norm of $\mathcal{Z}_\mu$, corresponding to ε there is $N \in \mathbb{N}$ such that

$$\begin{aligned} \mu(a_n)|(W_{\psi,\varphi}F_n)''(a_n)| &= \mu(a_n)\big|\psi''(a_n)F_n(\varphi(a_n)) \\ &\quad + F_n'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\big) \\ &\quad + \psi(a_n)\varphi'(a_n)^2 F_n''(\varphi(a_n))\big| \\ &< \varepsilon, \end{aligned} \tag{2.25}$$

for each $n \geq N$. By condition (VII) and (2.25), and since $A_2(\psi,\varphi) = A_3(\psi,\varphi) = 0$, for all n sufficiently large,

$$\begin{aligned} \mu(a_n)|\psi''(a_n)F_n(\varphi(a_n))| &< \varepsilon + \mu(a_n)|F_n'(\varphi(a_n))(2\psi'(a_n)\varphi'(a_n) \\ &\quad +\psi(a_n)\varphi''(a_n))| + \mu(a_n)|\psi(a_n)\varphi'(a_n)^2 F_n''(\varphi(a_n))| \\ &\leq \varepsilon + C\Big(\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)) \\ &\quad + \mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n))\Big) \\ &< C\varepsilon. \end{aligned}$$

Hence, for all n sufficiently large,

$$\mu(a_n)|\psi''(a_n)|\big(K(\varphi(a_n)) - \varepsilon\big) \leq \mu(a_n)|\psi''(a_n)|2|F(\varphi(a_n))| < C\,\varepsilon.$$

Since K is bounded below by δ on the range of φ, we deduce

$$\mu(a_n)|\psi''(a_n)| \leq \frac{1}{\delta}\mu(a_n)|\psi''(a_n)|K(\varphi(a_n)).$$

Combining the last two inequalities, for all n sufficiently large, we obtain

$$\begin{aligned} \mu(a_n)|\psi''(a_n)|K(\varphi(a_n)) &\leq \frac{1}{\delta}\varepsilon \sup_{w\in\mathbb{D}} \mu(w)|\psi''(w)|K(\varphi(w)) + C\varepsilon \\ &= \frac{1}{\delta}\varepsilon M_1 + C\varepsilon \\ &= C\,\varepsilon, \end{aligned}$$

proving that $A_1(\psi,\varphi) = 0$.

Conversely, suppose $A_j(\psi,\varphi) = 0$, for $j = 1, 2, 3$. By Lemma 1.4.1, to prove the compactness of the operator $W_{\psi,\varphi}$, it suffices to show that if $\{f_n\}$ is a sequence in X converging to zero uniformly on compact subsets of $\mathbb{D}$ with norms bounded by some constant $L > 0$, then $\|W_{\psi,\varphi} f_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to \infty$.

Let $\{f_n\}$ be such a sequence and fix $\varepsilon > 0$. Choose $s \in (0,1)$ such that

$$\begin{aligned} &\mu(z)|\psi''(z)|K(\varphi(z)) < \frac{\varepsilon}{3L}, \\ &\mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{1-|\varphi(z)|^2}K(\varphi(z)) < \frac{\varepsilon}{3C}, \\ &\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z)) < \frac{\varepsilon}{3C}, \end{aligned} \tag{2.26}$$

whenever $|\varphi(z)| > s$, where C is the constant in condition (VII).

Since $\{f_n\}$ converges to zero uniformly on $\{|w| \leq s\}$, so do $\{f_n'\}$ and $\{f_n''\}$. Thus, there exists a positive integer N such that for all $n \geq N$ and all complex numbers w with $|w| \leq s$, $|f_n(w)| < \varepsilon$, $|f_n'(w)| < \varepsilon$, and $|f_n''(w| < \varepsilon$. Hence, with M_1, M_2, and M_3 as in

Theorem 2.1.1, if $n \geq N$ and $|\varphi(z)| \leq s$, then

$$\mu(z)|\psi''(z)|K(\varphi(z))|f_n(\varphi(z))| < M_1\varepsilon,$$

$$\mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{1 - |\varphi(z)|^2}K(\varphi(z))|f_n'(\varphi(z))| < M_2\,\varepsilon,$$

$$\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1 - |\varphi(z)|^2)^2}K(\varphi(z))|f_n''(\varphi(z))| < M_3\,\varepsilon.$$

Therefore, by (2.10), for $|\varphi(z)| \leq s$,

$$\begin{aligned}
\mu(z)|(W_{\psi,\varphi}f_n)''(z)| \leq\ & \mu(z)|f_n(\varphi(z))||\psi''(z)| \\
& + \mu(z)|f_n'(\varphi(z))|\frac{\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|}{1 - |\varphi(z)|^2} \\
& + \mu(z)|f_n''(\varphi(z))|\frac{|\psi(z)\varphi'(z)^2|}{(1 - |\varphi(z)|^2)^2} \\
\leq\ & \frac{1}{\delta}\mu(z)|f_n(\varphi(z))||\psi''(z)|K(\varphi(z)) \\
& + \frac{1}{\delta}\mu(z)|f_n'(\varphi(z))|\frac{\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|}{1 - |\varphi(z)|^2}K(\varphi(z)) \\
& + \frac{1}{\delta}\mu(z)|f_n''(\varphi(z))|\frac{|\psi(z)\varphi'(z)^2|}{(1 - |\varphi(z)|^2)^2}K(\varphi(z)) \\
<\ & \frac{1}{\delta}(M_1 + M_2 + M_3)\varepsilon. \qquad (2.27)
\end{aligned}$$

If $|\varphi(z)| > s$, then by (1.2), condition (VII) for $j = 1, 2$ and inequalities (2.26), for all n

sufficiently large, we have

$$\begin{aligned}\mu(z)|(W_{\psi,\varphi}f_n)''(z)| &= \mu(z)\big|\psi''(z)f_n(\varphi(z))\\ &\quad +f_n'(\varphi(z))\big(2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\big)\\ &\quad +\psi(z)\varphi'(z)^2 f_n''(\varphi(z))\big|\\ &\le L\mu(z)|\psi''(z)|K(\varphi(z))\\ &\quad +C\mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{1-|\varphi(z)|^2}K(\varphi(z))\\ &\quad +C\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z))<\varepsilon. \qquad (2.28)\end{aligned}$$

Finally, since $\{f_n\}$ and $\{f_n'\}$ converge to zero uniformly on compact subsets of $\mathbb{D}$, they converge pointwise. Thus, $|\psi(0)(f_n\circ\varphi)(0)|+|(\psi(f_n\circ\varphi))'(0)|\to 0$, as $n\to\infty$. By this, (2.27) and (2.28), we obtain $\|W_{\psi,\varphi}f_n\|_{\mathcal{Z}_\mu}\to 0$ as $n\to\infty$. Hence $W_{\psi,\varphi}$ is compact. □

2.2 The essential norm of the weighted composition operators mapping into $\mathcal{Z}_\mu$

In this section, we provide an approximation of the essential norm of the weighted composition operators acting on a large class of Banach space X of analytic functions on the open unit disk into the weighted Zygmund space $\mathcal{Z}_\mu$. With $A_j(\psi,\varphi)$ $(j=1,2,3)$ defined as in the statement of Theorem 2.1.2, the following result complements Theorem 3.2 in [8].

Part of the proof of Theorem 2.2.1 below is based on the use of Lemma 1.4.2. We note that while part (c) of Lemma 1.4.2 requires condition (IV), which we do not assume in the statement of the theorem, reading carefully the details of the proof one can see that assumption (2.10) is sufficient to apply the lemma.

Theorem 2.2.1. *Let X be a reflexive Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (V)-(VII)*, together with either* (I) *or* (VIII)*. Let μ be a weight, $\psi\in H(\mathbb{D})$ and*

$\varphi \in S(\mathbb{D})$ *satisfying* (2.10) *and such that* $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ *is bounded. Then*

$$\|W_{\psi,\varphi}\|_e \asymp A_1(\psi,\varphi) + A_2(\psi,\varphi) + A_3(\psi,\varphi).$$

Remark 2.2.1. The assumption that the space X is reflexive is used only to prove the lower estimate of the essential norm.

Proof. We begin by showing that

$$\|W_{\psi,\varphi}\|_e \geq C\big(A_1(\psi,\varphi) + A_2(\psi,\varphi) + A_3(\psi,\varphi)\big).$$

If $\|\varphi\|_\infty < 1$, it follows immediately that $A_j(\psi,\varphi) = 0$ for $j = 1,2,3$. So assume $\|\varphi\|_\infty = 1$, and prove that $A_j(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e$, for $j = 1,2,3$, for some $C > 0$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$ and

$$A_3(\psi,\varphi) := \lim_{n\to\infty} \mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n)).$$

Fix $\varepsilon > 0$ and for each $n \in \mathbb{N}$ choose $f_n \in X$ with $\|f_n\| \leq 1$ satisfying (2.11). Since by the reflexivity assumption, the unit ball is compact under the topology of uniform convergence on compact subsets of $\mathbb{D}$, arguing as in the proof of Theorem 2.1.2, the sequence $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$ and the sequence $\{F_n\}$ defined in (2.12) is bounded in X and converges to zero uniformly on compact subsets of $\mathbb{D}$. The sequence $\{H_n\}$ defined by $H_n = F_n\alpha^2_{\varphi(a_n)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ and by condition (VI) is bounded in X. Recalling $H_n(\varphi(a_n)) = H_n'(\varphi(a_n)) = 0$ and the expression of $H''(\varphi(a_n))$ given in (2.15), we have

$$
\begin{aligned}
|(W_{\psi,\varphi}H_n)''(a_n)| &= \big|\psi''(a_n)H_n(\varphi(a_n)) + 2\psi'(a_n)\varphi'(a_n)H_n'(\varphi(a_n)) \\
&\quad + \psi(a_n)\big(\varphi''(a_n)H_n'(\varphi(a_n)) + \varphi'(a_n)^2 H_n''(\varphi(a_n))\big)\big| \\
&= \frac{|2\psi(a_n)\varphi'(a_n)^2 F_n(\varphi(a_n))|}{(1-|\varphi(a_n)|^2)^2}. \qquad (2.29)
\end{aligned}
$$

Let T be a compact operator, so $W_{\psi,\varphi} - T$ is bounded and by Proposition 1.4.1 and Lemma 1.4.1, $\|TH_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to \infty$. Using (2.29), we have

$$
\begin{aligned}
C\|W_{\psi,\varphi} - T\| &\geq \limsup_{n\to\infty} \|H_n\| \|W_{\psi,\varphi} - T\| \\
&\geq \limsup_{n\to\infty} \|(W_{\psi,\varphi} - T)H_n\|_{\mathcal{Z}_\mu} \\
&= \limsup_{n\to\infty} \|W_{\psi,\varphi}H_n\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi}H_n)''(a_n)| \\
&= \limsup_{n\to\infty} \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2 F_n(\varphi(a_n))|}{(1-|\varphi(a_n)|^2)^2}. \qquad (2.30)
\end{aligned}
$$

From (2.11) and (2.13), we obtain

$$
\begin{aligned}
&\mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}\big(K(\varphi(a_n)) - \varepsilon\big) \\
&\quad \leq \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|f_n(\varphi(a_n))| \\
&\quad \leq \mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}|F_n(\varphi(a_n))|. \qquad (2.31)
\end{aligned}
$$

Using (2.30) and (2.31) and arguing as in the proof of Theorem 2.1.2 we see that

$$\begin{aligned} A_3(\psi,\varphi) &= \lim_{n\to\infty}\mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n)) \\ &\le C\|W_{\psi,\varphi}-T\|. \end{aligned} \tag{2.32}$$

Taking the infimum over all compact operators $T : X \to \mathcal{Z}_\mu$, it follows that

$$A_3(\psi,\varphi) \le C\|W_{\psi,\varphi}\|_e. \tag{2.33}$$

We next show that $A_2(\psi,\varphi) \le C\|W_{\psi,\varphi}\|_e$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$ and

$$A_2(\psi,\varphi) := \lim_{n\to\infty}\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)).$$

As shown in the proof of Theorem 2.1.2, the sequence $\{G_n\}$ defined by $G_n = F_n\alpha_{\varphi(a_n)}$ is bounded in X and converges to zero uniformly on compact subsets of $\mathbb{D}$.

Let $T : X \to \mathcal{Z}_\mu$ be a compact operator. By Proposition 1.4.1 and Lemma 1.4.1, $\|TG_n\|_\mu \to 0$ as $n \to \infty$. Using (2.20), and recalling that $G_n(\varphi(a_n)) = 0$, we have

$$\begin{aligned} C\|W_{\psi,\varphi}-T\| &\ge \limsup_{n\to\infty}\|(W_{\psi,\varphi}-T)G_n\|_{\mathcal{Z}_\mu} \\ &\ge \limsup_{n\to\infty}\|W_{\psi,\varphi}G_n\|_{\mathcal{Z}_\mu} \\ &\ge \limsup_{n\to\infty}\mu(a_n)|(W_{\psi,\varphi}G_n)''(a_n)| \\ &\ge \limsup_{n\to\infty}\mu(a_n)\Big(\big|G_n'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)\big)\big| \\ &\qquad -\psi(a_n)\big|\varphi'(a_n)^2G_n''(\varphi(a_n))\big|\Big). \end{aligned}$$

Hence, by condition (VII) for $j = 2$ and (2.20), we have

$$\limsup_{n\to\infty} \mu(a_n)\Big|G_n'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\big)\Big|$$

$$\leq C\|W_{\psi,\varphi} - T\| + \limsup_{n\to\infty} \mu(a_n)\big|\psi(a_n)\varphi'(a_n)^2 G_n''(\varphi(a_n))\big|.$$

$$\leq C\|W_{\psi,\varphi} - T\|$$

$$+ C\limsup_{n\to\infty} \mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n))|. \tag{2.34}$$

By (2.21), (2.32), and (2.34), it follows that

$$\limsup_{n\to\infty} \mu(a_n)\frac{|(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n))F_n(\varphi(a_n))|}{1-|\varphi(a_n)|^2}$$

$$\leq C\|W_{\psi,\varphi} - T\| + CA_3(\psi,\varphi). \tag{2.35}$$

Fix $\varepsilon > 0$. By (2.35) and arguing as in the proof of Theorem 2.1.2, we obtain

$$\limsup_{n\to\infty} \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n))$$

$$\leq C\|W_{\psi,\varphi} - T\| + C\varepsilon + CA_3(\psi,\varphi).$$

Since ε is arbitrary, combining this with (2.33), we obtain

$$\begin{aligned} A_2(\psi,\varphi) &= \lim_{n\to\infty} \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)) \\ &\leq C\|W_{\psi,\varphi} - T\| + C\|W_{\psi,\varphi}\|_e. \end{aligned}$$

Taking the infimum over all compact operators $T : X \to \mathcal{Z}_\mu$, it follows that

$$A_2(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e. \tag{2.36}$$

Finally, we show that $A_1(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e$. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ and

$$A_1(\psi,\varphi) := \lim_{n\to\infty} \mu(a_n)|\psi''(a_n)|K(\varphi(a_n)).$$

Let $T : X \to \mathcal{Z}_\mu$ be a compact operator. Then, by Lemma 1.4.1 applied to the sequence $\{F_n\}$, we have $\|TF_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}
C\|W_{\psi,\varphi} - T\| &\geq \limsup_{n\to\infty} \|F_n\|\|W_{\psi,\varphi} - T\| \\
&\geq \limsup_{n\to\infty} \|(W_{\psi,\varphi} - T)F_n\|_{\mathcal{Z}_\mu} \\
&= \limsup_{n\to\infty} \|W_{\psi,\varphi}F_n\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi}F_n)''(a_n)| \\
&\geq \limsup_{n\to\infty} \mu(a_n)\Big(\big|\psi''(a_n)F_n(\varphi(a_n))\big| \\
&\qquad - \big|\big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\big)F_n'(\varphi(a_n))\big| \\
&\qquad - \big|\psi(a_n)\varphi'(a_n)^2F_n''(\varphi(a_n))\big|\Big).
\end{aligned} \tag{2.37}$$

By (2.37) and condition (VII), we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \mu(a_n)|\psi''(a_n)F_n(\varphi(a_n))| \\
&\qquad \le C\|W_{\psi,\varphi} - T\| \\
&\qquad\quad + \limsup_{n\to\infty} \mu(a_n)|F_n'(\varphi(a_n))(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n))| \\
&\qquad\quad + \limsup_{n\to\infty} \mu(a_n)|\psi(a_n)\varphi'(a_n)^2 F_n''(\varphi(a_n))| \\
&\qquad \le C\|W_{\psi,\varphi} - T\| \\
&\qquad\quad + C\limsup_{n\to\infty} \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2} K(\varphi(a_n)) \\
&\qquad\quad + C\limsup_{n\to\infty} \mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2} K(\varphi(a_n)) \\
&\qquad \le C\big(\|W_{\psi,\varphi} - T\| + A_2(\psi,\varphi) + A_3(\psi,\varphi)\big).
\end{aligned} \tag{2.38}
$$

Fix $\varepsilon > 0$. Again, arguing as in the proof of Theorem 2.1.2, and using (2.36), and (2.33), from (2.38) we obtain

$$
A_1(\psi,\varphi) \le C\big(\|W_{\psi,\varphi} - T\| + \varepsilon + \|W_{\psi,\varphi}\|_e + \|W_{\psi,\varphi}\|_e\big).
$$

Since ε is arbitrary, taking the infimum over all compact operators $T : X \to \mathcal{Z}_\mu$, we have $A_1(\psi,\varphi) \le C\|W_{\psi,\varphi}\|_e$, which, combined with (2.36) and (2.33), yields the desired lower estimate on the essential norm.

To prove the upper estimate, fix $\varepsilon > 0$ and $s \in (0,1)$ and choose $r \in (0,1)$ as in parts (b) and (c) of Lemma 1.4.2 and Remark 1.4.2. Since $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is bounded and, by condition (V), the operator $\mathcal{T}_r : X \to X$ is compact, the composition product

$W_{\psi,\varphi}\mathcal{T}_r : X \to \mathcal{Z}_\mu$ is also compact. Thus,

$$\begin{aligned} \|W_{\psi,\varphi}\|_e &\leq \|W_{\psi,\varphi} - W_{\psi,\varphi}\mathcal{T}_r\| \\ &= \sup_{\|f\|\leq 1} \|W_{\psi,\varphi}(I - \mathcal{T}_r)f\|_{\mathcal{Z}_\mu} \\ &\leq L_0 + L_1 + L_2 + L_3, \end{aligned} \tag{2.39}$$

where

$$
\begin{aligned}
L_0 &= \sup_{\|f\|\le 1} |\psi(0)(I-\mathcal{T}_r)f(\varphi(0))| + |\psi'(0)(I-\mathcal{T}_r)f(\varphi(0)) \\
&\qquad + \psi(0)\varphi'(0)((I-\mathcal{T}_r)f)'(\varphi(0))|, \\
L_1 &= \sup_{\|f\|\le 1}\sup_{z\in\mathbb{D}} \mu(z)|\psi''(z)(I-\mathcal{T}_r)f(\varphi(z))|, \\
&\le \sup_{\|f\|\le 1}\sup_{|\varphi(z)|\le s} \mu(z)|\psi''(z)(I-\mathcal{T}_r)f(\varphi(z))| \\
&\qquad + \sup_{\|f\|\le 1}\sup_{|\varphi(z)|> s} \mu(z)|\psi''(z)(I-\mathcal{T}_r)f(\varphi(z))|, \\
L_2 &= \sup_{\|f\|\le 1}\sup_{z\in\mathbb{D}} \mu(z)|\big(2\psi'(z)\varphi'(z)+\varphi''(z)\psi(z)\big)((I-\mathcal{T}_r)f)'(\varphi(z))|, \\
&\le \sup_{\|f\|\le 1}\sup_{|\varphi(z)|\le s} \mu(z)|\big(2\psi'(z)\varphi'(z)+\varphi''(z)\psi(z)\big)((I-\mathcal{T}_r)f)'(\varphi(z))| \\
&\qquad + \sup_{\|f\|\le 1}\sup_{|\varphi(z)|> s} \mu(z)|\big(2\psi'(z)\varphi'(z) \\
&\qquad\quad + \varphi''(z)\psi(z)\big)((I-\mathcal{T}_r)f)'(\varphi(z))|, \\
L_3 &= \sup_{\|f\|\le 1}\sup_{z\in\mathbb{D}} \mu(z)|\varphi'(z)^2((I-\mathcal{T}_r)f)''(\varphi(z))| \\
&\le \sup_{\|f\|\le 1}\sup_{|\varphi(z)|\le s} \mu(z)|\varphi'(z)^2((I-\mathcal{T}_r)f)''(\varphi(z))| \\
&\qquad + \sup_{\|f\|\le 1}\sup_{|\varphi(z)|> s} \mu(z)|\big(2\psi'(z)\varphi'(z) \\
&\qquad\quad +\varphi''(z)\big)((I-\mathcal{T}_r)f)'(\varphi(z))|.
\end{aligned}
$$

Then,

$$\begin{aligned} L_0 &= \sup_{\|f\|\leq 1} |\psi(0)(I-\mathcal{T}_r)f(\varphi(0))| \\ &\qquad + |\psi'(0)(I-\mathcal{T}_r)f(\varphi(0)) + \psi(0)\varphi'(0)((I-\mathcal{T}_r)f)'(\varphi(0))| \\ &\leq |\psi(0)|\varepsilon + |\psi'(0)|\varepsilon \\ &\qquad +|\psi(0)\varphi'(0)((I-\mathcal{T}_r)f)'(\varphi(0))|K(\varphi(0))^{-1}K(\varphi(0)) \\ &\leq \big(|\psi(0)| + |\psi'(0)| + |\psi(0)\varphi'(0)|K(\varphi(0))\big)\varepsilon. \end{aligned} \tag{2.40}$$

By condition (2.10), Lemma 1.4.2, Remark 1.4.2, Theorem 2.1.1, and condition (VII), we have

$$
\begin{aligned}
L_1 &\leq \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} \mu(z)|\psi''(z)K(\varphi(z))||K(\varphi(z))^{-1}(I-\mathcal{T}_r)f(\varphi(z))| \\
&\qquad + \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|> s} \mu(z)|\psi''(z)K(\varphi(z))||K(\varphi(z))^{-1}(I-\mathcal{T}_r)f(\varphi(z))| \\
&\leq M_1 \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} C|(I-\mathcal{T}_r)f(\varphi(z))| + C \sup_{|\varphi(z)|> s} \mu(z)|\psi''(z)K(\varphi(z))| \\
&\leq CM_1\varepsilon + C \sup_{|\varphi(z)|> s} \mu(z)|\psi''(z)|K(\varphi(z)), \\
L_2 &= \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\varphi''(z)\psi(z)|}{1-|\varphi(z)|^2}K(\varphi(z)) \\
&\qquad\qquad \times K(\varphi(z))^{-1}|((I-\mathcal{T}_r)f)'(\varphi(z))|(1-|\varphi(z)|^2) \\
&\qquad + \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|> s} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\varphi''(z)\psi(z)|}{1-|\varphi(z)|^2}K(\varphi(z)) \\
&\qquad\qquad \times K(\varphi(z))^{-1}|((I-\mathcal{T}_r)f)'(\varphi(z))|(1-|\varphi(z)|^2) \\
&\leq M_2\,\varepsilon + C \sup_{|\varphi(z)|> s} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\varphi''(z)\psi(z)|}{1-|\varphi(z)|^2}K(\varphi(z)), \\
L_3 &\leq \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} \mu(z)\frac{|\psi(z)\varphi'(z^2)|}{(1-|\varphi(z)|^2)^2}K(\varphi(z)) \\
&\qquad\qquad \times K(\varphi(z))^{-1}|((I-\mathcal{T}_r)f)''(\varphi(z))|(1-|\varphi(z)|^2)^2 \\
&\qquad + \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|> s} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z)) \\
&\qquad\qquad \times K(\varphi(z))^{-1}|((I-\mathcal{T}_r)f)''(\varphi(z))|(1-|\varphi(z)|^2)^2 \\
&\leq M_3\,\varepsilon + C \sup_{|\varphi(z)|> s} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z)),
\end{aligned}
$$

Therefore, combining (2.39), (2.40) and (2.41), we obtain

$$\|W_{\psi,\varphi}\|_e \leq C\Big(\varepsilon + \sup_{|\varphi(z)|>s} \mu(z)|\psi''(z)|K(\varphi(z))$$

$$+ \sup_{|\varphi(z)|>s} \mu(z)\frac{|2\psi'(z)\varphi'(z) + \varphi''(z)\psi(z)|}{1-|\varphi(z)|^2}K(\varphi(z))$$

$$+ \sup_{|\varphi(z)|>s} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z))\Big).$$

If $\|\varphi\|_\infty < 1$ and since ε is arbitrary, we see that $\|W_{\psi,\varphi}\|_e = 0$ and the operator is compact. If $\|\varphi\|_\infty = 1$ and since ε is arbitrary, letting $s \to 1$, we obtain

$$\|W_{\psi,\varphi}\|_e \leq C\big(A_1(\psi,\varphi) + A_2(\psi,\varphi) + A_3(\psi,\varphi)\big).$$

□

2.3 Boundedness and compactness of weighted composition operators mapping into the little weighted Zygmund spce $\mathcal{Z}_{\mu,0}$

We conclude this chapter by focusing on the weighted composition operators mapping a Banach space of analytic functions into $\mathcal{Z}_{\mu,0}$. In particular, we provide equivalent conditons for the boundedness and the compactness of weighted composition operators acting on a large class of Banach spaces X of analytic functions on $\mathbb{D}$ into the little weighted Zygmund spce $\mathcal{Z}_{\mu,0}$, where μ is a given weight. The following result complements Theorem 3.4 in [8].

Theorem 2.3.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (II), (VI) *and* (VII). *Suppose μ is a weight, $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ satisfying* (2.10). *The following conditions are equivalent:*

(a) $W_{\psi,\varphi} : X \to \mathcal{Z}_{\mu,0}$ *is compact.*

(b) $W_{\psi,\varphi} : X \to \mathcal{Z}_{\mu,0}$ *is bounded.*

(c) $\lim_{|z|\to 1} \mu(z)|\psi''(z)|K(\varphi(z)) = 0,$

$$\lim_{|z|\to 1} \mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{1-|\varphi(z)|^2}K(\varphi(z)) = 0,$$

$$\lim_{|z|\to 1} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^2}K(\varphi(z)) = 0.$$

Proof. The implication $(a) \Rightarrow (b)$ follows from the fact that every compact operator is bounded. To prove $(b) \Rightarrow (c)$, suppose $W_{\psi,\varphi} : X \to \mathcal{Z}_{\mu,0}$ is bounded.

Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|a_n| \to 1$ as $n \to \infty$, fix $f \in X$ such that $\|f\| \leq 1$, and for $n \in \mathbb{N}$ and $z \in \mathbb{D}$, let

$$g_n(z) = \alpha^2_{\varphi(a_n)}(z)f(z), \quad \text{and} \quad h_n(z) = \alpha_{\varphi(a_n)}(z)f(z).$$

By condition (VI), $\{g_n\}$ and $\{h_n\}$ are bounded sequences in X. Note that $g_n(\varphi(a_n)) = g'_n(\varphi(a_n)) = 0$, and $g''_n(\varphi(a_n)) = -\frac{2f(\varphi(a_n))}{(1-|\varphi(a_n)|^2)^2}$. Since $W_{\psi,\varphi}g_n \in \mathcal{Z}_{\mu,0}$, as $n \to \infty$,

$$\mu(a_n)\frac{|2\psi(a_n)\varphi'(a_n)^2 f(\varphi(a_n))|}{(1-|\varphi(a_n)|^2)^2} = \mu(a_n)|(W_{\psi,\varphi}g_n)''(a_n)| \to 0.$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2), we have

$$\lim_{n\to\infty} \mu(a_n)\frac{|\psi(a_n)\varphi'(a_n)^2|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n)) = 0. \tag{2.41}$$

Similarly, since $W_{\psi,\varphi}h_n \in \mathcal{Z}_{\mu,0}$, $h_n(\varphi(a_n)) = 0$, and

$$h_n'(\varphi(a_n)) = \frac{f(\varphi(a_n))}{|\varphi(a_n)|^2 - 1},$$

it follows that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mu(a_n)|(W_{\psi,\varphi}h_n)''(a_n)| \\ &= \lim_{n\to\infty} \mu(a_n)\Big| - \frac{f(\varphi(a_n))}{1-|\varphi(a_n)|^2}\Big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\Big) \\ &\qquad + \psi(a_n)\varphi'(a_n)^2 h_n''(\varphi(a_n))\Big|. \end{aligned}$$

Thus, by (VII) for $j = 2$ and (2.41), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \mu(a_n)\left|\frac{f(\varphi(a_n))}{1-|\varphi(a_n)|^2}\Big(2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)\Big)\right| \\ &= \lim_{n\to\infty} \mu(a_n)\big|\psi(a_n)\varphi'(a_n)^2 h_n''(\varphi(a_n))\big| \\ &\le \lim_{n\to\infty} C\mu(a_n)\frac{\big|\psi(a_n)\varphi'(a_n)^2\big|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n)) \\ &= 0. \end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \le 1$, by (1.2), we have

$$\lim_{n\to\infty} \mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n) + \psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n)) = 0. \tag{2.42}$$

Again, with f as above, since $W_{\psi,\varphi}f \in \mathcal{Z}_{\mu,0}$,

$$\begin{aligned}\lim_{n\to\infty}\mu(a_n)|(W_{\psi,\varphi}f)''(a_n)| &= \lim_{n\to\infty}\mu(a_n)\big|\psi''(a_n)f(\varphi(a_n))\\ &\quad +f'(\varphi(a_n))\big(2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)\big)\\ &\quad +\psi(a_n)f''(\varphi(a_n))\varphi'(a_n)^2\big|\\ &= 0.\end{aligned}$$

By this, (VII) for $j = 1, 2$, (2.42), and (2.41) we obtain

$$\begin{aligned}&\lim_{n\to\infty}\mu(a_n)\big|\psi''(a_n)f(\varphi(a_n))\big|\\ &\quad\le \lim_{n\to\infty}C\mu(a_n)\frac{|2\psi'(a_n)\varphi'(a_n)+\psi(a_n)\varphi''(a_n)|}{1-|\varphi(a_n)|^2}K(\varphi(a_n))\\ &\qquad+\lim_{n\to\infty}C\mu(a_n)\frac{\big|\psi(a_n)\varphi'(a_n)^2\big|}{(1-|\varphi(a_n)|^2)^2}K(\varphi(a_n))\\ &\quad=0.\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \le 1$, by have

$$\lim_{n\to\infty}\mu(a_n)|\psi''(a_n)|K(\varphi(a_n)) = 0.$$

To prove that $(c) \Rightarrow (a)$, suppose (c) holds. By Theorem 2.1.2, the operator $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is compact. Thus, to prove that $W_{\psi,\varphi} : X \to \mathcal{Z}_{\mu,0}$ is compact, it suffices to show that $W_{\psi,\varphi}f \in \mathcal{Z}_{\mu,0}$ for any $f \in X$.

Let $f \in X$. By (1.2) and (VII) for $j = 1, 2$, we have

$$\begin{aligned}
\mu(z)|(W_{\psi,\varphi}f)''(z)| &= \mu(z)\big|\psi''(z)f(\varphi(z)) \\
&\quad + f'(\varphi(z))\big(2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big) \\
&\quad + \psi(z)f''(\varphi(z))\varphi'(z)^2\big| \\
&\leq \|f\|\mu(z)|\psi''(z)|K(\varphi(z)) \\
&\quad + C\|f\|\mu(z)\frac{\big|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big|}{1 - |\varphi(z)|^2}K(\varphi(z)) \\
&\quad + C\|f\|\mu(z)\frac{\big|\psi(z)\varphi'(z)^2\big|}{(1 - |\varphi(z)|^2)^2}K(\varphi(w)) \\
&\to 0,
\end{aligned}$$

as $|z| \to 1$. Thus, $W_{\psi,\varphi}f \in \mathcal{Z}_{\mu,0}$, as desired. $\square$

Chapter 3: Weighted composition operators into the weighted-type Banach spaces $\mathcal{V}_n$ for $n \geq 3$

In this chapter, we characterize the bounded and compact weighted composition operators from a large class of Banach space X of analytic functions on the open unit disk into the weighted-type Banach spaces $\mathcal{V}_n$ for $n \geq 3$. Under more restrictive conditions, we provide an approximation of the essential norm of such operators. We also show that all bounded weighted composition operators from X to the little weigted-type $\mathcal{V}_{n,0}$ spaces are compact and characterize such operators.

We shall make use of the following result that was proven by Stević.

Lemma 3.0.1. ([24], Lemma 4) *For $\psi, f \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$, and $z \in \mathbb{D}$,*

$$\begin{aligned}(W_{\psi,\varphi}f)^{(n)}(z) &= (\psi(f \circ \varphi))^{(n)}(z) \\ &= \sum_{k=0}^{n} f^{(k)}(\varphi(z)) \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \varphi''(z), ..., \varphi^{(\ell-k+1)}(z)\big),\end{aligned}$$

where

$$B_{\ell,k}\big(\varphi'(z), \varphi''(z), ..., \varphi^{(\ell-k+1)}(z)\big) = \sum_{k_1,k_2,...,k_\ell} \frac{\ell!}{k_1!k_2!...k_\ell!} \prod_{j=1}^{l} \frac{\varphi^{(j)}(z)}{j!},$$

and the sum is taken over all $k_1, \ldots, k_\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ satisfying $k = k_1 + k_2 + \cdots + k_\ell$, and $k_1 + 2k_2 + \cdots + \ell k_\ell = \ell$.

Remark 3.0.1. A straightforward computation yields

$$B_{n,n}(\varphi'(z)) = (\varphi'(z))^n, \quad \text{and for} \quad n \geq 2$$

$$B_{n,n-1}(\varphi'(z), \varphi''(z)) = \frac{n(n-1)}{2}(\varphi'(z))^{n-2}\varphi''(z).$$

Fix $a \in \mathbb{D}$ and recall the disk automorphism that interchanges 0 and a:

$$\alpha_a(z) = \frac{a-z}{1-\overline{a}z} \text{ for } z \in \mathbb{D}.$$

Lemma 3.0.2. *Fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \leq 1$. For $k \in \mathbb{N}$ and $\varphi \in S(\mathbb{D})$, let us consider the function defined on $\mathbb{D}$ by*

$$H_{k,w}(z) = [\alpha_{\varphi(w)}(z)]^k f(z). \tag{3.1}$$

Then for $j = 0, 1, ..., k-1$, $H_{k,w}^{(j)}(\varphi(w)) = 0$, and

$$H_{k,w}^{(k)}(\varphi(w)) = \frac{(-1)^k k! f(\varphi(w))}{(1-|\varphi(w)|^2)^k}.$$

Proof. For $k = 1$ it is immediate to see that $H_{k,w}(\varphi(w)) = 0$ and

$$H'_{k,w}(\varphi(w)) = -\frac{f(\varphi(w))}{1-|\varphi(w)|^2}.$$

Arguing by induction, assume the result holds for some $k \geq 1$. Then

$$H_{k+1,w}(z) = H_{k,w}(z)\alpha_{\varphi(w)}(z).$$

Differentiating n times, substituting the value $\varphi(w)$ and using the inductive hypothesis, we obtain

$$\begin{aligned}(H_{k+1,w})^{(k+1)}(\varphi(w)) &= \sum_{j=0}^{k+1}\binom{k+1}{j}H_{k,w}^{(j)}(\varphi(w))\alpha_{\varphi(w)}^{(k+1-j)}(\varphi(w))\\ &= \binom{k+1}{k}H_{k,w}^{(k)}(\varphi(w))\alpha'_{\varphi(w)}(\varphi(w))\\ &\quad + \binom{k+1}{k+1}H_{k,w}^{(k+1)}(\varphi(w))\alpha_{\varphi(w)}(\varphi(w))\\ &= (k+1)\frac{(-1)^k k! f(\varphi(w))}{(1-|\varphi(w)|^2)^k}\Big(\frac{-1}{1-|\varphi(w)|^2}\Big)\\ &= \frac{(-1)^{k+1}(k+1)! f(\varphi(w))}{(1-|\varphi(w)|^2)^{k+1}},\end{aligned}$$

proving the result. □

3.1 Bounded and compact weighted composition operators mapping into weighted-type Banach spaces $\mathcal{V}_n$

We begin the section with a characterization of boundedness that extends Theorem 3.1 in [2]. We use Lemma 3.0.1 to determine the quantities that are required for such characterization.

Theorem 3.1.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ and let K be as in* (1.1) *satisfying conditions* (VI) *and* (VII)*. Let μ be a weight on $\mathbb{D}$, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is bounded if and only if for each $k = 0, 1, ..., n$ the quantity A_k defined below is finite:*

$$A_k := \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\varphi''(z),...,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k}K(\varphi(z)).$$

Proof. First, we prove that A_n is finite. Fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \le 1$ and consider the function

$$H_{n,w}(z) = [\alpha_{\varphi(w)}(z)]^n f(z).$$

By a repeated application of condition (VI), we see that $H_{n,w} \in X$ and

$$\|H_{n,w}\| = \|\alpha_{\varphi(w)}^n f\| \leq C\|f\| \leq C.$$

Thus, by Lemma 3.0.1, Lemma 3.0.2, and Remark 3.0.1

$$\begin{aligned}
&(1-|w|^2)|(W_{\psi,\varphi}H_{n,w})^{(n)}(w)| \\
&\quad = \mu(w)\big|\big(\psi'(g_w\circ\varphi)+\psi\varphi'(g_w'\circ\varphi)\big)'(w)\big| \\
&\quad = (1-|w|^2)\Big|\sum_{k=0}^{n} H_{n,w}^{(k)}(\varphi(w)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\ldots,\varphi^{(\ell-k+1)}(w)\big)\Big| \\
&\quad = (1-|w|^2)\Big|\frac{n!f(\varphi(w))}{(1-|\varphi(w)|^2)^n}\psi(w)B_{n,n}\big(\varphi'(w)\big)\Big| \\
&\quad = (1-|w|^2)\frac{|n!\psi(w)(\varphi'(w))^n f(\varphi(w))|}{(1-|\varphi(w)|^2)^n}.
\end{aligned}$$

Thus, since $W_{\psi,\varphi}$ is bounded, we have

$$\begin{aligned}
(1-|w|^2)\frac{|n!\psi(w)(\varphi'(w))^n f(\varphi(w))|}{(1-|\varphi(w)|^2)^n} &\leq \|W_{\psi,\varphi}H_{n,w}\|_{\mathcal{V}_n} \\
&\leq \|H_{n,w}\|\|W_{\psi,\varphi}\| \\
&\leq C\|W_{\psi,\varphi}\|.
\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2) we obtain

$$A_n = \sup_{w \in \mathbb{D}} (1 - |w|^2) \frac{|\psi(w)\varphi'(w)^n|}{(1 - |\varphi(w)|^2)^n} K(\varphi(w)) \leq C \|W_{\psi,\varphi}\|.$$

We next prove that A_{n-1} is finite by fixing $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \leq 1$ and considering the function $H_{n-1,w}(z) = [\alpha_{\varphi(w)}(z)]^{n-1} f(z)$. Then, for $k = 0, 1, ..., n-2$,By condition (VI), we see that $H_{n-1,w} \in X$ and

$$\|H_{n-1,w}\| = \|\alpha_{\varphi(w)}^{n-1} f\| \leq C\|f\| \leq C.$$

Thus, by Lemma 3.0.1, Lemma 3.0.2 and Remark 3.0.1

$$
\begin{aligned}
&(1-|w|^2)|(W_{\psi,\varphi}H_{n-1,w})^{(n)}(w)| \\
&\quad = (1-|w|^2)\Big|\sum_{k=0}^{n} H_{n-1,w}^{(k)}(\varphi(w)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\varphi''(w),...,\varphi^{(\ell-k+1)}(w)\big)\Big| \\
&\quad = (1-|w|^2)\Big|H_{n-1,w}^{(n-1)}(\varphi(w))\binom{n}{n-1}\psi'(w)B_{n-1,n-1}(\varphi'(w)) \\
&\qquad + H_{n-1,w}^{(n-1)}(\varphi(w))\binom{n}{n}\psi(w)B_{n,n-1}(\varphi'(w),\varphi''(w)) \qquad (3.2) \\
&\qquad + H_{n-1,w}^{(n)}(\varphi(w))\binom{n}{n}\psi(w)B_{n,n}(\varphi'(w))\Big| \\
&\quad = (1-|w|^2)\Big|H_{n-1,w}^{(n-1)}(\varphi(w))\Big(n\psi'(w)(\varphi'(w))^{n-1} \\
&\qquad + \psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)\Big) \\
&\qquad + H_{n-1,w}^{(n)}(\varphi(w))\psi(w)(\varphi'(w))^{n}\Big|.
\end{aligned}
$$

Thus, by the triangle inequality and the boundedness of the operator $W_{\psi,\varphi}$, from (3.2) we obtain

$$
\begin{aligned}
&(1-|w|^2)\Big|H_{n-1,w}^{(n-1)}(\varphi(w))\Big(n\psi'(w)(\varphi'(w))^{n-1} \\
&\qquad\qquad + \psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)\Big)\Big| \\
&\qquad \leq C\|W_{\psi,\varphi}\| + (1-|w|^2)\Big|H_{n-1,w}^{(n)}(\varphi(w))\psi(w)(\varphi'(w))^{n}\Big|.
\end{aligned}
$$

Using Lemma 3.0.2, and condition (VII) applied to $j = n$, we obtain

$$(1-|w|^2)\frac{(n-1)!|f(\varphi(w))|}{(1-|\varphi(w)|^2)^{n-1}}\Big|n\psi'(w)(\varphi'(w))^{n-1}$$

$$+\psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)\Big|$$

$$\leq C\|W_{\psi,\varphi}\| + C(1-|w|^2)\frac{|\psi(w)(\varphi'(w))^n|}{(1-|\varphi(w)|^2)^n}K(\varphi(w)).$$

After dividing by $(n-1)!$ and modifying the constant C accordingly, taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2) we obtain

$$(1-|w|^2)\frac{|n\psi'(w)(\varphi'(w))^{n-1}+\psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)|}{(1-|\varphi(w)|^2)^{n-1}}K(\varphi(w))$$

$$\leq C\|W_{\psi,\varphi}\| + C(1-|w|^2)\frac{|\psi(w)(\varphi'(w))^n|}{(1-|\varphi(w)|^2)^n}K(\varphi(w)).$$

Taking the supremum over all $w \in \mathbb{D}$, we have

$$A_{n-1} \leq C\|W_{\psi,\varphi}\| + CA_n < \infty.$$

By applying a similar process for each m from $n-2$ down to 1 and considering the functions $H_{m,w}$, we obtain that A_m is finite.

Finally, to prove that A_0 is finite, again fix $w \in \mathbb{D}$ and $f \in X$ such that $\|f\| \leq 1$. By

Lemma 3.0.1, we have

$$
\begin{aligned}
&(1-|w|^2)|(W_{\psi,\varphi}f)^{(n)}(w)| \\
&\quad = (1-|w|^2)\Big|\sum_{k=0}^{n} f^{(k)}(\varphi(w)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\ldots,\varphi^{(\ell-k+1)}(w)\big)\Big| \\
&\quad = (1-|w|^2)\Big|f(\varphi(w))\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,0}\big(\varphi'(w),\ldots,\varphi^{(\ell+1)}(w)\big) \\
&\qquad + \sum_{k=1}^{n} f^{(k)}(\varphi(w))\sum_{\ell=k}^{n}\binom{n}{l}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\ldots,\varphi^{(\ell-k+1)}(w)\big)\Big|.
\end{aligned}
$$

From this, by the boundedness of the operator, and condition (VII), we obtain

$$
\begin{aligned}
&(1-|w|^2)\Big|f(\varphi(w))\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,0}\big(\varphi'(w),\ldots,\varphi^{(\ell+1)}(w)\big)\Big| \\
&\quad \le C\|W_{\psi,\varphi}\| + (1-|w|^2)\sum_{k=1}^{n}|f^{(k)}(\varphi(w))|\Big| \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\ldots,\varphi^{(\ell-k+1)}(w)\big)\Big| \\
&\quad \le C\|W_{\psi,\varphi}\| + C\sum_{k=1}^{n}(1-|w|^2) \\
&\qquad \times \frac{\big|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\ldots,\varphi^{(\ell)}(w)\big)\big|}{(1-|\varphi(w)|^2)^k}K(\varphi(w)) \\
&\quad \le C\|W_{\psi,\varphi}\| + C\sum_{k=1}^{n}A_k.
\end{aligned}
$$

Hence, taking the supremum over all $f \in X$ with $\|f\| \leq 1$ and using (1.2) followed by the supremum over all $w \in \mathbb{D}$, we obtain $A_0 < \infty$, completing the proof of the necessity.

Conversely, suppose $A_k < \infty$, for all $k = 0, 1, \ldots, n$. Let $f \in X$ with $\|f\| \leq 1$, and fix $w \in \mathbb{D}$. Then, by (1.1), condition (VII), Lemma 3.0.1, and Remark 3.0.1 we have

$$(1 - |w|^2)|(W_{\psi,\varphi} f)^{(n)}(w)| = (1 - |w|^2)\Big|\sum_{k=0}^{n} f^{(k)}(\varphi(w))$$

$$\times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(w) B_{\ell,k}\big((\varphi'(w), \ldots, \varphi^{(\ell-k+1)}(w)\big)\Big|$$

$$\leq C \sum_{k=0}^{n} (1 - |w|^2)$$

$$\times \frac{\big|\sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(w) B_{\ell,k}\big(\varphi'(w), \ldots, \varphi^{(\ell-k+1)}(w)\big)\big|}{(1 - |\varphi(w)|^2)^k} K(\varphi(w))$$

$$\leq C \sum_{k=0}^{n} A_k.$$

Taking the supremum over all $w \in \mathbb{D}$, we obtain

$$\sup_{w \in \mathbb{D}} (1 - |w|^2)|(W_{\psi,\varphi} f)^{(n)}(w)| < \infty. \tag{3.3}$$

On the other hand, by (1.1), and condition (VII), we have

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \left|(W_{\psi,\varphi} f)^{(k)}(0)\right| \\
&\quad \le \sum_{k=0}^{n-1} |f^{(k)}(\varphi(0))| \left| \sum_{\ell=k}^{n-1} \binom{n-1}{l} \psi^{(n-1-l)}(0) B_{\ell,k}\big(\varphi'(0), \dots, \varphi^{(\ell-k+1)}(0)\big) \right| \\
&\quad \le C \sum_{k=0}^{n-1} \frac{\left| \sum_{\ell=k}^{n-1} \binom{n-1}{\ell} \psi^{(n-1-\ell)}(0) B_{\ell,k}\big(\varphi'(0), \dots, \varphi^{(\ell-k+1)}(0)\big) \right|}{(1-|\varphi(0)|^2)^k} K(\varphi(0)) \\
&\quad \le C \sum_{k=0}^{n-1} A_k, \qquad (3.4)
\end{aligned}
$$

which is finite. Thus by the definition of norm in $\mathcal{V}_n$, combining (3.3) and (3.4), we see that $\|W_{\psi,\varphi} f\|_{\mathcal{V}_n}$ is finite, proving the boundedness of $W_{\psi,\varphi}$. □

Corollary 3.1.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ and let K be as in* (1.1) *satisfying conditions* (VI) *and* (VII). *Let μ be a weight on $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then $M_\psi : X \to \mathcal{V}_n$ is bounded if and only if for each $k = 0, 1, ..., n$ the quantity $A_{M,k}$ defined below is finite:*

$$A_{M,k} := \sup_{z\in\mathbb{D}} \frac{\binom{n}{k}\psi^{(n-k)}(z)|}{(1-|z|^2)^{k-1})^k} K(z).$$

Corollary 3.1.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ and let K be as in* (1.1) *satisfying conditions* (VI) *and* (VII). *Let μ be a weight on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then $C_\varphi : X \to \mathcal{V}_n$ is bounded if and only if the quantity $A_{C,n}$ defined below is finite:*

$$A_{C,n} := \sup_{z\in\mathbb{D}} (1-|z|^2) \frac{|\varphi'(z)|^n}{(1-|\varphi(z)|^2)^n)^k} K(\varphi(z)).$$

Next, we turn our attention to the study of the compact weighted composition operators from a Banach space X of analytic functions into the weighted-type Banach spaces $\mathcal{V}_n$ for $n \geq 2$ under certain conditions on X. The following result complements Theorem 3.2 in [2].

Theorem 3.1.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ with K be as in (1.1) and satisfying conditions* (II), (VI), *and* (VII). *Suppose $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is bounded and suppose*

$$\delta := \inf_{z\in\mathbb{D}} K(\varphi(z)) > 0. \tag{3.5}$$

Then $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is compact if and only if for each $k = 0, 1, 2, ...n$ the quantity $\mathcal{A}_k(\psi, \varphi)$ defined by

$$\mathcal{A}_k(\psi,\varphi) := \lim_{|\varphi(z)|\to 1}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k}K(\varphi(z))$$

is zero.

Proof. Suppose $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is compact. If $\|\varphi\|_\infty < 1$, then $\mathcal{A}_k(\psi, \varphi) = 0$ for all $k = 0, 1, ..., n$, and we are done. So assume $\|\varphi\|_\infty = 1$. Let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $1/2 < |\varphi(a_m)| \to 1$ as $m \to \infty$ and

$$\mathcal{A}_n(\psi,\varphi) = \lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)B_{n,n}(\varphi'(a_m))|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)).$$

Thus, by Remark (3.0.1)

$$\mathcal{A}_n(\psi,\varphi) = \lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)).$$

Fix $\varepsilon > 0$. By (1.2), for each $m \in \mathbb{N}$ there exists $f_m \in X$ with $\|f_m\| \leq 1$ such that

$$|f_m(\varphi(a_m))| > K(\varphi(a_m)) - \varepsilon. \tag{3.6}$$

Since the sequence $\{\|f_m\|\}$ is bounded and by Proposition 1.4.1, the mapping $z \mapsto K(z)$ is bounded on compact subsets of $\mathbb{D}$. Using (1.2) we see that $\{f_m\}$ is uniformly bounded on compact sets.

For each $m \in \mathbb{N}$ and $z \in \mathbb{D}$, define

$$F_m(z) = \frac{\big(1 - |\varphi(a_m)|^2\big)z}{1 - \overline{\varphi(a_m)}z} f_m(z) = \big(\varphi(a_m) - \alpha_{\varphi(a_m)}(z)\big) f_m(z). \tag{3.7}$$

Then

$$F_m(\varphi(a_m)) = \varphi(a_m) f_m(\varphi(a_m)), \tag{3.8}$$

so that $|F_m(\varphi(a_m))| \leq |f_m(\varphi(a_m))|$. By (1.2), for $z \in \mathbb{D}$ we have

$$|F_m(z)| \leq \frac{1 - |\varphi(a_m)|^2}{1 - |z|} K(z).$$

Thus, F_m converges to zero uniformly on compact subsets of $\mathbb{D}$ since, as noted above, $z \mapsto K(z)$ is bounded on compact subsets of $\mathbb{D}$.

By (VI), we see that $F_m \in X$ and

$$\begin{aligned} \|F_m\| &\leq |\varphi(a_m)|\|f_m\| + \|\alpha_{\varphi(a_m)} f_m\| \\ &\leq \|f_m\| + C\|f_m\| \\ &\leq C\|f_m\| \\ &\leq C. \end{aligned} \tag{3.9}$$

For $m \in \mathbb{N}$ and $z \in \mathbb{D}$, define

$$G_{n,m}(z) = \big(\alpha_{\varphi(a_m)}(z)\big)^n F_m(z). \tag{3.10}$$

By a repeated application of condition (VI), we see that $G_{n,m} \in X$ and

$$\|G_{n,m}\| = \|\alpha^n_{\varphi(a_m)} F_m\| \leq C\|F_m\| \leq C,$$

and by Lemma 3.0.2, for $j = 0, 1, ..., n-1$,

$$G^{(j)}_{n,m}(\varphi(a_m)) = 0 \quad \text{and} \quad G^{(n)}_{n,m}(\varphi(a_m)) = \frac{n! F_m(\varphi(a_m))}{(1 - |\varphi(a_m)|^2)^n}. \tag{3.11}$$

Since $G_{n,m}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, by Lemma 1.4.1 and Remark 1.4.1, which may be applied since condition (II) holds, it follows that $\|W_{\psi,\varphi} G_{n,m}\|_{\mathcal{V}_n} \to 0$ as $m \to \infty$. Hence

$$\lim_{m\to\infty} (1 - |\varphi(a_m)|^2)|(W_{\psi,\varphi} G_{n,m})^{(n)}(a_m)| = 0. \tag{3.12}$$

Using (3.11), from (3.12), Remark 3.0.1 and Lemma 3.0.1, we obtain

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n F_m(\varphi(a_m))|}{(1-|\varphi(a_m)|^2)^n} \\
&= \lim_{m\to\infty}\frac{1}{n!}|\psi(a_m)(\varphi'(a_m))^n G_{n,m}^{(n)}(\varphi(a_m))| \\
&= \lim_{m\to\infty}\frac{1}{n!}|\psi(a_m)B_{n,n}(\varphi'(a_m))G_{n,m}^{(n)}(\varphi(a_m))| \\
&= \frac{1}{n!}\lim_{m\to\infty}(1-|a_m|^2)\sum_{k=0}^{n}G_{n,m}^{(k)}(\varphi(w)) \\
&\quad\times\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,k}\big(\varphi'(w),\varphi''(w),...,\varphi^{(\ell-k+1)}(w)\big)\Big| \\
&= \frac{1}{n!}\lim_{m\to\infty}(1-|a_m|^2)|(W_{\psi,\varphi}G_{n,m})^{(n)}(a_m)| \\
&= 0. \qquad (3.13)
\end{aligned}$$

From (3.6), (3.7), (3.13), and the fact that $|\varphi(a_m)|\to 1$ as $m\to\infty$, we obtain

$$\begin{aligned}
&(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}\big(K(\varphi(a_m))-\varepsilon\big) \\
&\le (1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|f_m(\varphi(a_m))| \\
&= (1-|a_m|^2)\frac{1}{|\varphi(a_m)|}\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|F_m(\varphi(a_m))| \\
&\to 0 \qquad (3.14)
\end{aligned}$$

as $m\to\infty$. Since $W_{\psi,\varphi}$ is bounded, by Theorem 3.1.1, the quantities A_j for $j=0,\dots,n$ are finite.

By condition (3.5), we have

$$
\begin{aligned}
(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n} &\le \sup_{w\in\mathbb{D}}(1-|w|^2)\frac{|\psi(w)(\varphi'(w))^n|}{(1-|\varphi(w)|^2)^n} \\
&\le \frac{1}{\delta}\sup_{w\in\mathbb{D}}(1-|w|^2)\frac{|\psi(w)(\varphi'(w))^n|}{(1-|\varphi(w)|^2)^n}K(\varphi(w)) \\
&= \frac{1}{\delta}A_n. \qquad (3.15)
\end{aligned}
$$

Therefore, from (3.14) and (3.15), we obtain

$$
\begin{aligned}
&(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)) \\
&\le (1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}\,\varepsilon \\
&\quad + (1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|F_m(\varphi(a_m))| \\
&\le \frac{1}{\delta}A_n\varepsilon + (1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|F_m(\varphi(a_m))| \\
&\to \frac{1}{\delta}A_n\,\varepsilon \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

Since ε is arbitrary, it follows that $\mathcal{A}_n(\psi,\varphi)=0$.

Let us next show that $\mathcal{A}_{n-1}(\psi,\varphi)=0$. Let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $\frac{1}{2} < |\varphi(a_m)| \to 1$ as $m\to\infty$ and

$$
\begin{aligned}
\mathcal{A}_{n-1}(\psi,\varphi) = &\lim_{m\to\infty}(1-|a_m|^2) \\
&\times \frac{|\sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,n-1}(\varphi'(z),\ldots,\varphi^{(\ell-(n-1)+1)}(a_m))|}{(1-|\varphi(a_m)|^2)^{n-1}}K(\varphi(a_m)).
\end{aligned}
$$

Then, by Remark 3.0.1, the above limit equals

$$\lim_{m\to\infty}(1-|a_m|^2)\frac{|n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}K(\varphi(a_m)).$$

Fix $\varepsilon>0$ and for each $m\in\mathbb{N}$, choose $f_m\in X$ such that $\|f_m\|\le 1$ and (3.6) holds relative to this ε. Define F_m in terms of f_m as done above, and set

$$G_{n-1,m}(z)=(\alpha_{\varphi(a_m)}(z))^{n-1}F_m(z). \tag{3.16}$$

Then the sequence $\{G_{n-1,m}\}_{m\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ and by Lemma 3.0.2, for $j=0,1,...,n-2$,

$$G_{n-1,m}^{(j)}(\varphi(a_m))=0 \text{ and } G_{n-1,m}^{(n-1)}(\varphi(a_m))=\frac{(n-1)!F_m(\varphi(a_m))}{(1-|\varphi(a_m)|^2)^{n-1}}. \tag{3.17}$$

By (3.9) and condition (VI) we have

$$\|G_{n-1,m}\|=\|(\alpha_{\varphi(a_m)})^{n-1}F_m\|\le C\|F_m\|\le C. \tag{3.18}$$

By Lemma 1.4.1, the compactness of the operator $W_{\psi,\varphi}$ implies that $\|W_{\psi,\varphi}G_{n-1,m}\|_{\mathcal{V}_n}\to 0$ as $m\to\infty$. Hence

$$\lim_{m\to\infty}(1-|a_m|^2)|(W_{\psi,\varphi}G_{n-1,m})^n(a_m)|=0. \tag{3.19}$$

Thus, by (3.17), corresponding to ε, there is $N\in\mathbb{N}$ such that for all $m>N$, we have

$$
\begin{aligned}
&(1-|a_m|^2)|(W_{\psi,\varphi}G_{n-1,m})^{(n)}(a_m)| \\
&\quad = (1-|a_m|^2)\Big|G^{(n-1)}_{n-1,m}(\varphi(a_m))\binom{n}{n-1}\psi'(a_m)B_{n-1,n-1}(\varphi'(a_m)) \\
&\qquad + G^{(n-1)}_{n-1,m}(\varphi(a_m))\psi(a_m)B_{n,n-1}(\varphi'(a_m),\varphi''(a_m)) \\
&\qquad + G^{(n)}_{n-1,m}(\varphi(a_m))\psi(a_m)B_{n,n}(\varphi'(a_m))\Big| \\
&\quad = (1-|a_m|^2)\Big|G^{(n-1)}_{n-1,m}(\varphi(a_m))\Big(n\psi'(a_m)(\varphi'(a_m))^{n-1} \\
&\qquad + \psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)\Big) \\
&\qquad + G^{(n)}_{n-1,m}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^{n}\Big| \\
&\quad < \varepsilon.
\end{aligned}
$$

Hence, by condition (VII) for $j = n$, for every $m > N$, we have

$$
\begin{aligned}
&(1-|a_m|^2)\Big|G^{(n-1)}_{n-1,m}(\varphi(a_m))\Big(n\psi'(a_m)(\varphi'(a_m))^{n-1} \\
&\qquad + \psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)\Big)\Big| \\
&\quad < \varepsilon + (1-|a_m|^2)\big|G^{(n)}_{n-1,m}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^{n}\big| \\
&\quad < \varepsilon + C(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^{n}|}{(1-|\varphi(a_m)|^2)^{n}}K(\varphi(a_m)). \qquad (3.20)
\end{aligned}
$$

By (3.17) and since we have shown above that $\mathcal{A}_n(\psi,\varphi)=0$, we have

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)|(n\psi'(a_m)(\varphi'(a_m))^{n-1}\\
&\qquad +\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m))(n-1)!F_m(\varphi(a_m))|\\
&\qquad \times\frac{1}{(1-|\varphi(a_m)|^2)^{n-1}}\\
&\quad \le\ \varepsilon + C\lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m))\\
&\quad =\ \varepsilon + C\mathcal{A}_n(\psi,\varphi)\\
&\quad =\varepsilon. \qquad (3.21)
\end{aligned}$$

By (3.6) and (3.8), and since $1/2<|\varphi(a_m)|$, we have

$$\begin{aligned}
K(\varphi(a_m))-\varepsilon &\le |f_m(\varphi(a_m))|\\
&= \frac{|F_m(\varphi(a_m))|}{|\varphi(a_m)|}\\
&\le 2|F_m(\varphi(a_m))|. \qquad (3.22)
\end{aligned}$$

Thus, for all m sufficiently large, from (3.21), we obtain

$$\begin{aligned}
&(1-|a_m|^2)\frac{|n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}\\
&\qquad \times(K(\varphi(a_m))-\varepsilon)\\
&\quad \le\ (1-|a_m|^2)\frac{|n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}\\
&\qquad \times 2|F_m(\varphi(a_m))|\\
&\quad <\ 2\varepsilon. \qquad (3.23)
\end{aligned}$$

Moreover, recalling condition (3.5), we have

$$
\begin{aligned}
&(1-|a_m|^2)\frac{|n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}} \\
&\quad \le \sup_{w\in\mathbb{D}}(1-|w|^2)\frac{|n\psi'(w)(\varphi'(w))^{n-1}+\psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)|}{(1-|\varphi(w)|^2)^{n-1}} \\
&\quad \le \frac{1}{\delta}\sup_{w\in\mathbb{D}}(1-|w|^2)\frac{|n\psi'(w)(\varphi'(w))^{n-1}+\psi(w)\frac{n(n-1)}{2}(\varphi'(w))^{n-2}\varphi''(w)|}{(1-|\varphi(w)|^2)^{n-1}}K(\varphi(w)) \\
&\quad = \frac{1}{\delta}A_{n-1}, \qquad (3.24)
\end{aligned}
$$

Therefore, by (3.21), (3.23), and (3.24), taking the limits as $m\to\infty$, we obtain

$$
\mathcal{A}_{n-1}(\psi,\varphi)\le \varepsilon\frac{1}{\delta}\,A_{n-1}+2\varepsilon=C\varepsilon.
$$

Since ε is arbitrary, we deduce that $\mathcal{A}_{n-1}(\psi,\varphi)=0$.

Finally, to prove that $\mathcal{A}_0=0$, let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $1/2<|\varphi(a_m)|\to 1$ as $m\to\infty$ and $\mathcal{A}_0(\psi,\varphi)$ equals the limit of

$$
(1-|a_m|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m))\Big|K(\varphi(a_m)).
$$

Fix $\varepsilon>0$ and for each $m\in\mathbb{N}$, let f_m and F_m be constructed as above in terms of ε and the sequence $\{a_m\}$, which, we recall, form bounded sequences in X. By Lemma 1.4.1, the uniform convergence of $\{F_m\}$ to zero on compact subsets of $\mathbb{D}$ guarantees that $\|W_{\psi,\varphi}F_m\|_{\mathcal{V}_n}\to$

0 as $m \to \infty$. Thus, corresponding to ε there is $N \in \mathbb{N}$ such that

$$\begin{aligned}
&(1-|a_m|^2)|(W_{\psi,\varphi}F_m)^{(n)}(a_m)| \\
&\quad = (1-|a_m|^2)\Big|\sum_{k=0}^{n} F_m^{(k)}(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-k+1)}(a_m)\big)\Big| \\
&\quad = (1-|a_m|^2)\Big|F_m(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\ldots,\varphi^{(\ell+1)}(a_m)\big) \\
&\qquad + \sum_{k=1}^{n} F_m^{(k)}(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-k+1)}(a_m)\big)\Big| \\
&\quad < \varepsilon, \qquad (3.25)
\end{aligned}$$

for each $m \geq N$. By (VII) for $j = 1, \ldots, n$, and since $\mathcal{A}_k(\psi, \varphi) = 0$ for each $k = 1, \ldots, n$, thus by (3.25) for all $m \geq N$, we have

$$
\begin{aligned}
&(1-|a_m|^2)\Big|F_m(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m)\big)\Big| \\
&\qquad \leq \varepsilon + C\sum_{k=1}^{n}(1-|a_m|^2) \\
&\qquad \times \frac{\big|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\dots,\varphi^{(\ell-k+1)}(a_m)\big)\big|}{(1-|\varphi(a_m)|^2)^k}K(\varphi(a_m)) \\
&\qquad \leq \varepsilon + C\sum_{k=1}^{n}\mathcal{A}_k(\psi,\varphi) \\
&\qquad = \varepsilon.
\end{aligned}
$$

Hence, for all m sufficiently large, and by (3.22)

$$
\begin{aligned}
&(1-|a_m|^2)\big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\varphi''(a_m),\dots,\varphi^{(\ell+1)}(a_m)\big)\big| \\
&\qquad\qquad \times (K(\varphi(a_m))-\varepsilon) \\
&\qquad \leq 2(1-|a_m|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m)\big)\Big| \\
&\qquad\qquad \times |F_m(\varphi(a_m))| \\
&\qquad \leq 2\varepsilon.
\end{aligned}
$$

By (3.5), we deduce that

$$
\begin{aligned}
(1-|a_m|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m)\big)\Big| \\
\le \frac{1}{\delta}(1-|a_m|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m)\big)\Big| \\
\times K(\varphi(a_m)).
\end{aligned}
$$

Combining the last two inequalities, for all m sufficiently large, we obtain

$$
\begin{aligned}
&(1-|a_m|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m)\dots,\varphi^{(\ell+1)}(a_m)\big)\Big|K(\varphi(a_m)) \\
&\le \varepsilon\frac{1}{\delta}\sup_{w\in\mathbb{D}}(1-|w|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(w)B_{\ell,0}\big(\varphi'(w),\dots,\varphi^{(\ell+1)}(w)\big)\Big| \\
&\qquad\times K(\varphi(w))+2\varepsilon \\
&=\varepsilon\, A_0+2\varepsilon \\
&=C\,\varepsilon,
\end{aligned}
$$

proving that $\mathcal{A}_0(\psi,\varphi)=0$.

Conversely, suppose $\mathcal{A}_k(\psi,\varphi)=0$, for $k=0,\dots,n$. By Lemma 1.4.1 and Remark 1.4.1, to prove the compactness of the operator $W_{\psi,\varphi}$, it suffices to show that if $\{f_m\}$ is a sequence in X converging to zero uniformly on compact subsets of $\mathbb{D}$ with norms bounded by some constant $L>0$, then $\|W_{\psi,\varphi}f_m\|_{\mathcal{V}_n}\to 0$ as $m\to\infty$.

Let $\{f_m\}$ be such a sequence and fix $\varepsilon>0$. Since $\mathcal{A}_k(\psi,\varphi)=0$, for $k=0,...,n$, there exists $s\in(0,1)$ such that whenever $|\varphi(z)|>s$,

$$
(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z))\ell}{(1-|\varphi(z)|^2)^k}K(\varphi(z))<\varepsilon. \tag{3.26}
$$

By condition (IV), there is a positive constant δ such that $K(w) > \delta$ whenever $|w| \leq s$. Since $\{f_m\}$ converges to zero uniformly on $\{|w| \leq s\}$, so does $\{f_m^{(k)}\}$ for each $k = 1, \ldots, n$. Thus, there exists a positive integer N such that for all $m \geq N$ and all $k = 0, \ldots, n$, $|f_m^{(k)}(w)| < \varepsilon$ whenever $|w| \leq s$. Hence, with A_k, for $k = 0, ..., n$ as in Theorem 3.1.1, if $m \geq N$ and $|\varphi(z)| \leq s$, then

$$(1-|z|^2)\frac{|\sum_{l=k}^{n}\binom{n}{l}\psi^{(n-l)}(z)B_{l,k}(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z))|}{(1-|\varphi(z)|^2)^k}K(\varphi(z))|f_m^{(k)}(\varphi(z))|$$

$$< A_k\varepsilon,$$

for all $k = 0, \ldots, n$. Therefore, for $|\varphi(z)| \leq s$,

$$\begin{aligned}
&(1-|z|^2)|(W_{\psi,\varphi}f_m)^{(n)}(z)| \\
&\quad = (1-|z|^2)\Big|\sum_{k=0}^{n} f_m^{(k)}(\varphi(z)) \\
&\qquad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big((\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\quad \leq (1-|z|^2)\sum_{k=0}^{n}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-l)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k} \\
&\qquad \times |f_m^{(k)}(\varphi(z))| \\
&\quad \leq \frac{1}{\delta}(1-|z|^2)\sum_{k=0}^{n}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k} \\
&\qquad \times K(\varphi(z))|f_m^{(k)}(\varphi(z))| \\
&\quad \leq \frac{\varepsilon}{\delta}\sum_{k=0}^{n}A_k. \qquad (3.27)
\end{aligned}$$

If $|\varphi(z)| > s$, then by (1.2), condition (VII) for $j = 1, ..., n$ and (3.26), for all m

sufficiently large, we have

$$
\begin{aligned}
&(1-|z|^2)(W_{\psi,\varphi}f_m)^{(n)}(z)\\
&\le C(1-|z|^2)\sum_{k=1}^{n}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-n+2)}(z))|}{(1-|\varphi(z)|^2)^{(k)}}\\
&\qquad\qquad \times K(\varphi(z))\\
&\qquad + (1-|z|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}(\varphi'(z),\ldots,\varphi^{(\ell+1)}(z))\Big|\\
&\qquad\qquad \times K(\varphi(z))\\
&\le C\varepsilon. \qquad (3.28)
\end{aligned}
$$

Finally, since the sequences $\{f_m\}$, $\{f'_m\}$,..., $\{f_m^{(n-1)}\}$ converge to zero uniformly on compact subsets of $\mathbb{D}$, they converge pointwise. Thus, $|\psi(0)(f_m\circ\varphi)(0)|+|(\psi(f_m\circ\varphi))'(0)|+\cdots+|(\psi(f_m\circ\varphi))^{(n-1)}(0)|\to 0$, as $m\to\infty$. By this, (3.27) and (3.28), we obtain $\|W_{\psi,\varphi}f_m\|_{\mathcal{V}_n}\to 0$ as $m\to\infty$. Hence $W_{\psi,\varphi}$ is compact. □

Corollary 3.1.3. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ with K be as in (1.1) and satisfying conditions* (II), (VI), *and* (VII). *Suppose $\psi\in H(\mathbb{D})$ such that $M_\psi : X\to\mathcal{V}_n$ is bounded and suppose*

$$\delta := \inf_{z\in\mathbb{D}} K(z) > 0. \qquad (3.29)$$

Then $M_\psi : X\to\mathcal{V}_n$ is compact if and only if for each $k=0,1,2,...n$ the quantity $\mathcal{A}_{M,k}(\psi)$ defined by

$$\mathcal{A}_{M,k}(\psi) := \lim_{|z|\to 1}\frac{\binom{n}{k}\psi^{(n-k)}(z)|}{(1-|z|^2)^{k-1})^k}K(z),$$

is zero.

Corollary 3.1.4. *Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$ with K be as in (1.1) and satisfying conditions* (II), (VI), *and* (VII). *Suppose φ an analytic self-map of $\mathbb{D}$ such that $C_\varphi : X \to \mathcal{V}_n$ is bounded and suppose (3.5) holds.Then $C_\varphi : X \to \mathcal{V}_n$ is compact if and only if the quantity $\mathcal{A}_{C,n}(\psi)$ defined by*

$$\mathcal{A}_{C,n}(\varphi) := \lim_{|\varphi(z)|\to 1} 1 - |z|^2) \frac{|\varphi'(z)|^n}{(1 - |\varphi(z)|^2)^n)^k} K(\varphi(z)),$$

is zero.

3.2 The essential norm of the weighted composition operators mapping into weighted-type Banach spaces $\mathcal{V}_n$

In this section, we provide an approximation of the essential norm of the weighted composition operators acting on a large class of Banach space X of analytic functions on the open unit disk into the weighted space $\mathcal{V}_n$. The following result complements Theorem 3.3 in [2].

Theorem 3.2.1. *Let X be a reflexive Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (V)-(VII) *together with either* (I) *or* (VIII). *Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$ satisfying* (3.5) *and such that $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is bounded. Then*

$$\|W_{\psi,\varphi}\|_e \asymp \sum_{k=0}^{n} \mathcal{A}_k(\psi, \varphi),$$

where $\mathcal{A}_k(\psi, \varphi)$ for $k = 0, 1, ..., n,$ as in Theorem 3.1.2.

Proof. We begin by showing that

$$\|W_{\psi,\varphi}\|_e \geq C \sum_{k=0}^{n} \mathcal{A}_k(\psi, \varphi).$$

If $\|\varphi\|_\infty < 1$, choose s such that $\|\varphi\|_\infty < s < 1$. Thus $A_k(\psi, \varphi) = 0$ for $k = 0, 1, \ldots, n$, since the supremum over an empty set is zero, so

$$\|W_{\psi,\varphi}\|_e \geq 0 = C \sum_{k=0}^{n} \mathcal{A}_k(\psi, \varphi).$$

Therefore, we shall assume $\|\varphi\|_\infty = 1$, and prove that $A_k(\psi, \varphi) \leq C\|W_{\psi,\varphi}\|_e$, for $k = 0, 1, \ldots, n$, for some positive constant C.

Let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_m)| \to 1$ as $m \to \infty$ and

$$\mathcal{A}_n(\psi, \varphi) = \lim_{m\to\infty} (1 - |a_m|^2) \frac{|\psi(a_m) B_{n,n}(\varphi'(a_m))|}{(1 - |\varphi(a_m)|^2)^n} K(\varphi(a_m)).$$

Thus, by Remark (3.0.1)

$$\mathcal{A}_n(\psi, \varphi) = \lim_{m\to\infty} (1 - |a_m|^2) \frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1 - |\varphi(a_m)|^2)^n} K(\varphi(a_m)).$$

Fix $\varepsilon > 0$. By (1.2), for each $m \in \mathbb{N}$ there exists $f_m \in X$ with $\|f_m\| \leq 1$ such that (3.6) holds. Since by the reflexivity assumption, the unit ball is compact under the topology of uniform convergence on compact subsets of $\mathbb{D}$, then $\{f_m\}$ converges to some function in X uniformly on compact subsets of $\mathbb{D}$, so that $\{f_m\}$ is uniformly bounded on compact sets.

For each $m \in \mathbb{N}$, let F_m be as in (3.7). Then, the sequence $\{F_m\}$ is bounded in X and converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $G_{n,m}$ be as in (3.10), then by (VI), $\|G_{n,m}\|$ is bounded by some constant $C > 0$, and $G_{n,m}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. By (3.11), we have

$$
\begin{aligned}
&(1-|a_m|^2)|(W_{\psi,\varphi}G_{n,m})^{(n)}(a_m)| \\
&\qquad = (1-|a_m|^2)\Big|G_{n,m}^{(n)}(\varphi(a_m))\psi(a_m)B_{n,n}((\varphi'(a_m))\Big| \\
&\qquad = (1-|a_m|^2)\Big|G_{n,m}^{(n)}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n\Big| \\
&\qquad = (1-|a_m|^2)\Big|\frac{n!F_m(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n}{(1-|\varphi(a_m)|^2)^n}\Big|. \qquad (3.30)
\end{aligned}
$$

Let T be a compact operator, so $W_{\psi,\varphi}-T$ is bounded and by Proposition 1.4.1 and Lemma 1.4.1, $\|TG_{n,m}\|_{\mathcal{V}_n} \to 0$ as $m\to\infty$. Then, using (3.30), we have

$$
\begin{aligned}
&C\|W_{\psi,\varphi}-T\| \\
&\quad \geq \limsup_{m\to\infty} \|G_{n,m}\|\|W_{\psi,\varphi}-T\| \\
&\quad \geq \limsup_{m\to\infty} \|(W_{\psi,\varphi}-T)G_{n,m}\|_{\mathcal{V}_n} \\
&\quad = \limsup_{m\to\infty} \|W_{\psi,\varphi}G_{n,m}\|_{\mathcal{V}_n} \\
&\quad \geq \limsup_{n\to\infty} (1-|a_m|^2)|(W_{\psi,\varphi}G_{n,m})^{(n)}(a_m)| \\
&\quad = \limsup_{m\to\infty} (1-|a_m|^2)\frac{|n!F_m(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}. \qquad (3.31)
\end{aligned}
$$

Using (3.6) and (3.8), we obtain

$$
\begin{aligned}
(1-|a_m|^2)&\frac{|n!\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}\big(K(\varphi(a_m))-\varepsilon\big) \\
&\le (1-|a_m|^2)\frac{|n!\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|f_m(\varphi(a_m))| \\
&\le (1-|a_m|^2)\frac{|n!\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}|F_m(\varphi(a_m))|. \qquad (3.32)
\end{aligned}
$$

Using (3.31) and (3.32), and arguing as in the proof of Theorem 3.1.2, we see that

$$
\begin{aligned}
\mathcal{A}_n(\psi,\varphi) &= \lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)) \\
&\le C\|W_{\psi,\varphi}-T\|. \qquad (3.33)
\end{aligned}
$$

Taking the infimum over all compact operators $T : X \to \mathcal{V}_n$, it follows that

$$
\mathcal{A}_n(\psi,\varphi) \le C\|W_{\psi,\varphi}\|_e. \qquad (3.34)
$$

We next show that $\mathcal{A}_{n-1}(\psi,\varphi) \le C\|W_{\psi,\varphi}\|_e$. Let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_m)| \to 1$ as $m \to \infty$ and

$$
\begin{aligned}
\mathcal{A}_{n-1}(\psi,\varphi) &= \lim_{m\to\infty}(1-|a_m|^2) \\
&\times \frac{|\sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,n-1}(\varphi'(a_m),...,\varphi^{(\ell-(n-1)+1)}(a_m))|}{(1-|\varphi(a_m)|^2)^{n-1}}K(\varphi(a_m)).
\end{aligned}
$$

Then, by Remark (3.0.1), the above limit equals

$$\lim_{m\to\infty}(1-|a_m|^2)\frac{|n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}$$

$$\times K(\varphi(a_m)).$$

Let $G_{n-1,m}$ be as in (3.16). Then, as shown in the proof of Theorem 3.1.2, $G_{n-1,m}$ is bounded in X and converges to zero uniformly on compact subsets of $\mathbb{D}$.

Let $T : X \to \mathcal{V}_n$ be a compact operator. By Proposition 1.4.1 and Lemma 1.4.1, $\|TG_{n-1,m}\|_{\mathcal{V}_n} \to 0$ as $m \to \infty$. Since $\|G_{n-1,m}\| \le C$, then by (3.17), we have

$$\begin{aligned}
C\|W_{\psi,\varphi}-T\| &\ge \limsup_{m\to\infty}\|G_{n-1,m}\|\|W_{\psi,\varphi}-T\| \\
&\ge \limsup_{m\to\infty}\|(W_{\psi,\varphi}-T)G_{n-1,m}\|_{\mathcal{V}_n} \\
&\ge \limsup_{m\to\infty}\|W_{\psi,\varphi}G_{n-1,m}\|_{\mathcal{V}_n} \\
&\ge \limsup_{m\to\infty}(1-|a_m|^2)|(W_{\psi,\varphi}G_{n-1,m})^{(n)}(a_m)| \\
&\ge \limsup_{m\to\infty}(1-|a_m|^2) \\
&\quad \times \Big|G_{n-1,m}^{(n-1)}(\varphi(a_m))\binom{n}{n-1}\psi'(a_m)B_{n-1,n-1}((\varphi'(a_m)) \\
&\quad + G_{n-1,m}^{(n-1)}(\varphi(a_m))\psi(a_m)B_{n,n-1}((\varphi'(a_m),\varphi''(a_m)) \\
&\quad + G_{n-1,m}^{(n)}(\varphi(a_m))\psi(a_m)B_{n,n}((\varphi'(a_m))\Big| \\
&= \limsup_{m\to\infty}(1-|a_m|^2)\Big|G_{n-1,m}^{(n-1)}(\varphi(a_m))\Big(n\psi'(a_m)(\varphi'(a_m))^{n-1} \\
&\quad + \psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)\Big) \\
&\quad + G_{n-1,m}^{(n)}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n\Big|
\end{aligned}$$

Since $G_{n-1,m}$ is bounded by some constant $C > 0$, by condition (VII) for $j = n$, we have

$$\begin{aligned}
&\limsup_{m\to\infty}(1-|a_m|^2)\Big|G_{n-1,m}^{(n-1)}(\varphi(a_m)) \\
&\qquad\qquad \times\Big(n\psi'(a_m)(\varphi'(a_m))^{n-1}+\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)\Big)\Big| \\
&\quad\le C\|W_{\psi,\varphi}-T\| \\
&\qquad +\limsup_{m\to\infty}(1-|a_m|^2)\Big|G_{n-1,m}^{(n)}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n\Big| \\
&\quad\le C\|W_{\psi,\varphi}-T\| \\
&\qquad +\limsup_{m\to\infty}(1-|a_m|^2)\frac{|G_{n-1,m}^{(n)}(\varphi(a_m))\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m))
\end{aligned}$$

By (3.17) , we have

$$\begin{aligned}
&\limsup_{m\to\infty}(1-|a_m|^2) \\
&\quad\times\frac{|(n-1)!F_m(\varphi(a_m))(n\psi'(a_m)(\varphi'(a_m))^{n-1}\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}} \\
&\le C\|W_{\psi,\varphi}-T\|+C\mathcal{A}_n.
\end{aligned} \tag{3.35}$$

Fix $\varepsilon > 0$. By (3.35) and arguing as in the proof of Theorem 3.1.2, we obtain

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2) \\
&\quad\times\frac{|(n-1)!(n\psi'(a_m)(\varphi'(a_m))^{n-1}\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}K(\varphi(a_m)) \\
&\le C\|W_{\psi,\varphi}-T\|+C\varepsilon+C\mathcal{A}_n(\psi,\varphi).
\end{aligned}$$

Combining this with (3.34), we obtain

$$\begin{aligned}\mathcal{A}_{n-1}(\psi,\varphi) &= \lim_{m\to\infty}(1-|a_m|^2)\\ &\times\frac{|(n-1)!(n\psi'(a_m)(\varphi'(a_m))^{n-1}\psi(a_m)\frac{n(n-1)}{2}(\varphi'(a_m))^{n-2}\varphi''(a_m)|}{(1-|\varphi(a_m)|^2)^{n-1}}\\ &\times K(\varphi(a_m))\\ &\leq C\|W_{\psi,\varphi}-T\|+C\|W_{\psi,\varphi}\|_e,\end{aligned}$$

since ε is arbitrary. Taking the infimum over all compact operators $T : X \to \mathcal{V}_n$, it follows that

$$\mathcal{A}_{n-1}(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e. \tag{3.36}$$

By applying a similar argument for $k = n-2, \ldots, 1$ we obtain that

$$\mathcal{A}_k(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e. \tag{3.37}$$

Finally, we show that $\mathcal{A}_0 \leq C\|W_{\psi,\varphi}\|_e$. Since $\|\varphi\|_\infty = 1$, let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $1/2 < |\varphi(a_m)| \to 1$ as $m \to \infty$ and

$$\begin{aligned}\mathcal{A}_0(\psi,\varphi) &= \lim_{m\to\infty}(1-|a_m|^2)\\ &\times\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}(\varphi'(a_m),\ldots,\varphi^{(\ell+1)}(a_m))\Big|K(\varphi(a_m)).\end{aligned}$$

Let $T : X \to \mathcal{V}_n$ be a compact operator. Then, by Proposition 1.4.1 and Lemma 1.4.1,

$\|TF_m\|_{\mathcal{V}_n} \to 0$ as $m \to \infty$. Thus,

$$
\begin{aligned}
&C\|W_{\psi,\varphi} - T\| \\
&\quad \geq \limsup_{m\to\infty} \|F_m\|\|W_{\psi,\varphi} - T\| \\
&\quad \geq \limsup_{m\to\infty} \|(W_{\psi,\varphi} - T)F_m\|_{\mathcal{V}_n} \\
&\quad = \limsup_{m\to\infty} \|W_{\psi,\varphi}F_m\|_{\mathcal{V}_n} \\
&\quad \geq \limsup_{m\to\infty}(1 - |a_m|^2)|(W_{\psi,\varphi}F_m)^{(n)}(a_m)| \\
&\quad \geq \limsup_{m\to\infty}(1 - |a_m|^2)\Big|\sum_{k=0}^{n} F_m^{(k)}(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(a_m) B_{\ell,k}\big(\varphi'(a_m), \ldots, \varphi^{(\ell-k+1)}(a_m)\big)\Big|. \qquad (3.38)
\end{aligned}
$$

By (VII) for $j = 1, \ldots, n$ and (3.38), we have

$$
\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)F_m(\varphi(a_m))\\
&\qquad\times\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}(\varphi'(a_m),\dots,\varphi^{(\ell+1)}(a_m))\Big|\\
&\le C\|W_{\psi,\varphi}-T\|\\
&\quad+\limsup_{m\to\infty}(1-|a_m|^2)\Big|\sum_{k=1}^{n}F_m^{(k)}(\varphi(a_m))\\
&\qquad\times\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\dots,\varphi^{(\ell-k+1)}(a_m)\big)\Big|\\
&\le C\|W_{\psi,\varphi}-T\|\\
&\quad+C\limsup_{m\to\infty}(1-|a_m|^2)\\
&\qquad\times\sum_{k=1}^{n}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}(\varphi'(a_m),\dots,\varphi^{(\ell-k+1)}(a_m))|}{(1-|\varphi(a_m)|^2)^k}\\
&\qquad\times K(\varphi(a_m))\\
&\le C\|W_{\psi,\varphi}-T\|+C\sum_{k=1}^{n}\mathcal{A}_k(\psi,\varphi). \qquad (3.39)
\end{aligned}
$$

Fix $\varepsilon>0$. Again, arguing as in the proof of Theorem 3.1.2, and using (3.34), (3.36), and (3.37), from (3.39) we obtain

$$
\mathcal{A}_0(\psi,\varphi)\le C\Big(\|W_{\psi,\varphi}-T\|+\varepsilon+n\|W_{\psi,\varphi}\|_e\Big).
$$

Since ε is arbitrary, taking the infimum over all compact operators $T:X\to\mathcal{V}_n$, we have

$\mathcal{A}_0(\psi,\varphi) \leq C\|W_{\psi,\varphi}\|_e$, which combined with (3.34), (3.36), and (3.37), yields

$$\|W_{\psi,\varphi}\|_e \geq C\sum_{k=0}^{n}\mathcal{A}_k(\psi,\varphi).$$

To prove the reverse inequality, fix $\varepsilon > 0$, $s \in (0,1)$ such that $|\varphi(z)| < s$ and choose $r \in (0,1)$ as in parts (b) and (c) of Lemma 1.4.2 and Remark 1.4.2. Since $W_{\psi,\varphi} : X \to \mathcal{V}_n$ is bounded and, by condition (V), the operator $\mathcal{T}_r : X \to X$ is compact, the composition product $W_{\psi,\varphi}\mathcal{T}_r : X \to \mathcal{V}_n$ is also compact. Thus,

$$\|W_{\psi,\varphi}(I-\mathcal{T}_r)\| \leq \sup_{\|f\|\leq 1}\|W_{\psi,\varphi}(I-\mathcal{T}_r)f\|_{\mathcal{V}_n} \leq C_0 + \sum_{k=0}^{n} L_k, \tag{3.40}$$

where

$$C_0 = \sup_{\|f\|\leq 1}\sum_{j=0}^{n-1}\left|\big(\psi(I-\mathcal{T}_r)(f\circ\varphi)\big)^{(j)}(0)\right|,$$

and for $k = 0,\ldots,n$,

$$L_k = \sup_{\|f\|\leq 1}\sup_{z\in\mathbb{D}}(1-|z|^2)\Big|(f-f_r)^{(k)}(\varphi(z))$$

$$\times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z)\big)\Big|. \tag{3.41}$$

Observe that, for $k = 0, \ldots, n$,

$$L_k \leq \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} (1-|z|^2)\Big|(f-f_r)^{(k)}(\varphi(z))$$

$$\times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\Big|$$

$$+ \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|> s} (1-|z|^2)\Big|(f-f_r)^{(k)}(\varphi(z))$$

$$\times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\Big|.$$

By Theorem 3.1.1, condition (3.5) (which guarantees that $K(\varphi(z)) > 0$), Lemma 1.4.2, Remark 1.4.2, and condition (VII), for $k = 0, \ldots, n$, we have

$$\begin{aligned}
L_k \leq\ & \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|\leq s} \Big((1-|z|^2) \\
& \times \frac{\big|\sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^k} K(\varphi(z)) \\
& \times K(\varphi(z))^{-1} |(f-f_r)^{(k)}(\varphi(z))| (1-|\varphi(z)|^2)^k\Big) \\
& + \sup_{\|f\|\leq 1} \sup_{|\varphi(z)|> s} \Big((1-|z|^2) \\
& \times \frac{\big|\sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^k} K(\varphi(z)) \\
& \times K(\varphi(z))^{-1} |(f-f_r)^{(k)}(\varphi(z))| (1-|\varphi(z)|^2)^k\Big) \\
\leq\ & A_k\,\varepsilon + C \sup_{|\varphi(z)|> s} (1-|z|^2) \\
& \times \frac{\big|\sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^k} K(\varphi(z)).
\end{aligned}$$

Finally, by Lemma 3.0.1 applied to $(I-\mathcal{T}_r)f$, Lemma 1.4.2 and Remark 1.4.2, we have

$$\begin{aligned} C_0 &= \sup_{\|f\|\le 1}\sum_{j=0}^{n-1}\big|\big(\psi(I-\mathcal{T}_r)(f\circ\varphi)\big)^{(j)}(0)\big|K(\varphi(0))^{-1}K(\varphi(0)) \\ &\le \varepsilon\, K(\varphi(0)). \end{aligned} \tag{3.42}$$

Therefore, from (3.40) and (3.42) we have

$$\begin{aligned} \|W_{\psi,\varphi}\|_e &\le \varepsilon\Big(K(\varphi(0))+\sum_{k=0}^{n}A_k\Big) \\ &\quad + C\sum_{k=0}^{n}\sup_{|\varphi(z)|>s}(1-|z|^2) \\ &\quad \times\frac{\big|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^k}K(\varphi(z)). \end{aligned}$$

If $\|\varphi\|_\infty<1$ and since ε is arbitrary, we see that $\|W_{\psi,\varphi}\|_e=0$ and the operator is compact. If $\|\varphi\|_\infty=1$ and since ε is arbitrary, letting $s\to 1$, we obtain

$$\|W_{\psi,\varphi}\|_e\le C\sum_{k=0}^{n}\mathcal{A}_k(\psi,\varphi),$$

completing the proof of the upper estimate. □

Corollary 3.2.1. *Let X be a reflexive Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (V)-(VII) *together with either* (I) *or* (VIII)*. Let $\psi\in H(\mathbb{D})$ and* (3.29) *be satisfied. Suppose that $M_\psi: X\to\mathcal{V}_n$ is bounded. Then*

$$\|M_\psi\|_e\asymp\sum_{k=0}^{n}\mathcal{A}_{M,k}(\psi),$$

where $\mathcal{A}_k(\psi)$ for $k = 0, 1, ..., n,$ as in Corollary 3.1.3.

Corollary 3.2.2. *Let X be a reflexive Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (V)-(VII) *together with either* (I) *or* (VIII)*. Let φ an analytic self-map of $\mathbb{D}$ satisfying* (3.5) *and such that $C_\varphi : X \to \mathcal{V}_n$ is bounded. Then*

$$\|C_{\psi,\varphi}\|_e \asymp \mathcal{A}_{C,n}(\varphi),$$

where $\mathcal{A}_{C,n}(\varphi)$ as in Corolaary 3.1.4.

3.3 Boundedness and compactness of weighted composition operators mapping into weighted-type Banach spaces $\mathcal{V}_{n,0}$

We conclude the section by focusing on the weighted composition operators mapping a Banach space of analytic functions into $\mathcal{V}_{n,0}$. In particular, we provide equivalent conditons for the boundedness and the compactness of weighted composition operators acting on a large class of Banach spaces X of analytic functions on $\mathbb{D}$ into the little weighted spce $\mathcal{V}_{n,0}$. The following result complements Theorem 3.1 in [2].

Theorem 3.3.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (II)*,* (VI) *and* (VII)*. Suppose $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$ satisfying* (3.5)*. The following conditions are equivalent:*

(*a*) *$W_{\psi,\varphi} : X \to \mathcal{V}_{n,0}$ is compact.*

(*b*) *$W_{\psi,\varphi} : X \to \mathcal{V}_{n,0}$ is bounded.*

(*c*) *For each $k = 0, \dots, n,$ then*

$$\lim_{|z|\to 1}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k}$$

$$\times K(\varphi(z)) = 0.$$

Proof. The implication $(a) \Rightarrow (b)$ is clear. To prove $(b) \Rightarrow (c)$, suppose $W_{\psi,\varphi} : X \to \mathcal{V}_{n,0}$ is bounded. Let $\{a_m\}$ be a sequence in $\mathbb{D}$ such that $|a_m| \to 1$ as $m \to \infty$, fix $f \in X$ such that $\|f\| \leq 1$, and for $m, n \in \mathbb{N}$, let

$$H_{n,m} := \left(\alpha_{\varphi(a_m)}\right)^n f = H_{n,a_m,f},$$

as defined in Lemma 3.0.2. Thus this family of functions is bounded in X and satisfies

$$H_{n,m}^{(j)}(\varphi(a_m)) = 0 \quad \text{and} \quad H_{n,m}^{(n)}(\varphi(a_m)) = \frac{(-1)^n n! f(\varphi(a_m))}{(1 - |\varphi(a_m)|^2)^n}, \tag{3.43}$$

for $j = 0, \ldots, n-1$. Since $W_{\psi,\varphi} H_{n,m} \in \mathcal{V}_{n,0}$, by (3.43), Lemma 3.0.1, and Remark 3.0.1, we have

$$\begin{aligned}
&(1 - |a_m|^2)|(W_{\psi,\varphi} H_{n,m})^{(n)}(a_m)| \\
&= (1 - |a_m|^2)\Big| \sum_{k=0}^{n} H_{n,m}^{(k)}(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(a_m) B_{\ell,k}\big(\varphi'(a_m), \ldots, \varphi^{(\ell-n+1)}(a_m)\big)\Big| \\
&= (1 - |a_m|^2)\Big| H_{n,m}^{(n)}(\varphi(a_m)) \\
&\qquad \times \sum_{\ell=n}^{n} \binom{n}{\ell} \psi(a_m) B_{\ell,n}\big(\varphi'(a_m), \ldots, \varphi^{(\ell-k+1)}(a_m)\big)\Big| \\
&= (1 - |a_m|^2) \frac{n! |f(\varphi(a_m))\psi(a_m) B_{n,n}\big(\varphi'(a_m)\big)|}{(1 - |\varphi(a_m)|^2)^n} \\
&= (1 - |a_m|^2) \frac{n! |\psi(a_m)(\varphi'(a_m))^n f(\varphi(a_m))|}{(1 - |\varphi(a_m)|^2)^n} \to 0, \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2), we have

$$\begin{aligned}
\lim_{m\to\infty}(1-|a_m|^2)\frac{|\psi(a_m)(\varphi'(a_m))^n|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)) \\
&= \frac{1}{n!}\lim_{m\to\infty}(1-|a_m|^2) \\
&\qquad \times \frac{\left|\psi(a_m)B_{n,n}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)\right|}{(1-|\varphi(a_m)|^2)^n}K(\varphi(a_m)) \\
&= 0. \qquad (3.44)
\end{aligned}$$

Similarly, for $m, n \in \mathbb{N}$ and $f \in X$ such that $\|f\| \leq 1$, consider the function $H_{n-1,m} = \big(\alpha_{\varphi(a_m)}\big)^{n-1} f$ defined in (3.1). Then,

$$H^{(k)}_{n-1,m}(\varphi(a_m)) = \begin{cases} 0 & \text{for } k = 0,\ldots,n-2, \\ \frac{(-1)^{n-1}(n-1)!f(\varphi(a_m))}{(1-|\varphi(a_m)|^2)^{n-1}} & \text{for } k = n-1. \end{cases} \qquad (3.45)$$

By condition (VI), we see that $\{H_{n-1,m}\}_{m=0}^{\infty}$ is bounded in X. Since $W_{\psi,\varphi}H_{n-1,m} \in \mathcal{V}_{n,0}$, it follows from (3.45) that as $m \to \infty$,

$$
\begin{aligned}
&(1-|a_m|^2)|(W_{\psi,\varphi}H_{n-1,m})^{(n)}(a_m)| \\
&\quad = (1-|a_m|^2)\Big|\sum_{k=0}^{n} H_{n-1,m}^{(k)}(\varphi(a_m)) \\
&\qquad\quad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)\Big| \\
&\quad = (1-|a_m|^2)\Big|H_{n-1,m}^{(n-1)}(\varphi(a_m)) \\
&\qquad\quad \times \sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,n-1}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big) \\
&\qquad + (1-|a_m|^2)\Big|H_{n-1,m}^{(n)}(\varphi(a_m)) \\
&\qquad\quad \times \sum_{\ell=n}^{n}\binom{n}{\ell}\psi(a_m)B_{\ell,n}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)\Big| \\
&\quad = (1-|a_m|^2)\Big|\frac{(n-1)!f(\varphi(a_m))}{(1-|\varphi(a_m)|^2)^{n-1}} \\
&\qquad\quad \times \sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi(a_m)B_{\ell,n-1}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big) \\
&\qquad + (1-|a_m|^2)H_{n-1,m}^{(n)}(\varphi(a_m))\psi(a_m)B_{n,n}\big(\varphi'(a_m)\big)\Big| \to 0.
\end{aligned}
$$

Thus, by (VII) for $j = n$, and (3.44), we obtain

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)\frac{|f(\varphi(a_m))|}{(1-|\varphi(a_m)|^2)^{n-1}} \\
&\qquad\times\Big|\sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi(a_m)B_{\ell,n-1}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)\Big| \\
&\quad\leq \lim_{m\to\infty}(1-|a_m|^2)\Big|H^{(n)}_{n-1,m}(\varphi(a_m))\psi(a_m)B_{n,n}\big(\varphi'(a_m)\big)\Big| \\
&\quad\leq \lim_{m\to\infty}\frac{(1-|a_m|^2)}{(1-|\varphi(a_m)|^2)^n-1} \\
&\qquad\times\sum_{\ell=n}^{n}\binom{n}{\ell}\psi(a_m)B_{\ell,n-1}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)K(\varphi(a_m)) \\
&\quad= 0.
\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \leq 1$, by (1.2), we have

$$\begin{aligned}
&\lim_{m\to\infty}\frac{(1-|a_m|^2)}{(1-|\varphi(a_m)|^2)^{n-1}} \\
&\qquad\times\Big|\sum_{\ell=n-1}^{n}\binom{n}{\ell}\psi(a_m)B_{\ell,n}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-(n-1)+1)}(a_m)\big)\Big| \\
&\qquad\times K(\varphi(a_m)) = 0. \qquad (3.46)
\end{aligned}$$

By applying similar processes for $k = n-2,\ldots,k = 1$, we obtain that

$$\lim_{|z|\to 1}(1-|z|^2)\sum_{\ell=k}^{n}\frac{\ell\binom{n}{\ell}\psi^{(n-l)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k}K(\varphi(z)) = 0,$$

Finally, for $k = 0$, with f as above. Since $W_{\psi,\varphi} f \in \mathcal{V}_{n,0}$, it follows that

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)|(W_{\psi,\varphi}f)^{(n)}(a_m)| \\
&= \lim_{m\to\infty}(1-|a_m|^2)\Big|\sum_{k=0}^{n} f^{(k)}(\varphi(a_m)) \\
&\quad\times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,k}\big(\varphi'(a_m),\ldots,\varphi^{(\ell-n+1)}(a_m)\big)\Big| \\
&= 0.
\end{aligned} \tag{3.47}$$

By this, (VII) for $j = 1, \ldots, n$, and (3.47), (3.46), and (3.44) we obtain

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2)\Big|f(\varphi(a_m)) \\
&\quad\times \sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\ldots,\varphi^{(\ell+1)}(a_m)\big)\Big| \\
&\le C\lim_{m\to\infty}(1-|a_m|^2) \\
&\quad\times \sum_{k=1}^{n}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(a_m),\ldots,\varphi^{(\ell-k+1)}(a_m))|}{(1-|\varphi(a_m)|^2)^k} \\
&\qquad \times K(\varphi(a_m)) = 0.
\end{aligned}$$

Taking the supremum over all $f \in X$ with $\|f\| \le 1$, by (1.2), we have

$$\begin{aligned}
&\lim_{m\to\infty}(1-|a_m|^2) \\
&\times\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(a_m)B_{\ell,0}\big(\varphi'(a_m),\ldots,\varphi^{(\ell+1)}(a_m)\big)\Big|K(\varphi(a_m)) = 0,
\end{aligned}$$

proving (c). To prove that $(c) \Rightarrow (a)$, suppose (c) holds. By Theorem 3.1.2, the operator

$W_{\psi,\varphi} : X \to \mathcal{V}_n$ is compact. Thus, to prove that $W_{\psi,\varphi} : X \to \mathcal{V}_{n,0}$ is compact, it suffices to show that $W_{\psi,\varphi} f \in \mathcal{V}_{n,0}$ for any $f \in X$. Let $f \in X$. By (1.2) and (VII) for $j = 1, \ldots, n$, we have

$$\begin{aligned}
(1-|z|^2)|(W_{\psi,\varphi}f)^{(n)}(z)| &= (1-|z|^2)\Big|\sum_{k=0}^{n} f^{(k)}(\varphi(z)) \\
&\quad \times \sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-l)}(z) B_{\ell,k}\big(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\le (1-|z|^2) C\|f\| \\
&\quad \times \sum_{k=0}^{n} \frac{|\sum_{\ell=k}^{n} \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}(\varphi'(z), \ldots, \varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^k} \\
&\quad \times K(\varphi(z)) \\
&\to 0, \quad \text{as } |z| \to 1.
\end{aligned}$$

Thus, $W_{\psi,\varphi} f \in \mathcal{V}_{n,0}$, as desired. □

Corollary 3.3.1. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (II), (VI) *and* (VII). *Suppose $\psi \in H(\mathbb{D})$ and* (3.29) *be satisfied. The following conditions are equivalent:*

(*a*) *$M_\psi : X \to \mathcal{V}_{n,0}$ is compact.*

(*b*) *$M_\psi : X \to \mathcal{V}_{n,0}$ is bounded.*

(*c*) *For each $k = 0, \ldots, n$, then*

$$\lim_{|z|\to 1} \frac{\binom{n}{k} |\psi^{(n-k)}(z)|}{(1-|z|^2)^{k-1}} K(z) = 0.$$

Corollary 3.3.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions* (II), (VI) *and* (VII). *Suppose φ an analytic self-map of $\mathbb{D}$ satisfying* (3.5). *The following conditions are equivalent:*

(a) $C_\varphi : X \to \mathcal{V}_{n,0}$ *is compact.*

(b) $C_\varphi : X \to \mathcal{V}_{n,0}$ *is bounded.*

(c)

$$\lim_{|z|\to 1}(1-|z|^2)\frac{|\varphi'(z)|^n}{(1-|\varphi(z)|^2)^n}K(\varphi(z)) = 0.$$

Chapter 4: Applications

In this chapter we show that our results in the previous chapters are applicable for the weighted Bergman space A^p_α for $1 \le p < \infty$, $\alpha > -1$, and the Hardy space H^p for $1 \le p \le \infty$.

4.1 Hypotheses hold for some Hardy and Bergman spaces

4.1.1 The weighted Bergman space A^p_α, $1 \le p < \infty$, $\alpha > -1$

The following result is due to Vukotic [28].

Lemma 4.1.1. *Suppose* $1 \le p < \infty$, $\alpha > -1$, *and* $z \in \mathbb{D}$. *Then*

$$K(z) = \sup\{|f(z)| : f \in A^p_\alpha, \|f\|_{p,\alpha} \le 1\} = \frac{1}{(1-|z|^2)^{(2+\alpha)/p}}.$$

Observe that the hypotheses of Lemma 1.4.1 are satisfied when $X = A^p_\alpha$ for $1 < p < \infty$, Y is a weighted-type Banach space of analytic functions and T is a weighted composition operator. Indeed, conditions (1) and (3) of the lemma are straightforward, while condition (2) follows from Proposition 1.4.1 since A^p_α is reflexive.

Lemma 4.1.2. *For a nonnegative integer* n, $1 \le p < \infty$, $\alpha > -1$, $f \in A^p_\alpha$ *and* $z \in \mathbb{D}$,

$$(1-|z|^2)^n |f^{(n)}(z)| \le 2^{n+(2+\alpha)/p} \frac{n! \|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}.$$

Proof. Suppose $f \in A^p_\alpha$ and $z \in \mathbb{D}$. By the Cauchy integral formula applied to the circle

$|w - z| = r$, where $r = \frac{1-|z|}{2}$, and Lemma 4.1.1, we obtain

$$
\begin{aligned}
|f^{(n)}(z)| &\leq \frac{n!}{2\pi r^n} \int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta \leq \frac{n!}{r^n} \frac{\|f\|_{p,\alpha}}{(1 - r - |z|)^{(2+\alpha)/p}} \\
&= 2^{n+(2+\alpha)/p} \frac{n! \|f\|_{p,\alpha}}{(1 - |z|^2)^{n+(2+\alpha)/p}}
\end{aligned}
$$

Multiplication by $(1 - |z|^2)^n$ yields the result. □

4.1.2 The Hardy space H^p, $1 \leq p \leq \infty$

We state below a well-known explicit formula for $K(z)$ for the Hardy space H^p.

Lemma 4.1.3. ([10], Lemma 4.1) *Suppose $1 \leq p < \infty$ and $z \in \mathbb{D}$, then*

$$
K(z) = \frac{1}{(1 - |z|^2)^{1/p}}.
$$

The proof of the following result is similar to the proof of Lemma 4.1.2 when $\alpha = -1$ and will be omitted.

Lemma 4.1.4. *For a nonnegative integer n, $1 \leq p < \infty$, $f \in H^p$ and $z \in \mathbb{D}$,*

$$
(1 - |z|^2)^n |f^{(n)}(z)| \leq 2^{n+1/p} \frac{n! \|f\|_{H^p}}{(1 - |z|^2)^{1/p}}.
$$

4.2 Applications for Hardy and Bergman spaces

By the previous section, we deduce that Theorems 2.1.1, 2.2.1, 2.3.1, 3.1.1, 3.2.1, and 3.3.1 are applicable for H^p for $1 \leq p \leq \infty$, and A^p_α for $1 \leq p < \infty$ and $\alpha > -1$. Indeed, we summarize these results as following.

Theorem 4.2.1. *Let $1 \leq p \leq \infty$, μ be a weight, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of*

$\mathbb{D}$. *Then* $W_{\psi,\varphi} : H^p \to \mathcal{Z}_\mu$ *is bounded if and only if the following quantities are finite:*

$$\begin{aligned}
\mathcal{Q}_1 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{1/p}}, \\
\mathcal{Q}_2 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}}, \quad \textit{and} \\
\mathcal{Q}_3 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+1)/p}},
\end{aligned}$$

in which case $\|W_{\psi,\varphi}\| \asymp \mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3$.

Under the boundedness assumption, $W_{\psi,\varphi} : H^p \to \mathcal{Z}_\mu$ *is compact if and only if the following limits are 0:*

$$\begin{aligned}
\mathcal{S}_1 &= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{1/p}}, \\
\mathcal{S}_2 &= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{1-|\varphi(z)|^{(p+1)/p}}, \quad \textit{and} \\
\mathcal{S}_3 &= \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+1)/p}}.
\end{aligned}$$

Moreover, $\|W_{\psi,\varphi}\|_e \asymp \mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3$ *for any choice of weight* μ *if* $1<p<\infty$.

For $1 \le p < \infty$, *the operator* $W_{\psi,\varphi} : H^p \to \mathcal{Z}_{\mu,0}$ *is bounded if and only if it is compact if and only if*

$$\begin{aligned}
\lim_{|z|\to 1} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{1/p}} &= 0, \\
\lim_{|z|\to 1} \frac{\mu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} &= 0, \quad \textit{and} \\
\lim_{|z|\to 1} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+1)/p}} &= 0.
\end{aligned}$$

Theorem 4.2.2. *Let $1 \le p \le \infty$, μ be a weight, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : H^p \to \mathcal{V}_n$ is bounded if and only if the following quantities for each $k = 0, \dots, n$ are finite:*

$$\mathcal{O}_{k,p} = \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+1)/p}},$$

in which case $\|W_{\psi,\varphi}\| \asymp \sum_{k=0}^{n}\mathcal{O}_{k,p}$.

Under the boundedness assumption on the operator, $W_{\psi,\varphi} : H^p \to \mathcal{V}_n$ is compact if and only if $\mathcal{N}_{k,p} = 0$ for $k = 0, \dots, n$, where the quantity $\mathcal{N}_{k,p}$ defined as

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+1)/p}}.$$

The operator $W_{\psi,\varphi} : H^p \to \mathcal{V}_{n,0}$ is bounded if and only if it is compact if and only if

$$\lim_{|z|\to 1}\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+1)/p}} = 0.$$

For $1 < p < \infty$, if $W_{\psi,\varphi} : H^p \to \mathcal{V}_n$ is bounded then $\|W_{\psi,\varphi}\|_e \asymp \sum_{k=0}^{n}\mathcal{N}_{k,p}$

The following proof is for both Theorem 4.2.1 and Theorem 4.2.2.

Proof. We begin by observing that the mapping $z \mapsto K(z)$ is the constant 1 and conditions (II)-(VI) and (VIII) clearly hold for the space H^∞. Moreover, since H^∞ is continuously embedded into the Bloch space, which satisfies the condition

$$(1-|z|^2)^j|f^{(j)}(z)| \le C\|f\|_{\mathcal{B}}, \qquad \text{for } z \in \mathbb{D}, f \in \mathcal{B},$$

for any $j \in \mathbb{N}$ (see [29], Theorem 5.4), where C is a positive constant independent of z and f, condition (VII) also holds for H^∞.

On the other hand, by Lemmas 4.1.3 and 4.1.4, conditions (VI) and (VII) hold for $j \in \mathbb{N}$. Thus, the approximation of the operator norm is an immediate consequence of Theorems 2.1.1 and 3.1.1.

To prove the essential norm estimates, observe that conditions (II)-(V) and (VIII) hold for H^p (see [10], p. 103). In the general case of an arbitrary weight μ, since H^p is a reflexive Banach space if $1 < p < \infty$, the hypotheses of Theorems 2.2.1 and 3.2.1. hold for H^p.

The result for the case when the target spaces of $W_{\psi,\varphi}$ are $\mathcal{Z}_{\mu,0}$ and $\mathcal{V}_{n,0}$ follows at once from Theorems 2.3.1 and 3.3.1. □

Theorem 4.2.3. *Let $1 \leq p < \infty$, $\alpha > -1$, μ be a weight, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded if and only if the following quantities are finite:*

$$\mathcal{R}_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}},$$

$$\mathcal{R}_2 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{(p+2+\alpha)/p}}, \text{ and}$$

$$\mathcal{R}_3 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+2+\alpha)/p}},$$

in which case $\|W_{\psi,\varphi}\| \asymp \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$.

Under the boundedness assumption, $W_{\psi,\varphi}$ is compact if and only if $\mathcal{P}_j = 0$ for $j = 1, 2, 3$, where

$$\mathcal{P}_1 = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}},$$

$$\mathcal{P}_2 = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{1-|\varphi(z)|^{(p+2+\alpha)/p}}, \text{ and}$$

$$\mathcal{P}_3 = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+2+\alpha)/p}}.$$

Moreover, $\|W_{\psi,\varphi}\|_e \asymp \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$, *for any choice of weight* μ *if* $1 < p < \infty$.

For $1 \le p < \infty$, $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ *is bounded if and only if it is compact and this holds if and only if*

$$\begin{aligned} \lim_{|z|\to 1} \frac{\mu(z)|\psi''(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} &= 0, \\ \lim_{|z|\to 1} \frac{\mu(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{(p+2+\alpha)/p}} &= 0, \quad \textit{and} \\ \lim_{|z|\to 1} \frac{\mu(z)|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{(2p+2+\alpha)/p}} &= 0. \end{aligned}$$

Theorem 4.2.4. *Let* $1 \le p < \infty$, $\alpha > -1$, μ *be a weight,* $\psi \in H(\mathbb{D})$, *and* φ *an analytic self-map of* $\mathbb{D}$. *Then* $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{V}_n$ *is bounded if and only if the following quantities for each* $k = 0, \dots, n$ *are finite:*

$$\mathcal{O}_{k,p,\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2) \frac{|\sum_{\ell=k}^n \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}(\varphi'(z), \dots, \varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+2+\alpha)/p}},$$

in which case $\|W_{\psi,\varphi}\| \asymp \sum_{k=0}^n \mathcal{O}_{k,p,\alpha}$.

Under the boundedness assumption on the operator, $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{V}_n$ *is compact if and only if* $\mathcal{N}_{k,p,\alpha} = 0$ *for* $k = 0, \dots, n$, *where the quantity* $\mathcal{N}_{k,p,\alpha}$ *defined as*

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} (1-|z|^2) \frac{|\sum_{\ell=k}^n \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}(\varphi'(z), \dots, \varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+2+\alpha)/p}}.$$

The operator $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{V}_{n,0}$ *is bounded if and only if it is compact if and only if*

$$\lim_{|z|\to 1} \frac{|\sum_{\ell=k}^n \binom{n}{\ell} \psi^{(n-\ell)}(z) B_{\ell,k}(\varphi'(z), \dots, \varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{(kp+2+\alpha)/p}} = 0.$$

For $1 < p < \infty$ *if* $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{V}_n$ *is bounded then* $\|W_{\psi,\varphi}\|_e \asymp \sum_{k=0}^n \mathcal{N}_{k,p,\alpha}$.

The following proof is for both Theorem 4.2.3 and Theorem 4.2.4.

Proof. By Lemmas 4.1.1 and 4.1.2, conditions (VI) and (VII) hold for $j \in \mathbb{N}$. Thus, the operator norm estimate is an immediate consequence of Theorems 2.1.1 and 3.1.1.

To prove the essential norm estimates, observe that conditions (I)-(V) hold for A^p_α (see [10], p. 106). In the general case of an arbitrary weight μ, since A^p_α is a reflexive Banach space if $1 < p < \infty$, the hypotheses of Theorems 2.2.1 and 3.2.1. The results for the weighted composition operator mapping into $\mathcal{Z}_{\mu,0}$ and $\mathcal{V}_{n,0}$ are an immediate consequence of Theorems 2.3.1 and 3.3.1 . $\square$

Chapter 5: Weighted composition operators from the space S^p into Zygmund-type spaces $\mathcal{Z}_\mu$ and the weighted-type Banach spaces $\mathcal{V}_n$.

In this chapter, we characterize the bounded and compact weighted composition operators from the space S^p into Zygmund-type spaces and the weighted-type Banach spaces $\mathcal{V}_n$ since our results in the previous chapters are not applicable to the space S^p. In particular, the following example shows that condition (IV) which is needed to be satisfied for our results fails for S^p.

Example 5.1. Fix $a \in \mathbb{D}$, let

$$S_a(z) = \frac{a - z}{1 - \overline{a}z},$$

and $f(z) = z$. Then $f \in S^p$, $\|f\|_{S^p} = 1$, and $f_a = S_a f$ has derivative

$$f_a'(z) = \frac{a + \overline{a}z^2 - 2z}{(1 - \overline{a}z)^2},$$

whose norm in H^p is unbounded as $|a| \to 1$. Thus, no constant C independent of a can exist for which

$$\|S_a f\|_{S^p} = \|f_a'\|_{H^p} \leq C\|f\|_{S^p}.$$

Proposition 3.1 in [11] shows that for each $z \in \mathbb{D}$, the quantity

$$K(z) = \sup\{|f(z)| : f \in S^p, \|f\|_{S^p} \leq 1\},$$

is bounded. In particular, it proves that $K(z)$ for S^p satisfies

$$1 \leq K(z) \leq \max\left\{1, \frac{p}{p-1}\left(1-(1-|z|)^{1-1/p}\right)\right\},$$

and

$$\|f\|_\infty \leq \frac{p}{p-1}\|f\|_{S^p}, \tag{5.1}$$

for all $f \in S^p$.

Lemma 5.0.1. *For* $|a| < 1$ *and* $0 \leq r < 1$,

$$\int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^2} = \frac{2\pi}{1-r^2|a|^2}.$$

Proof. Without loss of generality, we may assume a to be a positive real number. Then by letting $\zeta = e^{i\theta}$ and denoting by $\mathbb{T}$ the unit circle centered at the origin, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{|1-are^{i\theta}|^2} &= \int_0^{2\pi} \frac{d\theta}{1-2ar\cos\theta + a^2r^2} \\
&= \int_{\mathbb{T}} \frac{d\zeta}{\left(1-2ar(\frac{\zeta^2+1}{2\zeta})+a^2r^2\right)i\zeta} \\
&= \frac{1}{i}\int_{\mathbb{T}} \frac{d\zeta}{-ar\zeta^2+(1+a2r^2)\zeta - ar} \\
&= i\int_{\mathbb{T}} \frac{d\zeta}{ar\zeta^2-(1+a2r^2)\zeta+ar}.
\end{aligned}$$

Denote by f the function defined by

$$f(\zeta) = \frac{1}{ar\zeta^2-(1+a^2r^2)\zeta+ar}.$$

Noting that ar is the root inside $\mathbb{D}$ of the polynomial

$$ar\zeta^2 - (1 + a^2r^2)\zeta + ar,$$

we have

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{|1 - \overline{a}re^{i\theta}|^2} &= i(2\pi i)\mathrm{Res}(f, ar) \\ &= -2\pi\Big(\frac{1}{a^2r^2 - 1}\Big) \\ &= \frac{2\pi}{1 - r^2|a|^2},\end{aligned}$$

proving the result. □

Lemma 5.0.2. *Let φ be an analytic self-map of $\mathbb{D}$, for a fixed $n \in \mathbb{N}$ and $w \in \mathbb{D}$,the function*

$$\mathcal{F}_{n,w}(z) = \int_0^z \frac{(1 - |\varphi(w)|^2)^{n-1/p}}{(1 - \overline{\varphi(w)}\zeta)^n} d\zeta, \tag{5.2}$$

belongs to the space S^p, and

$$\|\mathcal{F}_{n,w}\|_{S^p} = \|\mathcal{F}'_{n,w}\|_{H^p} \leq 2^{n-2/p}. \tag{5.3}$$

Proof. Using the Lemma 5.0.1, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}|\mathcal{F}'_{n,w}(re^{i\theta})|^p d\theta \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-|\varphi(w)|^2)^{np-1}}{|1-\overline{\varphi(w)}re^{i\theta}|^{np}}d\theta \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-|\varphi(w)|^2)^{np-2}}{|1-\overline{\varphi(w)}re^{i\theta}|^{np-2}}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}re^{i\theta}|^2}d\theta \\
&\quad\le \frac{2^{np-2}}{2\pi}\int_0^{2\pi}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}re^{i\theta}|^2}d\theta \\
&\quad= 2^{np-2}\frac{1-|\varphi(w)|^2}{1-r^2|\varphi(w)|^2} \\
&\quad\nearrow 2^{np-2},
\end{aligned}$$

as $r \to 1$. Therefore, $\mathcal{F}_{n,w} \in S^p$ and

$$\|\mathcal{F}_{n,w}\|_{S^p} = \|\mathcal{F}'_{n,w}\|_{H^p} \le 2^{n-2/p},$$

as desired. □

Lemma 5.0.3. (Lemma 3.2, [11]) *Every sequence in S^p bounded in norm has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in S^p.*

Lemma 5.0.4. *Let $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi} : S^p \to \mathcal{V}_n$ for $n \in \mathbb{N}_0$ is bounded. Then $W_{\psi,\varphi}$ is compact if and only if $\|W_{\psi,\varphi}f_m\|_{\mathcal{V}_n} \to 0$ as $m \to \infty$ for any norm-bounded sequence $\{f_m\}$ in S^p converging to 0 uniformly in $\overline{\mathbb{D}}$.*

Proof. By Remark 1.4.1, it suffices to verify conditions (i) and (ii) of Lemma 1.4.1. Condition (i) is satisfied when $Y = \mathcal{V}_n$ for $n \in \mathbb{N}_0$ by the remarks preceding Corollary 4.1 in [6]). Condition (ii) is an immediate consequence of Lemma 5.0.3. □

Remark 5.0.1. Lemma 5.0.4 holds for the target space $\mathcal{Z}_\mu$ when μ is a bounded weight. Indeed, Colonna and Hmidouch in [6] showed that condition (i) is satisfied for $\mathcal{V}_n$ for $n \geq 2$ by Montel's Theorem and Proposition 2.1. The proof for that case (which corresponds to the Bergman weight $(1-|z|^2)$) can be applied to $\mathcal{Z}_\mu$ whenever μ is bounded.

5.1 Bounded and compact weighted composition operators from the space S^p into the Zygmund-type space

The following theorem is to characterize the boundedness of weighted composition operators from the space S^p into the Zygmund-type space.

Theorem 5.1.1. *Let μ be a weight, $\psi \in H(\mathbb{D})$, and φ an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $W_{\psi,\varphi} : S^p \to \mathcal{Z}_\mu$ *is bounded.*

(b) $\psi \in \mathcal{Z}_\mu$*, and the following quantities are finite:*

$$N_{\mathcal{Z}_\mu(1)} := \sup_{z\in\mathbb{D}} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}},$$

$$N_{\mathcal{Z}_\mu(2)} := \sup_{z\in\mathbb{D}} \mu(z)\frac{|\psi(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{1+1/p}}.$$

Proof. To prove (a)$\Longrightarrow$ (b), suppose $W_{\psi,\varphi} : S^p \to \mathcal{Z}_\mu$ is bounded. Since the constants functions are in S^p, $\psi = W_{\psi,\varphi}1 \in \mathcal{Z}_\mu$.

Fix $w \in \mathbb{D}$ and consider the functions $g_w = \mathcal{F}_{2,w}$ and $h_w = \mathcal{F}_{3,w}$ defined in Lemma 5.0.2.

Thus, for $z \in \mathbb{D}$,

$$\begin{aligned}
g_w(z) &= \int_0^z \frac{(1-|\varphi(w)|^2)^{2-1/p}}{(1-\overline{\varphi(w)}\zeta)^2} d\zeta, \\
\|g_w\|_{S^p} &= \|g_w'\|_{H^p} \le 2^{2-2/p}, \qquad (5.4) \\
h_w(z) &= \int_0^z \frac{(1-|\varphi(w)|^2)^{3-1/p}}{(1-\overline{\varphi(w)}\zeta)^3} d\zeta, \quad \text{and} \\
\|h_w\|_{S^p} &= \|h_w'\|_{H^p} \le 2^{3-2/p}. \qquad (5.5)
\end{aligned}$$

Observe that

$$\begin{aligned}
g_w'(z) = \frac{(1-|\varphi(w)|^2)^{2-1/p}}{(1-\overline{\varphi(w)}z)^2}, \quad g_w''(z) = \frac{2\overline{\varphi(w)}(1-|\varphi(w)|^2)^{2-1/p}(1-\overline{\varphi(w)}z)}{(1-\overline{\varphi(w)}z)^4}, \\
h_w'(z) = \frac{(1-|\varphi(w)|^2)^{3-1/p}}{(1-\overline{\varphi(w)}z)^3}, \quad h_w''(z) = \frac{3\overline{\varphi(w)}(1-|\varphi(w)|^2)^{3-1/p}(1-\overline{\varphi(w)}z)^2}{(1-\overline{\varphi(w)}z)^6}.
\end{aligned} \qquad (5.6)$$

Thus

$$g_w'(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{1/p}}, \quad g_w''(\varphi(w)) = \frac{2\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+1/p}}, \qquad (5.7)$$

$$h_w'(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{1/p}}, \quad h_w''(\varphi(w)) = \frac{3\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+1/p}}. \qquad (5.8)$$

We begin by proving that $N_{\mathcal{Z}_\mu(2)} < \infty$. By (5.7), we have

$$\begin{aligned}\big(W_{\psi,\varphi}g_w\big)''(w) &= \psi''(w)g_w(\varphi(w)) + \big(2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big)g_w'(\varphi(w)) \\ &\quad + \psi(w)\varphi'(w)^2 g_w''(\varphi(w)) \\ &= \psi''(w)g_w(\varphi(w)) + \frac{2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)}{(1-|\varphi(w)|^2)^{1/p}} \\ &\quad + \frac{2\overline{\varphi(w)}\psi(w)\varphi'(w)^2}{(1-|\varphi(w)|^2)^{1+1/p}}. \end{aligned} \tag{5.9}$$

Similarly, using (5.8), we have

$$\begin{aligned}\big(W_{\psi,\varphi}h_w\big)''(w) &= \psi''(w)h_w(\varphi(w)) + \big(2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)\big)h_w'(\varphi(w)) \\ &\quad + \psi(w)\varphi'(w)^2 h_w''(\varphi(w)) \\ &= \psi''(w)h_w(\varphi(w)) + \frac{2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)}{(1-|\varphi(w)|^2)^{1/p}} \\ &\quad + \frac{3\overline{\varphi(w)}\psi(w)\varphi'(w)^2}{(1-|\varphi(w)|^2)^{1+1/p}}. \end{aligned} \tag{5.10}$$

Subtracting (5.9) from (5.10), we obtain

$$\begin{aligned}\big(W_{\psi,\varphi}h_w\big)''(w) - \big(W_{\psi,\varphi}g_w\big)''(w) &= \psi''(w)h_w(\varphi(w)) - \psi''(w)g_w(\varphi(w)) \\ &\quad + \frac{\overline{\varphi(w)}\psi(w)\varphi'(w)^2}{(1-|\varphi(w)|^2)^{1+1/p}}. \end{aligned} \tag{5.11}$$

By (5.11), the boundedness of $W_{\psi,\varphi}$, (5.1), and estimates (5.4) and (5.5), we have

$$
\begin{aligned}
\mu(w)\frac{|\overline{\varphi(w)}\psi(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{1+1/p}} &\leq \|W_{\psi,\varphi}\|\big(\|h_w\|_{S^p}+\|g_w\|_{S^p}\big) \\
&\quad + \mu(w)\big|\psi''(w)h_w(\varphi(w))\big| + \mu(w)\big|\psi''(w)g_w(\varphi(w))\big| \\
&\leq \|W_{\psi,\varphi}\|\big(\|h_w\|_{S^p}+\|g_w\|_{S^p}\big) + \|\psi\|_{\mathcal{Z}_\mu}\big(\|h_w\|_\infty+\|g_w\|_\infty\big) \\
&\leq \Big(\|W_{\psi,\varphi}\| + \|\psi\|_{\mathcal{Z}_\mu}\frac{p}{p-1}\Big)\big(\|h_w\|_{S^p}+\|g_w\|_{S^p}\big) \\
&\leq \Big(\|W_{\psi,\varphi}\| + \|\psi\|_{\mathcal{Z}_\mu}\frac{p}{p-1}\Big)\big(2^{2-2/p}+2^{3-2/p}\big). \qquad (5.12)
\end{aligned}
$$

Let $P_n(z)=z^n$. Then $P_n \in S^p$, and

$$
\big(W_{\psi,\varphi}P_1\big)''(z) = (\psi\varphi)''(z) = (\psi''\varphi + 2\psi'\varphi' + \psi\varphi'')(z)
$$

Thus,

$$
(2\psi'\varphi' + \psi\varphi'')(z) = \big(W_{\psi,\varphi}P_1\big)''(z) - (\psi''\varphi)(z),
$$

whence

$$
\begin{aligned}
\big(W_{\psi,\varphi}P_2\big)''(z) &= (\psi\varphi^2)''(z) \\
&= \big(\psi''\varphi^2 + 2\psi\varphi'^2 + 2(2\psi'\varphi' + \psi\varphi'')\varphi\big)(z) \\
&= \big(\psi''\varphi^2 + 2\psi\varphi'^2 + 2\big((W_{\psi,\varphi}P_1)'' - \psi''\varphi\big)\varphi\big)(z) \\
&= \big(2\psi\varphi'^2 + 2\big(W_{\psi,\varphi}P_1\big)''\varphi - \psi''\varphi^2\big)(z).
\end{aligned}
$$

Thus,

$$(\psi\varphi'^2)(z) \;=\; \frac{1}{2}\big(W_{\psi,\varphi}P_2\big)''(z) - \big((W_{\psi,\varphi}P_1)''\varphi\big)(z) + \frac{1}{2}(\psi''\varphi^2)(z).$$

Hence

$$\begin{aligned}\sup_{w\in\mathbb{D}}\mu(w)|(\psi\varphi'^2)(w)| &\le \frac{1}{2}\|W_{\psi,\varphi}P_2\|_{\mathcal{Z}_\mu} + \sup_{w\in\mathbb{D}}\Big(\|W_{\psi,\varphi}P_1\|_{\mathcal{Z}_\mu}|\varphi(w)| \\ &\qquad + \frac{1}{2}\|\psi\|_{\mathcal{Z}_\mu}|\varphi^2(w)|\Big) \\ &\le \|W_{\psi,\varphi}P_2\|_{\mathcal{Z}_\mu} + \|W_{\psi,\varphi}P_1\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu}. \qquad (5.13)\end{aligned}$$

Fix $r \in (0,1)$ and consider $w \in \mathbb{D}$ such that $0 \le |\varphi(w)| \le r < 1$. Then by (5.13), we have

$$\begin{aligned}\mu(w)\frac{\big|\overline{\varphi(w)}\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^{1+1/p}} &\le \mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^{1+1/p}} \\ &\le \mu(w)\frac{\big|\psi(w)\varphi'(w)^2\big|}{(1-r^2)^{1+1/p}} \\ &\le \frac{\|W_{\psi,\varphi}P_2\|_{\mathcal{Z}_\mu} + \|W_{\psi,\varphi}P_1\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu}}{(1-r^2)^{1+1/p}}. \qquad (5.14)\end{aligned}$$

Now, consider $w \in \mathbb{D}$ such that $r < |\varphi(w)| < 1$. Then by (5.12), we have

$$\begin{aligned}\mu(w)\frac{r\big|\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^{1+1/p}} &\le \mu(w)\frac{\big|\overline{\varphi(w)}\psi(w)\varphi'(w)^2\big|}{(1-|\varphi(w)|^2)^{1+1/p}} \\ &\le \Big(\|W_{\psi,\varphi}\| + \|\psi\|_{\mathcal{Z}}\frac{p}{p-1}\Big)\big(2^{2-2/p} + 2^{3-2/p}\big). \qquad (5.15)\end{aligned}$$

Therefore, by (5.14) and (5.15) we have that

$$N_{\mathcal{Z}_\mu}(2) = \sup_{z\in\mathbb{D}} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}} \leq \max\{C_1, C_2\},$$

where

$$C_1 = \frac{\|W_{\psi,\varphi}P_2\|_{\mathcal{Z}_\mu} + \|W_{\psi,\varphi}P_1\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu}}{(1-r^2)^{1+1/p}},$$

and

$$C_2 = \frac{1}{r}\Big(\|W_{\psi,\varphi}\| + \frac{p\|\psi\|_{\mathcal{Z}}}{p-1}\Big)\big(\|h_w\|_{S^p} + \|g_w\|_{S^p}\big),$$

proving that $N_{\mathcal{Z}_\mu}(2) < \infty$. Next we prove that $N_{\mathcal{Z}_\mu}(1) < \infty$. Fix $w \in \mathbb{D}$.

By (5.7), the boundedness of $W_{\psi,\varphi}$, (5.1), and (5.4), we have

$$\begin{aligned}
\mu(w)\frac{|2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)|}{(1-|\varphi(w)|^2)^{1/p}} &= \mu(w)\big|\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)g_w'(\varphi(w))\big| \\
&= \mu(w)\big|\big(2\psi'(w)\varphi'(w)+\psi(w)\varphi''(w)\big)g_w'(\varphi(w)) \\
&\quad +\psi''(w)g_w(\varphi(w))+\psi(w)\varphi'(w)^2g_w''(\varphi(w)) \\
&\quad -\psi''(w)g_w(\varphi(w))-\psi(w)\varphi'(w)^2g_w''(\varphi(w))\big| \\
&\leq \mu(w)\big|\big(W_{\psi,\varphi}g_w\big)''(w)\big|+\mu(w)\big|\psi''(w)g_w(\varphi(w))\big| \\
&\quad +\mu(w)\big|\psi(w)\varphi'(w)^2g_w''(\varphi(w))\big| \\
&\leq \|W_{\psi,\varphi}g_w\|_{\mathcal{Z}_\mu}+\|\psi\|_{\mathcal{Z}_\mu}\|g_w\|_\infty \\
&\quad +\mu(w)\frac{2|\overline{\varphi(w)}\psi(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{1+1/p}} \\
&\leq \|W_{\psi,\varphi}\|\|g_w\|_{S^p}+\|\psi\|_{\mathcal{Z}_\mu}\|g_w\|_{S^p}\frac{p}{p-1} \\
&\quad +2\mu(w)\frac{|\psi(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{1+1/p}} \\
&\leq 4^{1-1/p}\Big(\|W_{\psi,\varphi}\|+\frac{p}{p-1}\|\psi\|_{\mathcal{Z}_\mu}\Big)+2N_2^{\mathcal{Z}_\mu}.
\end{aligned}$$

Taking the supremum over all $w\in\mathbb{D}$, we obtain $N_1^{\mathcal{Z}_\mu}<\infty$.

Finally, to show (b)$\Longrightarrow$ (a), suppose that $\psi\in\mathcal{Z}_\mu$ and $N_1^{\mathcal{Z}_\mu}, N_2^{\mathcal{Z}_\mu}<\infty$. Let $f\in S^p$. Then, by Lemma 4.1.3, (5.1), and recalling that $\|f'\|_{H^p}=\|f\|_{S^p}-|f(0)|$, for $z\in\mathbb{D}$, we have

$$
\begin{aligned}
\mu(z)|(W_{\psi,\varphi}f(z))''| &= \mu(z)|(\psi(z)f(\varphi(z))''| \\
&= \mu(z)\big|\psi''(z)f(\varphi(z)) + 2\psi'(z)\varphi'(z)f'(\varphi(z)) \\
&\quad + \psi(z)\big(\varphi''(z)f'(\varphi(z)) + \varphi'(z)^2 f''(\varphi(z))\big)\big| \\
&\le \mu(z)|\psi''(z)f(\varphi(z))| \\
&\quad + \mu(z)\big|\big(2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big)f'(\varphi(z))\big| \\
&\quad + \mu(z)\big|\varphi'(z)^2 f''(\varphi(z))\big| \\
&\le \mu(z)|\psi''(z)|\|f\|_\infty \\
&\quad + \mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}}\|f'\|_{H^p} \\
&\quad + \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}}\|f'\|_{H^p} \\
&\le \|\psi\|_{\mathcal{Z}_\mu}\frac{p}{p-1}\|f\|_{S^p} + \big(N_1^{\mathcal{Z}_\mu} + N_2^{\mathcal{Z}_\mu}\big)\big(\|f\|_{S^p} - |f(0)|\big) \\
&\le \Big(\frac{p}{p-1}\|\psi\|_{\mathcal{Z}_\mu} + N_1^{\mathcal{Z}_\mu} + N_2^{\mathcal{Z}_\mu}\Big)\|f\|_{S^p}.
\end{aligned}
$$

Taking the supremum over all $z \in \mathbb{D}$ proves the boundedness of $W_{\psi,\varphi}$. □

Remark 5.1.1. By Lemmas 4.1.3 and 4.1.4 and since S^p contains the constant functions, the equivalence of the following statements (A) and (B) is an immediate result of Theorem 5.3 in [10].

(A) $W_{\psi\varphi',\varphi} : H^p \to \mathcal{B}_\mu$ is bounded.

(B) The following quantities are finite:

$$\sup_{z\in\mathbb{D}} \mu(z)\frac{|\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}},$$

$$\sup_{z\in\mathbb{D}} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}}.$$

Comparing this equivalence with Theorem 5.1.1, we see that one of the characterizing conditions of boundedness (the second one) is identical, while the other is very similar, but there is a factor 2 in one of the terms in the first formula which is absent in part (B) above. We have not been able to prove or disprove whether these conditions are equivalent.

Next, we characterize the compactness of weighted composition operators from the space S^p into the Zygmund-type space.

Theorem 5.1.2. *Let μ be a positive continuous bounded function on $\mathbb{D}$, $\psi \in H(\mathbb{D})$, φ an analytic self-map of $\mathbb{D}$ such that $\|\varphi\|_\infty = 1$, and $W_{\psi,\varphi} : S^p \to \mathcal{Z}_\mu$ is bounded. Then the following statements are equivalent.*

(a) $W_{\psi,\varphi} : S^p \to \mathcal{Z}_\mu$ *is compact.*

(b) $\mathcal{N}_{\mathcal{Z}_\mu}(1) = \mathcal{N}_{\mathcal{Z}_\mu}(2)$, *where*

$$\mathcal{N}_{\mathcal{Z}_\mu}(1) := \lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}},$$

$$\mathcal{N}_{\mathcal{Z}_\mu}(2) := \lim_{s\to 1} \sup_{|\varphi(z)|>s} \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}}.$$

Proof. To prove (a)$\Longrightarrow$ (b), suppose $W_{\psi,\varphi} : S^p \to \mathcal{Z}_\mu$ is compact. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $0 < |\varphi(z_n)| \to 1$. We need to show that

$$\lim_{n\to\infty} \mu(z_n) \frac{|2\psi'(z_n)\varphi'(z_n) + \psi(z_n)\varphi''(z_n)|}{(1 - |\varphi(z_n)|^2)^{1/p}} = 0,$$

$$\lim_{n\to\infty} \mu(z_n) \frac{|\psi(z_n)\varphi'(z_n)^2|}{(1 - |\varphi(z_n)|^2)^{1+1/p}} = 0.$$

For each $n \in \mathbb{N}$, let $G_n := \mathcal{F}_{2,z_n}$ and $H_n := \mathcal{F}_{3,z_n}$ as in Lemma 5.0.2. Thus

$$\begin{aligned} G_n(z) &= \int_0^z \frac{(1 - |\varphi(z_n)|^2)^{2-1/p}}{(1 - \overline{\varphi(z_n)}\zeta)^2} d\zeta, \\ \|G_n\|_{S^p} &= \|G_n'\|_{H^p} \le 2^{2-2/p}, \end{aligned} \tag{5.16}$$

$$\begin{aligned} H_n(z) &= \int_0^z \frac{(1 - |\varphi(z_n)|^2)^{3-1/p}}{(1 - \overline{\varphi(z_n)}\zeta)^3} d\zeta, \quad \text{and} \\ \|H_n\|_{S^p} &= \|H_n'\|_{H^p} \le 2^{3-2/p}. \end{aligned} \tag{5.17}$$

First, observe that

$$\begin{aligned}
G_n(\varphi(z_n)) &= \int_0^{\varphi(z_n)} \frac{(1-|\varphi(z_n)|^2)^{2-1/p}}{(1-\overline{\varphi(z_n)}\zeta)^2} d\zeta, \\
&= \frac{(1-|\varphi(z_n)|^2)^{2-1/p}}{\overline{\varphi(z_n)}} (1-\overline{\varphi(z_n)}\zeta)^{-1}\Big|_0^{\varphi(z_n)} \\
&= \frac{(1-|\varphi(z_n)|^2)^{2-1/p}}{\overline{\varphi(z_n)}} \Big((1-|\varphi(z_n)|^2)^{-1} - 1\Big) \\
&= \frac{(1-|\varphi(z_n)|^2)^{2-1/p}}{\overline{\varphi(z_n)}} \frac{(1-1+|\varphi(z_n)|^2)}{(1-|\varphi(z_n)|^2)} \\
&= \frac{(1-|\varphi(z_n)|^2)^{1-1/p}}{\overline{\varphi(z_n)}} \big(\varphi(z_n)\overline{\varphi(z_n)}\big) \\
&= \varphi(z_n)(1-|\varphi(z_n)|^2)^{1-1/p}. \qquad (5.18)
\end{aligned}$$

Moreover,

$$G_n'(\varphi(z_n)) = \frac{1}{(1-|\varphi(z_n)|^2)^{1/p}}, \quad \text{and} \quad G_n''(\varphi(z_n)) = \frac{2\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{1+1/p}}. \qquad (5.19)$$

Second, note that

$$
\begin{aligned}
H_n(\varphi(z_n)) &= \int_0^{\varphi(z_n)} \frac{(1-|\varphi(z_n)|^2)^{3-1/p}}{(1-\overline{\varphi(z_n)}\zeta)^3} d\zeta, \\
&= \frac{(1-|\varphi(z_n)|^2)^{3-1/p}}{2\overline{\varphi(z_n)}} (1-\overline{\varphi(z_n)}\zeta)^{-2}\Big|_0^{\varphi(z_n)} \\
&= \frac{(1-|\varphi(z_n)|^2)^{3-1/p}}{2\overline{\varphi(z_n)}} \Big((1-|\varphi(z_n)|^2)^{-2}-1\Big) \\
&= \frac{(1-|\varphi(z_n)|^2)^{3-1/p}}{2\overline{\varphi(z_n)}} \Big(\frac{1-(1-|\varphi(z_n)|^2)^2}{(1-|\varphi(z_n)|^2)^2}\Big) \\
&= \frac{(1-|\varphi(z_n)|^2)^{1-1/p}}{2\overline{\varphi(z_n)}} \big(1-(1-|\varphi(z_n)|^2)\big)\big(1+(1-|\varphi(z_n)|^2)\big) \\
&= \frac{(1-|\varphi(z_n)|^2)^{1-1/p}}{2\overline{\varphi(z_n)}} \big(\varphi(z_n)\overline{\varphi(z_n)}\big)\big(2-|\varphi(z_n)|^2\big) \\
&= \frac{1}{2}\varphi(z_n)\big(2-|\varphi(z_n)|^2\big)\big(1-|\varphi(z_n)|^2\big)^{1-1/p}, \qquad (5.20)
\end{aligned}
$$

Furthermore,

$$
H_n'(\varphi(z_n)) = \frac{1}{(1-|\varphi(z_n)|^2)^{1/p}} \quad \text{and} \quad H_n''(\varphi(z_n)) = \frac{3\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{1+1/p}}. \qquad (5.21)
$$

By (5.18) and (5.20), we see that the sequences $\{G_n\}$ and $\{H_n\}$ converge to 0 uniformly in $\overline{\mathbb{D}}$. Since $W_{\psi,\varphi}$ is compact, by Remark 5.0.1, it follows that

$$
\begin{aligned}
\|W_{\psi,\varphi}G_n\|_{\mathcal{Z}_\mu} &\to 0, \quad \text{and} \qquad (5.22) \\
\|W_{\psi,\varphi}H_n\|_{\mathcal{Z}_\mu} &\to 0, \qquad (5.23)
\end{aligned}
$$

as $n \to 0$.

Arguing as in the proof of Theorem 5.1.1 for the functions G_n and H_n in place of g_w and h_w (see (5.11)), using (5.19), (5.21), we have

$$\begin{aligned}\big(W_{\psi,\varphi}H_n\big)''(z_n) - \big(W_{\psi,\varphi}G_n\big)''(z_n) &= \psi''(z_n)H_n(\varphi(z_n)) - \psi''(z_n)G_n(\varphi(z_n)) \\ &\quad + \frac{\overline{\varphi(z_n)}\psi(z_n)\varphi'(z_n)^2}{(1-|\varphi(z_n)|^2)^{1+1/p}}. \end{aligned} \tag{5.24}$$

Therefore,

$$\begin{aligned}\mu(z_n)\frac{|\overline{\varphi(z_n)}\psi(z_n)\varphi'(z_n)^2|}{(1-|\varphi(z_n)|^2)^{1+1/p}} &\leq \|W_{\psi,\varphi}H_n\|_{\mathcal{Z}_\mu} + \|W_{\psi,\varphi}G_n\|_{\mathcal{Z}_\mu} \\ &\quad + |\psi''(z_n)H_n(\varphi(z_n))| + |\psi''(z_n)G_n(\varphi(z_n))| \end{aligned} \tag{5.25}$$

Since $|\varphi(z_n)| \to 1$, and $\psi \in \mathcal{Z}_\mu$ by Theorem 5.1.1, then by (5.18), (5.20), (5.25), (5.22), and (5.23), we have

$$\lim_{n\to\infty} \mu(z_n)\frac{|\psi(z_n)\varphi'(z_n)^2|}{(1-|\varphi(z_n)|^2)^{1+1/p}} = 0. \tag{5.26}$$

Again, arguing as in the proof of Theorem 5.1.1 for G_n in place of g_w, we have

$$\begin{aligned}\big(W_{\psi,\varphi}G_n\big)''(z_n) &= \psi''(z_n)G_n(\varphi(z_n)) + \frac{2\psi'(z_n)\varphi'(z_n) + \psi(z_n)\varphi''(z_n)}{(1-|\varphi(z_n)|^2)^{1/p}} \\ &\quad + \frac{2\overline{\varphi(z_n)}\psi(z_n)\varphi'(z_n)^2}{(1-|\varphi(z_n)|^2)^{1+1/p}}. \end{aligned} \tag{5.27}$$

Hence

$$\mu(z_n)\frac{|2\psi'(z_n)\varphi'(z_n)+\psi(z_n)\varphi''(z_n)|}{(1-|\varphi(z_n)|^2)^{1/p}} \leq \mu(z_n)|(W_{\psi,\varphi}G_n)''(z_n)| + \mu(z_n)\big|\psi''(z_n)G_n(\varphi(z_n))\big| + \mu(z_n)\frac{2|\overline{\varphi(w)}\psi(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{1+1/p}}. \tag{5.28}$$

Since $|\varphi(z_n)| \to 1$, and $\psi \in \mathcal{Z}_\mu$ by Theorem 5.1.1, then by (5.22), (5.18), and (5.28) we have

$$\lim_{n\to\infty}\mu(z_n)\frac{|2\psi'(z_n)\varphi'(z_n)+\psi(z_n)\varphi''(z_n)|}{(1-|\varphi(z_n)|^2)^{1/p}} = 0. \tag{5.29}$$

Hence, by (5.26) and (5.29), $\mathcal{N}_{\mathcal{Z}_\mu}(1) = \mathcal{N}_{\mathcal{Z}_\mu}(2) = 0$.

Finally to prove (b) $\Longrightarrow$ (a), suppose $\mathcal{N}_{\mathcal{Z}_\mu}(1) = \mathcal{N}_{\mathcal{Z}_\mu}(2) = 0$. Fix $\varepsilon > 0$ and choose $s \in (0,1)$ such that

$$\mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}} < \varepsilon, \quad \text{and} \quad \mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}} < \varepsilon, \tag{5.30}$$

whenever $|\varphi(z)| > s$.

Let $\{f_n\}$ be a sequence of functions in S^p converging to zero uniformly in $\overline{\mathbb{D}}$ with S^p norms bounded by some positive constant L. By Remark 5.0.1, as uniform convergence implies points, it suffices to show that $\|W_{\psi,\varphi}f_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to \infty$. Since $|\psi(0)f_n(\varphi(0))| \to 0$ and $|\psi(0)f_n'(\varphi(0))| \to 0$ as $n \to \infty$, it suffices to show that $\sup_{z\in\mathbb{D}}\mu(z)|(W_{\psi,\varphi}f_n)''(z)| \to 0$ as $n \to \infty$.

Choose $N \in \mathbb{N}$ such that $\|f_n\|_\infty < \varepsilon$ for all $n \geq N$.

Suppose first that $|\varphi(z)| > s$. By (5.30), Lemma 4.1.3, (5.1), $\psi \in \mathcal{Z}_\mu$ by Theorem 5.1.1, and recalling that $\|f_n'\|_{H^p} = \|f_n\|_{S^p} - |f_n(0)| \le \|f_n\|_{S^p}$, for $n \ge N$, we have

$$\begin{aligned}
\mu(z)|(W_{\psi,\varphi}f_n(z))''| &= \mu(z)\big|\psi''(z)f_n(\varphi(z)) + 2\psi'(z)\varphi'(z)f_n'(\varphi(z)) \\
&\qquad + \psi(z)\big(\varphi''(z)f_n'(\varphi(z)) + \varphi'(z)^2 f_n''(\varphi(z))\big)\big| \\
&\le \mu(z)|\psi''(z)f_n(\varphi(z))| \\
&\qquad + \mu(z)\big|f_n'(\varphi(z))\big(2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big)\big| \\
&\qquad + \mu(z)|f_n''(\varphi(z))\varphi'(z)^2| \\
&\le \|\psi\|_{\mathcal{Z}_\mu}|f_n(\varphi(z))| + \|f_n'\|_{H^p}\mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}} \\
&\qquad + \|f_n'\|_{H^p}\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}} \\
&\le \|\psi\|_{\mathcal{Z}_\mu}\|f_n\|_\infty + \|f_n\|_{S^p}\mu(z)\frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}} \\
&\qquad + \|f_n\|_{S^p}\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}} \\
&\le (\|\psi\|_{\mathcal{Z}_\mu} + 2L)\varepsilon.
\end{aligned}$$

On the other hand, for $|\varphi(z)| \le s$, then $|f_n'(\varphi(z))| < \varepsilon$ and $|f_n''(\varphi(z))| < \varepsilon$ for all n sufficiently large. Thus, by Theorem 5.1.1, for all such n, we have

$$\begin{aligned}
\mu(z)|(W_{\psi,\varphi}f_n(z))''| &= \mu(z)|f_n(\varphi(z))\psi''(z)| \\
&\quad +\mu(z)\big|f_n'(\varphi(z))\big(2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)\big)\big| \\
&\quad +\mu(z)\big|f_n''(\varphi(z))\varphi'(z)^2\big| \\
&\le \|\psi\|_{\mathcal{Z}_\mu}\varepsilon+\mu(z)\frac{|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}} \\
&\qquad \times(1-|\varphi(z)|^2)^{1/p}|f_n'(\varphi(z))| \\
&\quad +\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}} \\
&\qquad \times(1-|\varphi(z)|^2)^{1+1/p}|f_n''(\varphi(z))| \\
&\le (\|\psi\|_{\mathcal{Z}_\mu}+N_{\mathcal{Z}_\mu}(1)+N_{\mathcal{Z}_\mu}(2))\varepsilon
\end{aligned}$$

Hence, $\|W_{\psi,\varphi}f_n\|_{\mathcal{Z}_\mu}\to 0$ as $n\to\infty$, thus $W_{\psi,\varphi}$ is compact. □

Remark 5.1.2. By Lemmas 4.1.3 and 4.1.4 and since S^p contains the constant functions, the equivalence of the following statements (A) and (B) is an immediate consequence of Theorem 5.5 in [10].

(A) $W_{\psi\varphi',\varphi}:H^p\to\mathcal{B}_\mu$ is compact.

(B) The following quantities are zero:

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\mu(z)\frac{|\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1/p}},$$

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\mu(z)\frac{|\psi(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{1+1/p}}.$$

Comparing this equivalence with Theorem 5.1.2, we see that one of the characterizing conditions of compactness (the second one) is identical, while the other is very similar, but there is a factor 2 in one of the terms in the first formula which is absent in part (B) above. We have not been able to prove or disprove whether these conditions are equivalent.

5.2 Bounded and compact weighted composition operators from the space S^p into the weighted-type Banach spaces $\mathcal{V}_n$.

The following result provides sufficient conditions for the boundedness of a weighted composition operator from the space S^p into the weighted-type Banach space $\mathcal{V}_n$ for $n \geq 3$.

Theorem 5.2.1. *Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : S^p \to \mathcal{V}_n$ is bounded if the following quantities are finite:*

$$N_{\mathcal{V}_n}(0) := \sup_{z\in\mathbb{D}}(1-|z|^2)\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}(\varphi'(z),\ldots,\varphi^{(\ell+1)}(z))\Big|,$$

$$N_{\mathcal{V}_n}(k) := \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{k-1+1/p}},$$

for $k = 1,\ldots,n$.

Proof. Suppose $N_{\mathcal{V}_n}(k)$for $k = 0, 1, \ldots, n$. Let $f \in S^p$. Then, by Lemma 4.1.4, (5.1), and recalling that $\|f'\|_{H^p} \le \|f\|_{S^p}$, for $z \in \mathbb{D}$, we have

$$\begin{aligned}
(1-|z|^2)|(W_{\psi,\varphi}f)^{(n)}(z)| &= (1-|z|^2)\Big|\sum_{k=0}^{n} f^{(k)}(\varphi(z)) \\
&\quad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\le (1-|z|^2)|f(\varphi(z))| \\
&\quad \times\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\quad +(1-|z|^2)\|f'\|_{H^p} \\
&\quad \times\frac{|\sum_{k=1}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)|}{(1-|\varphi(z)|^2)^{k-1+1/p}} \\
&\le \|f\|_\infty N_{\mathcal{V}_n}(0) + \|f\|_{S^p}\sum_{k=1}^{n} N_{\mathcal{V}_n}(k) \\
&\le \Big(\frac{p}{p-1}N_{\mathcal{V}_n}(0) + \sum_{k=1}^{n} N_{\mathcal{V}_n}(k)\Big)\|f\|_{S^p} \qquad (5.31)
\end{aligned}$$

Then, taking the supremum over all $z \in \mathbb{D}$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(W_{\psi,\varphi}f)^{(n)}(z)| \le C\|f\|_{S^p},$$

where $C = \dfrac{p}{p-1}N_{\mathcal{V}_n}(0) + \sum\limits_{k=1}^{n} N_{\mathcal{V}_n}(k)$.

Observe that the estimate (5.31) holds also for the j-th derivative of $(W_{\psi,\varphi}f)(0)$ where $j = 0,\dots,n-1$. Namely,

$$|(W_{\psi,\varphi}f)^{(j)}(0)| \le \Big(\frac{p}{p-1}N_{\mathcal{V}_n}(0) + \sum_{k=1}^{j} N_{\mathcal{V}_n}(k)\Big)\|f\|_{S^p} = C\|f\|_{S^p} \qquad (5.32)$$

Thus by the definition of $\|\cdot\|_{\mathcal{V}_n}$, (5.31), and (5.32) we have

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathcal{V}_n} &\leq \sum_{j=0}^{n-1}|(W_{\psi,\varphi}f)^{(j)}(0)| + C\|f\|_{S^p} \\ &\leq 2C\|f\|_{S^p}.\end{aligned}$$

proving the boundedness of $W_{\psi,\varphi}$. □

In the following theorem we provide sufficient conditions for the compactness of weighted composition operators from the space S^p into the weighted-type Banach spaces $\mathcal{V}_n$.

Theorem 5.2.2. *Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : S^p \to \mathcal{V}_n$ is compact if $N_{\mathcal{V}_n}(k)$ is finite and*

$$\mathcal{N}_{\mathcal{V}_n}(k) := \lim_{s\to 1}\sup_{|\varphi(z)|>s}(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{k-1+1/p}} = 0,$$

for $k = 1,\ldots,n$, where the quantities $N_k^{\mathcal{V}_n}$ are defined in Theorem 5.2.1.

Proof. Assume $N_k^{\mathcal{V}_n} < \infty$ for $k = 0,1,\ldots,n$ and $\mathcal{N}_{\mathcal{V}_n}(k) = 0$ for $k = 1,\ldots,n$.

Fix $\varepsilon > 0$ and choose $s \in (0,1)$ such that

$$(1-|z|^2)\frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z))|}{(1-|\varphi(z)|^2)^{((k-1)p+1)/p}} < \varepsilon, \tag{5.33}$$

whenever $|\varphi(z)| > s$ for each $k = 1,\ldots,n$.

Let $\{f_m\}$ be a sequence of functions in S^p converging to zero uniformly in $\overline{\mathbb{D}}$ with S^p norms bounded by some positive constant L. By Lemma 5.0.4, it suffices to show that $\|W_{\psi,\varphi}f_n\|_{\mathcal{V}_n} \to 0$ as $m \to \infty$. Since for $j = 0,\ldots,n-1$, $\left|(\psi(0)f_m(\varphi(0)))^{(j)}\right| \to 0$ as $m \to \infty$, it suffices to show that $\sup_{z\in\mathbb{D}}(1-|z|^2)|(W_{\psi,\varphi}f_m)^{(n)}(z)| \to 0$ as $m \to \infty$. Choose

$N \in \mathbb{N}$ such that $\|f_m\|_\infty < \varepsilon$ for all $m \geq N$. Suppose first that $|\varphi(z)| > s$. By (5.33), Lemma 4.1.4, (5.1), since $N_0^{\mathcal{V}_n} < \infty$, and recalling that $\|f_m'\|_{H^p} \leq \|f_m\|_{S^p}$, for $m \geq N$, we have

$$\begin{aligned}
(1-|z|^2)|(W_{\psi,\varphi}f_m)^{(n)}(z)| &= (1-|z|^2)\Big|\sum_{k=0}^{n} f_m^{(k)}(\varphi(z)) \\
&\quad \times \sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\leq (1-|z|^2)|f_m(\varphi(z))| \\
&\quad \times\Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}\big(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\quad +(1-|z|^2)\|f_m'\|_{H^p} \\
&\quad \times\frac{|\sum_{k=1}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\ldots,\varphi^{(\ell-k+1)}(z)\big)|}{(1-|\varphi(z)|^2)^{k-1+1/p}} \\
&\leq \|f_m\|_\infty N_0^{\mathcal{V}_n} + \|f_m\|_{S^p}\sum_{k=1}^{n}\mathcal{N}_{\mathcal{V}_n}(k) \\
&\leq (N_0^{\mathcal{V}_n} + L)\varepsilon.
\end{aligned}$$

On the other hand, for $|\varphi(z)| \leq s$, $|f_m^{(k)}(\varphi(z))| < \varepsilon$ for all m sufficiently large. Thus, for all such m, we have

$$\begin{aligned}
(1-|z|^2)|(W_{\psi,\varphi}f_m)^{(n)}(z)| &= (1-|z|^2)|f_m(\varphi(z))| \\
&\quad \times \Big|\sum_{\ell=0}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,0}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)\Big| \\
&\quad +(1-|z|^2)|\sum_{k=1}^{n}|f_m^{(k)}(\varphi(z))| \\
&\quad \times \frac{|\sum_{\ell=k}^{n}\binom{n}{\ell}\psi^{(n-\ell)}(z)B_{\ell,k}\big(\varphi'(z),\dots,\varphi^{(\ell-k+1)}(z)\big)|}{(1-|\varphi(z)|^2)^{k-1+1/p}} \\
&\quad \times(1-|\varphi(z)|^2)^{k-1+1/p} \\
&\leq \Big(\sum_{k=0}^{n}N_{\mathcal{V}_n}(k)\Big)\varepsilon.
\end{aligned}$$

Hence, $\|W_{\psi,\varphi}f_n\|_{\mathcal{V}_n}\to 0$ as $m\to\infty$, proving the compactness of $W_{\psi,\varphi}$. □

Remark 5.2.1. Theorems 5.2.1 and 5.2.2 generalize the sufficient conditions for boundedness and compactness for the weighted Zygmund space $\mathcal{Z}_\mu$ when $\mu(z) = 1-|z|^2$, and $\psi\in\mathcal{Z}_\mu$ is equivalent to $N_{\mathcal{V}_2}(0)<\infty$. Indeed

$$\begin{aligned}
N_{\mathcal{V}_2}(0) &= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big|\sum_{\ell=0}^{2}\binom{n}{\ell}\psi^{(2-\ell)}(z)B_{\ell,0}(\varphi'(z),\dots,\varphi^{(\ell+1)}(z))\Big| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z)|.
\end{aligned}$$

Bibliography

Bibliography

[1] Robert F. Allen, Katherine C. Heller, and Matthew A. Pons, *Multiplication operators on $S^2(\mathbb{D})$*, Acta Sci. Math. (Szeged) **81** (2015), no. 3-4, 575–587. MR 3443772

[2] Shams Alyusof and Flavia Colonna, *Weighted composition operators from analytic function spaces into zygmund-type spaces*, preprint.

[3] ______, *Weighted composition operators from S^p spaces into iterated weighted-type spaces*, preprint.

[4] Boo Rim Choe, Hyungwoon Koo, and Wayne Smith, *Composition operators on small spaces*, Integral Equations Operator Theory **56** (2006), no. 3, 357–380. MR 2270842

[5] Flavia Colonna, *New criteria for boundedness and compactness of weighted composition operators mapping into the Bloch space*, Cent. Eur. J. Math. **11** (2013), no. 1, 55–73. MR 2988782

[6] Flavia Colonna and Nacir Hmidouch, *Weighted composition operators on iterated weighted-type Banach spaces of analytic functions*, Complex Anal. Oper. Theory **13** (2019), no. 4, 1989–2016. MR 3957026

[7] Flavia Colonna and Maria Tjani, *Weighted composition operators from the Besov spaces into the weighted-type space H^∞_μ*, J. Math. Anal. Appl. **402** (2013), no. 2, 594–611. MR 3029174

[8] ______, *Essential norms of weighted composition operators from Hilbert function spaces into Zygmund-type spaces*, Mediterr. J. Math. **12** (2015), no. 4, 1357–1375. MR 3416865

[9] ______, *Essential norms of weighted composition operators from reproducing kernel Hilbert spaces into weighted-type spaces*, Houston J. Math. **42** (2016), no. 3, 877–903. MR 3570716

[10] ______, *Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions*, J. Math. Anal. Appl. **434** (2016), no. 1, 93–124. MR 3404550

[11] ______, *Weighted composition operators from Banach spaces of analytic functions into Bloch-type spaces*, Problems and recent methods in operator theory, Contemp. Math., vol. 687, Amer. Math. Soc., Providence, RI, 2017, pp. 75–95. MR 3635754

[12] John B. Conway, *Functions of one complex variable*, second ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York-Berlin, 1978. MR 503901

[13] ______, *A course in functional analysis*, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1985. MR 768926

[14] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397026

[15] Peter Duren and Alexander Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004. MR 2033762

[16] Richard J. Fleming and James E. Jamison, *Isometries on Banach spaces: function spaces*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 129, Chapman & Hall/CRC, Boca Raton, FL, 2003. MR 1957004

[17] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000. MR 1758653

[18] Katherine Cason Heller, *Composition Operators on S2(D)*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–University of Virginia. MR 2844199

[19] Qinghua Hu and Xiangling Zhu, *Essential norm of weighted composition operators from H^∞ to the Zygmund space*, J. Nonlinear Sci. Appl. **9** (2016), no. 7, 5082–5092. MR 3558464

[20] A. N. Kolmogorov and S. V. Fomīn, *Introductory real analysis*, Dover Publications, Inc., New York, 1975, Translated from the second Russian edition and edited by Richard A. Silverman, Corrected reprinting. MR 0377445

[21] Thomas L. Kriete, III and Barbara D. MacCluer, *Composition operators on large weighted Bergman spaces*, Indiana Univ. Math. J. **41** (1992), no. 3, 755–788. MR 1189910

[22] Barbara D. MacCluer, *Elementary functional analysis*, Graduate Texts in Mathematics, vol. 253, Springer, New York, 2009. MR 2462971

[23] Shûichi Ohno and Karel Stroethoff, *Weighted composition operators from reproducing Hilbert spaces to Bloch spaces*, Houston J. Math. **37** (2011), no. 2, 537–558. MR 2794562

[24] Joel H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. (2) **125** (1987), no. 2, 375–404. MR 881273

[25] Allen L. Shields, *Weighted shift operators and analytic function theory*, Topics in operator theory, 1974, pp. 49–128. Math. Surveys, No. 13. MR 0361899

[26] Stevo Stević, *On an integral operator from the Zygmund space to the Bloch-type space on the unit ball*, Glasg. Math. J. **51** (2009), no. 2, 275–287. MR 2500751

[27] Maria Tjani, *Compact composition operators on Besov spaces*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4683–4698. MR 1990767

[28] Dragan Vukotić, *A sharp estimate for A^p_α functions in* $\mathbf{C}^n$, Proc. Amer. Math. Soc. **117** (1993), no. 3, 753–756. MR 1120512

[29] Kehe Zhu, *Operator theory in function spaces*, second ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007. MR 2311536

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