GENERALIZED DEPTH AND ASSOCIATED PRIMES IN THE PERFECT CLOSURE R^{∞}

by

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Dedication

To Mom and Dad.

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Abstract

GENERALIZED DEPTH AND ASSOCIATED PRIMES IN THE PERFECT CLOSURE R^{∞}

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Letting $(S, \mathfrak{n})$ be a Noetherian local ring, and M be a finitely generated S-module, the notions of $\operatorname{depth}_S(M)$ and associated primes over M, denoted $\operatorname{Ass}_S(M)$, are fundamental concepts in commutative algebra. However, if S is non-Noetherian, both of these notions become more subtle. Prime ideals in this scenario may then be categorized as associated primes, weakly associated primes, strong Krull primes, and Krull primes, respectively $\operatorname{Ass}_S(M)$, $\widetilde{\operatorname{Ass}}_S(M)$, $\operatorname{sK}_S(M)$, and $\operatorname{K}_S(M)$. Likewise, any study of depth must distinguish between $\operatorname{c\,depth}_S(M)$, $\operatorname{k\,depth}_S(M)$, and $\operatorname{r\,depth}_S(M)$.

Now let $(R, \mathfrak{m})$ be a reduced Noetherian local ring of characteristic $p > 0$. By reduced, we mean there exists no element $r \in R$ such that $r^e = 0$ for some $e \in \mathbb{N}$. If we extend R to a ring which contains $r^{\frac{1}{p^e}}$ for all $r \in R$ and all $e \in \mathbb{N}$, we obtain the perfect closure R^{∞}. This extension shares many properties with R, however it will in general no longer be Noetherian. If we begin with a finitely generated R-module M, we can therefore investigate these more subtle notions of associated primes and depth over the R^{∞}-module $R^{\infty} \otimes_R M$.

In order to investigate any relationships between these measures over R^{∞} and the simpler measures over R, we consider the Frobenius functor. Given an R-module M, this

functor yields another R-module $F(M)$. In this thesis we establish relationships between the generalized associated primes over $R^\infty \otimes_R M$, and associated primes over iterations $F^e(M)$ of this functor, modulo a particular submodule. Then we assume a condition on R called F-purity which ensures a stable depth value of $F^e(M)$ over R for large values of e. The final section of this thesis establishes this stable depth measure, and we establish its relationships with the above mentioned generalized depths of $R^\infty \otimes_R M$ over R^∞. Under both lines of investigation, we obtain results for arbitrary finitely generated M, but we take additional care to discuss cyclic modules $M = R/I$, for which ideal theoretic proofs yield some more explicit arguments.

Chapter 1: Introduction

1.1 Conventions and Basics

All rings are commutative with unity, and all modules are unital.

The topic of this thesis is rings of characteristic $p > 0$, which we define below. Throughout the document, when we write p, we are referring to a prime number $p \in \mathbb{N}$.

We will be discussing both Noetherian and non-Noetherian rings. In particular, the subject matter of this project will be non-Noetherian extensions of Noetherian rings. But all rings will be local, meaning they will contain one and only one maximal ideal. As a convention, we will write $(S, \mathfrak{n})$ to denote arbitrary local rings with maximal ideal $\mathfrak{n}$, and $(R, \mathfrak{m})$ to denote Noetherian local rings with maximal ideal $\mathfrak{m}$. Respectively, we will write $(S, \mathfrak{n}, l)$ or $(R, \mathfrak{m}, k)$ if we discuss the corresponding fields $l := S/\mathfrak{n}$ or $k := R/\mathfrak{m}$.

All rings will be reduced. By saying S is reduced, we mean that there exist no non-zero elements $s \in S$ such that $s^n = 0$ for some $n > 1$ (i.e. $\sqrt{0} = 0$).

If S is a ring, we write $\operatorname{Spec}(S)$ to denote the set of prime ideals of S. This set is equipped with a topology, the *Zariski Topology* in which the closed sets are of the form $V(I) := \{P \in \operatorname{Spec}(S) \mid I \subseteq P\}$, where $I \subset S$ is an ideal.

In many contexts we will discuss p^e, denoting powers of a prime number p. For simplicity we will sometimes use q to represent p^e. When we say q varies, we refer to the values p^e as $e \in \mathbb{N}$ grows.

At varioius points, we will write $e \gg 0$ as e varies over $\mathbb{N}$. If we use this notation for some $\mathbb{N}$-indexed set, we are saying that there exists an $e_0 \geq 0$ where all elements of the set indexed by $e \geq e_0$ satisfy some condition. Similarly we will use $i \gg 0$ and $q \gg 0$, the latter referring to all $q \geq q_0$ for some $q_0 = p^{e_0}$.

If S is a ring and M is an S-module, we say M is *finitely generated* over S , or simply finite, if M can be generated by a finite set. That is, there exists some $\{m_i\}_{i=1}^{n} \subseteq M$ where every $m \in M$ can be written $m = \sum_{i=1}^{n} s_i m_i$ for $s_i \in S$. For example, the direct sum $M := \bigoplus_{i=1}^{n} S$ for $n \in \mathbb{N}$ is finite over S, while $M' := \bigoplus_{i=1}^{\infty} S$ cannot be finitely generated over S. A special case of finitely generated modules is *cyclic modules*, which can be generated by only one element. All cyclic modules are isomorphic to S/I for some ideal $I \subseteq S$.

If S is a ring and $n \in \mathbb{N}$, we will write S^n to denote the cartesian product $S^n := \Pi_{i=1}^{n} S = S \times S \times \ldots \times S$. This product is up to isomorphism the complete description of a finitely generated S-module called a *free module*, where the generating set forms a basis, or maximal linearly independent set. The cardinality n of its generating set is referred to as the module's *rank*.

We will be using many general constructions in commutative algebra. For more general and/or basic constructions, see the appendix A. Presently, however, we include any definitions which are more directly relevant to our scenario. Unless otherwise specified, such as perfect closure, Frobenius closure, and f-sequences, all definitions throughout the introduction section can be found in standard commutative algebra texts. If the reader requires a more thorough treatment, please see [BH97]. Some specific constructions vary among texts, such as the Koszul complex, however the definitions given here will serve adequately for the present discussion.

Throughout the discussion, let S be an arbitrary commutative ring, and let M and N be unital S-modules.

1.2 Koszul Complex

We begin with the Koszul complex $K_\bullet(\mathbf{x}; M)$ for a finite sequence $\mathbf{x} \subset S$ over an S-module M. As with any chain complex, the homology groups, and the information they yield, are a key goal of this construction. We introduce an explicit definition which can be found in

[Mat86, Chapter 6, Section 16].

Let $\mathbf{x} = x_1, \ldots, x_n$ be a finite sequence of elements in S, and let M be an S-module. Let $K_\bullet(\mathbf{x}; S)$ be the chain complex:

$$\ldots \longrightarrow 0 \longrightarrow S^{\binom{n}{n}} \xrightarrow{d_n} S^{\binom{n}{n-1}} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} S^{\binom{n}{0}} \longrightarrow 0$$

where each $S^{\binom{n}{k}}$ is the free module over S with rank $\binom{n}{k}$. Let this corresponding free module be generated by free generators indexed by all k-sized subcollections of $1, 2, \ldots, n$, as in $\mathbf{e}_{j_1,\ldots,j_k}$. The differentials d_k are defined on these free generators by:

$$d_k \ : \ \mathbf{e}_{j_1,\ldots,j_k} \longrightarrow \sum_{i=1}^{k} (-1)^{i-1} x_{j_i} \mathbf{e}_{j_1,\ldots,\hat{j}_i,\ldots j_k}$$

Here $j_1, \ldots, \hat{j}_i, \ldots j_k$ denotes the $k-1$-subcollection consisting of the same elements as $j_1, \ldots, j_k$, though omitting j_i.

We define the Koszul complex for $\mathbf{x}$ on M as $K_\bullet(\mathbf{x}; M) := K_\bullet(\mathbf{x}; S) \otimes_S M$. Explicitly,

$$\ldots \ \longrightarrow \ 0 \ \longrightarrow \ S^{\binom{n}{n}} \otimes_S M \ \xrightarrow{d_n \otimes_S 1} \ S^{\binom{n}{n-1}} \otimes_S M \ \xrightarrow{d_{n-1} \otimes_S 1} \ \ldots \ \xrightarrow{d_1 \otimes_S 1} \ S^{\binom{n}{0}} \otimes_S M \ \longrightarrow \ 0$$

Below we will use the homology groups of $K_\bullet(\mathbf{x}; M)$ to uncover features of $\mathbf{x}$. Namely, we obtain a measure of how the sequence $\mathbf{x} \subset S$ acts on M.

1.3 Regular Sequences

If S is any ring and M is an S-module, an element $s \in S$ is called M-*regular* if it annihilates no element of M, i.e. the map $M \xrightarrow{\cdot s} M$ is injective. A sequence $s_1, \ldots, s_n \in S$ is an M-*regular sequence* if:

1. $M/(s_1, \ldots, s_n)M \neq 0$

2. Each $\overline{s}_i$ is regular over $M/(s_1, \ldots, s_{i-1})M$

An M-regular sequence $\mathbf{x} \subset S$ is called maximal if there exists no $y \in S \setminus \mathbf{x}$ such that $\mathbf{x}, y$ is still an M-regular sequence.

One fact about finite sequences in S over M is that $s_1, \ldots, s_n$ is regular over M if and only if $s_1^{e_1}, \ldots, s_n^{e_n}$ is a regular sequence over M if and only if $s_1^{\frac{1}{c_1}}, \ldots, s_n^{\frac{1}{c_n}}$ is a regular sequence over M (provided each $s_i^{\frac{1}{c_i}} \in S$ exists) [Nor76, Theorem 5.1.3]. Here each $e_i, c_j \in \mathbb{N}$.

Regular sequences feature in the notion of grade of an ideal over a module, which we define in chapter 3. As we shall see, this notion requires a more thorough discussion in the non-Noetherian context than in Noetherian rings. As such, we reserve this discussion until later.

1.4 Prime avoidance and Countable Prime Avoidance

The prime avoidance lemma is a fundamental result in commutative algebra which holds for all rings. The lemma states that for any ring S, ideal $I \subset S$, and finite collection of prime ideals $P_1, \ldots, P_n \subset S$, if $I \subseteq \bigcup_{i=1}^n P_i$, then $I \subseteq P_i$ for some i [Kap70, section 2.2].

This result is relevant when discussing finitely many prime ideals. However, some research in commutative ring theory has investigated *countable* prime avoidance, which is the the identical lemma, except it allows for $\{P_i\}$ to be a countably infinite set. This condition is satisfied by $i)$ any complete local ring, or $ii)$ any ring which contains uncountably many elements $\{u_\lambda\}_{\lambda \in \Lambda}$ for which $u_\lambda - u_\mu$ is a unit for $\lambda \neq \mu$. In particular, any ring which contains an uncountable field is an example of this second condition (see [LW12, Lemma 13.2], [Bur72, Lemma 3], [SV85], [HH00]).

The stronger result states that in any ring S which satisfies $i)$ or $ii)$ above, for any ideal $I \subset S$, and any countable collection of prime ideals in $P_1, P_2, \ldots \subset S$, if $I \subseteq \bigcup_{i=1}^\infty P_i$, then $I \subseteq P_i$ for some i. This result will prove relevant in theorem 12 below.

To give an example of a ring which fails to satisfy countable prime avoidance, let $R =$

$\mathbb{Q}[x, y]_{(x,y)}$, which is a two-dimentional local ring. The only height 0 ideal is the zero ideal since R is a domain, and the only height 2 ideal is the maximal $\mathfrak{m} = (x, y)$. But there exist a countably infinite number of height 1 primes, namely all principal ideals generated by irreducible polynomials with constant term 0. But the union of all such ideals is precisely (x, y), which is clearly not contained in any particular height 1 prime.

1.5 Characteristic $p > 0$

We now narrow our focus to the subject matter of this thesis. Let S be a reduced commutative ring of characteristic $p > 0$. In such rings, p is the least value for which $1 + 1 + \ldots + 1 = p \equiv 0$, where $p \in \mathbb{N}$ is some prime number. Alternately if successive sums of 1 never equal 0, we say the ring has characteristic zero, such as any ring containing the rationals numbers $\mathbb{Q}$.

An example of a ring of characteristic $p > 0$ is the polynomial ring $S := \mathbb{F}_p[x_1, \ldots, x_n]$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p-elements and $x_1, \ldots, x_n$ are variables. Explicitly, this ring consists of all polynomials $g(x_1, \ldots, x_n)$ in n variables whose coefficients are all elements of $\mathbb{F}_p$, and whose domain is $(\mathbb{F}_p)^n$. Addition and multiplication in this ring are standard polynomial addition and multiplication.

The *Frobenius endomorphism* is central to the study of algebra in characteristic $p > 0$. Let $f : S \to S$ denote the map defined by $f(s) := s^p$ for all $s \in S$. In this context, this map is a ring homomorphism since $f(rs) = (rs)^p = r^p s^p = f(r)f(s)$, and

$$
\begin{aligned}
f(r+s) = (r+s)^p &= \sum_{i=0}^{p} \binom{p}{i} r^{p-i} s^i \\
&= r^p + \left(\sum_{i=1}^{p-1} \left(\frac{p!}{(i)!(p-i)!} \right) r^{p-i} s^i \right) + s^p \\
&= r^p + p\left(\sum_{i=1}^{p-1} \left(\frac{(p-1)!}{(i)!(p-i)!} \right) r^{p-i} s^i \right) + s^p \\
&\equiv r^p + 0\left(\sum_{i=1}^{p-1} \left(\frac{(p-1)!}{(i)!(p-i)!} \right) r^{p-i} s^i \right) + s^p \\
&= r^p + 0 + s^p \\
&= r^p + s^p \\
&= f(r) + f(s)
\end{aligned}
$$

Henceforth, whenever a map $f : S \to S$ appears, we refer to the Frobenius endomorphism.

1.6 Frobenius powers and the Frobenius closure

In a ring S of characteristic $p > 0$ the p^eth *Frobenius power* of an ideal $I = (x_1, \ldots, x_n)$ is the ideal generated by p^eth powers of the generators of I. That is, $I^{[p^e]} := (x_1^{p^e}, \ldots, x_n^{p^e})$. Frobenius powers of ideals feature centrally in many closure operations in rings of characteristic $p > 0$.

Below we will also discuss the ideal $I^{[\frac{1}{p^e}]} := (x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}})$, which may or may not exist in a given ring S, as some p^eth roots of some x_i's may not exist in S. However it will exist in the extension S^{∞} which we discuss in the next section. When such an ideal occurs, we will write $(\mathbf{x})^{[\frac{1}{q}]}S^{\infty}$.

Henceforth, for ease of notation we will use the convention $q := p^e$ to denote higher powers of p. For example, $I^{[q]}$ refers to the $p^e - th$ Frobenius power $I^{[p^e]}$ for some $e \in \mathbb{N}$ and some ideal I.

In Noetherian rings R of characteristic $p > 0$, we focus on the *Frobenius closure* (or F-closure) on ideals $I \subset R$, denoted I^F, where $I^F := \{r \in R \mid r^q \in I^{[q]} \text{ for some } q = p^e > 0\}$. This operation forms a closure operator, and I^F is also an ideal in S (see [Eps11]).

1.7 The Perfect Closure of a Reduced Ring of Characteristic $p > 0$

Henceforth we will use R to denote arbitrary Noetherian rings for any definitions. Furthermore, for our results throughout this project we will begin with a Noetherian local reduced $(R, \mathfrak{m})$. If we require a different assumption, we will specify.

Since all of our rings R are reduced, we know that the Frobenius $f : R \to R$ is an injective map as no $r \in R$ will yield $f(r) = r^p = 0$. However it will not be surjective in general. For example, if $R := \mathbb{F}_p[x]$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the variable x has no p-th root. We can therefore extend R by abstractly adjoining all p-th roots of elements of R.

Remark 1.7.1. In order to construct this extension, given $r \in R$ we include a justification for the existence of $r^{\frac{1}{p}}$. First we show that the total quotient ring of R is a finite direct product of fields. That is, $Q(R) \cong \Pi_{i=1}^{n} k_i$ for fields k_i.

Since R is a reduced Noetherian ring, all the associated primes of R are all minimal, or $\text{Ass}(R) = \min(R)$ (associated primes are defined below, see chapter 2). Letting $W = R \setminus \bigcup \text{Ass}(R) = R \setminus \bigcup \min(R)$ be the collection of all non-zero divisors of R, then the total quotient ring of R is given by $Q(R) = W^{-1}R$. $Q(R)$ is then Noetherian and zero dimensional, and therefore Artinian. Thus $Q(R) \cong \Pi_{i=1}^{n} R_i$ for some n, where each R_i is Artinian local. But since R is reduced, each R_i is a field.

Now let $(r_1, \ldots, r_n)$ denote the image of r in $Q(R)$, where $r_i \in k_i$. Then for each i, $r_i^{\frac{1}{p}} \in \overline{k}_i$. That is, $r_i^{\frac{1}{p}}$ is in the algebraic closure of k_i since it is the root of the polynomial $x^p - r_i$. We therefore see that $r^{\frac{1}{p}} = (r_1^{\frac{1}{p}}, \ldots, r_n^{\frac{1}{p}})$ exists in the extension $R \subseteq Q(R) \subseteq \Pi_{i=1}^n \overline{k}_i$.

Having established that pth roots of all $r \in R$ exist in an extension of R, let $R \hookrightarrow R^{\frac{1}{p}}$ denote the extension formed by adjoining all such roots to R. Thus $f : R^{\frac{1}{p}} \to R$ is a surjection. We can repeat this process and form an ascending chain, each isomorphic to one another as rings:

$$R \subseteq R^{\frac{1}{p}} \subseteq R^{\frac{1}{p^2}} \subseteq R^{\frac{1}{p^3}} \subseteq \ldots$$

We now have a directed system $(R^{\frac{1}{p^e}}, \varphi_{ee'})$ in the category of commutative rings, where $\subseteq$ is our pre-order and $\mathbb{N}$ is its index. The maps $\varphi_{ee'} : R^{\frac{1}{q}} \hookrightarrow R^{\frac{1}{q'}}$ are the embedding maps. We define the *perfect closure* or *perfection* of S as the direct limit of this chain:

$$R^\infty := \varinjlim R^{\frac{1}{p^e}}$$

which contains all p^e-th roots of any $s \in R^\infty$. Hence the Frobenius homomorphism is a surjection, and therefore an automorphism, on R^∞. Greenberg showed that given R we can always form R^∞ [Gre65]. For an explicit construction of a non-commutative twisted polynomial ring S, which then contains a commutative subring which is isomorphic to R^∞, see [Jor82], and a description in [NS04]. This construction also addresses how to construct the extension R^∞ given a non-reduced ring R.

Remark 1.7.2. With R^∞ now defined, we now have an alternate description of the Frobenius closure of an ideal. If $I = (x_1, \ldots, x_n) \subset R$ is an ideal in a reduced ring R of characteristic $p > 0$, we now have $I^F := IR^\infty \cap R$. Here IR^∞ is the ideal in R^∞ generated by the same generating set as I. We se e this equivalence because if $x = \sum_{i=1}^n x_i r_i^{\frac{1}{q_i}} \in IR^\infty$, then $x^q = \sum_{i=1}^n x_i^q r_i^{\frac{q}{q_i}} \in I^{[q]}$ for $q \geq \max\{q_i\}$. Conversely, if $x^q = \sum_{i=1}^n x_i^q r_i \in I^{[q]}$, then $x = \sum_{i=1}^n x_i r_i^{\frac{1}{q}} \in IR^\infty$.

Though R is a Noetherian ring, R^∞ will rarely be Noetherian as well. For an example of this fact, let $R := \mathbb{F}_p[x]$, whence $R^\infty := \mathbb{F}_p[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots]$. Here we can see that the chain,

$$(x) \subset (x^{\frac{1}{p}}) \subset (x^{\frac{1}{p^2}}) \subset \ldots$$

never stabilizes.

Specifically, R^∞ will be Noetherian if and only if R is a direct product of finitely many fields [NS04, Theorem 6.3]. However despite this difference, many well-determined relationships between R and R^∞ make the study of one ring closely intertwined with the other. Two particular such relationships which feature prominently in the ensuing discussion are the following:

1. There exists a bijective correspondence between f-sequences of R and ideals of R^∞ [NS04, Corollary 3.2]. The definition of f-sequences, as well as the explicit bijection, are defined below.

2. There exists an order isomorphism between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R^\infty)$. That is, the contraction map $\varphi : \mathrm{Spec}(R^\infty) \to \mathrm{Spec}(R)$ given by $Q \to Q \cap R$ for $Q \in \mathrm{Spec}(R^\infty)$ is an order-preserving bijection [NS04, Theorem 6.1, i)].

In light of condition 2), henceforth we will use the notation P and P^∞ to denote corresponding prime ideals in R an R^∞ respectively.

With such correspondences in mind, the overarching goal of this project is to explore such relationships, determine any further ones that may exist, and use features of both R and R^∞ to discover structural details of one another.

We begin by showing a stronger statement for 2), namely that $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R^\infty)$ are homeomorphic as topological spaces.

Theorem 1. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and let R^∞ be its perfect closure. Then the contraction map $\varphi : \mathrm{Spec}(R^\infty) \to \mathrm{Spec}(R)$ is a homeomorphism with respect to the Zariski topology.*

Proof. φ is already known to be a bijection by [NS04, Theorem 6.1, i)]. We first show that φ is continuous. Let $V(I) \subset \mathrm{Spec}(R)$ be a closed set for an ideal $I \subset R$, and we must show $\varphi^{-1}(V(I))$ is closed.

We claim $\varphi^{-1}(V(I)) = V(IR^\infty)$, which is also closed. Fix $P \in V(I)$, i.e. $I \subseteq P$, and hence $IR^\infty \cap R = I^F \subseteq P$. For any $x^{\frac{1}{q}} \in IR^\infty$, $x \in (I^{[q]}R^\infty \cap R) \subseteq (IR^\infty \cap R) \subseteq P = (P^\infty \cap R) \subseteq P^\infty$, whereby $x^{\frac{1}{q}} \in P^\infty$. Hence $IR^\infty \subseteq P^\infty$ and $P^\infty \in V(IR^\infty)$. Conversely, if $IR^\infty \subseteq P^\infty$, then $I \subseteq IR^\infty \cap R \subseteq P^\infty \cap R = P$, and $P \in V(I)$.

Next we must show that $\varphi^{-1} : \mathrm{Spec}(R) \to \mathrm{Spec}(R^\infty)$ is continuous. Let $V(J) \subset \mathrm{Spec}(R^\infty)$ be a closed set for some ideal $J \subset R^\infty$. We claim $\varphi(V(J)) = V(J \cap R)$, and hence its pre-image under φ^{-1} is closed. If $J \subseteq P^\infty$, then clearly $J \cap R \subseteq P^\infty \cap R = P$, whence $P \in V(J \cap R)$. Conversely, if $J \cap R \subseteq P$, fix $x^{\frac{1}{q}} \in J$. Then $x \in (J^{[q]}R^\infty \cap R) \subseteq (J \cap R) \subseteq P = (P^\infty \cap R) \subseteq P^\infty$. Hence $x^{\frac{1}{q}} \in P^\infty$, and $J \subseteq P^\infty$. $\square$

1.8 Frobenius Functor

Given a reduced ring R of characteristic $p > 0$, we now introduce the Frobenius functor on an R-module M. The image of M under this functor is denoted $F(M)$, and the relationship between M and $F(M)$ is a generalization of the relationship between R/I and $R/I^{[p]}$ for an ideal $I \subset R$ (see remark 1.8.1). In the literature this functor is typically defined using the R-R bimodule R^f which can be found in many resources such as [BH97]. However, we will define the R-R bimodule $R^{\frac{1}{f}}$, which is an equivalent construction. This alternate point of view yields a compatible definition of the module $R^\infty \otimes_R M$ as an R^∞ module. Equipped with such an object, we can then investigate this module in terms of non-Noetherian generalized depth measures which will be defined below.

First we introduce the standard R^f construction. Let R^f denote an R-R bimodule, which is isomorphic to R itself. However, let the action by R be as follows, for $a \in R^f$, and $r, s \in R$:

- Left action is performed in the obvious way, $r \cdot a = ra$
- Right action is given, $a \bullet s = af(s) = as^p$

To present our point of view, now let $R^{\frac{1}{f}}$ denote an R-R bimodule which as an abelian group is identical to the ring $R^{\frac{1}{p}}$. However the two-sided actions by R are as follows, for $a^{\frac{1}{p}} \in R^{\frac{1}{f}}$ and $r, s \in R$:

- Left action is given by $r \circ a^{\frac{1}{p}} = f^{-1}(r)a^{\frac{1}{p}} = r^{\frac{1}{p}}a^{\frac{1}{p}}$
- Right action is performed in the obvious way, $a^{\frac{1}{p}} \cdot s = a^{\frac{1}{p}}s$

Note that we reuse the "$\circ$" notation here, which in a previous section referred to a composition of maps. The usage of this symbol will always be made clear.

We claim that R^f and $R^{\frac{1}{f}}$ with their respective actions are isomorphic as R-R bimodules. In order to see this equivalence, consider the Frobenius homomorphism $f : R^{\frac{1}{f}} \to R$, through which $f(a^{\frac{1}{p}}) = (a^{\frac{1}{p}})^p = a$ for all $a \in R$. Recall that this map is an isomorphism since R is reduced. We then have,

$$r \circ a^{\frac{1}{p}} \cdot s = r^{\frac{1}{p}}a^{\frac{1}{p}}s \xrightarrow{f} ras^p = r \cdot a \bullet s$$

With this isomorphism established, we will henceforth use the $R^{\frac{1}{f}}$ construction.

Iterating, we have $R^{\frac{1}{f^e}}$ which as an R-module is isomorphic to $R^{\frac{1}{q}}$ for $q = p^e$, and left and right action by R is $r \circ a^{\frac{1}{q}} \cdot s = r^{\frac{1}{q}}a^{\frac{1}{q}}s$. Note that for any $q' \leq q$, any arbitrary element of $R^{\frac{1}{f}}$ can be of the form $a^{\frac{1}{q'}}$ for $a \in R$. However for any such element, we can rewrite as $(a')^{\frac{1}{q}}$, where $a' = a^{\frac{q}{q'}}$. Hence, without loss of generality we will write arbitrary elements of

$R^{\frac{1}{f^e}}$ in the form $a^{\frac{1}{q}}$. While there will be specific situations below in which the exponent of the element will not be equal the corresponding iteration $\frac{1}{f^e}$, all such situations will be specified.

We can also consider $R^{\frac{1}{f^e}}$ as a right $R^{\frac{1}{q}}$-module with the action performed in the obvious way. That is, $a^{\frac{1}{q}} \cdot s^{\frac{1}{q}} = a^{\frac{1}{q}} s^{\frac{1}{q}}$, which is compatible with the right action by R since $R \subseteq R^{\frac{1}{q}}$. Hence $R^{\frac{1}{f^e}} \in {}_R\mathrm{Mod}_{R^{1/q}}$, or $R^{\frac{1}{f^e}}$ is in the category of modules which are left defined over R, and right defined over $R^{\frac{1}{q}}$. Again in particular, objects in ${}_R\mathrm{Mod}_{R^{1/q}}$ are also in the category ${}_R\mathrm{Mod}_R$ because $R \subseteq R^{\frac{1}{q}}$, and hence they possess a right action by R.

Having established $R^{\frac{1}{f}}$, we can define the *Frobenius functor* $F(M)$, which is a covariant right exact functor in both the categories of left and right R-modules. These facts follow from properties of tensor products. We define:

$$F(M) := R^{\frac{1}{f}} \otimes_R M$$

and if $\varphi : M \to N$ is an R-module map, then:

$$F(\varphi) := 1_{R^{1/f}} \otimes_R \varphi$$

$F(M)$ is equipped with an R-$R^{\frac{1}{p}}$ bimodule structure, and we describe here the left and right action on a simple tensor. Note that we write such a simple tensor in the form $(a^{\frac{1}{p}} \otimes_R m^{[p]})$. See remark 1.8.2 for an explanation of the exponentiation $m^{[p]}$ used in this notation. We now have,

$$\begin{aligned} r \circ (a^{\frac{1}{p}} \otimes_R m^{[p]}) \cdot s^{\frac{1}{p}} &:= (r \circ a^{\frac{1}{p}} \cdot s^{\frac{1}{p}} \otimes_R m^{[p]}) \\ &= (r^{\frac{1}{p}} a^{\frac{1}{p}} s^{\frac{1}{p}} \otimes_R m^{[p]}) \end{aligned}$$

Further iterations are then given by $F^e(M) := R^{\frac{1}{f^e}} \otimes_R M$, with likewise left and right action, which is equivalent to $F(F(\dots F(M)))$ composed with itself e times. Hence, $F^e(M) \in {}_R\operatorname{Mod}_{R^{1/q}}$ for all e. And in particular $F^e(M)$ is an R-R bimodule since $R \subseteq R^{\frac{1}{q}}$.

Remark 1.8.1. We make a note regarding an explicit characterization of $F^e(M)$, where $M = R/I$ is a cyclic module. Above, we stated that relationship between M and $F(M)$ is a generalization of the relationship between R/I and $R/I^{[p]}$. We justify this claim for all $F^e(M)$.

By definition, $F^e(R/I) := R^{\frac{1}{f^e}} \otimes_R R/I \cong R^{\frac{1}{q}}/IR^{\frac{1}{q}}$, with this last equivalence holding since $R^{\frac{1}{f^e}} \cong R^{\frac{1}{q}}$ as a left R-module with left action $\circ$. But we claim this module is isomorphic to the cyclic R-module $R/I^{[q]}$. To see this fact, consider the maps $R^{\frac{1}{q}} \xrightarrow{f^e} R \xrightarrow{g} R/I^{[q]}$, where f is the e-th iteration of the Frobenius homomorphism, which is an isomorphism since R is reduced, and g is the natural projection. Letting $h := g \circ f$ be the composition map (to distinguish this use of the notation "$\circ$" from the left module action by R), the first isomorphism theorem states that $R^{\frac{1}{q}}/\ker(h) \cong R/I^{[q]}$. But $\ker(h) = IR^{\frac{1}{q}}$. To establish this fact, letting $I = (x_1, \dots, x_n)$, we have,

$$
\begin{aligned}
z \in \ker(h) &\Leftrightarrow z^q \in I^{[q]} \\
&\Leftrightarrow z^q = \sum_{i=1}^{n} x_i^q r_i \text{ for some } r_1, \dots, r_n \in R \\
&\Leftrightarrow z = \sum_{i=1}^{n} x_i r_i^{\frac{1}{q}} \text{ for some } r_1^{\frac{1}{q}}, \dots, r_n^{\frac{1}{q}} \in R^{\frac{1}{q}} \\
&\Leftrightarrow z \in IR^{\frac{1}{q}}
\end{aligned}
$$

Remark 1.8.2. Returning to the general case for R-modules M, note that in describing an arbitrary element of $F(M)$ above, we wrote $(a^{\frac{1}{p}} \otimes_R m^{[p]})$ with the exponent $[p]$ for $m \in M$. Likewise, we will denote $(a^{\frac{1}{q}} \otimes_R m^{[q]})$ for an arbitrary element of $R^{\frac{1}{f^e}}$. That is, the exponent

$[q] = [p^e]$ will indicate which Frobenius iteration $F^e(M)$ we are discussing. This notation will become relevant when some element of $F^{e'}(M)$ may be of the form $(a^{\frac{1}{q}} \otimes_R m^{[q']})$ for $q < q'$. For example, fix $(a^{\frac{1}{q}} \otimes_R m^{[q]}) \in F^e(M)$, and for $e \leq e'$ consider the natural map $\varphi : F^e(M) \to F^{e'}(M)$ induced by the inclusion $R^{\frac{1}{f^e}} \hookrightarrow R^{\frac{1}{f^{e'}}}$. This element's image under φ is denoted $(a^{\frac{1}{q}} \otimes_R m^{[q']})$.

To give another example of such discrepancy in exponents, note that for a simple tensor of the form $(s \otimes_R m^{[q]}) \in F^e(M)$, it is identical to $s^q \circ (1 \otimes_R m^{[q]})$ due to the left action of R. In general, for any $q \leq q'$ simple tensors of the form $(s^{\frac{1}{q}} \otimes_R m^{[q']})$ can be written $(s^{\frac{1}{q}} \otimes_R m^{[q']}) = (s^{\frac{1}{q}})^{q'} \circ (1 \otimes_R m^{[q']}) = s^{\frac{q'}{q}} \circ (1 \otimes_R m^{[q']}) \in F^{e'}(M)$. This fact will be relevant below.

Since $F^e(M) := (R^{\frac{1}{f^e}} \otimes_R M)$, we now have a directed system $\big(F^e(M), (\psi_{ee'} \otimes_R 1_M)\big)$, where the maps $\psi_{ee'} : R^{\frac{1}{f^e}} \hookrightarrow R^{\frac{1}{f^{e'}}}$ are embeddings. Recall that $R^\infty := \varinjlim R^{\frac{1}{q}}$, and $R^{\frac{1}{q}} \cong R^{\frac{1}{f^e}}$ as abelian groups. Therefore,

$$(R^\infty \otimes_R M) := \varinjlim F^e(M).$$

Since each $F^e(M)$ is defined as a right $R^{\frac{1}{q}}$-module as e grows, this limit $(R^\infty \otimes_R M)$ is now defined as a right R^∞-module, and hence also a right $R \subseteq R^\infty$ module, in the obvious way (and with obvious notation $m^{[\infty]}$):

$$(a^{\frac{1}{q}} \otimes_R m^{[\infty]}) \cdot s^{\frac{1}{q'}} := (a^{\frac{1}{q}} \cdot s^{\frac{1}{q'}} \otimes_R m^{[\infty]}) = (a^{\frac{1}{q}} s^{\frac{1}{q'}} \otimes_R m^{[\infty]})$$

Equipped with the Frobenius functor, we also can define the *Frobenius closure* on sub-modules, which is analogous to the Frobenius closure of ideals in a ring. If M is an R-module, and $N \subseteq M$ is a sub-module, we define the Frobenius closure of N in M as $N^F_M := \{m \in M \mid (1 \otimes_R m^{[q]}) \in N^{[q]} \subseteq F^e(M) \text{ for } e \gg 0\}$. Here $N^{[q]} := F^e(i)(N)$, where $i : N \to M$ is the embedding map, and $F^e(i)$ is its image under the e-th iteration of the

Frobenius functor. Equivalently $N_M^F := \{m \in M \mid (1 \otimes_R \overline{m}^{[q]}) = 0 \in F^e(M/N) \text{ for } e \gg 0\}$. This fact will prove relevant below.

Specifically, we discuss below the Frobenius closure of 0 in $F^e(M)$,

$$0^F_{F^e(M)} := \{(s^{\frac{1}{q}} \otimes_R m^{[q]}) \in F^e(M) \mid (s^{\frac{1}{q}} \otimes_R m^{[q']}) = 0 \in F^{e'}(M) \text{ for } e' \gg e\}$$

Remark 1.8.3. Note that since we are discussing a directed system, by definition $0 \in (R^\infty \otimes_R M)$ is precisely the image of the elements of $0^F_{F^e(M)}$ as e varies. That is,

$$(s^{\frac{1}{q}} \otimes_R m^{[q]}) \in 0^F_{F^e(M)} \Leftrightarrow (s^{\frac{1}{q}} \otimes_R m^{[q']}) = 0 \in F^{e'}(M) \text{ for some } e \leq e'$$

$$\Leftrightarrow (s^{\frac{1}{q}} \otimes_R m^{[\infty]}) = 0 \text{ in } R^\infty$$

In particular, for $M = R/I$, we have that $0^F_{R/I} = I^F/I$. Fix $m \in R$, and we can see this equivalence as follows:

$$\begin{aligned}
\overline{m} \in 0^F_{R/I} &\Leftrightarrow (1 \otimes_R \overline{m}^{[q]}) = 0 \text{ in } F^e(R/I) \text{ for } e \gg 0, \\
&\text{where } F^e(R/I) = R^{\frac{1}{q}} \otimes_R R/I \cong R^{\frac{1}{q}}/IR^{\frac{1}{q}} \\
&\Leftrightarrow \overline{m} = 0 \text{ in } R^{\frac{1}{q}}/IR^{\frac{1}{q}} \\
&\Leftrightarrow \overline{m}^q = 0 \text{ in } R/I^{[q]} \text{ for } e \gg 0 \\
&\Leftrightarrow m^q \in I^{[q]} \text{ for } q \gg 0 \\
&\Leftrightarrow m \in I^F
\end{aligned}$$

And by identical reasoning, $0^F_{F^e(R/I)} = (I^{[q]})^F/I^{[q]}$.

1.9 f-sequences

In a reduced ring R of characteristic $p > 0$, an *f-sequence* is a descending chain of ideals

$$\dots I_{e-1} \supseteq I_e \supseteq I_{e+1} \supseteq \dots$$

such that $f^{-1}(I_{e+1}) \cap R = I_e$ for every e, where f denotes the Frobenius endomorphsim which is defined on both R and R^∞. In discussing R^∞ this concept arises naturally. Recall that the set of ideals of R^∞ is in order-preserving bijective correspondence with the set of f-sequences of R. For an ideal $J \subset R^\infty$, this correspondence is given explicitly by:

$$\Gamma \;:\; J \to \big(J_e\big)_{e\in\mathbb{N}} \qquad \text{where for every } e \in \mathbb{N},\;\; J_e := \{r \in R \mid r^{\frac{1}{p^e}} \in J\} \qquad \text{[NS04]},$$
$$\text{or alternately stated } J_e = f^e(J) \cap R$$

In particular, note that $J_0 = J \cap R$.

For an example of a particular correspondence, if $I \subset R$ is an ideal, let $J = IR^\infty$ be the ideal generated in R^∞ by the same generators of I. Then its corresponding f-sequence in R is $\{(I^{[q]})^F\}$. Note that we take the Frobenius closure of the Frobenius powers of I. It is a fact that every ideal in an f-sequence is F-closed [NS04]. Hence given an ideal $I \subset R$ we can view $S := \{(I^{[q]})^F\}$ as a minimal f-sequence, since any f-sequence which contians I at some stage must therefore contain S.

Some examples of f-sequences are $\{J_e\} = \{(I^{[q]})^*\}$ and $\{J_e\} = \{(I^{[q]})^+\}$, where "+" and"$*$" denote plus closure and tight closure respectively, and $I \subset R$ is any ideal (for a discussion of these closure operations, see [SN04, Lemmas 5.1 and 5.2]). We list some additional examples, along with their corresponding ideals in R^∞.

Example 1.9.1. Let R be a Noetherian ring of characteristic $p > 0$. The following sequences $\{J_e\} \subset R$ are f-sequences with corresponding ideals $J \subset R^\infty$.

1. $\{J_e\} = \{(I^{[q]})^F\}$, $J = IR^\infty$, where $I \subset R$ is any ideal.

2. Letting $R = k[x,y]$, $\{J_e\} = \{(x,y)^{[q]}\} = \{(x^q, y^q)\}$, $J = (x,y)R^\infty$, where k is any field of characteristic $p > 0$.

3. Letting $R = k[x,y]$, $\{J_e\} = \{(x, y^q)\}$, $J = (x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots, y)R^\infty$

4. $\{J_e\} = \{P, P, \ldots\}$ for $P \in \operatorname{Spec}(R)$, $J = P^\infty$

Note that example 2 is of the form $\{I^{[q]}\}$, but examples $1, 3$ and 4 show that such a form does not characterize all f-sequences. Furthermore, in general a sequence of Frobenius iterates $\{I^{[q]}\}$ for some $I \subset R$ will not yield an f-sequence since $I^{[q]}$ may not be F-closed as q varies. However, there do exist rings for which such a construction will always do so. In order for such a scenario to hold, we will require the condition that all ideals $I \subset R$ are F-closed. Below we will discuss F-pure rings, where $I = I^F$ for all ideals $I \subset R$.

Recall that we study the scenario where our ring R is Noetherian. For any f-sequence $\{I_e\}$ in R, we now have an ascending chain

$$\ldots \supseteq I_{e-2} \supseteq I_{e-1} \supseteq I_e$$

for any given $e \in \mathbb{N}$. This chain must therefore stabilize. In [NS04, Remark 4.2, iv)] it is stated that I_e can be extended "downwards" given e. However, the Noetherianness of R yields a stabilizing ideal regardless of e.

Suppose now the chain stabilizes to some ideal I for which $I = f^{-1}(I)$. We ask whether we can explicitly find this stabilizing ideal. We begin with a lemma which shows that such an I is a radical ideal.

Lemma 1.9.2. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and let $I \subset R$ be an ideal. Then the following are all equivalent*

1. $f^{-1}(I) = I$

2. $f^{-e}(I) = I$ *for all* e

3. $I = \sqrt{I}$

Proof. <u>1 ⇔ 2:</u>

Clearly if $f^{-e}(I) = I$ for all e, then $f^{-1}(I) = I$. Conversely, if $f^{-1}(I) = I$, then

$$\begin{aligned} f^{-e}(I) &= f^{-1}(f^{-1}(\dots f^{-1}(f^{-1}(I)))) \\ &= f^{-1}(f^{-1}(\dots f^{-1}(I))) \\ &\vdots \\ &= f^{-1}(I) \\ &= I \end{aligned}$$

<u>2 ⇒ 3:</u> Suppose $r^n \in I$ for some n, and let $q > n$. Then $r^q \in I$. But then $f^{-e}(r^q) = r$, which is in I since $f^{-e}(I) = I$.

<u>3 ⇒ 1:</u> Suppose $r \in f^{-1}(I)$. Then $r^p \in I$, and $r \in I$ since $I = \sqrt{I}$.

□

It is a known fact that every term in an f-sequence has the same radical, i.e. $\sqrt{I_e} = \sqrt{I_{e'}}$ for all $e, e' \in \mathbb{N}$ [NS04, Remark 4.2, v)]. In fact, this radical ideal coincides with the stabilizing ideal.

Theorem 2. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and Let $\{I_e\}_{e\in\mathbb{N}}$ be an f sequence in R. Let $I \subset R$ be the ideal such that the ascending chain*

$$\dots \supseteq I_{e-2} \supseteq I_{e-1} \supseteq I_e \supseteq \dots$$

stabilizes, for $e \in \mathbb{N}$. Then,

$$I = \sqrt{I_e} \textit{ for all } e$$

Proof. Fix e, and clearly $\sqrt{I_e} \subseteq \sqrt{I} = I$, with this containment holding because $I_e \subseteq I$, and "$\sqrt{}$" is a closure operation [Eps11], while the equality holds by lemma 1.9.2.

Conversely, first note that for some $e' \geq e$, $f^{-e'}(I_e) = I$. Hence $I_e \supseteq f^{e'}(I)$. Now fix $r \in I$, and $r^{q'} = f^{e'}(r) \in I_e$, and $r \in \sqrt{I_e}$. $\square$

Chapter 2: Generalized Associated Prime Ideals

Let S be an arbitrary ring, and let M be any S-module. We say $P \in \operatorname{Spec}(S)$ is an *associated prime ideal* of M, or $P \in \operatorname{Ass}_S(M)$, if P is the annihilator in S of some non-zero element $m \in M$. In such a case we write $P = \operatorname{ann}_S(m)$. That is, $pm = 0$ for all $p \in P$, and if $p'm = 0$ then $p' \in P$. An equivalent definition states that $P \in \operatorname{Ass}_S(M)$ if the cyclic module S/P can be embedded into M, with $\varphi : S/P \hookrightarrow M$ denoting the map $\varphi : 1 \mapsto m$. As a consequence of this alternate definition, if $P \in \operatorname{Ass}_S(M)$, then $\operatorname{Hom}_S(S/P, M) \neq 0$.

For an example, let $R = \mathbb{Z}$, and $M = \mathbb{Z}/4\mathbb{Z}$. Then $(2) \in \operatorname{Ass}_{\mathbb{Z}}(M)$, with $(2) = \operatorname{ann}_{\mathbb{Z}}(\overline{2})$, since $2 \times \overline{2} = \overline{4} = \overline{0}$. And we see that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \neq 0$, since we have the embedding map φ defined by $\varphi : \overline{1} \mapsto \overline{2}$.

If S is a non-Noetherian ring, however, the notion of an associated prime ideal over a module becomes more subtle. We state three distinct subsets of $\operatorname{Spec}(S)$:

Definition. Let S be any commutative ring with identity. Let M be an S-module. Let $P \in \operatorname{Spec}(S)$ be a prime ideal.

1. $P \in \widetilde{\operatorname{Ass}}_S(M)$ is a *weakly associated prime*, or weak Bourbaki prime of M if it is minimal over some $\operatorname{ann}_S(m)$ for some $m \in M$. That is, $\operatorname{ann}_S(m) \subseteq P$, and if there exists a prime ideal $Q \subset S$ such that $\operatorname{ann}_S(m) \subseteq Q \subseteq P$, then $Q = P$.

2. $P \in \operatorname{sK}_S(M)$ is a *strong Krull prime* of M if for any finitely generated sub-ideal $I \subseteq P$ we have $I \subseteq \operatorname{ann}_S(m) \subseteq P$ for some $m \in M$.

3. $P \in \operatorname{K}_S(M)$ is a *Krull prime* of M if for any element $x \in P$ we have $x \in \operatorname{ann}_S(m) \subseteq P$ for some $m \in M$.

For any ring S and module M, we always have the containments $\mathrm{Ass}_S(M) \subseteq \widetilde{\mathrm{Ass}}_S(M) \subseteq \mathrm{sK}_S(M) \subseteq \mathrm{K}_S(M)$, with none of the containments reversible. For an overview see [IR84]. It is possible that $\mathrm{Ass}_S(M)$ is empty, and we provide an example.

Example 2.0.3. Let $R = \mathbb{F}_p[x]$, and hence $R^\infty = \mathbb{F}_p[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots]$. We claim $\mathrm{Ass}_{R^\infty}(R^\infty/(x)R^\infty) = \emptyset$.

To show this claim, first note that $(x)^\infty$ is minimal over $(x)R^\infty$, and hence it is the only possible associated prime of $R^\infty/(x)R^\infty$. Suppose $(x)^\infty = ((x)R^\infty :_{R^\infty} r)$ for some $r \in R^\infty \setminus (x)R^\infty$, where $r = \Sigma_{i=0}^n a_i x^{\frac{i}{q}}$ for some q, and $a_i \in \mathbb{F}_p$ for all i. Then $x^{\frac{1}{pq}} r \notin (x)R^\infty$. We can see this fact by multiplying:

$$x^{\frac{1}{pq}} r = x^{\frac{1}{pq}} \Sigma_{i=0}^n a_i x^{\frac{i}{q}}$$

$$= \Sigma_{i=0}^n a_i x^{\frac{1}{pq}} x^{\frac{i}{q}}$$

$$= \Sigma_{i=0}^n a_i x^{\frac{ip+1}{qp}}$$

But since $r \notin (x)R^\infty$, for at least one i, the corresponding monomial is non-zero, and $i < q$. But then $\frac{ip+1}{qp} < 1$, and we see that $x^{\frac{1}{pq}} r \notin (x)R^\infty$.

Hence $(x)^\infty$ cannot be an associated prime of $R^\infty/(x)R^\infty$, and hence $\mathrm{Ass}_{R^\infty}(R^\infty/(x)R^\infty) = \emptyset$.

While $\mathrm{Ass}_S(M)$ can be an empty set, if S is a Noetherian ring and $M \neq 0$, there always exists at least one associated prime of M. In this case the first three sets are equal, i.e. $\mathrm{Ass}_S(M) = \widetilde{\mathrm{Ass}}_S(M) = \mathrm{sK}_S(M)$. For an example of why $\mathrm{K}_S(M)$ is omitted in this equality, see [ES14, Remark 2.2]. However, if S is Noetherian *and* M is finitely generated, then all four sets are equal, and we uniformly discuss the set of associated primes of M. In such a scenario $\mathrm{Ass}_S(M)$ is a finite set, and if we let $\mathcal{Z}(\mathcal{M}) \subset S$ denote all elements of S which annihilate *any* element of M, we have $\mathcal{Z}(\mathcal{M}) = \bigcup \mathrm{Ass}_S(M)$ [BH97, Section 1.2]. That is, if

$s \notin \bigcup \operatorname{Ass}_S(M)$, then s is regular over M.

We return to our context where $(R, \mathfrak{m})$ is a Noetherian local ring of characteristic $p > 0$. Recall from above that a prime $P \in \operatorname{Spec}(R)$ corresponds to one and only one prime ideal in $\operatorname{Spec}(R^\infty)$ [NS04, Theorem 6.1, i)], which we are denoting P^∞. We investigate associated prime ideals in $P \subset R$ over a module M and its Frobenius iterates $F^e(M)$, and we determine some correspondences with their counterparts in $P^\infty \subset R^\infty$ over $(R^\infty \otimes_R M)$. In this study, the relationship between $\bigcup 0^F_{F^e(M)}$ and $0 \in (R^\infty \otimes_R M)$ reveals that we must consider associated primes over the modules $F^e(M)/0^F_{F^e(M)}$.

2.0.1 Results for Generalized Associated Prime Ideals

Given a module M over a regular local ring R, Epstein and Shapiro showed a characterization of the strong Krull primes of $R^\infty \otimes_R M$.

Theorem 3. *[ES14, Corollary 4.9] Let R be a regular Noetherian ring of prime characteristic $p > 0$. Let L be any R-module. Then*

$$\operatorname{sK}_{R^\infty}(L \otimes_R R^\infty) = \bigcup_{\mathfrak{q} \in \operatorname{Ass}_R L} \operatorname{sK}_{R^\infty}(R^\infty/\mathfrak{q}R^\infty).$$

Omitting the hypothesis of regularity of R, we show a characterization of these prime ideals in relation to the associated primes of $F^e(M)/0^F_{F^e(M)}$. Additionally we find that the strong Krull primes of $R^\infty \otimes_R M$ coincide with $\widetilde{\operatorname{Ass}}_R(R^\infty \otimes_R M)$.

Theorem 4. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and let R^∞ be its perfect closure. Let P and P^∞ be corresponding prime ideals in* $\operatorname{Spec}(R)$ *and* $\operatorname{Spec}(R^\infty)$ *respectively. Let M be an R-module. Then the following are all equivalent:*

1. *$P \in \bigcup \operatorname{Ass}_R \left(F^e(M)/0^F_{F^e(M)}\right)$ via left action by R*

2. *$P^\infty \in \widetilde{\operatorname{Ass}}_R(R^\infty \otimes_R M)$*

3. $P^\infty \in \mathrm{sK}_R(R^\infty \otimes_R M)$

Proof. $\underline{1 \Rightarrow 2:}$ Suppose $P \in \bigcup \mathrm{Ass}_R(F^e(M)/0^F_{F^e(M)})$ with $P = \big(0^F_{F^e(M)} :_R \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$. I.e. for all $r \in P$, we have $r \circ \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) = \sum_i (r_i^{\frac{1}{q}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) \in 0^F_{F^e(M)}$. But then by Remark 1.8.3, $\sum_i r_i^{\frac{1}{q}} s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]} = 0$ in R^∞. Thus $PR^\infty \subset P^{[\frac{1}{q}]}R^\infty \subseteq \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big)$.

On the other hand, fix some ${s'}^{\frac{1}{q'}} \in \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big)$. If $q' \leq q$, then ${s'}^{\frac{q}{q'}} \in R$ and ${s'}^{\frac{q}{q'}} \circ (\sum_i s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) = (\sum_i {s'}^{\frac{1}{q'}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) \in 0^F_{F^e(M)}$. Hence ${s'}^{\frac{q}{q'}} \in P$, and ${s'}^{\frac{1}{q'}} \in P^\infty$.

However if $q' > q$, consider the image of $s' \circ (\sum_i s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) = \sum_i ({s'}^{\frac{1}{q}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})$ under the map $\psi_{ee'} = \varphi_{ee'} \otimes_R 1 : R^{\frac{1}{f}} \otimes_R M \to R^{\frac{1}{f'}} \otimes_R M$. The corresponding left actions by R on $F^e(M)$ and $F^{e'}(M)$ yield,

$$
\begin{aligned}
\psi_{ee'}\Big(\sum_i ({s'}^{\frac{1}{q}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\Big) &= \sum_i ({s'}^{\frac{1}{q}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q']}) \\
&= \sum_i {s'}^{\frac{q'}{q}} \circ (s_i^{\frac{1}{q}} \otimes_R m_i^{[q']}) \\
&= \sum_i {s'}^{(\frac{q'}{q}-1)} s' \circ (s_i^{\frac{1}{q}} \otimes_R m_i^{[q']}) \\
&= {s'}^{(\frac{q'}{q}-1)} \circ \Big(\sum_i ({s'}^{\frac{1}{q'}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q']})\Big)
\end{aligned}
$$

But $\sum_i (s'^{\frac{1}{q'}} s_i^{\frac{1}{q}} \otimes_R m_i^{[q']}) \in 0^F_{F^{e'}(M)}$ because $s'^{\frac{1}{q'}} \in \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big)$. Thus $s' \in \big(0^F_{F^e(M)} :_R (\sum_i s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$, whereby $s' \in P$ and $s'^{\frac{1}{q'}} \in P^\infty$.

We have now shown that $PR^\infty \subseteq \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big) \subseteq P^\infty$. Since $\sqrt{PR^\infty} = P^\infty$, we know that P^∞ is minimal over PR^∞. Hence P^∞ is minimal over $\big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big)$ as well. This fact shows that P^∞ is minimal over the annihilator of an element of $(R^\infty \otimes_{R^\infty} M)$, and therefore $P^\infty \in \widetilde{\mathrm{Ass}}_R(R^\infty \otimes_R M)$.

<u>$2 \Rightarrow 3$:</u> Always true, as stated above.

<u>$3 \Rightarrow 1$:</u> Suppose $P^\infty \in \mathrm{sK}_R(R^\infty \otimes_R M)$, then PR^∞ is a finitely generated sub-ideal, and we have $PR^\infty \subseteq \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big) \subseteq P^\infty$. Hence $P \subseteq \big(0^F_{F^e(M)} :_R \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$ by definition (see Remark 1.8.3).

Conversely, fix $s \in \big(0^F_{F^e(M)} :_R \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$. Again by definition (Remark 1.8.3), $s \in \big(0 :_{R^\infty} \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\big) \subseteq P^\infty$. Thus $s \in P^\infty \cap R = P$. Therefore $P \supseteq \big(0^F_{F^e(M)} :_R \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$.

We now have $P = \big(0^F_{F^e(M)} :_R \sum_i (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big)$, and $P \in \mathrm{Ass}_R\left(F^e(M)/0^F_{F^e(M)}\right)$.

□

Furthermore, we have a similar theorem regarding cyclic modules. Its proof relies on different techniques, and we must discuss some preliminaries before we proceed.

Remark 2.0.4. Suppose M is a finitely generated R-module, and suppose the ideal J is the annihilator of M. Then by [Kap70, Theorem 86], if $P \in \mathrm{Spec}(R)$ is minimal over J, then P is the annihilator of some non-zero element of M.

In particular, now let N be a finitely generated R-module, and let $n \in N$ be some non-zero element such that $\text{ann}_R(n) \neq 0$. Let $J = ann_R(n)$, and let P be a minimal prime over J. We claim that $P = ann_R(rn)$ for some $r \in R$, i.e. P annihilates some multiple of n. In order to show this claim, let $M = R/J$, whereby P is minimal over the annihilator of M, and therefore P anihilates some non-zero element of M. Thus for some $r \in R \setminus J$, $P = (J :_R r)$. But since we said $J = (0 :_R n)$ for $n \in N$, we now have $P = \big((0 :_R n) :_R r)\big) = (0 :_R rn) = \text{ann}_R(rn)$, where $rn \in N$.

One special case of this particular scenario holds when $N = R/I$ is *any* cyclic module, whereby if P is minimal over $\text{ann}_R(\overline{n})$ for some $\overline{n} \in R/I$, we have that P is minimal over $(I :_R n)$. Thus $P = (I :_R rn)$ for some $r \in R$.

Recall that if $\{J_e\}$ is an f-sequence in R, we have that $\text{Ass}_R(R/J_e) \subseteq \text{Ass}_R(R/J_{e+1})$ for all e. In light of Remark 2.0.4, if $P \in \text{Ass}_R(R/J_e)$ for some e with $P = (J_e :_R s)$, we now have an explicit description of the element it annihilates over R/J_{e+1}. The following lemma is probably known, however we include a proof for the convenience of the reader.

Lemma 2.0.5. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and let $\{J_e\}$ be an f-sequence in R. Fix $P \in \text{Ass}_R(R/J_e)$ with $P = (J_e :_R s)$ for some $s \in R$. Then $P = (J_{e+1} :_R rs^p)$ for some $r \in R$.*

Proof. Fix $P \in \text{Ass}_R(R/J_e)$ with $P = (J_e :_R s)$ for some $s \in R$. We first show that $P \supseteq (J_{e+1} :_R s^p)$. Suppose $xs^p \in J_{e+1}$ for $x \in R$. Then certainly $x^p s^p \in J_{e+1}$, and $xs \in J_e = f^{-1}(J_{e+1})$. Hence $x \in P$.

Furthermore, note that P is minimal over $(J_{e+1} :_R s^p)$. Suppose there exists some $P \supseteq Q \supseteq (J_{e+1} :_R s^p)$ for some prime ideal Q. Fix $x \in P$, then $xs \in J_e$, and $x^p s^p \in J_{e+1}$, whereby $x^p \in (J_{e+1} :_R s^p) \subseteq Q$. Hence $x \in Q$, and therefore $P \subseteq Q$, and $P = Q$.

Finally, since P is minimal over $(J_{e+1} :_R s^p)$, by Remark 2.0.4, $P = (J_{e+1} :_R rs^p)$ for some $r \in R$. $\square$

By an inductive argument, if $e' > e$, then $P = (J_{e'} :_R rs^{\frac{q'}{q}})$ for some $r \in R$. We are now ready to prove the theorem for cyclic modules.

Theorem 5. *Let R be a reduced Noetherian ring of characteristic $p > 0$, and let R^∞ be its perfect closure. Let $J \subset R^\infty$ be an ideal, and let $\{J_e\}$ be its corresponding f-sequence in R. Let P and P^∞ be corresponding prime ideals in $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R^\infty)$ respectively. Then the following are all equivalent:*

1. $P \in \bigcup \mathrm{Ass}_R(R/J_e)$
2. $P^\infty \in \widetilde{\mathrm{Ass}}_R(R^\infty/J)$
3. $P^\infty \in \mathrm{sK}_R(R^\infty/J)$

Proof. <u>$1 \Rightarrow 2$:</u> Let $P \in \mathrm{Ass}_R(R/J_e)$, with $P = (J_e :_R s)$ for some $s \in R$. Then $P^{[\frac{1}{q}]}R^\infty \subseteq (J :_{R^\infty} s^{\frac{1}{q}})$ for $q = p^e$, since $ps \in J_e \Leftrightarrow p^{\frac{1}{q}}s^{\frac{1}{q}} \in f^{-e}(J_e) \Leftrightarrow p^{\frac{1}{q}}s^{\frac{1}{q}} \in J$. We claim $(J :_{R^\infty} s^{\frac{1}{q}}) \subseteq P^\infty$ as well.

To prove this claim, suppose $x^{\frac{1}{q'}}s^{\frac{1}{q}} \in J$ for some $x^{\frac{1}{q'}} \in R^\infty$. If $q' \leq q$, then $x^{\frac{q}{q'}}s \in J_e$, whereby $x^{\frac{q}{q'}} \in P$ and $x^{\frac{1}{q'}} \in P^\infty$. However if $q' > q$, then $xs^{\frac{q'}{q}} \in J_{e'}$, and clearly for any $y \in R$, $x(ys^{\frac{q'}{q}}) = y(xs^{\frac{q'}{q}}) \in J_{e'}$. Since $P \in \mathrm{Ass}_R(R/J_e)$, growth of associated primes over an f-sequence [SN04, Remark 4.2, *iv*)] states that $P \in \mathrm{Ass}_R(R/J_{e'})$ as well, and specifically $P = (J_{e'} :_R zs^{\frac{q'}{q}})$ for some $z \in R$ by Lemma 2.0.5. Hence $x(zs^{\frac{q'}{q}}) \in J_{e'}$, $x \in P$, and $x^{\frac{1}{q'}} \in P^\infty$.

We now have $P^{[\frac{1}{q}]}R^\infty \subseteq (J :_{R^\infty} s^{\frac{1}{q}}) \subseteq P^\infty$. Hence P^∞ is minimal over $(J :_{R^\infty} s^{\frac{1}{q}})$, which is to say that $P^\infty \in \widetilde{\mathrm{Ass}}_R(R^\infty/J)$.

<u>$2 \Rightarrow 3$</u>, Known to be true, as stated above.

<u>$3 \Rightarrow 1$:</u> Suppose $P^\infty \in \mathrm{sK}_{R^\infty}(R^\infty/J)$. Then PR^∞ is a finitely generated sub-ideal, and hence $PR^\infty \subseteq (J :_{R^\infty} s^{\frac{1}{q}}) \subseteq P^\infty$ for some $s \in R$. Now fix $x \in P$, and we have $xs^{\frac{1}{q}} \in J$,

and $x^q s = (xs^{\frac{1}{q}})^q = f^e(xs^{\frac{1}{q}}) \in f^e(J) \cap R = J_e$. Therefore $x \in (J_e :_R s)$. But since $x \in P$ was arbitrary, we know $P^{[q]} \subseteq (J_e :_R s)$.

Moreover, fix some $x \in (J_e :_R s)$, i.e. $xs \in J_e = f^e(J) \cap R$, and hence $x^{\frac{1}{q}} s^{\frac{1}{q}} \in J$. Hence $x^{\frac{1}{q}} \in (J :_{R^\infty} s^{\frac{1}{q}})$, which shows that $x^{\frac{1}{q}} \in P^\infty$ and $x \in P$.

We now have that $P^{[q]} \subseteq (J_e :_R s) \subseteq P$, whereby P is minimal over $(J_e :_R s)$, and $P \in \widetilde{\mathrm{Ass}}_R(R/J_e) = \mathrm{Ass}_R(R/J_e)$. This equality holds since R is a Noetherian ring. $\square$

Furthermore, for an ideal $I \subset R$, and for the f-sequence $\{(I^{[q]})^F\}$, we know the corresponding ideal in R^∞ is IR^∞. We then have a special case of these results.

Corollary 2.0.6. *Let R be a reduced Noetherian ring of characteristic $p > 0$, let $I \subset R$ be an ideal, and let R^∞ be its perfect closure. Let P and P^∞ be corresponding prime ideals in* $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(R^\infty)$ *respectively (that is, let $P = P^\infty \cap R$). Then the following are all equivalent:*

1. $P \in \bigcup \mathrm{Ass}_R(R/(I^{[q]})^F)$
2. $P^\infty \in \widetilde{\mathrm{Ass}}_R(R^\infty/IR^\infty)$
3. $P^\infty \in \mathrm{sK}_R(R^\infty/IR^\infty)$

Proof. Letting $J = IR^\infty$, and $J_e = (I^{[q]})^F$ for all e, we see this corollary is a special case of Theorem 5.

Alternately, letting $M = R/I$, we see the corollary is a special case of Theorem 4. $\square$

Chapter 3: Generalized Depth

Let S be any ring, again not necessarily Noetherian. We have the notion of *grade* of an ideal over a module M, $\mathrm{gr}_S(I, M)$, which is another ring theoretic concept which generalizes into multiple concepts in the non-Noetherian context. Above we have defined M-regular sequences and the Koszul complex, while the appendix contains a section discussing Ext. All three of these constructions feature presently. An explanation of generalized grade can be found in [Hoc74] and [Bar72].

Definition Let S be any commutative ring with identity, let $I \subset R$ be an ideal, and let M be an S-module.

1. $\mathrm{c}\ \mathrm{gr}_S(I, M) := \sup\{|\mathbf{x}|\}$ where $\mathbf{x} \subset I$ is a finite M-regular sequence contained in I.

2. $\mathrm{k}\ \mathrm{gr}_S(I, M) := \sup\{n - h\}$ where $\mathbf{x} = x_1, \ldots, x_n \subset I$ is a finite set, and h is index of the highest non-zero homology group of the Koszul complex $K_\bullet(x_1, \ldots, x_n; M)$. If $\mathbf{x}$ is a finite sequence, we can discuss the koszul grade on $\mathbf{x}$, $\mathrm{k}\ \mathrm{gr}_S(\mathbf{x}, M) := n - h$ with n and h as above.

3. $\mathrm{r}\ \mathrm{gr}_S(I, M) := \inf\{i \mid \mathrm{Ext}^i_R(S/I, M) \neq 0\}$ (for definition of $\mathrm{Ext}^i_R(S/I, M)$, see A.5).

For arbitrary $S, I,$ and M, we always have $\mathrm{c}\ \mathrm{gr}_S(I, M) \leq \mathrm{k}\ \mathrm{gr}_S(I, M) \leq \mathrm{r}\ \mathrm{gr}_S(I, M)$, while each inequality can be strict. See [Bar72, 2] for an example where $\mathrm{k}\ \mathrm{gr}_S(I, M) < \mathrm{r}\ \mathrm{gr}_S(I, M)$. See [ES14, Remark 2.2] for an example where $\mathrm{c}\ \mathrm{gr}_S(I, M) = 0 < \mathrm{k}\ \mathrm{gr}_S(I, M)$. The inequality in this second example holds by Remark 3.0.7.

Additionally if $I \subseteq J$ are ideals in S, then for each notion _ grade, we have _ $\mathrm{gr}_S(I, M) \leq$ _ $\mathrm{gr}_S(J, M)$, [Bar72]. For example, $\mathrm{c}\ \mathrm{gr}_S(I, M) \leq \mathrm{c}\ \mathrm{gr}_S(J, M)$.

None of these measures are necessarily finite. For example let $S = k[x_1, x_2, \ldots]$ be the non-Noetherian polynomial ring over some field k with infinitely many variables, let $I = (x_1, x_2, \ldots)$, and $M = S$. Then $x_1, x_2, \ldots$ is an infinite regular sequence over M. Hence $\text{c gr}_S(I, M)$ is infinite, as are the other two measures since $\text{c gr}_S(I, M) \leq \text{k gr}_S(I, M) \leq \text{r gr}_S(I, M)$.

If S is a Noetherian ring, then $\text{c gr}_S(I, M) \leq \text{k gr}_S(I, M) = \text{r gr}_S(I, M)$, and all three concepts are indeed finite. If S is Noetherian and furthermore M is finitely generated as an S-module, then all three measures $\_\_\ \text{gr}_S(I, M)$ coincide. In this case, some textbooks such as [BH97] state the third construction as the definition of $\text{gr}_S(I, M)$, while the other two values are proven as equivalences. In such a case, all maximal M-sequences in I have the same length.

Specifically if $(S, \mathfrak{n})$ is local, for each notion __ grade we define the __ *depth* of S on M, where each notion is defined for the unique maximal ideal $\mathfrak{n}$. That is, $\_\_\ \text{depth}_S(M) := \_\_\ \text{gr}_S(\mathfrak{n}, M)$. For example, $\text{c depth}_S(M) := \text{c gr}_S(\mathfrak{n}, M)$. In particular if S is Noetherian and M is finitely generated over S, then $\text{depth}_S(M) := \text{gr}_S(\mathfrak{n}, M)$. Furthermore, in such a case, let $\text{depth}_S(M) = n$. Then for all $i = 1, \ldots, n$, we have that $\text{depth}_S\big(M/(x_1, \ldots, x_i)M\big) = \text{depth}_S(M) - i$, where $x_1, \ldots, x_n$ is a maximal M-sequence.

With these three distinct values, and in light of the discussion in the previous chapter, we now have the generalized notions of both associated primes and depth for a module over a non-Noetherian local ring. For a given local ring $(S, \mathfrak{n})$, and S-module M, these concepts' relationships with one another follow as an exercise from their respective definitions. For the convenience of the reader, we include a proof here.

Remark 3.0.7. Let $(S, \mathfrak{n})$ be a local ring, and let M be an S-module. Then,

1. $\mathfrak{n} \in \text{Ass}_S(M)$ if and only if $\text{r depth}_S(M) = 0$

2. $\mathfrak{n} \in \text{sK}_S(M)$ if and only if $\text{k depth}_S(M) = 0$

3. $\mathfrak{n} \in \text{K}_S(M)$ if and only if $\text{c depth}_S(M) = 0$

Proof. 1): Since $(S, \mathfrak{n})$ is a local ring, we have:

$$\begin{aligned}\mathfrak{n} \in \operatorname{Ass}_S(M) &\Leftrightarrow 0 \neq \operatorname{Hom}_S(S/\mathfrak{n}, M) \cong \operatorname{Ext}^0_S(S/\mathfrak{n}, M)\\ &\Leftrightarrow \text{r depth}_S(M) = 0, \text{ by definition}\end{aligned}$$

2):

$$\begin{aligned}\mathfrak{n} \in \operatorname{sK}_S(M) &\Leftrightarrow \text{for all finite } \mathbf{x} = x_1, \ldots, x_n \subset \mathfrak{n}, (\mathbf{x}) \subseteq \operatorname{ann}_S(m) \subseteq \mathfrak{n} \text{ for some nonzero}\\ &\quad m \in M\\ * \qquad &\Leftrightarrow \text{ for all finite } \mathbf{x} = x_1, \ldots, x_n \subset \mathfrak{n}, H_n(\mathbf{x}; M) \neq 0\\ &\Leftrightarrow \sup\{\text{k gr}_S(\mathbf{x}; M) \mid \mathbf{x} \subset \mathfrak{n} \text{ is a finite set}\} = 0\\ &\Leftrightarrow \text{k depth}_S(M) = 0, \text{ by definition}\end{aligned}$$

Note the line labeled $*$. The statement in this line is true if and only if for all finite $\mathbf{x} = x_1, \ldots, x_n \subset \mathfrak{n}, (\mathbf{x}) \subseteq \operatorname{ann}_S(m)$ for some nonzero $m \in M$. But since $(S, \mathfrak{n})$ is a local ring, we also have the containment $\operatorname{ann}_S(m) \subseteq \mathfrak{n}$.

3):

$$\begin{aligned}\mathfrak{n} \in \operatorname{K}_S(M) &\Leftrightarrow \text{for all } x \in \mathfrak{n},\ x \in \operatorname{ann}_S(m) \subseteq \mathfrak{n} \text{ for some } m \in M,\ \text{i.e., there exist no}\\ &\quad M\text{-regular elements in } \mathfrak{n}\\ ** \qquad &\Leftrightarrow \sup\{|\mathbf{x}| \mid \mathbf{x} \subset \mathfrak{n} \text{ is a regular sequence}\} = 0\\ &\Leftrightarrow \text{c depth}_S(M) = 0, \text{ by definition}\end{aligned}$$

Similarly, the"if and only if" statement on the line labeled $**$ is true since $(S, \mathfrak{n})$ is a local ring. $\square$

Note that we have no equivalence for $\widetilde{\operatorname{Ass}}_S(M)$. These generalized depths do not currently relate to weakly associated primes in an obvious way.

Also note that if S is Noetherian and M is finitely generated, then all concepts coincide. In such a situation we can therefore uniformly say:

$$\mathfrak{n} \in \operatorname{Ass}_S(M) \text{ if and only if } \operatorname{depth}_S(M) = 0$$

3.1 Depth Behavior Over f-sequences

Returning to our context, let $(R, \mathfrak{m})$ be a reduced commutative Noetherian local ring of characteristic $p > 0$ and let $(R^\infty, \mathfrak{m}^\infty)$ be its corresponding perfect closure. Let $\{J_e\}$ be an f-sequence in R. Since our ring is Noetherian, and since any cyclic module is finitely generated, we have one uniform notion of associated prime ideals over R/J_e for any e. It is a fact that $\operatorname{Ass}_R(R/J_e) \subseteq \operatorname{Ass}_R(R/J_{e+1})$ for all e [SN04, Remark 4.3, *iv*)], i.e. terms in an f-sequence can gain associated primes but not lose any as e grows. Additionally, any regular element over a module must exist in the complement of the union of associated primes of the module [BH97, Section 1.2]. Hence later terms in an f-sequence can lose regular elements over the corresponding cyclic module, but cannot gain any such elements. This fact shows,

Theorem 6. *Let $(R, \mathfrak{m})$ be a local commutative Noetherian ring of characteristic $p > 0$, and let $\{J_e\}_{e\in\mathbb{N}}$ be an f-sequence in R. If $\operatorname{depth}_R(R/J_{e+1}) \leq 1$ for some e, then,*

$$\operatorname{depth}_R(R/J_e) \geq \operatorname{depth}_R(R/J_{e+1}).$$

Proof. If $\operatorname{depth}_R(R/J_e) = 0$, then $\mathfrak{m} \in \operatorname{Ass}_R(R/J_e) \subseteq \operatorname{Ass}_R(R/J_{e+1})$. Then $\operatorname{depth}_R(R/J_{e+1}) = 0$. It is therefore impossible that $\operatorname{depth}_R(R/J_e) = 0$ while $\operatorname{depth}_R(R/J_{e+1}) = 1$. $\square$

Since this hypothesis allows for an extremely restrictive case, we ask whether the same feature is true for regular sequences over R/J_{e+1} of length longer than 1. I.e., suppose we omit the hypothesis in this result that $\operatorname{depth}_R(R/J_{e+1}) \leq 1$, where again we have one uniform notion of depth since R is Noetherian. To give a specific context for this investigation, we investigate the minimal f-sequences $\{(I^{[q]})^F\}$ discussed above. For an ideal $I \subseteq R$ we ask whether the $\operatorname{depth}_R(R/(I^{[q]})^F)$ would decrease as q grows. For an arbitrary ring R of characteristic $p > 0$, this fact seems unlikely.

However, we can modify this question by invoking the notion of F-purity for our rings. Recall from above that every term in an f-sequence is Frobenius closed. However, if we demand the condition that *all* ideals in our rings are F-closed, we obtain an affirmative answer to this question. All the following results in this section require this assumption on our rings, which we define here. We list the general definition of F-pure rings, as well as two other consequences.

3.2 F-purity

Let S be any ring, and let A and B be S-modules. We say an S-module map $\varphi : A \to B$ is *pure* if the map $\varphi \otimes_S 1_M : A \otimes_S M \to B \otimes_S M$ is injective for all S-modules M. Alternately we say that A is a pure submodule of B.

We say that a reduced Noetherian ring R of characteristic $p > 0$ is F-*pure* if the embedding $R \hookrightarrow R^{\frac{1}{p}}$ is a pure map. Equivalently, for any R-module M the induced map $M \to F(M) := R \otimes_R M \to R^{\frac{1}{p}} \otimes_R M$ is an injective map. Note that if R is F-pure, then for any q the embeddings $R \hookrightarrow R^{\frac{1}{q}}$, as well as the embedding $R \hookrightarrow R^{\infty}$ are pure maps as well.

If a ring R is F-pure, then for any R-module M we have that $M \subseteq F(M)$ is a pure submodule as well. This statement holds since given $M \hookrightarrow F(M) := (R \otimes_R M) \hookrightarrow (R^{\frac{1}{p}} \otimes_R M)$, and given any R-module N, we have:

$$\begin{aligned}\big(M \otimes_R N \to F(M) \otimes_R N\big) &= \big((R \otimes_R M) \otimes_R N \to (R^{\frac{1}{p}} \otimes_R M) \otimes_R N\big)\\ &= \big(R \otimes_R (M \otimes_R N) \to R^{\frac{1}{p}} \otimes_R (M \otimes_R N)\big)\\ &= \big(M \otimes_R N \to F(M \otimes_R N)\big)\end{aligned}$$

which is an injection by F-purity of R.

Remark 3.2.1. If R is an F-pure ring, then every ideal $I \subset R$ is F-closed. For the convenience of the reader, we include a proof that $I^F = I$ for all such I.

Proof. Suppose R is an F-pure ring, and suppose $I \subsetneq I^F$ with $r \in I^F \setminus I$, and $I = (x_1, \ldots, x_n) \subset R$. Then there exists some q such that $r^q \in I^{[q]}$. Therefore $r \in IR^{\frac{1}{q}}$, since if $r^q = \Sigma_{i=1}^n a_i x_i^q \in I^{[q]}$ for $a_i \in R$, then $r = \Sigma_{i=1}^n a_i^{\frac{1}{q}} x_i \in IR^{\frac{1}{q}}$. However, we now have the map $R/I \to F^e(R/I) = R^{\frac{1}{q}} \otimes_R R/I \cong R^{\frac{1}{q}}/IR^{\frac{1}{q}}$ is not injective since $x \mapsto 0$. Hence R is not F-pure, which is a contradiction. $\square$

A particular case of prime characteristic rings is *F-finite* rings R, for which $R^{\frac{1}{p}}$ is finitely generated as a module over R, or $R^{\frac{1}{f}}$ is finitely generated as a *right* R-module. These rings are commonly discussed in the literature, and in such a case F-purity is equivalent to *F-splitness* [HR76, Corollary 5.3]. A reduced ring R of characteristic $p > 0$ is F-split if the embedding $R \hookrightarrow R^{\frac{1}{p}}$ splits. As a consequence, $R^{\frac{1}{p}} \cong R^a \oplus N$ for some free rank $a \in \mathbb{N}$, and some R-module N. We will not explicitly discuss this characterization. However the reader will be advised that under the assumption of F-finiteness, F-splitness is a sufficient condition in order for all results below to hold.

All F-pure rings are reduced, and we therefore need not mention the reduced hypothesis in our results below. To see this fact, suppose there exists some $r \in R$ such that $r^n = 0$, then let q be sufficiently large so that $q \geq n$. Then $r \notin (0)$, but $r^q \in (0)^{[q]}$, which contradicts the fact that the ideal (0) is F-closed.

Some examples of F-pure rings are regular rings, such as the polynomial ring $S = k[x_1, \ldots, x_n]$, where k is a field of characteristic $p > 0$. Additionally, a class of F-pure examples can be given if $(S, \mathfrak{n})$ is any F-finite regular local ring and $R = S/I$ is any quotient. In such a scenario R is F-pure if and only if $(I^{[q]} :_R I) \nsubseteq \mathfrak{n}^{[q]}$ for all q. This result is known as *Fedder's criterion.*

A non-example of an F-pure ring is $R := [x, y, z]/(x^p - yz^p)$. We can see that the ideal (z) is not F-closed. Certainly $x \notin (z)$, but $x^p = yz^p \in (z)^{[p]}$, and hence $x \in (z)^F \setminus (z)$. Alternatively, we can see $x = y^{\frac{1}{p}} z \in (z)R^\infty \cap R = (z)^F$.

3.2.1 Results Requiring F-purity

In an F-pure ring, since *all* ideals $I \subseteq R$ are F-closed by Remark 3.2.1, $\{(I^{[q]})^F\} = \{I^{[q]}\}$ is always an f-sequence. We now ask a question modified from the one stated above: in an *F-pure ring*, is $\operatorname{depth}_R(R/I^{[q]})$ non-increasing as q grows? This assumption on our rings provides a key piece of information regarding the behavior of depth over such sequences, as the answer to the question is yes. Additionally, we obtain several relationships with some of the generalized depth values of $(R^\infty \otimes_R M)$ over R^∞. Our first result in this section answers our question, and as we shall see, provides a context for further inquiry:

Lemma 3.2.2. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, and let $I \subset R$ be an ideal. If $\mathbf{x} = x_1, \ldots, x_n \subset R$ is a regular sequence on $R/I^{[p]}$, then it is regular on R/I.*

Proof. First recall that by the properties of regular sequences, $\mathbf{x} = x_1, \ldots, x_n$ is an $R/I^{[p]}$ sequence if and only if $\mathbf{x}^{[p]} = x_1^p, \ldots, x_n^p$ is a regular $R/I^{[p]}$ sequence as well [Nor76, Theorem 5.1.3].

We will proceed by induction on the length of $\mathbf{x}$. If $|\mathbf{x}| = 1$, let x and hence x^p be regular over $R/I^{[p]}$. We claim that x is R/I regular. If we suppose not, then $xz \in I$ for some $z \notin I$. Due to the F-purity of R, we know that $I = I^F$ by Remark 3.2.1, and hence $z^p \notin I^{[p]}$. But then $x^p z^p \in I^{[p]}$, which contradicts our assumption about of regularity of x^p over $R/I^{[p]}$.

Now assume the statement is true for length $\leq n-1$, and $|\mathbf{x}| = |\mathbf{x}^{[p]}| = n$, both of which are regular over $R/I^{[p]}$. By our inductive hypothesis, $\mathbf{x}' = x_1, \ldots, x_{n-1}$ is R/I - regular. But now assume that $\mathbf{x}$ is not R/I - regular, and x_n is not regular over $R/(I, \mathbf{x}')$, with $z \notin (I, \mathbf{x}')$ and $x_n z \in (I, \mathbf{x}')$. Now in R, $(I, \mathbf{x}') = (I, \mathbf{x}')^F$ by Remark 3.2.1, and hence $z^p \notin (I, \mathbf{x}')^{[p]}$. But then $x_n^p z^p \in (I, \mathbf{x}')^{[p]}$, whence x_n^p is not regular over $(I, \mathbf{x}')^{[p]}$, which is a contradiction. □

Note that the proof of this lemma requires all ideals in R to be F-closed, while the statement is unknown if R is not F-pure. But in this context the following corollary provides an affirmative answer to our question.

Theorem 7. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, and let $I \subset R$ be an ideal. Then,*

$$\operatorname{depth}_R(R/I) \geq \operatorname{depth}_R(R/I^{[p]}).$$

Proof. Let $\operatorname{depth}_R(R/I^{[p]}) = n$, and $\mathbf{x} = x_1, \ldots, x_n \subset R$ be a maximal $R/I^{[p]}$-sequence. Then by lemma 3.2.2 $\mathbf{x}$ is R/I-regular, and hence $\operatorname{depth}_R(R/I) \geq n$. □

Having answered the question for cyclic modules, we quickly can prove a stronger statement, and we can do so through a far simpler argument. Recall by remark 1.8.1 that $R/I^{[p]} \cong R^{\frac{1}{q}}/IR^{\frac{1}{q}} \cong (R^{\frac{1}{p}} \otimes_R R/I) = F(R/I)$. We have thus shown $\operatorname{depth}_R(R/I) \geq \operatorname{depth}_R\big(F(R/I)\big)$. We can ask whether this statement is true in greater generality. We indeed find that over arbitrary finitely generated R-modules M, we can make the identical statement for M and $F(M)$. However, we must first state a lemma addressing the different left and right actions of R on $F(M)$. We show that for a regular sequence $\mathbf{x}$, we can discuss its regularity over $F(M)$ without specifying left or right action. Additionally, this fact enables us to study $\operatorname{depth}_R\big(F(M)\big)$, which is therefore a two-sided concept.

Lemma 3.2.3. *Let R be an F-pure Noetherian ring of characteristic $p > 0$. Let M be an R-module, and let $\mathbf{x} = x_1, \ldots, x_n \subset R$ be a finite sequence. Then for any $e \in \mathbb{N}$, $\mathbf{x}$ is a regular left $F^e(M)$-sequence if and only if $\mathbf{x}$ is a regular right $F^e(M)$-sequence.*

Proof.

Fix $e \in \mathbb{N}$. It is easy to see that $\mathbf{x}$ is a left-regular $F^e(M)$-sequence if and only if $\mathbf{x}^{[q]} = x_1^q, \ldots, x_n^q$ is a left-regular $F^e(M)$-sequence if and only if $\mathbf{x}$ is a right-regular $F^e(M)$-sequence.

The first equivalence is true by properties of regular sequences [Nor76, Theorem 5.1.3], whereby $\mathbf{x}$ is a left-regular $F^e(M)$ if and only if $\mathbf{x}^{[q]}$ is a left-regular $F^e(M)$ sequence.

The second equivalence is true because for any $\sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) \in F^e(M)$, the left action by any x_j^q for $j = 1, \ldots, n$ is element-wise equivalent to right action by x_j. That is,

$$\begin{aligned} x_j^q\big(\textstyle\sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big) &= \textstyle\sum_i^k (x_j s_i^{\frac{1}{q}} \otimes_R m_i^{[q]}) \\ &= \textstyle\sum_i^k (s_i^{\frac{1}{q}} x_j \otimes_R m_i^{[q]}) \\ &= \big(\textstyle\sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[q]})\big) x_j \end{aligned}$$

□

We then have an immediate corollary.

Lemma 3.2.4. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$. Let M be an R-module, and let $F^e(M)$ denote the image of M under the e-th iteration of the Frobenius functor for $e \in \mathbb{N}$. Then* left $\operatorname{depth}_R F^e(M) =$ right $\operatorname{depth}_R F^e(M)$. *The two-sided* $\operatorname{depth}_R F^e(M)$ *is therefore a well-defined concept.*

Proof. By lemma 3.2.3, $\mathbf{x} \subset R$ is left $F^e(M)$-regular if and only if it is right $F^e(M)$-regular. Therefore all maximal such sequences have the same length under right and left action. □

We next prove an additional lemma. This fact is probably known, however we include a proof for the convenience of the reader.

Lemma 3.2.5. *Let $(R, \mathfrak{m})$ be a reduced Noetherian local ring. Let $M \subseteq N$ be a pure inclusion of R-modules such that $M \neq 0$ is finitely generated over R. Let the finite set $\mathbf{x} \subset R$ be an N-regular sequence. Then $\mathbf{x}$ is regular over M as well.*

Proof. We proceed by induction on $|\mathbf{x}|$. Let $\mathbf{x} = x$ be a singleton. Then $M/xM \neq 0$ by Nakayama's lemma since $x \in \mathfrak{m}$ and $M \neq 0$. Furthermore, x clearly annihilates no element of $M \subset N$.

Now assume the statment is true for $|\mathbf{x}| = n - 1$. Let $|\mathbf{x}| = n$, and let $\mathbf{x}' = x_1, \ldots, x_{n-1}$ denote the first $n-1$ elements of $\mathbf{x}$. Again, $M/\mathbf{x}M \neq 0$ by Nakayama's lemma since $\mathbf{x} \subset \mathfrak{m}$ and $M \neq 0$.

Additionally, $M/\mathbf{x}'M$ embeds into $N/\mathbf{x}'N$. In order to see this fact, recall that $M/\mathbf{x}'M \cong M \otimes_R R/(\mathbf{x}')$, and $N/\mathbf{x}'N \cong N \otimes_R R/(\mathbf{x}')$, and since M is a pure sub N-module we have the embedding $M \otimes_R R/(\mathbf{x}') \hookrightarrow N \otimes_R R/(\mathbf{x}')$.

Thus since x_n annihilates no element of $N/\mathbf{x}'N$, it will therefore annihilate no element of the submodule $M/\mathbf{x}'M$. Therefore, $\mathbf{x}$ is an M-regular sequence. $\square$

We can now show the identical statement as above for the more general setting for arbitrary finitely generated M and $F(M)$.

Lemma 3.2.6. *Let $(R, \mathfrak{m})$ be a local Noetherian F-pure ring of characteristic $p > 0$, and let M be a finitely generated R-module. If $\mathbf{x} = x_1, \ldots, x_n$ is regular over $F(M)$, then it is regular over M.*

Proof. $M \subseteq F(M)$ is a pure submodule. Hence by 3.2.5, if $\mathbf{x}$ is regular over $F(M)$, it is also regular over M. $\square$

We then have the immediate corollary regarding the non-increasing nature of depth under the Frobenius functor.

Theorem 8. *Let $(R, \mathfrak{m})$ be a local Noetherian F-pure ring of characteristic $p > 0$, and let M be a finitely generated R-module. Then,*

$$\operatorname{depth}_R(M) \geq \operatorname{depth}_R\big(F(M)\big).$$

Proof. If $\operatorname{depth}_R\big(F(M)\big) = d$ and $x_1, \ldots, x_d$ is a maximal $F(M)$-sequence, then by lemma 3.2.6 $x_1, \ldots, x_d$ is M-regular, and $\operatorname{depth}_R(M) \geq d$. $\square$

Remark 3.2.7. Note that the finite generation of M provides for this inequality since lemma 3.2.5 requires that hypothesis. We can then consider the pure submodule inclusion $F^e(M) \subseteq F^{e+1}(M)$ for $e > 0$. However, since we are not assuming that R is F-finite, $F^e(M)$ may not be finitely generated as a right R-module. Yet it *will* be finitely generated as a left R-module, since the f^{-e} action is equivalent to obvious action on the left by $R^{\frac{1}{q}}$ on $R^{\frac{1}{f}} \otimes_R M$. Recall that by lemma 3.2.3 we can consider the left action when discussing the regularity of a sequence over $F^e(M)$. Hence by lemma 3.2.5, if $\mathbf{x}$ is a maximal regular $F^{e+1}(M)$-sequence, then it is a regular $F^e(M)$-sequence.

We now have that in an F-pure Noetherian ring R of characteristic $p > 0$, for a finitely generated R-module M,

$$\operatorname{depth}_R(M) \geq \operatorname{depth}_R\big(F(M)\big) \geq \operatorname{depth}_R\big(F^2(M)\big) \geq \ldots$$

I.e., $\operatorname{depth}_R\big(F^e(M)\big)$ is non-increasing e grows. Furthermore, it is bounded below by 0, and the value must therefore stabilize, meaning there must exist some $d \in \mathbb{N}$ for which $\operatorname{depth}_R\big(F^e(M)\big) = d$ for $e \gg 0$. This fact yields a definition.

Definition: Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, and let M be a finitely generated R-module. Let e_0 be the value at which $\operatorname{depth}_R\big(F^{e_0}(M)\big)$ stabilizes. We define the stabilizing depth:

$$\operatorname{s\,depth}_R(M) := \operatorname{depth}_R\big(F^e(M)\big) \text{ for all } e \geq e_0.$$

In particular for an ideal $I \subset R$,

$$\operatorname{s\,depth}_R(R/I) := \operatorname{depth}_R(R/I^{[q]}) \text{ for all } q \geq q_0.$$

In order to show an example where this definition yields distinct values, we now include a class of a cyclic modules such that $\operatorname{depth}_R(R/I) > \operatorname{s\,depth}_R(R/I)$. First we cite such a module for which s depth drops to 0 for q large, then we construct a module over an extension of the base ring for which s depth drops to $t > 0$.

Example 3.2.8. Let $(A, \mathfrak{a})$ be a local Noetherian ring F-pure ring of characteristic $p > 0$, and let $J \subset A$ be an ideal such that $\operatorname{depth}_A(A/J) > \operatorname{s\,depth}_A(A/J) = 0$, i.e. $\operatorname{depth}_A(A/J) \neq 0$, while $\operatorname{depth}_A(A/J^{[q]}) = 0$ for $q \gg 0$. For an example for such a cyclic module, first see [SS04] for a ring S where the sets $\operatorname{Ass}_S(S/J^{[q]})$ grow as q increases. If we then fix $P \in \operatorname{Ass}_S(S/J^{[q]}) \setminus \operatorname{Ass}_S(S/J)$ for some q, we can then localize at P. Letting $A = S_P$ and $\mathfrak{a} = PA$ we have a local ring $(A, \mathfrak{a})$. Therefore, we can discuss the depth of modules over A, and specifically $\operatorname{depth}_A(A/JA) > 0$, while $\operatorname{depth}_A(A/J^{[q]}A) = \operatorname{s\,depth}_A(A/JA) = 0$.

Now let $R := A[\![x_1, \ldots, x_t]\!]$ for variables x_i, which is local with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_t) + \mathfrak{a}R$ [Mat86, Discussion on Page 4]. Let $I = JR$. Then for all q, $\operatorname{depth}_R(R/I^{[q]}) = \operatorname{depth}_A(A/J^{[q]}) + t$ (follows from [BH97, 1.2.16]). Hence $\operatorname{depth}_R(R/I) = \operatorname{depth}_A(A/J) + t > t$, but $\operatorname{depth}_R(R/I^{[q]}) = \operatorname{depth}_A(A/J^{[q]}) + t = 0 + t = t$ for $q \gg 0$. Thus $\operatorname{depth}_R(R/I) > \operatorname{s\,depth}_R(R/I) > 0$.

3.2.2 Depth Type Comparisons

With this new measure of a finitely generated module M over an F-pure R, we investigate how it relates to the other depth measures of $(R^\infty \otimes_R M)$ over the non-Noetherian extension R^∞. In particular for a cyclic module $M = R/I$, we would like to know these depth measures of R^∞/IR^∞ compare to $\operatorname{s\,depth}_R(R/I)$.

In order to compare $\operatorname{s\,depth}_R(M)$ with $\operatorname{c\,depth}_{R^\infty}(R^\infty \otimes_R M)$, we begin with a lemma regarding regular sequences from R.

Lemma 3.2.9. *Let R be an F-pure Noetherian ring of characteristic $p > 0$, and let R^∞ be its perfect closure. Let M be an R-module, and let $\mathbf{x} = x_1, \ldots, x_n \subset R$ be a finite* *sequence. Then,* $\mathbf{x}$ *is a regular* $F^e(M)$*-sequence for all* $e \in \mathbb{N}$ *if and only if* $\mathbf{x}$ *is a regular* $(R^\infty \otimes_R M)$*-sequence.*

Proof. If $\mathbf{x}$ is a regular $(R^\infty \otimes_R M)$-sequence, then by remark 3.2.7, $\mathbf{x}$ is a regular $F^e(M)$-sequence since $F^e(M) \hookrightarrow (R^\infty \otimes_R M)$ is a pure submodule by F-purity of R.

Conversely, we proceed by induction on $|\mathbf{x}|$. If $\mathbf{x} = x$ is a singleton, suppose x is not $(R^\infty \otimes_R M)$-regular, with $\left(\sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[\infty]})\right)x = \sum_i^k (s_i^{\frac{1}{q}} x \otimes_R m_i^{[\infty]}) = 0$. Then $0 = \sum_i^k (s_i^{\frac{1}{q}} x \otimes_R m_i^{[q']}) = \left(\sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[q']})\right)x$ in some $F^{e'}(M)$, and $0 \neq \sum_i^k (s_i^{\frac{1}{q}} \otimes_R m_i^{[q']})$, since it was not zero in $(R^\infty \otimes_R M) = \varinjlim F^e(M)$. Thus x is not $F^{e'}(M)$-regular.

Now suppose $\mathbf{x} = x_1, \ldots, x_n \subset R$ is $F^e(M)$-regular for all e, and let $\mathbf{x}' = x_1, \ldots, x_{n-1}$, which is by hypothesis regular over $(R^\infty \otimes_R M)$. We must show that x_n is regular over

$$
\begin{aligned}
(R^\infty \otimes_R M)/(R^\infty \otimes_R M)\mathbf{x}' &\cong (R^\infty \otimes_R M) \otimes_R R/\mathbf{x}'R \\
&\cong R^\infty \otimes_R (M \otimes_R R/\mathbf{x}'R) \\
&\cong R^\infty \otimes_R M/\mathbf{x}'M
\end{aligned}
$$

We then apply the identical argument to the regular element x_n over the module $F^e(M/\mathbf{x}'M)$, which is therefore is also regular over the module $(R^\infty \otimes_R M/\mathbf{x}'M)$. $\square$

If $\mathbf{x}$ is a maximal regular sequence over $F^e(M)$ for some *specific* e, and $\mathbf{y}$ is a maximal regular sequence over $F^e(M)$ for *all* e, clearly $|\mathbf{x}| \geq |\mathbf{y}|$. Hence In light of lemma 3.2.9, we can compare c $\text{depth}_{R^\infty}(R^\infty \otimes_R M)$ to s $\text{depth}_R(M)$.

Theorem 9. *Let* R *be an* F*-pure Noetherian ring of characteristic* $p > 0$*, let* R^∞ *be its perfect closure, and let* M *be a finitely generated* R*-module. Then* c $\text{depth}_{R^\infty}(R^\infty \otimes_R M)$ *is finite, and more specifically*

$$\text{s depth}_R(M) \geq \text{c depth}_{R^\infty}(R^\infty \otimes_R M)$$

In particular, if $M = R/I$ is a cyclic module,

$$\mathrm{s\ depth}_R(R/I) \geq \mathrm{c\ depth}_{R^\infty}(R^\infty/IR^\infty)$$

Proof. Let $\mathbf{x}' = x_1^{\frac{1}{q_1}}, \ldots, x_n^{\frac{1}{q_n}} \subset R^\infty$ be an $(R^\infty \otimes_R M)$-regular sequence. Then by properties of regular sequences [Nor76, Theorem 5.1.3], $\mathbf{x} := x_1, \ldots, x_n \subset R \subseteq R^\infty$ is $(R^\infty \otimes_R M)$-regular as well. By lemma 3.2.9, $\mathbf{x}$ is $F^e(M)$-regular for all e. Hence any maximal $(R^\infty \otimes_R M)$-sequence can have at most the length of $\mathrm{s\ depth}_R(M) = \mathrm{depth}_R\big(F^e(M)\big)$ for $e \gg 0$.

The particular case is true since $R^\infty \otimes_R R/I \cong R^\infty/IR^\infty$.

□

The particular case of the cyclic $\mathrm{c\ depth}_{R^\infty}(R^\infty/IR^\infty)$ in this theorem can be proven using a different technique. We include an alternate proof here here in order to display this differing point of view and the different techniques it incorporates. Again we begin with similar lemma.

Lemma 3.2.10. *Let R be an F-pure Noetherian ring of characteristic $p > 0$, and let R^∞ be its perfect closure. Let $I \subset R$ be an ideal, and let $\mathbf{x} = x_1, \ldots, x_n \subset R$ be a finite sequence. Then the following are equivalent:*

1. *$\mathbf{x}$ is an R^∞/IR^∞ - regular sequence.*
2. *$\mathbf{x}$ is an $R^\infty/I^{[q]}R^\infty$ - regular sequence for all q.*
3. *$\mathbf{x}$ is an $R/I^{[q]}$ - regular sequence for all q.*

Proof. $\underline{1 \Rightarrow 2:}$ We proceed by induction on the length of $\mathbf{x}$. If $|\mathbf{x}| = 1$, suppose the singleton x is not $R^\infty/I^{[q]}R^\infty$ - regular for some q and $zx \in I^{[q]}R^\infty$ for some $z \notin I^{[q]}R^\infty$, and hence $z^{\frac{1}{q}} \notin IR^\infty$. Then $z^{\frac{1}{q}}x^{\frac{1}{q}} \in IR^\infty$ and $x^{\frac{1}{q}}$ is not IR^∞ - regular, hence neither is x by properties of regular elements.

Suppose the statement is true for sequences of length $\leq n-1$, and $|\mathbf{x}| = n$. Let

$\mathbf{x}' = x_1, \ldots x_{n-1}$ (i.e. the same sequence as $\mathbf{x}$ but without the last element), which is regular over $R^\infty/I^{[q]}R^\infty$ for all q by hypothesis. But suppose for some q, x_n is not regular over $(I^{[q]}, \mathbf{x}')R^\infty$ with $zx_n \in (I^{[q]}, \mathbf{x}')R^\infty$. Then $z^{\frac{1}{q}}x_n^{\frac{1}{q}} \in (I, (\mathbf{x}')^{[\frac{1}{q}]})R^\infty$. Then $\mathbf{x}^{[\frac{1}{q}]}$ is not a regular R^∞/IR^∞ sequence, and neither is $\mathbf{x}$ by properties of regular sequences.

<u>$2 \Rightarrow 3$:</u> We again proceed by induction on the length of $\mathbf{x}$. If $|\mathbf{x}| = 1$, suppose the singleton x is not $R/I^{[q]}$ - regular for some q, and for some $z \notin I^{[q]}$ we have $zx \in I^{[q]} = I^{[q]}R^\infty \cap R$, with this equality due to F-purity. Hence $zx \in I^{[q]}R^\infty$ with $z \in R^\infty \setminus I^{[q]}R^\infty$. Then x is not regular over $R^\infty/I^{[q]}R^\infty$.

Now suppose the statement is true for sequences of length $\leq n-1$, and $|\mathbf{x}| = n$. Let $\mathbf{x}'$ be as above. Suppose for some q, x_n is not regular over $(I^{[q]}, \mathbf{x}')$ with $z \notin (I^{[q]}, \mathbf{x}')$ and $zx_n \in (I^{[q]}, \mathbf{x}') = (I^{[q]}, \mathbf{x}')R^\infty \cap R$ (again due to F-purity). Now $zx_n \in (I^{[q]}, \mathbf{x}')R^\infty$ with $z \in R^\infty \setminus (I^{[q]}, \mathbf{x}')R^\infty$. Hence $\mathbf{x}$ is not regular over $R^\infty/I^{[q]}R^\infty$.

<u>$3 \Rightarrow 1$:</u> Again we first suppose that $|\mathbf{x}| = 1$ and the singleton x is not R^∞/IR^∞ - regular. Let $z^{\frac{1}{q}}x \in IR^\infty$ for some $z^{\frac{1}{q}} \in R^\infty \setminus IR^\infty$. Then $zx^q \in I^{[q]}R^\infty \cap R = I^{[q]}$ (F-purity), with $z \notin I^{[q]}$. Hence x^q is not $R/I^{[q]}$ - regular and neither is x by properties of regular elements.

Let the statement be true for sequences of length $\leq n-1$, and $|\mathbf{x}| = n$. Let $\mathbf{x}'$ be as above. Suppose x_n is not regular over $R^\infty/(I, \mathbf{x}')R^\infty$ with $z^{\frac{1}{q}}x_n \in (I, \mathbf{x}')R^\infty$ for $z^{\frac{1}{q}} \in R^\infty \setminus (I, \mathbf{x}')R^\infty$. Then $z \in R \setminus (I^{[q]}, \mathbf{x}'^{[q]})$, but $zx_n^q \in (I^{[q]}, \mathbf{x}'^{[q]})R^\infty \cap R = (I^{[q]}, \mathbf{x}'^{[q]})$, due to F-purity. Then $\mathbf{x}^{[q]}$ is not a regular $R/I^{[q]}$ sequence, and neither is $\mathbf{x}$ by properties of regular sequences. $\square$

We now can prove the particular case, where the proof uses identical reasoning. This proof can invoke either lemma 3.2.9 or lemma 3.2.10.

Theorem 10. *Let R be an F-pure Noetherian ring of characteristic $p > 0$, let R^∞ be its perfect closure, and let $I \subset R$ be an ideal. Then* c $\text{depth}_{R^\infty}(R^\infty/IR^\infty)$ *is finite, and in particular*

$$\text{s depth}_R(R/I) \geq \text{c depth}_{R^\infty}(R^\infty/IR^\infty)$$

Proof. Let $\mathbf{x}' = x_1^{\frac{1}{q_1}}, \ldots, x_n^{\frac{1}{q_n}} \subset R^\infty$ be an R^∞/IR^∞-regular sequence. Then by properties of regular sequences, $\mathbf{x} := x_1, \ldots, x_n \subset R \subseteq R^\infty$ is R^∞/IR^∞-regular as well. By either lemma 3.2.9 or lemma 3.2.10, $\mathbf{x}$ is $R/I^{[q]}$-regular for all q. Hence any maximal (R^∞/IR^∞)-sequence can have at most the length of $\text{s depth}_R(R/I) = \text{depth}_R(R/I^{[q]})$ for $q \gg 0$. □

We now return to the case where M is an arbitrary finitely generated R-module. After c depth, the next largest depth value of $(R^\infty \otimes_R M)$ over R^∞ is the Koszul depth. Somewhat surprisingly we find that it coincides with our new notion.

Before we discuss our next result, we make note of some notation used throughout its proof. If for some q, $\mathbf{y} \subset R^{\frac{1}{q}} \subseteq R^\infty$ is some finite sequence (perhaps even $\mathbf{y} \subset R$), we can construct the koszul complex $K_\bullet(\mathbf{y}; F^{e'}(M))$ for any $e' \geq e$, since $R^{\frac{1}{q}} \subseteq R^{\frac{1}{q'}}$. But we can *also* construct $K_\bullet(\mathbf{y}; R^\infty \otimes_R M)$. Since both such complexes are constructed from the same finite sequence $\mathbf{y}$, the matrices defining the differentials d_j are identical in each construction. Therefore, when we discuss the differentials for the Koszul complex over $F^e(M)$ we write d_j^q, while the differentials over $R^\infty \otimes_R M$ are denoted d_j^∞.

Similarly, when we discuss an arbitrary element of a direct sum of copies of $F^e(M)$ we write $z^{[q]}$ or $y^{[q]}$. While their images in the corresponding direct sum of copies of $R^\infty \otimes_R M$ are are denoted $z^{[\infty]}$ and $y^{[\infty]}$ respectively.

Lastly, before we state our next theorem, we require a lemma.

Lemma 3.2.11. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, let $(R^\infty, \mathfrak{m}^\infty)$ be its perfect closure, and let M be a finitely generated R-module. Then,*

$$\text{k depth}_{R^\infty}(R^\infty \otimes_R M) = \sup\{\text{k gr}_{R^\infty}\left(\mathfrak{m}^{[\frac{1}{q}]}R^\infty, (R^\infty \otimes_R M)\right)\} \text{ for all } e \in \mathbb{N}, \text{ where } q = p^e.$$

In particular if $M = R/I$ is a cyclic module, then

$$\text{k depth}_{R^\infty}(R^\infty/IR^\infty) = \sup\{\text{k gr}_{R^\infty}(\mathfrak{m}^{[\frac{1}{q}]}R^\infty, R^\infty/IR^\infty)\} \text{ for all } e \in \mathbb{N}, \text{ where } q = p^e.$$

Proof. We know that $\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M) = \sup\{\mathrm{k\ gr}_{R^\infty}\big(I, (R^\infty \otimes_R M)\big)\}$, where $I \subset R^\infty$ is *any* finitely generated sub-ideal. We therefore must show that the ideals $\mathfrak{m}^{[\frac{1}{q}]}R^\infty$ determine the Koszul depth.

Recall from above that for ideals $I \subseteq J \subset R^\infty$, we have $\mathrm{k\ gr}_{R^\infty}(I, M) \leq \mathrm{k\ gr}_{R^\infty}(J, M)$.

Let $I = (y_1^{\frac{1}{q_1}}, \ldots, y_n^{\frac{1}{q_n}}) \subset R^\infty$ be a finitely generated ideal. Then $I \subseteq \mathfrak{m}^\infty$ since $(R^\infty, \mathfrak{m}^\infty)$ is a local ring. Letting $q \geq \max\{q_i\}$, we see that $I \subseteq \mathfrak{m}^{[\frac{1}{q}]}R^\infty$. Hence for *any* finitely generated $I \subset R^\infty$, we have $\mathrm{k\ gr}_{R^\infty}(I, M) \leq \mathrm{k\ gr}_{R^\infty}(\mathfrak{m}^{[\frac{1}{q}]}R^\infty, M)\big)$ for $q \gg 0$, since it is a sub-ideal. Therefore $\sup\{\mathrm{k\ gr}_{R^\infty}\big(I, (R^\infty \otimes_R M)\big)\}$ is achieved by ideals of the form $\mathfrak{m}^{[\frac{1}{q}]}R^\infty$.

The particular $M = R/I$ scenario holds since $R^\infty \otimes_R R/I \cong R^\infty / IR^\infty$.

□

We are now ready to prove the relationship between $\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M)$ and $\mathrm{s\ depth}_R(M)$.

Theorem 11. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, let R^∞ be its perfect closure, and let M be a finitely generated R-module. Then $\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M)$ is finite, and more specifically*

$$\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M) = \mathrm{s\ depth}_R(M)$$

In particular if $M = R/I$ is a cyclic module, then

$$\mathrm{k\ depth}_{R^\infty}(R^\infty / IR^\infty) = \mathrm{s\ depth}_R(R/I)$$

Proof. To show $\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M) \geq \mathrm{s\ depth}_R(M)$, let $\mathbf{x} = x_1, \ldots, x_n$ be minimal system of generators for $\mathfrak{m}$. We claim that $\mathrm{k\ gr}_{R^\infty}\big((\mathbf{x})R^\infty, (R^\infty \otimes_R M)\big) \geq \mathrm{s\ depth}_R(M)$, which will prove the inequality, since $\mathrm{k\ depth}_{R^\infty}(R^\infty \otimes_R M) \geq \mathrm{k\ gr}_{R^\infty}\big((\mathbf{x})R^\infty, (R^\infty \otimes_R M)\big)$.

To prove this claim, let $\mathrm{depth}_R\big(F^e(M)\big) \geq d$ for all e, and let $j > n - d$. Fix e, and $q = p^e$. Recall that since R is Noetherian, $\mathrm{k\ depth}_R(F^e(M)) = \mathrm{depth}_R(F^e(M))$. Thus $H_j\big(\mathbf{x}; F^e(M)\big) = 0$, and we must also show $H_j\big(\mathbf{x}; (R^\infty \otimes_R M)\big) = 0$. Fix $z^{[\infty]} \in \ker(d_j^\infty)$.

Then $z^{[q]} \in \ker(d_j^q)$. By exactness at this j-th homology group, there exists a non-zero $y^{[q]} \in \big(F^e(M)\big)^{\binom{n}{j+1}}$ such that $d_{j+1}^q(y^{[q]}) = z^{[q]}$. By F-purity of R, we have an injection $F^e(M)^{\binom{n}{j+1}} \hookrightarrow (R^\infty \otimes_R M)^{\binom{n}{j+1}}$, hence the image $y^{[\infty]}$ in $(R^\infty \otimes_R M)^{\binom{n}{j+1}}$ is non-zero. But now $d_{j+1}^\infty(y^{[\infty]}) = z^{[\infty]}$.

Since $z^{[\infty]} \in \ker(d_j^\infty)$ was chosen arbitrarily, therefore $\ker(d_j^\infty) = \operatorname{image}(d_{j+1}^\infty)$, and $H_j\big(\mathbf{x}; (R^\infty \otimes_R M)\big) = 0$.

To show the reverse inequality, again let $\mathbf{x} = x_1, \ldots, x_n$ be minimal system of generators for $\mathfrak{m}$, and recall that $\text{k depth}_{R^\infty}(R^\infty \otimes_R M) = \sup\{\text{k gr}_{R^\infty}\big((\mathbf{x})^{[\frac{1}{q}]}R^\infty, (R^\infty \otimes_R M)\big)\}$ by lemma 3.2.11. We claim that for any e, $\text{k gr}_{R^\infty}\big((\mathbf{x})^{[\frac{1}{q}]}R^\infty, (R^\infty \otimes_R M)\big) \le \operatorname{depth}_R\big(F^e(M)\big)$, where again $q = p^e$. Hence $\text{k depth}_{R^\infty}(R^\infty \otimes_R M)$ is finite, and since the statement is true for $e \gg 0$, the inequality is true.

To prove this claim, fix e. Let $\text{k gr}_{R^\infty}\big((\mathbf{x})^{[\frac{1}{q}]}R^\infty, (R^\infty \otimes_R M)\big) = d$, and $j > n - d$, whereby $H_j\big(\mathbf{x}^{\frac{1}{q}}; (R^\infty \otimes_R M)\big) = 0$. We must show that $H_j\big(\mathbf{x}; F^e(M)\big) = 0$ as well, where $\mathbf{x}$ acts on the left, which is equivalent to right action by $\mathbf{x}^{\frac{1}{q}}$. Therefore, we can equivalently ask whether $H_j\big(\mathbf{x}^{[\frac{1}{q}]}; F^e(M)\big) = 0$ via right action. Fix $z^{[q]} \in \ker(d_j^q)$, whereby $z^{[\infty]} \in \ker(d_j^\infty)$. Then $\overline{z^{[\infty]}} = 0 \in \operatorname{coker}(d_{j+1}^\infty)$ by exactness of $H_\bullet\big(\mathbf{x}^{\frac{1}{q}}; (R^\infty \otimes_R M)\big)$ at the j-th position. But $\operatorname{coker}(d_{j+1}^\infty) = \operatorname{coker}(d_{j+1}^q \otimes_R 1_{R^\infty}) = \operatorname{coker}(d_{j+1}^q) \otimes_R R^\infty$, and since R is F-pure we have an injection $\operatorname{coker}(d_{j+1}^q) \hookrightarrow \operatorname{coker}(d_{j+1}^q) \otimes_R R^\infty$. Since $\overline{z^{[\infty]}} = 0$ in the image of this map, therefore its preimage $\overline{z^{[q]}} = 0$ as well.

Since $z^{[q]} \in \ker(d_j^q)$ was chosen arbitrarily, $H_\bullet\big(\mathbf{x}; F^e(M)\big)$ is exact at the j-th position.

The particular case is true since $R^\infty \otimes_R R/I \cong R^\infty / I R^\infty$.

$\square$

Combining the previous results, we now have the comparison:

$$\text{k depth}_{R^\infty}(R^\infty \otimes_R M) = \text{s depth}_R(M) \geq \text{c depth}_{R^\infty}(R^\infty \otimes_R M)$$

And for cyclic $M = R/I$, we have:

$$\text{k depth}_{R^\infty}(R^\infty/IR^\infty) = \text{s depth}_R(R/I) \geq \text{c depth}_{R^\infty}(R^\infty/IR^\infty)$$

In order to find a condition under which this last inequality is an equality, currently we require a further supposition on our rings. Namely, we require rings which satisfy countable prime avoidance, which we introduce above. Alternately, a specific application of the prime avoidance lemma is sufficient.

Theorem 12. *Let $(R, \mathfrak{m})$ be an F-pure Noetherian local ring of characteristic $p > 0$, and let M be a finitely generated R-module. If either of the following two conditions hold:*

1. *R satisfies countable prime avoidance*
2. *$\bigcup\bigcup_e \operatorname{Ass}_R \left(F^e(M)/F^e(M)\mathbf{y}\right)$ contains finitely many prime ideals, where $\mathbf{y} \subset R$ is a maximal $(R^\infty \otimes_R M)$ sequence in R^∞*

then,

$$\text{s depth}_R(M) = \text{c depth}_{R^\infty}(R^\infty \otimes_R M)$$

In particular if $M = R/I$ is a cyclic module, if either of the following two conditions hold:

1. *R satisfies countable prime avoidance*
2. *$\bigcup\bigcup_e \operatorname{Ass}_R \left(R/(I, \mathbf{y})^{[q]}\right)$ contains finitely many prime ideals, where $\mathbf{y} \subset R$ is a maximal (R^∞/IR^∞) sequence in R^∞*

then,

$$\text{s depth}_R(R/I) = \text{c depth}_{R^\infty}(R^\infty/IR^\infty)$$

Proof. We have already shown that s $\operatorname{depth}_R(M) \geq$ c $\operatorname{depth}_{R^\infty}(R^\infty \otimes_R M)$. In order to prove the converse, suppose that s $\operatorname{depth}_R(M) >$ c $\operatorname{depth}_{R^\infty}(R^\infty \otimes_R M) = d$. Let $\mathbf{y} = y_1, \ldots, y_d \subset R$ be a maximal $(R^\infty \otimes_R M)$ sequence. Then for all e, $\operatorname{depth}_R \big(F^e(M)/F^e(M)\mathbf{y}\big) > 0 =$ c $\operatorname{depth}_{R^\infty} \big((R^\infty \otimes_R M)/(R^\infty \otimes_R M)\mathbf{y}\big)$. We claim that $\mathfrak{m} = \bigcup_e \bigcup \operatorname{Ass}_R \big(F^e(M)/F^e(M)\mathbf{y}\big)$. Note that this union consists of at most countably many prime ideals since it is a countable union of finite sets.

Clearly the union is contained in $\mathfrak{m}$, since each such associated prime ideal is a subideal of the maximal ideal in a local ring. Conversely, suppose that there exists some $z \in \mathfrak{m} \setminus \bigcup_e \bigcup \operatorname{Ass}_R \big(F^e(M)/F^e(M)\mathbf{y}\big)$. Then z is not contained in any associated prime of $F^e(M)/F^e(M)\mathbf{y}$ for any e. Hence z is regular over $F^e(M)/F^e(M)\mathbf{y}$ for all e. Then $\{\mathbf{y}, z\}$ is a right regular $F^e(M)$-sequence for all e, and hence it is $(R^\infty \otimes_R M)$-regular by lemma 3.2.9. But this fact contradicts the maximality of $\mathbf{y}$.

Now suppose either condition of the theorem holds. Since $\mathfrak{m}$ is contained in this union of prime ideals, then $\mathfrak{m} \subseteq P$ for some $P \in \operatorname{Ass}_R \big(F^e(M)/F^e(M)\mathbf{y}\big)$ and for some e. But since $\mathfrak{m}$ is maximal, $\mathfrak{m} = P$. We now have the maximal ideal of R shown to be an associated prime of $F^e(M)/F^e(M)\mathbf{y}$, and hence $\operatorname{depth}_R \big(F^e(M)/F^e(M)\mathbf{y}\big) = 0$. We have therefore contradicted our assumption that $\operatorname{depth}_R \big(F^e(M)/F^e(M)\mathbf{y}\big) > 0$.

The particular case is true since $R^\infty \otimes_R R/I \cong R^\infty/IR^\infty$, and

$$
\begin{aligned}
F^e(R/I)/F^e(R/I)\mathbf{y} &= (R^{\frac{1}{f^e}} \otimes_R R/I)/(R^{\frac{1}{f^e}} \otimes_R R/I)\mathbf{y} \\
&\cong (R^{\frac{1}{f^e}} \otimes_R R/I) \otimes_R R/\mathbf{y}R \\
&\cong R^{\frac{1}{f^e}} \otimes_R (R/I \otimes_R R/\mathbf{y}R) \\
&\cong R^{\frac{1}{f^e}} \otimes_R \big(R/(I + (\mathbf{y})\big) \\
&\cong R^{\frac{1}{f^e}} \otimes_R R/(I, \mathbf{y}) \\
&\cong R^{\frac{1}{f^e}}/\big(R^{\frac{1}{f^e}}(I, \mathbf{y})\big) \\
&\cong R/(I, \mathbf{y})^{[q]}
\end{aligned}
$$

□

In sum, if R is a Noetherian F-pure ring of characteristic $p > 0$, R^∞ is its perfect closure, and M is an R-module, we always have:

$$\text{k depth}_{R^\infty}(R^\infty \otimes_R M) = \text{s depth}_R(M) \geq \text{c depth}_{R^\infty}(R^\infty \otimes_R M)$$

Additionally, if either condition of theorem 12 holds,

$$\text{k depth}_{R^\infty}(R^\infty \otimes_R M) = \text{s depth}_R(M) = \text{c depth}_{R^\infty}(R^\infty \otimes_R M)$$

In particular if $M = R/I$ is a cyclic module, then

$$\text{k depth}_{R^\infty}(R^\infty/IR^\infty) = \text{s depth}_R(R/I) \geq \text{c depth}_{R^\infty}(R^\infty/IR^\infty)$$

And if either condition of theorem 12 holds,

$$\text{k depth}_{R^\infty}(R^\infty/IR^\infty) = \text{s depth}_R(R/I) = \text{c depth}_{R^\infty}(R^\infty/IR^\infty)$$

Appendix A:

Let S be a commutative ring with identity, not necessarily Noetherian, and let M be an S-module.

A.1 Tensor Products

We begin with the definition of the *tensor product* of two S-modules M and N over S, which is written $M \otimes_S N$. As a group, this new module consists the free abelian group G on pairs (m, n) which are subject to conditions. We define a subgroup H of this group defined by the following generators,

- $(m + m', n) - (m, n) - (m', n)$
- $(m, n + n') - (m, n) - (m, n')$
- $(m \cdot s, n) - (m, s \cdot n)$

where $m, m' \in M$, $n, n' \in N$, and $s \in S$. The tensor product of M and N over S is now defined $M \otimes_S N := G/H$. By construction, this resulting module now has the following three properties (which taken together are regarded as bilinearity over S):

- $(m + m', n) = (m, n) + (m', n)$
- $(m, n + n') = (m, n) + (m, n')$
- $(m \cdot s, n) = (m, s \cdot n)$

$M \otimes_S N$ is an S-module as well, as multiplication is performed in accordance with the actions defined over M and N. Tensor products are fundamental to commutative algebra, and feature centrally in constructions above. From them we define the Frobenius functor in characteristic $p > 0$. Functors over S-modules will appear shortly, while the Frobenius functor in particular is defined above.

Arbitrary elements of $M \otimes_S N$ are of the form $\sum_{i=1}^{k}(m_i \otimes_S n_i)$, which sometimes cannot be written in simpler terms if for each $i \neq j$, $m_i \neq m_j$ and $n_i \neq n_j$. Any isolated term of the form $(m \otimes_S n)$ is referred to as a *simple tensor*.

Tensor products of S-modules satisfy associativity. That is, $(L \otimes_S M) \otimes_S N \cong L \otimes_S (M \otimes_S N)$ for any S-modules L, M and N. Additionally they satisfy symmetry, whereby $M \otimes_S N \cong N \otimes_S M$.

A.2 Hom

An S*-linear map*, or an S*-homomorphism*, of S-modules is a function $\varphi : M \to N$ for which $\varphi(sm + s'm') = s\varphi(m) + s'\varphi(m')$ for $s, s' \in S$ and $m, m' \in M$. We write $\mathrm{Hom}_S(M, N)$ to denote the collection of all S-homomorphisms from M to N. This set is itself a module over S, where addition and scalar multiplication are given by,

- $(\varphi + \varphi')(m) := \varphi(m) + \varphi'(m)$ for all $m \in M$
- $(s \cdot \varphi)(m) := s \cdot \varphi(m) = s\varphi(m) = \varphi(sm)$ for $s \in S$, and for all $m \in M$

A.3 Functors, Tensor, Hom

We now define functors in the category of modules for over a ring S. Omitting a full introduction to category theory, we merely concern ourselves with functors which map to and from the category of S-modules. We first state the general definitions, then we focus on the specific functors relevant to the current project, using the above discussion of Hom and tensor products.

Let Mod_S and ${}_S\mathrm{Mod}$ denote the set of all right and left S-modules respectively. Additionally, the notation ${}_S\mathrm{Mod}_S$ denotes all modules both left and right defined over S. It should be noted, however, that left and right action by S may not be identical, as is the case in the Frobenius functor. Lastly, if $\varphi : R \to S$ is a ring homomorphism, we can discuss

${}_S\operatorname{Mod}_R$ or ${}_R\operatorname{Mod}_S$ which are respectively left or right defined over S and R. Throughout, whenever the difference in action is relevant, it is always mentioned explicitly.

Without loss of generality, in the present section we will discuss ${}_S\operatorname{Mod}$, while all definitions correspondingly generalize to all module categories mentioned in the previous paragraph.

A *functor* $F : {}_S\operatorname{Mod} \to {}_S\operatorname{Mod}$ is a mapping which associates each $M \in {}_S\operatorname{Mod}$ to some other $F(M) \in {}_S\operatorname{Mod}$. If $M, N \in {}_S\operatorname{Mod}$ and $f : M \to N$ is any map of S-modules (called a morphism), then F associates f to a new map $F(f) : {}_S\operatorname{Mod} \to {}_S\operatorname{Mod}$ such that,

1. $F(id_{{}_S\operatorname{Mod}}) = id_{{}_S\operatorname{Mod}}$, which is the identity map

2. If $g : N \to L$ is another map of S-modules, one and only one of the following is true:

   (a) F preserves the action of f and g, and we obtain:

   $F(g \circ f) = F(g) \circ F(f)$

   (b) F reverses the action of f and g, and we obtain:

   $F(f) : F(N) \to F(M)$

   $F(g) : F(L) \to F(N)$

   $F(g \circ f) = F(f) \circ F(g)$

If F satisfies $2a$), then F is called a *covariant* functor. If F satisfies $2b$), it is called a *contravariant* functor.

Let A be a fixed S-module. Below we must define the right-derived functors Ext^i_S, as well as the Frobenius functor in characteristic $p > 0$. In order to do so, we must first establish tensor products $A \otimes_S -$, and the Hom functors $\operatorname{Hom}_S(A, -)$ and $\operatorname{Hom}_S(-, A)$.

For any $M \in {}_S\operatorname{Mod}$ we can define the functor $F(M) := A \otimes_S M$, which is an S-module as well. Tensor products are covariant, since if $f : M \to N$ is a map of S-modules, we have $F(f) := 1 \otimes_S f$, which is a map $1 \otimes_S f : A \otimes_S M \to A \otimes_S N$. Additionally this functor is right exact, which we define below.

Again let $A \in {}_S\,\mathrm{Mod}$ be fixed, and let $M \in {}_S\,\mathrm{Mod}$ be arbitrary. We define the functor $F(M) := \mathrm{Hom}_S(A, M)$, which represents all S-homomorphisms from A to M. This functor is covariant, since if $f : M \to N$ is a map, $F(f) : \mathrm{Hom}_S(A, M) \to \mathrm{Hom}_S(A, N)$ is defined by $F(f) := f \circ \varphi$ for any $\varphi \in \mathrm{Hom}_S(A, M)$.

Similarly, we can define $F(M) := \mathrm{Hom}_S(M, A)$ which represents all S-module homomorphisms from M to A, which is again itself an S-module. However, this functor is now *contra*variant. If $f : M \to N$ is a map, then $F(f) : \mathrm{Hom}_S(N, A) \to \mathrm{Hom}_S(M, A)$ is defined by $F(f) := \varphi \circ f$ for any $\varphi \in \mathrm{Hom}_S(N, A)$.

Both Hom functors are left exact, which we define below.

We use the functors $A \otimes_S -$ and $\mathrm{Hom}_S(-, A)$ respectively to define the Frobenius functor over characteristic $p > 0$, and the right-derived functor Ext.

A.4 Direct Limits

A *directed set* is a set X together with a binary relation $\leq$ which for any $A, B, C \in X$ satisfies:

1. *Reflexivity*: $A \leq A$
2. *Transitivity*: If $A \leq B$ and $B \leq C$, then $A \leq C$
3. Every two elements of X must share an upper bound. That is, if $A, B \in X$, then there must exist some $C \in X$ such that $A \leq C$ and $B \leq C$.

The first two conditions without 3) on $\leq$ form a *pre-order*.

Let $(X, \leq)$ be a directed set, let I be some index, and let $\{Y_i \ : \ i \in I\}$ be a family of objects in X. For every $Y_i \leq Y_j$ let g_{ij}, be a map which maps Y_i to Y_j. These maps g_{ij} have the properties:

1. $g_{ii} : Y_i \to Y_i$ is the identity map

2. if $g_{ij} : Y_i \to Y_j$ and $g_{jk} : Y_j \to Y_k$ are maps for $i \leq j \leq k$, then $g_{ik} := g_{jk} \circ g_{ij}$ which maps Y_i to Y_k

The pair (Y_i, g_{ij}) is called a *directed system.*

Let (Y_i, g_{ij}) be a directed system. The *direct limit* of the system, $Y = \varinjlim Y_i$, is an element of the set X whereby for each Y_i we have a map $\varphi_i : Y_i \to Y$ where for each $Y_i \leq Y_j$, $\varphi_i = \varphi_j \circ g_{ij}$, where"$\circ$" denotes the composition map (this notation will be used differently below to denote an action by rings on a module. It will be clear throughout which usage is being discussed). Furthermore, suppose there exists some other $Y' \in X$ such that there exists a family of maps $\{\varphi_i'\}_{i \in I}$ from each Y_i to Y' with the same such properties as $\{\varphi_i\}_{i \in I}$. Then there exists a unique map $\psi : Y \to Y'$ such that $\varphi_i' = \psi \circ \varphi_i$ for all i. The direct limit Y may not exist given a category X, but it will exist for our directed systems throughout.

A.5 Ext

We now establish the Ext^i_S functors, which are referred to as a right-derived functor for reasons that will be apparent below. Left-derived functors can be defined using similar machinery, however they will not feature in our discussion.

Before proceeding, we note that our point of view while constructing the modules $\mathrm{Ext}^i_S(S/I, M)$ is not unique. They can alternately be defined using injective resolutions of S-modules in conjuctions with the covariant functor $\mathrm{Hom}_S(S/I, -)$. However we will utilize the free resolution point of view in conjunction with the contravariant $\mathrm{Hom}_S(-, M)$ functor.

In order to proceed we first must define chain complexes. Let S be a commutative ring. A *chain complex* $C_\bullet$ of modules over S is a sequence,

$$\ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots$$

where $\text{image}(d_{i+1}) \subseteq \ker(d_i)$ for all i. Hence each composition map has the property $d_i \circ d_{i+1} = 0$. The maps d_i are referred to as *differentials*. Furthermore, if $\text{image}(d_{i+1}) = \ker(d_i)$ for some i, we say that $C_\bullet$ is *exact* at the ith position. If $C_\bullet$ is exact for all i, then it an *exact sequence*. Note that we indexed our M_i in a descending manner, however the indexing can ascend as well.

Given a chain complex $C_\bullet$ of S-modules, since $\text{image}(d_{i+1}) \subseteq \ker(d_i)$ we define the i-th *homology* module of $C_\bullet$ by the quotient:

$$H_i^C := \ker(d_i)/\text{image}(d_{i+1})$$

In particular if $C_\bullet$ is exact at the i-th position, note that $H_i^C = 0$. If $C_\bullet$ is an exact sequence, all the homology groups are 0.

Letting A be any S-module, we can now define right and left exactness respectively for tensor products $A \otimes_S -$ and the Hom functors $\text{Hom}_S(-, A)$ and $\text{Hom}_S(A, -)$. Let $0 \to M_2 \to M_1 \to M_0 \to 0$ be an exact sequence of S-modules, called a *short exact sequence*. Then the following chain complexes are all exact at the 0-th position:

1. $\ldots \to A \otimes_S M_1 \to A \otimes_S M_0 \to 0$
2. $0 \to \text{Hom}_S(A, M_2) \to \text{Hom}_S(A, M_1) \to \ldots$
3. $0 \to \text{Hom}_S(M_0, A) \to \text{Hom}_S(M_1, A) \to \ldots$

1) describes the right exactness of tensor products, while 2) and 3) describe the left exactness of the Hom functors. This third condition in particular is necessary in our construction of Ext modules, since we will be applying the contravariant Hom to free resolutions of S/I for ideals $I \subset S$.

Let S be a ring and let M be an S-module. An augmented *free resolution* of M is an exact sequence of free modules,

$$\ldots \longrightarrow S^{n_1} \longrightarrow S^{n_0} \longrightarrow M \longrightarrow 0$$

where each S^{n_i} is a free module with the corresponding free ranks n_i. For arbitrary S and M, the n_i's may be infinite cardinals. But for Noetherian S, and if M is finitely generated over S, then each $n_i \in \mathbb{N}$. Furthermore, the modules S^{n_i} may or may not terminate, i.e. possibly $S^{n_i} = 0$ for $i \gg 0$. In this case, we say M has finite projective dimension. If no resolution exists with this property, we say M has infinite projective dimension.

We require one minor adjustment of our augmented free resolution of M before we define the functors Ext^i_S. We omit M in the last position, thus obtaining the *deleted free resolution*,

$$\ldots \longrightarrow S^{n_1} \longrightarrow S^{n_0} \longrightarrow 0$$

which is still exact in each position, except the 0-th homology group which is isomorphic to M. We will use this latter sequence in our forthcoming construction.

It is a fact that if S is a Noetherian, and M is a finitely generated S-module, there always exists a minimal free resolution. By minimal, we mean that if $C_\bullet$ is such a resolution, then each S^{n_i} cannot be generated by any strictly smaller collection of generators while maintaining that $C_\bullet$ is a free resolution.

Now let $I \subset S$ be an ideal. Finally we can define the functors Ext^i_S, and more concretely the modules $\mathrm{Ext}^i_S(S/I, M)$ for some S -module M, and $i \in \mathbb{N}$. Let

$$\ldots \xrightarrow{d_2} S^{n_1} \xrightarrow{d_1} S \xrightarrow{\epsilon} S/I \longrightarrow 0$$

be a free resolution of the cyclic module S/I (since S/I is a cyclic module, meaning it requires only one generator, note that $n_0 = 1$). As mentioned above, we will need the deleted resolution:

$$\ldots \xrightarrow{d_2} S^{n_1} \xrightarrow{d_1} S \longrightarrow 0$$

where again $H_0^{C_\bullet} \cong S/I$.

If M is any S-module, we now apply the contravariant functor $F(-) := \mathrm{Hom}_S(-, M)$ to this deleted free resolution of S/I. The action of this functor now reverses the direction of the chain complex. The ascending index extends to the right, hence the term "right derived" functor we use to describe Ext. Due to left exactness, the chain is now:

$$0 \longrightarrow \mathrm{Hom}_S(S, M) \xrightarrow{F(d_1)} \mathrm{Hom}_S(S^{n_1}, M) \xrightarrow{F(d_2)} \dots$$

This resulting complex will no longer be exact in general, and the resulting non-zero homology modules are the goal of this construction. We use them to measure certain features of how an ideal $I \subset S$ acts on M. The i-th *Ext functor* $\mathrm{Ext}^i_S(S/I, -) : {}_S\mathrm{Mod} \to {}_S\mathrm{Mod}$ maps the module M to the i-th *Ext module* $\mathrm{Ext}^i_S(S/I, M)$, which is defined as the i-th homology module of this complex.

One particular homology module of note is the 0-th $\mathrm{Ext}^0_S(S/I, M)$, which is isomorphic to $\mathrm{Hom}_S(S/I, M)$.

A.6 Closure Operator

Closure operators are a general notion used throughout mathematics, and they feature prominently in the study of ideals in commutative rings. Recall that given some set X, a *closure operator* c is an operation which maps subsets $U \subseteq X$ to some other "c-closed" $U^c \subseteq X$. Furthermore, for any $U, V \subseteq X$, we have:

1. $U \subseteq U^c$
2. $(U^c)^c = U^c$
3. If $U \subseteq V$, then If $U^c \subseteq V^c$

We see in [Eps11] that closure operations on ideals in rings are a thriving subject of study in commutative algebra. Throughout, we focus on the Frobenius closure in characteristic $p > 0$.

Bibliography

[Bar72] Floyd Barger, *A theory of grade for commutative rings*, Proc. Amer. Math. Soc. **36** (1972), no. 2, 365–368.

[BH97] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics, no. 39, Cambridge Univ. Press, Cambridge, 1997.

[Bur72] Lindsay Burch, *Codimension and analytic spread*, Proc. Cambridge Philos. Soc. **72** (1972), 369–373.

[Eps11] Neil Epstein, *A guide to closure operations in commutative algebra*, arXiv:math.AC/1106.1119v1, 2011.

[ES14] Neil Epstein and Jay Shapiro, *Strong Krull primes and flat modules*, J. Pure Appl. Algebra **218** (2014), no. 9, 1712–1729.

[Gre65] Marvin J. Greenberg, *Perfect closures of rings and schemes*, Proc. Amer. Math. Soc. **16** (1965), 313–317.

[HH00] Melvin Hochster and Craig Huneke, *Localization and test exponents for tight closure*, Michigan Math. J. **48** (2000), 305–329.

[Hoc74] Melvin Hochster, *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. (3) **29** (1974), 55–76.

[HR76] Melvin Hochster and Joel L. Roberts, *The purity of Frobenius and local cohomology*, Adv. Math. **21** (1976), no. 2, 117–172.

[IR84] Juana Iroz and David E. Rush, *Associated primes in non-Noetherian rings*, Canad. J. Math. **36** (1984), no. 2, 344–360.

[Jor82] D.A. Jordan, *Bijective extensions of injective ring endomorphisms*, J. London Math. Soc. (2) **25** (1982), 435–448.

[Kap70] Irving Kaplansky, *Commutative rings*, Allyn and Bacon Inc., Boston, 1970.

[LW12] G. Leuschke and R. Wiegand, *Cohen-macaulay representations*, second ed., Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, no. 8, Cambridge Univ. Press, Cambridge, 1986, Translated from the Japanese by M. Reid.

[Nor76] Douglas G. Northcott, *Finite free resolutions*, Cambridge Tracts in Mathematics, no. 71, Cambridge Univ. Press, Cambridge, 1976.

[NS04] Nicole Nossem and Rodney Y. Sharp, *Ideals in a perfect closure, linear growth of primary decompositions, and tight closure*, Trans. Amer. Math. Soc. **356** (2004), no. 9, 3687–3720.

[SN04] Rodney Y. Sharp and Nicole Nossem, *Ideals in a perfect closure, linear growth of primary decompositions, and tight closure*, Trans. Amer. Math. Soc. **356** (2004), no. 9, 3687–3720.

[SS04] Anurag K. Singh and Irena Swanson, *Associated primes and local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. **33** (2004), 1703–1733.

[SV85] Rodney Y. Sharp and Peter Vámos, *Baire's category theorem and prime avoidance in complete local rings*, Arch. Math. (Basel) (1985), no. 44, 243–248.

Curriculum Vitae

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