Essays on Preference Extensions

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Dedication

I dedicate this dissertation to my parents Tatiana and Lev.

Acknowledgments

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Abstract

ESSAYS ON PREFERENCE EXTENSIONS

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Individual preferences are only partially observable. We only observe choices for those opportunities that are available. However, the design and evaluation of policy many times requires the ability to extrapolate these preferences to new environments. In this dissertation I discuss the issue of extending preference relations from observed behavior in several contexts. The dissertation is organized into three chapters.

The first chapter presents a general representation theorem for the extension of preference relations. That is, it provides the conditions under which the existence of a utility-representable complete extension of an observed incomplete relation can be tested. This allows us to revise the existing revealed preference theory by showing that every revealed preference test can be represented as a set of internal and external consistency conditions.

The second chapter presents criteria under which a set of observed choices can be generated by a complete, transitive, monotone and quasi-linear preference relation. I test the empirical content of the quasi-linearity of preferences by conducting a laboratory experiment on individual decisions among goods and money. I show that while subjects generally satisfy the generalized axiom of revealed preference, they are not generally consistent with hypothesis of quasi-linear preferences.

The third chapter analyzes collective choice from a revealed preference perspective. A collective choice function is Pareto rationalizable if there are complete preference relations for each player (satisfying additional desired properties if necessary) such that observed choices are the Pareto efficient outcomes. I characterize the set of Pareto rationalizable single-valued collective choice functions.

Chapter 1: A Representation Theorem for General Revealed Preferences

1.1 Introduction

Rational behavior is commonly modeled in economics in three different ways. A long tradition, going back to the founders of neoclassical economics if not even earlier,[1] describes rational behavior as the maximization of an objective (utility) function. Another approach, pioneered by Frisch (1957) and developed and popularized by Debreu (1954), identifies rational behavior with the existence of a complete and transitive binary (preference) relation over the objects of choice. A third strand, pioneered by Samuelson (1938), describes rational behavior as the satisfaction of congruence (revealed preference) conditions on finite sets of observed choices.

The connection between the different approaches to rational behavior has been the object of attention of a theoretical literature in economics starting with the contributions of Debreu (1954) on the problem of representing preference relation by means of utility functions and Afriat (1967) on the problem of the construction of utility functions on the basis of finite data sets. A seminal contribution by Richter (1966) considers the connection between the three different approaches, providing a general equivalence result between congruence conditions on finite sets of observed choices and the existence of preference relations, and an equivalence result between congruence conditions on sets of observed choices from competitive (linear) budgets and the representation of the underlying preferences by means of a utility function. Further efforts to connect these notions have been taken by Jaffray (1975) and Bossert et al. (2002) in terms of constructing an upper semi-continuous utility-representable extension of a given preference relation.

[1] See e.g. Stigler (1950) for a historical summary.

In this chapter we seek to connect the three models of rational behavior in a parsimonious way. That is, we seek for criteria under which a preference relation implied by a finite set of choice observations has a complete extension that can in turn be represented by a utility function. To this end, we build on the functional approach of Duggan (1999) and Demuynck (2009), and introduce the notion of a rational closure as a mapping from (possibly incomplete) preference relations to (possibly incomplete) preference relations whose fixed points are transitive and that preserves separability properties of the original preference relation. We show that the transitive closure considered previously by Duggan (1999) and Demuynck (2009) and others is an example (not unique) of a rational closure.

Our main result is a representation theorem. We show that an incomplete preference relation has a complete, utility-representable extension that is a fixed point of the rational closure if and only if a simple set-theoretic consistency requirement between the rational closure and the incomplete preference relation is fulfilled. Intuitively, we think of the original preference relation as the information on preferences that has been obtained from (not necessarily finite) choice observations. The consistency requirement is then a general congruence condition guaranteeing that the observed behavior can be represented by a utility function. The fact that existence is obtained as a fixed point of a particular mapping can be exploited to obtain desirable properties of the utility function, as discussed below.

We then consider a revealed preference experiment. Intuitively, a revealed preference experiment represents a situation in which information on strict preferences has been obtained from finite, consecutive choice observations. With no restrictions on budget sets or the consumption space, we show that a revealed preference experiment can be rationalized by a utility function if and only if two conditions are satisfied: (1) the different observations do not directly contradict each other, and (2) the consistency requirement identified in the main theorem is satisfied by the union of the consecutive observations with respect to the transitive closure. In our formulation, condition (1) is equivalent to a general version of WARP, and conditions (1) and (2) are jointly equivalent to a general version of SARP. As a corollary, without restrictions on budget sets or the consumption space, a general version of SARP is necessary and sufficient for a revealed preference experiment to be rationalized by a utility function.

Similarly, we show that a revealed preference experiment can be rationalized by a strictly increasing utility function if and only if conditions similar to (1) and (2) above are satisfied, with the monotone closure (a mapping that incorporates both transitivity and monotonicity criteria) substituting for the transitive closure. This illustrate the point that the techniques in the chapter can be put to us to provide tests for the existence of utility functions representing choice observations that satisfy additional properties. An exception is continuity, which is not compatible with the properties of a rational closure–mappings that satisfy continuity do not induce transitivity and lead to inferences on preferences over pairs of alternatives that cannot be obtained from information about a finite sample of preferences over pairs of alternatives.

We also observe that a rationalization by a strictly increasing utility function while allowing for observed choices to be indifferent to some other alternatives in the budget set, as in Varian (1982), requires only a minor relaxation of condition (1). As a corollary a general version of GARP is necessary and sufficient for a revealed preference experiment to be rationalized allowing for indifferences of observed choices by a strictly increasing utility function.

The connections between the different approaches to rational behavior have been an object of attention of the literature for a long time. As mentioned above, the general connection between utility functions and preference relations was originally studied[2] by Debreu (1954) in the context of continuous utility functions. Rader (1963) and Jaffray (1975) relaxed the assumption of continuity and obtained semi-continuous utility rationalization results that were generalized by Bosi and Mehta (2002). Peleg (1970) shown the sufficient condition for existence of a continuous utility representation for incomplete preference relation. More recently Ok (2002), Evren and Ok (2011) have investigated a problem of existence of a vector-valued utility representation of preference relations.

[2]Earlier representation theorems under additional assumptions were provided by Cantor (1895) for completely ordered sets and Von Neumann and Morgenstern (1947) for lotteries.

The basic result connecting the set of choices and preference relations was proven by Szpilrajn (1930). Szpilrajn shown that any acyclic preference relation has a complete and transitive extension. Demuynck (2009) generalized the result by providing a condition to test for existence of complete extension that has properties usually assumed by economists.

The connection between finite data sets and the utility functions was originally investigated by Afriat (1967). Afriat (1967) uses linear budgets in a Euclidean consumption space and obtains the existence of a concave, monotone and continuous[3] utility function that is congruous with the observations. Subsequent literature has constructed tests for consistency of the finite consumption data with various utility maximization hypothesis. Kannai (1977), Matzkin (1991), Matzkin and Richter (1991) and Forges and Minelli (2009) address the question of testing concavity of the utility representation. Varian (1983), Diewert and Parkan (1985), Echenique and Saito (2015) and Polisson et al. (2015) develop tests for separable utility representations including the context of choices under uncertainty. Crawford (2010) propose a test for habit formation models. Reny (2015) extends Afriat (1967) results using linear budgets to infinite data sets–as in our case, the extra generality implies giving up on deriving continuity from the data. Chambers et al. (2010) characterize the testable implications of revealed preference theory. Chambers and Echenique (2016) provide a general, systematic overview of revealed preference results.

1.2 Preliminaries

1.2.1 Alternatives

We consider an arbitrary set of alternatives X, with elements denoted x, y, etc. Some examples of interest are (a) a finite set, representing job offers available to a worker, houses available to a buyer, etc., (b) the positive orthant of a finite Euclidean space $\mathbb{R}^m_+$, representing bundles of m commodities, (c) the set of sequences $(c_0, c_1, \ldots, c_t, \ldots)$ where $c_t \in \mathbb{R}^m_+$, representing consumption plans potentially available to a long lived agent, and (d) the set

[3] That is, with respect to the usual Euclidean topology.

of probability measures over $\mathbb{R}$, representing lotteries with monetary rewards or losses.

We introduce additional structure on the set of alternatives as needed. In particular, in order to define continuous preferences, we let (X, τ) be a topological space for some topology τ.[4] The classical continuous representation results of Debreu (1954) and Rader (1963) relay on the topological space (X, τ) being second countable, that is, having a countable base.[5] Continuity is of course an attractive property when there are infinite alternatives as in examples (b), (c) and (d). The Euclidean space of example (b), equipped with the usual Euclidean topology, is a second countable space. There are different topological spaces of interest for examples (c) and (d), and not all of them are second countable; see e.g. Mas-Colell (1986) and Stokey and Lucas (1989). This illustrates the usefulness of representation results for arbitrary sets of alternatives.

1.2.2 Preference Relations

A set $R \subseteq X \times X$ is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$. We denote the inverse relation $R^{-1} = \{(x, y) | (y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \backslash I(R)$. We denote the incomparable part by $N(R) = X \times X \backslash (R \cup R^{-1})$.

Definition 1.1. *Given a preference relation and any alternative in X, the* ***lower contour set*** *and the* ***upper contour set*** *of x are, respectively,*

$$L_R(x) = \{y | (x, y) \in P(R)\} \quad \textit{and} \quad U_R(x) = \{y | (y, x) \in P(R)\}.$$

We list below some properties of a preference relation:

Definition 1.2. *A preference relation R is said to be*

1. ***complete*** *if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \varnothing$).*

[4]Recall that a topology on X is a collection of subsets of X, called open sets, that includes $\varnothing$ and X, and that is closed under arbitrary unions and finite intersections.

[5]Recall that a base for a topology τ on X is a collection $\mathcal{B}$ of open sets, such that every $x \in X$ and every open set U containing x, there is $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

2. ***transitive*** *if* $(x, y) \in R$ *and* $(y, z) \in R$ *implies* $(x, z) \in R$ *for all* $x, y, z \in X$.

3. Z***-separable*** *for given* $Z \subseteq X$ *if for any* $(x, y) \in P(R)$ *there is* $z \in Z$ *such that* $(x, z) \in R$ *and* $(z, y) \in R$.

4. ***upper semi-continuous*** *if* (X, τ) *is a topological space and* $L_R(x)$ *for all* $x \in X$ *are open.*

5. ***continuous*** *if* (X, τ) *is a topological space and* $L_R(x)$ *and* $U_R(x)$ *for all* $x \in X$ *are open.*

Completeness and transitivity are the usual desirable properties of preference relations. Separability and continuity play a key role in classical representation results to which we appeal later on.

Examples. Suppose that $X = \mathbb{R}^m_+$, with the usual Euclidean topology. **(a)** It is well known that if R is complete, transitive and continuous, then R is $\mathbb{Q}^m_+$-separable, that is for any $(x, y) \in P(R)$ there is a bundle z all whose components are rational numbers such that $(x, z) \in R$ and $(z, y) \in R$ (see e.g. Kreps (2012), Proposition 1.15). Since $\mathbb{Q}^m_+$ is countable, it follows that R is separable with respect to any collection of subsets of $\mathbb{R}$ that includes $\mathbb{Q}^m_+$. **(b)** Denote by L the lexicographic preference relation, i.e. $(x, y) \in L$ if there is $k \in \{1, \ldots, m\}$ such that $x_i = y_i$ for $i < k$ and $x_k > y_k$. It is well-known that this relation is not $\mathbb{Q}^m_+$-separable (see e.g. Mas-Colell et al. (1995), Example 3.C.1). However, L is $\mathbb{R}^m_+$-separable.

A driving idea in this chapter is to extend incomplete preference relations including additional comparisons of pairs of alternatives while preserving the asymmetric part of the original preference relation:

Definition 1.3. *A preference relation* R' *is an* ***extension*** *of* R*, denoted* $R \preceq R'$*, if* $R \subseteq R'$ *and* $P(R) \subseteq P(R')$.

1.2.3 Functions over Preference Relations

In this section we consider general functions $F : \mathcal{R} \to \mathcal{R}$ defined over the set of preference relations which may be used to extend an incomplete preference relation.

Definition 1.4. *For any given function $F : \mathcal{R} \to \mathcal{R}$, we let*

1. $\mathcal{R}_F = \{R \in \mathcal{R} | R \preceq F(R)\}$,

2. $\mathcal{R}_F^Z = \{R \in \mathcal{R}$ *and R is Z-separable* $|R \preceq F(R)\}$,

$\mathcal{R}_F$ and $\mathcal{R}_F^Z$ are different sets of preference relations that are extended by F.

We list below some properties of a function over the set of preference relations:

Definition 1.5. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

1. ***monotone*** *if for all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,*

2. ***closed*** *if for all $R \in \mathcal{R}$, $R \subseteq F(R)$,*

3. ***idempotent*** *if for all $R \in \mathcal{R}$, $F(F(R)) = F(R)$,*

4. ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*

5. ***expansive*** *if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$ and $P(R) = P(R \cup S)$,*

6. ***transitive-inducing*** *if any preference relation satisfying $R = F(R)$ is transitive,*

7. ***separability-preserving*** *if there is a countable set Q_F such that for any countable set Z and $R \in \mathcal{R}_F^{Q_F \cup Z}$, $F(R)$ is $(Q_F \cup Z)$-separable.*

8. ***upper semi-continuous*** *if (X, τ) is a topological space and $R \in \hat{\mathcal{R}}_F$ implies $F(R)$ is upper semi-continuous,*

9. ***continuous*** *if (X, τ) is a topological space and $R \in \tilde{\mathcal{R}}_F$ implies $F(R)$ is continuous.*

Any function $F : \mathcal{R} \to \mathcal{R}$ that is monotone, closed and idempotent is called a **closure**. A closure is algebraic as defined above if any element of the closure can be obtained from applying the closure to a finite subset of the original relation.[6]

Expansiveness and transitivity impose conditions on fixed points of F. Expansiveness of F, in particular, means that we can add some indifference pairs to any fixed point $R = F(R)$ that is not complete, such that the new relation will be in $\mathcal{R}_F$.

Separability preserving implies that if R is separable with respect to *any* countable set Z and is extended by F, then we can augment Z so that F preserves separability with respect to the augmented set. This is because $R \in \mathcal{R}_F^Z$ implies $R \in \mathcal{R}_F^{Q_F \cup Z}$.

Gathering the first seven properties, we can define the following:

Definition 1.6. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be a **rational closure** if it is an expansive algebraic closure that induces transitivity and preserves separability.*

Intuitively, a rational closure is a rule which may be useful to extend some original incomplete preference relation, and that satisfies certain desirable criteria. Those criteria include not losing information contained in the original preference relation (closeness and monotonicity), being thorough in using that information (idempotence), using finite sets of information contained in the original preference relation to make each binary comparison (algebraicity), being able to incorporate some indifferences (expansiveness), and inducing transitivity and preserving separability of the original preference relation, both of which are useful to build a utility representation.

The **transitive closure** provides a natural example of a function over preference relations. Denote it by

$$T : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, \ldots, s_n$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \ldots, n-1$, and $s_1 = x$ and $s_n = y$.

We claim:

[6] See e.g. Davey and Priestley (2002), definition 7.12.

Lemma 1.1. *The transitive closure $T : \mathcal{R} \to \mathcal{R}$ is a rational closure.*

Proof. It is easy to check that T is an algebraic closure and that it induces transitivity.

To prove that T is separability-preserving, recall that by definition for any $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, s_2, \ldots, s_n$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \ldots, n-1$, and $s_1 = x$ and $s_n = y$. This implies that for any $(x, y) \in P(T(R))$ there is some $k \in \{1, \ldots, n-1\}$ such that $(s_k, s_{k+1}) \in P(R)$. Now suppose the R is Z-separable; this implies that there is some $z \in Z$ such that $(s_k, z), (z, s_{k+1}) \in R$. But then $(x, z), (z, y) \in T(R)$. Thus, $T(R)$ is also Z-separable. (That is, in terms of the definition or separability-preservation, $Q_T = \varnothing$.)

To prove that T is expansive, consider a relation $R = T(R)$ and assume that $N(R) \neq \varnothing$. Take any element $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y), (y, x)\}$. We claim that $R' \preceq T(R')$, which would prove that T is expansive. It is clear that $R' \subseteq T(R')$. Therefore, we only need to show that $P(R') \subseteq P(T(R'))$. Assume, on the contrary, that there are elements z and w for which $(z, w) \in P(R')$ and $(w, z) \in T(R')$, and note that $(x, y) \neq (z, w) \neq (y, x)$. From the definition of T, we know that there is some finite sequence $s_1, \ldots, s_n$ such that $s_1 = w$, $s_n = z$, and $(s_j, s_j + 1) \in R'$ for each $j = 1, \ldots, n-1$. Let m be the minimal integer such that there is such sequence of length m, and let S be any such sequence of length m.

Given a sequence S as described above, there is some j such that either $(s_j, s_{j+1}) = (x, y)$ or $(s_j, s_{j+1}) = (y, x)$ for some $1 < j < m-1$; otherwise $(w, z) \in T(R) = R$, contradicting $(z, w) \in P(R')$. Suppose without loss of generality that $(s_j, s_{j+1}) = (x, y)$ for some $1 < j < m-1$; then there is no $k \neq j$ such that $(s_k, s_{k+1}) = (y, x)$ or $(s_k, s_{k+1}) = (x, y)$, otherwise S would not be the shortest sequence from w to z such that every consecutive pair is in R'. Since $(z, w) \in P(R')$, we have $(z, w) \in R'$. Now consider the finite sequence $y, s_{j+2}, \ldots, s_{m-1}, z, w, s_1, \ldots, s_{j-1}, x$. Note that every pair of consecutive elements of the sequence is in R' and is different from (x, y) and (y, x), so every pair of consecutive elements of the sequence is in R. But then $(y, x) \in T(R) = R$, contradicting $(x, y) \in N(R)$. $\square$

Note that separability-preserving holds for the transitive closure in a very simple form. That is, if R is Z-separable for any Z and extended by T, then $T(R)$ is also Z-separable. We are giving more latitude in the definition of a separability-preserving function to accommodate other useful rational closures. In particular, the monotone closure, described in Section 4, requires to expand Z judiciously in order for the monotone closure to preserve separability with respect to the augmented set.

It is simple to check that the transitive closure is not upper semi-continuous and therefore not continuous for arbitrary topological space (X, τ). As an illustration of a closure that is continuous–and therefore, a fortiori upper semi-continuous–for arbitrary topological space (X, τ), consider the **continuity closure** given by

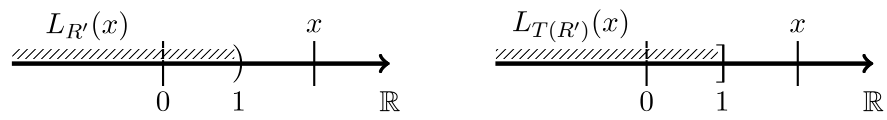
$$C : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in C(R)$ if and only if there is a sequence $(x_n, y_n) \in R$, such that $x_n \to x$ and $y_n \to y$. Unfortunately, the continuity closure is neither algebraic nor transitive, as shown by means of examples below.

Examples. **(a)** Consider the preference relation R' over $X = \mathbb{R}$, equipped with the Euclidean topology, such that $(x, y) \in I(R')$ if and only if either $x, y > 1$, or $x, y < 1$, or $x, y = 0, 1$, and $(x, y) \in P(R')$ if and only $x > 1 > y$. We can check that $(x, y) \in I(T(R'))$ if and only if either $x, y > 1$ or $x, y \leqslant 1$ and $(x, y) \in P(T(R'))$ if and only $x > 1 \geqslant y$. Note that $L_{R'}(x)$ is open for $x > 1$, but $L_{T(R')}(x)$ is not. That is, $T(R')$ is transitive and extends R', but it is not upper semi-continuous. (See Figure 1.) We can also check that $(x, y) \in I(C(R'))$ if and only if $x, y \geqslant 1$ or $x, y \leqslant 1$, and $(x, y) \in P(C(R'))$ if and only if $x > 1 > y$. That is, $C(R')$ is continuous and extends R', but it is not transitive.

(b) Consider the preference relation $R'' = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ over $X = \mathbb{R}$, equipped with the Euclidean topology. Note that $(0, 1) \in C(R'')$, but there is no finite sub-relation $R \subset R''$ such that $(0, 1) \in C(R)$, showing that the continuous closure is not algebraic.

A condition to apply the classical representation results of Debreu (1954) and Rader



Open lower contour set for R' Closed lower contour set for $T(R')$

Figure 1.1: Upper semi-continuous R' but not upper semi-continuous $T(R')$

(1963) is that (X, τ) is a second countable topological space for some topology such that contour sets are open. If (X, τ) is second countable, in fact, upper semi-continuity of F implies that F is separability-preserving with respect to the base of the topology. That is, for a given second countable topological space, separability preserving with respect to the base of the topology, upper semi-continuity, and continuity are increasingly demanding conditions on F. However, as illustrated by the discussion above, upper semi-continuity is in fact too demanding to be of interest for our approach as it conflicts with other desirable properties.

1.2.4 Consistency

As shown below, the following is a necessary and sufficient condition for $F(R)$ to be an extension of R:

Definition 1.7. *Given a function $F : \mathcal{R} \to \mathcal{R}$, a preference relation R is said to be **F-consistent** if $F(R) \cap P^{-1}(R) = \varnothing$.*

Example. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5\}$ and consider the preference relation $R = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$. This relation is not transitive and is not T-consistent (see Figure 1.2) because $(x_1, x_3) \in T(R)$ and $(x_3, x_1) \in P(R)$. On other hand $R' = \{(x_1, x_2), (x_2, x_3), (x_4, x_5)\}$ is not transitive but it is T-consistent. Note that transitivity of R is sufficient but not necessary for T-consistency of R.

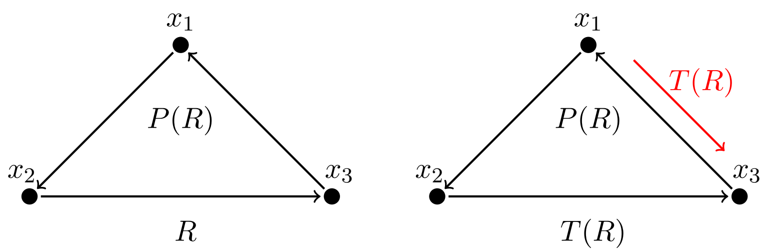


Figure 1.2: Violation of T-consistency

From idempotence and transitivity-preserving it follows that, if F is a rational closure, then $F(R)$ is transitive for any R. Thus, if F is a rational closure, $F(R) \supseteq T(R)$, so that F-consistency implies T-consistency. That is, transitivity is a minimum requirement for an extension rule intended to lead to a utility representation, but F may incorporate other desiderata.

1.3 A Representation Theorem

Our main result is a theorem providing conditions for the existence of a utility function that represents the complete extension of a given preference relation. We do it by showing the existence of a complete relation that is a fixed point of a rational closure. Following Debreu (1954), we define as a **natural topology** for a given complete and transitive preference relation R any topology such that R is continuous.

Theorem 1.1. *Let F be a rational closure and let $R \in \mathcal{R}$ be Z-separable for some countable set Z. R has a complete extension $R^* = F(R^*)$ that can be represented by a utility function if and only if R is F-consistent. Moreover, the utility function is continuous in any natural topology.*

To prove Theorem 1.1 we need several supplementary results. We use the following result (contained in Lemma 1 in Demuynck (2009)) repeatedly in the proofs:

Lemma 1.2. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, then $R \in \mathcal{R}_F$ if and only if R is F-consistent.*

Proof. Since $R \subseteq F(R)$ by assumption, we only need to show that $P(R) \subseteq P(F(R))$ if and only if R is F-consistent. If $(x, y) \in P(R)$ then $(x, y) \in R$ and therefore $(x, y) \in F(R)$. Thus, $(x, y) \in P(F(R))$ for every $(x, y) \in P(R)$ if and only if $(y, x) \notin F(R)$ for every $(x, y) \in P(R)$, or equivalently if and only if $F(R) \cap P^{-1}(R) = \varnothing$. □

The next two results are useful in order to apply Zorn's lemma and show that there is a complete extension of the original preference relation.

Lemma 1.3. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, monotone and algebraic, then for any countable Z and every chain*

$$R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$$

such that $R_\alpha \in \mathcal{R}_F^Z$ for all α, we have $\cup_{\alpha \geqslant 0} R_\alpha \in \mathcal{R}_F^Z$.

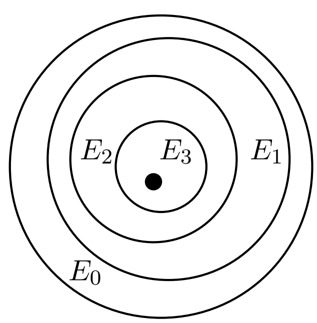


Figure 1.3: Illustration for Lemma 1.3. $E_i \subset \mathcal{R}_F^Z$ is the of Z-separable F-consistent extensions of R_i. If every element in the sequence $\{E_i\}$ is non-empty, then the limit relation $\cup_{\alpha \geqslant 0} R_\alpha$ also has a non-empty set of Z-separable F-consistent extensions.

Proof. Let $B = \cup_{\alpha \geqslant 0} R_\alpha$. If the chain is finite B is itself an element (the last element) of the chain, so that $B \in \mathcal{R}_F$ is immediate. Thus, we only need to be concerned with infinite chains. We know that each element R_α of the chain is F-consistent (from Lemma 1.2) and Z-separable, and we only need to show that B is F-consistent and Z-separable.

For consistency of B, assume that there is $(x, y) \in F(B)$ but $(y, x) \in P(B)$. By construction of B we know that $(y, x) \in R_a$ for some relation R_a (with finite index a), and therefore $(y, x) \in R_\alpha$ for $\alpha \geqslant a$. Since F is algebraic, there is some finite relation $R' \subseteq B$ such that $(x, y) \in F(R')$. Moreover, since R' is finite, there is some R_b (with finite index b) in the chain such that $R' \subseteq R_b$. Since F is monotone, $F(R') \subseteq F(R_b)$ and therefore $(x, y) \in F(R_b)$. By monotonicity again, $(x, y) \in F(R_\alpha)$ for $\alpha \geqslant b$. Hence, there is a finite $c = \max\{a, b\}$ such that R_c is not F-consistent, a contradiction.

For Z-separability of B, suppose that $(x, y) \in P(B)$. By construction of B we know that $(x, y) \in R_d$ for some relation R_d (with finite index d), and $(y, x) \notin R_\alpha$ for any α. Hence $(x, y) \in P(R_d)$. From Z-separability of R_d, there is $z \in Z$ such that $(x, z) \in R_d$ and $(z, y) \in R_d$. But then $(x, z) \in B$ and $(z, y) \in B$. □

Lemma 1.4. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, idempotent, separability preserving, and expansive, then for any countable $Z \supseteq Q_F$ and for every $R \in \mathcal{R}_F^Z$ such that $N(R) \neq \varnothing$ there is a non-empty subset S of $N(R)$ such that $R \cup S \in \mathcal{R}_F^Z$.*

Proof. Consider first the case $R \neq F(R)$, and let $S = F(R) \backslash R$. Note that $S \neq \varnothing$ since F is closed, and by construction $R \cup S = F(R)$. Since F is separability preserving, then $R \in \mathcal{R}_F^Z$ implies $R \cup S$ is Z-separable. Since F is idempotent, $F(F(R)) = F(R)$ so $F(F(R)) \succeq F(R)$ and $F(R) = R \cup S \in \mathcal{R}_F^Z$.

Consider now the case $R = F(R)$. Since F is expansive, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S$ is F-consistent and $P(R \cup S) = P(R)$. Since R is Z-separable, it follows that $R \cup S$ is also Z-separable. Since $R \cup S$ is F-consistent and Z-separable, we get $R \cup S \in \mathcal{R}_F^Z$. □

In order to prove Theorem 1.1 we need also a classical result from Debreu (1954) included below for reference.

Lemma 1.5 (Lemma 2 from Debreu (1954))**.** *If R is a complete, transitive and Z-separable preference relation for some countable $Z \subseteq X$, then there is a utility function that represents R and is continuous in any natural topology.*

We turn to the main proof next.

Proof of Theorem 1.1. Assume throughout the proof that F is a rational closure and R a Z-separable preference relation for some countable Z. Define $Z' = Z \cup Q_F$ and note that R is $Z \cup Q_F$-separable. To prove necessity of the condition in the statement of the theorem, suppose first that R is not F-consistent, and let R' be any extension of R. From $R' \supseteq R$ and monotonicity of F we get $F(R') \supseteq F(R)$. From $P(R') \supseteq P(R)$ we get $P^{-1}(R') \supseteq P^{-1}(R)$. It follows that if R is not F-consistent, then R' is not F-consistent. Note that if R' is not F-consistent, then $P(R') \neq P(F(R'))$ and hence $R' \neq F(R')$. Thus, if R is not F-consistent, it cannot have an extension that is a fixed point of F.

To prove sufficiency, suppose R is F-consistent. Since F is separability-preserving and R is Z'-separable, then $F(R)$ is Z'-separable as well. Let

$$\Omega = \{R' \in \mathcal{R}_F^{Z'} | R \preceq R'\}$$

be the set of extensions of R that are themselves Z'-separable and extended by F. Note that by Lemma 1.2, $R \in \Omega$ if R is F-consistent, so Ω is nonempty.

We claim that every chain $R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$ of relations in Ω has an upper bound $B = \cup_{\alpha \geqslant 0} R_\alpha \in \Omega$. To see this, from Lemma 1.3, $B \in \mathcal{R}_F^{Z'}$. It remains to check that $R \preceq B$. Clearly, $R \subseteq B$. If $P(B) \not\supseteq P(R)$, then there are elements $x, y \in X$ such that $(x, y) \in P(R)$ and $(y, x) \in B$. But then there must be a relation R_α in the chain for which $(y, x) \in R_\alpha$, which contradicts the fact that $R \preceq R_\alpha$ and we conclude that $B \in \Omega$. Clearly, $\preceq$ is a partial order (reflexive, antisymmetric and transitive binary relation) on Ω

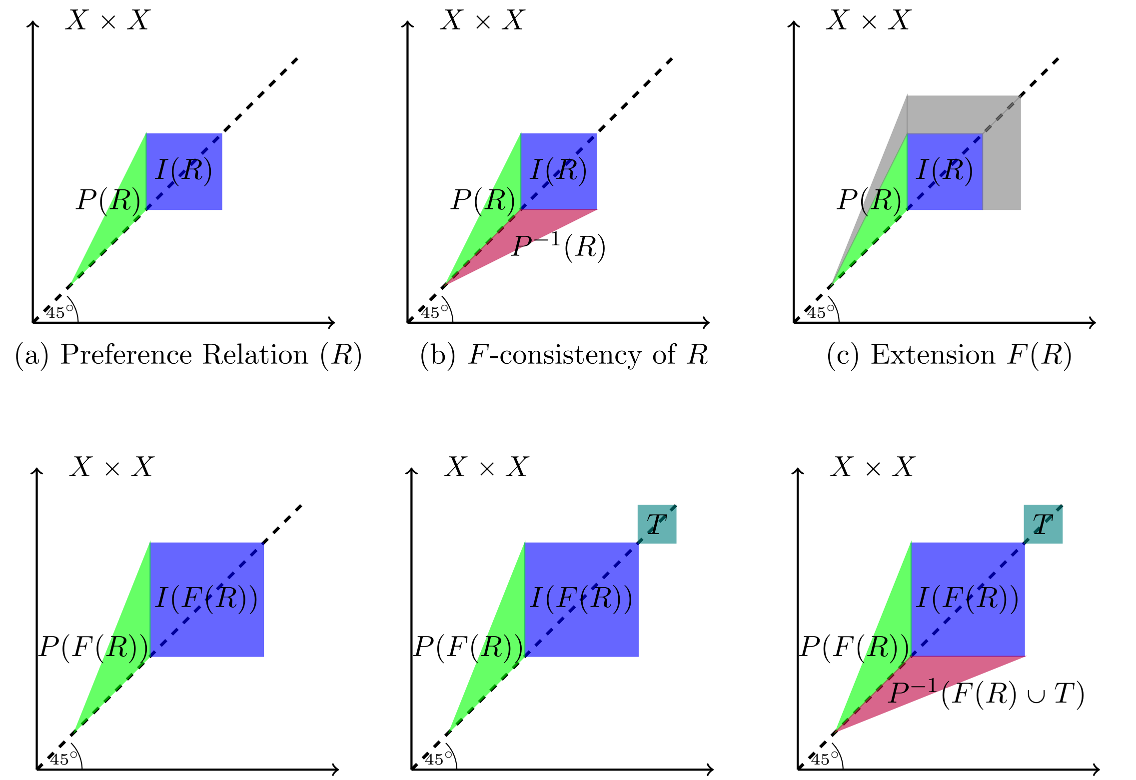
and we just showed that every chain has an upper bound. Hence, by Zorn's lemma, there is maximal element of Ω, and we can denote it by R^*.

We claim that R^* is complete. To see this, assume on the contrary that $N(R^*) \neq \emptyset$. From Lemma 1.4 and the fact that $Q_F \subseteq Z'$, we know that there is a nonempty $S \subseteq N(R^*)$ such that $R^* \cup S \in \mathcal{R}_F^{Z'}$. Since $R^* \succeq R$, we have $R^* \cup S \supset R^* \supset R$. Note that $(y,x) \in P(R)$ implies $(x,y) \notin R^*$ (since $R^* \succeq R$) and $(x,y) \notin S$ (since $S \subseteq N(R^*) \subseteq N(R)$). Hence, $P(R^* \cup S) \supseteq P(R)$. But then $R^* \cup S \succeq R$ and $R^* \cup S \in \mathcal{R}_F^{Z'}$, which contradicts that R^* is a maximal element of Ω.

We claim further that R^* is a fixed point of $F(R)$, i.e. $F(R^*) = R^*$. To see this, note that $R^* \subseteq F(R^*)$ follows from the fact that $R^* \preceq F(R^*)$. To get the reverse, assume that $(x,y) \in F(R^*)$ and $(x,y) \notin R^*$. From completeness of R^*, (y,x) must be an element of $P(R^*)$ which contradicts $R^* \preceq F(R^*)$. Therefore, $F(R^*) \subseteq R^*$.

We are left to show that there is a utility function that represents $R^* = F(R^*)$. We just showed that R^* is complete. Since F is a rational closure, R^* is transitive as well. As we already showed $R^* \in \Omega \subseteq \mathcal{R}_F^{Z'}$, it follows that R^* is Z'-separable. Hence, R^* satisfies the conditions from Lemma 1.5, i.e. there is utility function that represents R^*. Moreover, the utility function is continuous in natural topology. □

Intuitively, we can think of the rational closure F as helping to construct a complete extension of the original preference relation via an iterated algorithm. Starting with the original preference relation, the algorithm works as follows. If the preference relation at a given iteration is already a fixed point of F, but it is not a complete relation, the algorithm requires adding indifference pairs to R while keeping the new preference relation in $\mathcal{R}_F$. Adding indifference points in this manner is possible since F is expansive. If instead the preference relation at a given iteration is not a fixed point, the algorithm requires going from R to $F(R)$, which expands R while preserving its asymmetric part—since F is idempotent, the new preference relation is a fixed point of F. In this sense, the proof of Theorem 1.1 establishes that such algorithm converges to a complete fixed point as long as the original



(a) Preference Relation (R) (b) F-consistency of R (c) Extension $F(R)$

(d) Extended Relation $F(R)$ (e) Addition of T (f) F-consistency of $F(R) \cup T$

Figure 1.4: Intuition for the proof of Theorem 1.1. The dashed line is the diagonal. If (x, y) lies above the diagonal, then (y, x) is the symmetric point below the diagonal.

preference relation is F-consistent. Since the algorithm preserves separability properties of the preference relation, and fixed points of F are transitive, the complete fixed point is representable by a utility function. Of course, convergence need not occur after a finite number of iterations, and the proof relies on Zorn's Lemma to assert existence.

1.4 Revealed Preference Revisited

In this section we illustrate the techniques proposed in the chapter by revisiting the classical problem of the existence of a utility function rationalizing observations obtained from a finite number of budget sets. Formally, a **consumption experiment** is a finite vector $E = (x_i, B_i)_{i=1}^n \in (X \times 2^X)^n$ where for each $i = 1, \ldots, n$, $x_i \in B_i \subseteq X$. The interpretation is that x_i are chosen alternatives and B_i are budget sets, so that each x_i is (directly) revealed to be strictly preferred to each alternative in $B_i \backslash \{x_i\}$.

Given a consumption experiment $E = (x_i, B_i)_{i=1}^n$, for each $i = 1, \ldots, I$, let $R_i = \{(x_i, y) : y \in B_i \backslash \{x_i\}\}$, and let $R_E = \bigcup_i R_i$. We say that an experiment $E = (x_i, B_i)_{i=1}^n$ can be **rationalized** if there is a preference relation that is a complete extension of every element in the set $\{R_i\}$ and that can be represented by a utility function. We claim:

Proposition 1.1. *A consumption experiment can be rationalized if and only if (1) $R_E \geq R_i$ for $i \in \{1, \ldots, n\}$ and (2) R_E is T-consistent.*

Proof. To prove sufficiency of conditions 1 and 2, note that R_E is separable with respect to the finite set $\{x_1, \ldots, x_n\}$ and recall that the transitive closure is a rational closure (Lemma 1). From Theorem 1, then, R_E has a complete extension R^* that can be represented by a utility function if and only if R_E is T-consistent; that is, condition 2. Since $R^* \geq R_E$ and $\geq$ is a transitive relation, condition 1 is sufficient for $R^* \geq R_i$ for $i \in \{1, \ldots, n\}$.

To prove necessity of condition 1, note that by construction $P(R_i) = R_i$ for $i \in \{1, \ldots, n\}$. Hence, if there is some i such that $P(R_E) \not\supseteq P(R_i)$, it must be the case that there is some $j \neq i$ such that $(x_i, x_j) \in R_i = P(R_i)$ and $(x_j, x_i) \in R_j = P(R_j)$. But there cannot be any preference relation R^* satisfying $(x_i, x_j) \in P(R^*)$ and $(x_j, x_i) \in P(R^*)$.

To prove necessity of condition 2, suppose R_E is not T-consistent but condition 1 holds. Then for any preference relation R^* that extends every R_i, we can build a cycle of strict preference between three or more alternatives, which implies that R^* cannot be represented by a utility function. □

Two criteria to ascertain the rationality of the consumption experiment (developed first by Samuelson (1938) and Houthakker (1950)) are described below:

Definition 1.8. *The consumption experiment* $E = (x_i, B_i)_{i=1}^n$ *satisfies the Weak Axiom of Revealed Preference (**WARP**) if for every* $\{i, j\} \subseteq \{1, \ldots, n\}$, $x_j \in B^i$ *implies* $x_i = x_j$ *or* $x_i \notin B_j$.

Definition 1.9. *The consumption experiment* $E = (x_i, B_i)_{i=1}^n$ *satisfies the Strong Axiom of Revealed Preference (**SARP**) if for every integer* $m \leqslant n$ *and every* $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ *for* $j = 1, \ldots, m-1$ *implies* $x_{i_1} = x_{i_m}$ *or* $x_{i_1} \notin B_{i_m}$.

The interpretation of WARP is that two alternatives cannot be directly revealed to be strictly preferred to each other, while the interpretation of SARP that in addition two alternatives cannot be indirectly revealed to be strictly preferred to each other, via a chain of direct revelation. (See Figure 1.5.)

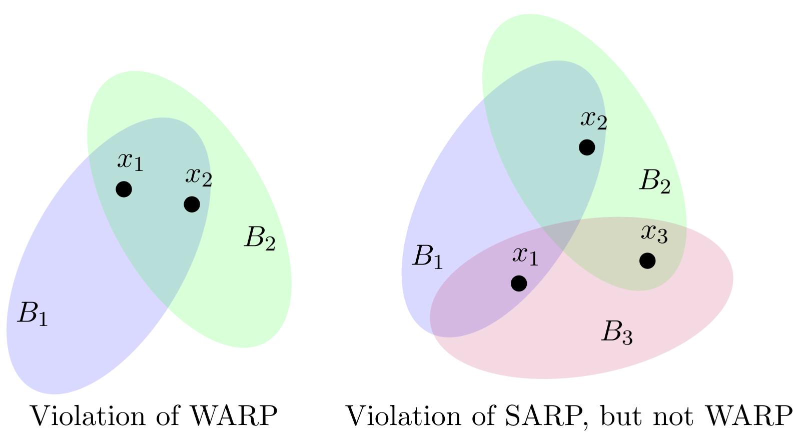


Figure 1.5: Relation between WARP and SARP

The following are immediate:

Lemma 1.6. $E = (x_i, B_i)_{i=1}^n$ *satisfies WARP if and only if* $R_E \geq R_i$ *for* $i \in \{1, \ldots, n\}$.

Lemma 1.7. *$E = (x_i, B_i)_{i=1}^n$ satisfies SARP if and only if (1) $R_E \succeq R_i$ for $i \in \{1, \ldots, n\}$ and (2) R_E is T-consistent.*

As a corollary of Proposition 1.1 and Lemma 1.7, a finite consumption experiment can be rationalized if and only if it satisfies SARP.

By working with other closures we can induce monotonicity as well as transitivity in the complete extension of the original preference relation. For this purpose we need to introduce more structure on X. Assume X is endowed with a transitive and reflexive relation $\geqslant$, with strict part denoted by $>$, and suppose there is a countable set Q that is dense in X with respect to $\geqslant$; that is, for all $x, y \in X$ such that $x > y$ there is $Z \in Q$ such that $x > z > y$. As an example, we have $X = \mathbb{R}^m$ and $Q = \mathbb{Q}^m$ for positive integer m. We say that a preference relation R is **monotone** if for all $x, y \in X$, $x > y$ implies $(x, y) \in P(R)$.

We define the **monotone closure** by $M : \mathcal{R} \to \mathcal{R}$, where $(x, y) \in M(R)$ if there is a finite sequence $s_1, \ldots, s_n$ such that $s_1 = x$ and $s_n = y$, and for any $j = 1, \ldots, n-1$ either (1) $(s_j, s_{j+1}) \in R$, or (2) $s_j > s_{j+1}$.

Lemma 1.8. *The monotone closure $M : \mathcal{R} \to \mathcal{R}$ is a rational closure.*

Proof. It is easy to check that M is an algebraic closure and that induces transitivity. Expansiveness of M can be proven in a similar way to the expansiveness of T, considering the fact that $R = M(R)$ is already monotone relation, i.e. all pairs $x \geqslant y$ are already in $M(R)$.

We are left to show that M is separability-preserving. Consider a preference relation R satisfying $R \in \mathcal{R}_M^Z$ for some countable set $Z \subseteq X$. We claim that $M(R)$ is $Z \cup Q$-separable, so that $Q_M = Q$. To see this, note that $(x, y) \in P(M(R))$ implies that there is a sequence $S = s_1, \ldots, s_n$, such that $s_1 = x$, $s_n = y$ and for any $j = 1, \ldots, n-1$ either $(s_j, s_{j+1}) \in R$, or $s_j > s_{j+1}$, with at least one $k \in \{1, \ldots, n-1\}$ such that either (1) $(s_k, s_{k+1}) \in P(R)$, or (2) $s_k > s_{k+1}$. If $(s_k, s_{k+1}) \in P(R)$, then there is $z \in Z$, such that $\{(x, z), (z, y)\} \subseteq M(R)$. If $s_k > s_{k+1}$ there is $z \in Q$ (since Q is dense with respect to $\geqslant$) such that $s_k \geqslant z \geqslant s_{k+1}$, i.e. $\{(x, z), (z, y)\} \subseteq M(R)$. □

We make the (relatively mild) assumption that budgets are **comprehensive**; that is for each $i \in \{1, \ldots, n\}$, $x \in B_i$ and $y < x$ imply $y \in B_i$.

We have:

Proposition 1.2. *A consumption experiment with comprehensive budgets can be rationalized by a strictly increasing utility function if and only if (1) $R_E \geq R_i$ for $i \in \{1, \ldots, n\}$ and (2) R_E is M-consistent.*

Proof. We prove sufficiency of conditions (1) and (2); necessity of each of the two conditions follows along the lines of the previous proposition.

From condition (2) and Lemma 1.2, we have that $M(R_E)$ extends R_E. From condition 1, then, $M(R_E)$ extends R_i for $i \in \{1, \ldots, n\}$. Since $(x_i, y) \in P(R_i)$ for all $y \in B_i \backslash \{x_i\}$, we must have $(x_i, y) \in P(M(R_E))$ for all $y \in B_i \backslash \{x_i\}$. But from the definition of M, $y > x_i$ implies $(y, x_i) \in M(R_E)$. It follows that $(x, y) \in R_i$ implies that it is not the case that $y > x$, and hence $(x, y) \in R_E$ implies that it is not the case that $x < y$.

We claim that $M(R_E)$ is monotone; that is $(x, y) \in P(M(R_E))$ for all $x > y$. To see this, from the definition of M, $x > y$ implies $(x, y) \in M(R_E)$. So we only need to show $x < y$ implies $(x, y) \notin M(R_E)$, or equivalently, $(x, y) \in M(R_E)$ implies that it is not the case that $x < y$. That is, it remains to be shown that if there there is a sequence $s_1, \ldots, s_n$ such that $s_1 = x$ and $s_n = y$, and for any $j = 1, \ldots, n-1$ either (i) $(s_j, s_{j+1}) \in R_E$, or (ii) $s_j > s_{j+1}$, then it cannot be the case that $x < y$.

Consider any such sequence as described in the previous paragraph. Trivially, if every consecutive pair in the sequence is of type (ii) we get $x > y$, so assume there is some consecutive pair of type (i) in the sequence, and let (s_k, s_{k+1}) be the last step of type (i). From the definition of E, this implies that s_k is equal to x_i for some $i \in \{1, \ldots, n\}$. If $k + 1 = n$, we get immediately $(s_k, y) \in R_i$. If $k + 1 < n$, using the fact that (s_k, s_{k+1}) is the last step of type (i) we get $y < s_{k+1}$, hence from comprehensiveness of budget sets $y \in B_i$. Since $y < s_{k+1} \in B_i$, we know $y \neq x_i$ and then $(s_k, y) \in R_i$. In either case, then, from condition (1), $(s_k, y) \in P(R_E)$. But if $y > x$, we can show $(y, s_j) \in M(R_E)$ using the sequence $y, x, s_1, \ldots, s_j$, which violates condition (2).

Note that $M(R_E)$ is (trivially) M-consistent, and it is separable with respect to the countable set $\{x^1, \ldots, x^\tau\} \cup Q$. Since, from Lemma 1.8, M is a rational closure, it follows from Theorem 1 that $M(R_E)$ has a complete extension $R^* = M(R^*)$ that can be represented by a utility function. Since $M(R_E)$ is monotone and R^* is an extension of $M(R_E)$, it follows that R^* is monotone. This in turn implies that any utility function representing R^* is strictly increasing.

Since R^* is an extension of $M(R_E)$, it follows from conditions (1) and (2) and Lemma 1.2 that it is an extension of R_i for $i \in \{1, \ldots, n\}$ as well. □

Note that consistency with the monotone closure implies that for every budget B_i there is no point above the chosen alternative x_i; we do not need budgets to be comprehensive to establish this. This implies that directly observed preferences do not contradict monotonicity. Comprehensive budgets help us in proving that consistency with the monotone closure implies that preferences built using the closure do not contradict monotonicity either.

1.5 Generalized Revealed Preference Revisited

Varian (1982) introduces an approach to revealed preference in which observed choices are revealed to be strictly preferred to alternatives that are in the budget set and that are cheaper than other alternatives in the budget set, and observed choices are revealed to be weakly preferred to alternatives that are in the budget set but are not cheaper than other alternatives. Intuitively, observed choices are possibly indifferent to other alternatives in the budget set.

To formalize this approach in our environment, we assume as in the previous section that X is endowed with a transitive and reflexive relation $\geqslant$, with strict part denoted by $>$, and suppose there is a countable set Q that is dense in X with respect to $\geqslant$; that is, for all $x, y \in X$ such that $x > y$ there is $Z \in Q$ such that $x > z > y$.

Given a consumption experiment $E = (x_i, B_i)_{i=1}^n$, for each $i = 1, \ldots, I$, let R_i and R_E be defined as in the previous section, and let $\tilde{R}_i = \{(x_i, y) : y \in B_i \backslash \{x_i\}$ and $y < x$ for some $x \in B_i\}$ and let $\tilde{R}_E = \bigcup_i \tilde{R}_i$.

We say that an experiment $E = (x_i, B_i)_{i=1}^n$ can be **rationalized with possibly indifferent choices** if there is a monotone preference relation that is a complete extension of every element in the set $\{\tilde{R}_i\}$ and of $T(R_E)$ and that can be represented by a utility function.

We have:

Proposition 1.3. *A consumption experiment with comprehensive budgets can be rationalized with possibly indifferent choices by a strictly increasing utility function if and only if (1') $T(R_E) \geq \tilde{R}_i$ for $i \in \{1, \ldots, n\}$ and (2') $T(R_E)$ is M-consistent.*

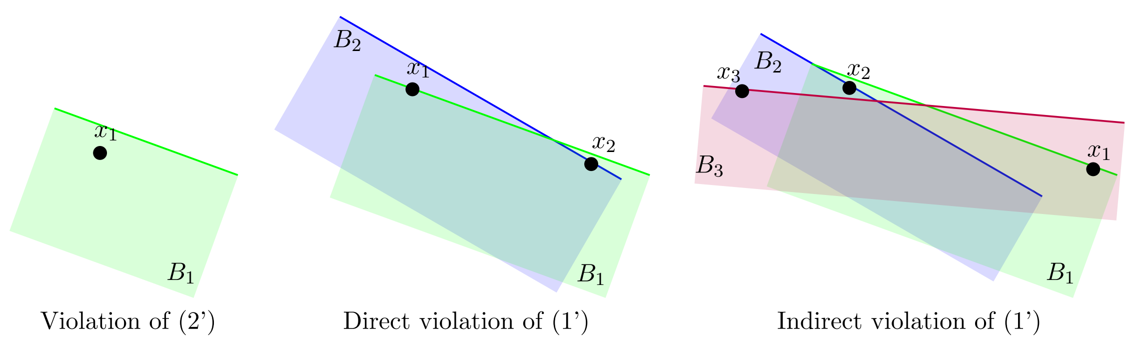


Figure 1.6: Relation between (1') and (2')

The proof is analogous to the proof of Proposition 1.2.

Adapting the formulation by Varian (1982) to our setting, we can define

Definition 1.10. *The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the Generalized Axiom of Revealed Preference (**GARP**) if $(x_i, x_j) \in T(R_E)$ implies that there is no y such that $y \in B_j$ and $y > x_i$.*

With comprehensive budgets, GARP can be restated as $(x_i, x_j) \in T(R_E) \Rightarrow (x_j, x_i) \notin \tilde{R}_j$. We claim that Conditions (1') and (2') in Theorem 1.3 are jointly equivalent to GARP plus the assumption that observed choices are maximal with each budget set, i.e. for any $i \in \{1, \ldots, n\}$ there is no $y \in B_i$, such that $y > x_i$. (See Figure 1.6.) Maximality of observed choices is assumed in the original paper by Varian (1982).

Lemma 1.9. *The consumption experiment $E = (x_i, B_i)_{i=1}^n$ with comprehensive budgets satisfies GARP and maximality of observed choices if and only if (1') $T(R_E) \geq \tilde{R}_i$ for $i \in \{1, \ldots, n\}$ and (2') $T(R_E)$ is M-consistent.*

Proof. To prove sufficiency, assume first that there is a violation of GARP, that is there is $(x_i, x_j) \in T(R_E)$ and $(x_j, x_i) \in \tilde{R}_j$. Therefore, $(x_j, x_i) \in P(\tilde{R}_j)$ and $(x_i, x_j) \in T(R_E) \cap P^{-1}(\tilde{R}_j)$. Hence, $\tilde{R}_j \nleq T(R_E)$, i.e. a violation of (1'). Assume instead there is a violation of maximality of observed choices, i.e. there is $y > x_i$ and $y \in B_i$. Then $(y, x_i) \in M(T(R_E))$ and $(x_i, y) \in P(T(R_E))$. Hence $T(R_E)$ is not M-consistent, i.e. there is a violation of (2').

To prove necessity of condition (1'), suppose it is violated. We have then that there is $(x_i, x_j) \in \tilde{R}_i$ and $(x_j, x_i) \in T(R_E)$. And $(x_i, x_j) \in \tilde{R}_i$ implies that there is $y > x_j$ such that $y \in B_i$; which violates GARP.

To prove necessity of condition (2'), suppose it is violated. Then there is $(x_i, y) \in P(T(R_E))$ and $(y, x_i) \in M(T(R_E))$. Since $T(R_E)$ is transitive and $>$ is transitive as well we can claim that $y > x_i$. At the same time $(x_i, y) \in T(R_E)$ implies that there is j, such that $y \in B_j$ and $x_j \in B_i$, hence $(x_i, x_j) \in T(R_E)$. Recall that budgets are comprehensive, therefore, $x_i \in B_j$, because $y > x_i$. Then $(x_j, x_i) \in \tilde{R}_j$; which violates GARP. □

As a corollary of Proposition 1.3 and Lemma 1.9, an extended version of GARP (including maximality of observed choices) is necessary and sufficient for rationalization with possibly indifferent choices by a strictly increasing utility function.

Table 1.1 above summarizes the relation between revealed preferences axioms and the conditions on extensions and closures we use to obtain representations. We think of row

Table 1.1: A cheat sheet of consistency conditions and revealed preference axioms

	$\varnothing$	T-consistency	M-consistency
$\{R_i \leq R_E\}$	WARP	SARP	$*$
$\{\tilde{R}_i \leq T(R_E)\}$	$**$	$***$	GARP

conditions as criteria regarding the internal consistency of the observed choices, i.e. the consistency of each of the observed choices with the complete dataset. By the same token, we think of column conditions as criteria regarding the external consistency of the dataset, i.e. its consistency with theories about how a complete preference ordering should look like. External consistency conditions grow more demanding as we move from left to right. Internal consistency conditions cannot be similarly ranked; while $P(\tilde{R}_i) \subseteq P(\tilde{R}_i)$, $P(R_E)$ is not necessarily a subset of $P(T(R_E))$. (See Figure 1.7.)

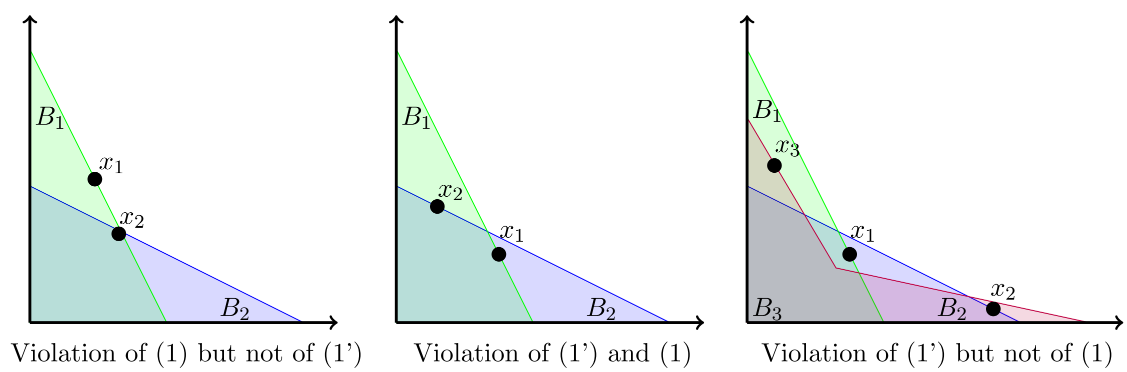


Figure 1.7: Relation between internal consistency conditions

The unnamed cells in Table 1 are of some interest. ($*$) gives us necessary and sufficient conditions for representation by an increasing utility function in Proposition 1.2, and thus is just a version of SARP. (∗∗) is the Weak version of GARP, requiring that observations do not directly contradict rationalization by monotone preferences allowing for indifference of observed choices. (∗∗∗) gives us necessary and sufficient conditions for representation by a utility function assuming all observations come from a region in the consumption space where preferences are monotone.

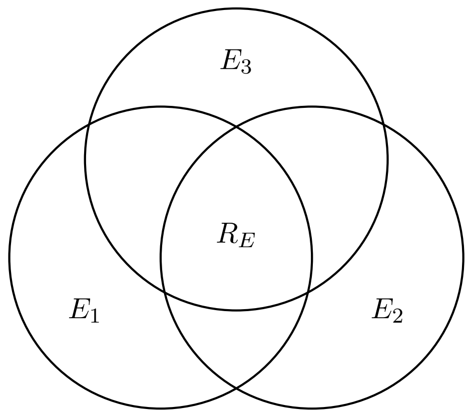
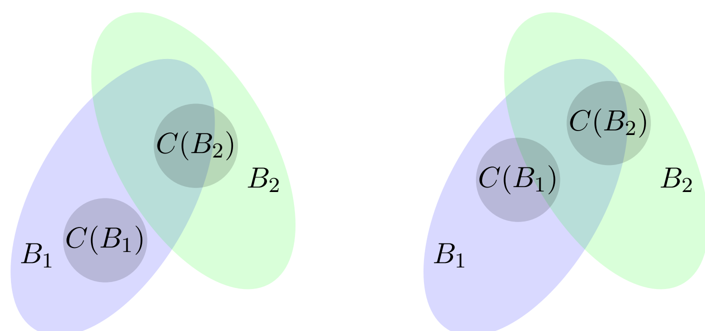


Figure 1.8: Consistency conditions. E_i is the set of F-consistent Z-separable extensions of R_i. Internal consistency requires the aggregated preference relation (R_E) to be a F-consistent Z-separable extension of R_i for all i.

Our approach allows to restate the revealed preference axioms in a very abstract and general form. Neither internal nor external consistency conditions require unique observed choices from budget sets. The internal consistency condition is built on the idea that there are some elementary observed relations (R_i and $\tilde{R}_i$ in the examples above), and that the aggregated revealed preference relation (R_E and $T(R_E)$ in the examples above) must be an extension of elementary ones. The external consistency condition, in turn, is built on the idea that the aggregated observed preference relation can be extended to a complete preference relation satisfying properties like transitivity and monotonicity. This framework allows is to consider, from the perspective of revealed preferences, observed behavior in environments beyond the classical one of the a sequence of single alternatives chosen from budgets sets. Observed behavior may be set-valued, for instance, in situations in which there are many opportunities to choose from the same budget (see Figure 1.9). And choice sets may be very different from linear budget sets (e.g. multiple price lists in experimental economics).



Internally consistent demand   Internally inconsistent demand

Figure 1.9: Internal consistency of set-valued demands

1.6 Conclusion

In this chapter we show that there is a complete extension of an incomplete preference relation that is fixed point of a mapping over preferences (the rational closure) and can be represented by a utility function if and only if the original preference relation satisfies a congruence condition related to the specific mapping. Intuitively, a rational closure is a rule that can be used to extend an incomplete preference relation. The proof of the theorem relies on the alternated application of the rational closure and the addition of indifference pairs to construct the extension.

An advantage of using an explicit rule to construct the complete extension of the original preference relation is that further desiderata on the utility function can be induced by the rule. We illustrate this point by revisiting the classical revealed preference problem of the existence of a rationalization of a sequence of observed choices by means of strictly increasing utility functions. We show, in particular, that classical revealed preference axioms can be recast in terms of extensions and closures in a very abstract way, encompassing situations that do not reduce to a sequence of single choices from linear budget sets.

Rational closures as defined in this chapter construct preferences over each ordered pair of alternatives not in the original preference relation employing only a finite number of observations regarding other pairs of alternatives. This "algebraic" requirement is necessary for the usage of Zorn's lemma in the proof of existence of the complete extension. However, this requirement is not compatible with a rule that induces continuity in order to construct the complete extension. Thus, constructing a continuous extension of the original preference relation seems an elusive goal in the general case,[7] i.e. without assumptions such as Euclidean consumption spaces and further constraints over budget sets.

[7]That is, for a fixed topology; continuity of the complete extension holds by assumption for natural topologies.

Chapter 2: A Revealed Preference Test of Quasi-Linear Preferences

2.1 Introduction

We provide criteria for a set of observed choices to be generated by a quasi-linear preference relation. When these conditions are satisfied there exists a complete, transitive, monotone and quasi-linear preference relation consistent with observed behavior. In the special case of linear budget sets, our condition implies the existence of a concave quasi-linear and continuous utility function that rationalizes the data.

It is difficult to overstate the importance of the assumption of quasilinear preferences in both the theoretical and empirical economics. It plays a crucial role in mechanism design, the theory of the household as well as applied welfare analysis. For instance, it is a necessary assumption for the Revenue Equivalence theorem (Krishna, 2009; Myerson, 1981), the existence of truth-revealing dominant strategy mechanism for public goods (Green and Laffont, 1977) and the Rotten Kids theorem (Becker, 1974; Bergstrom and Cornes, 1983). It is also an often invoked assumption in applied welfare analysis (Domencich and McFadden, 1975; Allcott and Taubinsky, 2015). This chapter is concerned with the testable implications of this assumption.

Above-mentioned applications differ greatly in the nature of the choice sets faced by agents and are many times silent about whether preferences are convex or not. A main advantage of our approach is that it does not rely in either the convexity of preferences or the linearity of budget sets. Our test is built on the following observation. If preferences are quasi-linear, say in good x, it must be true that $(x, y) \succeq (x', y')$ implies $(x+\alpha, y) \succeq (x'+\alpha, y')$ for all α. This property can be directly tested and requires making no assumptions on the shape (or existence) of the utility function or ancillary assumptions on the choice set. The added generality of our test makes it possible to device tests of quasi-linearity of preferences in strategic environments.

Brown and Calsamiglia (2007) provide a test for the existence of a quasi-linear, concave and monotone utility function that rationalizes choices over linear budget sets.[1] Relaxing these two assumptions implies that while we can guarantee the existence of a quasi-linear preference relation consistent with observed data, we cannot guarantee the existence of a quasi-linear utility function representing it we talk about preference relation.[2] We show, however, that our test is equivalent to Brown and Calsamiglia (2007) if choice sets are linear. This result is similar to Afriat (1967) who shows that if budgets are linear, convexity of preferences has no empirical content.[3]

The second contribution of the chapter is to provide an empirical test of the quasi-linearity of preferences using both lab and field data. In our experiment, we mimic different consumption groups by offering gift cards at discounted prices. We test if subjects preferences over 5 alternative goods in 30 different budgets can be rationalized by a quasi-linear preference ordering. We find that while support for the Generalized Axiom of Revealed Preferences is strong, the support for quasi-linearity of preferences is weak.[4] We conduct a second test of quasi-linearity of preferences using the Spanish Continuous Family Expenditure Survey (see Beatty and Crawford (2011)).[5] In this dataset, we find stronger support of the assumption of quasi-linearity of preferences against the alternative of random choice. Due to the low variability in prices and income in this dataset,however, we conclude that

[1]More recently, Cherchye et al. (2015b) proposed a test of generalized quasi-linear preferences (Bergstrom and Cornes, 1983). Their test is a generalization of Brown and Calsamiglia (2007) and also requires linear budget sets.

[2]We note that previous tests of quasi-linear preferences can be extended to more general budget sets if the assumption of concavity is maintained. In this case, the existence of a concave utility representation of preferences will be equivalent to the existence of a set of prices that contain observed choice sets and for which observed choices satisfy cyclical monotonicity.

[3]Convexity is also a crucial assumption in the test of separability of preferences, homotheticity (Varian, 1983), expected utility theory (Varian, 1983), preferences with habit formation (Crawford, 2010), collective model of household consumption (Cherchye et al., 2007) and subjective expected utility (Echenique and Saito, 2015).

[4]The test was designed to test quasi-linearity of preferences in money. We find that quasi-linearity of preferences fail even if we relax this requirement.

[5]Crawford (2010) aggregates the data into 14 categorie. We use a further aggregation into 5 categories as in Beatty and Crawford (2011).

further tests are needed to establish the empirical validity of the assumption of quasi-linear preferences.

Varian (1982) shows how to derive non-parametric bounds on welfare measures based solely on the satisfaction of the Generalized Axiom of Revealed Preferences. These bounds can be quite wide and uninformative (Hausman and Newey, 1995). We show that these bounds can be significantly narrowed under the assumption of quasi-linear preferences. This suggests that aggregate welfare measures can be derived without making assumptions about the unobserved heterogeneity of preferences.

2.2 Theoretical Framework

Let us start by defining the quasi-linearity of preferences. Preferences are said to be *quasi-linear in the i-th component* if x being better than y implies that a shift of x along the i-th axis ($z = x + \alpha e_i$) is better then the same shift of y along the same axis ($w = y + \alpha e_i$). Wherer, x, y, w and z are consumption bundles and e_i is the i-th unit vector, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th place.

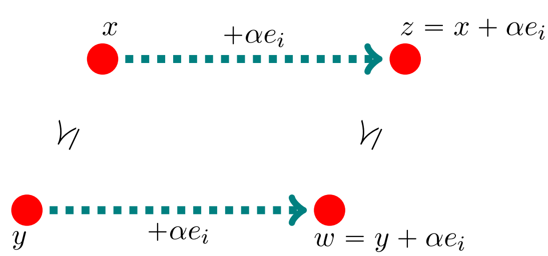


Figure 2.1: Quasilinear Preferences

Figure 2.1 shows this shift graphically. The dashed lines arrows represent shifts. If they are of the same length, then preferences are quasi-linear if x is better than y implies that

$z = x + \alpha e_i$ is better than $w = y + \alpha e_i$.

2.2.1 Idea of the Proof

Let us show the intuition for the necessary condition on quasi-linearity of preferences and later we formally prove that it is sufficient as well. Figure 2.2 shows preferences that are inconsistent with quasi-linearity. It is enough to have x strictly better than y and $w = y + \alpha e_i$ weakly better than $z = x + \alpha e_i$ for some $\alpha \in \mathbb{R}$.

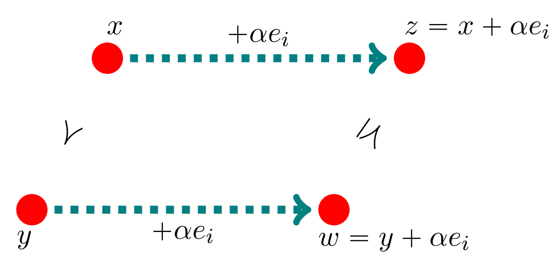


Figure 2.2: Contradiction of Quasi-linearity

Thus, if the revealed preference relation contains two pairs as in Figure 2.2, then the preference relation contains a contradiction of quasi-linearity. Clearly it is a necessary condition for quasi-linearity of preferences.

We intend to apply our test to a revealed preference relation derived from choice data. Hence, let us translate the case of violation of quasi-linearity to the context of revealed preferences. Let $(x^t, B^t)_{t=1,\dots,T}$ be the finite consumption experiments, where B^t are budgets and x^t are chosen points.

Now we will define a revealed preference relation. In Figure 2.3, bundle x^t is better than y if y lies in budget B^t (including the boundary of the budget) and x^t is strictly better than z if z lies strictly inside (in the interior) of budget B^t. Note that the only bundles that are revealed better than others are the chosen bundles (x^t). That is, we do not know if y is

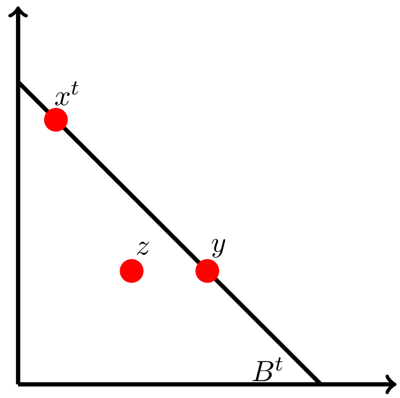


Figure 2.3: Revealed Preference Relation

better than z. Hence, to get a contradiction of quasi-linearity we need to consider at least two chosen points.

Figure 2.4(a) shows that bundle x^r is strictly better than the chosen bundle x^t. Let us explicitly show that this example contains the violation of quasi-linearity. Figure 2.4(b) shows that w is a shift[6] of the point x^r along the horizontal axis by the same amount as z is a shift of the point x^t. And from the definition of the revealed preference relation we know that x^t is strictly better than w, and x^r is strictly better than z. Figure 2.4(c) contains the case similar to one on Figure 2.2, thus the revealed preference relation violates quasi-linearity.

Figure 2.5(a) shows the condition necessary to eliminate the possibility of violations of quasi-linearity shown at the Figure 2.4(c). If the line from x^t (the strictly worse bundle) to z, [such that is still in the budget B^r (still less preferred than x^r)] is shorter than a line from x^r to the point w, [that lies in the budget B^r (is worse than x^t)], then there is a violation

[6]Note that we are not restricting the space to the positive orthant. Hence, the budget line (more general hyperplane) simply separates the space into two subspaces, and interior of budget is the subspace that lies "below" the budget line.

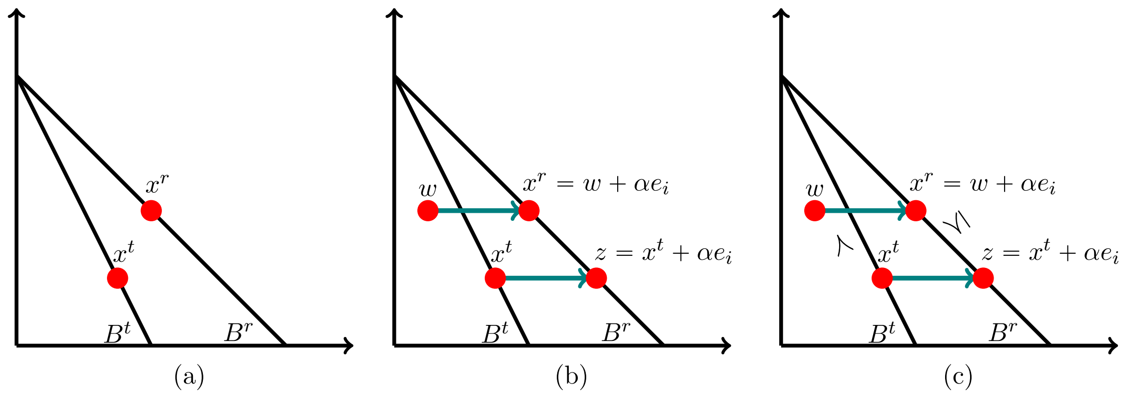


Figure 2.4: Constructing Test for Quasi-linearity

of quasi-linearity.

The blue line is the maximum distance (along the horizontal axis) from x^t to such point z that is still worse than x^t. The green line is the minimum distance from x^r to the y that is worse than x^t. If blue line is longer than green one then there is z' that is strictly worse than x^t and y' that is strictly worse than x^r. And these points can be equal shifts of x^r and x^t respectively. It is exactly the case shown in Figures 2.2 and 2.4(c). Hence, the test is simply checking that for any x^t that is strictly worse than x^r blue line must be shorter than the green one.

Since we would like to construct a preference relation that is transitive and quasi-linear, the test should take into account transitivity as well. We do this in the similar fashion as the transformation of Weak Axiom of Revealed Preferences (WARP) into General Axiom of Revealed Preference (GARP). Hence, consider a chain of elements such that each is less preferred than a previous one and is achieved by shifting some chosen point. Sum of shifts of elements in the chain should be less than the shift from the last to the first one. Let us illustrate this with a simple example. Suppose we have three budgets and three chosen

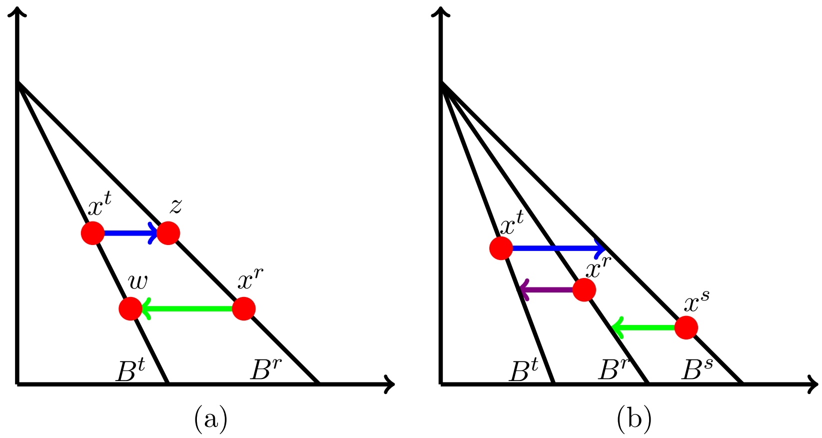


Figure 2.5: Test for Quasi-Linearity

points x^t, x^r and x^s as illustrated in Figure 2.5(b). Then, x^s is better than both x^r and x^t and x^r is better than x^t. Figure 2.5(b) illustrates how the test works, it is passed if sum of lengths of green and purple lines is greater than length of the blue one.

Let us show that the test shown on the Figure 2.5(a) is not capable of detecting violation of the joint hypothesis of quasi-linearity and transitivity, and we have to use use the case represented on the Figure 2.5(b) using the simple example. Figure 2.6(a) shows an example of a relation that passes the test of quasi-linearity (Figure 2.5(a)) but fails the joint test of quasi-linearity and transitivity (Figure 2.5(b)). Therefore, this set of choices cannot be generated by complete, transitive and quasi-linear preference relation. Distance from x^s to budget set B^t is less than distance from x^t to B^s, hence, there is no direct violation of quasi-linearity. Bundle x^t is preferred to u, so by quasi-linearity w is better than x^r and by transitivity w is better than v. Hence, applying quasi-linearity again we get that z is better than x^s, and at the same time x^s is strictly better than z, since z lies strictly inside of the budget set B^s. That is a violation of joint hypothesis of quasi-linearity and transitivity.

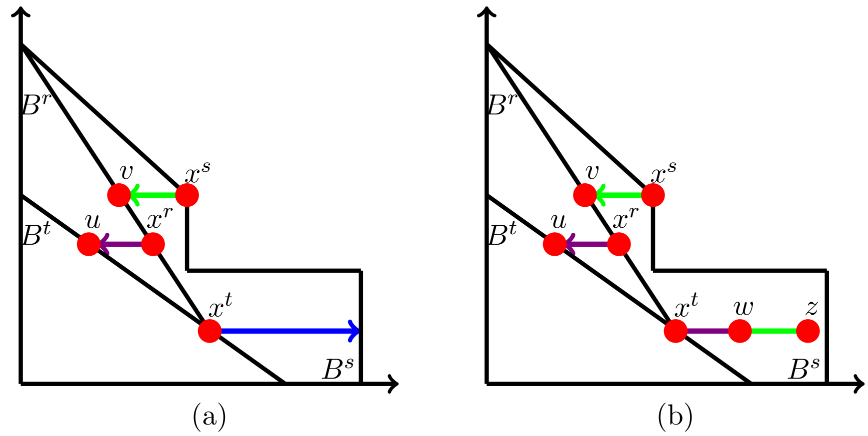


Figure 2.6: Violation of Quasi-linearity under Assumption of Transitivity

2.2.2 Test

Consider a set of alternatives $X \subseteq \mathbb{R}^N$. A set $R \subseteq X \times X$ is said to be a **preference relation**. We denote the set of all preference relations on X by $\mathcal{R}$. Denote the reverse preference relation by $R^{-1} = \{(x,y)|(y,x) \in R\}$, the symmetric part of R by $I(R) = R \cap R^{-1}$ and the asymmetric part by $P(R) = R \backslash I(R)$. Denote the non-comparable part by $N(R) = X \times X \backslash (R \cup R^{-1})$. A relation R' is said to be an **extension** of R, denoted by $R \leq R'$ if $R \subseteq R'$ and $P(R) \subseteq P(R')$.

Definition 2.1. *Preference relation is said to be:*

1. ***Complete*** *if* $N(R) = \varnothing$*;*

2. ***Transitive*** *if for any* $x, y, z \in X$*:* $(x,y) \in R$ *and* $(y,z) \in R$ *implies* $(x,z) \in R$*;*

3. ***Monotone*** *if for any* $x \gg y$ $(x,y) \in R$*;*

4. ***Quasi-linear in the i-th component*** *if for any* $(x,y) \in R$ *and any* $\alpha \in \mathbb{R}$ $(x + \alpha e_i, y + \alpha e_i) \in R$*.*

Let $(x^t, B^t)_{t=1...T}$ be the **finite consumption experiment** where x^t are chosen points and B^t are budgets. We assume all budgets to be compact and monotone.[7] Denote by R_v the **revealed preference** relation,[8] that is $(x^t, y) \in R_v$ if $y \in B^t$, $(x^t, x^t) \in I(R_v)$ and $(x, y) \in P(R_v)$ for any $y \in B^t \backslash \{x^t\}$. Denote by $T(R_v)$ the **transitive closure** of the relation, i.e. $(x, y) \in T(R_v)$ if there is a sequence $S = s_1, s_2, ..., s_n$ such that for any $j = 1...n-1$ $(s_j, s_{j+1}) \in R_v$. Note that $(x, y) \in P(T(R_v))$ if $(x, y) \in T(R_v)$ and there is $k = 1...n-1$ such that $(s_k, s_k + 1) \in P_v$. Denote by $C = \{x^1, x^2, ..., x^T\}$ the *set of all chosen points* in the finite consumption experiment $(x^t; B^t)_{t=1..T}$. A revealed preference relation is said to be **acyclic** if it satisfies SARP.[9]

Definition 2.2. *A revealed preference relation satisfies* ***QLSARP*** *with respect to the i-th component if for any sequence of distinct elements* $x^{k_1}, ..., x^{k_n} \in C$ *and* $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$*, such that* $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$ *and* $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$ *for* $j = 2, \ldots, n-1$*, then* $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^n \beta_j) e_i) \notin T(R_v)$.

QLSARP is simply the formal statement of the test from Figure 2.5. Note that α does not need to be positive, i.e. (x^{k_1}, x^{k_2}) may be an element of R_v, while none of further pairs $(x^{k_j}, x^{k_{j+1}})$ can be in the revealed preference relation. Given a sequence of chosen points $x^{k_1}, \ldots, x^{k_n}$, $(\alpha, \beta_3, \ldots, \beta_n)$ are shifts (to the left) of the chosen points generating a new sequence, such that each point in this new sequence is preferred to the next one. QLSARP fails if and only if there is a sequence of chosen points and vector of such shifts, such that the point obtained by shifting the initial chosen point (to the right) by $(\alpha + \sum_{j=3}^n \beta_j)$ is less preferred than the last point of the sequence of chosen points. Note that if the sequence consists of x^{k_1}, x^{k_2} only, then we need to check that if $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$, then $(x^{k_2}, x^{k_1} + \alpha e_i) \notin T(R_v)$, which exactly coincides with Figure 2.5(a).

[7] $x \in B^t$, then any $y \leqslant x$ is also in B^t. And since we work on $\mathbb{R}^N$ it will also include elements with negative coordinates.

[8] All the results below can be achieved for weak rationalization as well, but the conditions will be inelegant.

[9] The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the Strong Axiom of Revealed Preference (**SARP**) if for every integer $m \leqslant n$ and every $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ for $j = 1, \ldots, m-1$ implies $x_{i_1} = x_{i_m}$ or $x_{i_1} \notin B_{i_m}$.

Theorem 2.1. *An acyclic revealed preference relation R_v generated by a finite consumption experiment with monotone and compact budgets has an extension that is complete, transitive, monotone and quasilinear in the i-th component if and only if R_v satisfies QLSARP with respect to the i-th component.*

Proof of Theorem 2.1 is in Appendix A and the proof of the similar result for the case of weak rationalization is presented in Appendix B. The pseudo-code algorithmic implementation of QLSARP is presented in Appendix C.

Note that Theorem 2.1 does not guarantee the existence of a utility function. If we assume that the set of alternatives is a subset of $\mathbb{Q}^N$, then existence of utility would follow immediately since the space is countable and any complete and transitive relation over no more than countable set of alternatives can be represented by a utility function.[10] However, the utility function is not necessarily continuous.

2.3 Testing Quasi-Linearity

We test quasi-linearity in two contexts: quasi-linearity in goods[11] and quasi-linearity in money. By quasi-linearity in goods we mean the quasi-linearity in at least one of goods. It is usually assumed that if consumer has to make choices among large variety of goods, then his preferences are quasi-linear at least in one of them. Quasi-linearity in money is usually assumed, since money is a natural numeraire in which one can express the value of every good.

In order to test the assumption we have to use some benchmark. Our benchmark is rationality of preferences that is equivalent to the revealed preference relation to be consistent with GARP (see Varian (1982)). It is important to consider that people make mistakes.

[10]The proof for existence of utility representation of a preference relation over no more than countable set of alternatives can be found in Fishburn (1988). Moreover, the condition on the set of alternatives can be relaxed to being subset of $\mathbb{R}^{N-1} \times \mathbb{Q}$, using the result from Freer and Martinelli (2016).

[11]We relax the assumption and allow the goods to be different for different subjects. The detailed discussion on the assumptions of common versus individual numeraires is in Appendix.

Some people may therefore not pass QLSARP exactly even though their underlying preferences are quasi-linear. For the measure of distance from rationality we use the Critical Cost Efficiency Index (CCEI) introduced by Afriat (1973). Since X is a linear space and B^t are simply subsets of linear space, we can introduce $B^t(e) = \{x \in X : \frac{x}{e} \in B^t\}$. Then, $R_v(e)$ is a revealed preference relation generated by the finite consumption experiment $(x^t, B^t(e))$. The CCEI for QLSARP can be defined as the maximum $e \in (0, 1]$ such that $R_v(e)$ satisfies QLSARP.[12]

Changing e changes the probability that a set of random choices will pass QLSARP. To control this we use the **predictive success index** introduced by Selten (1991). The predictive success index is defined as the difference between share of people that satisfies axiom at the given e and the probability that uniform random choices will satisfy the axiom at the same e. This index ranges between -1 and 1, with -1 meaning no subject passes even though random choice would with probability one and 1 meaning every subjects passes even though random choice never would. To estimate the probability that uniform random choices will satisfy the axiom we use the Monte Carlo method with 1000 simulated random agents for each set of prices.

2.3.1 Quasi-Linearity in Goods

First, we test quasi-linearity of preferences in goods that is existence of the numeraire good. For this purpose we use data from Mattei (2000). It is experiment with 8 real consumption goods.[13] Each subject faced 20 different budgets and one (uniformly randomly chosen) consumer choice was implemented. The experiment was conducted with 20 economics and 100 business students from the University of Lausanne.[14] The payment was in goods and the average monetary equivalent of payment is \$30.5.

Figure 2.7 shows the distribution of CCEIs for GARP and QLSARP. Each consumer

[12]The CCEI can be defined similarly for any other axiom, e.g. GARP.

[13]The goods were: milk chocolate, biscuits, orange juice, iced tea, writing pads, plastic folders, diskettes, post it

[14]The paper also contains similar non-incentivized expenditure survey for 320 real consumers. We do not include the results from non-incentivized treatment.

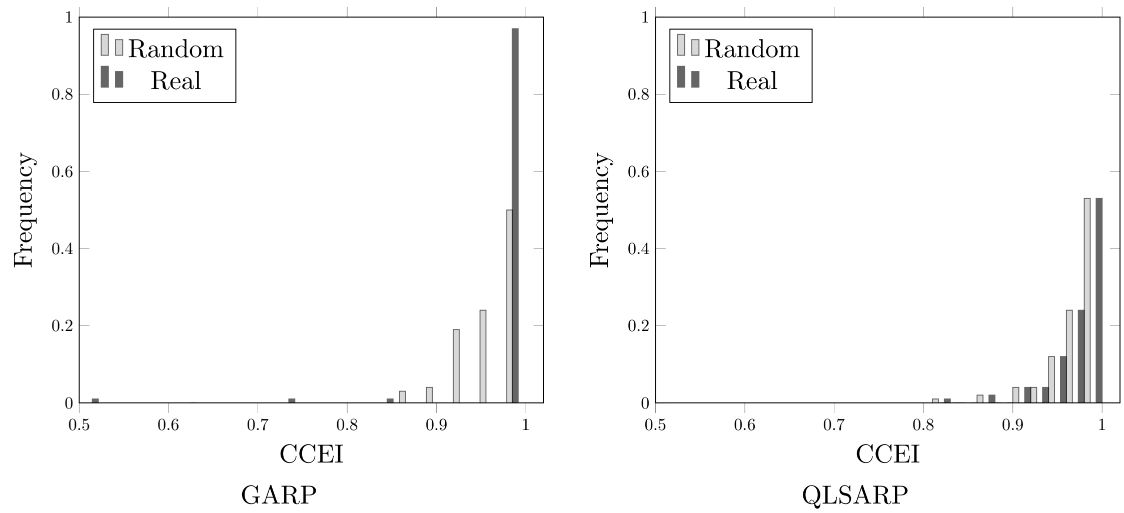
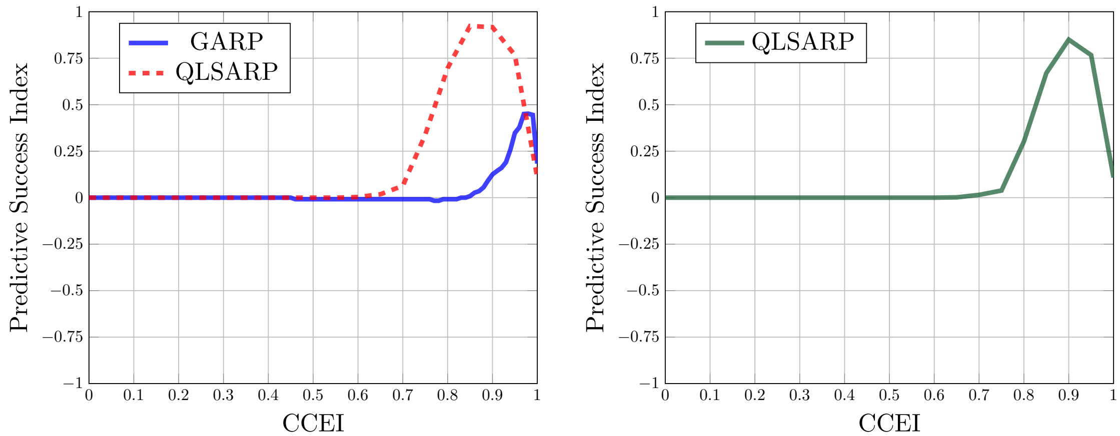


Figure 2.7: CCEI distributions for GARP and QLSARP

faced 8 goods, so, there will be 8 different CCEIs depending on the good in which we assume preference to be quasi-linear in. The CCEI for QLSARP is maximum of these eight different CCEIs for each consumer. The CCEI levels for QLSARP are lower than ones for GARP. This is predictable, since the QLSARP is a stricter test. Let us then consider the Predictive Success Index for these two axioms.

Figure 2.8 shows that quasi-linearity of preferences can not be rejected immediately. Figure 2.8a shows the predictive success index for QLSARP and GARP separately. It shows that at the low level of decision making error (high level of CCEI $\approx .95$) GARP outperforms QLSARP (predictive success index for GARP is higher than one for QLSARP). Hence the hypothesis of rational preferences is more favorable rather than a hypothesis of quasi-linear preferences. While at the higher level of decision making error (lower level of CCEI $\approx .85$) QLSARP outperforms GARP.

Figure 2.8b shows the predictive success index of QLSARP conditioning on random



(a) H_0: Random Choices

(b) H_0: Random Choices that Pass GARP

Figure 2.8: Predictive Success Index

choices that pass GARP, that is predictive success index equals 1 if none of the random choices that passed GARP pass QLSARP, while all real observations pass QLSARP; and predictive success index equals -1 if all random choices pass that pass GARP pass QLSARP as well, while none of real observations pass QLSARP. Note that difference between figures is small, since the original test for GARP is weak (see Figure 2.7). The idea of conditioning QLSARP Predictive Success Index on GARP, is that assume there is a person (random set of choices) that looks consistent with GARP (has rational preferences), what is the probability that this person will look consistent with QLSARP (has quasi-linear preferences) as well. Figure 2.8b shows that conditioning of QLSARP on random choices that pass GARP does not change the picture a lot. QLSARP still performs good enough at the lower level at CCEI ($\approx .9$).

Figure 2.8 shows the trade-off between assuming rational and quasi-linear preferences.

Agents are consistent with the assumption of rational preferences under the lower decision making error (higher CCEI). Assuming only rationality of preferences we gather very vague predictions of future behavior. While assuming quasi-linear preferences allows us to tighten these predictions significantly. However, assuming quasi-linear preferences requires assuming that decision-making error is higher.

2.3.2 Quasi-Linearity in Money

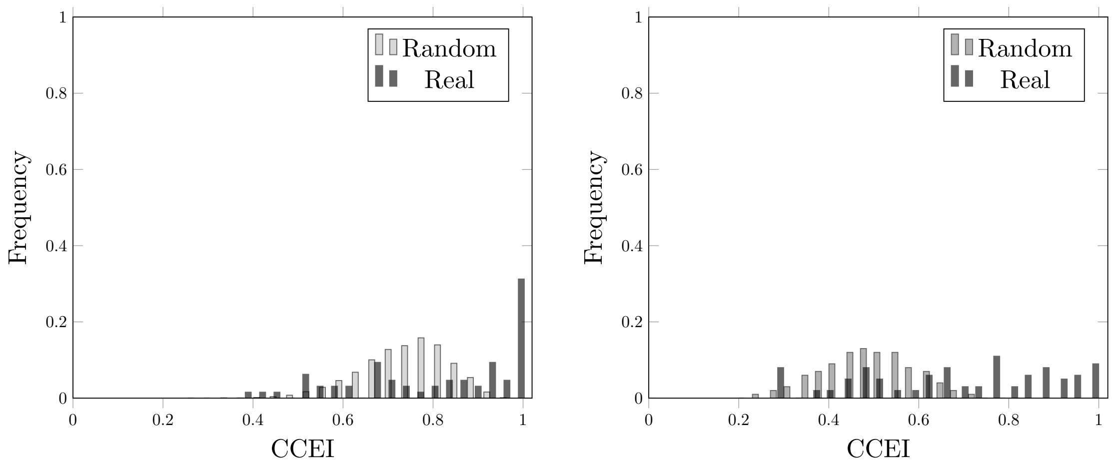
We now test quasi-linearity in money which is not possible to do using the data from Mattei (2000). To test quasi-linearity in money, we conduct an experiment[15] in which each subject had to allocate an endowment among five goods: Cash, Fandango Gift Card, Barnes and Noble gift card, Gap gift card and Mason Money. Each good stands for the category of goods and services on which subjects are expected to spend money. Fandango is movie theater ticket distributor and stands for entertainment spending. Barnes and Noble is book distributor and stands for necessities since textbooks are a required expenditure for students. Gap is clothing store and stands for durable goods. Mason Money is George Mason internal monetary system that can be used at any on-campus restaurant, therefore, it stands for food spending. The commodities are chosen as to minimize the transaction costs of consumption. The unit of measurement of each commodity is $1. Subjects are asked to allocated a 100 tokens between the above described goods facing prices that are denominated in tokens per dollar. Each subject faces 30 decision problems, one of which is chosen at random to be implemented. The experiment was conducted with 64 George Mason undergraduates.[16]

Figure 2.9 shows distribution of GARP and QLSARP (in money) CCEIs for the experimental data. Figure 2.9a shows distribution of GARP CCEIs for randomly simulated choices[17] and actual choices. Figure 2.9a shows that distribution of CCEIs for real choices

[15]For extensive description of experimental procedure, instructions and screen-shots see Appendix D.

[16]The complete explanation of the experimental design and procedures is in the Appendix.

[17]We firstly select a random order of five goods, then generate a share of income spent on the first of them using uniform random distribution. The share of the remaining income is then determined in equal manner for the second good. The same procedure is repeated for the third, fourth and fifth good.



(a) GARP (b) QLSARP

Figure 2.9: CCEI distributions for GARP and QLSARP

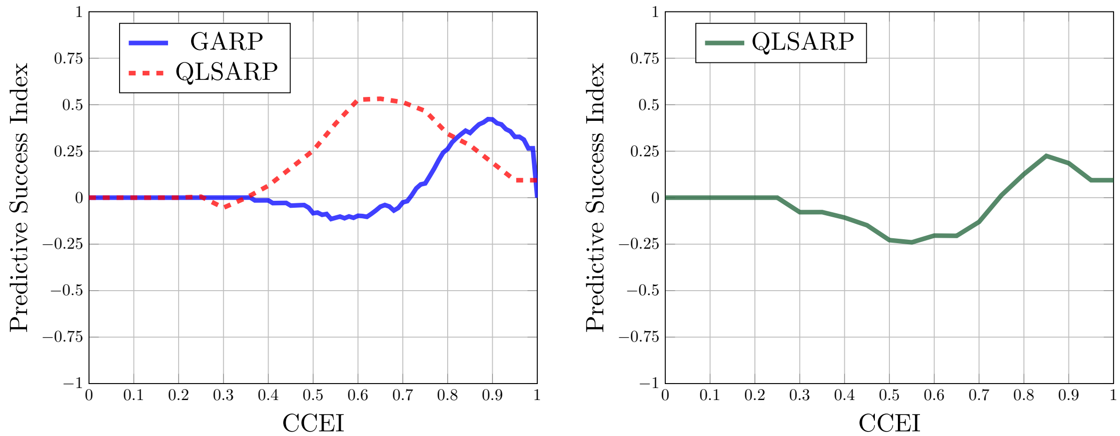
is shifted to the right in comparison to the distribution of random choices: mean GARP CCEI for real choices is .81 and for random choices is .75.[18] Figure 2.9b shows distribution of QLSARP CCEIs for randomly simulated choices and actual choices. The distribution of the QLSARP CCEIs for actual choices is shifted to the right and more dispersed than the distribution of QLSARP CCEIs for random choices. The mean QLSARP CCEI is .69 for actual choices and .49 for random choices.[19]

These results are similar if we compare actual choices to the choices of a "synthetic" subject. A "synthetic" subject is a collection of 30 budgets and associated decisions taken at random from all the menus produced by the experiment.[20]

[18]Distributions are different according to Kolmogorov-Smirnov test with $p < .001$. The difference in means is significant with $p < .001$ according to t-test. Median GARP CCEI for real choices is .85 and .76 for random choices, the difference is significant according to Wilcoxon rank sum test with $p < .001$.

[19]Distributions are different according to Kolmogorov-Smirnov test with $p < .001$. The difference in means is significant with $p < .001$ according to t-test. Median QLSARP CCEI for real choices is .73 and .49 for random choices, the difference is significant according to Wilcoxon rank sum test with $p < .001$.

[20]Results are available from the authors upon request.



(a) H_0: Random Choices

(b) H_0: Random Choices that Pass GARP

Figure 2.10: Predictive Success Index

Figure 2.10a shows the predictive success indexes for GARP and QLSARP. Figure 2.10b shows the predictive success index for QLSARP conditioning on random choices passing GARP. We observe that the predictive success index for QLSARP peaks for CCEIs levels between $.6 - .7$. This implies that to accept the hypothesis that subjects behave as if they had quasi-linear preference we would need to accept that they are willing to waste about 30 to 40% of their income. This is equivalent to losing an average of \$16 out the average experimental payments of \$40. Calculating the predictive success on random choices that pass GARP reduces the values of predictive success index. At the same time the predictive success index peaks at a CCEI level of .85. Our experimental evidence suggests that quasi-linearity of preferences in money is a strong assumption that deserves further investigation.

2.3.3 Quasi-Linearity in Panel Data

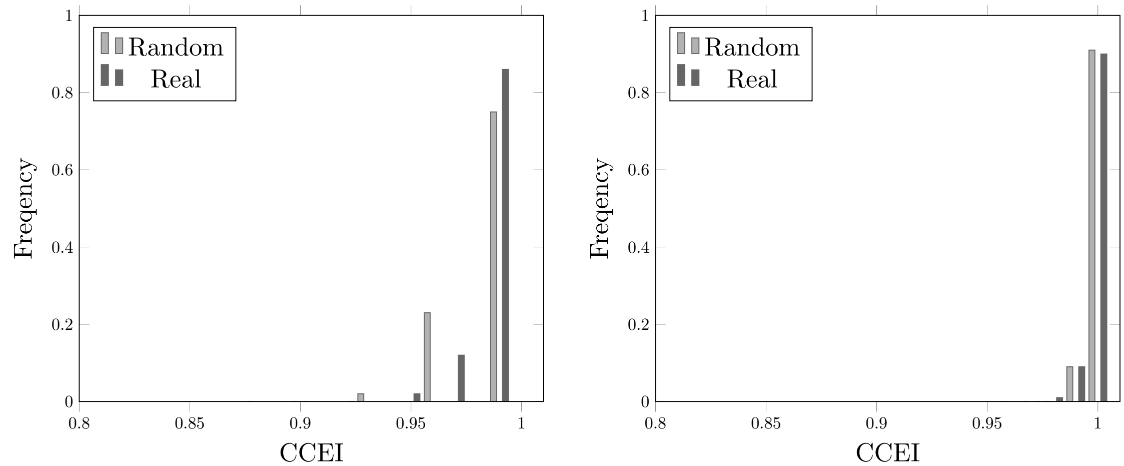
We use data from the Spanish Continuous Family Expenditure Survey (the Encuesta Continua de Presupuestos Familiares - ECPF). ECPF is a quarterly survey of households that are randomly rotated out at a rate of 12.5% per quarter. In this panel, household can be followed for up to eight consecutive periods and the years we use run from 1985 to 1997. For comparability with previous studies, we use the subsample used by Beatty and Crawford (2011) which comprises only two-adult households with a single income earner in the non-agricultural sector. The data set consists of 21,866 observations on 3,134 households which gives an average of seven consecutive periods per household. Expenditures of each households are aggregated into five groups: "Food, Alcohol and Tobacco","Energy and Services at Home","Non Durables","Travel" and "Personal Services". The price data are national consumer price indexes for the corresponding expenditure categories.

Note that the longitudinal nature of the data requires assuming that preferences are stable over time. Also, the data set imputes the same price index for all the household for a given quarter. Price variation across households is then obtained through their rotation in the sample.

A common problem in testing rationality with household expenditure data is the low power of the test due to limited price variation. The power of the test can be increasing by imposing additional assumptions like quasi-linearity of preferences. For instance, Heufer et al. (2014) shows that the hypothesis of homothetic preferences is well-powered in the ECPF panel. The same is true for the test of quasi-linear preferences.

We first determine the power of our test of quasi-linear preferences by obtaining CCEIs for agents choosing at random. We generate random choices over the budgets using the same procedure as in our experiment (see previous section).

Figure 2.11 shows the distributions of CCEIs for random and actual decisions, since there are five commodities and quasi-linearity of preferences could hold for each one of them, we report the distribution of maximum and minimum CCEI. For each agent we compute the CCEI for each commodity and take the maximum and minimum of them. These are the

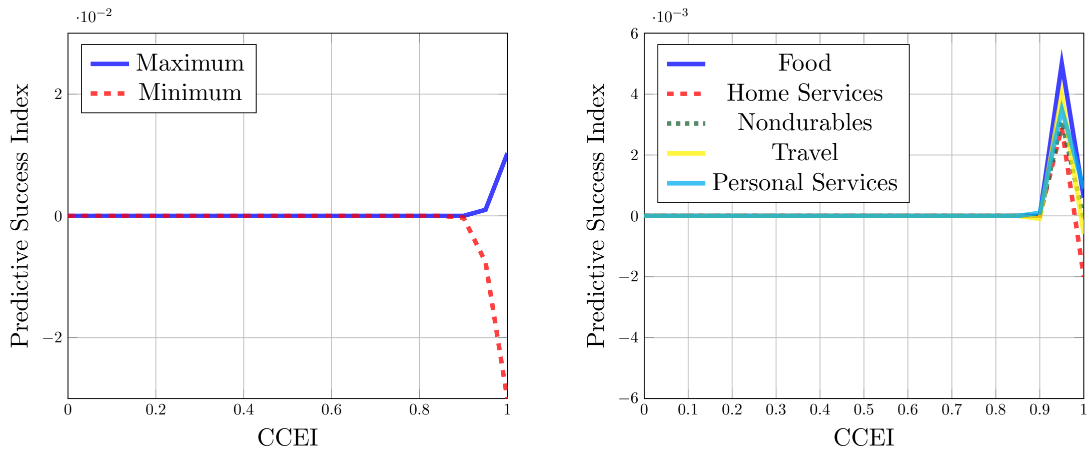


(a) Minimum CCEI (b) Maximum CCEI

Figure 2.11: CCEI Distributions for QLSARP

numbers used in Figure 2.11. This amounts to assuming that different agents might have quasi-linear preferences in different goods. Figure 2.11(a) shows the distributions of the maximum CCEIs for observed and random decisions. The mean of the maximum CCEIs for random decisions is .995 (the median is .997) and the mean of the maximum CCEIs for observed choices is .997 (the median is .997). Figure 2.11(b) shows the distributions of the minimum CCEIs for observed and random choices. The mean of the minimum CCEIs for random decisions is .979 (the median is .981) and the mean of the minimum CCEIs for random choices is .988 (the median is .989). If we consider CCEIs by commodity groups for random subjects, the mean is .988 (the median is .991) and for observed decisions the mean CCEI by the commodity groups is .992 (the median is .995).

Figure 2.12 shows the predictive success indexes. Figure 2.12(a) shows the minimum and maximum predictive success indexes conditioned on random choices. Note that due to the very small difference between the CCEI for random and real subjects the predictive success



(a) Bound for Predictive Succes Index (b) Predictive Sucess Indexes for Commodity Groups

Figure 2.12: Predictive Success Index

is zero for the most levels of the CCEI. A difference appears for a CCEI level close to one. At this level, observed decisions are consistent with QLSARP while random decisions are not. The predictive success index peaks at the CCEI level of 1 and provide weak positive evidence for quasi-linearity (the index is .1). Figure 2.12(b) shows the predictive success index conditioned on random choices by commodity groups. In this case, the predictive success is close zero at every level of CCEI. The maximum among these predictive success indexes peaks at .005 at the CCEI level of .95. Not surpringly, the hypothesis of quasi-linear preferences does no worse once we assume that different people have quasi-linear preferences in different goods.

Quasi-linearity in money is not testable in the ECPF dataset since no money leftover is recorded. However, quasi-linearity of preferences are testable if we allow for the existence of an unobserved commodity Cherchye et al. (2015b). We test for the joint hypothesis of quasi-linearity and the existence of an unobserved commodity and find that only 0.9% of households passed the test.[21] This suggests that our original findings are robust.

2.4 Nonparametric Welfare Analysis

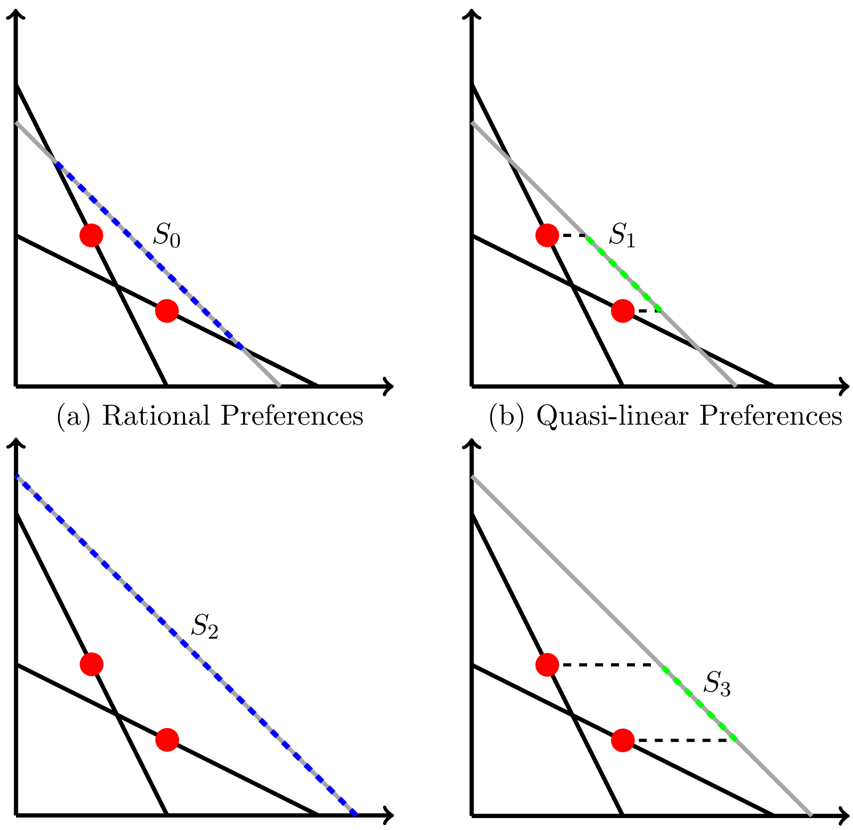
Afriat (1977) shows that revealed preference restrictions can be used to estimate the welfare effect of price changes. This idea was further developed by Varian (1982) to construct "support sets", i.e. the sets that include all the potential choices of consumer if the consumer is rational. We conduct a similar exercise to show how the assumption of quasi-linearity can help narrow prediction of choices in new budgets.

Figure 2.13 shows the extrapolation of behavior based on GARP alone. The red dots are the chosen bundles and the dashed areas, denoted by S_i, are the sets of possible choices over the new budgets. Figure 2.13(a) shows all possible choices if preferences are rational (S_0). Any bundle not in S_0 violates GARP. Figure 2.13(b) shows all possible choices if preferences are quasi-linear (S_1). Note that all the points in $S_0 \backslash S_1$ would violate QLSARP, but not GARP. Figure 2.13(c) shows all possible choices if preferences are rational (S_2). In this case S_2 is the entire budget line, since GARP imposes no restriction choices on budgets that do not intersect with previous ones. Figure 2.13(d) all possible choices if preferences are quasi-linear (S_3). Note that predictions under quasi-linearity remain the same regardless the new budget intersects with and old one or not.

Figure 2.14 shows that if preferences are quasi-linear even one observed choice can help narrow the set of possible choices in new choice sets. Figure 2.14(a) present the case of a change of the price for one of the goods. In this case, amount of good x_2 have to be "above" the amount consumed in the original choice set (S_4). Figure 2.14(b) is the case of a change income holding prices constant. In this case, the set of possible choices is a singleton (S_5).[22] Sharper predictions can be obtained if additional choices are observed. Figure 2.14(c) shows that predictions can be narrowed further in this case. Figure 2.14(c) shows the bounds for

[21] We cannot calculate how severe deviations from rationality are because income is not observed.

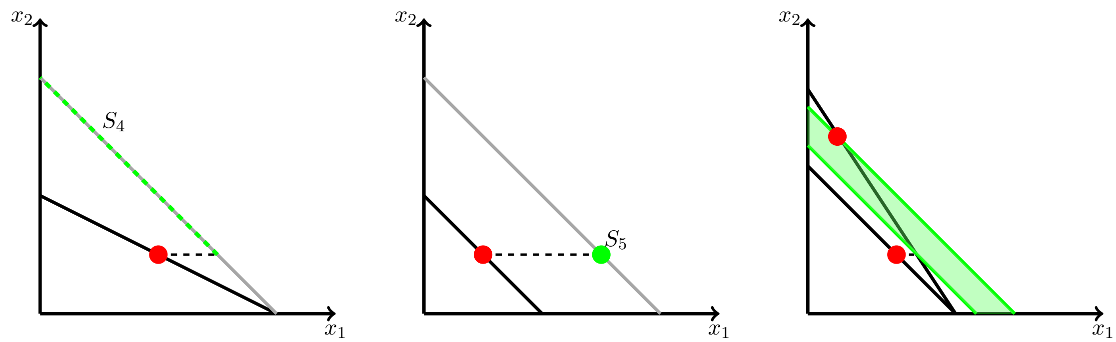
[22] Note that if we relax the requirements to QLGARP, then the support set in this case would include the entire new budget set.



(a) Rational Preferences (b) Quasi-linear Preferences

(c) Rational Preferences (d) Quasi-linear Preferences

Figure 2.13: Possible Choices: Comparison of Assumptions of Rational Preferences and Quasi-Linear Preferences



(a) Wide Forecast (b) Narrow Forecast (c) Compensated Variation

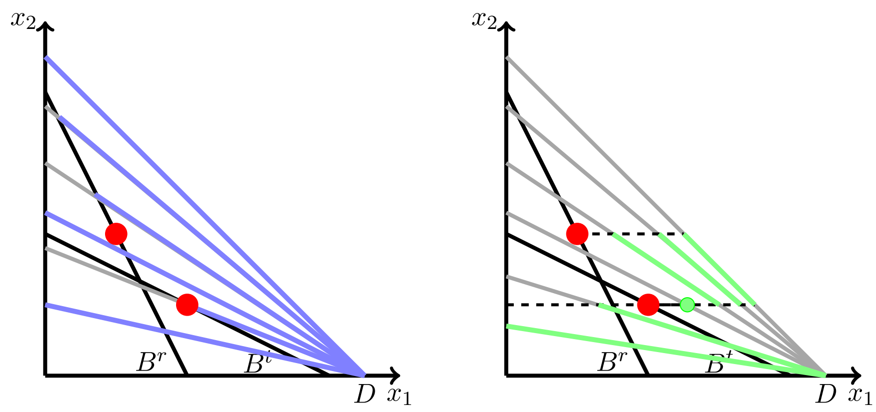
Figure 2.14: Possible Choices: Constructing a Forecast from a Single Observation

compensated variation for the point in the outer budget after an increase in the price of good 2. The outer green line represent the budget that attains the original choice at the new prices. The utility associated with this budget cannot be lower that the utility at the original prices and therefore represent an upper bound of the compensation necessary to maintain the original level of utility. To derive the lower bound of the compensating variation consider the case in which all the points on the original budget set that give at least as much of good 2 as the observed choice in the new budget are indifferent to the originally chosen bundle. This certainly does not contradict quasi-linearity. In this case, the needed compensation is defined by the lower green line.[23]

The next step is to construct bounds on demand functions and consumer surplus. Figure 2.15 provides a two-dimensional example. Since under quasi-linearity in x_1, estimates of consumer surplus can be approximated by changes in the demand of x_2 due to price changes, we construct bounds for the demand in x_2.

Figure 2.15 shows the procedure to bound the demand of x_2 according to GARP and

[23]This derivation is based on Figure 2.14(b), therefore, it would not be correct under the assumption of QLGARP instead of QLSARP.

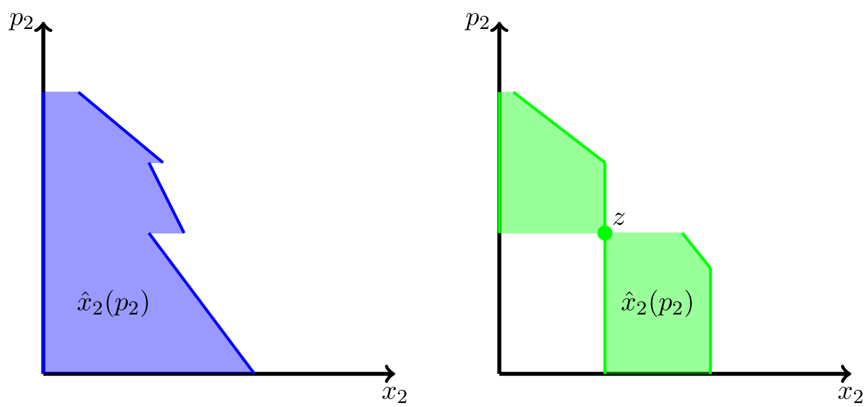


(a) for GARP (b) for QLSARP

Figure 2.15: Constructing Demand Bounds

QLSARP. Observed choices are represented by red dots the budgets B^t and B^r. Hypothetical budgets are represented by blue lines starting at point D. The only difference among these budgets is the price of x_2. Figure 2.15(a) shows bounds on the demand of x_2 in the case of rational preferences. Note that the bounds on demand will have two "jumps". One at the intersection of the new budget sets with B^t and one at the intersection of the new budget sets with B^r. Note that these extrapolation is done over a set of non previously observed choice sets.

Figure 2.15(b) repeats the same exercise under the assumption of quasi-linear preferences. There are three different cases. The first case corresponds to the budget sets that intersect the actual choice in B^t. In this case the bound on demand do not depend on prices. The second case corresponds to the case in which the new budget set is parallel to B^r. This case is similar to Figure 2.14(b) where a point-prediction is possible. The third case corresponds to the new budget sets for which the choice at B^r is not affordable. The prediction in this case is bounded above by the consumption of x_2 in B^r and strictly below this amount when this is not affordable.



(a) Rational Preferences (b) Quasi-linear Preferences

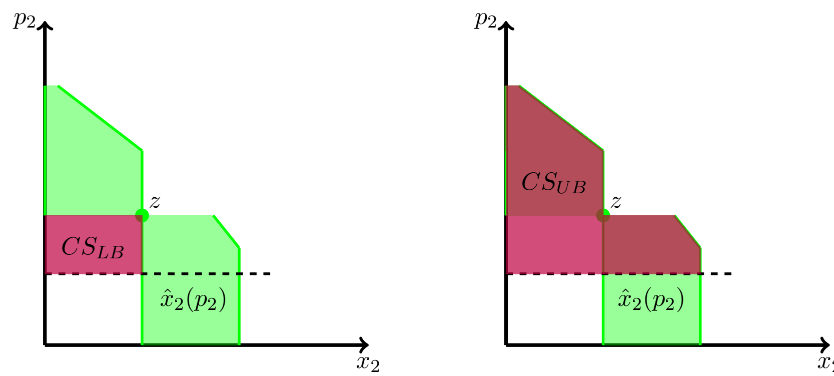
Figure 2.16: Bounded Demands

Figure 2.16 shows the bounds for the demand function constructed using the same procedure as the one used to construct Figure 2.15. Note that demand is not necessarily continuous. Two sectors on the bounds on demand deserve attention. Figure 2.16(a), which shows the bounds on demand based on rational preferences only, have jumps corresponding to the cases where choices have jumps as well.

Figure 2.16(b) shows bound for the demand correspondence given the assumption of quasi-linear preferences. In this case, there is a point at which bounds on demand shift to the left discontinuously. If the price is higher than the price at this point (z), then the lower bound for the demand is zero. If the price is lower than the price at z, then the lower bound for the demand is constant.

This exercise shows that the bounds under quasi-linearity can be tight than under the assumption of rational preferences alone.

The bounds on the demand function imply also bounds on consumer surplus. This can be done by using the upper bound of demand to construct the upper bound of the consumer surplus and the lower bound of demand to construct the lower bound of the consumer surplus. Note that upper bounds for demand functions are the right border lines of the green area. This boundary is weakly decreasing and therefore a valid demand function under quasi-linearity. This therefore provides the logical upper bound to all demand functions consistent with quasi-linearity and the original choices.



(a) Lower Bound for Consumer Surplus (b) Upper Bound for Consumer Surplus

Figure 2.17: Constructing Bounds for Compensated Variation and Consumer Surplus

Figure 2.17 shows the bounds for Consumer Surplus when preferences are quasi-linear. Figure 2.17(a) shows the lower bound for the consumer surplus and Figure 2.17(b) shows the upper bound for the consumer surplus. The bounds on the consumer surplus follow tightly the bounds on the demand functions explained above.

2.5 Connection to Previous Literature

In this section we show connection between QLSARP and the test from Brown and Calsamiglia (2007). We show that if budgets are linear, then QLSARP is equivalent to the condition from Brown and Calsamiglia (2007). Therefore, if a consumption experiment satisfies QLSARP, then there is a strictly concave, continuous, strictly monotone and quasi-linear utility function that rationalizes the consumption experiment. This implies that if budgets are linear, then continuity and concavity have no empirical content under the assumption of quasi-linearity of preferences. Moreover, it allows us to to develop the test for concavity of preferences under the assumption of quasi-linearity preferences, using non-linear budgets.

Brown and Calsamiglia (2007) use linear budgets and normalize a price of a good in which preferences are supposed to be quasi-linear (numeraire), so the linear consumption experiment can be defined as $((x^t, y^t), (p^t, 1))_{t=1}^T$, where $p^t \in \mathbb{R}_+^{N-1}$ is the vector of prices. Linear budget set can be defined as $B^t = \{x \in \mathbb{R}^{N-1}, y \in \mathbb{R} : (p^t, 1)(x, y) \leqslant (p^t, 1)(x^t, y^t)\}$. Following Brown and Calsamiglia (2007) a linear consumption experiment $(x^t, p^t)_{t=1}^T$ is said to be **strictly cyclically monotone** if for any subset $\{((x^m, y^m), (p^m, 1))\}_{m=1}^M$ $p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) > 0$. A utility function **rationalizes** the consumption experiment $((x^t y^t), (p^t, 1))_{t=1}^T$ if $u(x^t, y^t) \geqslant u(x, y)$ for any $p^t x + y \leqslant p^t x^t + y^t$ for any $t = 1, \ldots, T$.

Theorem 2.2 (Theorem 2.2 from Brown and Calsamiglia (2007))**.** *A finite consumption experiment $((x^t, y^t), (p^t, 1))_{t=1}^T$ can be rationalized by a strictly concave, continuous, quasi-linear and strictly monotone utility function if and only if it is strictly cyclically monotone.*

Lemma 2.1. *Let $((x^t, y^t), (p^t, 1))_{t=1}^T$ be a finite consumption experiment that generates a revealed preference relation R_v. Then, an acyclic R_v satisfies QLSARP if and only if the consumption experiment is strict cyclically monotone.*

Note that if there is a strictly concave, continuous, quasi-linear and strictly monotone utility function that rationalizes the consumption experiment, then there is a complete extension of the corresponding revealed preference relation that is complete, transitive and monotone and quasi-linear. Therefore, if the consumption experiment is strict cyclically monotone, then the revealed preference relation generated by the consumption experiment satisfies QLSARP. Therefore, we left to prove the reverse, i.e. if there is an acyclic R_v satisfies QLSARP, then the consumption experiment is strict cyclically monotone.

Proof. As in cyclical monotonicity for QLSARP we also need to consider any subset of budget experiment $\{((x^m, y^m), (p^m, 1))\}_{m=1}^{M}$ if for every $j < M$ $((x^j, y^j), (x^{j+1}, y^{j+1} - \beta^j) \in T(R_v)$, then $((x^M, y^M), (x^1, y^1 + \sum_{j=1}^{M-1} \beta^j)) \notin P(T(R_v))$. QLSARP implies this it should be true for any sequence[24] of β^j. However, checking the minimum β^j is enough. And this β^j can be defined as $\beta^j = p^j(x^{j+1} - x^j) + (y^{j+1} - y^j)$. Then $((x^M, y^M), (x^1, y^1 + \sum_{j=1}^{M-1} \beta^j)) \notin P(T(R_v))$ can be rewritten as $p^M x^1 + y^1 + \sum_{j=1}^{M-1} \beta^j > p^M x^M + y^M$. And simplifying the expression we get that $p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) + \underbrace{(y^2 - y^1) + (y^3 - y^2) + \ldots + (y^1 - y^M)}_{=0} = p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) > 0$ that is stict cyclical monotonicity. □

Hence, combining Theorem 2.2 and Lemma 2.1 we can get the following result.

Theorem 2.3. *A finite consumption experiment $((x^t, y^t), (p^t, 1))_{t=1}^{T}$ can be rationalized by a strictly concave, continuous, quasi-linear and strictly monotone utility function if and only if R_v is acyclic and satisfies QLSARP with respect to n-th component (y^t).*

Note that Theorem 2.2 uses strict concavity of the utility function as a crucial assumption. While from Theorem 2.1 we know that QLSARP is equivalent to the existence of an extension of the revealed preference relation that is just complete, transitive, monotone and quasi-linear. Therefore, if budgets are linear concavity and continuity has no empirical content under the assumption of quasi-linearity of preferences.

Now let us show how to construct the test for the concavity of the utility function under the assumption of quasi-lienarity of preferences using non-linear budgets. Following Forges and Minelli (2009) a function $g^t(x, y) : \mathbb{R}^L \to \mathbb{R}$ is said to be **gauge function** of the budget set B^t if $B^t = \{x \in X : g^t(x, y) \leqslant 0\}$. And with a finite consumption experiment one can associate the linearized experiment, if gauge function is differentiable at least at the optimal points. $((x^t, y^t), C^t)_{t=1}^{T}$ is a **linearized consumption experiment** associated with $((x^t, y^t), B^t)_{t=1}^{T}$ if $C^t = \{x \in X : \nabla g^t(x^t, y^t)(x, y) \leqslant \nabla g^t(x^t, y^t)(x^t, y^t)\}$. Denote by

[24] QLSARP specifies β^j to be positive, however, in Appendix A we show that to obtain quasi-linear representation, the similar condition should hold for any real sequence of β^j not necessarily positive.

R_v^C the revealed preference relation generated by the linearized consumption experiment $((x^t, y^t), C^t)_{t=1}^T$.

Theorem 2.4. *Let $((x^t, y^t), C^t)_{t=1}^T$ be a finite consumption experiment with gauge functions that are increasing, continuous, quasi-convex, differentiable at every (x^t, y^t) and can be represented as $g^t(x, y) = h^t(x) + y$. Then there is a continuous, strictly concave, strictly increasing and quasi-linear utility function $(u(x) + y)$ that rationalizes it if and only if R_v^C is acyclic and satisfies QLSARP.*

Since the proof is very similar to the proof of the Theorem 2.2 let us sketch it to emphasize the main differences. The proof can be done just by showing that the following are equivalent:

1. There is a continuous, concave, increasing and quasi-linear utility function that rationalizes R_v,
2. There are numbers u^t and u^s such that $u^s \leqslant u^t + \nabla h^t(x^t)(x^s - x^t)$ for any $t, s = 1, \ldots, T$,[25]
3. Linearized experiment satisfies cyclical monotonicity

(1) $\Rightarrow$ (2) From the first order condition for quasi-linear utility function we know that $\nabla u(x^t) = \nabla h^t(x^t)$ and from the fact that u is concave we can conclude that $\nabla h^t(x^t)$ is the super-gradient, i.e. $u(x) \leqslant u(x^t) + \nabla h^t(x^t)(x^s - x^t)$ for any $x \in X$.

The rest of the implications are the same as in the original proof, since we take $\nabla h^t(x^t)$ as linear prices, and as the result we will get the utility function that represents R_v^C and, hence, represents R_v. As we have already shown if prices are linear, then QLSARP is equivalent to strict cyclical monotonicity. That completes the proof. Hence, concavity of quasi-linear representation can be rejected if the consumption experiment satisfies QLSARP, while its

[25]For the case of not necessarily differentiable utility function one similarly can start from $u^s + y^s \leqslant u^t + y^t + (\nabla h^t(x^t), 1)((x^s, y^s) - (x^t, y^t) = u^t + y^t \nabla h^t(x^t)(x^s - x^t) + (y^s - y^t) = u^t + y^s + \nabla h^t(x^t)(x^s - x^t)$, that is equivalent to the original inequality.

linearized version does not. It would imply that there is a quasi-linear extension of the revealed preference relation, while there is no concave and quasi-linear utility function that rationalizes the consumption experiment.

2.6 Summary

We provide a necessary and sufficient condition for a set of observed choices to be consistent with the existence of complete, transitive, monotone and quasi-linear preference ordering. This condition applies to choices over compact and downward closed budget sets and does not require preferences to be convex. This condition does not guarantee existence of a utility function but only the existence of a preference relation, that is complete transitive, monotone, quasi-linear and generates the observed behavior. This condition is sufficient for the existence of quasi-linear utility function if budget sets are linear.

We conduct a laboratory experiment to test the hypothesis that preferences are quasi-linear in money. Our experiments show that while individual choices are generally consistent with GARP they are no more consistent with quasi-linearity in money than choices made at random. We also use the data from the Spanish Continuous Family Expenditure Survey to test for quasi-linearity using household consumption data. We find support for the hypothesis of quasi-linearity against the the alternative that choices are made at random. However, and due to low observe variation in prices, we find no support for the existence of quasi-linear preferences against the alternative of the existence of a smooth utility function.

We discuss how the assumption of quasi-linearity can be used to perform non-parametric welfare analysis. In particular, we show how to estimate the bounds for the demand correspondence, consumer surplus and compensated variation under the assumption of quasi-linear preferences. Note that this estimations requires only the existence of a complete, transitive, monotone and quasi-linear preference ordering. These bounds can be considerably tighter than nonparametric bounds derived under the assumption of rational preferences alone.

An important advantage of the test of quasi-linearity we present here is that it applies to a large class of problems. First, it applies to consumer problems in the presence of distortions due to taxes, subsidies or non-linear pricing. Second, it can also be extended to test for quasi-linearity of preferences in strategic situations where such assumption might be invoked (e.g. auction theory).

2.7 Appendix

2.7.1 Proof of Theorem 2.1

To prove Theorem 2.1 we use the notion of function over preference relations. A convenient example of such function is the *transitive closure*, which adds (x, z) to R for each $(x, y) \in R$ and $(y, z) \in R$. Being more precise $(x, y) \in T(R)$ if there is a sequence of elements $S = s_1, \ldots, s_n$, such that for every $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R$, where T stays for transitive closure. That is transitive closure of a preference relation is a transitive preference relation, since transitive closure is idempotent and applying transitive closure to the transitive closure of the relation does not add anything to the relation. The function we propose is a generalization of transitive closure that guarantees that every fixed point of it is transitive, monotone and quasi-linear preference relation.

The proof is organized as follows. First, we introduce the terminology and define the function over preference relations that implies the desirable properties while extending the preference relation. Further we show that the function is "well-behaved", i.e. the possibility of extension of preference relation by the function can be stated as a simple set-theoretical condition. Finally, we show that consistency of a preference relation with the function is equivalent to QLSARP.

Let $F : \mathcal{R} \to \mathcal{R}$ be a function over preference relations. For a given function $F(R)$, a preference relation R is said to be F**-consistent** if $F(R) \cap P^{-1}(R) = \varnothing$.[26]

Lemma 2.2. *A preference relation $R \subseteq F(R)$ is F-consistent if and only if $R \preceq F(R)$.*[27]

[26] Recall that $(x, y) \in P^{-1}(R)$ if $(y, x) \in R$ and $(y, x) \notin R$.
[27] Recall that $R \preceq F(R)$ ($F(R)$ is an extension of R) if $R \subseteq F(R)$ and $P(R) \subseteq P(F(R))$.

We omit the proof since it can be found in Demuynck (2009). Henceforth, further we use these notions as equivalent definitions. For any function $F : \mathcal{R} \to \mathcal{R}$, let $\mathcal{R}_F^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. We use the following definition throughout the section.

Definition 2.3. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be an **algebraic closure** if*

1. *For all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$, and,*
2. *For all $R \in \mathcal{R}$, $R \subseteq F(R)$, and,*
3. *For all $R \in \mathcal{R}$, $F(F(R)) \subseteq F(R)$, and,*
4. *For all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$, s.t. $(x, y) \in F(R')$.*

Properties (1) to (3) are those that define **closure** and are connected to the extrapolation of the relation by $F(R)$. Property (1) is *monotonicity*, it states that from larger amount of information we can get better extrapolation. Property (2) is *extensiveness*, that is function adds additional information about preference relation. Property (3) is *idempotence*, that is the function delivers all the information after the first application of it to the binary relation. Property (4) is one that makes the closure **algebraic** (for the formal definition in more general context see e.g. Crawley and Dilworth (1973)). This property is one that allows us to make our theory testable with finite data set, i.e. there is finite set of observations that is not F-consistent.

A function $F : \mathcal{R} \to \mathcal{R}$ is said to be **weakly expansive**[28] if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is $T \subseteq N(R)$ such that $R \cup T \in \mathcal{R}_F^*$. This property states that F is non-satiated, i.e. for any incomplete fixed point ($F(R) = R$) preference relation, there is F-consistent extension of this relation. It is important since we are used to assume that preferences are complete, i.e. any two bundles are comparable, and weak expansiveness guarantees that the set of assumptions is not contradictory with completeness axiom.

[28] Weak expansiveness is equivalent to the condition C7 in Demuynck (2009).

Theorem 2.5 (Theorem 2 from Demuynck (2009))**.** *Let F be a weakly expansive algebraic closure. Then, a relation $R \in \mathcal{R}$ has a complete extension $R^* = F(R^*)$ if and only if R is F-consistent.*

To provide the intuition for the proof of Theorem 2.5 let us show the algorithm that can is used to construct a fixed point complete extension. Denote $R_0 = R$, i.e. the relation we start from. Then, for any $a > 0$ if $R_a \neq F(R_a)$, then $R_{a+1} = F(R_a)$. If $R_a = F(R_a)$, then from expansiveness we know that there is T, such that $R_a \cup T \in \mathcal{R}^*_F$, so let $R_{a+1} = R_a \cup T$. The existence of the limit relation which is a fixed point of F is guaranteed by the fact that F is algebraic closure.

Let us specify the closure that will guarantee us existence of complete, monotone, transitive and quasi-linear in i-th component extension of the original relation.

Definition 2.4. *Denote the* ***Quasi-linear in the i-th component Monotone Transitive closure*** *by $QLiMT(R)$. Then, $(x, y) \in QLiMT(R)$ if there is a sequence $S = s_1, ..., s_n$, s.t. $s_1 = x$, $s_n = y$ and $\forall j = 1..n-1$*

1. $\exists \alpha_j \in \mathbb{R} : (s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$, *or*

2. $s_j \gg s_{j+1}$

We consider quasi-linearity[29] with monotonicity, since these notions are usually considered together (see e.g. Kreps (2012) who defines quasi-linearity as the definition we provide jointly with the monotonicity). However, monotonicity has no empirical content if we assume budgets to be monotone. In the case of non-monotone budgets QLSARP can be easily modified by using $TM(R)$ instead of $T(R)$.[30] $TM(R)$ is transitive and monotone closure, that can be defined as following: $(x, y) \in TM(R)$ if there is a sequence $S = s_1, \ldots, s_n$, such that for any $j = 1, \ldots, n-1$ either $(s_j, s_{j+1}) \in R$ or $s_j \geqslant s_{j+1}$. If one wants to separate the assumption of quasi-linearity from the monotonicity, monotonicity of budget sets can

[29] Recall that R is said to be **quasi-linear in the i-the component** if for any $(x, y) \in R$ and $\alpha \in \mathbb{R}$ $(x + \alpha e_i, y + \alpha e_i) \in R$.

[30] Recall that $(x, y) \in T(R)$ if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for any $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R$

be relaxed to monotonicity in the i-th component[31] (the same as it should be quasi-linear in). In this case QLSARP is equivalent to the existence of the extension of revealed preference relation that is quasi-linear in the i-th component, monotone in the i-th component, transitive and complete .

The proof of Theorem 2.1 consists of two parts:

1. Proposition 2.1 shows that $QLiMT(R)$ is a weakly expansive algebraic closure;
2. Proposition 2.2 shows that R_v satisfies QLSARP if and only if $T(R_v)$[32] is $QLiMT$-consistent.

After proving these Propositions we can apply Theorem 2.5 to complete the proof of Theorem 2.1.

Proof of Proposition 2.1

Proposition 2.1. *$QLiMT(R)$ is a weakly expansive algebraic closure.*

The proof of the Proposition 2.1 consists of the following lemmas:

1. Lemma 2.3 shows that $R = QLiMT(R)$ if and only if R is quasi-linear, transitive and monotone. This Lemma shows that any fixed point of $QLiMT$ is a quasi-linear, transitive and monotone relation. Moreover, we use it extensively in the further proofs.
2. Lemma 2.4 shows that $QLiMT$ is a closure, i.e. satisfies (1)-(3). Note that if it is a closure, it is algebraic by definition, because any element of $QLiMT(R)$ is added by adding the finite sequence of elements of R which generates it.
3. Lemma 2.5 shows that $QLiMT$ is weakly expansive

Lemma 2.3. *$R = QLiMT(R)$ if and only if R is quasi-linear, transitive and monotone.*

[31] The budget is monotone with respect to i-component if $x \in B$ implies that for any $\alpha \in \mathbb{R}_+$ $x - \alpha e_i \in B$

[32] Recall that R_v is a revealed preference relation obtained $(x^t, y) \in R_v$ if $y \in B^t$, where x^t is chosen bundle and B^t is corresponding budget. $T(R_v)$ is a transitive closure of revealed preference relation.

Proof. (⇒). R is transitive, because if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in QLiMT(R)$. R is monotone also by the definition of $QLiMT(R)$. R is quasi-linear in i-th component, i.e. if $(x + \alpha e_i, y + \alpha e_i) \in R$, then $(x, y) \in QLiMT(R)$.

(⇐). From the Definition of $QLiMT(R)$ it is clear that $R \subseteq QLiMT(R)$. Therefore, we need to show that $QLiMT(R) \subseteq R$ to show the equality of these sets. To prove this we need to show, that if there is $(x, y) \in QLiMT(R)$, then $(x, y) \in R$. To do this take $(x, y) \in QLiMT(R)$ and show that (x, y) is in R as well. We prove this by the induction on the length of chain that adds (x, y) to $QLiMT$. If length of the shortest chain is 2 it is immediately true since R is quasi-linear, transitive and monotone. Now suppose that every element $(x, y) \in QLiMT(R)$, such that it can be added to the $QLiMT(R)$ by a chain of the length no more than k is in R as well. To do the induction step, consider an element $(x, y) \in QLiMT(R)$ that is added to the $QLiMT(R)$ by a chain of the length of $k + 1$. Let us show that length of the chain can be reduced, i.e. the same element can be added to the $QLiMT(R)$ by a chain of the length no more than k. Take a $j \in \{1, \ldots, n-1\}$ and consider the four following cases:

Case 1: $s_{j-1} \gg s_j$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. Knowing that $s_{j-1} + \alpha_j e_i \gg s_j + \alpha_j e_i$ then by monotonicity, $(s_{j-1} + \alpha_j e_i, s_j + \alpha_j e_i) \in R$. By transitivity $(s_{j-1} + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. Therefore, length of the chain can be reduced.

Case 2: $s_{j-1} \gg s_j$ and $s_j \gg s_{j+1}$, then $s_{j-1} \gg s_{j+1}$. By monotonicity $(s_{j-1}, s_{j+1}) \in R$. Therefore, length of the chain can be reduced.

Case 3: $(s_{j-1} + \alpha_{j-1} e_i, s_j + \alpha_{j-1} e_i) \in R$ and $s_j \gg s_{j+1}$ Knowing that $s_j + \alpha_{j-1} e_i \gg s_{j+1} + \alpha_{j-1} e_i$ then by monotonicity, $(s_j + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. By transitivity $(s_{j-1} + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. Therefore, length of the chain can be reduced.

Case 4: $(s_{j-1} + \alpha_{j-1} e_i, s_j + \alpha_{j-1} e_i) \in R$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. By quasi-linearity $(s_j + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. By transitivity $(s_{j-1} + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. Therefore, length of the chain can be reduced.

This completes the induction. □

Lemma 2.4. *$QLiMT(R)$ is an algebraic closure*

Note that $QLiMT(R)$ is algebraic (satisfies condition (4)) by construction, since every element can be added using a finite sequence. According to the Lemma 5 from Demuynck (2009) $F(R)$ is a closure if and only if $F(R) = \bigcap\{Q \supseteq R : \; Q = F(Q)\}$. Hence, it is enough to prove that $QLiMT(R) = \bigcap\{Q \supseteq R : \; Q = QLiMT(Q)\}$

Proof. ($\subseteq$) Let $(x, y) \in QLiMT(R)$, then there is a sequence $S = s_1, \ldots, s_n$ such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$. Since, $R \subseteq Q$, then for the entire sequence $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in Q$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$. Hence, $(x, y) \in QLiMT(Q)$.

($\supseteq$) Note that from Lemma 2.3 we know that $QLiMT(QLiMT(R)) = QLiMT(R)$. Therefore $QLiMT(R) \in \bigcap\{Q \supseteq R : \; Q = QLiMT(Q)\}$. Thus, if $(x, y) \in Q$ for any $Q \in \bigcap\{Q \supseteq R : \; Q = QLiMT(Q)\}$, then $(x, y) \in QLiMT(R)$. ☐

Lemma 2.5. *$QLiMT(R)$ is weakly expansive.*

Proof. Take a point $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y)\}$. We need to show that $QLiMT(R') \cap P^{-1}(R') = \varnothing$. Assume to the contrary that there is a $(z, w) \in QLiMT(R') \cap P^{-1}(R') \neq \varnothing$.

Case 1: $(x, y) \neq (w, z)$**.** There is a chain $S = s_1, \ldots, s_n$ such that there is k such that $s_k + \alpha_k e_i = x$ and $s_{k+1} + \alpha_k e_i = y$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$ or $s_j \gg s_{j+1}$ for any $j \neq k$ and $(w, z) \in R$ by assumption. Let us consider the following sequence[33] $S' = s_{k+1}, \ldots, s_n, w, z, s_1, \ldots, s_k$ with:

$$(s_{k+1} + \alpha_{k+1} e_i, s_{k+2} + \alpha_{k+1} e_i), \ldots, (s_{n-1} + \alpha_{n-1} e_i, w + \alpha_{n-1} e_i),$$

$$(w, z), (z + \alpha_1 e_i, s_2 + \alpha_1 e_i), \ldots, (s_{k-1} + \alpha_{k-1} e_i, s_k + \alpha_{k-1} e_i)$$

[33] We omit elements of sequence that correspond to monotonicity, since $QLiMT(R) = R$ already implies that all monotonicity pairs are already in R.

elements of R.

From Lemma 2.3 we know that R is quasi-linear. Therefore, $(s_{k+1}, s_k) \in R$. Recall that by Lemma 2.3 R is quasi-linear. Hence, $(y, x) \in R$ since they can be obtained by adding $\alpha_k e_i$ to s_{k+1} and s_k respectively. Therefore, $(x, y) \notin N(R)$.

Case 2: $(x, y) = (w, z)$**.** Then, $(y, x) \in P(R')$ and $(x, y) \in QLiMT(R')$. If $(x, y) \in QLiMT(R)$, then there is direct contradiction, because $QLiMT(R) = R$ and this contradicts the fact that $(x, y) \in N(R)$. Hence, $(x, y) \in QLiMT(R') \backslash R$. Then, there is a chain $S = s_1, \ldots, s_n$ such that there is k such that $s_k + \alpha_k e_i = y$ and $s_{k+1} + \alpha_k e_i = x$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$. Therefore, there is a sequence $S' = s_1, \ldots, s_k$, which contains only the elements of R, this implies, that $(x, y) \in QLiMT(R)$. Therefore, $(x, y) \notin N(R)$.

□

This completes the proof of Proposition 2.1.

(ii) Proof of Proposition 2.2: R_v **satisfies QLSARP if and only if** $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \varnothing$

Recall the definition of QLSARP.

Definition 2.5. *A revealed preference relation satisfies* ***QLSARP*** *with respect to i-th component if for any sequence of distinct elements* $x^{k_1}, ..., x^{k_n} \in C$ *and* $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$*, such that* $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$ *and* $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$ *for* $j = 2, \ldots, n-1$*, then* $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i) \notin T(R_v)$*.*

And since $QLiMT(R)$ is a weakly expansive algebraic closure we need to show that QLSARP is equivalent to $QLiMT$-consistency for R_v. Recall that elements of $QLiMT$ are added through sequences, such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_j) \in T(R_v)$ or $s_j \gg s_{j+1}$. A sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$ that adds (x, y) to $QLiMT(R)$ is said to be (x, y)**-irreducible length sequence** if there is no shorter sequence $S' = s'_1, \ldots, s'_n$, $s'_1 = x$ and $s'_n = y$ that adds (x, y) to $QLiMT(R)$. For any sequence S denote by $\beta_j = \alpha_j - \alpha_{j-1},\ j = 2, \ldots, n$. From the definition of $QLiMT(R)$ we can be sure that for any $(x, y) \in QLiMT(R)$ there is a finite irreducible length sequence. Note that if $(x, y) \in R$ the (x, y)-irreducible length sequence will trivially be $s_1 = x$, $s_2 = y$. Moreover, for each element there is the shortest sequence, hence we need to show that QLSARP checks that for any pair (x, y) from the strict part of $P(T(R_v))$ there is no (y, x)-irreducible length sequence in $T(R_v)$ - $(y, x) \in QLiMT(T(R_v))$.

But QLSARP does not say anything about one element being greater than another. So let us show that no (x, y)-irreducible length sequence will contain $s_j \gg s_{j+1}$. Note that assumption of monotonicity of budget sets allows us to claim that $(x, y) \in R_v$ then for any $z \leqslant y$ $(x, z) \in R_v$ as well. And this fact can be generalized for the transitive closure of the relation.

Fact 2.1. *$(x, y) \in T(R_v)$ then $(x, z) \in T(R_v)$ for any $z \leqslant y$.*

Hence, none of (x, y)-irreducible length sequences will contain $s_j \gg s_{j+1}$, because s_j has to be a chosen point, hence $(s_j, s_{j+1}) \in T(R_v)$ and the length of the sequence can be reduced otherwise.

For further proof denote by $\beta_j = \alpha_j - \alpha_{j-1}$ for $j \in \{2, \ldots, n\}$. Originally $QLiMT(R)$ assumes β_j to be any real number while QLSARP considers only positive β_j. So, let us show that (x, y)-irreducible length sequence will not contain $\beta_j \leqslant 0$.

Lemma 2.6. *For any $(x, y) \notin T(R_v)$ and $(x, y) \in QLiMT(T(R_v))$, an (x, y)-irreducible length sequence has all $\beta_j > 0$.*

Proof. On the contrary assume that there is $j \in \{2, ..., n-1\}$ such that $\beta_j \leqslant 0$. And further we show that there is a shorter sequence $S' = s'_1, \ldots, s'_{j-1}, s'_{j+1}, \ldots, s'_n$, $s'_1 = x$ and $s'_n = y$, such that for $k = 1, \ldots, j-1, j+1, \ldots, n'-1$, $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) \in T(R_v)$. This contradicts the fact that the original sequence is (x, y)-irreducible length sequence.

If $\beta_j \leqslant 0$, then $\alpha_j \leqslant \alpha_{j-1}$, hence $s_j + \alpha_{j-1}e_i \geqslant s_j + \alpha_j e_i$. Therefore, $(s_{j-1} + \alpha_{j-1}e_i, s_j + \alpha_{j-1}e_i) \in T(R_v)$ (by construction), $(s_j + \alpha_{j-1}e_i, s_j + \alpha_j e_i) \in T(R_v)$ (by Fact 2.1, i.e. monotonicity of $T(R_v)$) and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$. Hence , by transitivity of $T(R_v)$ $(s_{j-1} + \alpha_{j-1}e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$.

So, we need to define s'_k and α'_k for all $k = 1, \ldots, j-1, j+1, \ldots, n$ to obtain $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) \in T(R_v)$. Let $s'_k = s_k$ for every $k \neq j+1$ and $s_{j+1} = s_{j+1} - \beta_j$. Let $\alpha'_k = \alpha_k$ for every $k \notin \{j, j+1\}$. Then for every $k \neq j+1$ $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) = (s_k + \alpha_k e_i, s_{k+1} + \alpha_k e_i) \in T(R_v)$. Let $\alpha'_{j+1} = \alpha_{j+1} + \beta_j \leqslant \alpha_{j+1}$, then (i) $(s_{j-1} + \alpha_{j-1}e_i, s'_{j+1} + \alpha_{j-1}e_i) \in T(R_v)$, and (ii) $s'_{j+1} + \alpha'_{j+1}e_i = s_{j+1} + \alpha_{j+1}e_i$. So, we only left to show that $(s_{j-1} + \alpha_{j-1}e_i, s'_{j+1} + \alpha_{j-1}e_i) \in T(R_v)$ to complete the proof. Since $\alpha'_{j+1} \leqslant \alpha_{j+1}$, then $s_{j+2} + \alpha'_{j+1}e_i \leqslant s_{j+2} + \alpha_{j+1}e_i$. Hence, by Fact 2.1 $(s_{j+1} + \alpha_{j+1}e_{i,j+2} + \alpha'_{j+1}e_i) \in T(R_v)$. Since $s'_{j+1} + \alpha'_{j+1}e_i = s_{j+1} + \alpha_{j+1}e_i$, then $(s_{j+1} + \alpha_{j+1}e_{i,j+2} + \alpha'_{j+1}e_i) = (s'_{j+1} + \alpha'_{j+1}e_{i,j+2} + \alpha'_{j+1}e_i) \in T(R_v)$.

□

Another difference between QLSARP and $QLiMT$ is that $QLiMT(R)$ does not require elements of sequence $S = s_1, \ldots, s_n$ to be distinct, while QLSARP implies there is no $s_j + \alpha_j e_i = s_k + \alpha_k e_i$.

Lemma 2.7. *For any $(x, y) \notin R_v$ and $(x, y) \in QLiMT(T(R_v))$ and (x, y)-irreducible length sequence S there is not $j \neq k$ such that $s_j + \alpha_j e_i = s_k + \alpha_k e_i$.*

Proof. Without loss of generality assume that $k > j$, then from Lemma 2.6 $\alpha_k > \alpha_j$. Hence $s_k \leqslant s_j$ and $(s_{j-1} + \alpha_{j-1}e_i, s_k + \alpha_{j-1}e_i) \in T(R_v)$. So, S is not (x, y)-irreducible length sequence. □

To prove Theorem 2.1 we need to show that R_v satisfies QLSARP with respect to i-th component if and only if $T(R_v)$ is $QLiMT$-consistent. Recall that $QLiMT$-consistency is equivalent to $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \varnothing$.

Proposition 2.2. *$QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \varnothing$ in and only if R_v satisfies QLSARP with respect to i-th component.*

Proof. ($\Leftarrow$) We assume that there is a contradiction of $QLiMT$-consistency and construct the contradiction of QLSARP from the contradiction of $QLiMT$-consistency. Assume that $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \varnothing$, thus there is $(y, x) \in P(T(R_v))$ and $(x, y) \in QLiMT(T(R_v))$. Since $(x, y) \in QLiMT(T(R_v))$, there is (x, y)-irreducible length sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$ and $s_1 = x$ and $s_n = y$. For $j = 1, \ldots, n$ let $x^{k_j} = s_j + \alpha_j e_i$ and for $j = 2, \ldots, n$ let $\beta_j = \alpha_j - \alpha_{j-1}$. Note that for $j = 1, \ldots, n-2$

(i) $x = x^{k_1} - \alpha_1 e_i$ implies $(y, x) = (y, x^{k_1} - \alpha_1 e_i) \in P(T(R_v))$ and $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$;

(ii) $x^{k_j} \in C$ (all x^{k_j} are chosen points) by construction of R_v;

(iii) All $\beta_j > 0$, by Lemma 2.6;

(iv) All x^{k_j} are distinct points, by Lemma 2.7;

(v) $\alpha_j = \sum_{r=1}^{j} \beta_r + \alpha_1$ and $s_n = y$.

Then $(x^{k_n}, y + (\sum_{r=1}^{j} \beta_r + \alpha_1) e_i) \in T(R_v)$. That is exactly a contradiction of QLSARP.

($\Rightarrow$) Assume that there is a violation of QLSARP, let us show that then $T(R_v)$ is not $QLiMT$-consistent - $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \varnothing$. Let $y = x^{k_1}$ and $x = x^{k_2} - \alpha e_i$, then $(y, x) \in P(T(R_v))$. Let $\alpha_1 = \alpha$, $\alpha_j = \sum_{r=1}^{j} \beta_r + \alpha_1$ and $s_j = x^{k_j} - \alpha_j e_i$ for $j = 2, \ldots, n$. Then $x^{k_1} = x$, $x^{k_n} = y$ and for any $j = 2, \ldots, n$ there is $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$. Therefore, $(x, y) \in QLiMT(T(R_v))$. This implies that $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \varnothing$, i.e. there is a violation of $QLiMT$-consistency.

□

This allows us to apply Theorem 2.5 to complete the proof of Theorem 2.1.

2.7.2 An Extension Theorem for Weak Rationalization

Since Afriat (1967), in revealed preference theory it is common to talk about "weak rationalization" as a weakened concept of the rationalization we mentioned above. Weak rationalization assume that chosen point is only weakly preferred to the points that lie on the upper boundary of the budget set. However, it provokes a gap between the theoretical definition of revealed preference relation and the consumption experiment, because strict part can not be determined in standard way. Let us refine QLSARP to test for the existence of complete, quasi-linear, transitive and monotone extension of the weak rationalization.

For a monotone and compact set B, let ∂B be the **upper boundary** if for any $y \in \partial B$ and $z > y$, $z \notin B$. Denote the weak rationalization generated by finite consumption experiment $(x^t, B^t)_{t=1,\ldots,T}$ by R_w, then $(x^t, y) \in R_w$ if and only if $y \in B^t$ and $(x^t, y) \in P(R_w)$[34] if and only if $y \in B^t \backslash \partial B^t$. Let $T_w(R_w)$ be the **transitive closure of weak rationalization** and $(x, y) \in P(T_w(R_w))$ if and only if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for every $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R_w$ and there is k such that $(s_k, s_k + 1) \in P(R_w)$; $(x, y) \in I(T_w(R_w))$ if and only if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for every $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R_w$ and $(x, y) \notin P(T_w(R_w))$. Then $T_w(R_w)$ is consistent with definition of preference relation and can be decomposed into disjoint strict and indifference parts, such that $T_w(R_w) = P(T_w(R_w)) \cup I(T_w(R_w))$. This allows us to apply similar construction procedure to $T_w(R_w)$ as to $T(R_v)$.

Definition 2.6. *A weak rationalization R_w satisfies **QLGARP** with respect to i-th component if for any sequence of distinct elements $x^{k_1}, ..., x^{k_n} \in C$ and $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$, such that $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T_w(R_w))$ and $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T_w(R_w)$ for $j = 2, \ldots, n-1$, then $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i) \notin T_w(R_w)$.*

A weak rationalization is said to be **weakly acyclic** if it satisfies GARP[35] and from

[34] Recall that by $P(R)$ we denote strict part of the relation

[35] The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the General Axiom of Revealed Preference

Afriat (1967) we know that rationalization is weakly acyclic if and only if it satisfies GARP. Then, the extension result can be immediately achieved from Theorem 2.1

Corollary 2.1. *A weakly acyclic weak rationalization R_w generated by a finite consumption experiment with monotone and compact budgets has an extension that is complete, transitive, monotone and quasilinear in i-the component if and only if R_w satisfies QLGARP with respect to the i-th component.*

The proof of Corollary 2.1 is straight-forward, since the $T_w(R_w)$ is monotone and transitive as well as $T(R_v)$, then Lemma 2.6 and Lemma 2.7 hold.

Proposition 2.3. *$QLiMT(T_w(R_w)) \cap P^{-1}(T(R_w)) = \varnothing$ in and only if R_w satisfies QL-GARP with respect to the i-th component.*

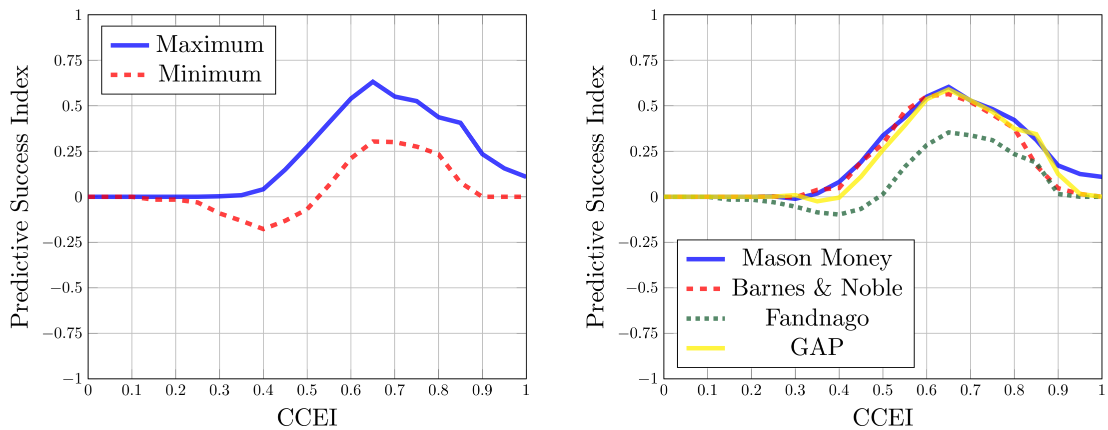
Then, Proposition 2.3 can be proven similarly to the Proposition 2.2. And the proof of Proposition 2.1 is done for an arbitrary preference relation. Then the proof of Corollary 2.1 follows immediately from Theorem 2.5.

2.7.3 More On Quasi-Linearity in Goods

Let us provide more evidence on quasi-linearity in goods. We start from showing the analysis for quasi-linearity in commodities from the experimental data we collected. Recall that using the data from Mattei (2000) and the survey data from ECPF we got the strong positive evidence for quasi-linearity in goods under marginally larger decision errors levels. However, the both data sets have the feature of not sufficient price variation, unlike the experiment we conducted.

Figure 2.18 shows the predictive success indexes for the assumption of quasi-linearity in goods. Figure 2.18(a) shows the predictive success index for the assumption of individual numeraires, that is the numeraire for each person can be different and chosen to either maximize on minimize the predictive success. Figure 2.18(b) shows the predictive success

(**GARP**) if for every integer $m \leqslant n$ and every $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ for $j = 1, \ldots, m-1$ implies $x_{i_1} = x_{i_m}$ or $x_{i_1} \notin B_{i_m} \backslash \partial B_{i_m}$.



(a) Maximum and Minimum: Individual Numeraire (b) By Commodities: Common Numeraire

Figure 2.18: Predictive Succes Index

index for the assumption of common numeraire, that is we suppose each of goods possible to be a numeraire and compute the predictive success index for it. Note that assumption of quasi-linearity in goods (especially assuming the individual numeraire) looks favorable comparing to the assumption of quasi-linearity in money (Figure 2.10). Note as well, that the CCEI at which predictive success index peaks for the assumption of quasi-linearity in goods is higher comparing to the CCEI at which predictive success peaks for quasi-linearity in money. This reduces, the "costs" of rationality, i.e. to look quasi-linear in money person has to give up 35% of income, while to look quasi-linear in goods only 30%.

The support for the assumption of quasi-linearity in goods in we see less support in the experimental data than in the case of experiment from Mattei (2000) and the ECPF (Figures 2.8 and 2.12 respectively). However, for the case of individual numeraires evidence is still supporting the assumption of quasi-linearity, since the lower bound of 95% confidence interval (at CCEI of .7) is still positive (about .15).[36] Therefore, the hypothesis of

[36]The confidence interval is estimated using the results from Demuynck (2014a).

quasi-linear preferences looks favorable comparing to the null hypothesis of uniform random choices. Note, that comparing to the previous evidence on quasi-linearity in goods we provided, this case requires allowing larger decision making error. This "change" in decision making quality should not be surprising, since if we compare it to the results we obtained from Mattei (2000) data, then we will see that this is due to the change of the power of test. Larger price variation and larger amount of budgets in our experiment allows us to have mean (and median) level of CCEI lower for uniform random subjects.

Note that Figure 2.18 compares as well the assumption of common versus individual numeraires. Both assumptions get the empirical support from the data, but the 95% confidence interval lower bound for the assumption of individual numeraires is three times of 95% confidence interval lower bound for the assumption of the common numeraire. While, there is no significant difference between the best predictive success for the assumption of common numeraires and the assumption of individual numeraires, assuming that different people can have different numraires still make the assumption of quasi-linearity in goods more convincing.

Chapter 3: On Pareto Rationalization of Collective Action

3.1 Introduction

This chapter provides a revealed preference characterization of single-valued Pareto rationalizable collective choice functions. A collective choice function is said to be Pareto rationalizable if there are complete individual preference relations (satisfying additional properties if necessary) such that the observed choices are the only Pareto efficient outcomes over the observed choice sets. We show that Pareto rationalizability of single-valued choice functions is equivalent to unitary rationalizability[1] By unitary rationalizability we mean that there is a complete preference relation (satisfying additional properties if necessary), such that the observed choices are the only maximal elements within the choice sets. We show that the result apply regardless of whether individual outcomes or only aggregated collective outcomes are observed. The characterization we construct implies a revealed preference test that can be applied to test the Pareto rationalizability using the observed (panel or experimental) data.

Pareto efficiency of collective choices is a particularly important assumption in household economics. Assuming Pareto efficiency of collective outcomes, Apps and Rees (1988) and Brett (1998) show that the welfare analysis of household behavior using the collective model differs fundamentally from the welfare analysis using the unitary model. Moreover, Chiappori (2010) provides testable implications of transferable utility, assuming household choices to be Pareto efficient.[2] Cherchye et al. (2015a) provide an identification of sharing rules for general collective consumption models assuming the collective choices to be Pareto

[1]We follow the name given by Browning and Chiappori (1998). Sprumont (2000) uses a term "team rationalizability".

[2]Cherchye et al. (2015b) provide a revealed preference test for the transferability of utility under the assumption of Pareto efficiency of household choice.

efficient.

The applications of the revealed preference approach to collective choice differs based on whether we assume individual outcomes are observed.[3] For the case of observed individual outcomes, Sprumont (2000) provides a sufficient condition for the Pareto rationalization and Echenique and Ivanov (2011) provides a characterization of Pareto rationalizable collective choice functions for the two-player case. For the case of unobserved individual outcomes (only aggregated collective outcomes are observed), Cherchye et al. (2010) proposes a test for the weak rationalization[4] of collective choice function.

3.2 Definitions

Let $N = \{1, \dots, n\}$ be the set of agents. For every $j \in N$ let X_j be the set of strategies. Let $X = \times_{j \in N} X_j$ be the set of outcomes (joint actions). Let $\mathcal{B} \subseteq 2^X$ be a **collection of budgets**, denoting every budget from $\mathcal{B}$ by $B \subseteq X$. A **joint choice function** $C : \mathcal{B} \to 2^X$ assigns every budget of $B \in \mathcal{B}$ as non-empty set of chosen points $C(B)$. A joint choice function $C : \mathcal{B} \to 2^X$ is **single-valued** if $|C(B)| = 1$ for every $B \in \mathcal{B}$.

3.2.1 Preferences

A set $R \subseteq X \times X$ is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$. We denote the inverse relation $R^{-1} = \{(x, y) | (y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \backslash I(R)$. We denote the incomparable part by $N(R) = X \times X \backslash (R \cup R^{-1})$. A preference relation R is said to be **complete** if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \varnothing$). A preference relation R is said to be **transitive** if $(x, y) \in R$ and

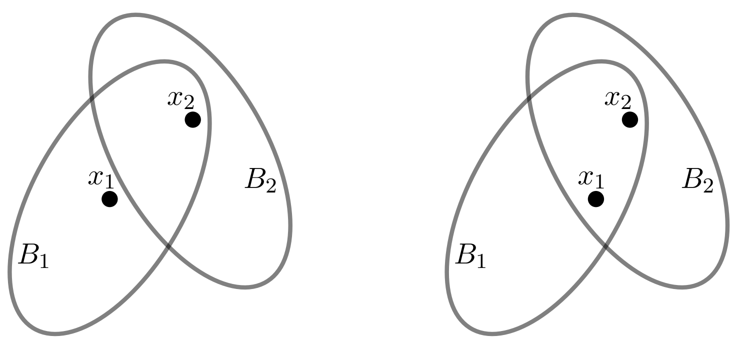
[3] Revealed preference approach pioneered by Samuelson (1938) assumes that the preferences of agents cannot be observed while we can observe their choices. Since Richter (1966) and Afriat (1967) this approach has been extensively used to construct tests of consistency of individual (and collective) behavior with various theories (see Chambers and Echenique (2016) for review).

[4] Unlike the strong rationalization we use, Cherchye et al. (2010) assume that there may be other Pareto efficient outcomes in the observed budgets.

$(y, z) \in R$ implies that $(x, z) \in R$.

Definition 3.1. *A preference relation R' is an* ***extension*** *of R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

Every joint choice function generates a preference relation. Denote an atomic (within budget) revealed preference relation by $R_C(B)$, such that $(x, y) \in R_C(B)$ if and only if $x \in C(B)$ and $y \in B$. Denote the **collective revealed preference relation** by $R_C = \bigcup_{B \in \mathcal{B}} R_C(B)$. A revealed preference relation satisfies **internal consistency** if and only if $R_C(B) \preceq R_C$ for every $B \in \mathcal{B}$.



(a) Internally consistent choice (b) Internally inconsistent choice

Figure 3.1: Internal consistency of collective choice function

Figure 3.1 illustrates the notion of internal consistency. Assume we observe only two budget, B_1 and B_2, and the choice function $C(B_1) = x_1$ and $C(B_2) = x_2$. Figure 3.1(a) provides an example of internally consistent preference relation. Figure 3.1(b) provides an example of preference relation that violates internal consistency, since $(x_1, x_2) \in P(R(B_1))$ and $(x_2, x_1) \in P(R(B_2))$. Hence, $(x_1, x_2) \in I(R_C)$, hence R_C cannot be extension of $R(B_1)$.

3.2.2 Notion of Rationality

We use functions over preference relations to impose the notion of rationality. The simplest example of such a function is *transitive closure* (T), which adds (x, z) to R, whenever there is a finite sequence $x = y_1, \ldots, y_n = z$, such that R contains (y_j, y_{j+1}) for every $j = 1, \ldots, n-1$. Transitive closure has several properties which allow every preference relation which can be extended by its transitive closure to have a complete and transitive extension (see Richter (1966)).

Definition 3.2. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

1. ***monotone*** *if for all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,*

2. ***closed*** *if for all $R \in \mathcal{R}$, $R \subseteq F(R)$,*

3. ***idempotent*** *if for all $R \in \mathcal{R}$, $F(F(R)) = F(R)$,*

4. ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*

5. ***weakly expansive*** *if for any $R = F(R)$ and $N(R) \neq \varnothing$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \preceq F(R \cup S)$.*

Any function $F : \mathcal{R} \to \mathcal{R}$ that is monotone, closed and idempotent is called a **closure**. A closure is algebraic as defined above if any element of the closure can be obtained from applying the closure to a finite subset of the original relation.[5] Weak expansiveness imposex conditions on the fixed points of F.[6] In particular, it guarantees that for every fixed point of F there is a set of non-comparable points (comparisons) which can be added to the fixed point, such that the enlarged relation can be extended by F.

Definition 3.3. *Given a function $F : \mathcal{R} \to \mathcal{R}$, a preference relation R is said to be* ***F-consistent*** *if $F(R) \cap P^{-1}(R) = \varnothing$.*

[5] See e.g. Davey and Priestley (2002), definition 7.12.

[6] Fixed point of F is such R, that $F(R) = R$.

As opposed to the internal consistency we introduced above, F-consistency is an "external consistency" condition. It tests whether a preference relation can be extended by a function F. This external consistency depends on the definition of function F, while internal consistency does not.

Example. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5\}$ and consider the preference relation $R = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$. This relation is not transitive and is not T-consistent (see Figure 3.2) because $(x_1, x_3) \in T(R)$ and $(x_3, x_1) \in P(R)$. On the other hand $R' = \{(x_1, x_2), (x_2, x_3), (x_4, x_5)\}$ is not transitive but is T-consistent. Note that transitivity of R is sufficient but not necessary for T-consistency of R.

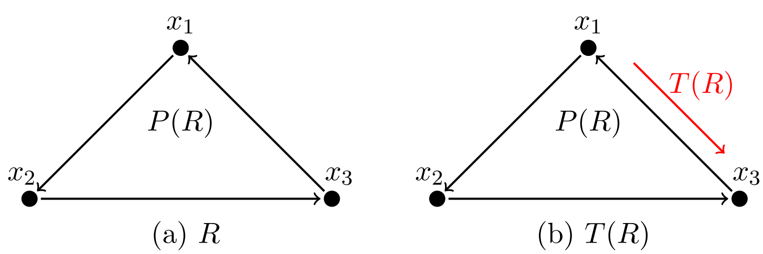


Figure 3.2: Violation of T-consistency

As we previously mentioned, the idea behind F is to impose the desired properties or the "notion of rationality". Further, we assume that every notion of rationality includes completeness and transitivity of preference relations. A function $F : \mathcal{R} \to \mathcal{R}$ induces transitivity if $T(F(R)) = F(R)$ for every $R \in \mathcal{R}$, or equivalently, $F(R)$ is transitive for every $R \in \mathcal{R}$.

3.2.3 Rationalization

Let R_j be the player $j \in N$ individual preference relations of players, then denote by Π the **Pareto relation**, $(x, y) \in \Pi$ if and only if $(x, y) \in R_j$ for all $j \in N$. Equivalently the Pareto relation can be defined as

$$\Pi = \bigcap_{j \in N} R_j.$$

Denote by $M(B, R) = \{x \in B : \forall y \in B,\ (y, x) \notin P(R)\}$ the set of **maximal elements** of budget B according to preference relation R.

Definition 3.4. *A joint choice function $C(\cdot)$ is said to be F**-Pareto rationalizable** if and only if there are complete individual preference relations $R_j = F(R_j)$ for every $j \in N$, such that*

$$C(B) = M(B, \Pi)$$

for every $B \in \mathcal{B}$, where $\Pi = \bigcap_{j \in N} R_j$.

3.3 Results

Theorem 3.1. *Let F be a weakly expansive algebraic closure. A single-valued joint choice function $C : \mathcal{B} \to X$ is F-Pareto rationalizable if and only if R_C satisfies internal consistency and F-consistency.*

Let us make several remarks about the main result of the chapter. First, it does not directly depend on the number of players, since the conditions are equivalent to the unitary rationalizability. By unitary rationalizability, we mean that there is a single-player ("dictator") who is making decisions for the group according to his or her preferences. In this sense, the result has some of the flavor of Arrow's theorem. Second, Theorem 3.1 assumes the individual outcomes to be observed. Later in the chapter, we show how the result can be applied for the case in which we observe only aggregated consumption.

Further, we consider two important cases of Theorem 3.1, which are dictated by two major notions of rationality. If outcomes are deterministic it is usually assumed that a rational individual has transitive and complete preferences. If outcomes are stochastic (lotteries), it is usually assumed that a rational individual has complete, transitive and independent preferences. We call the first type of rationalization T-Pareto rationalization and the second one TI-Pareto rationalization, named after two closures T and TI which impose the corresponding notions of rationality. Recall that to characterize each of the Pareto rationalizations, it is enough to define the correct F (the one that induces desired properties) and show that it is a weakly expansive algebraic closure.

3.3.1 T-Pareto Rationalization

Recall that a preference relation R is said to be **transitive** if $(x, y) \in R$ and $(y, z) \in R$ implies that $(x, z) \in R$. Denote the **transitive closure** by

$$T : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, \ldots, s_m$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \ldots, m-1$, and $s_1 = x$ and $s_m = y$. For the proof that transitive closure is a weakly expansive algebraic closure which induces transitivity see Demuynck (2009). Hence, the following corollary is immediate.

Corollary 3.1. *A single-valued joint choice function $C(\cdot)$ is T-Pareto rationalizable if and only if R_C satisfies internal consistency and is T-consistent.*[7]

3.3.2 TI-Pareto Rationalization

Further, we need to assume that X is a mixture space, since this structure is required to define the independence. A preference relation R satisfies independence if $(x, y) \in R$ implies that for every $x \in X$ and for every $\alpha \in [0, 1]$, $(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) \in R$. Denote the **transitive and independent closure** by

$$TI : \mathcal{R} \to \mathcal{R},$$

[7] Freer and Martinelli (2016) show that internal consistency and T-consistency of a preference relation are equivalent to the existence of the utility function that represents the complete fixed point extension. Moreover, if collective revealed preference relation is obtained from the finite amount of observations that this utility function is continuous.

where $(x, y) \in TI(R)$ if and only if there are finite sequences $x = s_1, \ldots, s_{m+1} = y$, $z_1, \ldots, z_m$ and $\alpha_1, \ldots, \alpha_m$ such that for each $j = 1, \ldots, m$ $\alpha \in [0,1]$ and $(\alpha s_j + (1-\alpha)z_j, \alpha s_{j+1} + (1-\alpha)z_j) \in R$. For the proof that $TI(R)$ is an algebraic closure that induces transitivity and independence, see Demuynck and Lauwers (2009). The proof that $TI(R)$ is weakly expansive is in the Appendix.

Corollary 3.2. *A single-valued joint choice function $C(\cdot)$ is TI-Pareto Rationalizable if and only if R_C satisfies internal consistency and is TI-consistent.*

3.3.3 Unobservable Individual Outcomes

Let us introduce additional notation to state the problem with unobserved individual outcomes. We follow the standard framework of this problem (see, for instance, Cherchye et al. (2010)) to obtain the revealed preference characterization. Let $\mathbb{R}^L_+$ be a set of aggregated outcomes. We assume budgets to be linear; that is, every budget is uniquely determined by the vector of prices $p \in \mathbb{R}^L_+$ and we assume income to be known.[8] A finite consumption experiment is a set $E = \{(x^t, p^t)\}_{t=1}^T$, where p^t are the prices which determine the budget, x^t is an aggregated outcome and $x \in B^t$ if $p^t x \leqslant p^t x^t$.

Definition 3.5. *A finite consumption experiment satisfies **Strong Axiom of Revealed Preference (SARP)** if for every sequence of distinct elemenents $x^{t_1} \in B^{t_2}, \ldots, x^{t_{m-1}} \in B^{t_m}$ implies $x^{t_m} \notin B^{t_1}$.*

Recall that individual outcomes are not observed; therefore, we need to be able to recover them using only observed aggregated outcome.

Definition 3.6. *A vector of individual outcomes $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ is said to be a **decomposition** of x if $\sum_{i=1}^n \hat{x}_i = x$.*

Then the set of collective outcomes is $(\mathbb{R}^L_+)^n$, where n is a number of agents. Note that $\hat{x} \in B^t$ if and only if $p^t x \leqslant p^t x^t$. This is a restrictive assumption which requires every

[8]Further result is also true if we relax the assumption and require budgets to be just compact and comprehensive.

decomposition which is affordable (at the aggregate level) to be in the budget set. A finite consumption experiment E exhibits **strong collective rationalizability** if and only if there are such decompositions of x^t and complete and transitive preference relations R_j (over $(\mathbb{R}^L_+)^n$) that $M(B^t, \Pi) = \hat{x}^t$.

Corollary 3.3. *A finite consumption experiment E exhibits strong collective rationalizability if and only if it satisfies SARP.*

Note that Corollary 3.3 states that collective strong rationalization does not have an additional empirical content comparing to unitary strong rationalization. Again this is not true for weak rationalization (see Chiappori (1988)).

3.4 Concluding Remarks

We consider only single-valued collective choice functions, leaving the question open for set-valued functions. However, a more general case is not computationally feasible for more than two players.[9] The complete proof can be found in Demuynck (2014b) and we further provide a sketch of the proof. Assume the set of alternatives is finite and consider a problem of T-Pareto rationalization. Moreover, we have observed all the possible choice sets and the observed relation is a partial order. Then, the solution candidate is a partial order. However, it is non-trivial question whether this partial order can be obtained as an intersection of no more than n complete orders. This is the dimension problem (introduced by Dushnik and Miller (1941)) and is not computationally feasible if there are more than two players (see Yannakakis (1982) for the proof).

3.5 Appendix

3.5.1 Proof of Theorem 3.1

Let us start by introducing some additional notations and preliminary results.

[9] By this we mean that problem is NP-complete.

Proposition 3.1 (Theorem 2 from Demuynck (2009)). *Let $F : \mathcal{R} \to \mathcal{R}$ be a weakly expansive algebraic closure. R has a complete extension $R^* = F(R^*)$ if and only if R is F-consistent.*

Proof can be found in Demuynck (2009).

The following lemma uses the notion of team rationalizability, introduced by Sprumont (2000) and assumes that all the players act as the team which is driven by the similar preference relation. A joint choice function $C(\cdot)$ is **F-team rationalizable** if and only if there exists a complete preference relation $R^* = F(R^*)$ such that $C(B) = M(B, R^*)$ for every $B \in \mathcal{B}$.

The proof consists of the following parts.

1. We show that F-Pareto rationalizability is equivalent for F-team rationalizability

2. We show that F-team rationalibility is equivalent to internal and F-consistency of R_C.

This implies that F-Pareto rationalizability, F-team rationalizability and consistency of R_C are all equivalent. Therefore, it is enough to conclude the proof.

(1) Let us start by showing the equivalence between F-Pareto and F-team rationalizability.

Lemma 3.1. *If $F : \mathcal{R} \to \mathcal{R}$ is a monotone function and $R_j \preceq F(R_j)$ for all $j \in N$, then $\Pi \preceq F(\Pi)$.*

Proof. Assume on the contrary that there is $(x, y) \in P^{-1}(\Pi) \cap F(\Pi)$. By definition of Π $(y, x) \in P^{-1}(\Pi)$ implies that $(y, x) \in R_j$ for every $j \in N$. Moreover, there is $i \in N$, such that $(y, x) \in P(R_i)$. Note that $\Pi \subseteq \mathbb{R}_i$, then by monotonicity, $(x, y) \in F(\Pi)$ implies that $(x, y) \in R_i$. This contradicts the fact that $R_j \preceq F(R_j)$ for all $j \in N$. □

Lemma 3.2. *If $|M(B, R)| = 1$ for every $B \in \mathcal{B}$ and $R \preceq R'$, then $M(B, R) = M(B, R')$ for every $B \in \mathcal{B}$.*

Proof. Recall that the set of maximal elements is the collection of points x such that there is no $y \in B$ and $(y, x) \in P(R)$. At the same time, if x is the unique maximum point, then for every other point $y \in B$ it should be the case that $(x, y) \in P(R)$. Now we can employ the fact that $R \preceq R'$, then $P(R) \subseteq P(R')$, therefore for every $B \in \mathcal{B}$ and $x \in M(B, R)$ and every $y \in B$ there is $(x, y) \in P(R')$, therefore, x is the unique maximal point of B according to R'. □

Lemma 3.3. *Let $C : \mathcal{B} \to 2^X$ be a single-valued choice function. C is F-Pareto Rationalizable if and only if it is F-team rationalizable.*

Proof. **(⇒)** Lemma 3.1 guarantees that Π is F-consistent, then Proposition 3.1 guarantees that there is a complete extension of Π, which is a fixed point of $F(R)$. Then, by Lemma 3.2 we know that $M(B, \Pi) = M(B, R)$ for every $B \in \mathcal{B}$. Recall that $M(B, \Pi) = C(B)$ for every $B \in \mathcal{B}$, hence, $M(B, R) = C(B)$ for every $B \in \mathcal{B}$.

(⇐) If there is a complete preference relation $F(R^*) = R^*$ that extends R_C, then assume for all $j \in N$ $R_j = R^* = F(R_j)$. Hence, the Pareto relation coincides with R^* and extends R_C. □

(2) Sufficiency of internal and F-consistency of R_C for team rationalizability follows from Lemma 3.2 and Proposition 3.1. Therefore, we concentrate on the necessity of conditions over R_C.

We need to prove two separate conditions. First, we show that internal consistency is necessary for F-team rationalizability of a single-valued joint choice function. Second, we show that $R_C \preceq R^*$ (for every R^* which guarantees F-team rationalizability). This allows us to claim that F-consistency is necessary, because F-team rationalizability implies that there is a complete fixed point extension of R_C, then R_C is F-consistent (see Proposition 3.1).

Lemma 3.4. *Let $C : \mathcal{B} \to 2^X$ be a single-valued choice function and let F be a weakly* *expansive algebraic closure. If $C : \mathcal{B} \to 2^X$ is F-team rationalizable, then R_C satisfies internal consistency.*

Proof. Assume to the contrary there is $B \in \mathcal{B}$ such that $(y,x) \in P(R_C(B))$ and $(x,y) \in R_C$. The first claim implies that $y \in C(B)$ and $x \in B$. At the same time, there is $B' \in B$ such that $x \in C(B')$ and $y \in B'$. Hence, there is no $z \in B$ which strictly dominates the y, at the same time by completeness of R^* this implies that $(y,z) \in P(R^*)$ for every $z \in B$. This contradicts the fact that $x \in C(B')$, because $y \in B'$ and $(y,x) \in P(R^*)$. □

Before we prove the following lemma, let us make an observation: if $C : \mathcal{B} \to 2^X$ is a single-valued joint choice function and R_C is internally consistent, then $R_C = P(R_C)$. Note that in general $R_C \backslash P(R_C) = I(R_C)$; thus, the only contradiction which can appear to the claim above is $x, y \in B$ and $x, y \in B'$ for some $B, B' \in \mathcal{B}$. Moreover, to make those points indifferent, x has to be chosen from one set and y has to be chosen from a different set. This directly implies a contradiction to internal consistency.

Lemma 3.5. *Let $C : \mathcal{B} \to 2^X$ be a single-valued choice function and let F be a weakly expansive algebraic closure. If $C : \mathcal{B} \to 2^X$ is F-team rationalizable, then $R_C \preceq R^*$.*

Proof. Recall that $C : \mathcal{B} \to 2^X$ is a single valued and R^* is a complete relation, therefore, for every $B \in \mathcal{B}$, $x = C(B)$ and $y \in B \backslash C(B)$ we have $(x,y) \in P(R^*)$, because R^* is complete and $C : \mathcal{B} \to 2^X$ is single-valued. Note that this implies that $P(R_C) \subseteq P(R^*)$. Recall that F-team rationalizability implies internal consistency of R_C (see Lemma 3.4), hence, $R_C = P(R_C) \subseteq R^*$. This completes the proof that $R_C \preceq R^*$. □

3.5.2 Proof of Corollary 3.2

As we previously mentioned, the only thing left to complete the proof of Corollary 3.2 is that $TI(R)$ is weakly expansive.

Lemma 3.6. *$TI : \mathcal{R} \to \mathcal{R}$ is weakly expansive.*

Proof. Take a point $(x, y) \in N(R)$ and consider relation $R' = R \cup \{(x, y)\}$. We need to show that $P^{-1}(R') \cap TI(R') = \varnothing$. Assume to the contrary that there is $(z, w) \in P^{-1}(R') \cap TI(R)$, that is $(w, z) \in P(R')$.

Case 1: $(w, z) \neq (x, y)$. At the same time $(z, w) \in TI(R')\backslash R$ implies that there are sequences $S = s_1, \ldots, s_n$, $L_1, \ldots, L_n$ and $\alpha_1, \ldots, \alpha_n$ such that for each $j = 1, \ldots, n$ $\alpha \in [0, 1]$ and $(\alpha s_j + (1-\alpha) z_j, \alpha s_{j+1} + (1-\alpha) z_j) \in R$. Then, there is m such that $s_m = x$ and $s_{m+1} = y$. So, we can reorder existing sequences in the following way:

$$(y + \alpha_{m+1} L_{m+1}, s_{m+2} + \alpha_{m+1} L_{m+1}), \ldots, (z, w), \ldots$$

$$\ldots, (s_{m-1} + \alpha_{m-1} L_{m-1}, x + \alpha_{m-1} L_{m-1})$$

This implies that $(y, x) \in TI(R) = R$, that is a contradiction to the fact, that $(x, y) \in N(R)$.

Case 2: $(w, z) = (x, y)$. Hence, $(y, x) \in TI(R')$, this implies that there are sequences $S = s_1, \ldots, s_n$, $L_1, \ldots, L_n$ and $\alpha_1, \ldots, \alpha_n$ such that for each $j = 1, \ldots, n$ $\alpha \in [0, 1]$ and $(\alpha s_j + (1 - \alpha) z_j, \alpha s_{j+1} + (1 - \alpha) z_j) \in R$. Then, if the sequence does not contain $s_j = x$ and $s_{j+1} = y$ we obtain the contradiction, since $(y, x) \in R = TI(R)$ (this implies that $(x, y) \notin N(R)$). If the sequence contains $s_j = x, s_{j+1} = y$, then we can find a subsequence from y to x, hence, $(y, x) \in R = TI(R)$ (this implies that $(x, y) \notin N(R)$). □

3.5.3 Proof of Corollary 3.3

Let a unitary revealed preference relation R_E contain (x, y) if and only if $x = x^t$ and $y \in B^t$.

Lemma 3.7. *R_E is internally consistent and T-consistent if and only if E satisfies SARP.*

Proof can be found in Freer and Martinelli (2016).

Let us start by proving sufficiency of the SARP for strong collective rationalizability with the following lemma.

Lemma 3.8. *If R_E is internally and T-consistent then there is a decomposition such that R_C is internally and T-consistent.*

Proof. Take an arbitrary decomposition for every $\hat{x}^t$ for every x^t (decomposition should be similar for similar x). This allows us to construct a collective revealed preference relation R_C. Hence, let us show that R_C is internally and T-consistent and apply Theorem 3.1. Note that $(\hat{x}, \hat{y}) \in R_C$ if and only if $(x, y) \in R_E$ or $x = y = x^t$ for some t.

Assume there is a violation of internal consistency; that is, $(\hat{x}, \hat{y}) \in R_C$ and $(\hat{y}, \hat{x}) \in P(R_C(B^t))$. This implies that $x = x^t$ for some t and $y = x^s$ for some s and $x^s \neq x^t$, hence, both pairs have to belong to R_E, but this contradicts the fact that R_E is internally consistent.

Assume there is a violation of T-consistency; then there is $(\hat{y}, \hat{x}) \in P(R_C)$ (this implies that $(y, x) \in P(R_E)$) and $(\hat{x}, \hat{y}) \in T(R_C)$. The latter implies that there is a sequence $S = \hat{s}_1, \ldots, \hat{s}_n$ such that $(\hat{s}_j, \hat{s}_{j+1}) \in R_C$. This implies that $(s_j, s_{j+1}) \in R_E$. Hence, $(x, y) \in T(R_E)$ contradicts the T-consistency of R_E. □

Let us move onto proving necessity. Note that Theorem 3.1 implies that if E exhibits strong collective rationalizability, then there is a decomposition such that R_C is internally and T-consistent.

Lemma 3.9. *R_C is internally and T-consistent then R_E is internally and T-consistent.*

Proof. Recall that $(\hat{x}, \hat{y}) \in R_C$ if and only if $(x, y) \in R_E$ or $x = y = x^t$ for some t.

Assume there is a violation of internal consistency of R_E, i.e. $x^s \in B^t$ and $x^t \in B^s$ for some $s \neq t$, this obviously implies violation of internal consistency for R_C, because $\hat{x}^s \in B^t$ and $\hat{x}^t \in B^s$.

Assume there is a violation of T-consistency of R_E, then there is a sequence $S = s_1, \ldots, s_n$ such that $s_{j+1} \in B^j$ and $s^n \in B^1$. Note that all elements of the sequence are chosen points, therefore, there is a unique mappint between s_j and $\hat{s}_j$, therefore, this causes violation of T-consistency for R_C. □

This concludes the proof.

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