Helly-type results for k-systems

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Abstract

HELLY-TYPE RESULTS FOR K-SYSTEMS

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Theorems of the form "if every m objects from a set has a certain property P, then the entire set has the property P" are known as Helly-type theorems, after Eduard Helly's theorem of 1923. Many transversal theorems on collections of bodies in $\mathbb{R}^n$ have been stated and proved in this Helly-type style. In this paper, we primarily focus on the planar case of transversals providing support to a collection of convex bodies.

Previous authors have shown that given a sufficiently large family of pairwise disjoint convex planar bodies with the property that any three bodies share a support line, the entire family shares at least one support line. Unfortunately, *sufficiently large* does not provide an exact number of bodies necessary to ensure that if any three bodies have the *support property*, then the entire family does.

Using what is called a (special) set-system of words and letters to model these disjoint convex planar bodies, the existing literature shows us that the minimum number of bodies necessary to ensure the aforementioned support property is no more than 143. In this paper we do not provide the exact number, but rather a generalized framework for which this planar case is a special case.

Consistent with earlier authors, we too use a set-system of words and letters. Within our framework, these systems are known as k-systems and *special* k-systems. We provide a few Helly-type results on both k-systems and special k-systems.

Chapter 1: Helly-type theorems and transversals

Eduard Helly (1884-1943) was born in Vienna. He received his PhD degree in 1907 under W. Wirtinger. In 1914 Helly joined the Austrian Army. Wounded by the Russians, he was taken as a prisoner to Siberia. He returned to Vienna in 1920 and married a year later. In 1938, Helly and family fled from Nazi controlled Austria and emigrated to the United States. Only after receiving a recommendation from Albert Einstein was Helly able to receive a faculty position at Paterson Junior College in New Jersey, in 1939. In 1943 Helly had a second, and fatal, heart attack.

> Had Helly succeeded in staying in the mainstream of mathematics, as an academician who published and participated in seminars, he would have undoubtedly have capitalised on his earlier contributions. He not only might have seen to it that proper credit should be ascribed, but it is likely that he would have extended his results further. ... In most careers there are some disappointments and failures, but Helly's career derailed early, and life never gave him a chance to get back on the right track. (Hochstadt [8])

Although results attributed to Helly are few, one of the theorems known as "Helly's Theorem" lies at the heart of combinatorial geometry.

Helly's Theorem. *Let $\mathcal{K}$ be a family of convex sets in $\mathbb{R}^d$, and suppose $\mathcal{K}$ is finite or each member of $\mathcal{K}$ is compact. If every $d+1$ or fewer members of $\mathcal{K}$ have a common point, then there is a point common to all members of $\mathcal{K}$.(Helly [7])*

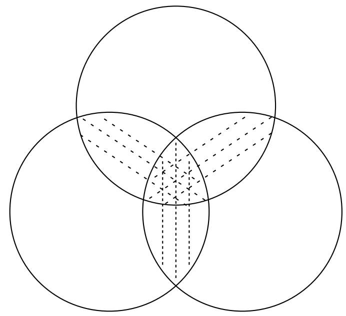


Figure 1.1: Illustration for Helly's Theorem (planar case)

Here we do not focus on specifically Helly's Theorem, but rather stylistically similar theorems. Theorems of the form "if every m objects from a set has a certain property P, then the entire set has the property P" are known as Helly-type theorems. This comes from the tradition following the mentioned Helly's Theorem, published in 1923. Consider the following examples which illustrate these types of theorems. The first is a very specific case of Helly's Theorem.

Example 1. If each two segments of a family of segments (in the line) have a common point, then all the segments of the family have a common point (Hadwiger [6]).

Example 2. If a family of ovals[1] is such that each two of its members have a common point, then through each point of the plane there is a line that intersects all the ovals of the family (Hadwiger [6]).

Definition 1.1. By a *line transversal* in the plane, we mean a line which intersects each member of a collection of sets in $\mathbb{R}^2$.

Many theorems regarding transversals can be stated in this Helly-type fashion. Here we present one such example which is similar to our later discussions.

[1] By an *oval* is meant a bounded, closed convex set.

Definition 1.2. A family of ovals, $\mathcal{F}$, in $\mathbb{R}^2$ is said to be *totally separable* if the following is true for each b_i, b_j, $i \neq j$, from $\mathcal{F}$:

1) $b_i \cap b_j = \emptyset$

2) there is a line that separates b_i, b_j without intersecting any oval in $\mathcal{F}$.

Example 3. If each three members of a totally separable system of ovals admit a common transversal, then there is a transversal common to all of the ovals of the system (Hadwiger [6]).

Considering the magnitude of existing literature on the theory of transversals (see for example, Danzer, Grünbaum, Klee [3], Eckhoff [4], Goodman, Pollack, Wenger [5], Holmsen [9]), a complete discussion of transversals is well beyond the scope of this paper. Instead, we focus our discussion on line transversals in the plane. Specifically, we focus on theorems stated in this Helly-type style with regards to a special kind of line-transversal in the plane.

Chapter 2: Support lines

The main result of this chapter is Theorem 2.1. Before proving this theorem, however, a few definitions are necessary. We also establish a few supporting Lemmas prior to the proof.

Theorem 2.1. *Given disjoint convex bodies A and B in the plane, there are exactly four lines supporting both A and B. Two of these lines separate A and B, and the other two support A and B from the same side.*

Figure 2.1 illustrates Theorem 2.1, showing us two disjoint convex planar bodies and their four shared supports.

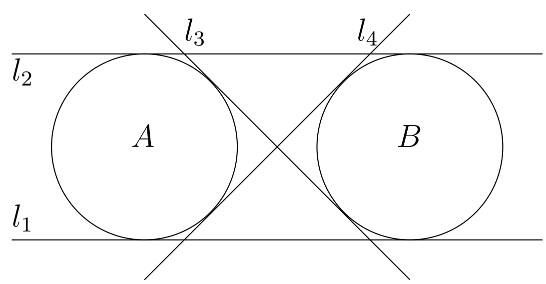
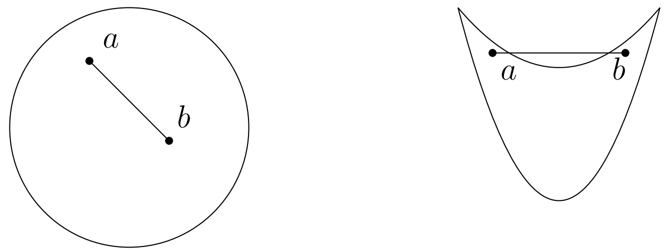


Figure 2.1: Four support lines for two disjoint convex bodies

Definition 2.1. A set $S \subset \mathbb{R}^2$, S is *convex* if for every $a, b \in S$, the entire line segment connecting a and b is also in S (Figure 2.2).

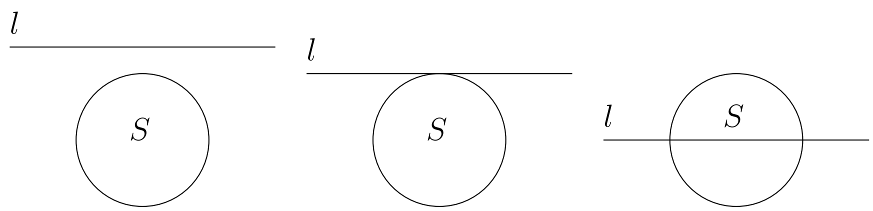
Definition 2.2. By a *planar convex body*, we mean a compact convex set in the plane with non-empty interior.

Definition 2.3. A line-transversal of a convex body is said to *support* the body if it does not intersect the interior of the body (Figure 2.3).



(a) A convex set (b) Not a convex set

Figure 2.2: Convex and non-convex sets



(a) l does not support S (b) l supports S (c) l does not support S

Figure 2.3: Support and non-support lines

Definition 2.4. We say that a support line provides *definite common support* to a pair of bodies if they lie in the same half-plane defined by the support line.

Definition 2.5. A support line provides *separating support* to a pair of bodies if the bodies lie in opposite half-planes defined by the support line.

Definition 2.6. A line is said to *strictly separate* two disjoint convex bodies in the plane if it defines two open half-planes each of which contains one of the two convex bodies (Figure 2.4).

Lemma 2.2. *Given disjoint convex bodies A and B in the plane, there are points $a \in A$ and $b \in B$ such that*

$$\|a - b\| = \min\{\|x - y\| \mid x \in A, y \in B\}.$$

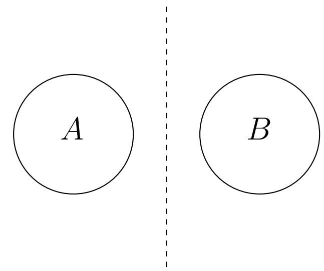


Figure 2.4: A line that strictly separates A and B

Proof. Since A and B are disjoint and non-empty, $\|x - y\| > 0$ for all $x \in A$ for all $y \in B$. Let

$$\delta = \inf\{\|x - y\| \mid x \in A, y \in B\}.$$

Thus, there are sequences of points $x_1, x_2, \ldots \in A$ and $y_1, y_2, \ldots \in B$ such that $\lim_{i\to\infty} \|x_i - y_i\| = \delta$. By the compactness of A and B, we know that there is a subsequence of $x_{i_1}, x_{i_2}, \ldots$ of $x_1, x_2, \ldots$ that converges to a point $a \in A$. Similarly, there is a subsequence $y_{j_1}, y_{j_2}, \ldots$ of $y_{i_1}, y_{i_2}, \ldots$ that converges to a point $b \in B$. So,

$$\delta = \lim_{k\to\infty} \|x_{j_k} - y_{j_k}\| = \|a - b\|. \qquad \square$$

We have said nothing of the uniqueness of the $a \in A$ and $b \in B$ in Lemma 2.2. Figure 2.5 gives an example of convex bodies A and B and distinct pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ such that

$$\|a_1 - b_1\| = \|a_2 - b_2\| = \min\{\|x - y\| \mid x \in A, y \in B\}.$$

Lemma 2.3. *Given disjoint convex bodies A and B in the plane, there is a line that strictly separates A and B.*

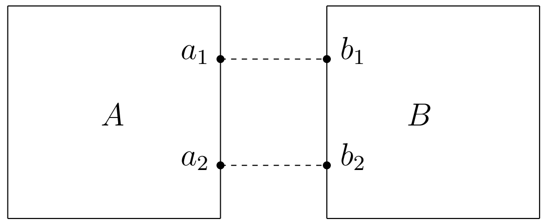


Figure 2.5: Closest points are not unique

Proof. By Lemma 2.2, we can select points $a \in A$ and $b \in B$ such that

$$\|a - b\| = \min\{\|x - y\| \mid x \in A, y \in B\}.$$

Consider the line segment connecting a and b. We claim that any line that perpendicularly intersects the open line segment (a, b) strictly separates A and B (Figure 2.6).

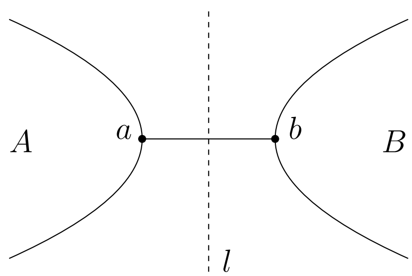


Figure 2.6: A line strictly separating A and B

It is the convexity of A and B that ensures neither intersects l. To see this, suppose to the contrary that there is an $x \in A$ which is also in the half-plane containing b, determined by l. Since A is convex, then $[a, x] \subset A$. This means that we can find a point $p \in [a, x]$ sufficiently close to a which is closer to b than a is (see Figure 2.7). This is in direct contradiction to our selection of a and b as,

$$\|a - b\| = \min\{\|x - y\| \mid x \in A, y \in B\}.$$

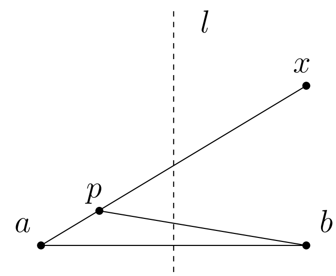


Figure 2.7: A cannot intersect l

Thus, A has no points in common with l (similarly for B). □

At this point, it is useful to introduce two closely related functions. Let $f_C(\alpha) : C \to \mathbb{R}$ and $g_C(\alpha) : C \to \mathbb{R}$, where C is a convex planar body in $\mathbb{R}^2$. Define $f_C(\alpha)$ to be the maximum value of the x-coordinate for any point in C after a rotation of the plane about some fixed point of angle α. Similarly, define $g_C(\alpha)$ to be the minimum value of the x-coordinate for any point in C after the same rotation. Figure 2.8 illustrates a rotation of 2 disjoint planar bodies, A and B, of angle π.

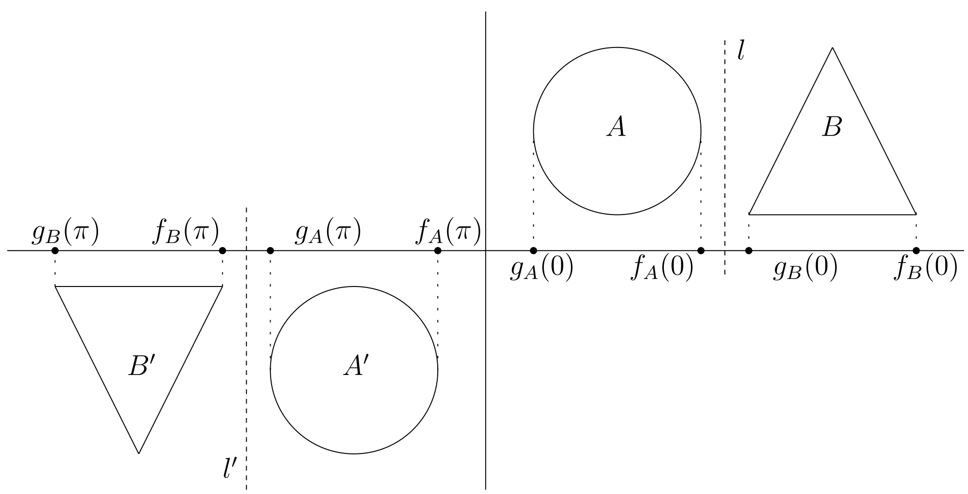


Figure 2.8: Rotating the plane by π

We now define four difference functions, $\delta_i(\alpha),\ i = 1, 2, 3, 4$, for two disjoint convex planar bodies A and B:

$$\delta_1(\alpha) = f_B(\alpha) - f_A(\alpha), \quad \delta_2(\alpha) = g_B(\alpha) - g_A(\alpha)$$

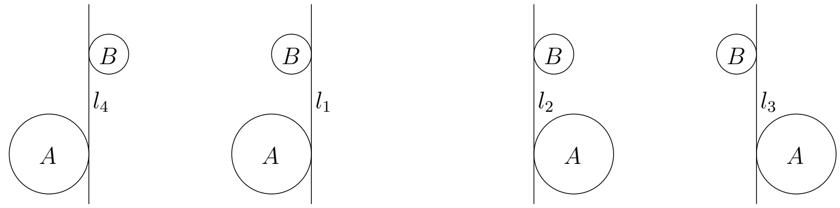
$$\delta_3(\alpha) = f_B(\alpha) - g_A(\alpha), \quad \delta_4(\alpha) = g_B(\alpha) - f_A(\alpha).$$

It is important to notice that since $f_C(\alpha)$ and $g_C(\alpha)$ are continuous functions, then so also are $\delta_i(\alpha),\ i = 1, 2, 3, 4.$

Given two disjoint convex planar bodies, A and B, oriented as in Figure 2.8, we let $\delta_i(0) = d_i$. From this we see that $\delta_i(\pi) = -d_i$. From calculus (and the continuity of $\delta_i(\alpha)$) we know that $\delta_i(\alpha)$ takes on every value between $-d$ and d. Thus, there is some α_i such that $\delta_i(\alpha_i) = 0$. These facts will be critical in the proof of Theorem 2.1.

Proof of Theorem 2.1. We first tackle the issue of existence of four supporting lines for disjoint convex bodies A and B. These four lines, however, are given to us from our $\delta_i(\alpha)$ functions.

By rotating the plane and stopping at the first instance where $\delta_1(\alpha) = 0$ we find a vertical line providing support to both A and B (at $x = f_B(\alpha) = f_A(\alpha)$). Proceeding similarly for $\delta_2(\alpha), \delta_3(\alpha),\ \delta_4(\alpha)$, we find four values of α indicating lines that provide support to both A and B. Due to the way $\delta_i(\alpha)$ is defined, we see that two of these lines provide definite common support, and two provide separating support (See Figure 2.9).



(a) $g_B(\alpha_4) - f_A(\alpha_4) = 0$ (b) $f_B(\alpha_1) - f_A(\alpha_1) = 0$ (c) $g_B(\alpha_2) - g_A(\alpha_2) = 0$ (d) $f_B(\alpha_3) - g_A(\alpha_3) = 0$

Figure 2.9: Four values for α yielding four support lines

After combining Figure 2.9 into one figure, we see illustrated in Figure 2.10 four support lines for two disjoint convex bodies in the plane.

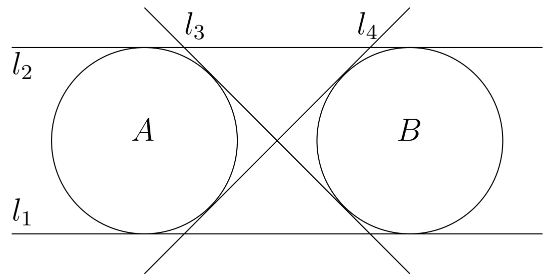


Figure 2.10: Four support lines

Here we claim that these are the only support lines for disjoint convex bodies A and B. We proceed with proof by contradiction. Assume to the contrary that there is at least one more line providing support. If this fifth line provides support, then it must provide definite common support or separating support to A and B.

First we check to see if our new support line can provide definite common support. Figure 2.11 illustrates the following construction. Let $a_1 \in l_1 \cap A$, $a_2 \in l_2 \cap A$, $b_1 \in l_1 \cap B$, and $b_2 \in l_2 \cap B$. Let w_1 be the unique line determined by a_1 and a_2. Let w_2 be the unique line determined by b_1 and b_2. Let l_c be a line containing two closest points of A and B (Lemma 2.2).

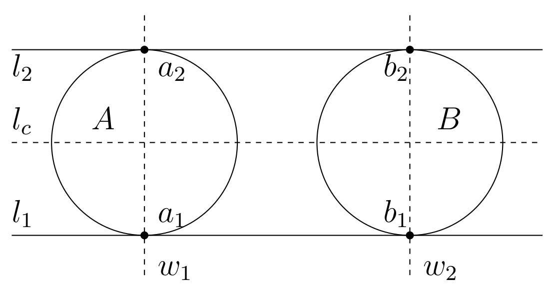


Figure 2.11: Illustration for proof of Theorem 2.1

If l_5 is a fifth line providing definite common support to A and B, then it must intersect w_1 and w_2 in the same half-plane determined by l_c. Clearly, $l_5 \cap w_1$ cannot occur in the interior of A, so must occur outside the interior. We also note that if l_5 is to provide support, it cannot intersect w_1 between a_1 and a_2 (similarly, l_5 cannot intersect w_2 between b_1 and b_2). Thus, the only eligible points that exist in $A \cap w_1 \setminus \text{int} A$ are a_1 and a_2 (similarly, the only eligible points that exist in $B \cap w_2 \setminus \text{int} B$ are b_1 and b_2). Thus, if l_5 provides definite common support to A and B, then either $a_1, b_1 \in l_5$ or $a_2, b_2 \in l_5$. In either case, l_5 is not a new support line and so cannot be a new line providing definite common support.

So, we are left to check whether l_5 can provide separating support to A and B. If l_5 is to provide separating support, then it must cross some line which connects two closest points (one from A and one from B). One of these lines is shown in Figure 2.11 as l_c.

Although we only consider the situation where l_5 separates A and B into the same half-planes as l_3, the same argument holds as if l_5 separated A and B in the same way as l_4. See Figure 2.12 for the following construction. Let $p_3 = l_3 \cap l_c$ and $p_5 = l_5 \cap l_c$. Let a, b be fixed closest points where $a \in A \cap l_c$ and $b \in B \cap l_c$. Let α_3 be the acute angle formed from l_3 and l_c. Let α_5 be the acute angle formed from l_5 and l_c.

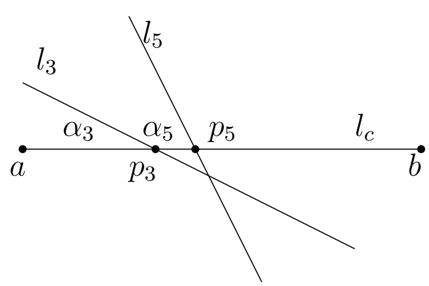


Figure 2.12: Construction for proof of Theorem 2.1

We now consider two cases (Figure 2.13).

Case I. $\text{dist}(a, p_5) \leq \text{dist}(a, p_3)$.

If $\alpha_5 \leq \alpha_3$, then l_5 intersects the interior of A. If $\alpha_5 > \alpha_3$, a simple comparison of triangles shows that l_5 does not provide support to B.

Case II. $\text{dist}(a, p_5) > \text{dist}(a, p_3)$.

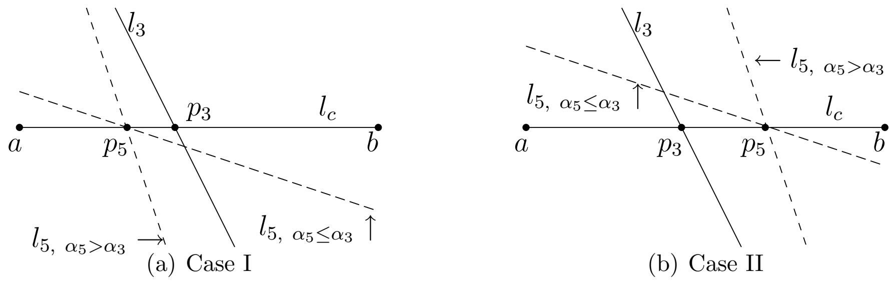


Figure 2.13: Cases for proof of Theorem 2.1

If $\alpha_5 \leq \alpha_3$, then l_5 intersects the interior of B. If $\alpha_5 > \alpha_3$, a simple comparison of triangles shows that l_5 does not provide support to A.

In either case, l_5 cannot be a new separating support line if $\alpha_3 \neq \alpha_5$ or $p_3 \neq p_5$. Thus, if l_5 is a separating support line, then $l_5 = l_3$ (or similarly $l_5 = l_4$). □

From Theorem 2.1, we have that any two disjoint convex bodies in the plane have exactly four support lines. This can be extended to larger collections of convex bodies as shown in Dawson [1]:

Theorem 2.4. *For any family of pairwise disjoint convex planar bodies,*

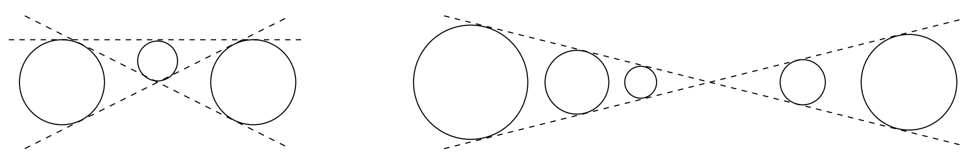
1) any two bodies have exactly four common support lines;

2) any three bodies have at most three common supporting lines;

3) any five bodies have at most two common support lines.

Figure 2.14 illustrates some possible configurations of bodies in the plane fitting parts 2 and 3 of Theorem 2.4.

These simply stated facts lie at the heart of the results of Chapter 3. In that chapter, we consider several Helly-type results concerning support lines to a collection of disjoint convex bodies. Specifically, if our family of bodies is *large enough*, and for any $m \geq 3$ such that any m bodies share a support line, then so must the entire family of bodies share one support line.



(a) 3 bodies sharing 3 supports (b) 5 bodies sharing 2 supports

Figure 2.14: Illustration for Theorem 2.4

Chapter 3: Helly-type results on support lines

In this chapter we present several Helly-type results regarding support lines for families of disjoint convex planar bodies.

Definition 3.1. We say that a family, $\mathcal{F}$, of convex bodies in the plane has the *support property* S; provided, that all members of $\mathcal{F}$ share a support line.

Definition 3.2. We say that a family, $\mathcal{F}$, of convex bodies in the plane has the property *S(m)* if every m members of $\mathcal{F}$ share a support line.

It should be clear that:

$$S \Rightarrow S(m), \text{ and } S(j) \Rightarrow S(i) \text{ for } j \geq i.$$

While it is clear that $S \Rightarrow S(m)$, we consider under what circumstances $S(m) \Rightarrow S$. We find that given a family, $\mathcal{F}$, of disjoint convex planar bodies with the property $S(m)$, as long as $|\mathcal{F}|$ is large enough, then $S(m) \Rightarrow S$. We also see that the larger m is, the smaller $|\mathcal{F}|$ can be, yet still ensuring that $S(m) \Rightarrow S$.

The results from Dawson [2] can be summarized by the following theorem.

Theorem 3.1. *For a family, $\mathcal{F}$, of pairwise disjoint convex planar bodies,*

1) $S(5) \Rightarrow S$*,*

2) if $|\mathcal{F}| \geq 7$ *then* $S(4) \Rightarrow S$*,*

3) if $|\mathcal{F}| \geq 237$ *then* $S(3) \Rightarrow S$*.*

Here Dawson gives us certain *threshold* numbers for the necessary size of $|\mathcal{F}|$ to ensure that $S(m) \Rightarrow S$. In general, if $|\mathcal{F}| < 7$ then $S(4) \not\Rightarrow S$. To see this, consider

Figure 3.1 which shows 6 bodies with the property $S(4)$, but not S. Through combinatoric methods, Dawson proves the first two results. By Dawson's own admission, the stated threshold number of 237 in part 3 of Theorem 3.1 is probably much too high.

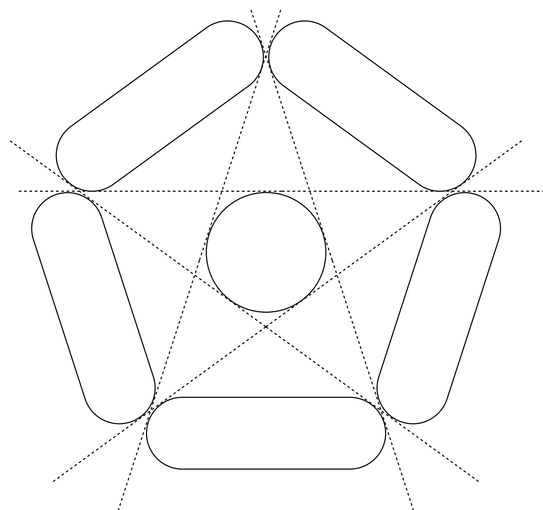


Figure 3.1: Illustration for Theorem 3.1, where $S(4) \nRightarrow S$ if $|\mathcal{F}| = 6$

A similar approach by Revenko & Soltan [10] refines some of Dawson's results and illustrates we need far fewer bodies than 237 to ensure $S(3) \Rightarrow S$. Revenko & Soltan show us through their use of *set-systems* that the threshold number to ensure $S(3) \Rightarrow S$ is no higher than 143. Additionally, they provide an arrangement of 16 bodies in the plane which has property $S(3)$ but not S. Reconstructed from Revenko & Soltan [10], Figure 3.2 shows us this family of 16 bodies where $S(3) \nRightarrow S$. The authors speculate that the true threshold number for a family of disjoint convex planar bodies to ensure $S(3) \Rightarrow S$ is much lower than 143.

Open problem. *The minimum number, m, of bodies with the $S(3)$ property, to ensure that $S(3) \Rightarrow S$ remains an open problem.*

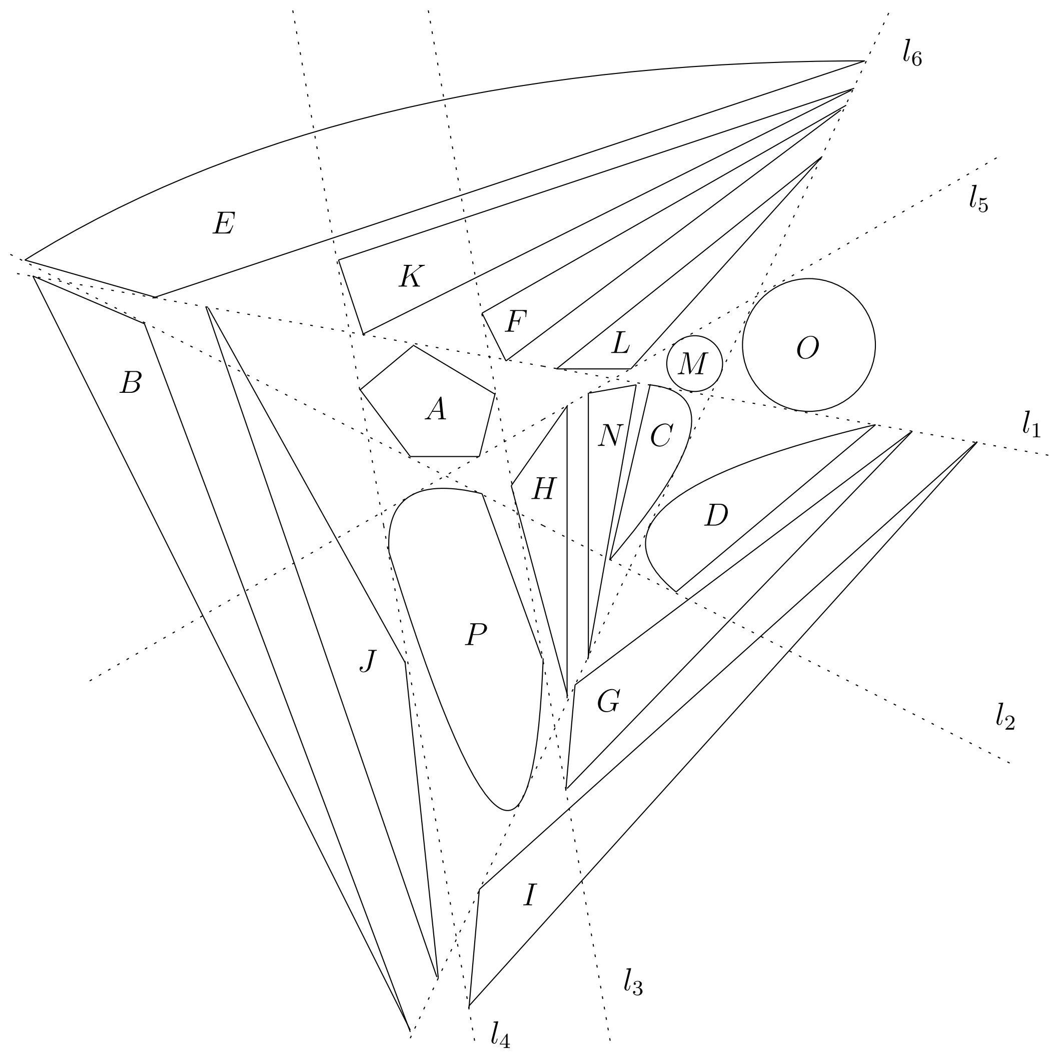


Figure 3.2: Family of 16 bodies where $S(3) \nRightarrow S$

Chapter 4: Set-systems

Set-systems have been successfully used to establish several results and provide a way to capture the underlying structure of these special transversals providing support. The results from the preceding chapter derive from these results on set-systems.

Definition 4.1. Let $\mathcal{S}$ be a set of distinct elements $A, B, C, \ldots$, which we call *letters*. A *set-system* $\mathcal{L}$ on $\mathcal{S}$ is a collection of subsets of $\mathcal{S}$. We call these subsets *words*[1]. We also require that any two words in a set-system be mutually non-inclusive.

Geometrically, we interpret this set-system on $\mathcal{S}$ as the collection of transversals providing support to a family of convex bodies in the plane. Each distinct word represents a transversal providing support to a subcollection of bodies from $\mathcal{S}$. Each distinct letter of a word corresponds to a distinct body being supported.

To better model the limitations of a subcollection of disjoint convex bodies regarding the number of shared supports, Revenko & Soltan [10] introduce the following definition of a *special set-system.*

Definition 4.2. A set-system $\mathcal{L}$ on the set $\mathcal{S}$ is called *special*, provided that it has the following properties:

(P1) no pair of letters in $\mathcal{S}$ is contained in more than four words;

(P2) no triple of letters in $\mathcal{S}$ is contained in more than three words;

(P3) no quintuple of letters in $\mathcal{S}$ is contained in more than two words.

[1] *Words* in our context are subsets of S where the order of letters is irrelevant.

Definition 4.3. A set-system $\mathcal{L}$ on the set $\mathcal{S}$ has the (transversal) property T if all the letters of $\mathcal{S}$ belong to a word. $\mathcal{S}$ has the property $T(m)$, where m is a given positive integer, if any m distinct letters in $\mathcal{S}$ belong to a word.

A great strength of this set-system modeling approach is that we can represent in tabular format the transversal structure of a given family of convex bodies. Once in this tabular format, one can use combinatoric methods to prove geometric results. Consider Table 4.1 and Figure 4.1 which illustrate the relationship between a set system of four words on 4 letters and the set of supporting transversals on four convex bodies which they model.

Here we see that if a line, w_i, provides support to some body, say A, then $+$ is inserted into the corresponding row and column of the table. Similarly, if a line does not provide support to a specific body, then $-$ is inserted into the corresponding row and column of the table.

Table 4.1: A special set-system of four words on four letters

	A	B	C	D
w_1	$+$	$+$	$+$	$-$
w_2	$+$	$+$	$-$	$+$
w_3	$+$	$-$	$+$	$+$
w_4	$-$	$+$	$+$	$+$

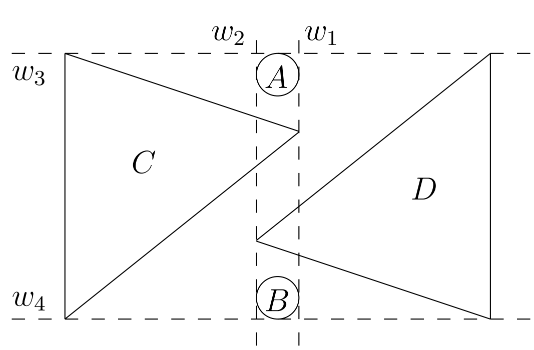


Figure 4.1: Four convex bodies and four supporting transversals

Theorems 4.1–4.4 give a summary of the results from Revenko & Soltan [10], and so the reader is directed there for the proofs. Similar results were found by Dawson [2], however Theorem 4.4 provides a refinement to Dawson's upperbound of 237.

Theorem 4.1. $T(6) \Rightarrow T$ *for any special set-system.*

Theorem 4.2. $T(5) \Rightarrow T$ *for any special set-system on at least* 7 *letters.*

Theorem 4.1 gives us that in any special set-system it is always the case that $T(6) \Rightarrow T$ (of course, considering special sets with less than 6 letters is trivial).

Theorem 4.2 gives us that if we have a special-set system with at least 7 letters, and there is a word containing *any* selection of 5 letters, then there is a word containing all of the letters. To see that, in general, $T(5) \nRightarrow T$ with fewer than 7 letters, consider Table 4.2 which shows a special set-system of 6 words on 6 letters with property $T(5)$ but not T. Thus, we can conclude that 7 is the lowest number of letters necessary to ensure that in a special set-system, $T(5) \Rightarrow T$.

It is important to note that the (transversal) property $T(m)$ in a set-system is not the same as the (support) property $S(m)$ for a family of convex planar bodies. As such, the special set-system in Table 4.2 is not geometrically realizable. If it were, it would contradict Dawson's result, Theorem 3.1.

Table 4.2: A special set-system of six words on six letters

	A	B	C	D	E	F
w_1	+	+	+	+	+	−
w_2	+	+	+	+	−	+
w_3	+	+	+	−	+	+
w_4	+	+	−	+	+	+
w_5	+	−	+	+	+	+
w_6	−	+	+	+	+	+

Theorem 4.3. $T(4) \Rightarrow T$ *for any special set-system on at least* 11 *letters.*

To see that, in general, $T(4) \not\Rightarrow T$ for a special set-system with fewer than 11 letters, consider Table 4.3, which shows a special set-system of 5 words on 10 letters which has property $T(4)$, but not T. As in our comment following Theorem 4.2, we may conclude that 11 is the lowest number of letters necessary to ensure that in a special set-system, $T(4) \Rightarrow T$.

Table 4.3: A special set-system of five words on ten letters

	A	B	C	D	E	F	G	H	I	J
w_1	+	+	+	+	+	+	+	+	−	−
w_2	+	+	+	+	+	+	−	−	+	+
w_3	+	+	+	+	−	−	+	+	+	+
w_4	+	+	−	−	+	+	+	+	+	+
w_5	−	−	+	+	+	+	+	+	+	+

Theorem 4.4. $T(3) \Rightarrow T$ *for any special set-system on at least* 143 *letters.*

Additionally, Revenko & Soltan [10] provide a lower bound for the minimum number of letters in a special set-system where $T(3) \not\Rightarrow T$. Their example of 10 words on 20 letters is shown in Table 4.4. Thus, we conclude that there is a smallest integer n, $21 \leq n \leq 143$ such that, for any special set-system with n letters, $T(3) \Rightarrow T$.

Table 4.4: A special set-system of ten words on twenty letters

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+	+
+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−	+	+	+
+	−	−	−	−	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+
+	−	−	−	−	−	−	−	−	+	+	+	+	−	−	−	−	+	−	−
+	−	−	−	−	−	−	−	−	−	−	−	−	+	+	+	+	+	−	−
−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	−
−	−	−	−	−	−	−	−	−	+	+	−	−	+	+	−	−	+	+	−
−	−	−	−	−	−	−	−	−	−	−	+	+	−	−	+	+	+	+	−
−	−	−	−	−	−	−	−	−	+	−	+	−	+	−	+	−	+	−	+
−	−	−	−	−	−	−	−	−	−	+	−	+	−	+	−	+	+	−	+

Chapter 5: k-systems

In slight departure from Revenko & Soltan [10] we introduce the idea of a k-system.

Definition 5.1. A *k-system* $\mathcal{L}$ on a set of letters S is a set-system of words on the letters of S, such that each pair of letters from S is contained in at least one word, but no more than k words.

Theorem 5.1. *$T(k+2) \Rightarrow T$ for any k-system, $k \geq 1$.*

Proof. We prove this by contradiction, assuming there is some k-system $\mathcal{L}$ on a set S of letters with the property $T(k+2)$ but not T. Clearly $|S| > k+2$ and S is not a word.

We choose any pair of letters, AB, in S. Due to property $T(k+2)$, we know there is a word containing AB and at least k other letters from S. Since we are in a k-system, there are at most k of these words. Let $w_1, w_2, \ldots, w_m$ $(1 \leq m \leq k)$ denote all the distinct words containing AB and at least k other letters from S. Since each w_i is distinct from S, we can find letters in $S \backslash w_i$ for each w_i. That is, we can find $l_i \notin w_i$ for each $1 \leq i \leq m \leq k$. Since $\mathcal{L}$ has property $T(k+2)$, there is another word, w, containing all of $A, B, l_1, l_2, \ldots, l_m$. Since w is distinct from each of $w_1, w_2, \ldots, w_m$ we obtain our contradiction. Thus, $T(k+2) \Rightarrow T$. □

It is natural to speculate whether $T(k+1) \Rightarrow T$ for any k-system. Theorem 5.4 gives us a condition on the set S ensuring $T(k+1) \Rightarrow T$. Lemmas 5.2, 5.3 are used in the proof of Theorem 5.4.

Lemma 5.2. *$T(3) \Rightarrow T$ for any 2-system on at least 5 letters.*

Proof. It suffices to show that $T(3) \Rightarrow T(5)$. Since $T(5) \Rightarrow T(4)$, by Theorem 5.1 we would have $T(3) \Rightarrow T$. We proceed by contradiction, assuming there is some 2-system, $\mathcal{L}$, on a set S with at least 5 letters, where $\mathcal{L}$ has property $T(3)$ but not T. Since $\mathcal{L}$ does not have property $T(5)$, we can find a subset M of S, with 5 letters which is not entirely contained in a word. Let $M = \{A, B, C, D, E\}$.

Claim 1: Any pair of letters from M is contained in a word from $\mathcal{L}$ which contains at least 4 letters from M. Arbitrarily select a pair of letters from M, say AB. There are no more than two words which cover triples of the form $\{ABX : X \in M\}$. Since $|M| = 5$, one of these words must contain 4 letters from M. Since AB were chosen arbitrarily, we see that any pair of letters from M is contained in a word from $\mathcal{L}$ containing 4 letters from M.

With *Claim 1*, we know there is a word containing AB and 2 more letters from M, say CD. That is, $ABCD \subset w_1$. There is a second word containing AE and 2 more letters from M (which are also in w_1). Without loss of generality, suppose $BC \subset w_2$. That is, $ABCE \subset w_2$. Lastly, we consider w_3, which contains each of the letters not in w_1, w_2 (in our example, DE) and 2 other letters from M. Regardless of which pair of additional letters we select from ABC, we find a pair of letters that is in each of w_1, w_2, w_3. Since we are in a 2-system, at least two of these words must be the same. Thus, M is contained in a word from $\mathcal{L}$, providing our contradiction. □

To see that, in general, $T(3) \not\Rightarrow T$ for a 2-system with with less than 5 letters, consider Table 5.1 which shows us a 2-system of 4 words on 4 letters with property $T(3)$ but not property T.

Lemma 5.3. *$T(4) \Rightarrow T$ for any 3-system on at least 6 letters.*

Proof. We claim that if $|S| \geq 6$, then $T(4) \Rightarrow T$. We prove this by contradiction, assuming there is some 3-system on a set with at least 6 letters and property $T(4)$, but

Table 5.1: A 2-system of four words on four letters

	A	B	C	D
w_1	$+$	$+$	$+$	$-$
w_2	$+$	$+$	$-$	$+$
w_3	$+$	$-$	$+$	$+$
w_4	$-$	$+$	$+$	$+$

not T. Select letters $\{A, B, C, D, E, F\}$ such that they do not all belong to a word. We know that we can select these letters, otherwise we would have the property $T(6)$ in a 3-system (by Theorem 5.1, this would ensure that we have property T). We then find words containing the 4-tuples as shown in Table 5.2. Since we have assumed that no word contains $\{A, B, C, D, E, F\}$, we can conclude that none of w_1, w_2, w_3 is equal to any of w_4, w_5, w_6. At least two of w_1, w_2, w_3, w_7 are the same, since we have a 3-system and BC is in each of the four words. Thus, these words together imply the existence of a word, $\bar{w}_1$, containing 5 letters from S. Similarly, w_4, w_5, w_6, w_7 imply the existence of a second word, $\bar{w}_2$, distinct from $\bar{w}_1$, containing 5 letters from S. It must be distinct from $\bar{w}_1$, otherwise we would have already found our contradiction.

Table 5.2: Seven words from a 3-system

	A	B	C	D	E	F
w_1	$+$	$+$	$+$	$+$		
w_2	$+$	$+$	$+$		$+$	
w_3	$+$	$+$	$+$			$+$
w_4			$+$	$+$	$+$	$+$
w_5		$+$		$+$	$+$	$+$
w_6	$+$			$+$	$+$	$+$
w_7		$+$	$+$	$+$	$+$	

We relabel our points so that:

$$\{A', B', C', D', E'\} \subset \bar{w}_1 \text{ and } \{A', B', C', D', F'\} \subset \bar{w}_2.$$

We now consider 3 more words r_1, r_2, r_3 containing the 4-tuples as shown in Table 5.3.

Table 5.3: Three additional words: r_1, r_2, r_3

	A'	B'	C'	D'	E'	F'
$\bar{w}_1$	+	+	+	+	+	−
$\bar{w}_2$	+	+	+	+	−	+
r_1	+	+			+	+
r_2	+		+		+	+
r_3	+			+	+	+

Since $A'F'$ occur in each of $\bar{w}_2, r_1, r_2, r_3$, we know that at least two of them are equal. Furthermore, we may assume $\bar{w}_2 \neq r_i,\ i = 1, 2, 3$; otherwise our contradiction is reached. So, for some $i \neq j,\ r_i = r_j$. This implies a third word, $\bar{w}_3$ distinct from $\bar{w}_1$ and $\bar{w}_2$ which also contains five letters from S.

Table 5.4: Three additional words: s_1, s_2, s_3

	A''	B''	C''	D''	E''	F''
$\bar{w}_1$	+	+	+	+	+	−
$\bar{w}_2$	+	+	+	+	−	+
$\bar{w}_3$	+	+	+	−	+	+
s_1	+			+	+	+
s_2		+		+	+	+
s_3			+	+	+	+

As before, we relabel our letters and construct 3 words, s_1, s_2, s_3 (see Table 5.4), and argue that these words must be distinct form $\bar{w}_1, \bar{w}_2, \bar{w}_3$, implying that there is a fourth word $\bar{w}_4$ containing five letters from S. We relabel our points so that $F''' \notin \bar{w}_1, E''' \notin \bar{w}_2, D''' \notin \bar{w}_3, C''' \notin \bar{w}_4$ (see Table 5.5).

At this point we witness our contradiction. Since we have a 3-system and $A'''B'''$ appears in each of $\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4$ then for some $i, j = 1, 2, 3, 4$ and $i \neq j, \bar{w}_i = \bar{w}_j$. This demands that there is a word containing all of our six letters, which were chosen

Table 5.5: Four distinct words contradicting a 3-system

	A'''	B'''	C'''	D'''	E'''	F'''
$\bar{w}_1$	+	+	+	+	+	−
$\bar{w}_2$	+	+	+	+	−	+
$\bar{w}_3$	+	+	+	−	+	+
$\bar{w}_4$	+	+	−	+	+	+

to not all belong to one word. □

Using Lemmas 5.2, 5.3 we generalize the result to any k-system.

Theorem 5.4. *$T(k+1) \Rightarrow T$ for any k-system, $k \geq 2$, on at least $k+3$ letters.*

Proof. Lemmas 5.2, 5.3 cover the specific cases when $k = 2, 3$. So, we are left only to check when $k \geq 4$.

Let $\mathcal{L}$ be any k-system on a set S which has at least $k+3$ letters. If $\mathcal{L}$ were to have property $T(k+2)$, then by Theorem 5.1 we would have that $\mathcal{L}$ also has property T. So, to obtain a contradiction we assume that $\mathcal{L}$ has property $T(k+1)$ but not $T(k+3)$, remembering that $T(k+3) \Rightarrow T(k+2)$.

Since $\mathcal{L}$ on set S does not have property $T(k+3)$, we can find a subset $M \subset S$ which has exactly $k+3$ letters that are not all contained in a single word. Select two letters, say AB, from M. There are no more than k words which contain all the $(k+1)$-tuples of the form $\{ABX_1X_2 \ldots X_{k-1} : X_i \in M\}$. There are exactly $\binom{k+1}{k-1} = \frac{k^2+k}{2}$ of these $(k+1)$-tuples.

Here we note that $\frac{k^2+k}{2} > 2k$ when $k \geq 4$. Since the number of $(k+1)$-tuples of this form is strictly greater than $2k$, we have that there is some word w from $\mathcal{L}$ which contains at least 3 of these distinct $(k+1)$-tuples from M. This forces w to contain all the letters from M, thus providing our contradiction. □

To see, in general, that $T(k+1) \not\Rightarrow T$ for a k-system on a set S with fewer than

$k+3$ letters, consider Table 5.6 which shows a k-system with $k+2$ words on $k+2$ letters that has property $T(k+1)$ but not T.

Table 5.6: A k-system on a set with $k+2$ letters where $T(k+1) \not\Rightarrow T$.

	l_1	l_2	l_3	$\dots$	l_k	l_{k+1}	l_{k+2}
w_1	+	+	+	$\dots$	+	+	−
w_2	+	+	+	$\dots$	+	−	+
w_3	+	+	+	$\dots$	−	+	+
$\vdots$	+	+	+	$\vdots$	+	+	+
w_k	+	+	−	$\dots$	+	+	+
w_{k+1}	+	−	+	$\dots$	+	+	+
w_{k+2}	−	+	+	$\dots$	+	+	+

Note. *In the statement of Theorem 5.4 we restricted $k \geq 2$. This was done because Theorem 5.4 fails when $k = 1$. If $k = 1$ and $|S| = n$, then we could form $\binom{n}{2}$ words, each of length 2. This collection would certainly provide a valid, if uninteresting k-system where $T(2) \not\Rightarrow T$.*

It is tempting to suggest that for any k-system there exists a finite c_k such that if $|S| \geq c_k$ and S has property $T(m)$, $m \geq 3$, then it also has T. Certainly this is true for $k = 1$ as Theorem 5.1 illustrates. This is not the case, however, without additional constraints on our k-system. Consider Table 5.7 which shows us an infinite set S of letters in a 3-system with 4 words where $T(3) \not\Rightarrow T$.

Table 5.7: An infinite set, S, where $T(3) \not\Rightarrow T$

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	$\dots$
w_1	+	+	+	−	+	+	+	−	+	+	+	−	$\dots$
w_2	+	+	−	+	+	+	−	+	+	+	−	+	$\dots$
w_3	+	−	+	+	+	−	+	+	+	−	+	+	$\dots$
w_4	−	+	+	+	−	+	+	+	−	+	+	+	$\dots$

We can strengthen this statement by showing that, in general, for a k-system,

$k \geq 3$, $T(k) \not\Rightarrow T$, regardless of how large S is. To see this, consider Table 5.8 which shows a k-system on an arbitrarily large set S with $k+1$ words.

Table 5.8: Arbitrarily large set, S, where $T(k) \not\Rightarrow T$

	A_1	A_2	$\dots$	A_k	A_{k+1}	B_1	B_2	$\dots$	B_k	B_{k+1}	$\dots$	M_1	M_2	$\dots$	M_k	M_{k+1}
w_1	+	+	$\dots$	+	−	+	+	$\dots$	+	−	$\dots$	+	+	$\dots$	+	−
w_2	+	+	$\dots$	−	+	+	+	$\dots$	−	+	$\dots$	+	+	$\dots$	−	+
$\vdots$	+	+	$\vdots$	+	+	+	+	$\vdots$	+	+	$\dots$	+	+	$\vdots$	+	+
w_k	+	−	$\dots$	+	+	+	−	$\dots$	+	+	$\dots$	+	−	$\dots$	+	+
w_{k+1}	−	+	$\dots$	+	+	−	+	$\dots$	+	+	$\dots$	−	+	$\dots$	+	+

In Table 5.8 we see an arbitrarily large set of letters. To be exact, we have that $|S| = m(k+1)$, $m \in \mathbb{Z}_+$. We see that our table has at least four words in it. We also see that each word contains all but the letters with a fixed subindex. That is, w_1 contains all letters except the ones subindexed as $k+1$; w_2 contains all letters but those subindexed with k. Proceeding thusly, we see that w_j, $1 \leq j \leq k+1$, contains all letters but the ones subindexed with $k+2-j$. Within this framework, we see that we can select any k letters and find a word containing them (since we have at least $k+1$ words in our set-system). Thus, we see that in general, for a k-system, $T(k) \not\Rightarrow T$, regardless the size of S.

Chapter 6: Special k-systems

In Chapter 4 we witness several theorems on a special set-system. These results, when interpreted, give us specific results on the support lines of a family of convex bodies in the plane. As of yet, we have not defined a *special k-system*. Here too, we make the distinction between a set-system and a special set-system in a given k-system (modified from Revenko & Soltan [10]).

Definition 6.1. A *special k-system* $\mathcal{L}$ on the set S is a k-system with the additional properties:

(P2) no triple of letters in S is contained in more than three words;

(P3) no quintuple of letters in S is contained in more than two words

From this definition, one can see that the special set-system discussed in Chapter 4 is what we now call a *special 4-system*.

We claim that given a special k-system on a set S, there exists a finite number, c_k, such that if $|S| \geq c_k$ then $T(3) \Rightarrow T$. Although only a rough approximation, Theorem 6.1 gives us an upper bound for such a number c_k.

Theorem 6.1. $T(3) \Rightarrow T$ *for any special k-system on at least* $5k^3 + 2$ *letters.*

Proof. We proceed with proof by contradiction, assuming there is a special k-system on a set S with at least $5k^3 + 2$ letters where $T(3) \not\Rightarrow T$.

We begin by selecting any two letters from S, say AB. Since there are at most k words containing AB, there is a word w_1 which contains AB and at least $\frac{5k^3}{k}$ other letters from S. Thus, $|w_1| \geq 5k^2+2$. Since we assumed that our special k-system does not have property T, there is some letter, $Z \notin w_1$. We now consider words containing AZ. Since there are at most k of these words, there is some word w_2 containing AZ and at least $\frac{5k^2}{k}$ letters from w_1. Again we find a letter $Y \notin w_2$ and examine words containing YZ. Since we are in a k-system, there is some word w_3 which contains YZ and at least $\frac{5k}{k}$ letters from both w_1 and w_2. Thus, we see that $|w_1 \cap w_2 \cap w_3| \geq 5$. This contradicts our definition for a special k-system in so much that no quintuple can be in more than two words. □

In terms of a special 4-system, the upper bound calculated in Theorem 6.1 is much higher than the upper bounds provided by both Dawson [2] and Revenko & Soltan [10]. However, the strength of this theorem is that it provides a straightforward finiteness proof of the existence of c_k for *any* k. That is, $c_k \leq 5k^3 + 2$.

The upper bound of 137 computed by Theorem 6.1 for c_3 is too high. Fortunately, we can establish a much better upper bound for c_3. By modifying the proof of Revenko & Soltan [10] for a special 3-system, Theorem 6.2 provides a much lower threshold number for c_3.

Theorem 6.2. *$T(3) \Rightarrow T$ for any special 3-system, on at least 74 letters.*

Proof. It suffices to show that $T(3) \Rightarrow T(4)$. By Theorem 5.4, we have that $T(4) \Rightarrow T$ and hence $T(3) \Rightarrow T$. To obtain a contradiction, we assume that S is not a word. We then show that every triple is contained in at least 2 words.

This situation would imply that for any four letters $ABCD$, from S, there are two distinct words, u_1, u_2, that contain ABC and two distinct words, v_1, v_2, that contain BCD. Since BC are in all four words (and we are in a 3-system), then at least one of $u_i = v_j$. Thus, we have a word containing $ABCD$. Since these four letters were chosen arbitrarily, we have that $T(3) \Rightarrow T(4) \Rightarrow T$.

We proceed to show by contradiction that every triple is contained in two distinct words by assuming that there is some triple, ABC, contained in only one word. So, $\{A, B, C\} \subset w_1$ and $w_1 \neq S$.

Case I: $|w_1| \geq 48$

Choose a letter $D \notin w_1$. The triples $\{ADX : X \in w_1 \backslash \{A\}\}$ are covered by at most 3 words each containing AD. Thus, one of these words, say w_2, contains A and at least 16 other letters of w_1. Since ABC is in only one word, w_1, we can assume without loss of generality that $B \notin w_2$. There are at most 3 words covering triples of the form $\{BDX : X \in w_1 \cap w_2\}$. Thus, one of these words, say w_3, contains at least 5 letters common to w_1 and w_2, contradicting that no quintuple be contained in more than 2 words in a special k-system.

Case II: $|w_1| \leq 31$

At most 2 more words, y_1, y_2, cover the triples $\{ABX : X \notin w_1\}$. Similarly, at most 4 new words, u_1, u_2 and v_1, v_2 cover the triples $\{ACX : X \notin w_1\}$ and $\{BCX : X \notin w_1\}$, respectively. Then the (at most) 8 intersections $y_i \cap u_j \cap v_l$ cover the set $S \backslash w_1$, which has at least $74 - 31 = 43$ letters. Thus, the largest of the intersections contains at least $\lceil \frac{43}{8} \rceil = 6$ letters, again contradicting that no quintuple be contained in more than 2 words in a special k-system.

Case III: $32 \leq |w_1| \leq 47$

At most 2 more words cover triples $\{ABX : X \notin w_1\}$. Thus, one of these words, w_2, contains AB and at least $\lceil \frac{74-47}{2} \rceil = 14$ letters from $S \backslash w_1$. At most 2 other words each containing AC cover triples $\{ACX : X \in w_2 \backslash w_1\}$, and we obtain a word, w_3, which contains AC and at least 7 other letters in $w_2 \backslash w_1$.

We now show that $w_1 \backslash \{A, B, C\} \subset w_2 \cup w_3$. Assume for a moment the existence of a letter $D \in w_1 \backslash \{A, B, C\}$ which is not in $w_2 \cup w_3$. Then at most 2 new words each containing AD cover triples $\{ADX : X \in (w_2 \cap w_3) \backslash w_1\}$. Thus, one of these words contains A and at least 4 other letters common to $(w_2 \cap w_3) \backslash w_1$, again contradicting that no quintuple be contained in more than 2 words in a special k-system. Thus, $w_1 \backslash \{A, B, C\} \subset w_2 \cup w_3$.

So $|w_1 \backslash \{A, B, C\}| \geq 29$, and one of w_2 (which contains AB) or w_3 (which contains AC) also contains at least 14 other letters of w_1. We assume that it is w_2, which contains AB and hence not C. Choose a letter $Z \notin w_1$, and consider the triples $\{CZX : X \in w_1 \cap w_2\}$. Since we are in a 3-system, these triples are covered by at most 3 words different from w_1 and w_2. Since the number of triples is at least 16, one of these words contains at least 5 letters common to both w_1 and w_2, once again contradicting that no quintuple be contained in more than 2 words in a special k-system.

In each of these cases we find that ABC must be contained in more than one word. Thus, $T(3) \Rightarrow T(4)$ for any special 3-system on at least 74 letters. Since we have more than 6 letters and by Theorem 5.4, $T(3) \Rightarrow T(4) \Rightarrow T$. □

Table 6.1 shows us a special 3-system of 4 words on 16 letters. This with Theorem 6.2 allow us to conclude, $17 \leq c_3 \leq 74$.

Table 6.1: A special 3-system of 4 words on 16 letters.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}	l_{15}	l_{16}
w_1	+	+	+	−	+	+	+	−	+	+	+	−	+	+	+	−
w_2	+	+	−	+	+	+	−	+	+	+	−	+	+	+	−	+
w_3	+	−	+	+	+	−	+	+	+	−	+	+	+	−	+	+
w_4	−	+	+	+	−	+	+	+	−	+	+	+	−	+	+	+

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Curriculum Vitae

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