STABILITY AND CLASSIFICATION OF POLYGON SPACES

by

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Dedication

For Happy and Dempsey.

Acknowledgments

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Table of Contents

List of Figures

Abstract

STABILITY AND CLASSIFICATION OF POLYGON SPACES

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We study the spaces of closed linkages of line segments in $\mathbb{R}^d$, called polygon spaces, and the action on them by the orthogonal and special orthogonal groups of matrices. A polygon space $V_d(\ell)$ is determined by an ordered list of edge lengths $\ell = (l_1, \ldots, l_n)$ and the dimension $d \geq 2$ of the ambient space. It is well-known [KM95] that the space of admissible edge lengths, given by a generalization of the triangle inequalities, is a combinatorial object whose components determine certain features of $V_d(\ell)$ and of the moduli space $M_d(\ell) = V_d(\ell)/SO(d)$. We expand upon this classification program by describing explicitly the variety $V_d(\ell)$ in terms of those components.

We define the "dimension" of a polygon to be the dimension of the smallest affine space containing the polygon's edges. The interplay between dimension of polygons and the dimension of the ambient space gives a new approach to the study of the moduli spaces $M_d(\ell)$. In particular, we show that these spaces form a directed system for increasing d, and that this system stabilizes at $d = n$, where n is the number of edges of the polygons in $V_d(\ell)$. As a tool toward this end we use a "diagonals" map that sends a polygon to its ordered list of diagonal lengths, and show that this map is injective on polygons of relatively small dimension.

We also take a detailed look at 4-gons, and construct the spaces $M_d(\ell)$ as CW-complexes for all possible ℓ and d. These constructions expand upon known constructions for low dimension. They also serve as an example of results presented earlier in the paper, and as evidence for conjectures presented later.

Chapter 1: Introduction

1.1 Background

The study of polygon spaces fits within the larger study of configuration spaces of linkages, which is extensive and dates back hundreds of years [KM02]. It is known that any compact real algebraic variety is the configuration space of a linkage [DO05], as is the interior of any compact manifold with boundary [Kou14]. There is an anecdote in which William Thurston says a linkage can "sign your name" [Kin99], meaning that there exists a linkage and a specified vertex of that linkage whose configuration space is an arbitrarily close approximation of your signature. Linkages also appear in the applied sciences in the studies of mechanical linkages [KM02] and of protein folding [DO05]. A linkage in which the initial and terminal vertices are identical is called a polygon. Polygon spaces appear in the study of symplectic geometry of Grassmannians [HK97] and a variation, hyperpolygon spaces, are related to the study of Higgs bundles [GM13].

We will use the term polygon space to refer to the entire configuration space of polygons with given edge lengths, whereas its corresponding "moduli space" is its quotient by rotations and/or reflections in the ambient Euclidean space. It is known that if the edge lengths are sufficiently generic, the corresponding space of polygons in $\mathbb{R}^d$ is smooth for arbitrarily high d. The same can be said for moduli spaces with sufficiently generic edge length vectors only if d is 2 or 3. Millson and Kapovich study moduli spaces of polygons in $\mathbb{R}^2$ [KM95], and also in $\mathbb{R}^3$ [KM96] where they admit a natural complex-analytic structure, whereas Farber and Fromm study smooth properties of polygon spaces in arbitrarily high dimension [FF13]. We offer new results for both moduli spaces and polygon spaces for arbitrary dimensions and all possible edge lengths.

1.2 Summary

In Chapter 2 we define polygon spaces $V_d(\ell)$ and show that they are real algebraic varieties. We also define the "dimension" of a polygon, as distinct from the dimension of the ambient space in which it lives, and show that the space $V_d(\ell)$ is "stratified" by the dimension of the polygons within it. We define the smooth and critical loci, $V_d^\circ(\ell)$ and $V_d^\wedge(\ell)$, of a polygon space. We show that the smooth locus is a manifold and that the critical locus is the dimension-1 stratum and is a subvariety. We then define a combinatorial object $D_n \subset \mathbb{R}^n$ that describes the possible edge lengths ℓ that give rise to nonempty polygon spaces. The object D_n is partitioned into a "border", "open chambers", and "walls". We show that when ℓ belongs to the border, $V_d(\ell)$ consists of only its critical locus; when ℓ belongs to an open chamber, $V_d(\ell)$ consists of only its smooth locus; and when ℓ lies on an intersection of k walls we give an explicit description of the critical locus $V_d^\wedge(\ell)$ as the disjoint union of k spheres.

In Chapter 3 we turn our attention to the moduli spaces $M_d(\ell) = V_d(\ell)/SO(d)$. We define the diagonals map on $M_d(\ell)$ that maps a polygon to its ordered list of diagonal lengths, and we show that this map is injective on polygons of dimension less than d. This is the crucial step to the main result of this chapter: for fixed ℓ the moduli spaces $M_d(\ell)$ form a direct system for increasing d, and this system stabilizes at $d = n$ where n is the length of ℓ.

In Chapter 4 we take a detailed look at the moduli spaces of 4-gons. Our results from Chapter 3 allow us to restrict our attention to $M_d(\ell)$ for $d = 2, 3, 4$. We also show that we may restrict our attention to only finitely many ℓ in D_4. We then construct all possible homeomorphism types of moduli spaces of 4-gons as CW-complexes that are determined by d and ℓ. In Chapter 5 we use our observations from Chapter 4 to motivate new conjectures.

1.3 Groundwork and notation

We will be working with powers of Euclidean space $\mathbb{R}^d$ such as $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$. We denote points in $\mathbb{R}^{dn}$ by $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,d}) \in \mathbb{R}^d$. Given $\mathbf{x} \in \mathbb{R}^d$ we let $|\mathbf{x}|$ denote the length of $\mathbf{x}$. Concretely, if $\mathbf{x} = (x_1, \ldots, x_d)$ then $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_d^2}$. We denote the coordinate functions on powers of $\mathbb{R}^d$ as $\mathbf{r}_i : \mathbb{R}^{dn} \to \mathbb{R}^d$, $\mathbf{r}_i(X) = \mathbf{x}_i$, and the standard coordinate functions on $\mathbb{R}^n$ as $r_i : \mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto x_i$. Let

$$\mathbb{R}[X] = \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_n] = \mathbb{R}[x_{1,1}, \ldots, x_{1,d}, \ldots, x_{n,1}, \ldots, x_{n,d}]$$

be the polynomial ring in $d \times n$ variables with coefficients in $\mathbb{R}$. Given a subset $S \subset \mathbb{R}[X]$, the *zero-locus* of S is

$$Z(S) = \left\{X \in \mathbb{R}^{dn} \,:\, f(X) = 0 \,\forall\, f \in S\right\}.$$

A *real algebraic variety* is a set of the form $Z(S)$. Given polynomials $f_1, \ldots, f_s \in \mathbb{R}[X]$, we let $\langle f_i \rangle_{i=1,\ldots,s}$ denote the ideal generated by the f_i. Given an ideal $I \subset \mathbb{R}[X]$ we let $\mathbf{V}(I)$ denote the real algebraic variety $Z(I)$. Given varieties $V_1 \subset \mathbb{R}^a$, $V_2 \subset \mathbb{R}^b$, a *regular map* $f : V_1 \to V_2$ is the restriction of a polynomial map $\mathbb{R}^a \to \mathbb{R}^b$. The varieties V_1 and V_2 are *isomorphic*, written $V_1 \cong V_2$, if there are regular maps $f : V_1 \to V_2$ and $g : V_2 \to V_1$ such that the compositions $g \circ f$ and $f \circ g$ are the identity maps on V_1 and V_2, respectively.

We let $\mathbf{0}$ denote the origin in Euclidean space, we let S^d denote the unit sphere in $\mathbb{R}^d$, and we let I_d denote the $d \times d$ identity matrix. We let $GL(d)$ denote the general linear group of $d \times d$ matrices. Given $T \in GL(d)$ we let T^t denote the transpose of T and let $\det(T)$ denote the determinant of T. We let $O(d)$ and $SO(d)$ denote the orthogonal and special orthogonal groups of $d \times d$ matrices. Concretely,

$$O(d) = \{T \in GL(d) : TT^t = I_d\} \quad \text{and} \quad SO(d) = \{T \in GL(d) : TT^t = I_d, \det(T) = 1\}.$$

We let $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ denote the standard basis for $\mathbb{R}^d$. Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^d$ we let $\mathrm{Span}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ denote their span as a real vector subspace of $\mathbb{R}^d$. Given a real vector subspace L of $\mathbb{R}^d$ we let $\dim(L)$ denote its dimension as a real vector space. An *affine subspace* of $\mathbb{R}^d$ is a translate of a linear subspace. The *dimension* of an affine subspace is the dimension of the corresponding linear subspace. Two affine subspaces are *orthogonal* if their corresponding linear subspaces are orthogonal. In particular, if A and B are orthogonal affine subspaces of $\mathbb{R}^d$ and $A \cap B = \mathbf{c}$, then for any $\mathbf{a} \in A$, $\mathbf{b} \in B$, the Pythagorean theorem gives

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{b}|^2.$$

The *affine hull* of a subset S of $\mathbb{R}^d$, denoted $\mathrm{Aff}(S)$, is the intersection of all affine subspaces of $\mathbb{R}^d$ containing S. A major result of Chapter 3 depends on intersections of spheres in $\mathbb{R}^d$, and we now define a particular kind of sphere toward that end. This definition is also used in Lemma 2.1.8.

Definition 1.3.1. *Given an affine subspace $A \subset \mathbb{R}^d$, an A*-sphere *is a set of the form*

$$S(A, \mathbf{c}, \rho) := \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in A, \ |\mathbf{x} - \mathbf{c}| = \rho\right\},$$

where $\mathbf{c} \in A$ and $\rho > 0$. The center *of $S(A, \mathbf{c}, \rho)$ is $\mathbf{c}$ and its* radius *is ρ.*

On several occasions we make use of the fact that the restriction of a continuous function is continuous, so we prove it here.

Lemma 1.3.2. *Let $f : A \to B$ be a continuous function of topological spaces, and let U be a subset of A. Then the restriction of f to U is a continuous function $f|_U : U \to f(U)$ under the subspace topology.*

Proof. Let $f : A \to B$ be a continuous function of topological spaces, and let U be a subset of A. Let $V \subset f(U)$ be open in the subspace topology, and let V' be open in B so that $V' \cap f(U) = V$. Then $f^{-1}(V')$ is open in A by continuity, so $f^{-1}(V') \cap U$

is open in U by the subspace topology. But $f^{-1}(V') \cap U$ is precisely $f|_U^{-1}(V)$. For the inclusion $f^{-1}(V') \cap U \subset f|_U^{-1}(V)$, let $a \in f^{-1}(V') \cap U$. Then $f(a) \in V'$ and $f(a) \in f(U)$, so $f(a) \in V' \cap f(U) = V$, so $a \in f^{-1}(V)$. But also $a \in U$ so $a \in f|_U^{-1}(V)$. For the inclusion $f|_U^{-1}(V) \subset f^{-1}(V') \cap U$, let $a \in f|_U^{-1}(V)$. Then $a \in U$. Also, $f(a) \in V \subset V'$, so $f(a) \in V'$ so $a \in f^{-1}(V')$. See the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ U \arrow[u, hook] \arrow[r, "f|_U"] & f(U) \arrow[u, hook] \end{tikzcd}$$

where the vertical arrows are inclusion maps. $\square$

Chapter 2: Polygon spaces

In this chapter we define our primary objects of study and state results about their differential-geometric and real-algebraic structure.

2.1 Polygons, varieties, and dimension

We begin with two definitions for polygon spaces, one in terms of edges and the other in terms of vertices.

Definition 2.1.1. *Given* $\ell = (l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$ *and* $d \geq 2$*, the corresponding* edge polygon space *is the set*

$$E_d(\ell) = \left\{ P = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathbb{R}^{dn} \,:\, |\mathbf{p}_i| = 1 \; \forall \, i = 1, \dots, n, \; \sum_{i=1}^{n} l_i \mathbf{p}_i = \mathbf{0} \right\}.$$

Definition 2.1.2. *Given* $\ell = (l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$ *and* $d \geq 2$*, the corresponding* vertex polygon space *is the set*

$$V_d(\ell) = \left\{ P = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \in \mathbb{R}^{d(n-1)} : |\mathbf{v}_i - \mathbf{v}_{i-1}| = l_i, \; i = 1, \dots, n \right\},$$

where $\mathbf{v}_0 = \mathbf{v}_n = \mathbf{0}$*. The vector* $\mathbf{v}_i$ *is called a* vertex *of* P *for all* $i = 0, \dots, n$*. Let* $i, j \in \{0, \dots, n-1\}$ *with* $i < j$*. If* $j = i + 1$ *or* $(i, j) = (0, n-1)$*, the vertices* $\mathbf{v}_i, \mathbf{v}_j$ *are called* adjacent*. Otherwise, the vector* $\mathbf{v}_j - \mathbf{v}_i$ *is called the* (i, j)*-th* diagonal *of* P*.*

The elements of $E_d(\ell)$ and $V_d(\ell)$ are called *polygons* or n*-gons* where n is the length of ℓ. Definition 2.1.1 is the working definition in [FF13], though they implicitly use Definition

2.1.2 in one of their major propositions. We will be using both throughout this dissertation as they lend themselves more naturally to different proofs and points-of-view. Figure 2.1 shows that the two definitions describe the same polygons, and Proposition 2.1.3 makes this relationship explicit.

Proposition 2.1.3. *Given $\ell = (l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$, the polygon spaces $E_d(\ell)$ and $V_d(\ell)$ are isomorphic real algebraic varieties.*

Proof. To show that $E_d(\ell)$ is a real algebraic variety define the polynomials

$$f_i : \mathbb{R}^{dn} \to \mathbb{R} \qquad (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto x_{i,1}^2 + \cdots + x_{i,d}^2 - 1, \qquad i = 1, \dots, n$$

$$g_j : \mathbb{R}^{dn} \to \mathbb{R} \qquad (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto l_1 x_{1,j} + \cdots + l_n x_{n,j}, \qquad j = 1, \dots, d.$$

Then $f_i((\mathbf{x}_1, \dots, \mathbf{x}_n)) = 0$ for all $i = 1, \dots, n$ if and only if $|\mathbf{x}_i| = 1$ for all $i = 1, \dots, n$, and $g_j((\mathbf{x}_1, \dots, \mathbf{x}_n)) = 0$ for all $j = 1, \dots, d$ if and only if $\sum_{i=1}^n l_i \mathbf{x}_i = \mathbf{0}$. Thus $E_d(\ell)$ is the variety $\mathbf{V}(I_\ell)$, where $I_\ell = \langle f_i, g_j \rangle_{i=1,\dots,n;\, j=1,\dots,d}$.

To show that $V_d(\ell)$ is a real algebraic variety define the polynomials

$$h_i : \mathbb{R}^{d(n-1)} \to \mathbb{R} \quad (\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \mapsto (x_{i,1} - x_{i-1,1})^2 + \cdots + (x_{i,d} - x_{i-1,d})^2 - l_i^2, \quad i = 1, \dots, n,$$

with the understanding that $x_{0,j} = x_{n,j} = 0$ for all $j = 1, \dots, d$. Then $h_i((\mathbf{x}_1, \dots, \mathbf{x}_{n-1})) = 0$ for all $i = 1, \dots, n$ if and only if $|\mathbf{x}_i - \mathbf{x}_{i-1}| = l_i$ for all $i = 1, \dots, n$, where $\mathbf{x}_0 = \mathbf{x}_n = \mathbf{0}$. Thus $V_d(\ell)$ is the variety $\mathbf{V}(I'_\ell)$, where $I'_\ell = \langle h_i \rangle_{i=1,\dots,n}$.

Now we show that $V_d(\ell)$ and $E_d(\ell)$ are isomorphic. Let $\ell = (l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$ and define the polynomial map

$$\psi : \mathbb{R}^{dn} \to \mathbb{R}^{d(n-1)}$$

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto (l_1 \mathbf{x}_1, l_1 \mathbf{x}_1 + l_2 \mathbf{x}_2, \dots, \sum_{i=1}^{n-1} l_i \mathbf{x}_i).$$

If $P = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in E_d(\ell)$ then for all $i = 1, \ldots, n$ we have

$$|\mathbf{r}_i(\psi(P)) - \mathbf{r}_{i-1}(\psi(P))| = \left| \sum_{j=1}^{i} l_j \mathbf{p}_j - \sum_{j=1}^{i-1} l_j \mathbf{p}_j \right| = |l_i \mathbf{p}_i| = l_i$$

so $\psi(P) \in V_d(\ell)$ and thus ψ restricts to a regular map of varieties. Abusing notation we say that $\psi : E_d(\ell) \to V_d(\ell)$. Now since no $l_i = 0$ we may define the polynomial map

$$\phi : \mathbb{R}^{d(n-1)} \to \mathbb{R}^{dn}$$

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \mapsto \left(\frac{\mathbf{x}_1 - \mathbf{x}_0}{l_1}, \ldots, \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{l_n} \right)$$

where we understand $\mathbf{x}_0 = \mathbf{x}_n = \mathbf{0}$. If $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d(\ell)$, then $|\mathbf{r}_i(\phi(V))| = \left| \frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{l_i} \right| = 1$ for all $i = 1, \ldots, n$, and $\sum_{i=1}^{n} l_i \mathbf{r}_i(\phi(V)) = \sum_{i=1}^{n} \mathbf{v}_i - \mathbf{v}_{i-1} = -\mathbf{v}_0 + \mathbf{v}_n = \mathbf{0}$, so $\phi(V) \in E_d(\ell)$ and thus ϕ restricts to a regular map of varieties $\phi : V_d(\ell) \to E_d(\ell)$. Finally, we have $\phi = \psi^{-1}$, as shown here:

$$\begin{aligned}
\phi \circ \psi(P) &= \phi \left(l_1 \mathbf{p}_1, l_1 \mathbf{p}_2 + l_2 \mathbf{p}_2, \ldots, \sum_{i=1}^{n-1} l_i \mathbf{p}_i \right) \\
&= \left(\frac{l_1 \mathbf{p}_1 - \mathbf{0}}{l_1}, \frac{l_1 \mathbf{p}_1 + l_2 \mathbf{p}_2 - l_1 \mathbf{p}_1}{l_2}, \ldots, \frac{\sum_{i=1}^{n} l_i \mathbf{p}_i - \sum_{i=1}^{n-1} l_i \mathbf{p}_i}{l_n} \right) \\
&= (\mathbf{p}_1, \ldots, \mathbf{p}_n) \\
&= P;
\end{aligned}$$

$$
\begin{aligned}
\psi \circ \phi(V) &= \psi\left(\frac{\mathbf{v}_1 - \mathbf{v}_0}{l_1}, \dots, \frac{\mathbf{v}_n - \mathbf{v}_{n-1}}{l_n}\right) \\
&= \left(l_1\left(\frac{\mathbf{v}_1 - \mathbf{v}_0}{l_1}\right), l_1\left(\frac{\mathbf{v}_1 - \mathbf{v}_0}{l_1}\right) + l_2\left(\frac{\mathbf{v}_2 - \mathbf{v}_1}{l_2}\right), \dots, \sum_{i=1}^{n-1} l_i\left(\frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{l_i}\right)\right) \\
&= (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \\
&= V.
\end{aligned}
$$

□

Part of our work concerns polygon spaces $V_d(\ell)$ for fixed d and varying ℓ. Lemma 2.1.4 greatly simplifies this task by stating that two polygon spaces are isomorphic if their corresponding edge lengths are permutations of each other.

Lemma 2.1.4. *Given a permutation $\sigma \in S_n$, and $\ell \in \mathbb{R}^n_{>0}$, let $\sigma(\ell) = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$. The polygon spaces $V_d(\ell)$ and $V_d(\sigma(\ell))$ are isomorphic as varieties.*

Proof. Given Proposition 2.1.3, we prove the equivalent statement that $E_d(\ell)$ and $E_d(\sigma(\ell))$ are isomorphic as varieties. An element $\sigma \in S_n$ induces a polynomial map

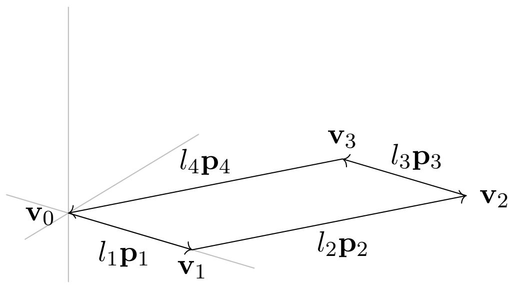
$$\psi_\sigma : \mathbb{R}^{dn} \to \mathbb{R}^{dn},\ (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto (\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(n)}).$$

If $P = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in E_d(\ell)$, then $\sum_{i=1}^n l_i \mathbf{p}_i = \mathbf{0}$, and thus $\sum_{i=1}^n l_{\sigma(i)} \mathbf{p}_{\sigma(i)} = \mathbf{0}$, so $\psi_\sigma(P) = (\mathbf{p}_{\sigma(1)}, \dots, \mathbf{p}_{\sigma(n)}) \in E_d(\sigma(\ell))$. Thus ψ_σ restricts to a regular map of varieties

$$\psi_\sigma : E_d(\ell) \to E_d(\sigma(\ell)),$$

and has regular inverse given by $\psi_{\sigma^{-1}} : E_d(\sigma(\ell)) \to E_d(\ell)$. □

The notion of "dimension" of a polygon, which is distinct from the dimension of the ambient space in which it lives, plays a large role in our work. We define it here.



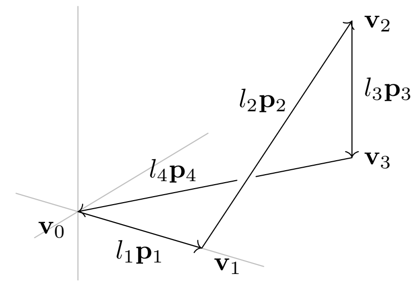


Figure 2.1: Pictures of a 2-dimensional polygon (left) and a 3-dimensional polygon (right). In both pictures, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V_3(\ell)$ and $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \in E_3(\ell)$, where $\ell = (l_1, l_2, l_3, l_4)$.

Definition 2.1.5. *The* dimension *of a polygon* $P = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \in V_d(\ell)$ *is*

$$\dim(P) := \dim(\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})).$$

The k-stratum *of* $V_d(\ell)$ *is defined to be* $V_d^k(\ell) := \{P \in V_d(\ell) : \dim(P) = k\}$.

Remark 2.1.6. *We define the dimension of a polygon* $P = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in E_d(\ell)$ *as*

$$\dim(P) := \dim(\mathrm{Span}(\mathbf{p}_1, \dots, \mathbf{p}_n)),$$

and the define the k-stratum *of* $E_d(\ell)$ *to be* $E_d^k(\ell) := \{P \in E_d(\ell) : \dim(P) = k\}$. *The isomorphisms of Proposition* 2.1.3 *preserve dimension.*

Lemma 2.1.7. *Let* $\ell = (l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$ *and let* $P \in V_d(\ell)$. *Then* $\dim(P) \le \min\{n-1, d\}$.

Proof. This follows immediately from the fact that $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ is a linear subspace of $\mathbb{R}^d$ spanned by $n-1$ vectors. □

The following lemma says that as long as the polygon space is not completely degenerate (a notion to be made precise later), it contains polygons of all possible dimensions.

Lemma 2.1.8. *Let* $\ell \in \mathbb{R}^n_{>0}$. *If* $V_d^2(\ell) \ne \varnothing$, *then* $V_d^k(\ell) \ne \varnothing$ *for all* $k = 2, \dots, \min\{n-1, d\}$.

Proof. It suffices to show for all $k = 2, \dots, \min\{n-1, d\} - 1$, if $V_d^k(\ell)$ is nonempty, then $V_d^{k+1}(\ell)$ is nonempty. Let $k \in \{2, \dots, \min\{n-1, d\} - 1\}$ and suppose $V_d^k(\ell) \neq \varnothing$. Let $P = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \in V_d^k(\ell)$. Given $i \in \{1, \dots, n-1\}$, let $U_i = \mathrm{Span}(\{\mathbf{v}_j\}_{j \neq i})$ and let $L_i = \mathrm{Aff}(\mathbf{v}_{i-1}, \mathbf{v}_{i+1})$. We claim there exists $i \in \{1, \dots, n-1\}$ such that $\mathbf{v}_i \in U_i \setminus L_i$. Since $k < n-1$ there exists a linear dependence among $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$, and thus there exists $i \in \{1, \dots, n-1\}$ so that $\mathbf{v}_i \in U_i$. Now if $\mathbf{v}_i \in L_i$, note that $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, $\mathbf{v}_{i+1}$ are colinear. But also, $\mathbf{v}_i \in \mathrm{Span}(\mathbf{v}_{i-1}, \mathbf{v}_{i+1})$, so $\mathbf{v}_{i+1} \in \mathrm{Span}(\mathbf{v}_{i-1}, \mathbf{v}_i)$, which is a subset of U_{i+1}. Thus $\mathbf{v}_{i+1} \in U_{i+1}$. Now if $\mathbf{v}_{i+1} \in L_{i+1}$, we have $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, $\mathbf{v}_{i+1}$, $\mathbf{v}_{i+2}$ are colinear. But also by the previous argument we have $\mathbf{v}_{i+2} \in U_{i+2}$. By continuing this process, we must eventually find $\mathbf{v}_j \in U_j \setminus L_j$, otherwise all of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ are colinear and thus $\dim(P) = 1$, contradicting $k \geq 2$. Now let $i \in \{1, \dots, n-1\}$ such that $\mathbf{v}_i \in U_i \setminus L_i$. Let $\mathbf{c}$ be the orthogonal projection of $\mathbf{v}_i$ onto L_i. Let $A = \mathrm{Aff}(U_i^{\perp} + \mathbf{c}, \mathbf{v}_i)$. Let $S = S(A, \mathbf{c}, |\mathbf{v}_i - \mathbf{c}|)$ be an A-sphere, and let $\mathbf{w} \in S \setminus U_i$. We claim that $P' = (\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{n-1}) \in V_d^{k+1}(\ell)$. See Figure 2.2. First we show that $P' \in V_d(\ell)$. It is enough to show $|\mathbf{w} - \mathbf{v}_{i-1}| = |\mathbf{v}_i - \mathbf{v}_{i-1}|$ and $|\mathbf{v}_{i+1} - \mathbf{w}| = |\mathbf{v}_{i+1} - \mathbf{v}_i|$. We note that A and L_i are orthogonal as affine subspaces of $\mathbb{R}^d$ and $A \cap L_i = \mathbf{c}$. Thus since $\mathbf{w} \in A$ and $\mathbf{v}_{i-1}, \mathbf{v}_{i+1} \in L_i$, we have

$$|\mathbf{w} - \mathbf{v}_{i-1}|^2 = |\mathbf{w} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{v}_{i-1}|^2,$$

$$|\mathbf{w} - \mathbf{v}_{i+1}|^2 = |\mathbf{w} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{v}_{i+1}|^2.$$

Also, since $\mathbf{c}$ is the orthogonal projection of $\mathbf{v}_i$ onto $\mathrm{Aff}(\mathbf{v}_{i-1}, \mathbf{v}_{i+1})$ we have

$$|\mathbf{v}_i - \mathbf{v}_{i-1}|^2 = |\mathbf{v}_i - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{v}_{i-1}|^2,$$

$$|\mathbf{v}_{i+1} - \mathbf{v}_i|^2 = |\mathbf{v}_{i+1} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{v}_i|^2.$$

Finally, since $\mathbf{w}$, $\mathbf{v}_i \in S$ we have

$$|\mathbf{w} - \mathbf{c}| = |\mathbf{v}_i - \mathbf{c}|.$$

It follows from the above equations that

$$\begin{aligned}|\mathbf{w}-\mathbf{v}_{i-1}| &= \sqrt{|\mathbf{w}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{v}_{i-1}|^2}\\ &= \sqrt{|\mathbf{v}_i-\mathbf{c}|^2+|\mathbf{c}-\mathbf{v}_{i-1}|^2}\\ &= |\mathbf{v}_i-\mathbf{v}_{i-1}|,\end{aligned}$$

and

$$\begin{aligned}|\mathbf{v}_{i+1}-\mathbf{w}| &= \sqrt{|\mathbf{v}_{i+1}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{w}|^2}\\ &= \sqrt{|\mathbf{v}_{i+1}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{v}_i|^2}\\ &= |\mathbf{v}_{i+1}-\mathbf{v}_i|.\end{aligned}$$

Thus $P' \in V_d(\ell)$. Lastly, since

$$k = \dim(\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_{n-1})) = \dim(\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\dots,\mathbf{v}_{n-1}))$$

and $\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\dots,\mathbf{v}_{n-1})) \subsetneq \mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{w},\mathbf{v}_{i+1},\dots,\mathbf{v}_{n-1}))$ we have

$$\dim(P') = \dim(\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{w},\mathbf{v}_{i+1},\dots,\mathbf{v}_{n-1})) = k+1,$$

so $P' \in V_d^{k+1}(\ell)$. $\square$

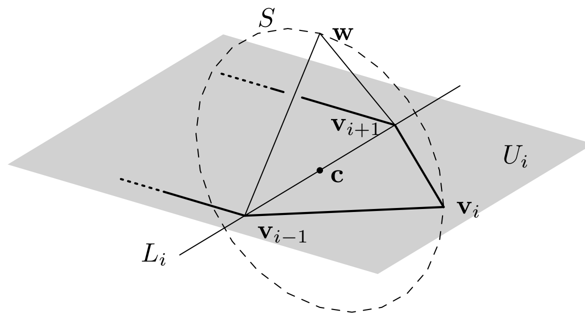


Figure 2.2: A picture of the process used in the proof of Lemma 2.1.8, to create a $(k+1)$-dimensional polygon from a k-dimensional polygon.

For $i = 1, \ldots, d$ define the *lower-i layer* of $V_d(\ell)$, denoted $\mathcal{S}^i$, to be the union $\bigcup_{k \le i} V_d^k(\ell)$, and define the *above-$i$ layer* of $V_d(\ell)$, denoted $V_d^{k>i}(\ell)$, to be $V_d(\ell) \setminus \mathcal{S}^i$.

Lemma 2.1.9. *The lower-i layer $\mathcal{S}^i$ of $V_d(\ell)$ is closed in $V_d(\ell)$*

Proof. The rank function rank : $M_{m \times n} \to \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers, that takes an $m \times n$ matrix to its rank is lower semicontinuous, and thus $\text{rank}^{-1}(\{1, \ldots, i\})$ is a closed subset of $M_{m \times n}$. For $P = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d(\ell)$, identify P with the $d \times (n-1)$ matrix $(\mathbf{v}_1^t, \ldots, \mathbf{v}_{n-1}^t)$. Then $\mathcal{S}^i = \text{rank}|_{V_d(\ell)}^{-1}(\{1, \ldots, i\})$. □

2.2 The space of admissible edge lengths

In this section we describe the set of ℓ for which the polygon space $V_d(\ell)$ is nonempty. We will see that this set has combinatorial properties that govern the structure of $V_d(\ell)$. Define the polygon inequalities on $\mathbb{R}^n$ to be

$$l_i \le \frac{1}{2} \sum_{k=1}^{n} l_k \quad \text{for} \quad i = 1, \ldots, n. \tag{2.1}$$

These inequalities are equivalent to

$$l_i \le \sum_{k \ne i} l_k \quad \text{for} \quad i = 1, \dots, n, \tag{2.2}$$

since

$$l_i \le \frac{1}{2}\sum_{k=1}^{n} l_k \iff \frac{1}{2} l_i \le \frac{1}{2}\sum_{k \ne i}^{n} l_k \iff l_i \le \sum_{k \ne i}^{n} l_k.$$

Note that the triangle inequalities are the special case when $n = 3$. The polygon inequalities define a polyhedral cone in $\mathbb{R}^n$. Let L_n be the solution set to the polygon inequalities intersected with the strictly positive orthant:

$$L_n = \left\{ \ell \in \mathbb{R}^n : l_i \le \sum_{k \ne i} l_k \text{ and } l_i > 0 \ \forall\, i = 1, \dots, n \right\}.$$

Then L_n is the cone defined by the polygon inequalities minus the faces $l_i = 0$, $i = 1, \dots, n$. See Figure 2.3.

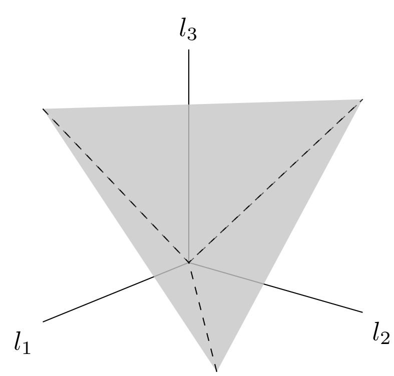


Figure 2.3: A picture of L_3 defined by the triangle inequalities. The dashed lines represent the faces $l_i = 0$, $i = 1, 2, 3$.

Proposition 2.2.1. *The polygon space $V_d(\ell)$ is non-emtpy if and only if $\ell \in L_n$.*

Proof. We prove the equivalent statement, that $E_d(\ell)$ is non-empty if and only if $\ell \in L_n$, following the proof in [KM95]. Inducting on n, the base case $n = 3$ is equivalent to the triangle inequalities, which are well-known to give necessary and sufficient conditions for three positive real numbers to be the edge lengths of a triangle. Now suppose $n \geq 4$ and suppose the proposition holds for $(n-1)$-gons. Let $\ell = (l_1, \ldots, l_n) \in \mathbb{R}^n_{>0}$ satisfy (2.1). We claim there exists $i \in \{1, \ldots, n\}$ such that $l_i + l_{i+1} \leq \frac{1}{2}\sum_{k=1}^n l_k$ (where we understand l_{n+1} to mean l_1). If not, for all $i \in \{1, \ldots, n\}$ we have $l_i + l_{i+1} > \frac{1}{2}\sum_{k=1}^n l_k$, and thus

$$2 \cdot \sum_{k=1}^n l_k = \sum_{k=1}^n l_k + l_{k+1} > n \cdot \frac{1}{2}\sum_{k=1}^n l_k \geq 2 \cdot \sum_{k=1}^n l_k,$$

which is a contradiction since $n \geq 4$. Given i such that $l_i + l_{i+1} \leq \frac{1}{2}\sum_{k=1}^n l_k$, we have $\ell' = (l_1, \ldots, l_{i-1}, l_i + l_{i+1}, l_{i+2}, \ldots, l_n)$ satisfies (2.1). Thus by the induction hypothesis there exists $(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \in E_d(\ell')$, so

$$l_1\mathbf{p}_1 + \ldots + l_{i-1}\mathbf{p}_{i-1} + (l_i + l_{i+1})\mathbf{p}_i + l_{i+2}\mathbf{p}_{i+1}, \ldots, l_n\mathbf{p}_{n-1} = \mathbf{0},$$

and thus

$$l_1\mathbf{p}_1 + \ldots + l_{i-1}\mathbf{p}_{i-1} + l_i\mathbf{p}_i + l_{i+1}\mathbf{p}_i + l_{i+2}\mathbf{p}_{i+1}, \ldots, l_n\mathbf{p}_{n-1} = \mathbf{0},$$

so $(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_i, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_{n-1}) \in E_d(\ell)$. See Figure 2.4.

For the other direction let $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in E_d(\ell)$. Let $i \in \{1, \ldots, n\}$ and let a and $\mathbf{q}$ be a scalar and unit vector, respectively, such that $l_i\mathbf{p}_i + l_{i+1}\mathbf{p}_{i+1} + a\mathbf{q} = \mathbf{0}$. Then $(\mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{q})$ is a triangle in $E_d((l_i, l_{i+1}, a))$, so by the triangle inequalities,

$$l_i \leq a + l_{i+1}. \tag{2.3}$$

Also, by definition we have $(\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{q}, \mathbf{p}_{i+2}, \dots, \mathbf{p}_n) \in E_d(\ell')$ where

$$\ell' = (l_1, \dots, l_{i-1}, a, l_{i+2}, \dots, l_n),$$

so by the induction hypothesis,

$$a \le l_1 + \dots + l_{i-1} + l_{i+2} + \dots + l_n. \tag{2.4}$$

Equations (2.3) and (2.4) together imply that $l_i \le l_1 + \dots + l_{i-1} + l_{i+1} + \dots + l_n$. Since i was chosen arbitrarily, it follows that ℓ satisfies the polygon inequalities (2.2). See Figure 2.5. □

We observe that if $\ell = (l_1, \dots, l_n) \in L_n$, then $\lambda\ell = (\lambda l_1, \dots, \lambda l_n) \in L_n$ for all $\lambda > 0$. The following lemma says the isomorphism type of $V_d(\ell)$ remains constant when scaling ℓ.

Lemma 2.2.2. *For $\lambda > 0$ let $\lambda(l_1, \dots, l_n) = (\lambda l_1, \dots, \lambda l_n)$. The polygon spaces $V_d(\ell)$ and $V_d(\lambda\ell)$ are isomorphic as varieties.*

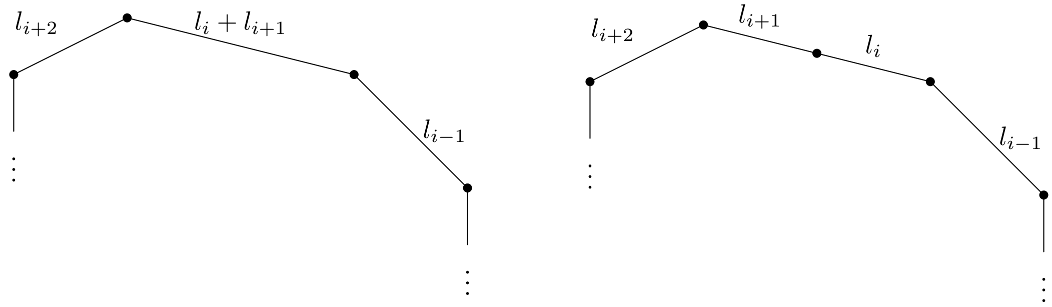
Proof. We show that the edge polygon spaces $E_d(\ell)$ and $E_d(\lambda\ell)$ are in fact identical as sets. The result then follows by Proposition 2.1.3. We have $P = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in E_d(\ell)$ if and only if $\sum_{i=1}^n l_i \mathbf{p}_i = \mathbf{0}$ if and only if $\lambda \sum_{i=1}^n l_i \mathbf{p}_i = \mathbf{0}$ if and only if $\sum_{i=1}^n \lambda l_i \mathbf{p}_i = \mathbf{0}$ if and only if $P \in E_d(\lambda\ell)$. □

Lemma 2.2.2 says that our task of studying isomorphism types of varieties as ℓ varies in L_n is reduced to studying those as ℓ varies in some slice of L_n. Consider the hyperplane

$$H = \left\{ \ell \in \mathbb{R}^n \,:\, \sum_{i=1}^n l_i = 1 \right\}.$$

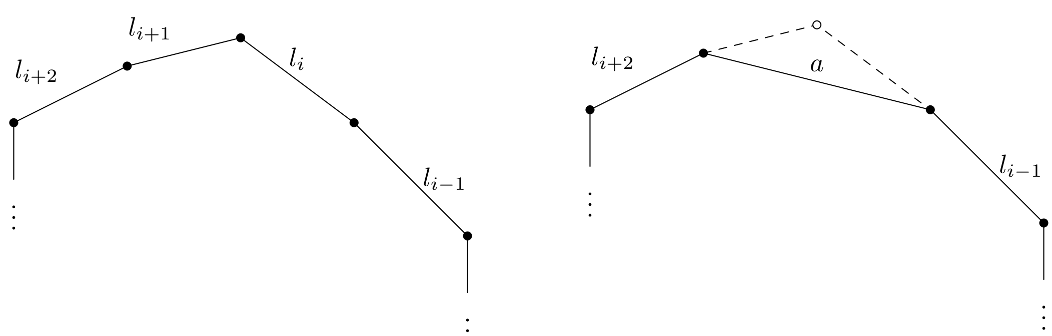
Definition 2.2.3. *The* normalized edge length space of n-gons *is*

$$D_n := L_n \cap H = \left\{ \ell \in \mathbb{R}^n : \sum_{i=1}^n l_i = 1,\, l_i \le \sum_{k \ne i} l_k \text{ and } l_i > 0 \;\forall\, i = 1, \dots, n \right\}$$



(a) (b)

Figure 2.4: If $\ell = (l_1, \dots, l_n)$ satisfies the polygon inequalities, then there is some i so that $(l_1, \dots, l_{i-1}, l_i + l_{i+1}, l_{i+2}, \dots, l_n)$ does as well. Then by the induction hypothesis there is a polygon with edge lengths $(l_1, \dots, l_{i-1}, l_i + l_{i+1}, l_{i+2}, \dots, l_n)$ (Subfigure 2.4a), and thus there is a polygon with edge lengths ℓ (Subfigure 2.4b).



(a) (b)

Figure 2.5: If a polygon has edge lengths $\ell = (l_1, \dots, l_n)$ (Subfigure 2.5a), then for every $i \in \{1, \dots, n\}$ there exists a polygon with edge lengths $\ell' = (l_1, \dots, l_{i-1}, a, l_{i+2}, \dots, l_n)$ where $l_i \leq a + l_{i+1}$ (Subfigure 2.5b). By the induction hypothesis, $a \leq l_1 + \cdots + l_{i-1} + l_{i+1} + \cdots + l_n$. In conjunction with $l_i \leq a + l_{i+1}$, and since i was chosen arbitrarily, we conclude ℓ satisfies the polygon inequalities (2.2).

If $\ell = (l_1, \ldots, l_n) \in L_n$, then $\lambda\ell \in D_n$ where $\lambda = 1/\sum_{i=1}^n l_i$. Thus by Lemma 2.2.2, every polygon space of n-gons is isomorphic to $V_d(\ell)$ for some $\ell \in D_n$. We note that the closure of D_n is

$$\mathrm{cl}(D_n) = \left\{ \ell \in \mathbb{R}^n : \sum_{i=1}^n l_i = 1,\ l_i \le \sum_{k \ne i} l_k \text{ and } l_i \ge 0\ \forall\, i = 1, \ldots, n \right\},$$

which is an $(n-1)$-dimensional polytope in $\mathbb{R}^n$, and that D_n is the polytope $\mathrm{cl}(D_n)$ minus the faces $l_i = 1$, $i = 1, \ldots, n$. The boundary of D_n is

$$\partial(D_n) = \left\{ \ell \in \mathbb{R}^n : \sum_{i=1}^n l_i = 1, \text{ and } l_i = \sum_{k \ne i} l_k \text{ or } l_i = 0 \text{ for some } i \in \{1, \ldots, n\} \right\}.$$

We define the *border* of a subset S of a topological space to be the intersection of S with the boundary of S. We let $b(S)$ denote the border of S. Then the border of D_n is

$$b(D_n) = \left\{ \ell \in \mathbb{R}^n : \sum_{i=1}^n l_i = 1, \text{ and } l_i = \sum_{k \ne i} l_k \text{ for some } i \in \{1, \ldots, n\} \right\}. \tag{2.5}$$

See Figure 2.6.

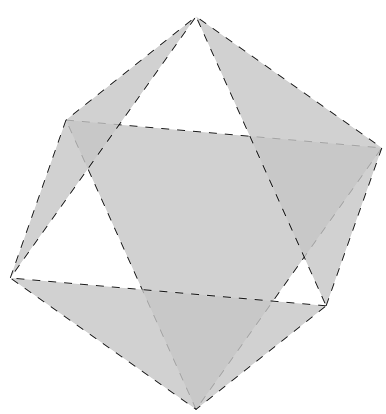


Figure 2.6: The closure of D_4 has the combinatorial type of a 3-dimensional cross-polytope [KM95]. This pictures shows the border $b(D_4)$, which is the union of the interiors of facets $l_i = \sum_{k \ne i} l_k$, $i = 1, 2, 3, 4$. The missing triangular facets are given by $l_i = 0$, $i = 1, 2, 3, 4$, and are not part of D_4.

The border is one part of a more general "critical" subset of D_n. We define the components of this subset here.

Definition 2.2.4. *For* $\mathbf{a} = (a_1, \dots, a_n) \in \{\pm 1\}^n$ *define* $I_{\mathbf{a}} = \{i \in \{1, \dots, n\} : a_i = 1\}$*. Then* $\mathbf{a}$ *gives a linear map*

$$f_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}, \qquad (x_1, \dots, x_n) \mapsto \sum_{i \in I} x_i - \sum_{i \notin I} x_i,$$

and the map $f_{\mathbf{a}}$ *defines a linear hyperplane* $H_{\mathbf{a}} = f_{\mathbf{a}}^{-1}(0)$*. A* wall *of* D_n *is an intersection* $W_{\mathbf{a}} := H_{\mathbf{a}} \cap D_n$ *for some* $\mathbf{a} \in \{\pm 1\}^n$*. Given* $\ell \in D_n$ *we define the* depth *of* ℓ*, denoted* $\operatorname{depth}(\ell)$*, to be the number of distinct walls* $W_{\mathbf{a}}$ *containing* ℓ*. Let* $A_\ell = \{\mathbf{a} \in \{\pm 1\}^n : a_1 = 1$ *and* $\ell \in W_{\mathbf{a}}\}$*. See Figure* 2.7

Remark 2.2.5. *The walls of* D_n *give a polytopal complex whose support is* $\operatorname{cl}(D_n)$*. This complex is addressed in Chapter* 5*.*

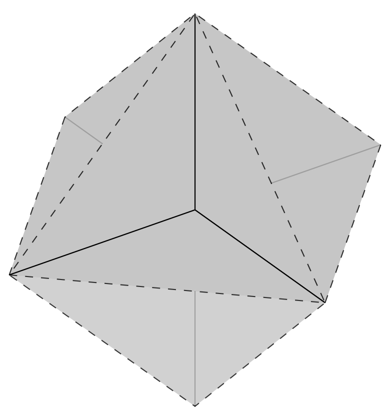


Figure 2.7: The picture shows the walls of D_4. The walls on the border are given by $l_i = \sum_{k \neq i} l_k$, $i = 1, 2, 3, 4$, and the walls that intersect the interior of D_4 are given by $l_1 + l_2 = l_3 + l_4$, $l_1 + l_3 = l_2 + l_4$, and $l_1 + l_4 = l_2 + l_4$ [KM95].

Lemma 2.2.6. *Let* $\ell \in D_n$*. Then* $\operatorname{depth}(\ell) = |A_\ell|$*.*

Proof. The hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ are identical if and only if $\mathbf{a} = \pm\mathbf{b}$, and thus $W_{\mathbf{a}} = W_{\mathbf{b}}$ if and only if $\mathbf{a} = \pm\mathbf{b}$. Thus every wall containing ℓ can be written $W_{\mathbf{a}}$ where $\mathbf{a} \in A_\ell$, and if $\mathbf{a}, \mathbf{b} \in A_\ell$ with $\mathbf{a} \neq \mathbf{b}$, then $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are distinct walls containing ℓ. □

2.3 Degenerate and non-degenerate loci

Consider the map $F : \mathbb{R}^{d(n-1)} \to \mathbb{R}^n$ defined by

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \mapsto (|\mathbf{x}_1 - \mathbf{x}_0|, \ldots, |\mathbf{x}_n - \mathbf{x}_{n-1}|),$$

where $\mathbf{x}_0 = \mathbf{x}_n = \mathbf{0}$. Let

$$\Omega = \{(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) : \mathbf{x}_i \neq \mathbf{x}_{i-1} \; \forall i = 1, \ldots, n\}.$$

Lemma 2.3.1. *The restriction $F|_\Omega : \Omega \to \mathbb{R}^n$ is a smooth map of manifolds, and $V_d(\ell) = F|_\Omega^{-1}(\ell)$.*

Proof. The conditions $\mathbf{x}_i = \mathbf{x}_{i-1}$, $i = 1, \ldots, n-1$ define a closed set of $\mathbb{R}^{d(n-1)}$, so Ω is open in $\mathbb{R}^{d(n-1)}$ and is thus a manifold. The coordinate functions of F are

$$f_k(X) := r_k(F(X)) = |\mathbf{x}_k - \mathbf{x}_{k-1}| = ((x_{k,1} - x_{k-1,1})^2 + \cdots + (x_{k,d} - x_{k-1,d})^2)^{1/2}$$

for $k = 1, \ldots, n$, and their partial derivatives have the form

$$\frac{\partial f_k}{x_{i,j}} = \frac{x_{k,j} - x_{k-1,j}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|} \qquad \text{if } i = k, \tag{2.6}$$

$$\frac{\partial f_k}{x_{i,j}} = \frac{-(x_{k,j} - x_{k-1,j})}{|\mathbf{x}_k - \mathbf{x}_{k-1}|} \qquad \text{if } i = k-1, \tag{2.7}$$

$$\frac{\partial f_k}{x_{i,j}} = 0 \qquad \text{otherwise.} \tag{2.8}$$

These and all higher partial derivatives $\frac{\partial^s f_k}{x_{i,j}^s}$ exist at $X = (\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ if and only if

$$|\mathbf{x}_k - \mathbf{x}_{k-1}| \neq \mathbf{0}$$

for all $k = 1, \ldots, n-1$, i.e., if and only if $X \in \Omega$. Thus $F|_\Omega : \Omega \to \mathbb{R}^n$ is a smooth map of manifolds. To see that $V_d(\ell)$ is a subset of Ω, we first note that $V_d(\ell) = F^{-1}(\ell)$ whenever $\ell \in \mathbb{R}^n_{>0}$. Let $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d(\ell)$. Then by the definition of $V_d(\ell)$ we have $|\mathbf{v}_i - \mathbf{v}_{i-1}| = l_i$ and $l_i > 0$ for all $i = 1, \ldots, n$, so $\mathbf{v}_i \neq \mathbf{v}_{i-1}$ for any $i = 1, \ldots, n$, and thus $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \subset \Omega$. $\square$

Definition 2.3.2. *Let $F : M \to N$ be a smooth map of manifolds where $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$, and let $\mathbf{m} \in M$. The* Jacobian *of F at $\mathbf{m}$, denoted $dF_{\mathbf{m}}$, is the $n \times m$ matrix whose entry in the i-th row and j-th column is $\frac{\partial f_i}{\partial x_j}$ evaluated at $\mathbf{m}$. The point $\mathbf{m} \in M$ is a* regular point *of F if* $\operatorname{rank}(dF_{\mathbf{m}}) = n$. *Otherwise, $\mathbf{m}$ is called a* critical point *of F.*

We let Ω° and $\Omega^\wedge$ denote the sets of regular points and critical points of F. Given the result $V_d(\ell) = F|_\Omega^{-1}(\ell)$ of Lemma 2.3.1, we may write $V_d(\ell) = F|_{\Omega^\circ}^{-1}(\ell) \sqcup F|_{\Omega^\wedge}^{-1}(\ell)$. Let $V_d^\circ(\ell)$ and $V_d^\wedge(\ell)$ denote $F|_{\Omega^\circ}^{-1}(\ell)$ and $F|_{\Omega^\wedge}^{-1}(\ell)$, respectively.

Proposition 2.3.3. *The set $V_d^\wedge(\ell)$ of critical points of F contained in $V_d(\ell)$ is the dimension-1 stratum $V_d^1(\ell)$ of $V_d(\ell)$. Equivalently, the set $V_d^\circ(\ell)$ of regular points of F contained in $V_d(\ell)$ is the union $\bigcup_{k>1} V_d^k(\ell)$ of dimension-k strata of $V_d(\ell)$ for $k > 1$.*

Proof. Given $X = (\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \in \Omega$, we will show that the Jacobian of F at X is not surjective if and only if the $\mathbf{x}_i$ are colinear, and therefore $P \in V_d^\wedge(\ell)$ if and only if $\dim(P) = 1$. We follow the proof of the related Proposition 3.1 in [FF13]. The entries of dF_X are given in Equations (2.6)-(2.8). Let $(dF_X)_k$ denote the k-th row vector of dF_X. By indexing the columns of dF_X as

$$(1,1), \ldots, (1,d), \ldots, (n-1,1), \ldots (n-1,d)$$

we may write

$$(dF_X)_k = \left(\mathbf{0}, \ldots, \mathbf{0}, \frac{-(\mathbf{x}_k - \mathbf{x}_{k-1})}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}, \frac{\mathbf{x}_k - \mathbf{x}_{k-1}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}, \mathbf{0}, \ldots, \mathbf{0}\right)$$

where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$, the vector $\frac{-(\mathbf{x}_k - \mathbf{x}_{k-1})}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}$ has column indices $(k-1,1), \ldots, (k-1,d)$, and the vector $\frac{\mathbf{x}_k - \mathbf{x}_{k-1}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}$ has column indices $(k,1),\dots,(k,d)$. Thus given a point

$$Y = \begin{bmatrix} \mathbf{y}_1^t \\ \vdots \\ \mathbf{y}_{n-1}^t \end{bmatrix} \in \mathbb{R}^{d(n-1)}, \qquad \text{where } \mathbf{y}_i = \begin{bmatrix} y_{i,1} \\ \vdots \\ y_{i,d} \end{bmatrix}$$

we have

$$\begin{aligned} (dF_X)_k(Y) &= \left\langle \frac{-(\mathbf{x}_k - \mathbf{x}_{k-1})}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}, \mathbf{y}_{k-1} \right\rangle + \left\langle \frac{\mathbf{x}_k - \mathbf{x}_{k-1}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}, \mathbf{y}_k \right\rangle \\ &= \left\langle \mathbf{y}_k - \mathbf{y}_{k-1}, \frac{\mathbf{x}_k - \mathbf{x}_{k-1}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|} \right\rangle. \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean dot product, and $\mathbf{y}_0 = \mathbf{y}_n = \mathbf{0} \in \mathbb{R}^d$. To simplify notation, define the unit vectors $\mathbf{u}_k = \frac{\mathbf{x}_k - \mathbf{x}_{k-1}}{|\mathbf{x}_k - \mathbf{x}_{k-1}|}$ for $k = 1,\dots,n$. Then for all $k = 1,\dots,n$ and all $Y \in \mathbb{R}^{d(n-1)}$ we have $(dF_X)_k(Y) = \langle \mathbf{y}_k - \mathbf{y}_{k-1}, \mathbf{u}_k \rangle$, and thus

$$\begin{aligned} dF_X(Y) &= ((dF_X)_1(Y),\dots,(dF_X)_n(Y)) \\ &= (\langle \mathbf{y}_1 - \mathbf{y}_0, \mathbf{u}_1 \rangle, \dots, \langle \mathbf{y}_n - \mathbf{y}_{n-1}, \mathbf{u}_n \rangle). \end{aligned}$$

The image $dF_X(\mathbb{R}^{d(n-1)})$ is a vector subspace of $\mathbb{R}^n$, so if dF_X is not surjective for some $X = (\mathbf{x}_1,\dots,\mathbf{x}_{n-1}) \in \Omega$, then there is a nonzero vector $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{R}^n$ such that for all $Y \in \mathbb{R}^{d(n-1)}$, $dF_X(Y)$ lies in the hyperplane orthogonal to $\mathbf{a}$. Thus for all $Y \in \mathbb{R}^{d(n-1)}$ we have

$$
\begin{aligned}
0 &= \langle \mathbf{a}, dF_X(Y) \rangle \\
&= \sum_{k=1}^{n} a_k \langle \mathbf{y}_k - \mathbf{y}_{k-1}, \mathbf{u}_k \rangle \\
&= \sum_{k=1}^{n} \langle \mathbf{y}_k - \mathbf{y}_{k-1}, a_k \mathbf{u}_k \rangle \\
&= (\langle \mathbf{y}_1, a_1\mathbf{u}_1 \rangle - \langle \mathbf{y}_0, a_1\mathbf{u}_1 \rangle) + (\langle \mathbf{y}_2, a_2\mathbf{u}_2 \rangle - \langle \mathbf{y}_1, a_2\mathbf{u}_2 \rangle) + \cdots + (\langle \mathbf{y}_n, a_n\mathbf{u}_n \rangle - \langle \mathbf{y}_{n-1}, a_n\mathbf{u}_n \rangle) \\
&= \sum_{k=0}^{n-1} \langle \mathbf{y}_k, a_k\mathbf{u}_k - a_{k+1}\mathbf{u}_{k+1} \rangle.
\end{aligned}
$$

where we understand $a_0\mathbf{u}_0$ to be 0. Since Y can be generally chosen, it must be that $a_k\mathbf{u}_k - a_{k+1}\mathbf{u}_{k+1} = \mathbf{0}$ for all $k = 1, \ldots, n-1$. Since $X \in \Omega$, $\mathbf{u}_k \neq \mathbf{0}$ for any $k \neq 0$. Thus if some $a_{k\neq 0} = 0$, then $a_{k-1} = a_{k+1} = 0$ and thus $\mathbf{a} = (0, \ldots, 0)$ contradicting our choice of $\mathbf{a}$. Thus no $a_{k\neq 0} = 0$ and we conclude that the $\mathbf{u}_k$, and thus the $\mathbf{x}_k$, are colinear. Conversely, if the $\mathbf{x}_k$ are colinear then so are the $\mathbf{u}_k$, so there is linear dependence among every pair $\{\mathbf{u}_k, \mathbf{u}_{k+1}\}$, and thus there exists some nonzero vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $a_k\mathbf{u}_k - a_{k+1}\mathbf{u}_{k+1} = \mathbf{0}$ for all $k = 1, \ldots, n-1$. Thus we have

$$
0 = \sum_{k=1}^{n-1} \langle \mathbf{y}_k, a_k\mathbf{u}_k - a_{k+1}\mathbf{u}_{k+1} \rangle = \langle \mathbf{a}, dF_X(Y) \rangle
$$

for all $Y \in \mathbb{R}^{d(n-1)}$, so the image $dF_X(\mathbb{R}^{d(n-1)})$ is contained in the hyperplane orthogonal to $\mathbf{a}$, and thus dF_X is not surjective. □

Proposition 2.3.4. *The set $V_d^\circ(\ell)$ is a manifold of dimension $d(n-1) - n$.*

Proof. We show that Ω° is an open subset of Ω which is in turn an open set of $\mathbb{R}^{d(n-1)}$, and thus Ω° is a manifold of dimension $d(n-1)$. Then since $V_d^\circ(\ell) = F|_{\Omega^\circ}^{-1}(\ell)$ consists of regular points of F, by the Implicit Function Theorem (Theorem 1.38 in [War83]) $V_d^\circ(\ell)$ is a submanifold Ω° of dimension $d(n-1)-n$. To see that Ω° is an open subset of Ω, consider the map $h:\Omega\to\mathbb{R}$, $X\mapsto\det(dF_X\cdot dF_X^t)$. The map h is continuous since matrix multiplication and the determinant map are continuous, and since the maps $X\to dF_X$ and $X\to dF_X^t$ are continuous by the definition of Ω. Thus $h^{-1}(\mathbb{R}\setminus\{0\})$ is open in Ω. A point $X\in\Omega$ is a regular point of F if and only if $\operatorname{rank}(dF_X)=n$, if and only if $\operatorname{rank}(dF_X\cdot dF_X^t)=n$, if and only if $\det(dF_X\cdot dF_X^t)\neq 0$, if and only if $X\in h^{-1}(\mathbb{R}\setminus\{0\})$. Thus the set of regular points Ω° of F is the open set $h^{-1}(\mathbb{R}\setminus\{0\})$ of Ω. □

Proposition 2.3.5. *The dimension-1 stratum $V_d^1(\ell)$ is isomorphic as a variety to the disjoint union of* $\operatorname{depth}(\ell)$ *many spheres.*

Proof. Let $\ell\in D_n$, and recall the set $A_\ell=\{\mathbf{a}\in\{\pm1\}^n : a_1=1 \text{ and } \ell\in W_{\mathbf{a}}\}$ from Definition 2.2.4. For each $\mathbf{a}\in A_\ell$, each $i=2,\dots,n$, and each $j=1,\dots,d$ define the map

$$h_{\mathbf{a},i,j}:\mathbb{R}^{dn}\to\mathbb{R}$$

$$(\mathbf{x}_1,\dots,\mathbf{x}_n)\mapsto x_{i,j}-a_ix_{1,j}.$$

Let $\mathbf{V_a}$ denote the variety $\mathbf{V}(\langle h_{\mathbf{a},i,j}\rangle_{i=2,\dots,n;\,j=1,\dots,d})$ and let $\Sigma_\ell=\bigcup_{\mathbf{a}\in A_\ell}\mathbf{V_a}$. Consider the variety

$$E_d(\ell)\cap\Sigma_\ell=\bigcup_{\mathbf{a}\in A_\ell}E_d(\ell)\cap\mathbf{V_a}.$$

We observe that if $\mathbf{a},\mathbf{b}\in A_\ell$ are distinct, then $\mathbf{V_a}\cap\mathbf{V_b}=\{(\mathbf{0},\dots,\mathbf{0})\}$. Thus since $(\mathbf{0},\dots,\mathbf{0})\notin E_d(\ell)$, we have

$$E_d(\ell)\cap\Sigma_\ell=\bigsqcup_{\mathbf{a}\in A_\ell}E_d(\ell)\cap\mathbf{V_a}.$$

By Remark 2.1.6 we have $V_d^1(\ell)\cong E_d^1(\ell)$, so it remains to show that $E_d^1(\ell)=E_d(\ell)\cap\Sigma_\ell$, and that for all $\mathbf{a}\in A_\ell$ we have $E_d(\ell)\cap\mathbf{V_a}\cong S^{d-1}$.

First we show that $E^1_d(\ell) = E_d(\ell) \cap \Sigma_\ell$. Note that for any $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{dn}$, if $X \in \mathbf{V_a}$ for some $\mathbf{a} \in A_\ell$ then $\mathbf{x}_i = a_i \mathbf{x}_1$ for all $i = 2, \ldots, n$, and thus the $\mathbf{x}_i$ are colinear. Thus if $P \in E_d(\ell) \cap \Sigma_\ell \subset \mathbb{R}^{dn}$ then $P \in E^1_d(\ell)$. Now let $P = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in E^1_d(\ell)$. Then the $\mathbf{p}_i$ are colinear so since the $\mathbf{p}_i$ are unit vectors there is some $\mathbf{a} \in \{\pm 1\}^n$ with $a_1 = 1$ such that $\mathbf{p}_i = a_i \mathbf{p}_1$ for all $i = 2, \ldots, n$. Since $P \in E_d(\ell)$ we have $\sum_{i=1}^n l_i \mathbf{p}_i = \mathbf{0}$, but since $\mathbf{p}_i = a_i \mathbf{p}_1$ for all $i = 2, \ldots, n$ we have $\sum_{i=1}^n l_i a_i \mathbf{p}_1 = \mathbf{0}$ and thus $\sum_{i=1}^n a_i l_i = 0$. Thus $\ell \in W_\mathbf{a}$ so $\mathbf{a} \in A_\ell$, and we have $P \in E_d(\ell) \cap \Sigma_\ell$.

Now we define an isomorphism $S^{d-1} \to E_d(\ell) \cap \mathbf{V_a}$ for each $\mathbf{a} \in A_\ell$. For all $\ell \in D_n$ and all $\mathbf{a} \in A_\ell$ define the polynomial map $\psi_\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{dn}$, $\mathbf{x} \mapsto (a_1 \mathbf{x}, \ldots, a_n \mathbf{x})$. We claim that $\psi_\mathbf{a}$ restricts to a regular map of varieties $S^{d-1} \to E_d(\ell) \cap \mathbf{V_a}$. Let $\mathbf{u} \in S^{d-1}$. To show that $\psi_\mathbf{a}(\mathbf{u}) \in E_d(\ell)$ we must show that $\psi_\mathbf{a}(\mathbf{u}) \in (S^{d-1})^n$ and $\sum_{i=1}^n l_i r_i(\psi_\mathbf{a}(\mathbf{u})) = \mathbf{0}$; to show that $\psi_\mathbf{a}(\mathbf{u}) \in \mathbf{V_a}$ it suffices to show that $r_i(\psi_\mathbf{a}(\mathbf{u})) = a_i r_1(\psi_\mathbf{a}(\mathbf{u}))$ for all $i = 1, \ldots, n$. Since $\mathbf{u} \in S^{d-1}$ and $\mathbf{a} \in \{\pm 1\}^n$, we have $\psi_\mathbf{a}(\mathbf{u}) \in (S^{d-1})^n$, and since $\mathbf{a} \in A_\ell$ we have $\sum_{i=1}^n l_i a_i = 0$ and thus $\sum_{i=1}^n l_i r_i(\psi_\mathbf{a}(\mathbf{u})) = \sum_{i=1}^n l_i a_i \mathbf{u} = \mathbf{0}$. Therefor $\psi_\mathbf{a}(\mathbf{u}) \in E_d(\ell)$. Now if $\mathbf{a} \in A_\ell$ then $a_1 = 1$ so $r_1(\psi_\mathbf{a}(\mathbf{u})) = \mathbf{u}$, so $r_i(\psi_\mathbf{a}(\mathbf{u})) = a_i r_1(\psi_\mathbf{a}(\mathbf{u}))$ for all $i = 1, \ldots, n$. Therefor $\psi_\mathbf{a}(\mathbf{u}) \in \mathbf{V_a}$. Thus $\psi_\mathbf{a}$ restricts to a regular map of varieties $S^{d-1} \to E_d(\ell) \cap \mathbf{V_a}$. To see that this map is an isomorphism, define the polynomial map $\phi : \mathbb{R}^{dn} \to \mathbb{R}^d$, $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \mapsto \mathbf{x}_1$. If $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in E_d(\ell)$ then $\mathbf{p}_1 \in S^{d-1}$ so ϕ restricts to a regular map of varieties $E_d(\ell) \cap \mathbf{V_a} \to S^{d-1}$. It remains to show that $\psi_\mathbf{a} \circ \phi$ and $\phi \circ \psi_\mathbf{a}$ are the identity maps on $E_d(\ell) \cap \mathbf{V_a}$ and S^{d-1}, respectively. Let $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in E_d(\ell) \cap \mathbf{V_a}$. Then $\mathbf{p}_i = a_i \mathbf{p}_1$ for all $i = 1, \ldots, n$, so

$$\psi_\mathbf{a} \circ \phi((\mathbf{p}_1, \ldots, \mathbf{p}_n)) = \psi_\mathbf{a}(\mathbf{p}_1) = (a_1 \mathbf{p}_1, \ldots, a_n \mathbf{p}_1) = (\mathbf{p}_1, \ldots, \mathbf{p}_n).$$

Now let $\mathbf{u} \in S^{d-1}$. If $\mathbf{a} \in A_\ell$ then $a_1 = 1$, so we have

$$\phi \circ \psi_\mathbf{a}(\mathbf{u}) = \phi((a_1 \mathbf{u}, \ldots, a_n \mathbf{u})) = a_1 \mathbf{u} = \mathbf{u}.$$

Thus for each $\mathbf{a} \in A_\ell$ we have $E_d(\ell) \cap \mathbf{V_a} \cong S^{d-1}$. This fact together with Equation **??** gives

$$V_d^1(\ell) \cong \bigsqcup_{\mathbf{a} \in A_\ell} E_d(\ell) \cap \mathbf{V_a} \cong \bigsqcup_{i=1}^{\mathrm{depth}(\ell)} S_i^{d-1},$$

completing the proof. □

Proposition 2.3.6. *If* $\mathrm{depth}(\ell) = 0$*, then* $V_d(\ell)$ *is a manifold of dimension* $d(n-1)-n$*. If* ℓ *is on the border of* D_n*, then* $V_d(\ell)$ *is isomorphic to a sphere.*

Proof. If $\mathrm{depth}(\ell) = 0$ then by Proposition 2.3.5 $V_d^1(\ell) = \varnothing$, so $V_d(\ell) = V_d^\circ(\ell)$, Thus by Proposition 2.3.4 $V_d(\ell)$ is a manifold of dimension $d(n-1)$.

Now suppose $\ell \in b(D_n)$. We will show that $V_d(\ell) = V_d^1(\ell)$, so that $V_d(\ell)$ is the disjoint union of $\mathrm{depth}(\ell)$ many spheres, and then we will show that $\mathrm{depth}(\ell) = 1$. To show that $V_d(\ell) = V_d^1(\ell)$, given Remark 2.1.6 we prove the equivalent statement $E_d(\ell) = E_d^1(\ell)$. Since $\ell \in b(D_n)$, from Equation 2.5 we have $l_k = \sum_{i \neq k} l_i$ for some $k \in \{1, \ldots, n\}$. If $P \in E_d(\ell)$ then $\mathbf{0} = \sum_{i=1}^n l_i \mathbf{p}_i$, and thus $l_k(-\mathbf{p}_k) = \sum_{i\neq k} l_i \mathbf{p}_i$. But since $l_k = \sum_{i \neq k} l_i$, we have

$$\sum_{i \neq k} l_i(-\mathbf{p}_k) = \sum_{i \neq k} l_i \mathbf{p}_i, \tag{2.9}$$

and thus $|\sum_{i\neq k} l_i \mathbf{p}_i| = \sum_{i \neq k} l_i$. Therefor $\{l_i \mathbf{p}_i : i \neq k\}$ is a collection of vectors whose sum has length equal to the sum of the lengths of the vectors, and thus they must all point the same direction, so there is some unit vector $\mathbf{u}$ such that $\mathbf{p}_i = \mathbf{u}$ for all $i \neq k$. Now (2.9) can be rewritten

$$\sum_{i \neq k} l_i(-\mathbf{p}_k) = \sum_{i \neq k} l_i \mathbf{u}. \tag{2.10}$$

Thus $\mathbf{u} = -\mathbf{p}_k$ so $\dim(P) = 1$. Since P was chosen arbitrarily in $E_d(\ell)$ we have $E_d(\ell) = E_d^1(\ell)$. It remains to show that $\mathrm{depth}(\ell) = 1$. Again by Equation 2.5 there exists $k \in \{1, \ldots, n\}$ such that $l_k = \sum_{i \neq k} l_i$, and thus $\ell \in W_\mathbf{a}$ where $I_\mathbf{a} = \{k\}$. Thus $\mathrm{depth}(\ell) \geq 1$. Now suppose for a contradiction depth$(\ell) > 1$. Then there exists some $\mathbf{b} \neq \pm\mathbf{a}$ so that

$$\sum_{i \in I_{\mathbf{b}}} l_i = \sum_{i \notin I_{\mathbf{b}}} l_i. \tag{2.11}$$

Without loss of generality we may assume $k \in I_{\mathbf{b}}$. Then the left hand side of (2.11) is $l_k + \sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i$, so we may write (2.11) as

$$l_k + \sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i = \sum_{i \notin I_{\mathbf{b}}} l_i. \tag{2.12}$$

Since $l_k = \sum_{i \neq k} l_i$ we may write $l_k = \sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i + \sum_{i \notin I_{\mathbf{b}}} l_i$, and thus rewrite (2.12) as

$$\sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i + \sum_{i \notin I_b} l_i + \sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i = \sum_{i \notin I_{\mathbf{b}}} l_i. \tag{2.13}$$

But (2.13) implies $2 \cdot \sum_{i \in I_{\mathbf{b}} \setminus \{k\}} l_i = 0$, contradicting $l_i > 0$ for all $i \in \{1, \ldots, n\}$. $\square$

Chapter 3: Orbit spaces

In this chapter we turn our attention to orbit spaces of polygons under the action of the orthogonal and special orthogonal groups. Our main result is that the orbit spaces form a direct system with a stable limit.

3.1 Descriptions

We follow the treatment of orbit spaces in [Bre72]. Let G be a topological group. A Hausdorff topological space X is a *G-space* if there exists a continuous map $\Theta : G \times X \to X$ such that:

(1) $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$ for all $g, h \in G$, $x \in X$;

(2) $\Theta(e, x) = x$ for all $x \in X$, where e is the identity in G.

An element $g \in G$ defines a homeomorphism $\theta_g : X \to X$, $x \mapsto \Theta(g, x)$, since $(\theta_g)^{-1} = \theta_{g^{-1}}$. The set $[x] = \{\theta_g(x) : g \in G\}$ is called the *orbit* of x under G. We let X/G denote the set of orbits of elements of X, and we let $\pi : X \to X/G$ be the map $x \mapsto [x]$. Then X/G endowed with the quotient topology is called the *orbit space* of X under G. Given an open set $U \subset X$ we have $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} \theta_g(U)$, a union of open sets. Thus $\pi^{-1}(\pi(U))$ is open, so by the definition of open sets in the quotient topology, $\pi(U)$ is open, and thus π is an open map.

Remark 3.1.1. *If X is a G space and H is a subgroup of G, then X is an H space.*

Lemma 3.1.2. *The groups $O(d)$ and $SO(d)$ are compact topological groups.*

Proof. The groups $O(d)$ and $SO(d)$ are topological subspaces of $\mathbb{R}^{d^2}$, and the multiplication and inverse maps are polynomial in $\mathbb{R}^{d^2}$ and are thus continuous, so $O(d)$ and $SO(d)$ are topological groups. To see they are compact consider the map $g : GL(d) \to GL(d)$ given by $T \mapsto TT^t$. Then $O(d)$ is the closed set $g^{-1}(I_d)$. Moreover the condition $TT^t = I_d$ implies $\det(T) \in \{\pm 1\}$ which is a polynomial bound on the entries of T, so $O(d)$ is closed and bounded in $\mathbb{R}^{d^2}$ and is thus compact. Now $SO(d)$ is the closed subset of $O(d)$ given by $\det|_{O(d)}^{-1}(1)$, so $SO(d)$ is compact as well. $\square$

Proposition 3.1.3. *$V_d(\ell)$ is an $SO(d)$-space and an $O(d)$-space.*

Proof. By Remark 3.1.1 it suffices to show that $V_d(\ell)$ is an $O(d)$ space. Given that $O(d)$ is a topological group and $V_d(\ell) \subset \mathbb{R}^{d(n-1)}$ is a Hausdorff topological space since subspaces of Hausdorff spaces are Hausdorff, it remains to define a continuous map $\Theta : O(d) \times V_d(\ell) \to V_d(\ell)$ that satisfies properties (1) and (2) above. Let T be an element of $O(d)$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Since T is a linear isometry we have $|T(\mathbf{x}) - T(\mathbf{y})| = |T(\mathbf{x} - \mathbf{y})| = |\mathbf{x} - \mathbf{y}|$. If $P = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d(\ell)$ then by definition we have $|\mathbf{v}_i - \mathbf{v}_{i-1}| = l_i$ for all $i = 1, \ldots, n$, and thus $|T(\mathbf{v}_i) - T(\mathbf{v}_{i-1})| = l_i$ for all $i = 1, \ldots, n$. Thus by defining $T(P) := (T(\mathbf{v}_1), \ldots, T(\mathbf{v}_{n-1}))$ we have a map $\Theta : O(d) \times V_d(\ell) \to V_d(\ell)$ defined by $\Theta(T, P) = T(P)$. Moreover conditions (1) and (2) above hold, as shown here:

(1)

$$\begin{aligned}
\Theta(T_1, \Theta(T_2, P)) &= \Theta(T_1, (T_2(\mathbf{v}_1), \ldots, T_2(\mathbf{v}_{n-1}))) \\
&= (T_1(T_2(\mathbf{v}_1)), \ldots, T_1(T_2(\mathbf{v}_{n-1}))) \\
&= (T_1T_2(\mathbf{v}_1), \ldots, T_1T_2(\mathbf{v}_{n-1})) \\
&= \Theta(T_1T_2, P);
\end{aligned}$$

(2)

$$\Theta(I_d, P) = (I_d(\mathbf{v}_1), \ldots, I_d(\mathbf{v}_{n-1})) = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) = P.$$

Since the $O(d)$ action is defined linearly, by viewing $O(d)$ as a subset of $(\mathbb{R}^d)^2$, and $V_d(\ell)$ as a subset of $\mathbb{R}^{d(n-1)}$ the map Θ is the restriction of a linear map $\Psi : (\mathbb{R}^d)^2 \times \mathbb{R}^{d(n-1)} \to \mathbb{R}^{d(n-1)}$. Thus Θ is continuous. $\square$

We will work mainly with $V_d(\ell)$ as an $SO(d)$ space, so the notation $[P]$ will be reserved for the $SO(d)$ orbit of P. If we wish to refer to the $O(d)$ orbit of P we will simply write $O(d)(P)$.

Definition 3.1.4. *The* moduli space of closed n-gons in $\mathbb{R}^d$ *is*

$$M_d(\ell) := V_d(\ell)/SO(d).$$

For $k = 1, \ldots, d$, the k-stratum of $M_d(\ell)$ is $M_d^k(\ell) := \{\pi(P) : P \in V_d^k(\ell)\}$. For $i = 1, \ldots, d-1$ the above-i layer *of $M_d(\ell)$ is $M_d^{>i}(\ell) := \bigcup_{k>i} M_d^k(\ell)$.*

Lemma 3.1.5. *The orbit space $M_d(\ell)$ is compact Hausdorff, and $\pi : V_d(\ell) \to M_d(\ell)$ is closed. Also, the actions of $O(d)$ and $SO(d)$ on $V_d(\ell)$ are proper.*

Proof. Theorem 3.1 in Chapter I of [Bre72] says, in part, that if X is a G-space and G is compact, then X/G is Hausdorff and $\pi : X \to X/G$ is closed. If, in addition, X is compact then X/G is compact. Since $SO(d)$ is compact we have $M_d(\ell)$ is Hausdorff and π is closed. Also, $V_d(\ell)$ is closed since it is an algebraic variety, and it is bounded by the conditions $|\mathbf{v}_i - \mathbf{v}_{i-1}| = l_i$ for $i = 1, \ldots, n$, so $V_d(\ell)$ is closed and bounded in $\mathbb{R}^{d(n-1)}$ and thus compact. Therefor $M_d(\ell)$ is compact. The actions of $O(d)$ and $SO(d)$ on $V_d^d(\ell)$ are proper since $V_d(\ell)$ is Hausdorff and $SO(d)$ and $O(d)$ are compact [Bou98]. $\square$

Lemma 3.1.6. *The actions of $O(d)$ and $SO(d)$ on $V_d^d(\ell)$ are free.*

Proof. It is enough to show that $O(d)$ acts freely on $V_d^d(\ell)$. Let $P = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d^d(\ell)$ and let $T \in O(d)$ such that $T(P) = P$. We aim to show that T is the identity. Since

$\dim(P) = d$ the set $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ contains a basis $(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_d})$ for $\mathbb{R}^d$. Since $T(\mathbf{v}_{i_j}) = \mathbf{v}_{i_j}$ for all $j = 1, \dots, d$, $T = I_d$. □

Proposition 3.1.7. *The stratum $M_d^1(\ell)$ is a discrete set of* $\operatorname{depth}(\ell)$*-many points, and the stratum $M_d^d(\ell)$ is a manifold.*

Proof. Note that $M_d^1(\ell)$ is empty if $\operatorname{depth}(\ell) = 0$ and $M_d^d(\ell)$ is empty if $d \geq n$, neither of which contradict the statement of the proposition. To see that $M_d^d(\ell)$ is a manifold, we note that by Proposition 2.1.9, $V_d^d(\ell)$ is an open subset of $V_d(\ell)$ and thus is open in $V_d^\circ(\ell)$, so $V_d^d(\ell)$ is a manifold. By Lemmas 3.1.6 and 3.1.5, the action on $V_d^d(\ell)$ by $SO(d)$ is free and proper. Thus since $SO(d)$ is a Lie group that acts smoothly on $\mathbb{R}^d$, and thus on $V_d(\ell)$, $M_d^d(\ell) = \pi\left(V_d^d(\ell)\right)$ is a manifold by the Quotient Manifold Theorem ([Lee13]). To see that $M_d^1(\ell)$ is a discrete set of $\operatorname{depth}(\ell)$-many points, we note that by Proposition 2.3.5 $V_d(\ell)$ is the disjoint union of $\operatorname{depth}(\ell)$-many spheres. The result then follows by the observation that polygons P and P' lie on the same sphere in $V_d(\ell)$ if and only if $[P] = [P']$. □

Lemma 3.1.8. *Given a permutation $\sigma \in S_n$, let $\sigma(\ell) = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$. The moduli spaces $M_d(\ell)$ and $M_d(\sigma(\ell))$ are homeomorphic.*

Proof. In Lemma 2.1.4 we showed that $V_d(\ell)$ and $V_d(\sigma(\ell))$ are isomorphic as varieties, by showing that $E_d(\ell)$ and $E_d(\sigma(\ell))$ are isomorphic as varieties. It remains to show that he isomorphism ψ_σ from the proof of Lemma 2.1.4 is $SO(d)$-equivariant. Let $P = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in E_d(\ell)$. Then we have

$$\psi_\sigma(T(P)) = \psi_\sigma(T(\mathbf{p}_1), \dots, T(\mathbf{p}_n)) = (T(\mathbf{p}_{\sigma(1)}), \dots, T(\mathbf{p}_{\sigma(n)})) = T(\psi_\sigma(P)).$$

□

3.2 Intersections of spheres

In this section we take a slight detour to develop a fact about the intersections of finitely many spheres in $\mathbb{R}^d$. This fact, stated as Proposition 3.2.3, plays a key role in leading to the main result of this chapter. Recall from Section 1.3 that given an affine subspace $A \subset \mathbb{R}^d$, an A-sphere is a set of the form

$$S(A, \mathbf{c}, \rho) := \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in A, |\mathbf{x} - \mathbf{c}| = \rho\right\}.$$

For the remainder of this section we simply say "sphere" to mean "A-sphere". We say a sphere $S(A, \mathbf{c}, \rho)$ is *full-dimensional* if $A = \mathbb{R}^d$, in which case we may write $S(\mathbf{c}, \rho)$ to mean $S(\mathbb{R}^d, \mathbf{c}, \rho)$. Given $S = S(A, \mathbf{c}, \rho)$ we write $\overline{S}$ to mean $S(\mathbf{c}, \rho)$, i.e., the smallest full dimensional sphere containing S. A sphere $S(A, \mathbf{c}, \rho)$ is *orthogonal* to an affine subspace B if $\mathbf{c} \in B$ and A and B are orthogonal as affine spaces. Lemma 3.2.1 shows that the intersection of a sphere and a plane is a sphere (or empty or a singleton), Lemma 3.2.2 uses this fact to show that the intersection of two spheres is a sphere that is orthogonal to the original spheres' centers (or empty or a singleton), and Proposition 3.2.3 uses induction to extend this result to finitely many spheres.

Lemma 3.2.1. *Let $S = S(A, \mathbf{c}, \rho)$ be a sphere in $\mathbb{R}^d$ and let B be an affine subspace of $\mathbb{R}^d$. Then $S \cap B$ is either empty, a singleton, or the sphere $S(A \cap B, \mathbf{c}', \rho')$ where $\mathbf{c}'$ is the orthogonal projection of $\mathbf{c}$ onto B and $\rho'^2 = \rho^2 - |\mathbf{c} - \mathbf{c}'|^2$.*

Proof. Let $S, B, \mathbf{c}', \rho'$ be as above. Since $\mathbf{c}'$ is the orthogonal projection of $\mathbf{c}$ onto B, then $|\mathbf{c} - \mathbf{c}'| = \rho$ if and only if $S \cap B = \{\mathbf{c}'\}$ and $|\mathbf{c} - \mathbf{c}'| > \rho$ if and only if $S \cap B = \varnothing$. Thus if $S \cap B$ is neither empty nor a singleton then $\rho' > 0$, and it remains to show that $S \cap B = S(A \cap B, \mathbf{c}', \rho')$. Note that for any $\mathbf{x} \in B$ we have

$$|\mathbf{x} - \mathbf{c}'|^2 + |\mathbf{c} - \mathbf{c}'|^2 = |\mathbf{x} - \mathbf{c}|^2. \tag{3.1}$$

Let $\mathbf{x} \in S \cap B$. Since $\mathbf{x} \in S$ we have $|\mathbf{x} - \mathbf{c}| = \rho$, thus by equation (3.1) we have $|\mathbf{x} - \mathbf{c}'|^2 = \rho^2 - |\mathbf{c} - \mathbf{c}'|^2 = \rho'^2$, so $\mathbf{x} \in S(A \cap B, \mathbf{c}', \rho')$. Now let $\mathbf{x} \in S(A \cap B, \mathbf{c}', \rho')$. Then $|\mathbf{x} - \mathbf{c}'| = \rho'$ so by equation (3.1), $|\mathbf{x} - \mathbf{c}|^2 = \rho'^2 + |\mathbf{c} - \mathbf{c}'|^2 = \rho^2$, so $\mathbf{x} \in S$. Since $\mathbf{x}$ is also in B, we have $\mathbf{x} \in S \cap B$. ☐

Lemma 3.2.2. *Let $S_1 = S(A_1, \mathbf{c}_1, \rho_1)$, $S_2 = S(A_2, \mathbf{c}_2, \rho_2)$ be spheres in $\mathbb{R}^d$. Then $S_1 \cap S_2$ is either empty, a singleton, or a sphere in $\mathbb{R}^d$ orthogonal to* $\mathrm{Aff}(\mathbf{c}_1, \mathbf{c}_2)$.

Proof. It suffices to prove the statement for full-dimensional spheres. To see why, note that if $S_1 \cap S_2$ is neither empty nor a singleton, then $\overline{S_1} \cap \overline{S_2}$ is neither empty nor a singleton. Thus if the statement holds for full-dimensional spheres, then $\overline{S_1} \cap \overline{S_2}$ is a sphere orthogonal to $\mathrm{Aff}(\mathbf{c}_1, \mathbf{c}_2)$. Now,

$$S_1 \cap S_2 = (\overline{S_1} \cap A_1) \cap (\overline{S_2} \cap A_2) = (\overline{S_1} \cap \overline{S_2}) \cap (A_1 \cap A_2),$$

and thus $S_1 \cap S_2$ is also a sphere by Lemma 3.2.1, and is orthogonal to $\mathrm{Aff}(\mathbf{c}_1, \mathbf{c}_2)$ since it is a subset of $\overline{S_1} \cap \overline{S_2}$. Thus we may assume S_1 and S_2 are full-dimensional spheres in $\mathbb{R}^d$. Without loss of generality let $\mathbf{c}_1 = \mathbf{0}$ so that

$$S_1 = S(\mathbf{0}, \rho_1) = \{(x_1, \dots, x_d) : x_1^2 + \cdots + x_d^2 = \rho_1^2\}, \tag{3.2}$$

and let $\mathbf{c}_2 = (c, 0, \dots, 0)$ for some $c \neq 0$ so that

$$S_2 = S(\mathbf{c}_2, \rho_2) = \{(x_1, \dots, x_d) : (x_1 - c)^2 + x_2^2 + \cdots + x_d^2 = \rho_2^2\}. \tag{3.3}$$

Suppose $S_1 \cap S_2 \neq \varnothing$ and let $\mathbf{x} = (x_1, \dots, x_d) \in S_1 \cap S_2$. From (3.2) and (3.3) we have $x_1 = \frac{\rho_1^2 - \rho_2^2 + c^2}{2c}$. Let $\gamma = \frac{\rho_1^2 - \rho_2^2 + c^2}{2c}$. Since $\mathbf{x} \in S_1$ we have $\gamma \leq \rho_1$; moreover if $\gamma = \rho_1$ then $\mathbf{x}$ is the only point on S_1 with $x_1 = \gamma$, and thus $S_1 \cap S_2 = \{\mathbf{x}\}$. Thus if $S_1 \cap S_2$ is neither empty nor a singleton we have $\gamma < \rho_1$. Let $\rho = \sqrt{\rho_1^2 - \gamma^2} > 0$ and let $A = \{(\gamma, x_2, \dots, x_d) : x_i \in \mathbb{R}\}$. We claim that $S_1 \cap S_2 = S(A, (\gamma, 0, \dots, 0), \rho)$. Note that whenever $\mathbf{x} \in A$ we have

$$|\mathbf{x} - \mathbf{0}|^2 = |\mathbf{x} - (\gamma, 0, \dots, 0)|^2 + |\mathbf{0} - (\gamma, 0, \dots, 0)|^2, \tag{3.4}$$

and

$$|\mathbf{x} - \mathbf{c}_2|^2 = |\mathbf{x} - (\gamma, 0, \dots, 0)|^2 + |\mathbf{c} - (\gamma, 0, \dots, 0)|^2. \tag{3.5}$$

Let $\mathbf{x} \in S_1 \cap S_2$. Then $\mathbf{x} \in A$ so by (3.4) we have

$$\begin{aligned} |\mathbf{x} - (\gamma, 0, \dots, 0)|^2 &= |\mathbf{x} - \mathbf{0}|^2 - |\mathbf{0} - (\gamma, 0, \dots, 0)|^2 \\ &= \rho_1^2 - \gamma^2, \end{aligned}$$

so $\mathbf{x} \in S(A, (\gamma, 0, \dots, 0), \rho)$. Now let $\mathbf{x} \in S(A, (\gamma, 0, \dots, 0), \rho)$. Then again $\mathbf{x} \in A$ so by (3.4) we have

$$\begin{aligned} |\mathbf{x} - \mathbf{0}|^2 &= |\mathbf{x} - (\gamma, 0, \dots, 0)|^2 + |\mathbf{0} - (\gamma, 0, \dots, 0)|^2 \\ &= \rho_1^2 - \gamma^2 + \gamma^2 \\ &= \rho_1^2, \end{aligned}$$

so $\mathbf{x} \in S_1$, and by (3.5) we have

$$\begin{aligned} |\mathbf{x} - \mathbf{c}_2|^2 &= |\mathbf{x} - (\gamma, 0, \dots, 0)|^2 + |\mathbf{c} - (\gamma, 0, \dots, 0)|^2 \\ &= \rho_1^2 - \gamma^2 + (c - \gamma)^2 \\ &= \rho_2^2, \end{aligned}$$

so $\mathbf{x} \in S_2$. Finally, A is clearly orthogonal to $\mathrm{Aff}(\mathbf{0}, \mathbf{c}_2) = \{(x_1, 0, \dots, 0) : x_1 \in \mathbb{R}\}$. □

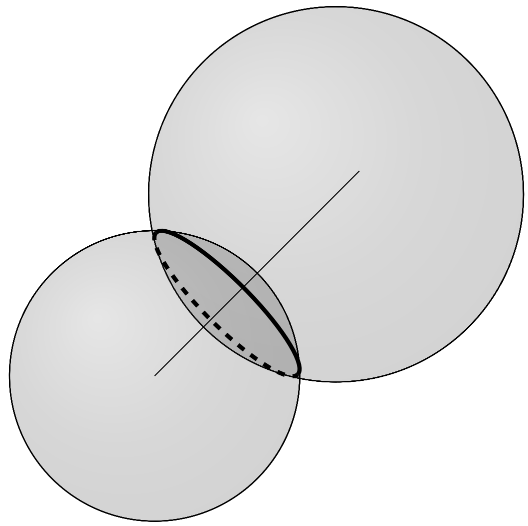


Figure 3.1: Two spheres and their intersection, which by Lemma 3.2.2 is another sphere orthogonal to the line through the original spheres' centers. This image is attributed to [Wol12].

Proposition 3.2.3. *Let $S_1, \dots, S_k$ be spheres in $\mathbb{R}^d$ where $S_i = (A_i, \mathbf{c}_i, \rho_i)$. Their intersection $\bigcap_{i=1}^k S_i$ is either empty, a singleton, or a sphere in $\mathbb{R}^d$ orthogonal to* $\mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_k)$.

Proof. We induct on k. The base case $k = 2$ is Lemma 3.2.2. For the inductive step let $S_1, \dots, S_k$ be as above and assume the Lemma holds for $k - 1$ spheres. Let

$$S = \bigcap_{i=1}^{k} S_i = \left(\bigcap_{i=1}^{k-1} S_i \right) \cap S_k.$$

If S is neither empty nor a singleton, then $\bigcap_{i=1}^{k-1} S_i$ is neither empty nor a singleton, thus $\bigcap_{i=1}^{k-1} S_i$ is a sphere orthogonal to $\mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$. Then by the base case, S is a sphere orthogonal to $\mathrm{Aff}(\mathbf{c}, \mathbf{c}_k)$, where $\mathbf{c}$ is the center of $\bigcap_{i=1}^{k-1} S_i$. Since $\bigcap_{i=1}^{k-1} S_i$ is orthogonal to $\mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$ and since $S \subset \bigcap_{i=1}^{k-1} S_i$, then S is orthogonal to $\mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$. Since S is orthogonal to both $\mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$ and $\mathrm{Aff}(\mathbf{c}, \mathbf{c}_k)$, S is orthogonal to $\mathrm{Aff}(\mathbf{c}, \mathbf{c}_1, \dots, \mathbf{c}_k)$. Finally, since $\mathbf{c}$ is the center of $\bigcap_{i=1}^{k-1} S_i$ we have $\mathbf{c} \in \mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_{k-1})$, and thus $\mathrm{Aff}(\mathbf{c}, \mathbf{c}_1, \dots, \mathbf{c}_k) = \mathrm{Aff}(\mathbf{c}_1, \dots, \mathbf{c}_k)$, completing the proof. ∎

In order to make use of Proposition 3.2.3 in the next section, we will need a fact about how the orthogonal group $O(d)$ acts on intersections of spheres. It is stated as Lemma 3.2.5.

Definition 3.2.4. *Given a subset $A \subset \mathbb{R}^d$, the* point-wise stabilizer of A in $O(d)$ *is*

$$O(d)_A \coloneqq \{T \in O(d) : T(\mathbf{x}) = \mathbf{x} \ \forall \, \mathbf{x} \in A\}.$$

Lemma 3.2.5. *Let S be a sphere in $\mathbb{R}^d$ orthogonal to a linear subspace $L \subset \mathbb{R}^d$. Then $O(d)_L$ acts transitively on S.*

Proof. Let S be a sphere in $\mathbb{R}^d$ with center $\mathbf{c}$ and radius ρ, orthogonal to a linear subspace $L \subset \mathbb{R}^d$, and let $\mathbf{x}, \mathbf{y} \in S$. Let $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ be an orthonormal basis for L. Since $\mathbf{x} - \mathbf{c}$ and $\mathbf{y} - \mathbf{c}$ are both orthogonal to L, we may extend $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ to two orthonormal bases $(\mathbf{u}_1, \dots, \mathbf{u}_k, \frac{\mathbf{x}-\mathbf{c}}{|\mathbf{x}-\mathbf{c}|}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_d)$ and $(\mathbf{u}_1, \dots, \mathbf{u}_k, \frac{\mathbf{y}-\mathbf{c}}{|\mathbf{y}-\mathbf{c}|}, \mathbf{t}_{k+2}, \dots, \mathbf{t}_d)$. Since $O(d)$ acts transitively on orthonormal bases for $\mathbb{R}^d$, there exists $T \in O(d)$ such that $T(\mathbf{u}_i) = \mathbf{u}_i$ for all $i = 1, \dots, k$ and $T(\frac{\mathbf{x}-\mathbf{c}}{|\mathbf{x}-\mathbf{c}|}) = \frac{\mathbf{y}-\mathbf{c}}{|\mathbf{y}-\mathbf{c}|}$. Since $T(\mathbf{u}_i) = \mathbf{u}_i$ for all $i = 1, \dots, k$, $T \in O(d)_L$. Since T is linear, $|\mathbf{x} - \mathbf{c}| = |\mathbf{y} - \mathbf{c}| = \rho$, and $\mathbf{c} \in L$, we have

$$T(\mathbf{x}) = T(\mathbf{x} - \mathbf{c}) + T(\mathbf{c}) = \rho \cdot T\left(\frac{\mathbf{x} - \mathbf{c}}{|\mathbf{x} - \mathbf{c}|}\right) + \mathbf{c} = \rho \cdot \frac{\mathbf{y} - \mathbf{c}}{|\mathbf{y} - \mathbf{c}|} + \mathbf{c} = \mathbf{y} - \mathbf{c} + \mathbf{c} = \mathbf{y}.$$

$\square$

3.3 The diagonals map

Let $P = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \in V_d(\ell)$. Let $i, j \in \{0, \ldots, n-1\}$ with $i < j-1$ and $(i,j) \neq (0, n-1)$. Then $\mathbf{v}_i, \mathbf{v}_j$ are nonadjacent vertices of $P \in V_d(\ell)$. Define the map

$$\mathrm{diag}^d_{i,j} : V_d(\ell) \to \mathbb{R}$$

$$P \mapsto |\mathbf{v}_j - \mathbf{v}_i|$$

that takes a polygon to its (i,j)-th diagonal length. We claim the map $\mathrm{diag}^d_{i,j}$ is continuous. Note that it is the restriction to $V_d(\ell)$ of the map $g : \mathbb{R}^{d(n-1)} \to \mathbb{R}$ defined by $g((\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})) = |(\mathbf{x}_i - \mathbf{x}_j)|$, and g is the composition of the linear map $(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \mapsto (\mathbf{x}_i - \mathbf{x}_j)$ with the distance map $\mathbf{x} \mapsto |\mathbf{x}|$. Thus g is continuous, and therefor so is $\mathrm{diag}^d_{i,j}$. Order the diagonals lexicographically so that $(i,j) < (k,l)$ if $i < j$ or if $i = k$ and $j < l$. There are $\binom{n}{2} - n$ diagonals of P since there are $\binom{n}{2}$ vertex pairs and n of them are adjacent pairs. The *diagonals map* on $V_d(\ell)$ is the map

$$\mathrm{diag}^d : V_d(\ell) \to \mathbb{R}^{\binom{n}{2}-n}$$

$$P \mapsto (\mathrm{diag}^d_{i,j}(P))$$

that sends a polygon to its ordered list of diagonal lengths (see Figure 3.2). Since the component maps of diag^d are continuous, diag^d is continuous.

Proposition 3.3.1 shows that the fibers of diag^d are the $O(d)$ orbits of $V_d(\ell)$. Proposition 3.3.2 then shows that the $O(d)$ orbit and the $SO(d)$ orbit of a polygon are identical if the polygon has small dimension relative to the ambient space, and thus for large enough d the fibers of diag^d are the $SO(d)$ orbits of $V_d(\ell)$. This allows the diagonals map to descend to an embedding on $M_d(\ell)$, as shown in Theorem 3.3.4.

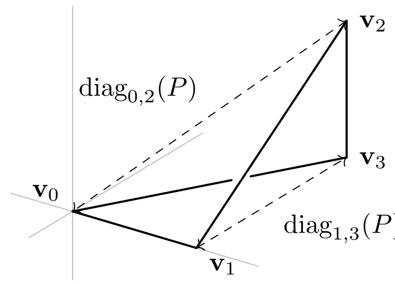


Figure 3.2: A 4-gon with its two diagonal lengths labeled.

Proposition 3.3.1. *For all* $P \in V_d(\ell)$, $\mathrm{diag}^{-1}(\mathrm{diag}(P)) = O(d)(P)$.

Proof. The inclusion $O(d)(P) \subset \mathrm{diag}^{-1}(\mathrm{diag}(P))$ is immediate since $O(d)$ consists of isometries, and thus $\mathrm{diag}(P) = \mathrm{diag}(Q)$ for all $Q \in O(d)(P)$. Now we prove the inclusion

$$\mathrm{diag}^{-1}(\mathrm{diag}(P)) \subset O(d)(P).$$

Let $Q \in \mathrm{diag}^{-1}(\mathrm{diag}(P))$ and write $P = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$, $Q = (\mathbf{w}_1, \dots, \mathbf{w}_{n-1})$. Since $\mathbf{v}_1$, $\mathbf{w}_1$ lie on the sphere centered at the origin $S(\mathbf{0}, l_1)$, there exists $T_1 \in O(d)$ such that $T_1(\mathbf{w}_1) = \mathbf{v}_1$. Let $Q_1 = T_1(Q) = (\mathbf{v}_1, T_1(\mathbf{w}_2), \dots, T_1(\mathbf{w}_{n-1}))$. Note that $\mathbf{v}_2$ and $T_1(\mathbf{w}_2)$ both lie on the intersection of spheres $S(\mathbf{0}, \mathrm{diag}_{0,2}(P)) \cap S(\mathbf{v}_1, l_2)$. By Proposition 3.2.3, this intersection is either a singleton or a sphere orthogonal to the linear subspace $L = \mathrm{Aff}(\mathbf{0}, \mathbf{v}_1)$. If a singleton, take $T_2 = I_d \in O(d)$; if not, by Lemma 3.2.5 there exists $T_2 \in O(d)_L$ such that $T_2(\mathbf{w}_2) = \mathbf{v}_2$. In either case let $Q_2 = T_2(Q_1) = (\mathbf{v}_1, \mathbf{v}_2, T_2 \circ T_1(\mathbf{w}_3), \dots, T_2 \circ T_1(\mathbf{w}_{n-1}))$. Next we note that $\mathbf{v}_3$ and $T_2 \circ T_1(\mathbf{w}_3)$ both lie on the intersection of spheres $S(\mathbf{0}, \mathrm{diag}_{0,3}(P)) \cap S(\mathbf{v}_1, \mathrm{diag}_{1,3}(P)) \cap S(\mathbf{v}_2, l_3)$. Again by Proposition 3.2.3, this intersection is either a singleton or a sphere orthogonal to the linear subspace $L = \mathrm{Aff}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$. If a singleton, take $T_3 = I_d \in O(d)$; if not, by Lemma 3.2.5 there exists $T_3 \in O(d)_L$ such that $T_3(T_2 \circ T_1(\mathbf{w}_3)) = \mathbf{v}_3$. In either case let $Q_3 = T_3(Q_2) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, T_3 \circ T_2 \circ T_1(\mathbf{w}_4), \dots, T_3 \circ T_2 \circ T_1(\mathbf{v}_{n-1}))$. Continuing in this way completes the proof, taking $T = T_{n-1} \circ \cdots \circ T_1$ as the element of $O(d)$ for which $T(Q) = P$. $\square$

Proposition 3.3.2. *Let $P \in V_d(\ell)$. Then $O(d)(P) = SO(d)(P)$ if and only if* $\dim(P) < d$.

Proof. Let $P \in V_d(\ell)$ with $\dim(P) = k < d$. Clearly $SO(d)(P) \subset O(d)(P)$ since $SO(d) \subset O(d)$. To see that $O(d)(P) \subset SO(d)(P)$ let $P = (\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$, and let $T(P) \in O(d)(P)$. We seek a $T' \in SO(d)$ so that $T'(P) = T(P)$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_k)$ be an orthonormal basis for $\mathrm{Span}(\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \subsetneq \mathbb{R}^d$. Then $(T(\mathbf{e}_1), \ldots, T(\mathbf{e}_k))$ is an orthonormal basis for $\mathrm{Span}(\mathbf{0}, T(\mathbf{v}_1), \ldots, T(\mathbf{v}_{n-1})) \subsetneq \mathbb{R}^d$. Since $SO(d)$ acts transitively on orthonormal bases of proper subspaces of $\mathbb{R}^d$, there exists $T' \in SO(d)$ such that $T'(\mathbf{e}_i) = T(\mathbf{e}_i)$ for $i = 1, \ldots, k$. Thus $T'(\mathbf{v}_i) = T(\mathbf{v}_i)$ for $i = 1, \ldots, n-1$, so $T'(P) = T(P)$. For the other direction, suppose $\dim(P) = d$. We want to show $SO(d)(P) \subsetneq O(d)(P)$. Let $T \in O(d) \setminus SO(d)$. We claim that $T(P) \in O(d)(P) \setminus SO(d)(P)$. Suppose for a contradiction there exists $T' \in SO(d)(P)$ such that $T'(P) = T(P)$. Then $T^{-1}T'(P) = P$. Since $\dim(P) = d$, Lemma 3.1.6 says $T^{-1}T' = I_d$, and thus $T = T'$, contradicting $T' \in SO(d)$ and $T \notin SO(d)$. $\square$

Since $SO(d)$ consists of isometries, the maps $\mathrm{diag}^d_{i,j}$ and diag^d on $V_d(\ell)$ descend to maps on $M_d(\ell)$:

$$[\mathrm{diag}^d_{i,j}] : M_d(\ell) \to \mathbb{R}^{\binom{n}{2}-n}$$

$$[P] \mapsto \mathrm{diag}^d_{i,j}(P);$$

$$[\mathrm{diag}^d] : M_d(\ell) \to \mathbb{R}^{\binom{n}{2}-n}$$

$$[P] \mapsto \mathrm{diag}^d(P).$$

We may omit the superscript d from the diagonals maps when the dimension of the ambient space $\mathbb{R}^d$ is clear from context. Let $\mathbf{D}_d(\ell)$ denote the image $[\mathrm{diag}^d](M_d(\ell)) = \mathrm{diag}^d(V_d(\ell))$.

Lemma 3.3.3. *The map* $[\mathrm{diag}] : M_d(\ell) \to \mathbb{R}^{\binom{n}{2}-n}$ *is continuous.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} V_d(\ell) \arrow[dr, "\mathrm{diag}"] \arrow[dd, "\pi"] & \\ & \mathbb{R}^{\binom{n}{2}-n} \\ M_d(\ell) \arrow[ur, "{[\mathrm{diag}]}"'] & \end{tikzcd}$$

Let $U \subset \mathbb{R}^{\binom{n}{2}-n}$ be open. Since diag is continuous, $\mathrm{diag}^{-1}(U)$ is open in $E_d(\ell)$. Since π is open, $\pi(\mathrm{diag}^{-1}(U))$ is open in $M_d(\ell)$. It remains to show $[\mathrm{diag}^d]^{-1}(U) = \pi(\mathrm{diag}^{-1}(U))$. Indeed both $[\mathrm{diag}^d]^{-1}(U)$ and $\pi(\mathrm{diag}^{-1}(U))$ are equal to the set $\{[P] : \mathrm{diag}(P) \in U\}$. $\square$

Theorem 3.3.4. *The diagonals map $[\mathrm{diag}^d] : M_d(\ell) \to \mathbb{R}^{\binom{n}{2}-n}$ is an embedding if and only if $d \geq n$.*

Proof. Since $[\mathrm{diag}^d]$ is continuous, $M_d(\ell)$ is compact, and $\mathbb{R}^{\binom{n}{2}-n}$ is Hausdorff, it suffices to show that $[\mathrm{diag}^d]$ is injective if and only if $d \geq n$. Suppose $d \geq n$ and suppose $[\mathrm{diag}^d]([P]) = [\mathrm{diag}]^d([Q])$. Then $\mathrm{diag}^d(P) = \mathrm{diag}^d(Q)$, so by Lemma 3.3.1, $O(d)(P) = O(d)(Q)$. Since $d \geq n$, $\dim(P) = \dim(Q) < d$ so by Proposition 3.3.2, $SO(d)(P) = SO(d)(Q)$, and thus $[P] = [Q]$. Now suppose $d < n$. By Lemma 2.1.7, $V_d^d(\ell) \neq \varnothing$. To show that $[\mathrm{diag}^d]$ is not injective, let $P \in V_d^d(\ell)$ and let $T \in O(d) \setminus SO(d)$. Then $[T(P)] \neq [P]$, but $[\mathrm{diag}^d]([P]) = [\mathrm{diag}^d]([T(P)]$. $\square$

3.4 Stabilization of orbit spaces

Fix $n \geq 3$. Given $(x_1, \ldots, x_d) \in \mathbb{R}^d$ let $(\mathbf{x}, 0) = (x_1, \ldots, x_d, 0) \in \mathbb{R}^{d+1}$. For all $d > 2$ define the map

$$a_d : \mathbb{R}^{d(n-1)} \to \mathbb{R}^{(d+1)(n-1)}$$

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \mapsto ((\mathbf{x}_1, 0), \ldots, (\mathbf{x}_{n-1}, 0)).$$

Clearly a_d is continuous. Moreover if $|\mathbf{v}_i - \mathbf{v}_{i-1}| = l_i$ for all $i = 1, \ldots, n$, then we have $|(\mathbf{v}_i, 0) - (\mathbf{v}_{i-1}, 0)| = l_i$ for all $i = 1, \ldots, n$, so the map a_d restricts to a continuous map $V_d(\ell) \to V_{d+1}(\ell)$.

Lemma 3.4.1. *If $O(d)(P) = O(d)(Q)$, then $O(d+1)(a_d(P)) = O(d+1)(a_d(Q))$.*

Proof. If $O(d)(P) = O(d)(Q)$ we may write $P = T(Q)$ for some $T \in O(d)$. Note that $T \oplus I_1 \in O(d+1)$, where I_1 is the 1×1 identity matrix, and we have

$$(T \oplus I_1)(a_d(Q)) = a_d(T(Q)) = a_d(P).$$

Thus $a_d(P) \in O(d+1)(a_d(Q))$, but since distinct $O(d+1)$ orbits are disjoint we have $O(d+1)(a_d(P)) = O(d+1)(a_d(Q))$. □

Now, for $d > 2$ define the map

$$\alpha_d : M_d(\ell) \to M_{d+1}(\ell) \tag{3.6}$$

$$[P] \mapsto [a_d(P)].$$

Lemma 3.4.2. *The map α_d is continuous for all $d > 2$.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} V_d(\ell) \arrow[r, "a_d"] \arrow[d, "\pi_d"] & V_{d+1}(\ell) \arrow[d, "\pi_{d+1}"] \\ M_d(\ell) \arrow[r, "\alpha_d"] & M_{d+1}(\ell) \end{tikzcd}$$

Let U be an open subset of $M_{d+1}(\ell)$. Since π_{d+1} and a_d are continuous and π_d is open, we have $\pi_d(a_d^{-1}(\pi_{d+1}^{-1}(U)))$ is open in $M_d(\ell)$, and it remains to show

$$\pi_d(a_d^{-1}(\pi_{d+1}^{-1}(U))) = \alpha_d^{-1}(U).$$

Note that commutativity gives us

$$\alpha_d(\pi_d(P)) = \pi_{d+1}(a_d(P)) \tag{3.7}$$

for all $P \in V_d(\ell)$. First we show $\alpha_d^{-1}(U) \subset \pi_d(a_d^{-1}(\pi_{d+1}^{-1}(U)))$. Let $[P] \in \alpha_d^{-1}(U)$. Then $\alpha_d(\pi_d(P)) \in U$, so by (3.7) we have $\pi_{d+1}(a_d(P)) \in U$. Thus $P \in a_d^{-1}(\pi_{d+1}^{-1}(U))$, so $[P] \in \pi(a_d^{-1}(\pi_{d+1}^{-1}(U)))$. For the other direction, let $[P] \in \pi(a_d^{-1}(\pi_{d+1}^{-1}(U)))$, and let $Q \in a_d^{-1}(\pi_{d+1}^{-1}(U))$ such that $\pi_d(Q) = [P]$. Since $Q \in a_d^{-1}(\pi_{d+1}^{-1}(U))$ we have $\pi_{d+1}(a_d(Q)) \in U$. Thus by (3.7) we have $\alpha_d(\pi_d(Q)) \in U$. But $\pi_d(Q) = [P]$, so $[P] \in \alpha_d^{-1}(U)$. □

Remark 3.4.3. *We point out here that a_d is an isometry, and thus $\operatorname{diag}^d(P) = \operatorname{diag}^{d+1}(a_d(P))$, and $[\operatorname{diag}^d]([P]) = [\operatorname{diag}^{d+1}](\alpha_d([P]))$ for all $P \in V_d(\ell)$.*

Lemma 3.4.4. *Let $[P] \in M_d(\ell)$ with $\dim(P) = k$. Then there is a representative*

$$(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \in [P]$$

such that $\operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) = \operatorname{Span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$.

Proof. If $k = d$ the statement is trivial. If $k < d$, let $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ be an orthonormal basis for $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \subsetneq \mathbb{R}^d$. Since $SO(d)$ acts transitively on orthonormal bases of proper subspaces of $\mathbb{R}^d$, there exists $T \in SO(d)$ such that $T(\mathbf{u}_i) = \mathbf{e}_i$ for all $i = 1, \dots, k$, where $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ is the standard basis on $\mathbb{R}^d$. Then $T(P) \subset \mathrm{Span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$. □

Proposition 3.4.5. *Let $\ell \in \mathrm{int}(D_n)$. The images $\mathbf{D}_d(\ell) = [\mathrm{diag}^d](M_d(\ell))$ for $d \geq 2$ form a chain of inclusions that stabilizes at $d = n-1$:*

$$\mathbf{D}_2(\ell) \hookrightarrow \mathbf{D}_3(\ell) \hookrightarrow \cdots \hookrightarrow \mathbf{D}_{n-1}(\ell) = \mathbf{D}_n(\ell) = \mathbf{D}_{n+1}(\ell) = \cdots$$

Proof. Remark 3.4.3 implies that $[\mathrm{diag}^d](M_d(\ell)) \subset [\mathrm{diag}^{d+1}](M_{d+1}(\ell))$. It remains to show that $[\mathrm{diag}^{d+1}](M_{d+1}(\ell)) \subset [\mathrm{diag}^d](M_d(\ell))$ if $d \geq n-1$. Suppose $d \geq n-1$ and let $[\mathrm{diag}^{d+1}]([P]) \in [\mathrm{diag}^{d+1}](M_{d+1}(\ell))$. Since $d+1 > n-1$, by Lemma 2.1.7, $k = \dim(P) < d+1$, so by Lemma 3.4.4 there is a representative $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \in [P]$ with $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \subset \mathrm{Span}(\mathbf{e}_1, \dots, \mathbf{e}_k) \subsetneq \mathbb{R}^{d+1}$. Thus if $p : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is projection onto the first d coordinates, we have $p(\mathbf{v}_i) \in \mathbb{R}^d$ and $|p(\mathbf{v}_i) - p(\mathbf{v}_{i-1})| = |\mathbf{v}_1 - \mathbf{v}_{i-1}| = l_i$ for all $i = 1, \dots, n$. Thus $[p(\mathbf{v}_1), \dots, p(\mathbf{v}_{n-1})] \in M_d(\ell)$. Since truncating zeros does not affect diagonal lengths, we have $[\mathrm{diag}^d]([p(\mathbf{v}_1), \dots, p(\mathbf{v}_{n-1})] = [\mathrm{diag}^{d+1}]([P])$, so $[\mathrm{diag}^{d+1}]([P]) \in [\mathrm{diag}^d](M_d(\ell))$. □

Theorem 3.4.6. *Let $\ell \in \mathrm{int}(D_n)$, and let $d > 2$. Then $M_d(\ell)$ and $M_{d+1}(\ell)$ are homeomorphic if and only if $d \geq n$.*

Proof. Let $\ell \in \mathrm{int}(D_n)$. We show that the map $\alpha_d : M_d(\ell) \to M_{d+1}(\ell)$ in Equation 3.6 is an embedding if and only if $d \geq n$. We first show that α_d is not injective if $d < n$. Since $\ell \notin b(D_n)$, Proposition 2.3.6 says there exist $P \in V_d(\ell)$ with $\dim(P) > 1$, so Lemma 2.1.8 says there exist $P \in V_d(\ell)$ with $\dim(P) = d$. Let $P \in V_d(\ell)$ with $\dim(P) = d$. Proposition 3.3.2 says the $SO(d)$ orbit of P is a proper subset of the $O(d)$ orbit of P, so there exists some $Q \in O(d)(P) \setminus [P]$, and thus $[P] \neq [Q]$. However, as we will see, $\alpha_d([P]) = \alpha_d([Q])$. Since $Q \in O(d)(P)$ we have $O(d)(P) = O(d)(Q)$. Thus by Lemma 3.4.1 we have

$$O(d+1)(a_d(P)) = O(d+1)(a_d(Q)). \tag{3.8}$$

But now $a_d(P)$ and $a_d(Q)$ are polygons of dimension d in $V_{d+1}(\ell)$, so Proposition 3.3.2 says their $O(d+1)$ orbits are equal to their $SO(d+1)$ orbits. Thus (3.8) says $[a_d(P)] = [a_d(Q)]$, and thus $\alpha_d([P]) = \alpha_d([Q])$.

Now we show that α_d is a homeomorphism if $d \geq n$. Consider the following commutative diagram.

$$\begin{tikzcd}
M_d(\ell) \arrow[rr, "\alpha_d"] \arrow[dr, "{[\mathrm{diag}^d]}"'] & & M_{d+1}(\ell) \arrow[dl, "{[\mathrm{diag}^{d+1}]}"] \\
& \mathbb{R}^{\binom{n}{2}-n} &
\end{tikzcd}$$

Remark 3.4.3 says $[\mathrm{diag}^d]([P]) = [\mathrm{diag}^{d+1}](\alpha_d([P]))$ for all $P \in V_d(\ell)$, and Theorem 3.3.4 says $[\mathrm{diag}^d]$ and $[\mathrm{diag}^{d+1}]$ are both embeddings, so the map $[\mathrm{diag}^{d+1}]^{-1} \circ [\mathrm{diag}^d]$ is a well-defined embedding equal to α_d. By Proposition 3.4.5 we have $[\mathrm{diag}^d](M_d(\ell)) = [\mathrm{diag}^{d+1}](M_{d+1}(\ell))$, so the embedding α_d is onto, and thus □

Corollary 3.4.7. *For all $2 \leq d < e$ define the map*

$$\alpha_{d,e} : M_d(\ell) \to M_e(\ell)$$

$$[P] \mapsto \alpha_{e-1} \circ \cdots \circ \alpha_d([P])$$

The pair $\langle M_d(\ell), \alpha_{d,e} \rangle$ defines a directed system of topological spaces. If $\ell \in \operatorname{int}(D_n)$

$$\varinjlim M_d(\ell) = M_n(\ell).$$

Proof. This is the content of Theorem 3.4.6 framed in the language of directed systems. $\square$

The relationship between the direct system of moduli spaces in Corollary 3.4.7 and the direct system of their diagonals images in Proposition 3.4.5 is summed up in the diagram below. The diagonals system stabilizes one dimension sooner than the moduli space system, because the diagonals map does not see the difference between $O(d)$ and $SO(d)$ orbits of polygons, which are still distinct in $M_{n-1}(\ell)$ but collapse in $M_n(\ell)$.

$$\begin{tikzcd}
M_2(\ell) \arrow[r] \arrow[d] & M_3(\ell) \arrow[r] \arrow[d] & \cdots \arrow[r] & M_{n-1}(\ell) \arrow[r] \arrow[d] & M_n(\ell) \arrow[r, equal] \arrow[d] & M_{n+1}(\ell) \arrow[r, equal] \arrow[d] & \cdots \\
\mathbf{D}_2(\ell) \arrow[r, hook] & \mathbf{D}_3(\ell) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbf{D}_{n-1}(\ell) \arrow[r, equal] & \mathbf{D}_n(\ell) \arrow[r, equal] & \mathbf{D}_{n+1}(\ell) \arrow[r, equal] & \cdots
\end{tikzcd}$$

Chapter 4: 4-gons

4.1 4-gons in $\mathbb{R}^2$

Millson and Kapovich proved the following theorem for 4-gons in $\mathbb{R}^2$.

Theorem 4.1.1 ([KM95]). *Let $\ell \in \mathrm{int}(D_4)$. If $\mathrm{depth}(\ell) = 0$, $M_2(\ell)$ is homeomorphic to a circle or the disjoint union of two circles. If $\mathrm{depth}(\ell) = 1$, $M_2(\ell)$ is homeomorphic to a bouquet of two circles. If $\mathrm{depth}(\ell) = 2$, $M_2(\ell)$ is homeomorphic to the union of two circles identified at two different points. If $\mathrm{depth}(\ell) = 3$, $M_2(\ell)$ is homeomorphic to the union of three circles in which each pair of circles has a common point.*

In this section we provide a constructive proof of Theorem 4.1.1 by presenting $M_2(\ell)$ as a CW-complex for all $\ell \in \mathrm{int}(D_4)$. In Subsection 4.1.1 we show that $M_2(\ell)$ is CW-complex. In Subsections 4.1.2, 4.1.3, 4.1.4, and 4.1.5, we describe the particular CW-complexes for $\mathrm{depth}(\ell) = 0, 1, 2$, and 3, respectively.

4.1.1 Building a CW-complex

In this subsection we show that $M_2(\ell)$ is a CW-complex of line segments. In Lemma 4.1.2 we define a line segment $[a, b]$ in terms of ℓ. In Lemma 4.1.4 we define a unique polygon in $V_d(\ell)$ for every value $t \in [a, b]$. Equations 4.2 use this unique polygon to define four "standard" polygons in $V_d(\ell)$ for every value $t \in [a, b]$. Lemma 4.1.6 and Corollary 4.1.7 show that these definitions give four embeddings of $[a, b]$ into $M_d(\ell)$. Lemmas 4.1.8 and 4.1.9 show that the embeddings fill all of $M_d(\ell)$ (except for a special case to be safely ignored until Subsection 4.1.5), and overlap only on their boundaries, if at all. This is enough to conclude that $M_d(\ell)$ is a CW-complex of line segments $[a, b]$.

Lemma 4.1.2. *Let $\ell = (l_1, l_2, l_3, l_4) \in \operatorname{int}(D_4)$. Then for all $d \geq 2$, the image $\operatorname{diag}^d_{0,2}(V_d(\ell))$ is a line segment $[a, b]$ with $0 \leq a < b$. In particular,*

$$\operatorname{diag}^d_{0,2}(V_d(\ell)) = [\max\{|l_1 - l_2|, |l_3 - l_4|\}, \min\{l_1 + l_2, l_3 + l_4\}].$$

Proof. We have $t \in \operatorname{diag}^d_{0,2}(V_d(\ell))$ if and only if $(l_1, l_2, t), (t, l_3, l_4) \in D_3$, that is, if and only if the triangle inequalities

$$t \leq l_1 + l_2, \quad t \leq l_3 + l_4, \quad t \geq l_1 - l_2, \quad t \geq l_3 - l_4, \quad t \geq l_2 - l_1, \quad t \geq l_4 - l_3 \tag{4.1}$$

are satisfied. Thus we have $\operatorname{diag}^d_{0,2}(V_2(\ell)) = [a, b]$ where

$$a = \max\{l_1 - l_2, l_2 - l_1, l_3 - l_4, l_4 - l_3\} = \max\{|l_1 - l_2|, |l_3 - l_4|\}, \quad b = \min\{l_1 + l_2,\, l_3 + l_4\}.$$

It remains to show $0 \leq a < b$. Clearly $a \geq 0$. If $a \geq b$ then either there exists $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$ such that $l_i - l_j \geq l_i + l_j$, contradicting $l_j > 0$, or there exist distinct $\{i, j\}, \{k, l\} \in \{\{1, 2\}, \{3, 4\}\}$ such that $l_i - l_j \geq l_k + l_l$, i.e. $l_i \geq l_j + l_k + l_l$, contradicting $\ell \in \operatorname{int}(D_4)$. Thus $a < b$, completing the proof. □

Remark 4.1.3. *It follows from the inequalities* (4.1) *that exactly one of $l_1 + l_2$ and $l_3 + l_4$ is in $[a, b]$. Thus if $l_1 + l_2 \in [a, b]$ then $b = l_1 + l_2$, and if $l_3 + l_4 \in [a, b]$ then $b = l_3 + l_4$. Similarly, exactly one of $|l_1 - l_2|$ and $|l_3 - l_4|$ is in $[a, b]$. Thus if $|l_1 - l_2| \in [a, b]$ then $a = |l_1 - l_2|$, and if $|l_3 - l_4| \in [a, b]$ then $a = |l_3 - l_4|$.*

Let $\mathbb{H}^1$ be the nonnegative ray in $\mathbb{R}$ and let $\mathbb{H}^2 = \mathbb{R} \times \mathbb{H}^1$ be the closed upper halfplane in $\mathbb{R}^2$.

Lemma 4.1.4. *Let $A = \mathbb{H}^2 \times (\mathbb{H}^1 \times \{0\}) \times \mathbb{H}^2 \subset \mathbb{R}^6$. For all $\ell \in \operatorname{int}(D_4)$ and all $t \in [a, b]$ not equal to* 0*, the fiber $(\operatorname{diag}_{0,2})^{-1}(t)$ contains a unique polygon $P \in A$. We denote this polygon $(\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t))$.*

Proof. Let $t \in [a,b]$ with $t \neq 0$. Given any $Q \in V_2(\ell)$ we have $O_2(Q) \subset V_2(\ell)$, and there is some $T \in O(3)$ such that $T(Q) \in A$. Thus if $P \in \mathrm{diag}_{0,2}^{-1}(t)$ there exists $T \in O(2)$ so that $T(P) \in A$ and since T is an isometry we have $T(P) \in \mathrm{diag}_{0,2}^{-1}(t)$, thus proving existence. To show uniqueness suppose $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in A \cap (\mathrm{diag}_{0,2}^2)^{-1}(t)$. Then $\mathbf{v}_2 = \mathbf{w}_2 = t\mathbf{e}_1$, so both $\mathbf{0}, \mathbf{v}_1, t\mathbf{e}_1$ and $\mathbf{0}, \mathbf{w}_1, t\mathbf{e}_1$ are vertices of triangles with $\mathbf{v}_1, \mathbf{w}_1 \in \mathbb{H}^2$. Thus since $t \neq 0$, $\mathbf{v}_1 = \mathbf{w}_1$. Similarly, both $\mathbf{0}, \mathbf{v}_3, t\mathbf{e}_1$ and $\mathbf{0}, \mathbf{w}_3, t\mathbf{e}_1$ are vertices of triangles with $\mathbf{v}_3, \mathbf{w}_3 \in \mathbb{H}^2$, so $\mathbf{v}_3 = \mathbf{w}_3$. See Figure 4.1. □

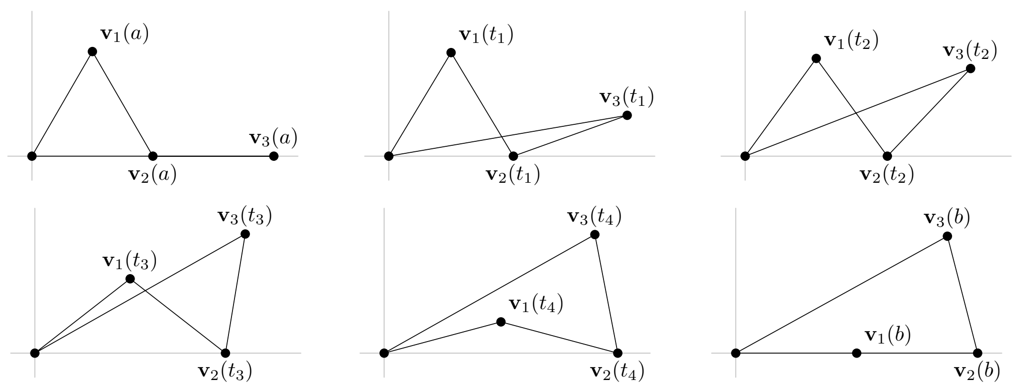


Figure 4.1: Given $\ell \in \mathrm{int}(D_4)$ so that $a \neq 0$, we show the unique polygon $(\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t))$ of Lemma 4.1.4 for six values $a < t_1 < t_2 < t_3 < t_4 < b \in [a,b]$.

Let $\overline{(x_1, x_2)} = (x_1, -x_2)$. Define the maps $f_i : [a,b] \to V_2(\ell), i \in \{1,2,3,4\}$ as follows:

$$\begin{aligned} f_1(t) &= (\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)) & \qquad f_2(t) &= (\mathbf{v}_1(t), \mathbf{v}_2(t), \overline{\mathbf{v}_3(t)}) \\ f_3(t) &= (\overline{\mathbf{v}_1(t)}, \mathbf{v}_2(t), \mathbf{v}_3(t)) & \qquad f_4(t) &= (\overline{\mathbf{v}_1(t)}, \mathbf{v}_2(t), \overline{\mathbf{v}_3(t)}) \end{aligned} \tag{4.2}$$

where $(\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t))$ is the unique polygon determined by t in Lemma 4.1.4 if $t \neq 0$, and $(\mathbf{v}_1(0), \mathbf{v}_2(0), \mathbf{v}_3(0)) = (l_1\mathbf{e}_2, \mathbf{0}, l_4\mathbf{e}_2)$. See Figure 4.2. We will be able to avoid the case $t = 0$ until Subsection 4.1.5.

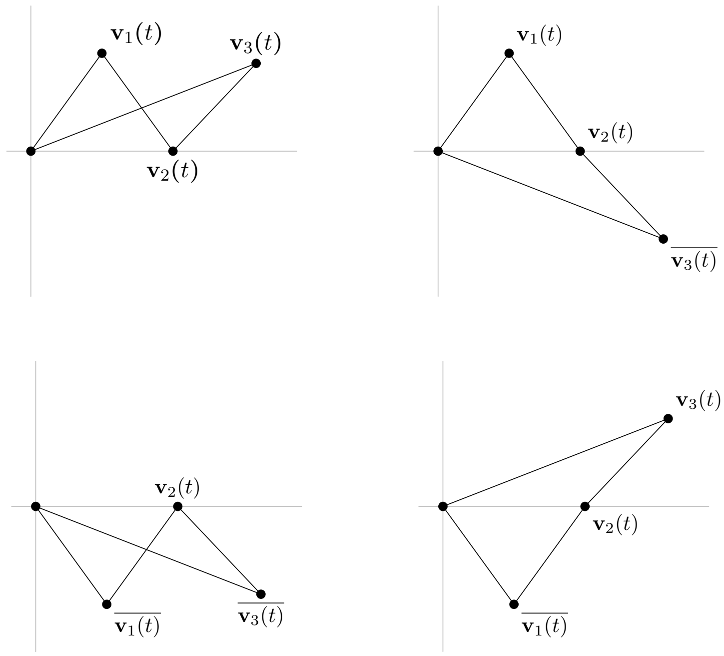


Figure 4.2: Given $\ell \in \operatorname{int}(D_4)$ and $t \in [a,b]$, $t \neq 0$, we show, clockwise from top left, the polygons $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$.

Remark 4.1.5. *Given $t \neq 0$, since $f_1(t)$ is the unique polygon in $\operatorname{diag}_{0,2}^{-1}(t) \cap A$, then $f_2(t) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is the unique polygon with $(\mathbf{v}_1, \mathbf{v}_2, \overline{\mathbf{v}_3}) \in \operatorname{diag}_{0,2}^{-1}(t) \cap A$, $f_3 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is the unique polygon with $(\overline{\mathbf{v}_1}, \mathbf{v}_2, \mathbf{v}_3) \in \operatorname{diag}_{0,2}^{-1}(t) \cap A$, and $f_4(t) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is the unique polygon with $(\overline{\mathbf{v}_1}, \mathbf{v}_2, \overline{\mathbf{v}_3}) \in \operatorname{diag}_{0,2}^{-1}(t) \cap A$.*

Lemma 4.1.6. *The maps f_i are embeddings $[a,b] \hookrightarrow V_2(\ell)$.*

Proof. It suffices to show that f_1 is an embedding. Note that f_1 has continuous inverse $\operatorname{diag}_{0,2} : \operatorname{im}(f_1) \to [a,b]$. Since continuous bijections from compact spaces to Hausdorff spaces are homeomorphisms, and since $[a,b]$ is Hausdorff, it remains to show that $\operatorname{im}(f_1)$ is a compact subspace of $V_2(\ell)$. Since $V_2(\ell)$ is itself compact it suffices to show that $\operatorname{im}(f_1)$ is closed in $V_2(\ell)$. The set A defined in Lemma 4.1.4 is closed in $\mathbb{R}^6$, so $A \cap V_2(\ell)$ is closed in $V_2(\ell)$, and $A \cap V_2(\ell)$ is precisely $\operatorname{im}(f_1)$. □

Corollary 4.1.7. *For $i = 1,2,3,4$, the map $[f_i] = \pi \circ f_i : [a,b] \to M_2(\ell)$ is an embedding.*

Proof. Since $f_i : [a,b] \to V_2(\ell)$ is an embedding it suffices to show that $\pi : \mathrm{im}(f_i) \to M_2(\ell)$ is an embedding. Since π is continuous it suffices to show that it is injective on $\mathrm{im}(f_i)$. If $\pi(f_i(t)) = \pi(f_i(s))$ then $f_i(t)$ and $f_i(s)$ must have the same diagonal lengths, so we have $t = \mathrm{diag}_{0,2}(f_i(t)) = \mathrm{diag}_{0,2}(f_i(s)) = s$, so $f_i(t) = f_i(s)$. □

Lemma 4.1.8. *If $a \neq 0$ then $M_2(\ell) = \bigcup_{i=1,2,3,4}[f_i]([a,b])$.*

Proof. Clearly $\bigcup_{i=1,2,3,4}[f_i]([a,b]) \subset M_2(\ell)$, so it remains to show $M_2(\ell) \subset \bigcup_{i=1,2,3,4}[f_i]([a,b])$. Let $[P] \in M_2(\ell)$ and choose a representative $P = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with $\mathbf{v}_2 = t\mathbf{e}_1$. Either $\mathbf{v}_1, \mathbf{v}_3 \in \mathbb{H}^2$, $\mathbf{v}_1, \overline{\mathbf{v}_3} \in \mathbb{H}^2$, $\overline{\mathbf{v}_1}, \mathbf{v}_3 \in \mathbb{H}^2$, or $\overline{\mathbf{v}_1}, \overline{\mathbf{v}_3} \in \mathbb{H}^2$. If $a \neq 0$ then $t \neq 0$, so by Remark 4.2.4 $P \in \{f_1(t), f_2(t), f_3(t), f_4(t)\}$, and thus $[P] \in \bigcup_{i=1,2,3,4}[f_i]([a,b])$. □

Lemma 4.1.9. *Given $i \neq j$ in $\{1,2,3,4\}$, if $f_i(t) = f_j(t)$ then $t \in \{a,b\}$.*

Proof. If $f_i(t) = f_j(t)$ then $\mathbf{v}_1(t) = \overline{\mathbf{v}_1(t)}$ or $\mathbf{v}_3(t) = \overline{\mathbf{v}_3(t)}$, so $\mathbf{v}_1(t) = \pm l_1\mathbf{e}_1$ or $\mathbf{v}_3(t) = \pm l_4\mathbf{e}_1$. If $\mathbf{v}_1(t) = \pm l_1\mathbf{e}_1$ then since $\mathbf{v}_2(t) = \mathbf{v}_1(t) + l_2\mathbf{p}_2$ we have $t\mathbf{e}_1 = \pm l_1\mathbf{e}_1 \pm l_2\mathbf{e}_1$, so $t \in \{l_1+l_2, |l_1-l_2|\}$. By Remark 4.1.3, if $t = l_1 + l_2$ then $t = b$ and if $t = |l_1 - l_2|$ then $t = a$. On the other hand if $\mathbf{v}_3(t) = \pm l_4\mathbf{e}_1$ then since $\mathbf{v}_2(t) = \mathbf{v}_3(t) - l_3\mathbf{p}_3$ we have $t\mathbf{e}_1 = \pm l_4\mathbf{e}_1 \pm l_3\mathbf{e}_1$ so $t \in \{l_3+l_4, |l_3-l_4|\}$. If $t = l_3 + l_4$ then $t = b$ and if $t = |l_3 - l_4|$ then $t = a$. □

Lemma 4.1.10. *Let $\ell \in \mathrm{int}(D_4)$ and let $[a,b] = \mathrm{diag}_{0,2}(V_2(\ell))$. Suppose $a \neq 0$. Then*

(i) *$a = |l_1 - l_2|$ if and only if $f_1(a) = f_3(a)$ and $f_2(a) = f_4(a)$;*

(ii) *$a = |l_3 - l_4|$ if and only if $f_1(a) = f_2(a)$ and $f_3(a) = f_4(a)$.*

(iii) *$b = l_1 + l_2$ if and only if $f_1(b) = f_3(b)$ and $f_2(b) = f_4(b)$;*

(iv) *$b = l_3 + l_4$ if and only if $f_1(b) = f_2(b)$ and $f_3(b) = f_4(b)$.*

Proof. We start by proving (i). The proofs of (ii), (iii), and (iv) will follow by permuting symbols. If $a = |l_1 - l_2|$ then $\mathbf{v}_2(a) = |l_1 - l_2|\mathbf{e}_1$, so $\mathbf{0}, \mathbf{v}_1(a), |l_1 - l_2|\mathbf{e}_1$ are the vertices of a triangle with edge lengths $l_1, l_2, |l_1 - l_2|$, and thus are colinear. Since $|l_1 - l_2| \neq 0$, $\mathbf{v}_1(a) = \pm l_1\mathbf{e}_1$ and thus $\mathbf{v}_1(a) = \overline{\mathbf{v}_1(a)}$. Therefore $f_1(a) = f_3(a)$ and $f_2(a) = f_4(a)$. Conversely, if $f_1(a) = f_3(a)$ and $f_2(a) = f_4(a)$, then $\mathbf{v}_1(a) = \overline{\mathbf{v}_1(a)}$ so $\mathbf{v}_1(a) = \pm l_1\mathbf{e}_1$. Thus $\mathbf{0}, \mathbf{v}_1(a), \mathbf{v}_2(a)$ are colinear vertices of a triangle with edge lengths, l_1, l_2, a, so $a = |l_1 - l_2|$. The proof of (ii) is identical to that of (i) after replacing l_1 by l_3, l_2 by l_4, f_1 by f_3, f_2 by f_4, and $\mathbf{v}_1$ by $\mathbf{v}_3$. The proof of (iii) is identical to that of (i) after replacing a with b and $l_1 - l_2$ with $l_1 + l_2$. The proof of (iv) is identical to that of (ii) after replacing a with b and $l_3 - l_4$ with $l_3 + l_4$. □

Proposition 4.1.11. *For all $\ell \in \operatorname{int}(D_4) \smallsetminus \left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}$, the moduli space $M_2(\ell)$ is a CW-complex of line segments.*

Proof. Let $\ell \in \operatorname{int}(D_4) \smallsetminus \left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}$. Since $\ell \neq \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ there is some permutation of the edge lengths so that $a \neq 0$, thus by Lemma 3.1.8 we may assume $a \neq 0$ without loss of generality. Lemma 4.1.8 says $M_2(\ell) = [f_1]([a,b]) \cup [f_2]([a,b]) \cup [f_3]([a,b]) \cup [f_4]([a,b])$. Lemma 4.1.6 says each $[f_i]$ is an embedding. Lemma 4.1.9 says that the $[f_i]([a,b])$ do not intersect on the interior of $[a,b]$. Together, these Lemmas present $M_2(\ell)$ as a CW-complex of line segments $[a,b]$, with characteristic maps $[f_i]$. □

In the following subsections we see how the depth of ℓ determines exactly which of the possibilities in Lemma 4.1.10 obtain, and thus determines the CW-complex $M_2(\ell)$. Recall the space D_4 and its walls from Figure 2.7, shown again in Figure 4.3

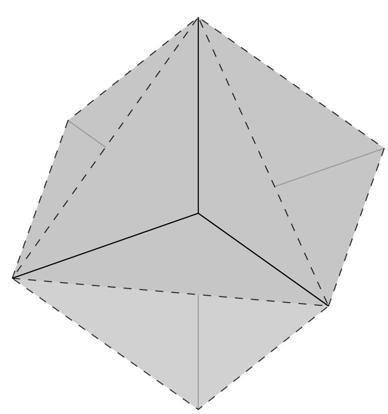


Figure 4.3: The space D_4 and some of its walls.

The walls of D_4 not on the border of D_4 are given by the equations

$$l_1 + l_2 = l_3 + l_4 \tag{4.3}$$

$$l_1 + l_3 = l_2 + l_4 \tag{4.4}$$

$$l_1 + l_4 = l_2 + l_3. \tag{4.5}$$

4.1.2 Depth 0

Lemma 4.1.12. *If $\ell \in D_4$ has depth 0, then $M_2(\ell)$ is homeomorphic to a circle or the disjoint union of two circles.*

Suppose depth$(\ell) = 0$. Since none of the wall equations (4.3), (4.4), (4.5) are satisfied, we have $l_1 + l_2 \neq l_3 + l_4$ and $|l_1 - l_2| \neq |l_3 - l_4|$. The four possibilities for the values of a and b, along with the gluing relations implied by Lemma 4.1.10 are listed here:

(i) $a = |l_1 - l_2| \neq |l_3 - l_4|$ and $b = l_1 + l_2 \neq l_3 + l_4 \implies f_1(a) = f_3(a) \neq f_2(a) = f_4(a)$ and $f_1(b) = f_3(b) \neq f_2(b) = f_4(b)$

(ii) $a = |l_3 - l_4| \neq |l_1 - l_2|$ and $b = l_3 + l_4 \neq l_1 + l_2 \implies f_1(a) = f_2(a) \neq f_3(a) = f_4(a)$ and $f_1(b) = f_2(b) \neq f_3(b) = f_4(b)$

(iii) $a = |l_1 - l_2| \neq |l_3 - l_4|$ and $b = l_3 + l_4 \neq l_1 + l_2 \implies f_1(a) = f_3(a) \neq f_2(a) = f_4(a)$ and $f_1(b) = f_2(b) \neq f_3(b) = f_4(b)$

(iv) $a = |l_3 - l_4| \neq |l_1 - l_2|$ and $b = l_1 + l_2 \neq l_3 + l_4 \implies f_1(a) = f_2(a) \neq f_3(a) = f_4(a)$ and $f_1(b) = f_3(b) \neq f_2(b) = f_4(b)$

The CW-complex in Case (i) is shown in Figure 4.4; the CW-complex in Case (ii) is identical to Case (i) after swapping f_2 with f_3. The CW-complex in Case (iii) is shown in Figure 4.5; the CW-complex in Case (iv) is identical to (iii) after swapping f_2 with f_3.

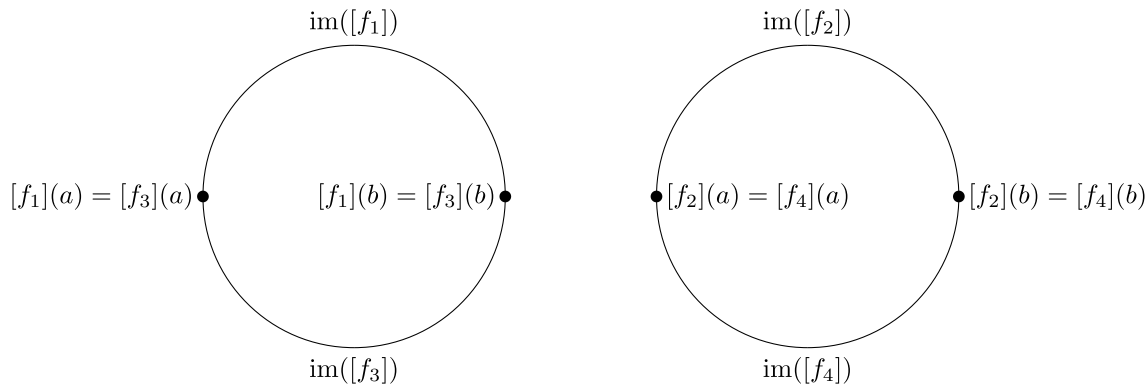


Figure 4.4: The CW-complex $M_2(\ell)$ when ℓ is in a depth-0 cell with $a = |l_1 - l_2|$ and $b = l_1 + l_2$.

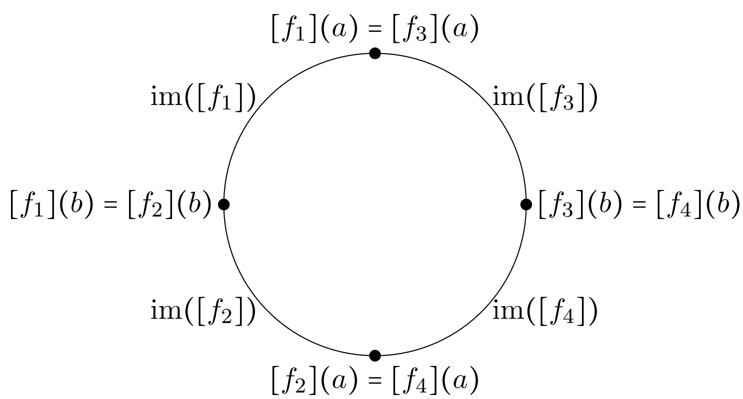


Figure 4.5: The CW-complex $M_2(\ell)$ when ℓ is in a depth-0 cell with $a = |l_1 - l_2|$ and $b = l_3 + l_4$.

4.1.3 Depth 1

Let ℓ be in a cell of depth 1 of D_4. Then exactly one of equations (4.3), (4.4), and (4.5) holds. By Lemma 3.1.8 it is enough to check the case when (4.3) holds. Then $b = l_1 + l_2 = l_3 + l_4$ so $f_1(b) = f_2(b) = f_3(b) = f_4(b)$. Also $|l_1 - l_2| \neq |l_3 - l_4|$, otherwise one of (4.4) or (4.5) would hold as well. Thus either $f_1(a) = f_3(a) \neq f_2(a) = f_4(a)$ if $a = |l_1 - l_2|$, or $f_1(a) = f_2(a) \neq f_3(a) = f_4(a)$ if $a = |l_3 - l_4|$. The CW-complex in the case $a = |l_1 - l_2|$ is shown in Figure 4.6; the case $a = |l_3 - l_4|$ is identical after swapping f_2 with f_3.

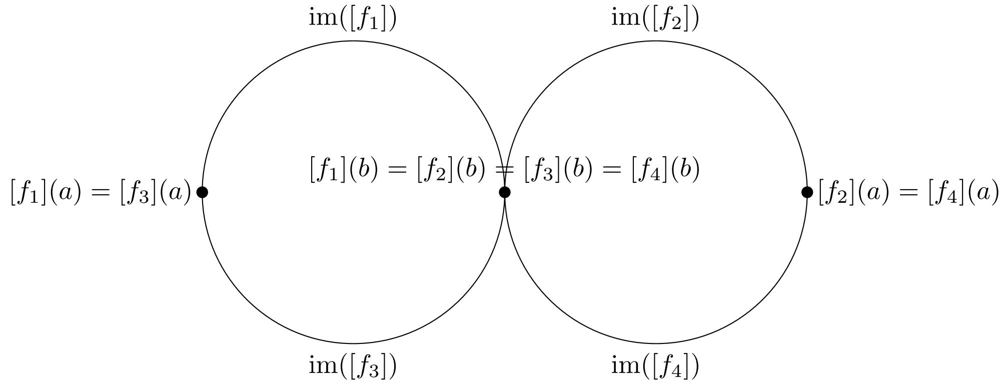


Figure 4.6: The CW-complex $M_2(\ell)$ when ℓ lies in a depth-1 cell with $l_1 + l_2 = l_3 + l_4$, and $a = |l_1 - l_2|$.

4.1.4 Depth 2

Given $\ell = (l_1, l_2, l_3, l_4)$ in a cell of depth 2 of D_4, exactly two of equations (4.3), (4.4), and (4.5) hold, and any two imply $b = l_1 + l_2 = l_3 + l_4$ and $a = |l_1 - l_2| = |l_3 - l_4|$. Thus we have $f_1(b) = f_2(b) = f_3(b) = f_4(b)$ and $f_1(a) = f_2(a) = f_3(a) = f_4(a)$. The corresponding CW-complex is shown in Figure 4.7.

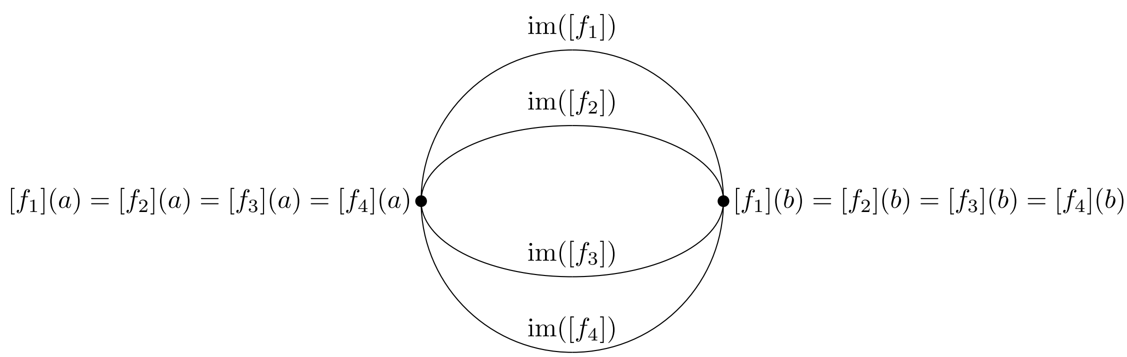


Figure 4.7: The polygon space $M_\ell(2)$ when ℓ lies in a depth-2 cell.

4.1.5 Depth 3

If depth$(\ell) = 0$ then ℓ satisfies all three of equations (4.3), (4.4), and (4.5), and thus $l_1 = l_2 = l_3 = l_4 = 1/4$, and in particular the line segment $[a, b]$ is $[0, 1/2]$. In this case the union of images of line segments $\bigcup_{i=1,2,3,4}[f_i]([a, b])$ does not make up all of $M_2(\ell)$.

We construct two additional embeddings of line segments into $M_2(\ell)$. Given $\theta \in [0, \pi]$ let $T_\theta \in SO(2)$ be counter-clockwise rotation through θ radians, and let $T_{-\theta} \in SO(2)$ be clockwise rotation through θ radians. Define two maps

$$g_\pm : [0, \pi] \to V_2(\ell)$$

$$\theta \mapsto \frac{1}{4}(\mathbf{e}_2, \mathbf{0}, T_{\pm\theta}(\mathbf{e}_2)),$$

where $\mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$. See Figure 4.8

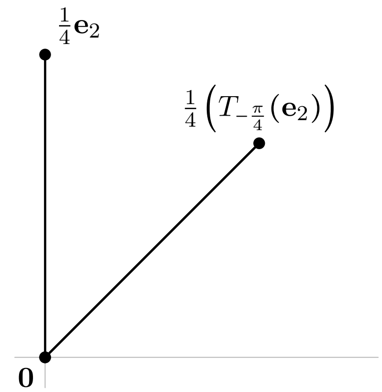


Figure 4.8: The polygon $g_i(\pi/4) \in V_d(\ell)$.

Clearly the maps $g_\pm$ are embeddings into $\mathbb{R}^6$ and thus into $V_2(\ell)$, so the maps $[g_\pm] : [a, b] \to M_2(\ell)$, $\theta \mapsto [g_\pm(\theta)]$ are embeddings as well. We also have $[g_+](0) = [g_-](0) = [f_2](0) = [f_3](0) = [(l\mathbf{e}_2, \mathbf{0}, l\mathbf{e}_2)]$ and $[g_+](\pi) = [g_-](\pi) = [f_1](0) = [f_3](0) = [(l\mathbf{e}_2, \mathbf{0}, -l\mathbf{e}_2)]$. Away from these points the pairwise intersections of $\mathrm{im}[g_\pm]$, $\mathrm{im}[f_i]$ are empty. We now have six embeddings of line segments

$$[0, \pi]_+,\ [0, \pi]_-,\ [a, b]_1,\ [a, b]_2,\ [a, b]_3,\ [a, b]_4$$

into $M_2(\ell)$ whose images comprise all of $M_2(\ell)$ and intersect only on their boundaries. The resulting CW-complex is shown in Figure 4.9.

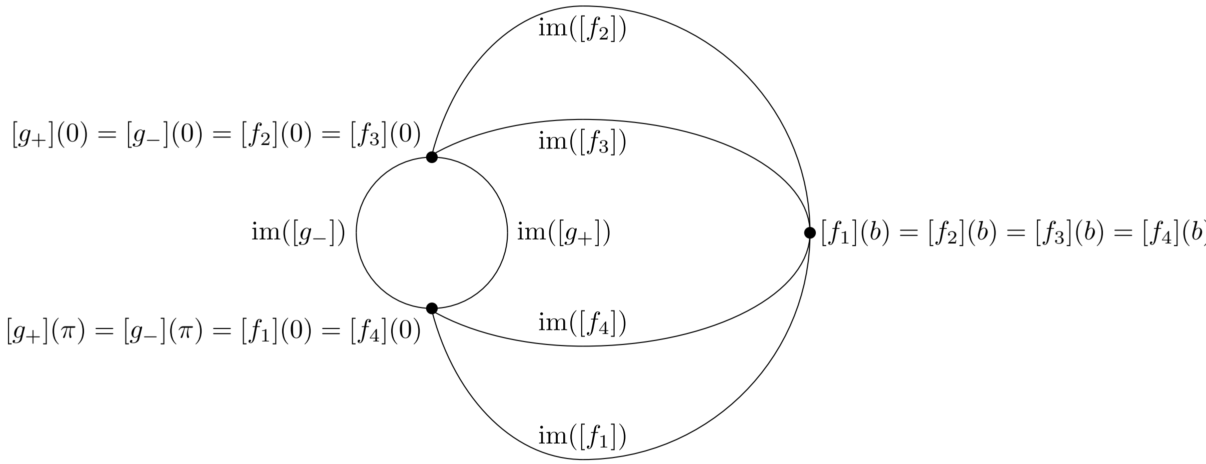


Figure 4.9: The CW-complex $M_2(\ell)$ when ℓ is the depth-3 cell $l_1 = l_2 = l_3 = l_4$.

4.2 4-gons in $\mathbb{R}^3$

Theorem 4.2.3 says that $M_3(\ell)$ is homeomorphic to a sphere for all $\ell \in \mathrm{int}(D_4)$. Compare this result to results developed in [HMM11] and [KM96]. In particular, Millson and Kapovich show that if $\ell \in \mathrm{int}(D_n)$ then $M_3(\ell)$ is complex-analytically isomorphic to the weighted quotient of $(S^2)^n$ by $PLS(2,\mathbb{C})$, and this quotient is isomorphic to S^2 when $n = 4$. [KM96]. We use the diagonals map to give a constructive proof in the special case $n = 4$ by showing that $M_3(\ell)$ is homeomorphic to two disks glued together along their boundary.

Proposition 4.2.1. *Let $\ell \in \mathrm{int}(D_4)$. Then $\mathbf{D}_3(\ell)$ is homeomorphic to a disk.*

Proof. Let $\ell \in \mathrm{int}(D_4)$ and recall from Lemma 4.1.2 that the set of $(0,2)$-diagonal lengths of polygons in $M_3(\ell)$ form a closed line segment $[\mathrm{diag}^3_{0,2}](M_3(\ell)) = [a,b]$. We prove the case $a = 0$ and $a \neq 0$ separately, starting with $a \neq 0$.

Let $\ell \in \mathrm{int}(D_4)$ and suppose $a \neq 0$. Given $t \in [a,b]$ recall the standard polygons $f_1(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t))$ and $f_2(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t), \overline{\mathbf{v}_3(t)})$ in $V_2(\ell)$ defined in Equations 4.2, and shown again in Figure 4.10.

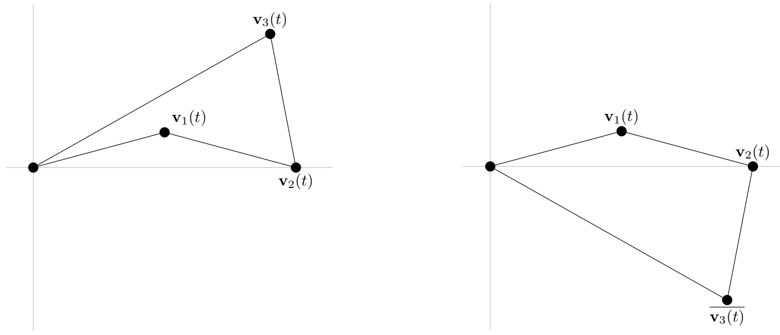


Figure 4.10: The polygons $f_1(t)$ and $f_2(t)$ for some $t \in [a, b]$.

Define the functions $c : [a, b] \to \mathbb{R}$, $t \mapsto \operatorname{diag}^2_{1,3}(f_1(t))$ and $d : [a, b] \to \mathbb{R}$, $t \mapsto \operatorname{diag}^2_{1,3}(f_2(t))$. See Figure 4.11.

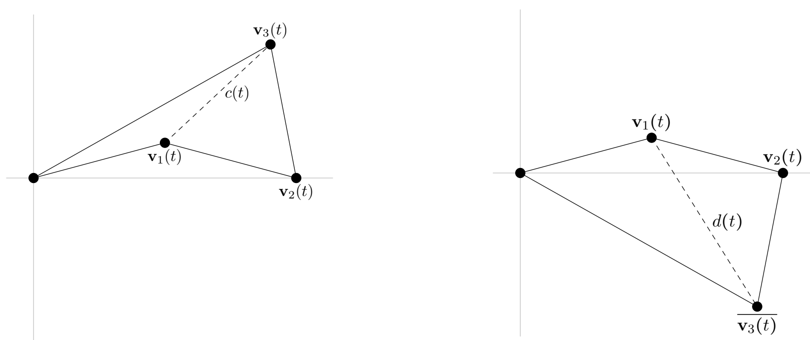
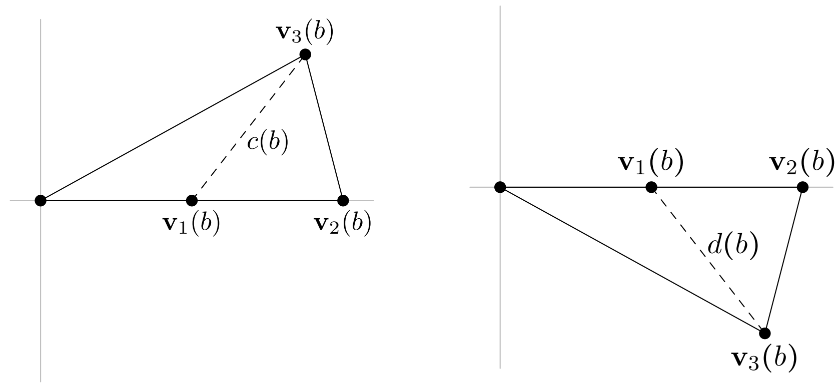
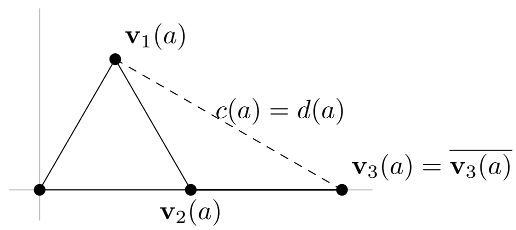


Figure 4.11: The polygons $f_1(t)$ and $f_2(t)$ for some $t \in [a, b]$, with their $(1, 3)$-diagonal lengths labeled.

The maps c and d are continuous since f_1, f_2 and $\operatorname{diag}^2_{1,3}$ are continuous. We observe that $c(t)$ is the distance between two points $\mathbf{v}_1(t)$, $\mathbf{v}_3(t)$ in the upper halfplane and $d(t)$ is the distance between $\mathbf{v}_1(t)$ and the reflection $\overline{\mathbf{v}_3(t)}$ in the lower halfplane. Thus $c(t) \leq d(t)$ for all $t \in [a, b]$, and $c(t) = d(t)$ if and only if one of $\mathbf{v}_1(t)$, $\mathbf{v}_3(t)$ lie on the intersection of the upper and lower halfplanes. Moreover, one of $\mathbf{v}_1(t)$, $\mathbf{v}_3(t)$ lie on the intersection of the upper and lower halfplanes if and only if $t \in \{a, b\}$. See Figure 4.12.



(a) If $\mathbf{v}_1(t)$ lies on the intersection of the upper and lower halfplanes, then $t \in \{a, b\}$ and $c(t) = d(t)$ In the example pictured we have $t = b$.



(b) If $\mathbf{v}_3(t)$ lies on the intersection of the upper and lower halfplanes, then $t \in \{a, b\}$ and $c(t) = d(t)$. In the example pictured we have $t = a$.

Figure 4.12: If $t \in \{a, b\}$ then either $\mathbf{v}_1(t)$ or $\mathbf{v}_3(t)$ lies on the intersection of the upper and lower halfplanes, and thus $c(t) = d(t)$. Conversely, if $c(t) = d(t)$ then either $\mathbf{v}_1(t)$ lies on the intersection of the upper and lower halfplanes (Figure 4.12a) or $\mathbf{v}_3(t)$ lies on the intersection of the upper and lower halfplanes (Figure 4.12b); in either case we have $t \in \{a, b\}$.

Thus c and d are continuous functions defined on the same closed interval $[a, b]$, with $c(t) < d(t)$ on (a, b) and $c(t) = d(t)$ on $\{a, b\}$, so the region in $R_\ell \subset \mathbb{R}^2$ bounded by the graphs of c and d is homeomorphic to a disc. See Figure 4.13.

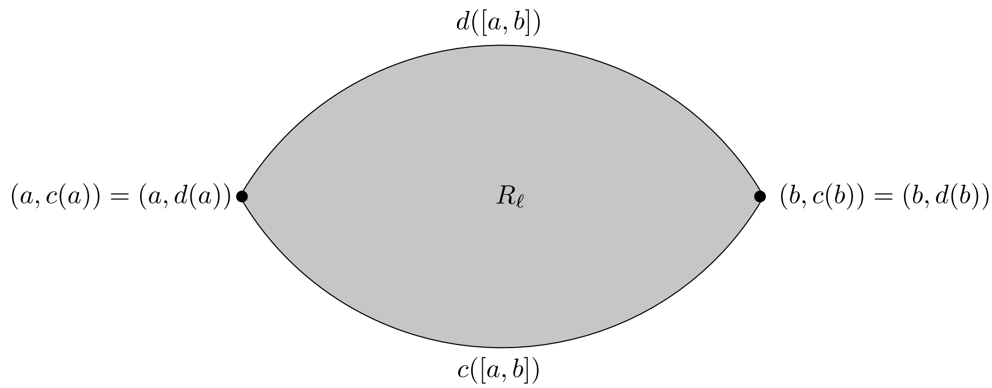


Figure 4.13: Given $\ell \in \mathrm{int}(D_4)$ the graphs of $c : [a,b] \to \mathrm{diag}^2_{1,3}(f_1(t))$ and $d : [a,b] \to \mathrm{diag}^2_{1,3}(f_2(t))$ bound a region R_ℓ homeomorphic to a disc if $a \neq 0$.

It remains to show that the diagonals image $\mathbf{D}_3(\ell)$ is equal to R_ℓ. Let $t \in [a,b]$ so that $f_1(t)$ and $f_2(t)$ are the polygons in $V_d(\ell)$ in Figure 4.10. Figure 4.14 shows their images in $V_3(\ell)$ under the map $a_2 : (\mathbb{R}^2)^4 \to (\mathbb{R}^3)^4$ from Section 3.4.

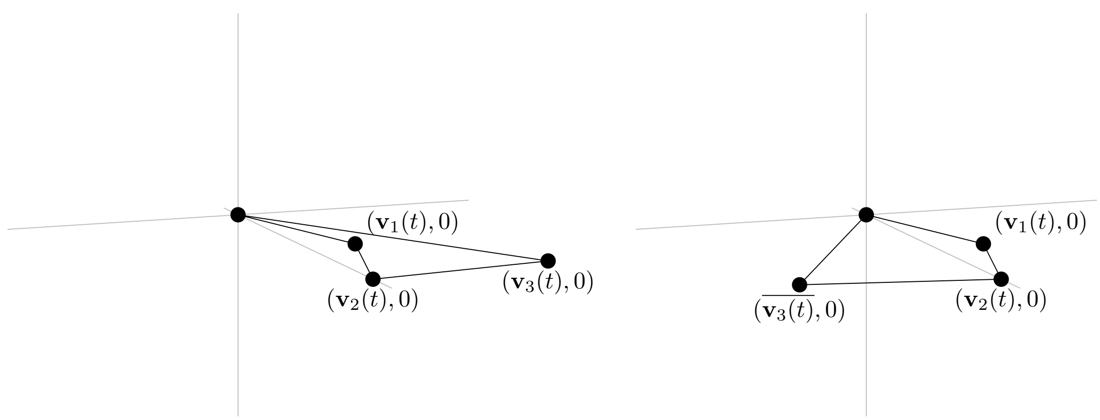


Figure 4.14: The polygons $a_2(f_1(t))$ and $a_2(f_2(t))$ in $V_3(\ell)$, where $f_1(t)$ and $f_2(t)$ are the polygons in $V_2(\ell)$ from Figure 4.10.

Since a_2 is an isometry (see Remark 3.4.3) we have $\mathrm{diag}^3_{1,3}(a_2(f_1(t))) = \mathrm{diag}^2_{1,3}(f_1(t)) = c(t)$ and $\mathrm{diag}^3_{1,3}(a_2(f_2(t))) = \mathrm{diag}^2_{1,3}(f_2(t)) = d(t)$ for all $t \in [a,b]$. Every $[P] \in M_3(\ell)$ with $(0,2)$-diagonal t has a representative $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ such that $\mathbf{v}_1 = (\mathbf{v}_1(t), 0)$ and $\mathbf{v}_2 = (\mathbf{v}_2(t), 0)$. Thus $\mathbf{v}_3$ must lie on the dashed circle in Figure 4.15.

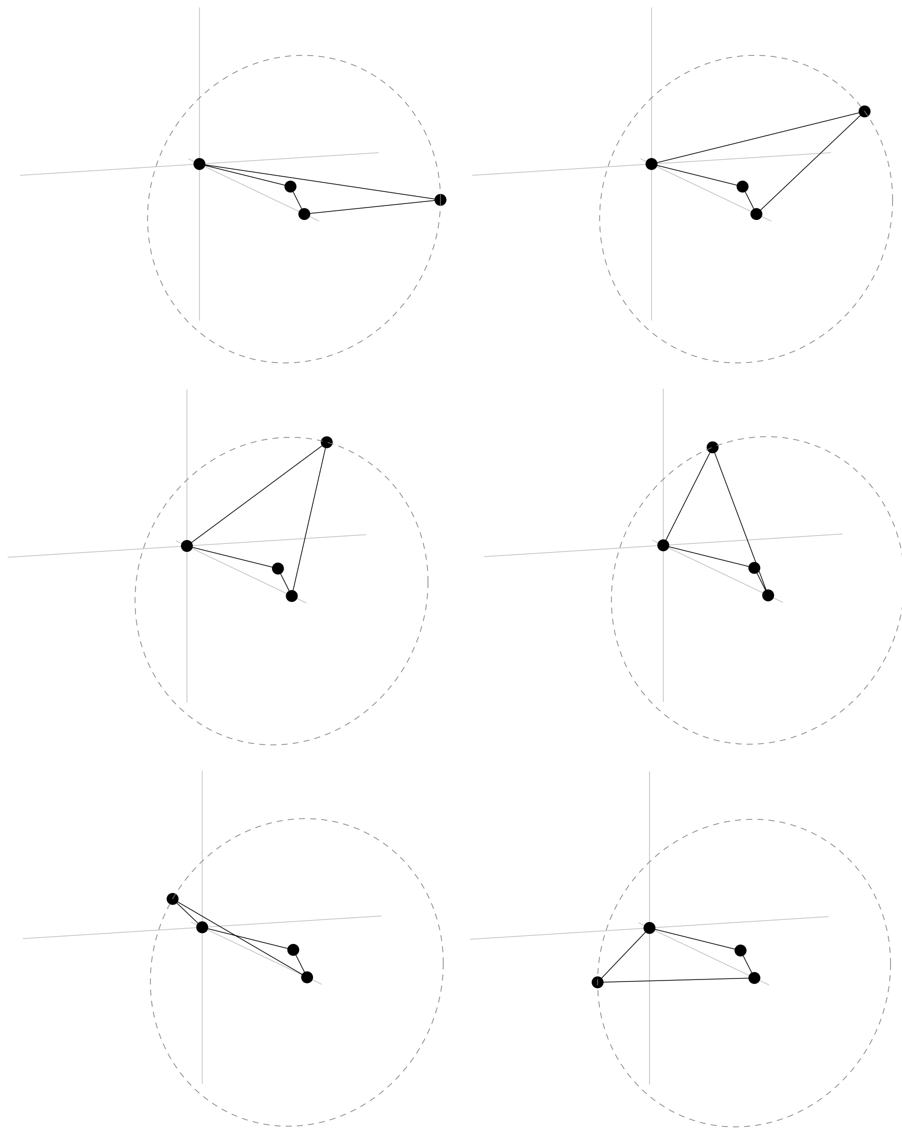


Figure 4.15: The set of all polygons in $(\text{diag}^3_{0,2})^{-1}(t)$ of the form $((\mathbf{v}_1(t), 0), (\mathbf{v}_2(t), 0), \mathbf{v}_3)$ is the set of $((\mathbf{v}_1(t), 0), (\mathbf{v}_2(t), 0), \mathbf{v}_3)$ such that $\mathbf{v}_3$ sits on a circle.

Conversely, every tuple of the form $((\mathbf{v}_1(t), 0), (\mathbf{v}_2(t), 0), \mathbf{v}_3)$ where $\mathbf{v}_3$ lies on the circle is a polygon in $V_3(\ell)$. Thus for all $t \in [a, b]$, the set of $(1,3)$-diagonal lengths of polygons in $M_3(\ell)$ with $(0,2)$-diagonal length t is the set of distances from $(\mathbf{v}_1(t), 0)$ to points on the dashed circle in Figure 4.15. This set of distances is precisely $[c(t), d(t)]$. Thus every point in the region R_ℓ is $[\text{diag}]([P])$ for some $[P] \in M_3(\ell)$, and every $[P] \in M_3(\ell)$ has $[\text{diag}]([P]) \in R_\ell$.

Most of what we have done so far applies to case $a = 0$. Let $\ell \in \mathrm{int}(D_4)$ such that $a = 0$. The functions $c : [0,b] \to \mathbb{R}$, $t \mapsto \mathrm{diag}^2_{1,3}(f_1(t))$ and $d : [0,b] \to \mathbb{R}$, $t \mapsto \mathrm{diag}^2_{1,3}(f_2(t))$ are still continuous functions with $c(t) < d(t)$ on $[0,b]$ and $c(b) = d(b)$. However $c(0) = |l_1 - l_3| \neq d(0) = l_1 + l_3$. See Figure 4.16.

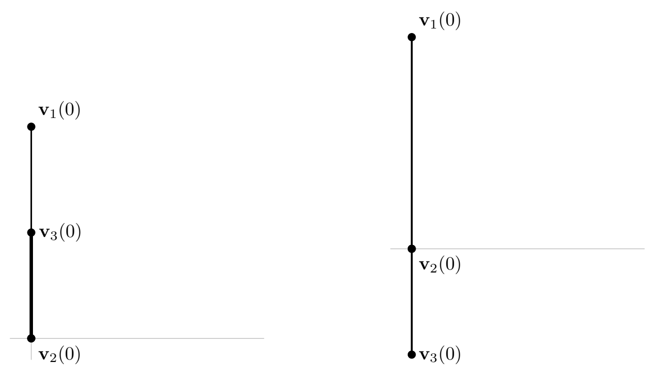


Figure 4.16: The polygons $f_1(0)$ (left) and $f_2(0)$ (right).

Thus the graphs of c and d do not bound a region in $\mathbb{R}^2$. However the graphs of c and d along with the line segment $[(0, c(0)), (0, d(0))]$ do bound a region $R_{\ell,a=0}$, and this region is again homeomorphic to a disk. See Figure

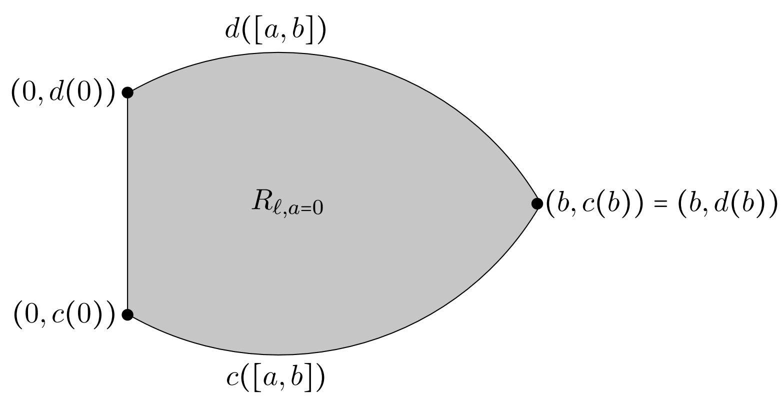


Figure 4.17: Given $\ell \in \mathrm{int}(D_4)$ such that $a = 0$, the graphs of c and d along with the line segment $[(0, c(0)), (0, d(0))]$ bound a region homeomorphic to a disk.

It remains to show that the diagonals image $\mathbf{D}_3(\ell)$ is equal to $R_{\ell,a=0}$, but again most of our work is done. If we let $\mathbf{D}_3(\ell)_{a\neq 0}$ denote the diagonals image of all polygons $P \in$

$M_3(\ell)$ with $\mathrm{diag}_{0,2}(P) \neq 0$, then $\mathbf{D}_3(\ell)_{a\neq 0}$ is equal to $R_{\ell,a=0}$ minus the boundary component $[(0,c(0),(0,d(0)]$. It remains to show that the diagonals image of all polygons $P \in M_3(\ell)$ with $\mathrm{diag}_{0,2} = 0$ is equal to $[(0,c(0),(0,d(0)]$, and this is clear from Figure

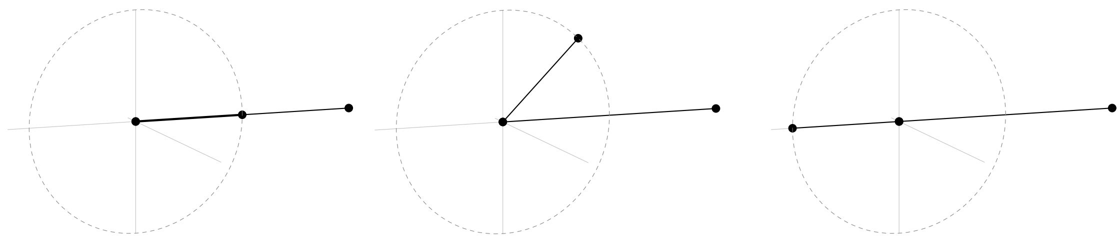


Figure 4.18: The diagonals image of all polygons $P \in M_3(\ell)$ with $\mathrm{diag}_{0,2} = 0$ is equal to $[(0,c(0)),(0,d(0))]$.

$\square$

Recall from Subsection 4.1.1 that $\mathbb{H}^1$ is defined to be the non-negative ray in $\mathbb{R}$ and $\mathbb{H}^2 = \mathbb{R} \times \mathbb{H}^1$ is the closed upper halfplane in $\mathbb{R}^2$. In addition let $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{H}^1$ be the closed upper halfspace in $\mathbb{R}^3$.

Lemma 4.2.2. *Let $B = (\mathbb{H}^2 \times \{0\}) \times (\mathbb{H}^1 \times \{(0,0)\}) \times \mathbb{H}^3 \subset (\mathbb{R}^3)^3$. Then for all $(t,s) \in \mathrm{diag}(V_3(\ell))$ with $t \neq 0$, the fiber $\mathrm{diag}^{-1}((t,s))$ contains a unique polygon $P \in B$. We denote this polygon $(\mathbf{v}_1(t,s),\mathbf{v}_2(t,s),\mathbf{v}_3(t,s))$.*

Proof. Let $(t,s) \in \mathrm{diag}(V_3(\ell))$ with $t \neq 0$. Given any $Q \in V_3(\ell)$ we have $O_3(Q) \subset V_3(\ell)$, and there is some $T \in O(3)$ such that $T(Q) \in B$. Thus if $P \in \mathrm{diag}^{-1}((t,s))$ there exists $T \in O(3)$ so that $T(P) \in B$ and since T is an isometry we have $T(P) \in \mathrm{diag}^{-1}((t,s))$, thus proving existence. To show uniqueness suppose $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3),(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3) \in B \cap (\mathrm{diag}^3)^{-1}((t,s))$. Then $\mathbf{v}_2 = \mathbf{w}_2 = t\mathbf{e}_1$, so both $\mathbf{0},\mathbf{v}_1,t\mathbf{e}_1$ and $\mathbf{0},\mathbf{w}_1,t\mathbf{e}_1$ are vertices of triangles with $\mathbf{v}_1,\mathbf{w}_1 \in \mathbb{H}^2 \times \{0\}$. Thus since $t \neq 0$, $\mathbf{v}_1 = \mathbf{w}_1$. Now, $(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$ and $(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}_3)$ are both triangular pyramids with base $\{\mathbf{0},\mathbf{v}_1,\mathbf{v}_2\} \subset \mathbb{R}^2 \times \{0\}$. Since the distances from the apex $\mathbf{v}_3$ to the base vertices $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$ are the same as the distances from the apex $\mathbf{w}_3$ to the base vertices $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$ (they are, respectively, l_4, s, and l_3), we conclude that $\mathbf{v}_3 = \mathbf{w}_3$. See Figure 4.19. $\square$

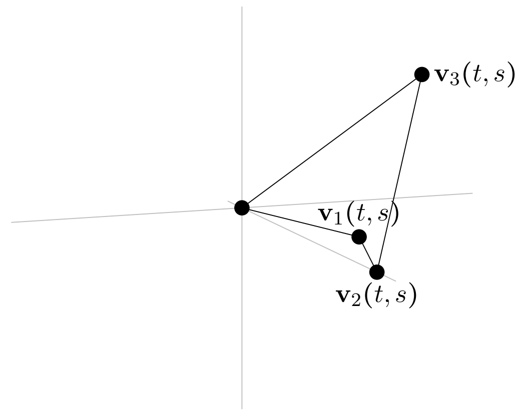


Figure 4.19: The unique polygon $P \in \operatorname{diag}^{-1}((t,s)) \cap B$ for some $(t,s) \in \operatorname{diag}(V_3(\ell))$.

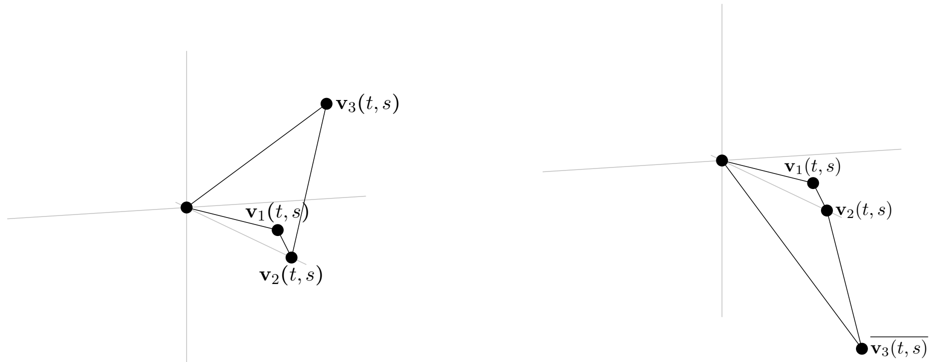
Theorem 4.2.3. *Given $\ell \in \operatorname{int}(D_4)$, the moduli space $M_3(\ell)$ is homeomorphic to S^2.*

Proof. Let $\overline{(x_1,x_2,x_3)} = (x_1,x_2,-x_3)$. Define the maps

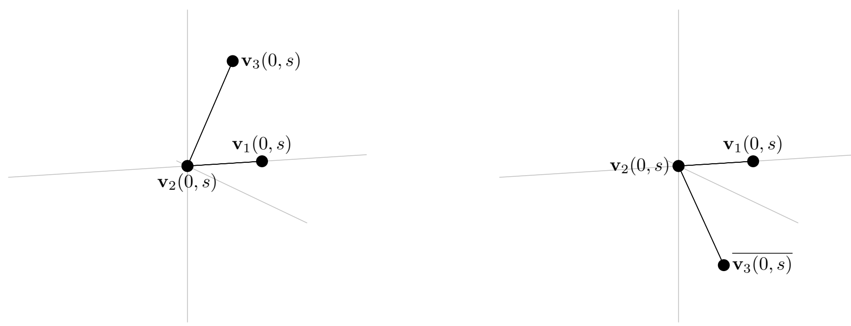
$$g_1 : \mathbf{D}_3(\ell) \to V_3(\ell) \qquad\qquad g_2 : \mathbf{D}_3(\ell) \to V_3(\ell)$$

$$(t,s) \mapsto (\mathbf{v}_1(t,s), \mathbf{v}_2(t,s), \mathbf{v}_3(t,s)) \qquad\qquad (t,s) \mapsto (\mathbf{v}_1(t,s), \mathbf{v}_2(t,s), \overline{\mathbf{v}_3(t,s)}),$$

where $(\mathbf{v}_1(t,s), \mathbf{v}_2(t,s), \mathbf{v}_3(t,s))$ is the unique polygon in Lemma 4.2.2 if $t \neq 0$, and if $t = 0$ set $\mathbf{v}_1(0,s) = l_1\mathbf{e}_2$, $\mathbf{v}_2(0,s) = \mathbf{0}$, and let $\mathbf{v}_3(0,s)$ be the point in the upper half of the coordinate hyperplane $\operatorname{Span}(\mathbf{e}_2,\mathbf{e}_3)$ uniquely determined by the closing conditions $|\mathbf{v}_k - \mathbf{v}_{k-1}| = l_k$. See Figure 4.20.



(a) The polygons $g_1((t,s)$ and $g_2((t,s))$ for some $(t,s) \in \operatorname{diag}(V_3(\ell))$ with $t \neq 0$.



(b) The polygons $g_1((0,s)$ and $g_2((0,s))$ for some $(0,s) \in \operatorname{diag}(V_3(\ell))$.

Figure 4.20: The polygons $g_1((t,s)$ and $g_2((t,s))$ for some $(t,s) \in \operatorname{diag}(V_3(\ell))$ with $t \neq 0$ (Figure 4.20a) and with $t = 0$ (Figure 4.20b).

Remark 4.2.4. *Since $g_1((t,s))$ is the unique polygon in $\operatorname{diag}^{-1}((t,s)) \cap B$, then $g_2((t,s)) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is the unique polygon with $(\mathbf{v}_1, \mathbf{v}_2, \overline{\mathbf{v}_3}) \in \operatorname{diag}^{-1}((t,s)) \cap B$.*

We claim that g_1 and g_2 are embeddings. It suffices to show that g_1 is an embedding. Note that g_1 has continuous inverse $\operatorname{diag} : \operatorname{im}(g_1) \to \mathbf{D}_3(\ell)$. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, and since $\mathbf{D}_3(\ell)$ is Hausdorff, it remains to show that $\operatorname{im}(g_1)$ is compact. Since $V_3(\ell)$ is compact it suffices to show that $\operatorname{im}(g_1)$ is closed in $V_3(\ell)$. But B is closed in $\mathbb{R}^9$ and $\operatorname{im}(g_1)$ is precisely the closed set $B \cap V_3(\ell)$.

Now we claim that $[g_i] = \pi \circ g_i : \mathbf{D}_3(\ell) \to M_3(\ell)$ is an embedding for $i = 1, 2$. Again, it suffices to show that $[g_1]$ is an embedding, and since $g_1 : [a,b] \to V_3(\ell)$ is an embedding it suffices to show that $\pi : \operatorname{im}(g_1) \to M_3(\ell)$ is an embedding. Since π is continuous it suffices to show that it is injective on $\mathrm{im}(g_1)$. If $\pi(g_1((t,s))) = \pi(g_1((u,v)))$ then $g_1((t,s))$ and $g_1((u,v))$ must have the same diagonal lengths, so we have $(t,s) = \mathrm{diag}(g_1((t,s))) = \mathrm{diag}(g_1((u,v)) = (u,v)$, so $g_1((t,s)) = g_1((u,v))$. If $t = 0$.

Finally $M_4(\ell) = [g_1](\mathbf{D}_3(\ell)) \cup [g_2](\mathbf{D}_e(\ell))$, and $[g_1]((t,s)) = [g_2]((t,s))$ if and only if t is on the boundary of $[a,b]$ and s is on the boundary of $[c_t, d_t]$; equivalently $[g_1]((t,s)) = [g_2]((t,s))$ if and only if $(t,s) \in \partial(\mathbf{D}_3(\ell))$. Thus $M_4(\ell)$ is the sphere $\mathbf{D}_3(\ell) \cup_{[g_i]} \mathbf{D}_3(\ell)$ $\square$

4.3 4-gons in $\mathbb{R}^4$

Theorem 4.3.1. *For all $\ell \in \mathrm{int}(D_4)$, and for all $d \geq 4$, $M_d(\ell)$ is homeomorphic to a disk.*

Proof. By Theorem 3.4.6, $M_d(\ell)$ is homeomorphic to $M_4(\ell)$ for all $d \geq 4$. By Theorem 3.3.4, $M_4(\ell)$ is homeomorphic to $[\mathrm{diag}^4](M_4(\ell))$. By Proposition 3.4.5, $[\mathrm{diag}^4](M_4(\ell)) = [\mathrm{diag}^3](M_3(\ell))$. By Proposition 4.2.1, $[\mathrm{diag}^3](M_3(\ell))$ is homeomorphic to a disk. $\square$

Chapter 5: Future directions

We first give a conjecture on the stratification of polygon spaces. Since $V_d(\ell)$ is an algebraic variety it is a stratified manifold. Since $SO(d)$ is compact, $M_d(\ell)$ is a semi-algebraic set and is thus also a stratified manifold.

Conjecture 5.0.1. *The stratifications of $V_d(\ell)$ and $M_d(\ell)$ by dimension are stratifications by manifolds.*

As evidence for Conjecture 5.0.1 we note that the diagonals image $\mathbf{D}_4(\ell)$ of $M_4(\ell)$, when $\ell = (1,1,1,1)$, is the region R in the non-negative orthant of $\mathbb{R}^2$ bounded by the circle of radius 2 centered at the origin. The interior of R is a 2-manifold and is equal to $[\mathrm{diag}^4](M_4^3(\ell))$. The interiors of the smooth boundary components of R are 1-manifolds and are together equal to $[\mathrm{diag}^4](M_4^2(\ell))$. The remaining boundary components are the points $(0,0)$, $(2,0)$, and $(0,2)$, which are together equal to $[\mathrm{diag}^4](M_4^1(\ell))$. Since $[\mathrm{diag}^4]$ is a homeomorphism onto its image in this case, the stratification $M_4(\ell) = M_4^1(\ell) \sqcup M_4^2(\ell) \sqcup M_4^3(\ell)$ is a stratification by manifolds.

The next conjecture seeks to generalize a result of Farber and Fromm. The walls $W_{\mathbf{a}}$ of D_n subdivide the closure of D_n into a polytopal complex $\mathcal{C}_n$. A *cell* of $\mathcal{C}_n$ is the interior of a polytope in $\mathcal{C}_n$. An *m-cell* of $\mathcal{C}_n$ is the interior of an m-dimensional polytope in $\mathcal{C}_n$. It is known that if ℓ and ℓ' lie in the same $(n-1)$-cell of $\mathcal{C}_n$, then $V_d(\ell)$ and $V_d(\ell')$ are $O(d)$-equivariantly diffeomorphic [FF13]. Conjecture 5.0.2 seeks to generalize this to cells of arbitrary dimension.

Conjecture 5.0.2. *If ℓ and ℓ' lie in the same cell of $\mathcal{C}_n$, then $V_d^\circ(\ell)$ and $V_d^\circ(\ell')$ are $O(d)$-equivariantly diffeomorphic.*

We observe that if ℓ and ℓ' lie in the same cell then $\mathrm{depth}(\ell) = \mathrm{depth}(\ell')$. Thus if Conjecture 5.0.2 holds we have decompositions $V_d(\ell) = V_d^\circ(\ell) \sqcup V_d^1(\ell)$ and $V_d(\ell) = V_d^\circ(\ell) \sqcup V_d^1(\ell)$, where $V_d^\circ(\ell)$ and $V_d^\circ(\ell')$ are diffeomorphic as manifolds, and $V_d^1(\ell)$ and $V_d^1(\ell')$ are isomorphic as varieties, since they are both the disjoint union of $\mathrm{depth}(\ell) = \mathrm{depth}(\ell')$ many spheres. The proof presented in [FF13] for the case $\mathrm{depth}(\ell) = 0$ uses cobordisms. A different proof, using Ehresmann's Fibration Theorem is presented in the masters thesis of Sean Lawton [Law03]. We conjecture that at least one of the proof methods, perhaps with some modifications, may be applied to the non-generic case $\mathrm{depth}(\ell) > 0$.

Our final conjecture refers to the stable limit of moduli spaces of ℓ-gons.

Conjecture 5.0.3. *The limit* $\varinjlim M_d(\ell)$ *is contractible.*

As evidence for Conjecture 5.0.3 we reference the case of 4-gons, in which $\varinjlim M_d(\ell)$ is homeomorphic to a disc.

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Biography

Jack Love graduated from John W. North High School in Riverside, CA in 1998. He attended the University of California San Diego from 1998 to 1999 and then took an extended academic hiatus. He returned to academia in 2005 at City College of San Francisco in California, and earned a BA in mathematics from UC Berkeley in 2010. He received his MA in mathematics from San Francisco State University in 2013, where he was a recipient of the NSF-funded Creating Momentum through Communicating Mathematics ($(CM)^2$) fellowship. As a $(CM)^2$ fellow he learned the value of mathematics research, mathematics education, and the intersection of the two. He began his PhD program in mathematics at George Mason University in Fairfax, VA in 2013. In 2015 he became Outreach Coordinator for the Mason Experimental Geometry Lab (MEGL), where the combination of research and education provided a natural extension to his time with $(CM)^2$. In 2018 he became an Instructor in the math department at GMU, and his role with MEGL shifted from Outreach Coordinator to Director of Outreach. Jack looks forward to a career of doing mathematics research and of sharing the joy and beauty of mathematics with others.